Sorting

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

Çağrı Çöltekin
ccoltekin@sfs.uni-tuebingen.de

University of Tübingen
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Why study sorting

- Sorting is one of the most studied (and common) problems in computing
- It is important to understand strengths and weaknesses of algorithms for sorting
- Many problems look like sorting. Learning sorting algorithms will help you solve other problems
- Available implementations are highly optimized (we are not just talking about asymptotic performance guarantees)
- In some (rare) cases, implementing your own sorting algorithm may be beneficial

Bubble sort

- We start with an ‘educational’ sorting algorithm
- Bubble sort is easy to understand, but performs bad – not used in practice
- We start from bubble sort, and see the improvements over it
- The idea is simple:
  - compare first two elements, swap if not in order
  - shift and compare the next two elements, again swap if needed
  - when you reach to the end, repeat the process from the beginning unless there were no swaps in the last iteration

Bubble sort

demonstration

89	67	88	12	72	76	93	57

```
swapped = True
n = len(seq)
while swapped:
  swapped = False
  for i in range(n - 1):
    if seq[i] > seq[i + 1]:
      seq[i], seq[i + 1]\
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Bubble sort

summary

- Worst case: $O(n^2)$
  $O(n^2)$ comparisons, $O(n^2)$ swaps
- Average case: $O(n^2)$
  $O(n^2)$ comparisons, $O(n^2)$ swaps
- Best case: $O(n)$
  $O(n)$ comparisons, $O(1)$ swaps
- Space complexity: $O(1)$

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- There are more concerns than performance
  - Many swaps
  - Bubble sort is *in-place*

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- Not practical – it is not used in practice

Insertion sort

- Insertion sort is one of the simpler sorting algorithms
- It is easy to understand, and reasonably fast for sorting short sequences
- On longer sequences, it performs worse than more advanced algorithms, like merge sort or quicksort (we will study those later)
- The general idea simple:
  - assume the elements arrive one by one, and we have a sorted sequence
  - insert the element to the correct position:
    - shift all elements larger than the new one to the right
    - place the new element in its correct place

Insertion sort

demonstration 1

i=1

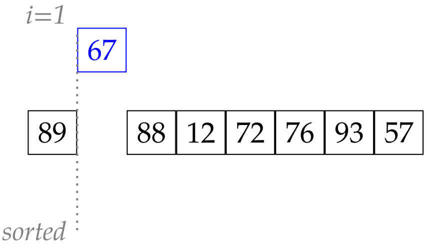
89	67	88	12	72	76	93	57

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
           and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 2

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Insertion sort

demonstration 3

i=1

67

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sorted

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Insertion sort

demonstration 4

i=2

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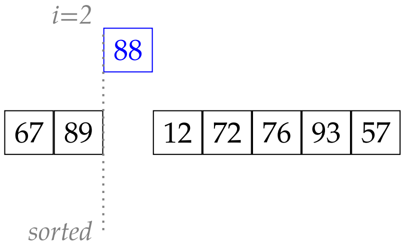
sorted

```
for i in range(1, len(seq)):
    cur = seq[i]
    j = i
    while seq[j - 1] > cur\
                and j in range(1,i+1):
        seq[j] = seq[j - 1]
        j -= 1
    seq[j] = cur
```

Insertion sort

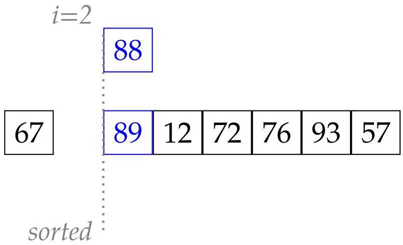
demonstration 5




```python
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 6




```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 7

i=3

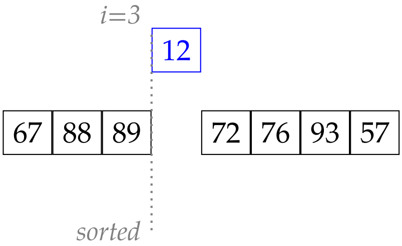
67	88	89	12	72	76	93	57

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 8




```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 9

i=3

12

67	88	89	72	76	93	57

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 10

i=4

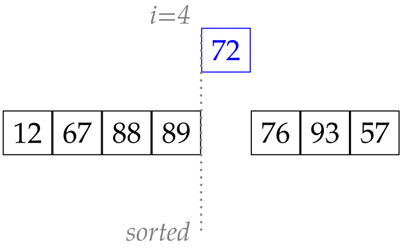
12	67	88	89	72	76	93	57

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
                and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

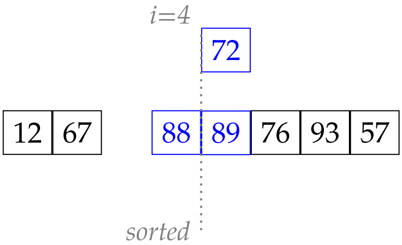
demonstration 11




```
for i in range(1, len(seq)):
    cur = seq[i]
    j = i
    while seq[j - 1] > cur\
               and j in range(1,i+1):
        seq[j] = seq[j - 1]
        j -= 1
    seq[j] = cur
```

Insertion sort

demonstration 12




```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
          and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 13

i=5

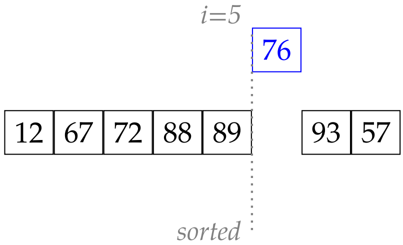
12	67	72	88	89	76	93	57

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
          and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

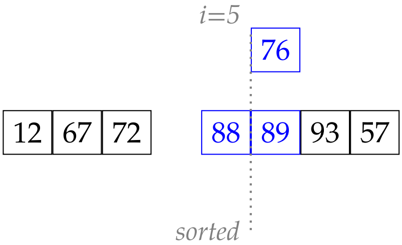
demonstration 14




```python
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

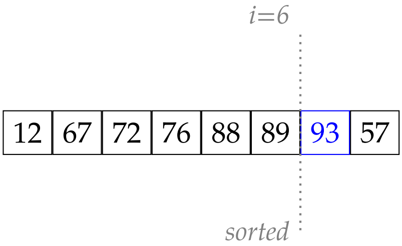
demonstration 15




```
for i in range(1, len(seq)):
    cur = seq[i]
    j = i
    while seq[j - 1] > cur\
              and j in range(1,i+1):
        seq[j] = seq[j - 1]
        j -= 1
    seq[j] = cur
```

Insertion sort

demonstration 16




```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
             and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 17

i=7

12	67	72	76	88	89	93	57

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
            and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 18

i=7

57

12	67	72	76	88	89	93

sorted

```
for i in range(1, len(seq)):
    cur = seq[i]
    j = i
    while seq[j - 1] > cur\
              and j in range(1,i+1):
        seq[j] = seq[j - 1]
        j -= 1
    seq[j] = cur
```

Insertion sort

demonstration 19

i=7

57

12		67	72	76	88	89	93

sorted

```
for i in range(1, len(seq)):
  cur = seq[i]
  j = i
  while seq[j - 1] > cur\
          and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

demonstration 20

12	57	67	72	76	88	89	93

sorted

```
for i in range(1, len(seq)):
    cur = seq[i]
    j = i
    while seq[j - 1] > cur\
                and j in range(1,i+1):
        seq[j] = seq[j - 1]
        j -= 1
    seq[j] = cur
```

Insertion sort

performance

- Worst case: $O(n^2)$
  $O(n^2)$ comparisons, $O(n^2)$ swaps
- Average case: $O(n^2)$
  $O(n^2)$ comparisons, $O(n^2)$ swaps
- Best case: $O(n)$
  $O(n)$ comparisons, $O(1)$ swaps
- Space complexity: $O(1)$
- In practice, insertion sort is faster than the bubble sort (and also selection sort)

```
for i in range(1, len(seq)):
  cur = seq[k]
  j = i
  while seq[j - 1] > cur\
           and j in range(1,i+1):
    seq[j] = seq[j - 1]
    j -= 1
  seq[j] = cur
```

Insertion sort

summary

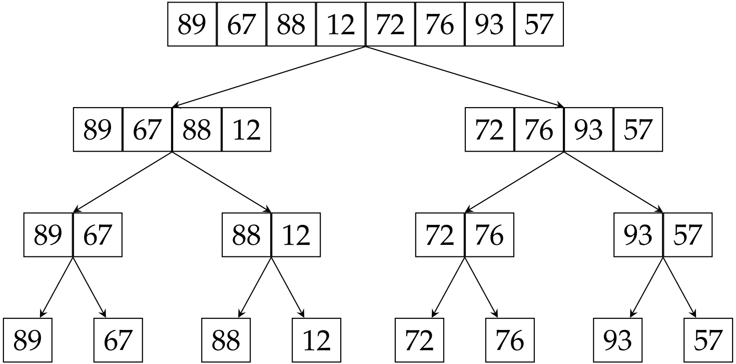
- Insertion sort is simple
- It is efficient for short sequences
- For long sequences it is much worse than more advanced algorithms like merge sort or quicksort (coming next)
- It is in-place
- It is *online*: it can sort items as they arrive
- It is *stable*: it does not swap elements with equal keys
- It is *adaptive*: faster if order of elements is closer to the sorted sequence

Merge sort
Introduction

- Merge sort is a divide-and-conquer algorithm for sorting
- It is relatively easy to understand (once you get your head around recursion)
- It has good asymptotic performance
- There are many practical cases where merge sort is used
- Basic idea is divide-and-conquer:
  - split the sequence
  - sort the subsequences
  - merge the sorted lists

Merge sort

demonstration – divide



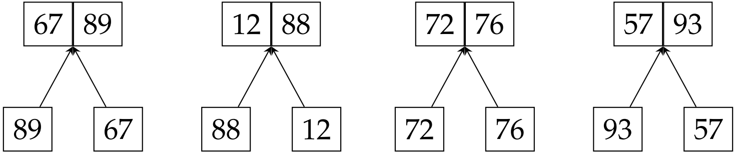
Merge sort

demonstration – combine

89	67	88	12	72	76	93	57

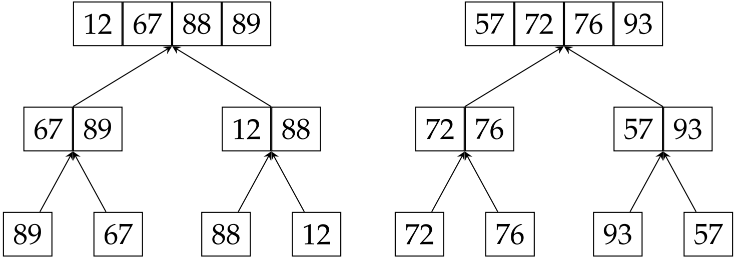
Merge sort

demonstration – combine



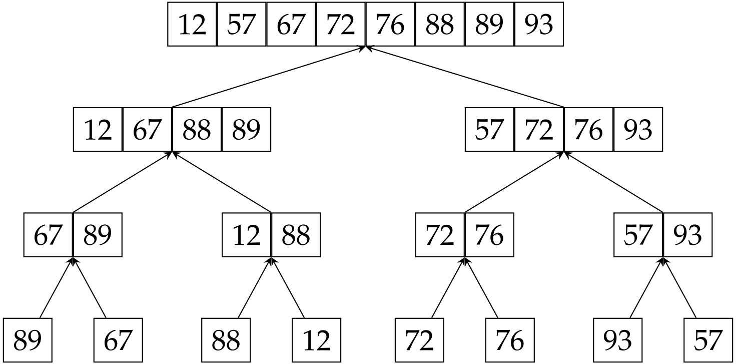
Merge sort

demonstration – combine



Merge sort

demonstration – combine

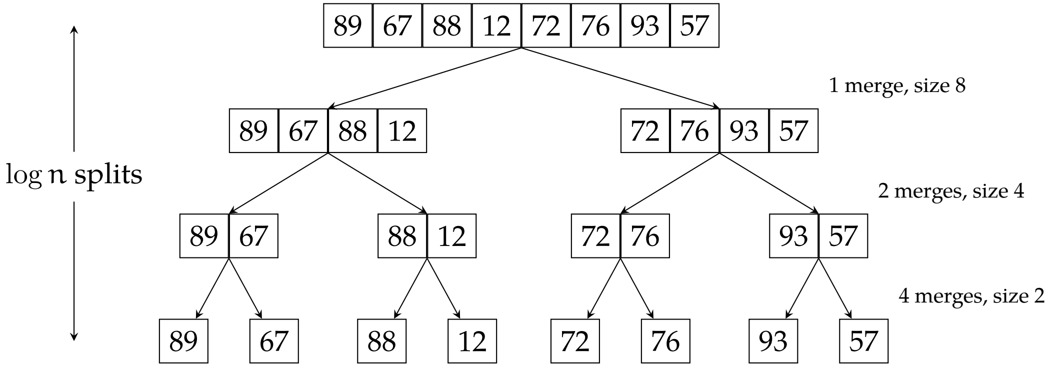


Merging sequences

```python
# s1, s2: sequences to be merged
# s: target sequence
i, j = 0, 0
n = len(s1) + len(s2)
while i + j < n:
  if j == len(s2) or \
      i < len(s1) and s1[i] < s2[j]:
    s[i+j] = s1[i]
    i += 1
  else:
    s[i+j] = s2[j]
    j += 1
```

- Keep two indices on both sequences, starting from the beginning
- Pick the smallest, place it in the target sequence
- The algorithm requires $O(n)$ steps to complete

Complexity of the merge sort



$$O(n) = n \log n$$

Merge sort

the implementation

```
def merge_sort(s):
  n = len(s)
  if n <= 1: return
  s1, s2 = s[:n//2], s[n//2:]
  merge_sort(s1)
  merge_sort(s2)
  merge(s1, s2, s)
```

- Once we have `merge()`, the rest is trivial:
  - Split the array into two
  - Recursively sort both sides
  - Stop when the input is length 1

Merge sort: summary

- Straightforward application of divide-and-conquer
- Worst case $O(n \log n)$ complexity (best/average cases are the same)
- Merge sort is not in-place: requires $O(n)$ additional space
- It is particularly useful for settings with low random-access memory, or sequential access
- Merge sort is stable
- It is a well studied algorithm, there are many variants (in-place, non-recursive)

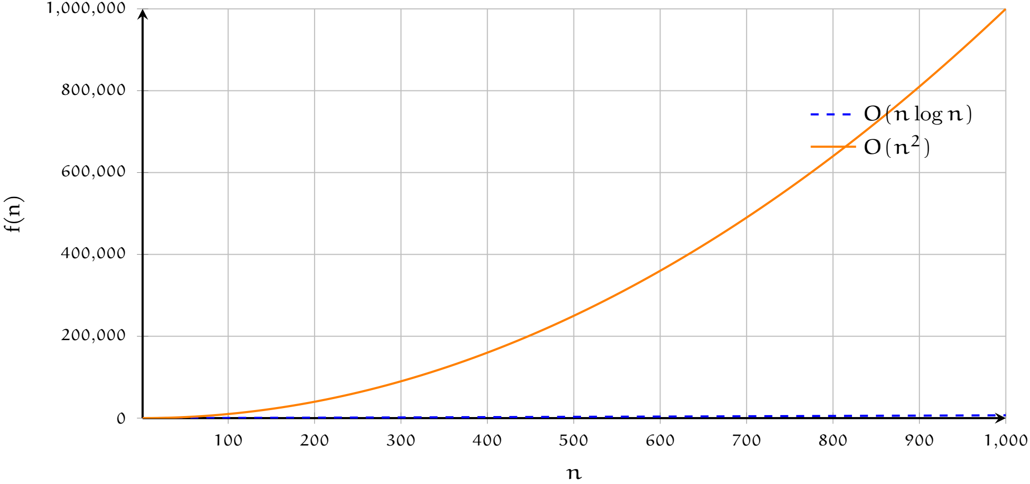
A short divergence to complexity

the difference between $O(n^2)$ and $n \log n$

n	$n \log n$	n^2
2	2	4
8	24	64
64	384	4096
1K	10 240	1 048 576
1M	20 971 520	1 099 511 627 776
1G	32 212 254 720	1 152 921 504 606 846 976

A short divergence to complexity

the difference between $O(n^2)$ and $n \log n$



Quicksort

introduction

- Quicksort is another popular divide-and-conquer sorting algorithm
- The main difference from the merge sort is that big the part of the work is done before splitting
- Its worse time complexity is $O(n^2)$, but in practice it performs better than merge sort on average
- General idea: pick a pivot p, and divide the sequence into three parts as
  - L smaller than a particular element p
  - G larger than a particular element p
  - E equal to a particular element p
- sort L and G recursively
- combination is simple concatenation

Quicksort

demonstration – divide

89	67	88	12	57	76	93	72

At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

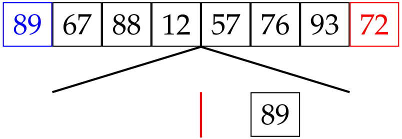
89	67	88	12	57	76	93	72

At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

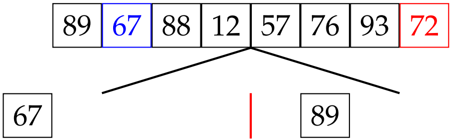


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

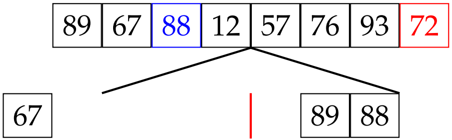


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

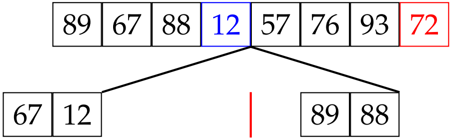


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

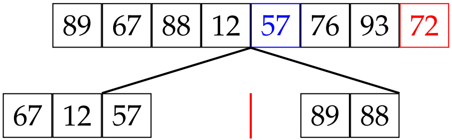


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

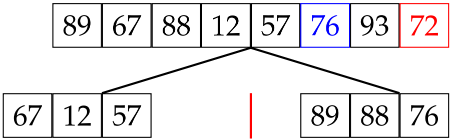


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

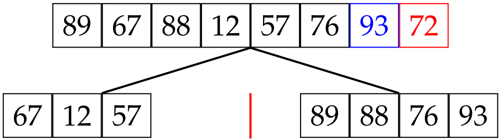


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

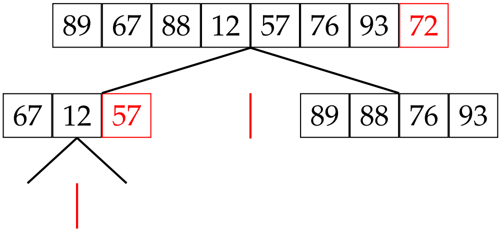


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide



At each divide step

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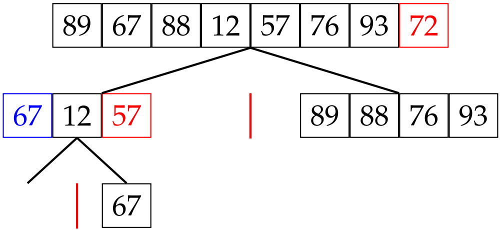
Quicksort

demonstration – divide



At each divide step

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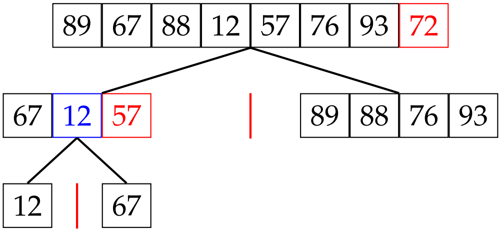
Quicksort

demonstration – divide

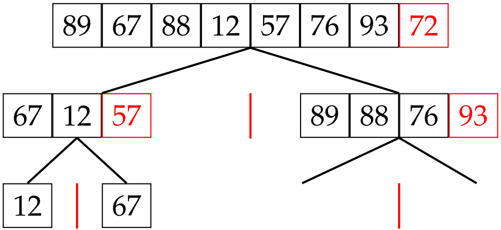


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

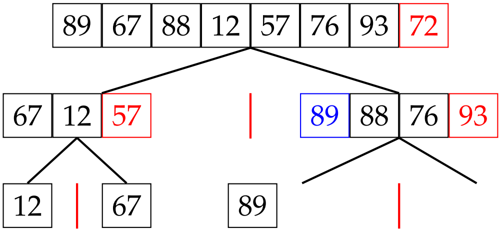


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide



At each divide step

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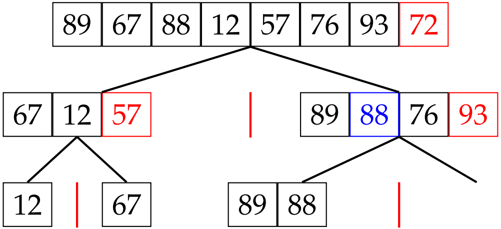
Quicksort

demonstration – divide

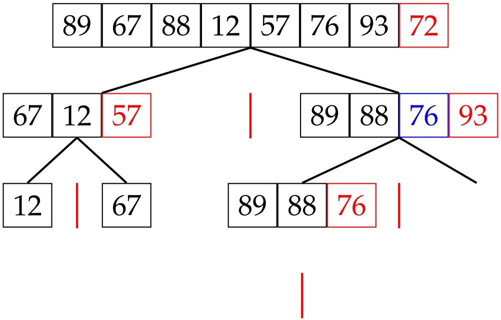


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide



At each divide step

- Pick a pivot
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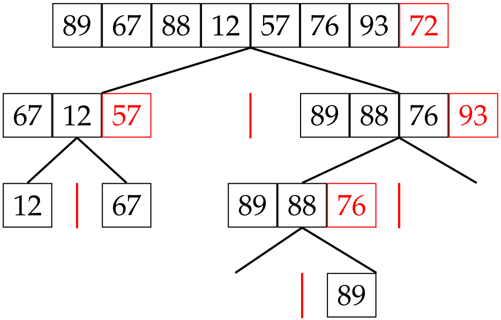
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide



At each divide step

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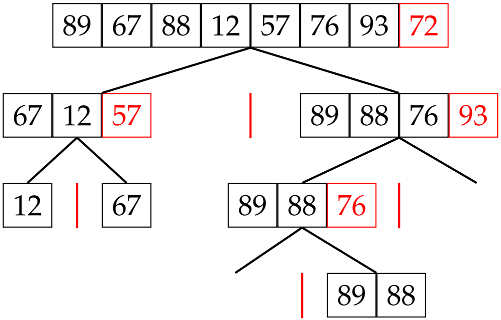
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide



At each divide step

- Pick a pivot
- Recursively call quicksort twice
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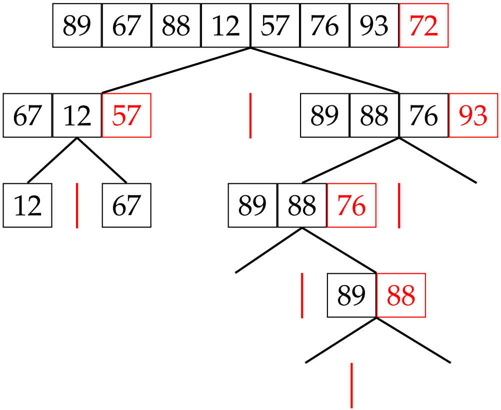
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide



At each divide step

- Pick a pivot
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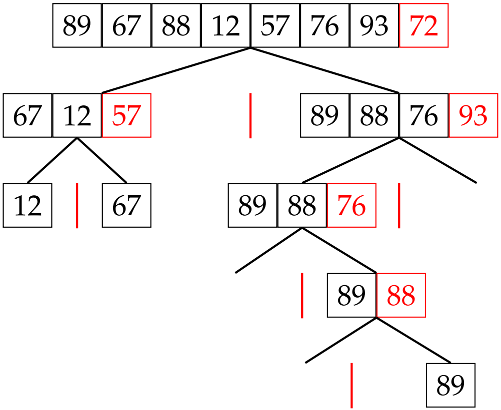
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – divide

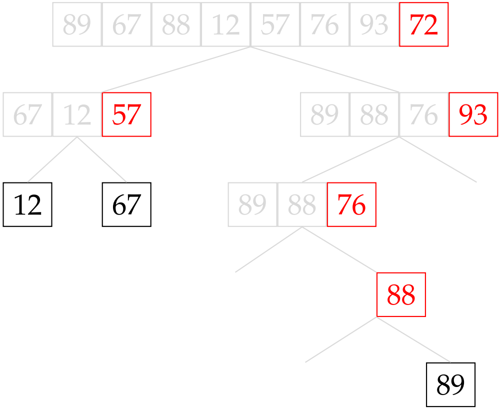


At each divide step

- Pick a pivot
- Recursively call quicksort twice
  - L for items less than the pivot
  - G for items greater than the pivot
- $O(n)$ operations

Quicksort

demonstration – combine

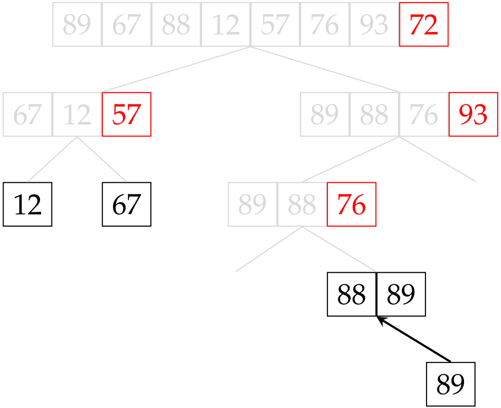


At each combine step:

- Simply concatenate
  - L the sorted items less than p
  - E items equal to p
  - G the sorted items greater than p
- No need for $O(n)$ merging

Quicksort

demonstration – combine



At each combine step:

- Simply concatenate
  - L the sorted items less than p
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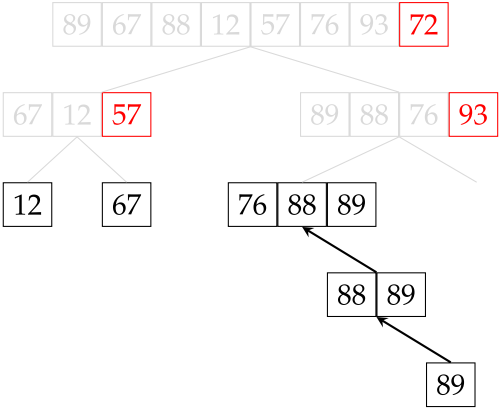
- No need for $O(n)$ merging

Quicksort

demonstration – combine

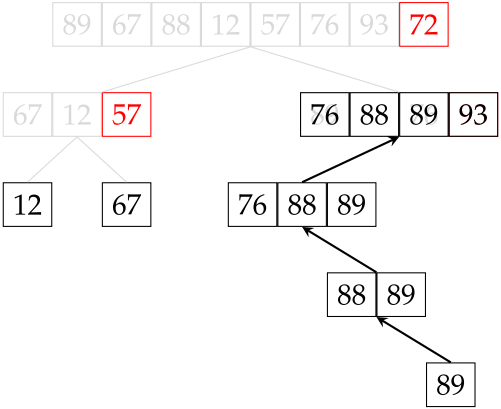


At each combine step:

- Simply concatenate
  - L the sorted items less than p
  - E items equal to p
  - G the sorted items greater than p
- No need for $O(n)$ merging

Quicksort

demonstration – combine



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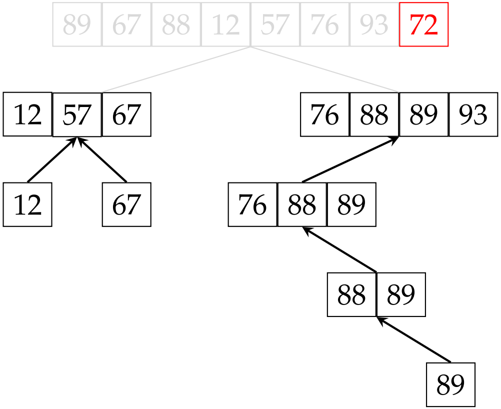

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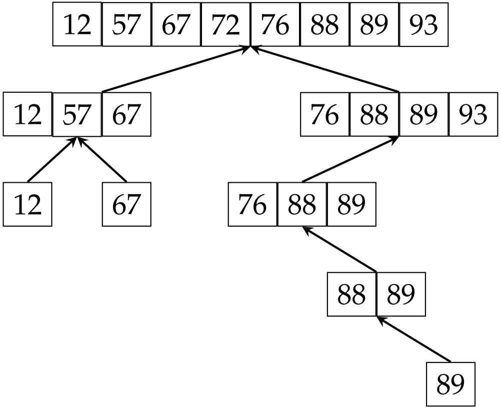

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Quicksort

Python three-liner implementation

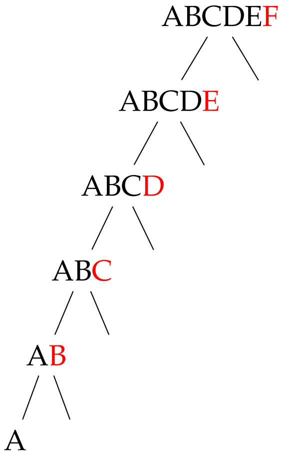
```
def qsort(seq):
  if len(seq <= 1): return seq
  return qsort([x for x in seq if x <  seq[-1]])\ # < p
       +       [x for x in seq if x == seq[-1]]\  # = p
       + qsort([x for x in seq if x >  seq[-1]])  # > p
```

- Practical implementations are not very different
- Common improvements include
  - in-place sorting
  - selecting the pivot more carefully

Quicksort

analysis

- Similar to the merge sort, quicksort performs $O(n)$ operations at each level in recursion
- The overall complexity is proportional to $n \times \ell$, where ℓ is depth of the tree
- The recursion tree of merge sort is balanced, so depth is $\log n$.
- For quicksort, we do not have a balanced-tree guarantee
- In the worst case, the depth of the tree can be n, resulting in $O(n^2)$ complexity



Quicksort

average-case complexity and preventing the worst case

- Worst case of the quicksort is when the input sequence is sorted
- If the input sequence is (approximately) random, the *expected* number of elements in each divide is $n/2$
- To reduce the probability of worst case, *randomized* quicksort picks the pivot randomly
- Best case happens if we pick the *median* of the sequence as the pivot, but finding median already requires $O(n \log n)$ (or $O(n)$, but not very practical)
- A common approach is picking three values (typically first, middle and last) from the sequence, and selecting the 'median of three' as the pivot

Quicksort

summary

- Complexity: $O(n \log n)$ average, $O(n^2)$ worst
- Despite its worst-case $O(n^2)$ complexity, quicksort is faster than merge sort on average (in practice)
- The algorithm can easily be implemented in-place (in-place version is more common)
- Quicksort is not stable
- Quicksort is one of the most-studied algorithms: there are many variants, its properties are well known

Sorting algorithms so far, and the lower bound

Algorithm	worst	average	best	memory	in-place	stable
Bubble sort	n^2	n^2	n	1	yes	yes
Insertion sort	n^2	n^2	n	1	yes	yes
Merge sort	$n \log n$	$n \log n$	$n \log n$	n	no	yes
Quicksort	n^2	$n \log n$	$n \log n$	$\log n$	yes	no

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- If our sorting algorithms requires comparing individual elements, the answer turns out to be 'no'

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- Lower bound of worst-case sorting is $\Omega(n \log n)$

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- Can we do better than $O(n \log n)$?
- If our sorting algorithms requires comparing individual elements, the answer turns out to be 'no'
- Lower bound of worst-case sorting is $\Omega(n \log n)$
- In some special cases, linear-time complexity is possible

Bucket sort

introduction

- Bucket sort puts elements of the input into a pre-defined number of ordered ‘buckets‘
- Elements in each bucket is sorted (typically using insertion sort)
- We can than retrieve the sorted elements by visiting each bucket
- The bucket sort *does not compare elements* to each other when deciding which bucket to place them
- In special cases, this results in $O(n)$ worst-case complexity

Bucket sort

demonstration

89	67	88	12	57	76	93	72	64	53	89	54	43	92	47	21	4

0-9
10-19
20-29
30-39
40-49
50-59
60-69
70-79
80-89
90-99

Bucket sort

demonstration

89	67	88	12	57	76	93	72	64	53	89	54	43	92	47	21	4

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- While placing the elements into the buckets, no comparisons between the keys
- Inside the buckets worst-case $O(n^2)$ (insertion sort)
- What if we had as many buckets as the keys?

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- While placing the elements into the buckets, no comparisons between the keys
- Inside the buckets worst-case $O(n^2)$ (insertion sort)
- What if we had as many buckets as the keys?
  - n insertion operations
  - n retrieval operations
  - $O(n)$ sorting time

Radix sort

- In a large number of cases, we want to sort objects with multiple keys
- In such cases, we define the order of key pairs as
  $(k_1, l_1) < (k_2, l_2)$ if $k_1 < k_2$, or $k_1 = k_2$ and $l_1 < l_2$
- This definition can be generalized to key tuples of any length
- This ordering is known as *lexicographic* or dictionary order
- Radix sort is the name for the technique that uses multiple stable bucket sorts for this purpose

Summary

- Sorting is an important and well-studied computational problem
- Most sorting algorithms/applications used in practice are highly optimized, often based on multiple basic algorithms
- Naive sorting algorithms run in $O(n^2)$ time
- Lower bound on worst-case sorting time is $\Omega(n \log n)$, divide-and-conquer algorithms achieve this
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 12)
- And a fun way to see sorting in action: `https://www.youtube.com/user/AlgoRythmics`

Next:

- Trees
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 8)

Acknowledgments, credits, references

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

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