Finite state automata

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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Why study finite-state automata?

- Finite-state automata are efficient models of computation
- There are many applications
  - Electronic circuit design
  - Workflow management
  - Games
  - Pattern matching
  - ...

  But more importantly ;-)
  - Tokenization, stemming
  - Morphological analysis
  - Spell checking
  - Shallow parsing/chunking
  - ...

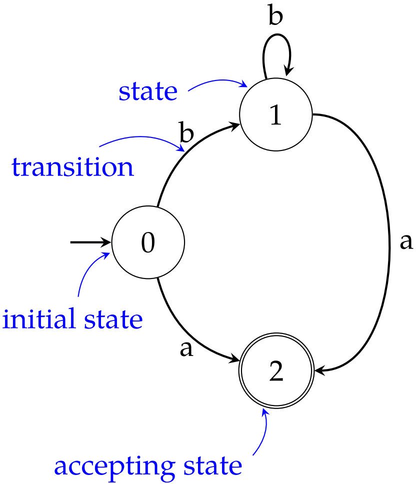
Finite-state automata (FSA)

- A finite-state machine is in one of a finite-number of states in a given time
- The machine changes its state based on its input
- Every regular language is generated/recognized by an FSA
- Every FSA generates/recognizes a regular language
- Two flavors:
  - *Deterministic finite automata* (DFA)
  - *Non-deterministic finite automata* (NFA)

  Note: the NFA is a superset of DFA.

FSA as a graph

- An FSA is a directed graph
- States are represented as nodes
- Transitions are labeled edges
- One of the states is the *initial state*
- Some states are accepting states



DFA: formal definition

Formally, a finite state automaton, M, is a tuple $(\Sigma, Q, q_0, F, \Delta)$ with

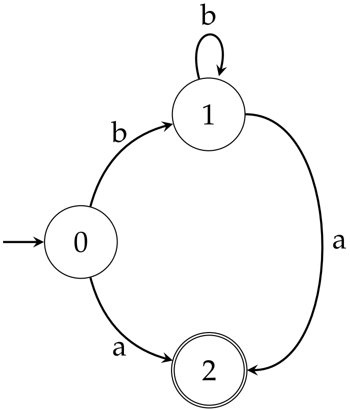
- Σ is the alphabet, a finite set of symbols
- Q a finite set of states
- q_0 is the start state, $q_0 \in Q$
- F is the set of final states, $F \subseteq Q$
- Δ is a function that takes a state and a symbol in the alphabet, and returns another state ($\Delta : Q \times \Sigma \to Q$)

At any state and for any input,
a DFA has a single well-defined action to take.

DFA: formal definition

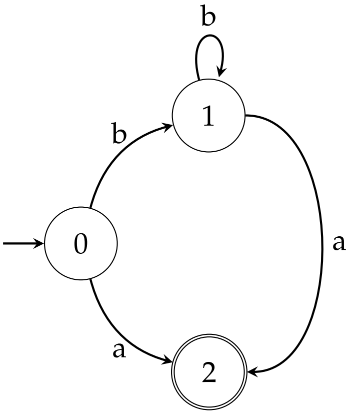
an example

$$
\begin{aligned}
\Sigma &= \{a, b\} \\
Q &= \{q_0, q_1, q_2\} \\
q_0 &= q_0 \\
F &= \{q_2\} \\
\Delta &= \{(q_0, a) \rightarrow q_2, \quad (q_0, b) \rightarrow q_1, \\
&\quad\ (q_1, a) \rightarrow q_2, \quad (q_1, b) \rightarrow q_1\}
\end{aligned}
$$



Another note on DFA

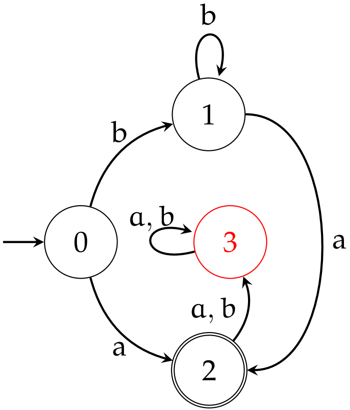
- Is this FSA deterministic?



Another note on DFA

error or sink state

- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state



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- Is this FSA deterministic?
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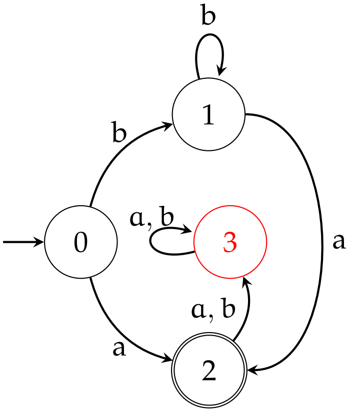
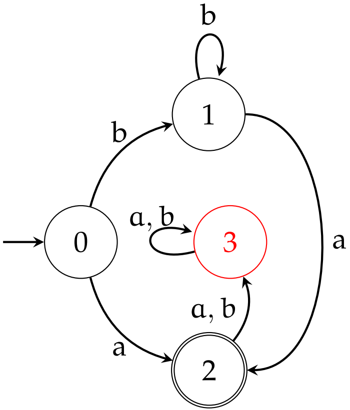
- For brevity, we skip the explicit error state



Another note on DFA

error or sink state

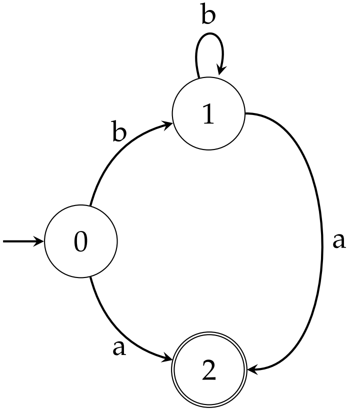
- Is this FSA deterministic?
- To make all transitions well-defined, we can add a sink (or error) state
- For brevity, we skip the explicit error state
  - In that case, when we reach a dead end, recognition fails



DFA: the transition table

transition table

		symbol	
		a	**b**
state	**→0**	2	1
	1	2	1
	***2**	∅	∅



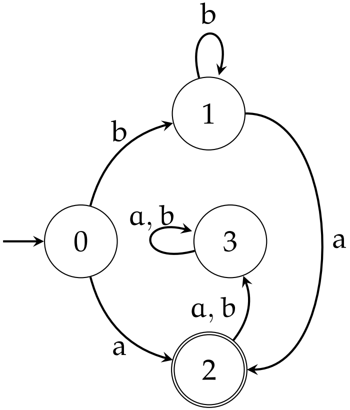
→ marks the start state

* marks the accepting state(s)

DFA: the transition table

transition table

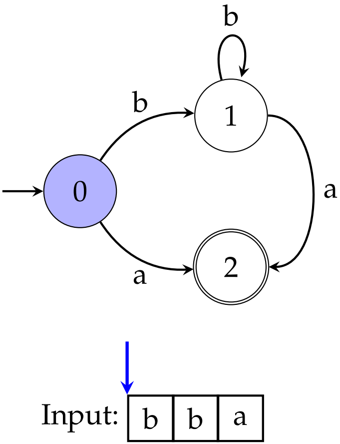
		symbol	
		a	**b**
state	→**0**	2	1
	1	2	1
	***2**	3	3
	3	3	3



- → marks the start state
- * marks the accepting state(s)

DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
3. Accept if in a final state at the end of the input



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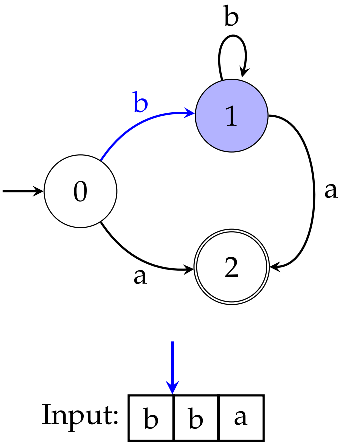

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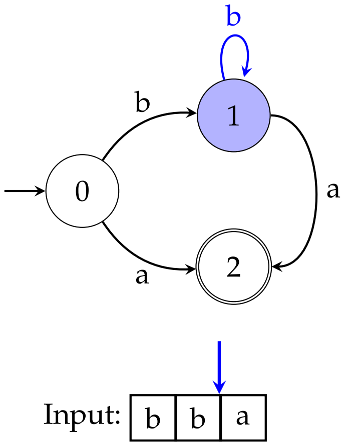

DFA recognition

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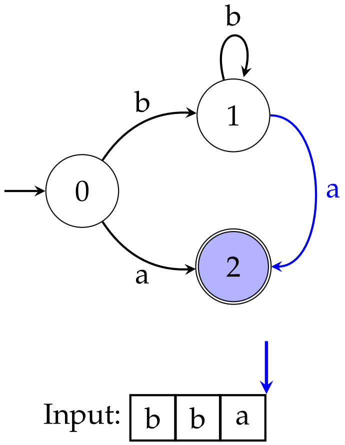

DFA recognition

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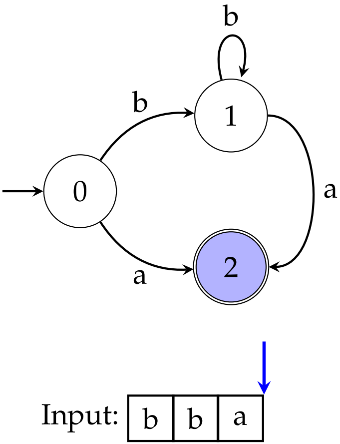

DFA recognition

1. Start at q_0
2. Process an input symbol, move accordingly
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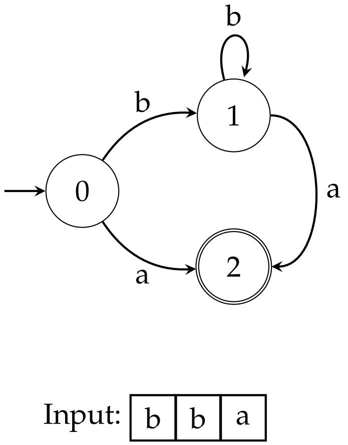
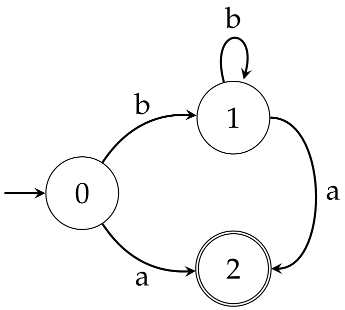


- What is the complexity of the algorithm?
- How about inputs:
  - bbbb
  - aa

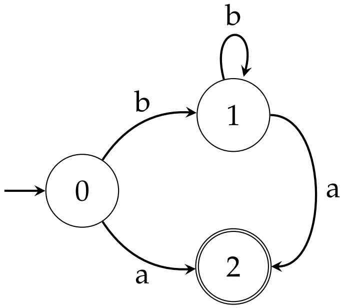
A few questions

- What is the language recognized by this FSA?



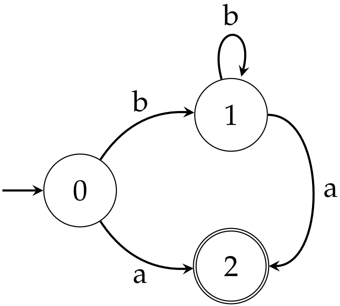
A few questions

- What is the language recognized by this FSA?
- Can you draw a simpler DFA for the same language?



A few questions

- What is the language recognized by this FSA?
- Can you draw a simpler DFA for the same language?
- Draw a DFA recognizing strings with even number of 'a's over $\Sigma = \{a, b\}$



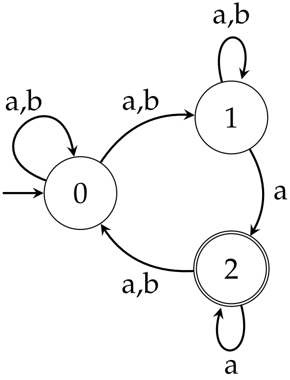
Non-deterministic finite automata

Formal definition

A non-deterministic finite state automaton, M, is a tuple $(\Sigma, Q, q_0, F, \Delta)$ with

- Σ is the alphabet, a finite set of symbols
- Q a finite set of states
- q_0 is the start state, $q_0 \in Q$
- F is the set of final states, $F \subseteq Q$
- Δ is a function from (Q, Σ) to $P(Q)$, power set of Q $(\Delta : Q \times \Sigma \to P(Q))$

An example NFA



transition table

		symbol	
		a	b
state	→**0**	0,1	0,1
	1	1,2	1
	***2**	0,2	0

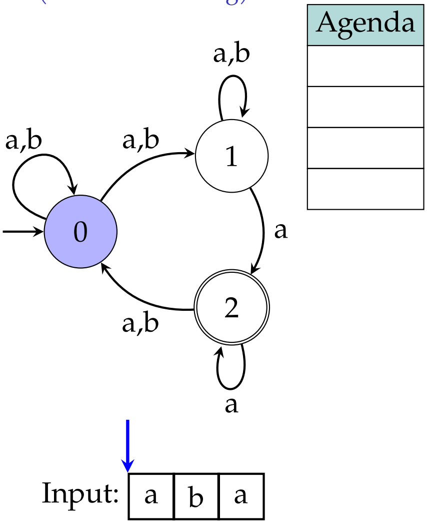
- We have nondeterminism, e.g., if the first input is a, we need to choose between states 0 or 1
- Transition table cells have *sets* of states

Dealing with non-determinism

- Follow one of the links, store alternatives, and *backtrack* on failure
- Follow all options in parallel

NFA recognition

as search (with backtracking)

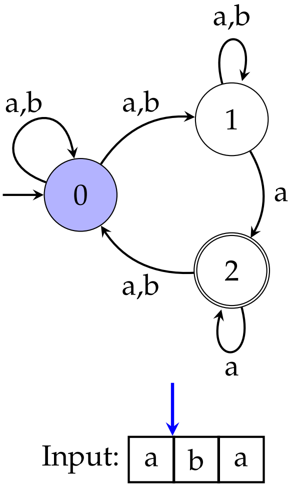


1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



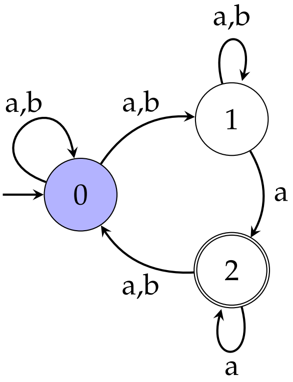
Agenda
$(q_0, 1)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 1)$
$(q_1, 1)$

Input:

a	b	a

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

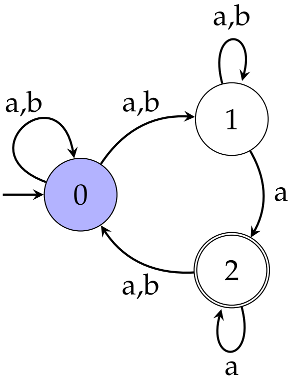
Accept if in an accepting state

Reject not in accepting state & agenda empty

Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 2)$
$(q_1, 2)$
$(q_1, 1)$

Input:

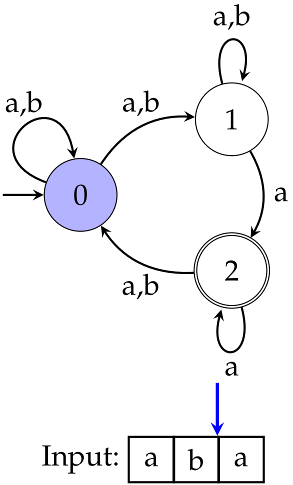
a	b	a

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept — if in an accepting state
Reject — not in accepting state & agenda empty
Backtrack — otherwise

NFA recognition

as search (with backtracking)



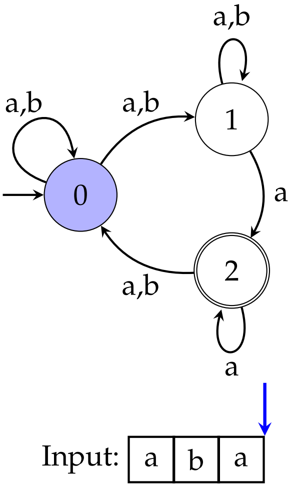
Agenda
$(q_0, 2)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



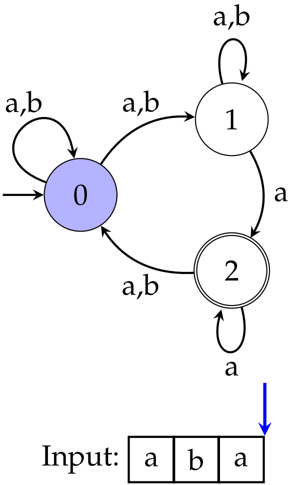
Agenda
$(q_0, 3)$
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_0, 3)$
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
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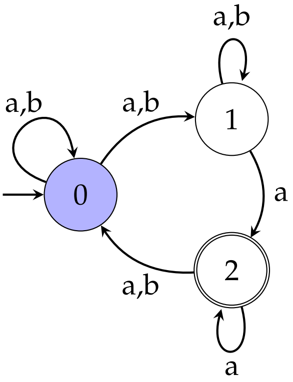
Backtrack	otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

Input:

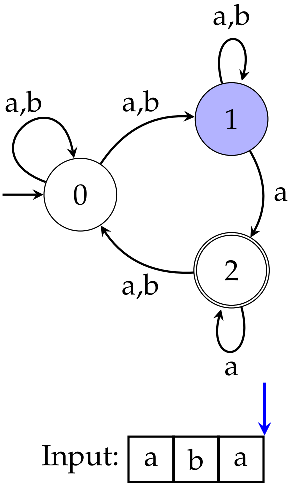
a	b	a

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 3)$
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
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4. At the end of input

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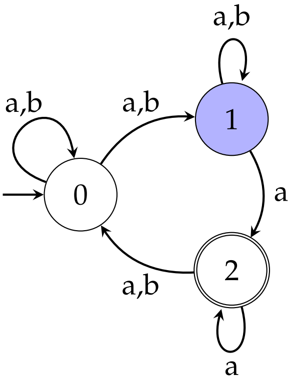
Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 2)$
$(q_1, 1)$

Input:

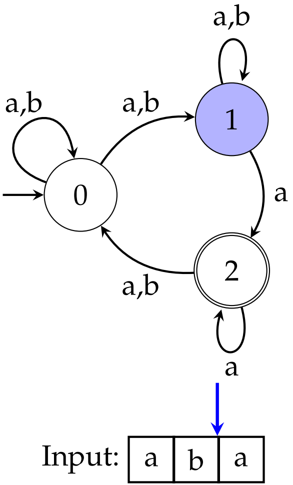
a	b	a

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 2)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

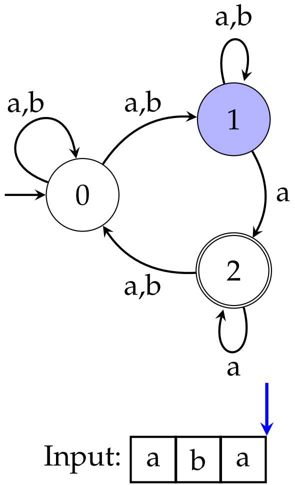
Accept if in an accepting state

Reject not in accepting state & agenda empty

Backtrack otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_2, 3)$
$(q_1, 3)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

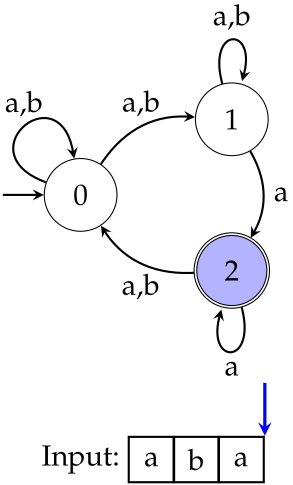
Accept: if in an accepting state

Reject: not in accepting state & agenda empty

Backtrack: otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_2, 3)$
$(q_1, 3)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
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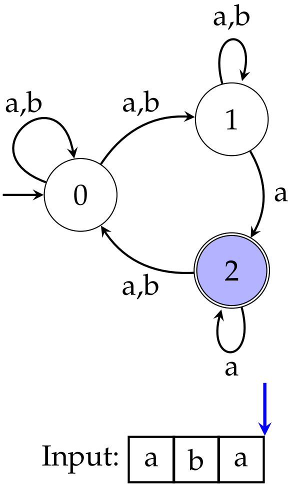
Backtrack: otherwise

NFA recognition

as search (with backtracking)



Agenda
$(q_1, 3)$
$(q_1, 1)$

1. Start at q_0
2. Take the next input, place all possible actions to an *agenda*
3. Get the next action from the agenda, act
4. At the end of input

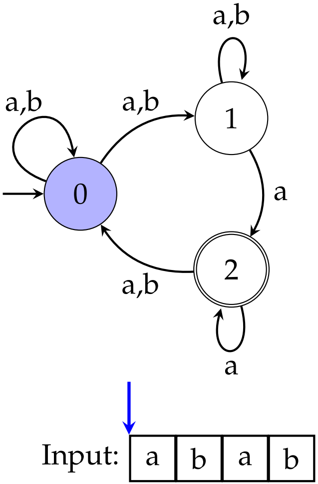
Accept	if in an accepting state
Reject	not in accepting state & agenda empty
Backtrack	otherwise

NFA recognition as search

summary

- Worst time complexity is exponential
  - Complexity is worse if we want to enumerate all derivations
- We used a stack as *agenda*, performing a depth-first search
- A queue would result in breadth-first search
- If we have a reasonable heuristic A* search may be an option
- Machine learning methods may also guide finding a fast or the best solution

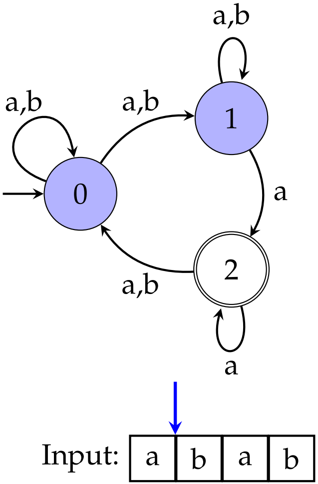
NFA recognition
parallel version



1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

NFA recognition

parallel version



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NFA recognition

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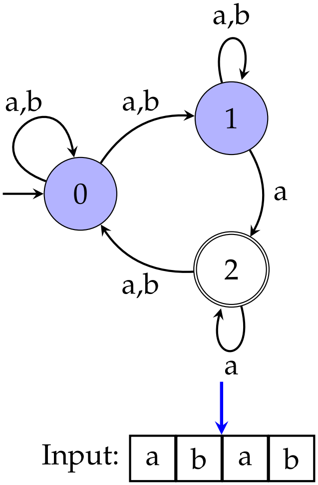

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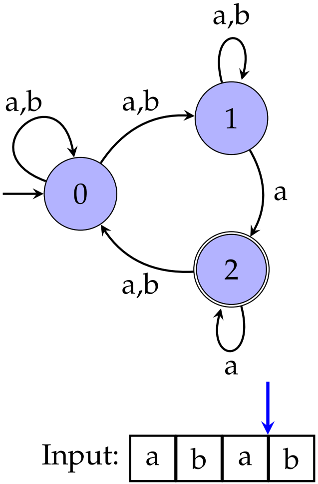

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NFA recognition

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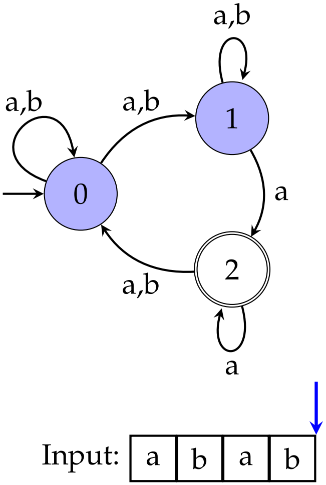

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NFA recognition

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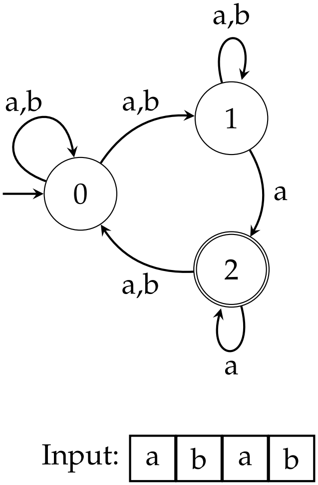


1. Start at q_0
2. Take the next input, mark all possible next states
3. If an accepting state is marked at the end of the input, accept

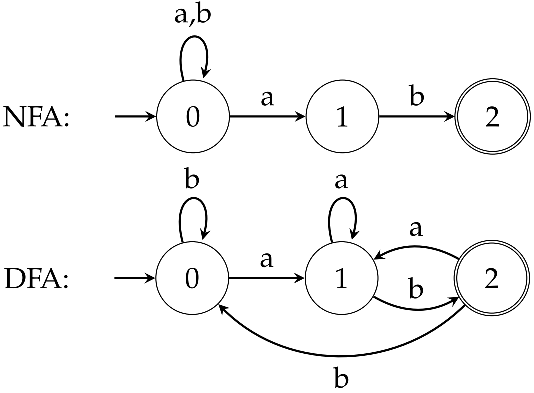
> Note: the process is *deterministic*, and *finite-state*.

An exercise

Construct an NFA and a DFA for the language over $\Sigma = \{a, b\}$ where all sentences end with ab.

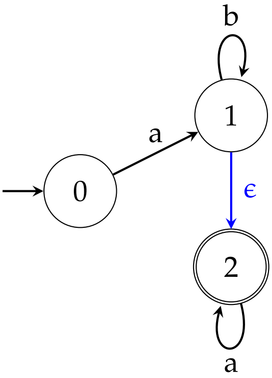
An exercise

Construct an NFA and a DFA for the language over $\Sigma = \{a, b\}$ where all sentences end with ab.



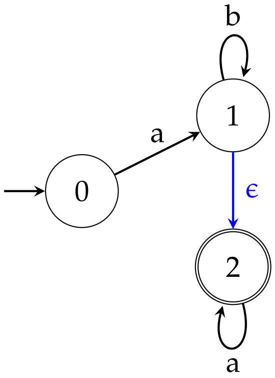
One more complication: ϵ transitions

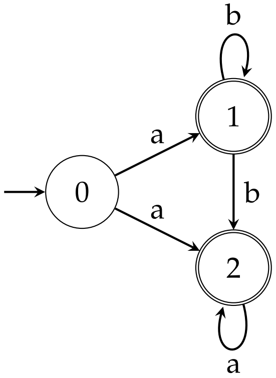
- An extension of NFA, ϵ-NFA, allows moving without consuming an input symbol, indicated by an ϵ-transition (sometimes called a λ-transition)
- Any ϵ-NFA can be converted to an NFA



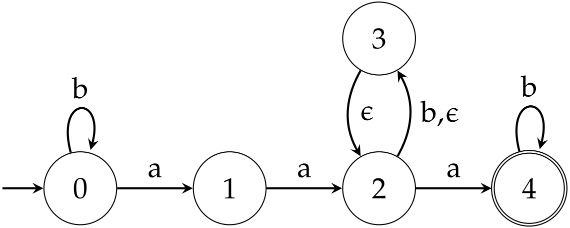
One more complication: ϵ transitions

- An extension of NFA, ϵ-NFA, allows moving without consuming an input symbol, indicated by an ϵ-transition (sometimes called a λ-transition)
- Any ϵ-NFA can be converted to an NFA





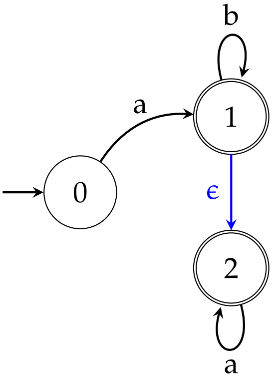
ϵ-transitions need attention



- How does the (depth-first) NFA recognition algorithm we described earlier work on this automaton?
- Can we do without ϵ transitions?

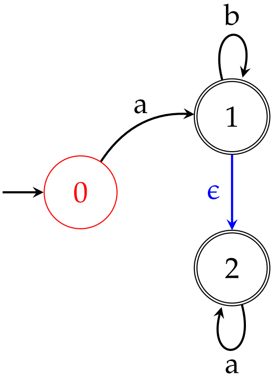
ϵ removal

- Intuition: if $i \xrightarrow{a} j \xrightarrow{\epsilon} k$, then $i \xrightarrow{a} k$.



ϵ removal

- Intuition: if $(i) \xrightarrow{a} (j) \xrightarrow{\epsilon} (k)$, then $(i) \xrightarrow{a} (k)$.
- We start with finding the ϵ-*closure* of all states



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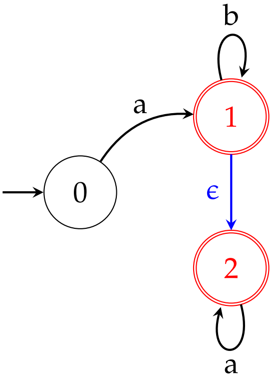
  - ϵ-closure$(q_0) = \{q_0\}$



ϵ removal

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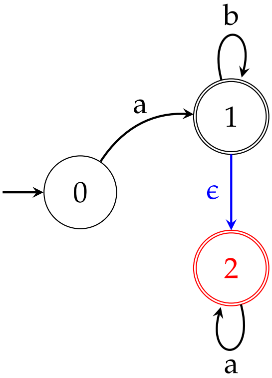

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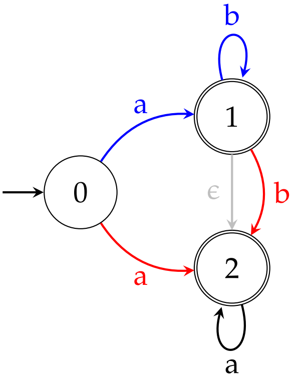

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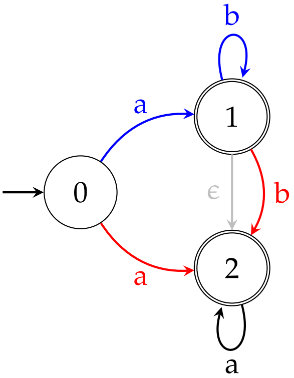
- For each incoming arc (q_i, q_j) to a node q_j with label ℓ
  - add a new arc (q_i, q_k) with label ℓ, for all $q_k \in \epsilon\text{-closure}(q_j)$
  - remove all ϵ transitions (q_j, q_k) for all $q_k \in \epsilon\text{-closure}(q_j)$



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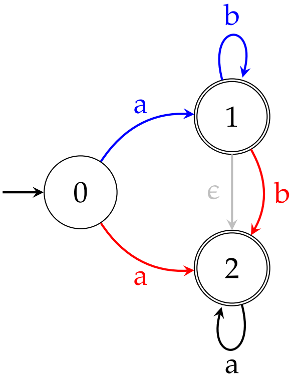
- ϵ-transitions from the initial state, and to/from the accepting states need further attention (next slide)



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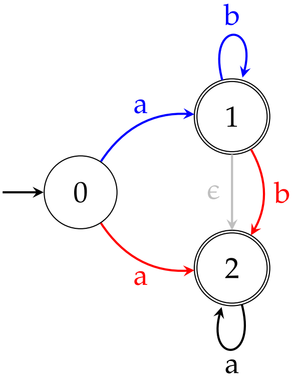
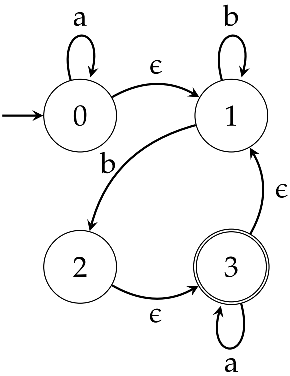
- Remove useless states, if any



ϵ removal

a(nother) example



ϵ removal

another (less trivial) example

- Compute the ϵ-closure:
  - ϵ-closure$(q_0) = \{q_0, q_1\}$
  - ϵ-closure$(q_1) = \{q_1\}$
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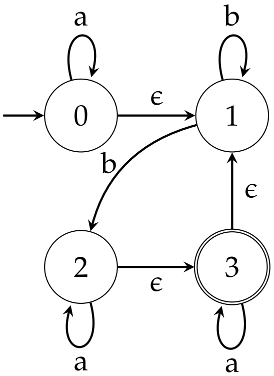

ϵ removal

another (less trivial) example

- Compute the ϵ-closure:
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  - $\epsilon\text{-closure}(q_1) = \{q_1\}$
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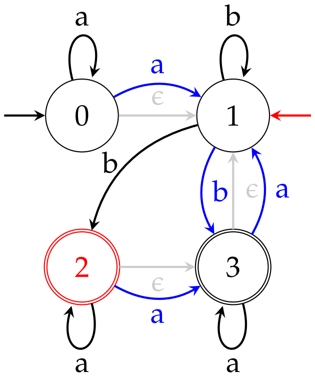
- For each incoming arc $\ell(q_i, q_j)$ to each node q_j
  - add $\ell(q_i, q_k)$ for all $q_k \in \epsilon\text{-closure}(q_j)$
  - if q_i is initial, mark q_j initial
  - if q_j is accepting, mark q_i accepting
  - remove all $\epsilon(q_j, q_k)$ for all $q_k \in \epsilon\text{-closure}(q_j)$



NFA–DFA equivalence

- The language recognized by every NFA is recognized by some DFA
- The set of DFA is a subset of the set of NFA (a DFA is also an NFA)
- The same is true for ϵ-NFA
- All recognize/generate regular languages
- NFA can automatically be converted to the equivalent DFA

Why do we use an NFA then?

- NFA (or ϵ-NFA) are often easier to construct
  - Intuitive for humans (cf. earlier exercise)
  - Some representations are easy to convert to NFA rather than DFA, e.g., regular expressions
- NFA may require less memory (fewer states)

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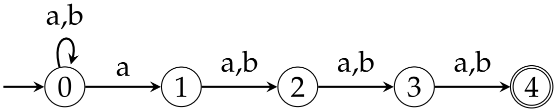

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A quick exercise – and a not-so-quick one

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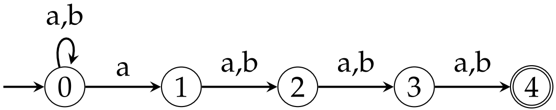

2. Construct a DFA for the same language

Summary

- FSA are efficient tools with many applications
- FSA have two flavors: DFA, NFA (or maybe three: ε-NFA)
- DFA recognition is linear, recognition with NFA may require exponential time
- Reading suggestion: Hopcroft and Ullman (1979, Ch. 2&3) (and its successive editions), Jurafsky and Martin (2009, Ch. 2)

Next:

- FSA determinization, minimization
- Reading suggestion: Hopcroft and Ullman (1979, Ch. 2&3) (and its successive editions), Jurafsky and Martin (2009, Ch. 2)

Acknowledgments, credits, references

- Hopcroft, John E. and Jeffrey D. Ullman (1979). *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley. ISBN: 9780201029888.
- Jurafsky, Daniel and James H. Martin (2009). *Speech and Language Processing: An Introduction to Natural Language Processing, Computational Linguistics, and Speech Recognition*. second edition. Pearson Prentice Hall. ISBN: 978-0-13-504196-3.

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