

String edit distance

Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

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Edit distance

- In many applications, we want to know how similar (or different) two string are
 - Comparing two files (e.g., source code)
 - Comparing two DNA sequences
 - Spell checking
 - Approximate string matching
 - Determining similarity of two languages
 - Machine translation
- The solution is typically formulated as the (inverse) cost of obtaining one of the strings from the other through a number of *edit operations*
- Once we obtain the optimal edit operations, we may (depending on the edit operations) also be able to determine the optimal alignment between the strings

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Hamming distance

a simple distance metric between two sequences

- The Hamming distance measures number of different symbols in the corresponding positions

h	y	g	i	e	n	e
h	i	g	i	e	n	e

0 + 1 + 0 + 0 + 0 + 0 + 0 = 1

h	y	g	i	e	n	e
h	i	y	g	e	i	n

0 + 1 + 1 + 1 + 0 + 1 + 1 = 5

- Very easy/efficient calculation
- But cannot handle sequences of different lengths (consider *hygiene* – *hygetine*)

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Introduction/motivation Longest common subsequence Levenshtein distance

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A family of edit distance problems

- The same overall idea applies to a number of well-known problems/solutions that differ in the type of operations allowed
 - Hamming distance: only replacements
 - Longest common subsequence (LCS) insertions and deletions
 - Levenshtein distance: insertions, deletions and substitutions
 - Levenshtein-Damerau distance: insertions, deletions and substitutions and transpositions (swap) of adjacent symbols
- Naïve solutions to all (except Hamming distance) have exponential time complexity
- Polynomial-time solution can be obtained using *dynamic programming*

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Longest common subsequence (LCS)

Problem definition

- A subsequence is an order-preserving (but not necessarily contiguous) sequence of symbols from a string (a version of the sequence where zero or more elements are removed)
 - hgg, gni, gene, hni, gne are subsequences of *hygiene*
- Note that a subsequence does not have to be a substring (substrings are contiguous)
 - hgg, gnic, ene are substrings of *hygiene*
- The LCS of two strings is the longest string that is a subsequence of both strings
 - LCS(*hygiene*, *hygien*) = *hygien*
 - LCS(*hygiene*, *hygiene*) = *hygiene* / *hygena*
- LCS is exactly the problem solved by the UNIX *diff* utility
- It has wide-ranging applications from source-code comparison to bioinformatics (e.g., DNA sequencing)

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LCS: a naive solution

- A simple solution is:
 - Enumerate all subsequences of the first string
 - Check if it is also a subsequence of the second string
- There are exponential number of subsequences of a string
 - the string *abc* has 8 subsequences:
 - abc*: nothing removed
 - ab, ac, bc*: individual elements are removed
 - a, b, c*: length-2 subsequences are removed
 - ε* (empty string): *abc* removed
 - For *abcde*, we have subsequences of *abc* once with, and once without *d*
 - Each additional symbol doubles the number of subsequences
- For strings of size *n* and *m*, the complexity of the brute-force algorithm is $O(2^m m)$

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LCS: recursive definition

- Consider two strings *Xx*, *Yy* and their LCS *Zz* (*X*, *Y*, *Z* are possibly empty strings, *x*, *y*, *z* are characters)
- If *x = y*, then this character has to be part of the LCS, *x = y = z*, and *Z* must be the LCS of *X* and *Y*
- If *x ≠ y*, there are three cases
 - x ≠ y ≠ z*: *Zz* is also the LCS of *X* and *Y*
 - x = z*: *Zz* is also the LCS of *Xx* and *Y*
 - y = z*: *Zz* is also the LCS of *X* and *Yy*
- This leads to following recursive definition:

$$\text{LCS}(Xx, Yy) = \begin{cases} \text{LCS}(X, Y)z & \text{if } x = y \\ \text{longer of } \text{LCS}(Xx, Y) \text{ and } \text{LCS}(X, Yy) & \text{otherwise} \end{cases}$$

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LCS: divide-and-conquer



- Note the *repeated computation*

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LCS: dynamic programming

general sketch

- To calculate $\text{LCS}(X_{1..i}, Y_{1..j})$, the LCS of string *X* up to index *i*, and the LCS of string *Y* up to index *j*, we (may) need
 - $\text{LCS}(X_{1..i-1}, Y_{1..j})$
 - $\text{LCS}(X_{1..i}, Y_{1..j-1})$
 - $\text{LCS}(X_{1..i-1}, Y_{1..j-1})$
- If we store the above three values, we need only $i \times j$ operations
- In the standard dynamic programming algorithm, we store the length of the LCS, in a matrix ℓ , where $\ell_{i,j}$ is the length of the $\text{LCS}(X_{1..i}, Y_{1..j})$
- Once we fill in the matrix, the $\ell_{n,m}$ is the length of the LCS
- We can trace back and recover the LCS using the dynamic programming matrix

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LCS with dynamic programming

demonstration

		0	1	2	3	4	5	6	7	8
		c	h	i	y	g	e	i	n	e
0	c	0	0	0	0	0	0	0	0	0
1	h	0	1	1	1	1	1	1	1	1
2	y	0	1	1	2	2	2	2	2	2
3	g	0	1	1	2	3	3	3	3	3
4	i	0	1	2	2	3	3	4	4	4
5	e	0	1	2	2	3	4	4	4	5
6	n	0	1	2	2	3	4	4	5	5
7	e	0	1	2	2	3	4	4	5	6

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Complexity of filling the LCS matrix

```
1 = np.zeros(shape=(n+1,m+1))
for i in range(1, n):
    for j in range(1, m):
        if X[i] == Y[j]:
            l[i, j] = l[i-1, j-1] + 1
        else:
            l[i, j] = max(l[i-1, j], l[i, j-1])
```

- Two loops up to *n* and *m*, the time complexity is $O(nm)$
- Similarly, the space complexity is also $O(nm)$

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Recovering the LCS from the matrix

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Transforming one string to another

- The table (back arrows) also gives a set of edit operations to transform one string to another
- For LCS, operations are:
 - copy (diagonal arrows in the demonstration)
 - insert (left arrows in the demo – assuming original string is the vertical one)
 - delete (up arrows in the demo)
- These also form an alignment between two strings
- Different set of edit operations recovered will yield the same LCS, but different alignments

LCS alignments

		0	1	2	3	4	5	6	7	8
		ε	h	i	y	g	e	i	n	e
0	ε	0	0	0	0	0	0	0	0	0
1	h	0	1	1	1	1	1	1	1	1
2	y	0	1	1	2	2	2	2	2	2
3	g	0	1	1	2	3	3	3	3	3
4	i	0	1	2	2	3	3	4	4	4
5	e	0	1	2	2	3	4	4	4	5
6	n	0	1	2	2	3	4	4	5	5
7	e	0	1	2	2	3	4	4	5	6

Alignments:

h-yg-i-ne
 ciccicicc
 hygei-ne
 h-yg-e-ne
 ciccicicc
 hyge-ne

LCS – some remarks

- We formulated the algorithm as maximizing the LCS
- Alternatively, we can minimize the costs associated with each operation:
 - copy = 0
 - delete = 1
 - insert = 1
- The cost settings above are the typical, e.g., as in *diff*
- In some applications we may want to have different costs for delete and insert (e.g., mapping lemmas to inflected forms of words)
- Similarly, we may want to assign different costs for different characters (e.g., higher cost to delete consonants in historical linguistics)

Levenshtein distance

definition

- Levenshtein distance between two strings is the total cost of *insertions*, *deletions* and *substitutions*
- With cost of 1 for all operations

$$\text{lev}(Xx, Yy) = \begin{cases} \text{len}(X) & \text{if } \text{len}(Yy) = 0 \\ \text{len}(Y) & \text{if } \text{len}(Xx) = 0 \\ \text{lev}(X, Y) & \text{if } x = y \\ 1 + \min \begin{cases} \text{lev}(X, Yy) \\ \text{lev}(Xx, Y) \end{cases} & \text{otherwise} \end{cases}$$

- Naive recursion (as defined above), again, is intractable
- But, the same dynamic programming method works

Levenshtein distance

demonstration

		0	1	2	3	4	5	6	7	8
		ε	h	i	y	g	e	i	n	e
0	ε	0	1	2	3	4	5	6	7	8
1	h	1	0	1	2	3	4	5	6	7
2	y	2	1	1	1	2	3	4	5	6
3	g	3	2	2	2	1	2	3	4	5
4	i	4	3	2	3	2	2	2	3	4
5	e	5	4	3	3	3	2	3	3	3
6	n	6	5	4	4	4	4	3	3	4
7	e	7	6	5	5	5	4	4	4	3

Levenshtein distance

edits and alignments

		0	1	2	3	4	5	6	7	8
		ε	h	i	y	g	e	i	n	e
0	ε	0	1	2	3	4	5	6	7	8
1	h	1	0	1	2	3	4	5	6	7
2	y	2	1	1	1	2	3	4	5	6
3	g	3	2	2	2	1	2	3	4	5
4	i	4	3	2	3	2	2	2	3	4
5	e	5	4	3	3	3	2	3	3	3
6	n	6	5	4	4	4	4	3	3	4
7	e	7	6	5	5	5	4	4	4	3

Edit distance: extensions and variations

- Another possible operation we did not cover is *swap* (or *transpose*), which is useful for applications like spell checking
- In some applications (e.g., machine translation, OCR correction) we may want to have one-to-many or many-to-one alignments
- Additional requirements often introduce additional complexity
- It is sometimes useful to learn costs from data

Summary

- Edit distance is an important problem in many fields including computational linguistics
- A number of related problems can be efficiently solved by dynamic programming
- Edit distance is also important for approximate string matching and alignment
- Reading suggestion: Goodrich, Tamassia, and Goldwasser (2013, chapter 13), Jurafsky and Martin (2009, section 3.11, or 2.5 in online draft)

Next:

- Algorithms on strings: tries
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 13),

Acknowledgments, credits, references

- Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. [9781118476734](https://doi.org/10.1111/9781118476734).
- Jurafsky, Daniel and James H. Martin (2009). *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics, and Speech Recognition*, second edition. Pearson Prentice Hall. [978-0-13-04196-3](https://doi.org/10.1111/9780130419633).

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