# FSA and regular languages

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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## Three ways to define a regular language

- A language is regular if there is regular grammar that generates/recognizes it
- A language is regular if there is an FSA that generates/recognizes it
- A language is regular if we can define a regular expressions for the language



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## Regular languages: some properties/operations

- $\mathcal{L}_1\mathcal{L}_2$ Concatenation of two languages $\mathcal{L}_1$ and $\mathcal{L}_2$: any sentence of $\mathcal{L}_1$ followed by any sentence of $\mathcal{L}_2$
- $\mathcal{L}^*$ Kleene star of $\mathcal{L}$: $\mathcal{L}$ concatenated with itself 0 or more times
- $\mathcal{L}^R$ Reverse of $\mathcal{L}$: reverse of any string in $\mathcal{L}$
- $\overline{\mathcal{L}}$ Complement of $\mathcal{L}$: all strings in $\Sigma^*_{\mathcal{L}}$ except the ones in $\mathcal{L}$ ($\Sigma^*_{\mathcal{L}} - \mathcal{L}$)
- $\mathcal{L}_1 \cup \mathcal{L}_2$ Union of languages $\mathcal{L}_1$ and $\mathcal{L}_2$: strings that are in any of the languages
- $\mathcal{L}_1 \cap \mathcal{L}_2$ Intersection of languages $\mathcal{L}_1$ and $\mathcal{L}_2$: strings that are in both languages

Regular languages are closed under all of these operations.



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## Regular expressions

- Every regular language (RL) can be expressed by a regular expression (RE), and every RE defines a RL
- A RE e defines a RL $\mathcal{L}(e)$
- Relations between RE and RL
  - $\mathcal{L}(\varnothing) = \varnothing$,
  - $\mathcal{L}(\epsilon) = \epsilon$,
  - $\mathcal{L}(a) = a$
  - $\mathcal{L}(ab) = \mathcal{L}(a)\mathcal{L}(b)$
  - $\mathcal{L}(a*) = \mathcal{L}(a)^*$
  - $\mathcal{L}(a|b) = \mathcal{L}(a) \cup \mathcal{L}(b)$ (some author use the notation a+b, we will use a|b as in many practical implementations)

  where, $a, b \in \Sigma$, $\epsilon$ is empty string, $\varnothing$ is the language that accepts nothing (e.g., $\Sigma^* - \Sigma^*$)
- Note: no standard complement and intersection in RE



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## Regular expressions

and some extensions

- Kleene star (`a*`), concatenation (`ab`) and union (`a|b`) are the basic operations
- Parentheses can be used to group the sub-expressions. Otherwise, the priority of the operators are as listed above: `a|bc*` = `a|(b(c*))`
- In practice some short-hand notations are common
  - `.` = `(a1|...|an)`, for $\Sigma = \{a_1, \ldots, a_n\}$
  - `a+` = `aa*`
  - `[a-c]` = `(a|b|c)`
  - `[^a-c]` = `.` − `(a|b|c)`
  - `\d` = `(0|1|...|8|9)`
  - ...
- And some non-regular extensions, like `(a*)b\1` (sometimes the term *regexp* is used for expressions with non-regular extensions)



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## Some properties of regular expressions

Useful identities for simplifying regular expressions

- `u|(v|w)` = `(u|v)|w`
- `u|v` = `v|u`
- `u(v|w)` = `uv|uw`
- `u|`$\varnothing$ = `u`
- `u`$\epsilon$ = $\epsilon$`u` = `u`
- $\varnothing$`u` = $\varnothing$
- `u(vw)` = `(uv)w`
- $\varnothing$`*` = $\epsilon$
- $\epsilon$`*` = $\epsilon$
- `(u*)*` = `u*`
- `u|u` = `u`
- `(u|v)*` = `(u*|v*)*`
- `u*|`$\epsilon$ = `u*`

**An exercise**

Simplify `a|ab*`

`a|ab*` = `a`$\epsilon$`|ab*`
= `a(`$\epsilon$`|b*)`
= `ab*`

Note: some of these are direct statements of Kleene algebra, others can be derived from them.



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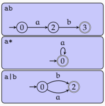
## Converting regular expressions to FSA




- For more complex expressions, one can replace the paths for individual symbols with corresponding automata
- Using $\epsilon$ transitions may ease the task
- The reverse conversion (from automata to regular expressions) is also easy:
  - identify the patterns on the left, collapse paths to single transitions with regular expressions



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## Exercise

convert `b((ab)*|a)` to an NFA



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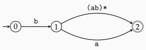




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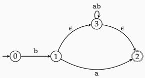
## Exercise

convert `b((ab)*|a)` to an NFA



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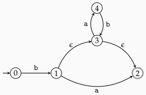
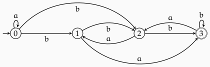
## Exercise

convert `b((ab)*|a)` to an NFA



## Converting FSA to regular expressions

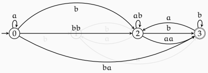



- The general idea: remove (intermediate) states, replacing edge labels with regular expressions





## Converting FSA to regular expressions

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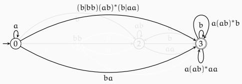
## Converting FSA to regular expressions

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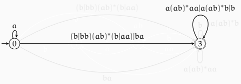
## Converting FSA to regular expressions

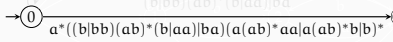



- The general idea: remove (intermediate) states, replacing edge labels with regular expressions





## Converting FSA to regular expressions

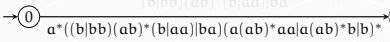



- The general idea: remove (intermediate) states, replacing edge labels with regular expressions





## Converting FSA to regular expressions




- The general idea: remove (intermediate) states, replacing edge labels with regular expressions

An exercise: simplify the resulting regular expressions

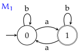




## Two example FSA

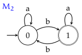
what languages do they accept?

$L_1 = \mathcal{L}(M_1)$




Odd number of a's over $\{a, b\}$.

$L_2 = \mathcal{L}(M_2)$




Odd number of b's over $\{a, b\}$.

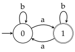
We will use these languages and automata for demonstration.



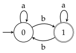


## Concatenation

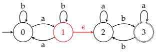
$L_1$




$L_2$




$L_1 L_2$

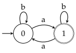




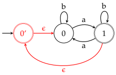



## Kleene star

$L_1$




$L_1^*$



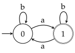

- What if there were more than one accepting states?



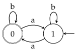


## Reversal

$L_1$




$L_1^R$

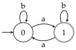




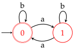



## Complement

$L_1$




$\overline{L_1}$

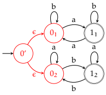




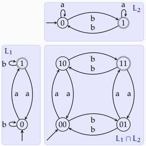



## Union

$L_1 \cup L_2$



## Intersection




...or

$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$





## Closure properties of regular languages

- Since results of all the operations we studied are FSA: Regular languages are closed under
  - Concatenation
  - Kleene star
  - Reversal
  - Complement
  - Union
  - Intersection





## Wrapping up

- FSA and regular expressions express regular languages
- Regular languages and FSA are closed under
  - Concatenation
  - Kleene star
  - Complement
  - Reversal
  - Union
  - Intersection
- To prove a language is regular, it is sufficient to find a regular expression or FSA for it
- To prove a language is not regular, we can use pumping lemma (see Appendix)

Next:

- FSTs





## Acknowledgments, credits, references

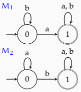
- The classic reference for FSA, regular languages and regular grammars is **hopcroft1979** (there are recent editions).





## Another exercise on intersection

Construct the intersection of the automata below (adapted from **hopcroft2007**, Fig. 4.4)








## Is a language regular?

— or not

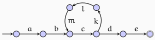
- To show that a language is regular, it is sufficient to find an FSA that recognizes it.
- Showing that a language is *not* regular is more involved
- We will study a method based on *pumping lemma*





## Pumping lemma

intuition




- What is the length of longest string generated by this FSA?
- Any FSA generating an infinite language has to have a loop (application of recursive rule(s) in the grammar)
- Part of every string longer than some number will include repetition of the same substring ('cklm' above)



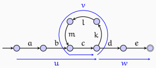


## Pumping lemma

definition

For every regular language $L$, there exist an integer $p$ such that a string $x \in L$ can be factored as $x = uvw$,

- $uv^iw \in L, \forall i \geqslant 0$
- $v \neq \epsilon$
- $|uv| \leqslant p$








## How to use pumping lemma

- We use pumping lemma to prove that a language is not regular
- Proof is by contradiction:
  - Assume the language is regular
  - Find a string $x$ in the language, for all splits of $x = uvw$, at least one of the pumping lemma conditions does not hold
    - $uv^iw \in L$ $(\forall i \geqslant 0)$
    - $v \neq \epsilon$
    - $|uv| \leqslant p$





## Pumping lemma example

prove $L = a^nb^n$ is not regular

- Assume $L$ is regular: there must be a $p$ such that, if $uvw$ is in the language
  1. $uv^iw \in L$ $(\forall i \geqslant 0)$
  2. $v \neq \epsilon$
  3. $|uv| \leqslant p$
- Pick the string $a^pb^p$
- For the sake of example, assume $p = 5$, $x = aaaaabbbbb$
- Three different ways to split

| | |
|---|---|
| $\underbrace{a}_{u}\underbrace{aaa}_{v}\underbrace{abbbbb}_{w}$ | violates 1 |
| $\underbrace{aaaa}_{u}\underbrace{ab}_{v}\underbrace{bbbb}_{w}$ | violates 1 & 3 |
| $\underbrace{aaaaab}_{u}\underbrace{bbb}_{v}\underbrace{b}_{w}$ | violates 1 & 3 |





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