Analysis of Prophet Inequalities for Subadditive Combinatorial Auctions

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Submitted in partial fulfillment
of the requirements for the degree of
Bachelor of Science in Engineering
Department of Computer Science
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April 2024

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Abstract

In this paper, we attempt to effectively design a posted price mechanism that achieves an $O(1)$ prophet inequality for general subadditive valuations. Precisely, we consider *static, anonymous,* and *item* prices which yields a very simple sequential auction model with a price menu. We present current state of the art mechanisms in [PDTKL17] (balanced pricing) and [KL20] which provide an $O(\log m)$ and an $O(\log \log m)$ prophet inequality respectively. Regarding improving their current guarantees to $o(\log m)$ and $o(\log \log m)$ respectively, we show impossibility results.

Moreover, in this paper, we consider a relaxation coined as bundled pricing where we no longer enforce item pricing in hopes of a better approximation guarantee. One such example of bundled prices was seen in Combinatorial Walrasian Equilibria from [FGL13] which achieve a 2-approximation in complete information. Unfortunately, this result does not extend to the Bayesian setting as we show by proving that the above bundled pricing rule is not balanced. Next, we consider the existence of balanced bundled prices that achieve constant factor approximations. In fact, we prove impossibility results regarding guarantees of the form $o(\log m)$ when $m \leq n$ using a reduction technique to balanced item prices. For the other case, we show existence of appropriate balanced bundled prices for general m and $n = 1$. We conclude with a conjecture regarding the relationship between the number of agents and the complexity of the valuation class for which there exist appropriate balanced bundled prices.

Acknowledgements

First and foremost, I would like to thank Professor Matthew Weinberg for advising me throughout the past two years. He introduced me to theoretical computer science, helped me navigate the research process, and provided reassurance when I thought graduate studies were beyond me, all of which has been instrumental in my growth academically and personally. I will miss the countless talks regarding completely arbitrary topics and our weekly meetings where we refined my proofs and discussed new research questions. My thesis and all the work accomplished in the past 2.5 years would not have been possible without his support and guidance.

I am also thankful to Professor Pedro Paredes for his role as my second reader, Professor Huacheng Yu for supporting my love for teaching and to the Computer Science Department for having provided a community of peers and faculty from whom I have learned so much. It has given me endless opportunities alongside the flexibility to pursue my interests outside of the department.

I owe a special thanks to Professors Jianqing Fan, Elizaveta Rebrova, and Emma Hubert from the Operations Research and Financial Engineering Department. Their willingness to spend hours discussing statistics, probability theory, and stochastic processes, and answering my questions with such care, has made this department a second home to me at Princeton. Their influence is a significant reason why I am excited to continue my graduate studies in Industrial Engineering and Operations Research at Columbia University.

Moreover, I would like to thank my family who have supported me through every part of my journey so far. Lastly, I would like to thank my friends that have shaped my Princeton experience tremendously. To my favorite person on campus, thank you for being my third reader, best friend, and constant companion, you made life here amazing. To my first COS partner, without you, I would not be a COS major. To the family I found within Dod 401, thank you for making Princeton a home.

Attributions

This senior thesis is an extension of my junior independent work. As such, Chapter $1-5$, and Chapter 7 were present in my junior independent work, although parts have been adapted and edited for this work. Chapter 6 contains entirely new work that was not present in my previous junior independent work and serves as the novel contribution to this thesis. In fact, it answers one of the few further open questions that we posed within my junior independent work. The presentation is as such in order to make the overall work entirely self contained.

To Dad, Mom, and Rimi

Contents

Chapter 1

Introduction

The ubiquitous challenge faced by theoretical computer scientists is designing *algorithmic mechanisms* to solve real world problems in a tractable manner. One may ask what algorithmic mechanisms are and, more importantly, how to use them to solve real world problems that often seem difficult to model. The first step in this design process is defining a set of formal modeling assumptions so that the problem may be mathematically solvable and subsequently analyzed.

These modeling assumptions must be reasonable as well as interpretable. Without such assumptions, we could end up designing models which give us no further understanding of the problem after mechanism design and analysis. As our primary example, we consider an auction, which we will delve into further as it is the main focus of this paper. The auction is designed to maximize welfare for a collection of agents, each having random valuations for items. In this model, one of the many underlying assumptions is that we can model an agent's general valuation as a random variable that follows a parametric probability distribution.

Upon successfully creating a mathematical model that captures a real world problem, we then design mechanisms to find a feasible solution. We revisit the setting of an auction. A basic mechanism to consider is for the allocator to design a menu of prices for each agent; note that these may be identical. After seeing these prices, our agents make decisions effectively realizing our final allocation of items.

We diverge from the topic at hand for a moment and examine the Princeton Farmer's Market, which occurs weekly in the spring. Consider a flower vendor at the farmer's market who has distributional knowledge of the customers he will attract; that is, students attending Princeton University as well as those in the area nearby. Moreover, he has knowledge of customers' value for the flowers on sale: i.e. seasonal flowers are often in high demand. However, a singular customer's valuations for items cannot be identified prior to their arrival. Furthermore, some customers desire a singular rose while others may want an assorted bouquet. Some customers are flower connoisseurs whereas others will only purchase under low prices. The vendor must use this knowledge in order to price his collection of flowers such that he can maximize his revenue alongside making the Princeton community happy (for future business of course). An immediate problem faced by the flower vendor is if he sets prices too low. In this case, he runs the risk of his most prized bouquets being purchased by those who arrive earlier in the day and who could potentially have lower value than someone who arrives later. Similarly, if he prices too high, nobody will buy his bouquets.[1]

Combinatorial auctions [FGL14, PDTKL17, KL20, CC22] are a framework that allows us to model situations like the one above. We find solutions to this framework through the development of algorithmic mechanisms (i.e. posted price mechanisms) whose performance can be analyzed thereafter. This class of problems goes hand in hand with another class known as stochastic online optimization. To analyze the performance of different algorithms designed to solve these problems, we use a tool known as prophet inequalities.

In general, combinatorial auctions have a wide range of real-world applications, such as the allocation of spectrum licenses, the sale of advertising slots, and the

[1]The following example is inspired by the one provided in [PDTKL17]

placement of infrastructure. For example, in spectrum auctions, the government auctions radio frequency spectrum to wireless providers. Prophet inequalities may be used to design mechanisms that optimize the allocation of the spectrum while ensuring that wireless providers are not incentivized to misreport their valuations. In the case of online advertising, where multiple slots to place advertisements are available, a mechanism that can optimally allocate the slots would greatly increase revenue for the advertiser. The same can be applied to the placement of infrastructure, such as wind turbines, where land must be divided to accommodate multiple structures.

1.1 History

The allocation problem first stemmed within the field of probability theory from the work of Krengel and Sucheston in 1978, where they provided guarantees on optimal stopping rules given a sequence of independent non-negative random variables [KS77, KS78]. A few years later, Samuel Cahn showed the relationship between thresholding rules and the maximum of a sequence of independent non-negative random variables $\{X_i\}_{i=1}^n$ [SC84]. A thresholding rule τ is defined by some constant $c > 0$, such that $\tau(c) = \inf\{i : X_i \geq c\}$. In other words, $X_{\tau(c)}$ is the first random variable that exceeds the specified threshold c. Cahn noted that while thresholding rules did not provide the tightest approximation to the expected maximum, they were often much easier to compute than optimal stopping rules.[2]

In order to understand Cahn's fundamental result, we define $X^{(1)} := \max_i X_i$ and m such that $\Pr\left[X^{(1)} \leq m\right] \geq 1/2$ and $\Pr\left[X^{(1)} \geq m\right] \geq 1/2$, i.e. m is median of distribution of $X^{(1)}$. Formally, he showed that $2\mathbb{E}[X_{\tau(m)}] \geq \mathbb{E}[X^{(1)}]$ which means that the median thresholding rule provides a 2-approximation to the expected maximum.

[2] The expected selection by the optimal stopping rule is formally defined as $\sup\{\mathbb{E}[X_\tau] : \tau \in \mathcal{T}_n\}$, where $\mathcal{T}_n$ is the set of stopping rules on $\{X_i\}_{i=1}^n$. Furthermore, even after the discovery of such an optimal rule, Cahn noted that it was significantly harder to implement than the simple thresholding rule.

At this point, the relevance of the above may come into question. What does finding the maximum have to do with notion of resource allocation and auction design? This problem may be thought of as a gambler attempting to select the maximal element irrevocably while the random variables are sequentially presented to him, with only knowledge of the distributions of all n random variables *a priori*. Thus, a natural interpretation of $\mathbb{E}[X_{\tau(m)}]$ is the expected selection of a gambler utilizing the median thresholding rule. Similarly, $\mathbb{E}[X^{(1)}]$ can be thought of as the expected performance of an all-seeing prophet, who had knowledge of the exact realizations of the sequence of n random variables *a priori*. Immediately, we may see that Cahn's result is clearly a 2-approximation of an agent looking to maximize his selection to the optimal expected selection.

Approximately 30 years later, Kleinberg and Weinberg developed a similar thresholding rule where $m = \mathbb{E}[X^{(1)}]/2$, which also provides the same 2-approximation [KW12]. However, this work was done in context of auctions with combinatorial constraints rather than as an exercise in probability theory.

The equivalence between the two problems can be seen by considering n agents, each with some non-negative random value for an item. The *allocator* prices the item such that the item is sold and the buyer has positive utility. Notice that setting the price by Cahn's median thresholding rule, where agents purchase the item as long as it is available and the purchase generates positive utility, immediately gives the classic 2-approximation. In fact, pricing by Kleinberg and Weinberg's mean thresholding rule [KW12] turns out to be part of a much broader class of prices known as *balanced prices*, which we will explore later.

So far, we have presented ideas that provide an accurate but naive depiction of what a model of the general resource allocation might look like. This is a result of our framework consisting only of a *singular item* as well as independent random variables representing the agents' value for the item. There is no discussion of multiple items, the intricacies to consider within modeling valuations, and the exponential increase in computational complexity of the general model. This naturally leads to the following question, which we work to answer in this paper.

Question 1. *How can we model the general resource allocation problem, where we have multiple items and agents, with the objective to maximize societal welfare while maintaining the constant factor asymptotic guarantee in expectation as shown above?*

1.2 Overview

The generalized resource allocation model is an auction with m items and n bidders. Each bidder assigns a value to every subset of m items, using a valuation function v_i for each $i \in [n]$, adding more sophistication to our previous model. In this paper, we work with the class of valuations functions that are subadditive. We will later see that subadditive valuations are a reasonable assumption to make in modeling preferences although it is a vexing open problem to design a pricing mechanism that achieves a constant factor asymptotic guarantee for them.

Consider the simple auction, a concept familiar to most. But now, rather than all agents bidding at the same time (i.e. Vickery auctions, First Price auctions, All Pay auctions, etc), we consider the case where the n bidders arrive sequentially, motivating the *online* nature of the problem. While this order may be arbitrary, assume without loss of generality that they arrive in the order $1, 2, \ldots, n$.[3] The allocator, now coined as the auctioneer, has knowledge of the distribution of the value functions, but not the realized individual valuation functions themselves prior to the auction, revealing the *stochastic* nature of the problem. Now, the auctioneer uses his knowledge of all n distributions to post prices as discussed before. As each agent i arrives, their valuation function v_i is revealed to the auctioneer upon which their most desirable remaining set

[3]This can be formalized by considering a map $\sigma : [n] \to N$, where σ maps the ordering to the set N of agents and the analysis still holds.

of items is awarded to them by our algorithm. Agent i thus pays the price of the set of items they were allocated. Doing so for every agent determines the allocation realized by our algorithm with respect to the valuations revealed throughout the process.

Note here we made no such claim regarding how the auctioneer prices the items, unlike the median or mean rule from the the work in [SC84, KW12]. In fact, pricing these sets of items such they provide ideal approximations has been a highly researched question, one we try to answer in this paper.

The generalization to multiple items was seen first in the realm of analyzing Bayesian Nash Equilibria and Price of Anarchy within the setting of auction theory [FGL14, RST16]. The framework of a Bayesian Nash Equilibrium is ideal as it allows for a clean way to analyze approximation guarantees. This is because Price of Anarchy is defined as the worst case ratio between the expected optimal welfare and expected welfare in an equilibrium. This ratio was exactly 2 in the examples provided earlier with a singular item. We will see the same ratio redefined later in terms of the elusive *prophet inequality* for our purposes. Furthermore, the assumption of playing in equilibrium is reasonable since best responding is rational. In fact, we will see that posted price mechanisms are more restrictive than Bayesian Games.[4]

Additionally, one can see a very clear connection to the notion of combinatorial auctions and prophet inequalities. Indeed, the authors of [FGL14] realized this and later devised a pricing strategy known as balanced prices [PDTKL17]. These prices are astonishingly advantageous and enjoy constant approximation guarantees for a variety of feasibility constraints such as combinatorial auctions, matroids, and knapsack for the classes of valuation functions XOS, Submodular, and Additive respectively. In this paper, we provide intuition and proofs as to why this is a great mechanism to set appropriate *static, anonymous, and item* prices for valuation classes such as XOS, but fail provide a bound of the form $o(\log m)$ for general subadditive valua-

[4] It can be shown that posted price mechanisms are dominant strategy truthful, and therefore produce a restricted class of trivial equilibria.

tion functions. They also enjoy several ideal properties that we examine in detail to conceive similar prices that provide a constant factor prophet inequality for general subadditive valuations.

Soon after the paper above, Dütting, Kesselheim, and Lucier published a result that established an $O(\log \log m)$ prophet inequality in [KL20]. They use a posted price mechanism which formulates the existence of prices that fulfills some constraint imposed by the authors. The proof of this involves the reformulation of the problem as a min-max game, where agents can take each item with probability at most q. However, this is shown to be inextensible to a bound of the form $o(\log \log m)$. We also examine why the posted price mechanism described in [KL20] fails to translate a pointwise result from the full information setting to a global guarantee within the Bayesian setting. This is a stark difference from the worse performing balanced price mechanism.

Very recently, a paper by José Correa and Andrés Cristi proved the existence of a $(6+\varepsilon)$ prophet inequality for subadditive valuations [CC22], achieving a major breakthrough from previous work in field. However, this method was non-constructive: while it showed that such a prophet inequality exists, there is no known way to implement it with posted price mechanisms. Moreover, they define "scores" which can be thought of as pseudo prices despite not satisfying some guarantees from posted price mechanisms, such as non-negative utility. For more detail, the interested reader may look at [CC22] or a survey of their work by Parashar and Saha in [SP22]. The authors also briefly provide a notion of bundled prices that achieve a 2-approximation on subadditive valuations in the deterministic case; however, we show that they are inextensible to the Bayesian case. We also present their subtle relations to the same balanced prices from [PDTKL17].

Motivated by the aforementioned work on bundled prices defined via Combinatorial Walrasian Equilibria in [FGL13], we work towards analyzing general bundled pricing rules while ensuring they are still static and anonymous. Specifically, we try to answer questions regarding the existence of bundled prices that are also balanced. In fact, we show, when there are at most as many items as people, it is impossible to construct bundled balanced prices that give a guarantee of the form $o(\log m)$. This impossibility guarantee is actually closely related to a nearly identical impossibility result we showed regarding item balanced prices earlier from [PDTKL17]. Furthermore, we notice a relationship between the number of agents and the "complexity" of the *welfare-maximizing* function for which there exist bundled prices admitting a constant factor guarantee. Specifically, we show results with 1 agent and a large number of agents, and how complex the function must be in either case to admit nice prices. We further conjecture using the notion of q-partitioning valuations from [BW23] as the metric of complexity to interpolate between the XOS and subadditive function classes.

In general, this constant factor prophet inequality has been a problem at the forefront of research within the last few years, as indicated by the steady flow of breakthroughs in the space. Thus, the goal of this paper is to design a posted price mechanism for subadditive combinatorial auctions that achieve a $O(1)$ or constant prophet inequality. In order to do so, we delve into each of the existing methods mentioned above and prove why they are inextensible to our desired result. We hope that this goal can be achieved via our future work, effectively answering the question posed in Section 1.1 for the very general class of functions known as subadditive valuations.

Chapter 2

Background

2.1 Model

Formally, we have a set N of n agents and a set M of m items. We notate X_i as the outcome space, or the set of possible allocations to each agent $i \in N$. Moreover, $\varnothing \in X_i$ for all $i \in N$, as we know each agent can simply receive nothing. Now, the joint outcome space, or the space of all possible allocations, can be denoted as $X = X_1 \times \cdots \times X_n$. Any allocation profile is notated as $\mathbf{x} = (x_1, \ldots, x_n) \in X$. We define $\mathbf{x}_S \in X$ as an allocation only to those buyers in the subset $S \subseteq N$. In other words, each agent $i \in S$ receives x_i, and agents $i \notin S$ receive $\varnothing$. Moreover, $\mathbf{x}_{[i-1]}$ is defined similarly where $S = \{1, 2, \ldots, i-1\}$. That is, we only allocate items to buyers 1 through $i-1$, and buyers i through n receive $\varnothing$. While these are all indeed allocations, we denote $\mathcal{F} \subseteq X$ as only those that are feasible. We require that if $\mathbf{x}$ is feasible then $\mathbf{x}_S$ is feasible or, more generally, $\mathcal{F}$ is downward closed.

Each agent i has a valuation function $v_i \in V_i$ where V_i is the set of possible valuations functions for agent i. We define the function as $v_i : 2^M \to \mathbb{R}_{\geq 0}$, or similarly $v_i : X_i \to \mathbb{R}_{\geq 0}$ since either 2^M or X_i notate the set of possible allocations to buyer i.[1] We assume these valuations are both normalized and monotone, which means

[1] Here 2^M represents the the power set of M, or in other words every subset of items.

$v_i(\varnothing) = 0$ and for $S \subseteq T \subseteq M$, we have $v_i(S) \leq v_i(T)$. Lastly, we assume that the values are finite, i.e. for all $i \in N$ and $S \subseteq M$, we have $\mathbb{E}[v_i(S)] < \infty$.

Furthermore, every v_i is drawn independently from some distribution $\mathcal{D}_i$.[2] We notate $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ as the product or joint distribution. Here, $\mathcal{D}_i$ is a distribution over the set of valuations V_i for any agent $i \in N$. Again, assume that the auctioneer has knowledge of every distribution but not the values themselves prior to the auction. Naturally, now that there are multiple items, we need to address the concern of how to model valuations for sets of items. The following classes of valuation functions satisfy certain properties in an attempt to encapsulate rational human behavior.

Definition 2.1.1. *The classes of common valuation functions in increasing order of generality are as follows:*

- ***Additive:*** *A valuation v is additive if for any $S \subseteq M$, we have that $v(S) = \sum_{j \in S} v(\{j\})$.*

- ***Submodular:*** *A valuation v is submodular if for any $S \subseteq T \subseteq M$ and $j \notin T$ we have that $v(S \cup \{j\}) - v(S) \geq v(T \cup \{j\}) - v(T)$. Such functions can be interpreted as representations of diminishing marginal utility, a common characteristic of human preferences.*

- ***XOS:*** *A valuation v is XOS if there are a collection of additive supports $a_1, \ldots, a_k$ such that for any $S \subseteq M$ we have $v(S) = \max\limits_{1 \leq i \leq k} a_i(S)$.*

  An equivalent definition says that a valuation v is XOS if for any $S \subseteq M$, and for any fractional covering $\{\lambda_i, T_i\}_{i=1}^k$ of S, we have $v(S) \leq \sum_{i=1}^k \lambda_i v(T_i)$. A fractional covering of S means for $\lambda_i > 0, T_i \subseteq M$ and for all $j \in S$ we have $\sum_{i: j \in T_i} \lambda_i \geq 1$.[3]

[2] The independence assumption is necessary, allowing for arbitrarily correlated valuations leads to a gap of $\Omega(n)$.

[3] Showing the two definitions to be equivalent requires use of duality and is left as exercise for the reader.

- ***Subadditive:*** *A valuation v is subadditive if for any $S, T \subseteq M$ we have that $v(S \cup T) \leq v(S) + v(T)$. This means that items are not necessarily more valuable together than they are apart, which is a very reasonable assumption in many real world settings.*

We notate $\mathbf{v} = (v_1, \ldots, v_n)$ as the vector of specific valuations, or a valuation profile. We specifically focus on subadditive valuation profiles. Utilities are quasi-linear and can be defined as $u_i = v_i(x_i) - \pi_i$ for buyer i. This is the price π_i buyer i pays for the items subtracted from the value for items received $v_i(x_i)$. The welfare is defined to be $\mathbf{v}(\mathbf{x}) = \sum_i v_i(x_i)$ for any allocation $\mathbf{x}$.

Additionally, we notate $\mathsf{OPT}(\mathbf{v}, \mathcal{F}) = \arg\max_{\mathbf{x} \in \mathcal{F}} \mathbf{v}(\mathbf{x})$ as the optimal allocation since it has the maximum welfare over all feasible allocations. For notational convenience, we omit the feasibility constraint and write $\mathsf{OPT}(\mathbf{v})$ instead. Thus, $\mathbf{v}(\mathsf{OPT}(\mathbf{v}))$ is the total value the allocation achieves. Similarly, we write $\mathsf{ALG}(\mathbf{v}) = (x_1, \ldots, x_n)$ as the algorithm's or mechanism's allocation and $\mathbf{v}(\mathsf{ALG}(\mathbf{v}))$ as the value it achieves. $\mathsf{ALG}_i(\mathbf{v})$ is the algorithm's allocation to buyer i for the valuation profile $\mathbf{v}$.

2.2 Posted Price Mechanisms

Earlier we discussed the notion of designing mechanisms to solve the mathematical problems. In this case, we consider a class of mechanisms known as posted price mechanisms to solve the combinatorial auction problem. Posted price mechanisms set pricing rules $p_i : 2^M \to \mathbb{R}_{\geq 0}$, which assign a non-negative price to each subset of items $S \subseteq M$ for all agents $i \in N$. Now we define three types of prices:

- *Static prices* are prices independent of the items that have already been sold. That is, for any $S \subseteq M$ and partial allocation $\mathbf{x}_{[i-1]}$ we have $p_i(S) = p_i(S|\mathbf{x}_{[i-1]})$.

- *Anonymous Prices* for an item are simply those that do not depend on the agent purchasing it. In other words, for any $i, j \in N$ and $S \subseteq M$, we have $p_i(S) = p_j(S)$.

- *Item prices* mean that the price of a set of items is the sum of the prices of the items within it, or $p_i(S) = \sum_{j \in S} p_j$.

Note that there are various types of posted price mechanisms. In this paper, we begin by exploring balanced prices from [PDTKL17] and then prices as defined in [KL20]. Furthermore, we mainly consider *static, anonymous, and item* prices, however, eventually we relax these assumptions in hopes of devising the constant prophet inequality via bundled prices [CC22]. Upon defining pricing rules, we may see how the mechanism operates.

Definition 2.2.1. *Below we define the entirety of the posted price mechanism:*

1. ***Offline Input:*** *The set of distributions for all n players $\{\mathcal{D}_i\}_{i=1}^n$.*

2. ***Online Input:*** *At each step $t \in [n]$, the new online input is the specific realization v_t. Thus, the total information available at step t is $(\{\mathcal{D}_i\}_{i=1}^n, \{v_i\}_{i=1}^t)$.*

3. ***Offline Output:*** *Output a set of n pricing rules $p_i(\cdot)$. In the case of item and anonymous prices, this is equivalent to setting a price for each item.*

4. ***Online Output:*** *In some situations, we allow for dynamic pricing rules where the pricing rules' output prior to the arrival of agents are subject to change based off of the online input. Then, based off of the pricing rule at step t, person t gets his most desirable set of remaining items notated as $\mathsf{ALG}_t(\mathbf{v})$ and pays the corresponding price $p_t(\mathsf{ALG}_t(\mathbf{v}))$.*

5. ***Output:*** *The final allocation is $\mathsf{ALG}(\mathbf{v})$.*

It has been shown that balanced prices generalize to dynamic pricing rules in [PDTKL17]. Such prices can be used to get good approximations of the optimal welfare by considering welfare to be comprised of utility and revenue. We bound each in order to evaluate the final welfare. This technique will be illustrated throughout the later sections. Lastly, a key property of these pricing mechanisms is that they are dominant strategy truthful, which is a result of agents having the choice to purchase their most desirable set causing it to be non-optimal to purchase anything but their mot favored set.

2.3 Prophet Inequalities

Prophet inequalities allow us to compare the worst case competitive stochastic ratio of our algorithm or mechanism, in our case posted price mechanisms, to the optimal allocation in hindsight:

$$\tau = \sup_{\mathcal{D}} \frac{\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[\mathbf{v}(\mathsf{OPT}(\mathbf{v}))]}{\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[\mathbf{v}(\mathsf{ALG}(\mathbf{v}))]}$$

It is evident that an equivalent form is:

$$\tau \cdot \mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[\mathbf{v}(\mathsf{ALG}(\mathbf{v}))] \geq \mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[\mathbf{v}(\mathsf{OPT}(\mathbf{v}))]$$

which would provide a τ approximation or a τ prophet inequality. In this case, τ could also be referred to as the competitive ratio of our algorithm. In general, these tools allow us to conveniently lower bound the relative performance of an algorithm to the optimum. Referring back to the work by Cahn and later Kleinberg and Weinberg, we realize that they both provide a 2 prophet inequality for the single item case [SC84, KW12]. One can also immediately see its relationship to Price of Anarchy within a Bayesian Game as mentioned earlier. The goal of this paper is to design an explicit posted price mechanism achieving a constant factor prophet inequality (i.e. $\tau = O(1)$) for general subadditive valuation profiles $\mathbf{v}$.

Chapter 3

Balanced Prices

In this chapter, we introduce a powerful posted price mechanism known as *balanced prices*. We walk through the intuition behind such prices, prove the guarantees they provide, and introduce their many implications. We will also show impossibility results regarding balanced prices. Furthermore, we use the quasi-linearity of utility to write welfare as the sum of revenue and utility generated by the mechanism. Much of these results are from the Feldman et al. [PDTKL17]. We finish with discussion of the more advanced results proved in [PDTKL17] and possible extensions.

3.1 Single Item Mechanisms

In order to provide intuition behind this mechanism, we consider the single item case, where each agent has a random value in $\mathbb{R}_{\geq 0}$ for the item. Furthermore, for any agent $i \in N$, their value v_i is drawn from $\mathcal{D}_i$ independently. The joint distribution is $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ and is known by the auctioneer ahead of time. Once again, agents arrive sequentially. As expected, this is the variant of the problem from Cahn's or Kleinberg and Weinberg's [SC84, KW12] work in the context of a combinatorial auction.

We notate $v^{(1)} := \max_i v_i$ and post the balanced price of $p = \mathbb{E}_{\mathbf{v} \sim \mathcal{D}}[v^{(1)}]/2$ where we notate q as the probability of the item being sold. Note that this is exactly the rule used in [KW12], however, the proof and work was done within a slightly different context. In general, this reinforces how balanced prices seem to encompass the work by Kleinberg and Weinberg alongside much more as we will see later.

In order to approximate the welfare, we use a combination of a revenue and utility bound. First, note that the expected revenue is simply $q \cdot \mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}]/2$. This is because with probability q, the item is sold yielding revenue $\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}]/2$ and with probability $1-q$, our revenue is 0. Notate the indicator random variables $X_i := \mathbb{I}(\text{item was still available when agent } i \text{ arrived})$ for all $i \in N$. Notate the event that the item was never sold to be $\{X_{n+1} = 1\}$. Then consider the utility bound:

$$\begin{aligned}\mathbb{E}[u(\mathbf{v})] &= \sum_{i\in N} \mathbb{E}[(v_i - \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}])_+ \cdot X_i] \\ &= \sum_{i\in N} \mathbb{E}[(v_i - \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}])_+] \cdot \mathbb{E}[X_i] \\ &= \sum_{i\in N} \mathbb{E}[(v_i - \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}])_+] \cdot \Pr[X_i = 1] \geq \sum_{i\in N} \mathbb{E}[(v_i - \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}])_+] \cdot (1-q)\end{aligned}$$

The last inequality holds since $1 - q = \Pr[X_{n+1} = 1] \leq \Pr[X_i = 1]$ for all $i \in [n]$. Combining our bounds yields the following:

$$\begin{aligned}\mathbb{E}[\mathsf{ALG}(\mathbf{v})] &\geq \mathbb{E}[r(\mathbf{v})] + \mathbb{E}[u(\mathbf{v})] \\ &\geq \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}] \cdot q + \sum_{i\in N} \mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[(v_i - \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}])_+] \cdot (1-q) \\ &\geq \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}] \cdot q + \mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[(v^{(1)} - \frac{v^{(1)}}{2})_+] \cdot (1-q) \\ &\geq \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}] \cdot q + \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}] \cdot (1-q) = \frac{1}{2}\mathbb{E}_{\mathbf{v}\sim\mathcal{D}}[v^{(1)}]\end{aligned}$$

In this analysis, we use that the maximum element of a sum lower bounds the sum, and furthermore the expression is clearly non-negative allowing us to drop $(\cdot)_+$. This yields a 2-approximation for the expected welfare with this technique of balancing the price. If the item was bought early, the algorithm does well by generating revenue and otherwise by generating utility. This thus illustrates the "balance" from posting such prices. Very soon, we will see how to generalize these prices to multiple item mechanisms, which is the main focus of this paper. Interestingly enough, the extension is very intuitive as we will see below.

3.2 Multiple Item Mechanisms

Returning to our multiple item setting, we first provide intuition then present the robust theorems and accompanying definitions from [PDTKL17] later. The general idea of balanced prices is that these prices are low enough such that if the optimal set U_i for agent i is available, then they are willing to purchase it: $\sum_{j \in U_i} p_j \leq v_i(U_i)$. Moreover, they must be high enough such that items do not get taken earlier, or so that loss in utility for agent i is at least made up by revenue from earlier. This can be written as: $\sum_{j \in T} p_j \geq v_i(U_i) - v_i(U_i \setminus T)$. For the remainder of the paper and this section, we consider a generalization of the above constraints and an accompanying theorem from Thomas Kesselheim's slides at EC 21 [KS21] shown below:

Definition 3.2.1. *A valuation function v_i admits (α, β)-balanced prices or is $\alpha\beta$ balanced if for any set $U \subseteq M$ there exist p_j^U for all $j \in U$ such that for all $T \subseteq U$:*

1. $\sum_{j \in T} p_j^U \geq \frac{1}{\alpha}(v_i(U) - v_i(U \setminus T))$
2. $\sum_{j \in U \setminus T} p_j^U \leq \beta \cdot v_i(U \setminus T)$

Theorem 3.2.2 ([KS21]). *If a class of valuation functions $\mathbf{v}$ allow for balanced prices as defined in Definition 3.2.1, then for any product distribution $\mathcal{D}$ there exists appropriate prices so that for the resulting allocation $\mathbf{x} = (x_1, \ldots, x_n)$ we have:*

$$\mathbb{E}_{\mathbf{v} \sim \mathcal{D}}[\mathbf{v}(\mathbf{x})] \geq \frac{1}{1+\alpha\beta} \cdot \mathbb{E}_{\mathbf{v} \sim \mathcal{D}}[\mathbf{v}(\mathit{OPT}(\mathbf{v}))]$$

The above definition generalizes the intuition highlighted at the beginning. Furthermore, this definition makes proving later impossibility claims easier due to its characterization of any arbitrary valuation function rather than a profile of functions. We proceed to provide the most robust result regarding balanced prices, characterizing it for the most general setting: a profile of valuation functions. This generalization once again highlights the flexibility and breadth of this pricing mechanism.

3.3 Robust definition of $\alpha\beta$ balanced prices

First, we define the term exchange compatible as the set of outcome profiles $\mathcal{H} \subseteq X$ where for a feasible allocation $\mathbf{x} \in \mathcal{F}$ and for all allocations $\mathbf{y} \in \mathcal{H}$, we may replace x_i, the allocation for any agent $i \in N$, with y_i such that the resulting allocation is still feasible. Mathematically, for all $\mathbf{y} \in \mathcal{H}$ and agents $i \in N$, we have that $(y_i, \mathbf{x}_{-i}) \in \mathcal{F}$. We may extend this to a family of sets $(\mathcal{F}_\mathbf{x})_{\mathbf{x}\in X}$ which are exchange compatible if the set $\mathcal{F}_\mathbf{x}$ is exchange compatible with allocation $\mathbf{x}$ for all $\mathbf{x} \in X$. Such a definition allows us to impose more general feasibility constraints as shown later. Now, we use the definition and accompanying theorem from [PDTKL17], which state the following:

Definition 3.3.1 (Key Balanced Prices Definition)**.** *Let $\alpha > 0, \beta \geq 0$. Given a set of feasible allocations $\mathcal{F}$ and a valuation profile $\mathbf{v}$, a pricing rule profile $\mathbf{p}$ is (α, β) balanced with respect to the allocation rule OPT [1], an exchange-compatible family of sets $(\mathcal{F}_\mathbf{x})_{\mathbf{x}\in X}$, and an indexing of the players $i \in [n]$ if for all $\mathbf{x} \in \mathcal{F}$:*

1. $\sum_{i\in N} p_i(x_i) \geq \frac{1}{\alpha}(\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}, \mathcal{F}_\mathbf{x})))$

2. *For all* $\mathbf{x}' \in \mathcal{F}_\mathbf{x}, \sum_i p_i(x'_i) \leq \beta(\mathbf{v}(\mathsf{OPT}(\mathbf{v}, \mathcal{F}_\mathbf{x})))$

[1] In the paper, this definition considers pricing rules to be balanced with respect to an arbitrary allocation rule ALG since it makes computation of prices easier. Here, we take the specific allocation rule to be the optimal allocation to highlight its relevance to prophet inequalities.

Theorem 3.3.2 ([PDTKL17]). *Suppose that the collection of pricing rules profiles $(\mathbf{p}^{\mathbf{v}})_{\mathbf{v}\in V}$ for feasible outcomes $\mathcal{F}$ and valuation profiles $\mathbf{v} \in V$ is (α, β)-balanced with respect to the allocation rule OPT and indexing of the players $i \in [n]$. Then for $\delta = \frac{\alpha}{1+\alpha\beta}$, the posted-price mechanism with pricing rule $\delta\mathbf{p}$, where $p_i(x_i) = \mathbb{E}_{\tilde{\mathbf{v}}}[p_i^{\tilde{\mathbf{v}}}(x_i)]$, generates welfare at least $\frac{1}{1+\alpha\beta} \cdot \mathbb{E}_{\mathbf{v}}[\mathbf{v}(\mathsf{OPT}(\mathbf{v}))]$ when approaching players in the order they are indexed.*

Proof. To demonstrate the above result, we once again will use a utility and revenue bound then combine them. To start, we have that $(\mathcal{F}_{\mathbf{x}})_{\mathbf{x}\in X}$ is an exchange-compatible family of sets with respect to which the pricing rules $(\mathbf{p}^{\mathbf{v}})_{\mathbf{v}\in V}$ are balanced. Throughout the proof, in order to be more concise, we use the following notation: $\mathbf{x}(\mathbf{v})$ is the allocation returned by our mechanism on the valuation profile $\mathbf{v}$. Furthermore, we notate $\mathbf{x}'(\mathbf{v}, \mathbf{v}') = \mathsf{OPT}(\mathbf{v}', \mathcal{F}_{\mathbf{x}(\mathbf{v})})$.

Utility Bound: We start by sampling $\mathbf{v}' \sim \mathcal{D}$. Next, since we want to lower bound utility, we consider letting agent i buy $\mathsf{OPT}_i((v_i, \mathbf{v}'_{-i}), \mathcal{F}_{\mathbf{x}(v'_i, \mathbf{v}_{-i})})$ [2] and letting them pay the respective posted price $\delta p_i(\mathsf{OPT}_i((v_i, \mathbf{v}'_{-i}), \mathcal{F}_{\mathbf{x}(v'_i, \mathbf{v}_{-i})}))$, which is indeed a feasible choice and thus a lower bound on utility. Doing so alongside the fact that $\mathbf{v}$, $\mathbf{v}'$ are identically and independently distributed, we can write in expectation for agent i:

$$\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] &\geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[v_i\left(\mathsf{OPT}_i((v_i, \mathbf{v}'_{-i}), \mathcal{F}_{\mathbf{x}(v'_i, \mathbf{v}_{-i})})\right) - \delta p_i\left(\mathsf{OPT}_i((v_i, \mathbf{v}'_{-i}), \mathcal{F}_{\mathbf{x}(v'_i, \mathbf{v}_{-i})})\right)\right] \\
&= \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[v'_i\left(x'_i(\mathbf{v}, \mathbf{v}')\right) - \delta p_i\left(x'_i(\mathbf{v}, \mathbf{v}')\right)\right]
\end{aligned}$$

The last line applies the notation we defined earlier alongside the fact that $(v_i, \mathbf{v}'_{-i})$ is i.i.d to $\mathbf{v}'$ and $(v'_i, \mathbf{v}_{-i})$ is i.i.d to $\mathbf{v}$, therefore equivalent in expectation. Such a trick is widely used in this literature. After summing across all agents and using linearity

[2] This is known to be the hallucination trick, and it allows one to move around expectations conveniently.

of expectation, we obtain:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\mathbf{v}'\left(\mathbf{x}'(\mathbf{v},\mathbf{v}')\right)\right] - \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\delta \cdot p_i\left(x_i'(\mathbf{v},\mathbf{v}')\right)\right]$$
$$= \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\mathbf{v}'\left(\mathsf{OPT}(\mathbf{v}',\mathcal{F}_{\mathbf{x}(\mathbf{v})})\right)\right] - \delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{i=1}^{n} p_i\left(x_i'(\mathbf{v},\mathbf{v}')\right)\right]$$

Note that second term on the RHS of the equation above can be reformulated by using the definition of the prices that were posted:

$$\delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{i=1}^{n} p_i\left(x_i'(\mathbf{v},\mathbf{v}')\right)\right] = \delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{i=1}^{n} \mathbb{E}_{\widetilde{\mathbf{v}}}[p_i^{\widetilde{\mathbf{v}}}\left(x_i'(\mathbf{v},\mathbf{v}')\right)]\right]$$
$$= \delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}',\widetilde{\mathbf{v}}}\left[\sum_{i=1}^{n} p_i^{\widetilde{\mathbf{v}}}\left(\mathsf{OPT}_i(\mathbf{v}',\mathcal{F}_{\mathbf{x}(\mathbf{v})})\right)\right]$$

Now we utilize the properties from Definition 3.3.1 in order to bound the term inside the expectation above: $\sum_{i=1}^{n} p_i^{\widetilde{\mathbf{v}}}\left(\mathsf{OPT}_i(\mathbf{v}',\mathcal{F}_{\mathbf{x}(\mathbf{v})})\right)$. To do so, we recall that property (1) is true for all $\mathbf{x} \in \mathcal{F}$. Consider $\mathbf{x} = \mathbf{x}(\mathbf{v})$. Once again, note that property (2) is true for all $\mathbf{x}' \in \mathcal{F}_{\mathbf{x}(\mathbf{v})}$. Consider $\mathbf{x}' = \mathsf{OPT}(\mathbf{v}',\mathcal{F}_{\mathbf{x}(\mathbf{v})})$. Consequently, we may claim the following:

$$\sum_{i=1}^{n} p_i^{\widetilde{\mathbf{v}}}\left(\mathsf{OPT}_i(\mathbf{v}',\mathcal{F}_{\mathbf{x}(\mathbf{v})})\right) \leq \beta(\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}},\mathcal{F}_{\mathbf{x}(\mathbf{v})}))) \tag{3.1}$$

Substituting the result from Equation 3.1, we find:

$$\delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{i=1}^{n} p_i\left(x_i'(\mathbf{v},\mathbf{v}')\right)\right] \leq \delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}',\widetilde{\mathbf{v}}}\left[\beta \cdot \widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}},\mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right] = \delta\beta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}}\left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}},\mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right]$$

We can drop the expectation over $\mathbf{v}'$ since there are no terms dependent on $\mathbf{v}'$ thus justifying the last equality. Substituting this result back in, our utility bound is:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\mathbf{v}'\left(\mathsf{OPT}(\mathbf{v}',\mathcal{F}_{\mathbf{x}(\mathbf{v})})\right)\right] - \delta\beta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}}\left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}},\mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right]$$

Replacing $\mathbf{v}'$ with $\widetilde{\mathbf{v}}$ in the first term of the RHS since they are i.i.d gives:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq (1-\delta\beta) \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}} \left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right] \tag{3.2}$$

Revenue Bound: This bound is simpler than the previous one. First, note that the revenue generated by agent i on the valuation profile $\mathbf{v}$ is $r_i(\mathbf{v}) = \delta \cdot p_i(x_i(\mathbf{v}))$. Summing this over all agents and taking an expectation, we find:

$$\begin{aligned}\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}\left[r_i(\mathbf{v})\right] &= \mathop{\mathbb{E}}_{\mathbf{v}}\left[\sum_{i=1}^{n} \delta \cdot p_i(x_i(\mathbf{v}))\right] \\ &= \mathop{\mathbb{E}}_{\mathbf{v}}\left[\sum_{i=1}^{n} \delta \cdot \mathbb{E}_{\widetilde{\mathbf{v}}}[p_i^{\widetilde{\mathbf{v}}}(x_i(\mathbf{v}))]\right] = \delta \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}}\left[\sum_{i=1}^{n} p_i^{\widetilde{\mathbf{v}}}(x_i(\mathbf{v}))\right]\end{aligned}$$

Above we use linearity of expectation and then apply the definition of the prices we posted. Now we utilize the properties from Definition 3.3.1 to bound $\sum_{i=1}^{n} p_i^{\widetilde{\mathbf{v}}}(x_i(\mathbf{v}))$. Once again, we recall that property (1) is true for all $\mathbf{x} \in \mathcal{F}$. Consider $\mathbf{x} = \mathbf{x}(\mathbf{v})$. Using this allows us to claim the following:

$$\sum_{i=1}^{n} p_i^{\widetilde{\mathbf{v}}}(x_i(\mathbf{v})) \geq \frac{1}{\alpha} \cdot \left(\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}})) - \widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right) \tag{3.3}$$

Substituting the result from Equation 3.3 into the initial bound gives us:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}\left[r_i(\mathbf{v})\right] \geq \frac{\delta}{\alpha} \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}} \left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}})) - \widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right] \tag{3.4}$$

Now we use the quasi-linearity property to bound welfare by summing overall utility and revenue. Combining the two bounds from Equations 3.2 and 3.4 yields:

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathbf{v}}[\mathbf{v}(\mathsf{ALG}(\mathbf{v}))] &\geq \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}\left[u_i(\mathbf{v})\right] + \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}\left[r_i(\mathbf{v})\right] \\ &\geq (1-\delta\beta) \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}} \left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right] + \frac{\delta}{\alpha} \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}} \left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}})) - \widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right]\end{aligned}$$

For $\delta = \frac{\alpha}{1+\alpha\beta}$, we see that $1 - \delta\beta = 1 - \frac{\alpha\beta}{1+\alpha\beta} = \frac{1}{1+\alpha\beta} = \frac{\delta}{\alpha}$. This allows us to write:

$$
\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{v}}[\mathbf{v}(\mathsf{ALG}(\mathbf{v}))] \geq (1-\delta\beta) \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}}\left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right] + \frac{\delta}{\alpha} \cdot \mathop{\mathbb{E}}_{\widetilde{\mathbf{v}}}\left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}))\right] \\
- \frac{\delta}{\alpha} \cdot \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}}\left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}, \mathcal{F}_{\mathbf{x}(\mathbf{v})}))\right] \\
= \frac{1}{1+\alpha\beta} \cdot \mathop{\mathbb{E}}_{\widetilde{\mathbf{v}}}\left[\widetilde{\mathbf{v}}(\mathsf{OPT}(\widetilde{\mathbf{v}}))\right] \\
= \frac{1}{1+\alpha\beta} \cdot \mathop{\mathbb{E}}_{\mathbf{v}}\left[\mathbf{v}(\mathsf{OPT}(\mathbf{v}))\right]
\end{aligned}
$$

The first step uses linearity of expectation, and we drop the expectation over $\mathbf{v}$ on the middle term since there are no terms dependent on $\mathbf{v}$ in it. Next, we substitute our results above. The last equality holds since both $\mathbf{v}$ and $\widetilde{\mathbf{v}}$ are i.i.d and therefore equivalent in expectation, proving the desired result. □

While Definition 3.3.1 is very robust, it is with respect to a family of sets $(\mathcal{F}_{\mathbf{x}})_{\mathbf{x}\in X}$. For our purposes, we define $\mathcal{F}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathcal{F}$ to be the set of allocations such that each $\mathbf{x}' \in \mathcal{F}_{\mathbf{x}}$ is disjoint of $\mathbf{x}$, each $\mathbf{x}'$ is feasible after $\mathbf{x}$ has been allocated, and $\mathbf{x} \cup \mathbf{x}' \in \mathcal{F}$. This leads to a special case of Definition 3.3.1 that is easier to interpret and will be used in the remainder of the paper. Lastly, note that by our precise choice of $(\mathcal{F}_{\mathbf{x}})_{\mathbf{x}\in X}$, it turns out that $\mathbf{v}(\mathsf{OPT}(\mathbf{v}, \mathcal{F}_{\mathbf{x}})) = \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))$, where $(\mathbf{v}|\mathbf{x})$ is the allocation that is feasible, disjoint of $\mathbf{x}$, and maximizes value.

Definition 3.3.3 (Special Case)**.** *Let $\alpha, \beta > 0$. A pricing rule profile $\mathbf{p}^{\mathbf{v}} = (p_1^{\mathbf{v}}, \ldots, p_n^{\mathbf{v}})$ defined by pricing rules (functions) $p_i^{\mathbf{v}} : 2^M \to \mathbb{R}_{\geq 0}$ is (α, β) balanced with respect to the valuation profile $\mathbf{v}$ if for all $\mathbf{x} \in \mathcal{F}$ and all $\mathbf{x}' \in \mathcal{F}$ with $\mathbf{x}, \mathbf{x}^{'}$ disjoint and $\mathbf{x} \cup \mathbf{x}' \in \mathcal{F}$,*

1. $\sum_i p_i^{\mathbf{v}}(x_i) \geq \frac{1}{\alpha}(\mathbf{v}(\mathit{OPT}(\mathbf{v})) - \mathbf{v}(\mathit{OPT}(\mathbf{v}|\mathbf{x})))$
2. $\sum_i p_i^{\mathbf{v}}(x_i') \leq \beta(\mathbf{v}(\mathit{OPT}(\mathbf{v}|\mathbf{x})))$

The intuition behind this definition is easier to see, since for any allocation $\mathbf{x}$ and $\mathbf{x}'$ that is feasible after $\mathbf{x}$, we know that $\mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))$ is the value that remains upon allocating $\mathbf{x}$. Similarly, we know that $\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))$ is the value lost from allocating $\mathbf{x}$.

The authors in [PDTKL17] showed that XOS valuations admit $(1,1)$-balanced prices and therefore Theorem 3.3.2 immediately provides a 2-approximation, which is the tightest current bound for XOS valuations [FGL14]. In the same paper, the authors also provide an explicit item pricing rule profile that is $(1,1)$-balanced with respect to general XOS valuations. This gives a $O(\log m)$ prophet inequality for subadditive valuations since they can be approximated via XOS valuations with a logarithmic penalty [BR11, Dob07]. In the next section, we see why there does not exist balanced item prices that provide a guarantee of $o(\log m)$.

3.4 Why do subadditive valuations give poor approximations?

We begin by introducing the notion of fractionally subadditive valuations as shown in the paper by Bhawalkar and Roughgarden [BR11] and later relating them to balanced prices.

Definition 3.4.1. *A valuation v on a set of items M is β-fractionally subadditive*[3] *if for each subset $T \subseteq M$, there is a price vector $\mathbf{a}$ that satisfies:*

1. *For all subsets $S \subseteq T$: $\sum_{j \in S} a_j \leq v(S)$*
2. *$\sum_{j \in T} a_j \geq v(T)/\beta$*

Furthermore, it is known that for every set M of m goods, every subadditive valuation v over M is also $\mathcal{H}_m$-fractionally subadditive, where $\mathcal{H}_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$.

[3] When $\beta = 1$, these are precisely XOS. When $\beta = O(\log m)$, these are subadditive valuations as shown by the remark above. This is why subadditive valuations can be approximated by XOS with a logarithmic penalty.

This definition is a relaxation of subadditivity for a valuation. We can also see the immediate resemblance to Definition 3.2.1. Thus, it allows us to bridge the gap between subadditive valuations and valuations being $\alpha\beta$ balanced or admitting (α, β)-balanced prices. We delve deeper into this relation next.

Proposition 3.4.2. *Let $\alpha, \beta > 0$. If a valuation function $v_i(\cdot)$ admits (α, β)-balanced prices from Definition 3.2.1 then it is true that $v_i(\cdot)$ is $\alpha\beta$-fractionally subadditive.*

Proof. We assume $v_i(\cdot)$ to admit (α, β)-balanced prices. This immediately implies, by relabeling Definition 3.2.1 for any set $T \subseteq M$, that there exists p_j for all $j \in T$, call it $\mathbf{p}^T$, such that for all $S \subseteq T$:

(a) $\sum_{j \in S} p_j^T \geq \frac{1}{\alpha}(v_i(T) - v_i(T \setminus S))$

(b) $\sum_{j \in T \setminus S} p_j^T \leq \beta \cdot v_i(T \setminus S)$

Instead, consider the following price vector $\mathbf{a} = \frac{1}{\beta} \cdot \mathbf{p}^T$. We can now write the following:

1. For all $S \subseteq T$, $\sum_{j \in T \setminus S} a_j \leq v_i(T \setminus S) \implies \sum_{j \in S'} a_j \leq v_i(S')$ by relabeling $S' = T \setminus S$

2. $\sum_{j \in T} a_j \geq \frac{v_i(T)}{\alpha\beta}$

The first line holds by dividing out β from both sides of property (b) listed above and consequently plugging in our price vector $\mathbf{a}$. Second, note that considering all $T \setminus S \subseteq T$ is equivalent to considering all $S \subseteq T$ thus the relabeling is valid. As to the second inequality, we look to property (a) and determine that we need only consider $S = T$ and divide by β on both sides to accommodate our new price vector $\mathbf{a}$. Lastly, notice that $v_i(T \setminus T) = 0$ which gives the result. The above characterization satisfies the formulation presented in Definition 3.4.1 and therefore shows that $v_i(\cdot)$ is in fact $\alpha\beta$-fractionally subadditive. $\square$

In order to show why balanced prices fail to provide a constant factor prophet approximation for general subadditive valuations, we first display the existence of a "bad" subadditive valuation. The existence of such a subadditive valuation was guaranteed in the work by Bhawalkar and Roughgarden [BR11]:

Lemma 3.4.3 ([BR11])**.** *For every set M of m goods, there exists a subadditive valuation v over M that is not β-fractionally subadditive for $\beta < \mathcal{H}_m$.*

We will use this Lemma, originally presented in [BR11] without proof. We require such a lemma as it provides us with an impossibility claim regarding fractionally subadditive valuations, which we showed earlier were connected to valuations admitting (α, β)-balanced prices. We will delve further into this connection next. The proof of this lemma uses a variant of an existing lower bound for the integrality gap of set cover linear programs – we defer the interested reader to [BR11].

Proposition 3.4.4. *There exists a subadditive valuation v, which is not $\alpha\beta$ balanced for $\alpha\beta < \mathcal{H}_m$.*

Proof. Consider the valuation v in Lemma 3.4.3. Assume for the sake of contradiction that it is (α, β) balanced for $\alpha\beta < \mathcal{H}_m$. By Proposition 3.4.2, we know that v is also $\alpha\beta$ fractionally subadditive for $\alpha\beta < \mathcal{H}_m$, a contradiction to Lemma 3.4.3. □

In an ideal world, for all subaddtive profiles $\mathbf{v}$ drawn from the product distribution $\mathcal{D}$, there would exist (α, β) balanced item prices for $\alpha\beta < \log m$. In this case, we could simply post such prices and therefore find a constructive constant factor prophet inequality for subadditive combinatorial auctions.

However, we display below that there exists a specific subadditive profile $\mathbf{v}$ such that they do not assume (α, β) balanced prices as defined in Definition 3.3.3 for $\alpha\beta < \mathcal{H}_m$. This would exhaustively show that the (α, β)-balanced prices are not a valid approach towards finding an constant factor prophet inequality for subadditive combinatorial auctions. As usual, we assume static, anonymous, and item prices.

Example 3.4.1. *Consider the following subadditive instance* $\mathbf{v} = (v_1, v_2)$ *where* $v_1(\cdot)$ *is the subadditive function given by Proposition 3.4.4.* $v_2(\cdot)$ *is an additive function such that it has high value for all items not in agent* 1*'s optimal set* $U \subseteq M$*:* $v_2(\{i\}) = C \cdot v_1(\{i\}), \forall i \notin U$ *and* 0 *otherwise, where we let* C *be some large constant. Additive functions are by definition subadditive, making this a valid subadditive instance. Here, there does not exist a pricing rule profile that is* (α, β)*-balanced with respect to* $\mathbf{v}$ *such that* $\alpha\beta < \mathcal{H}_m$*.*

Proof. Assume for the sake of contradiction that there exists a pricing rule profile $\mathbf{p}^{\mathbf{v}} = (p_1^{\mathbf{v}}, p_2^{\mathbf{v}})$ that is (α, β)-balanced with respect to $\mathbf{v}$ for $\alpha\beta < \mathcal{H}_m$.

First, we note that $\mathsf{OPT}(\mathbf{v}) = (U, M \setminus U)$. This is because U is the optimal valued set for agent 1. Furthermore, since agent 2's valuation is additive and only has non-zero value for items not in U, we find their optimal set to be $M \setminus U$. Now consider an $\mathbf{x}$ of the form $(T, \varnothing)$, where $T \subseteq U$. Consequently, $\mathsf{OPT}(\mathbf{v}|\mathbf{x}) = (U \setminus T, M \setminus U)$, since only T has been awarded so far. From property (1) of Definition 3.3.3, we can write the following:

$$\sum_{j \in T} p_j \geq \frac{1}{\alpha} \left(v_1(U) + v_2(M \setminus U) - v_1(U \setminus T) - v_2(M \setminus U) \right)$$

$$\sum_{j \in T} p_j \geq \frac{1}{\alpha} (v_1(U) - v_1(U \setminus T)) \tag{3.5}$$

Now consider $\mathbf{x} = (T, M \setminus U)$ and $\mathbf{x}' = (U \setminus T, \varnothing)$ for some $T \subseteq U$ as well as the same pricing rule $\mathbf{p}^{\mathbf{v}}$. We know that $\mathsf{OPT}(\mathbf{v}|\mathbf{x}) = (U \setminus T, \varnothing)$ since agent 2 received all their valued items and furthermore T was offered from agent 1's valued items. From property (2) of Definition 3.3.3:

$$\sum_{j \in U \setminus T} p_j \leq \beta v_1(U \setminus T) \tag{3.6}$$

Equations 3.5 and 3.6 satisfy both properties of Definition 3.2.1 showing that

$v_1(\cdot)$ is $\alpha\beta$ balanced and, by our assumption, we know that $\alpha\beta < \mathcal{H}_m$. This poses a contradiction to the definition of $v_1(\cdot)$ from Proposition 3.4.4. Therefore, there cannot exist (α, β) prices for this instance, which contradicts that for all subadditive instances there are (α, β)-balanced prices for $\alpha\beta < \mathcal{H}_m = \Omega(\log m)$, proving the desired result. □

This exhaustively proves that balanced item prices presented in [PDTKL17] fail to provide a prophet inequality of the form $o(\log m)$ for general subadditive valuations. We will see that bridging this gap requires posting different prices as presented in the next chapter.

Chapter 4

$O(\log \log m)$ Posted Price Mechanism

To bridge the asymptotic $O(\log m)$ gap with item prices, one needs to post different prices altogether. In 2020, Dütting, Lucier, and Kesselheim developed new prices that would bridge this gap to $O(\log \log m)$ [KL20]. The intuition for these prices stems from exactly what balanced prices fail to capture regarding subadditive valuations. Balanced prices have the unique property that the sum of the prices "nearly" approximates the optimal welfare, which leads to most items either being sold or affordable as prices are not too high. However, such a property can be detrimental to subadditive combinatorial auctions as explained below.

An example of a subadditive function is $f(x) = \sqrt{x}$, since $f(x+y) \leq f(x)+f(y)$ for all $x, y \geq 0$. The intuition here is that a large increase in the input does not correspond to a significant change in the output, due to the function flattening quickly. This explains why, with subadditive valuations, there are instances where it is beneficial to only consider a smaller subset of the items available for purchase that capture a majority of the optimal welfare. Dütting, Lucier, and Kesselheim achieve this by pricing items higher and targeting a specific q fraction of items. In fact, they proved that this specific fraction they target always exists.

4.1 Key Lemma for $O(\log\log m)$ Prophet Inequality

We begin by introducing a lemma that establishes the existence of appropriate prices in the complete information case *pointwise* to an agent i and their respective valuation v_i. We will see that this lemma is insufficient to prove a global guarantee and, in turn, the authors design different prices to do so, a stark difference from the balanced price analysis.

Lemma 4.1.1 ([KL20]). *For every $i \in N$, subadditive valuation $v_i \in V_i$, and set $U \subseteq M$ there exists prices p_j for $j \in U$ and a probability distribution δ over $S \subseteq U$ such that for all $T \subseteq U$:*

$$\sum_{j \in T} p_j + \sum_{S \subseteq U} \delta_S \left(v_i(S \setminus T) - \sum_{j \in S} p_j \right) \geq \frac{v_i(U)}{\alpha}$$

where $\alpha \in O(\log\log m)$.

Proof. The proof of this lemma was presented in [KL20]; however, we include it below for completeness. Consider any arbitrary agent i with valuation v_i, and a set $U \subseteq M$. We need to show that there exist prices p_j for $j \in U$, and a probability distribution δ over subsets $S \subseteq U$, such that for all $T \subseteq U$ we have:

$$\sum_{j \in T} p_j + \sum_{S \subseteq U} \delta_S \left(v_i(S \setminus T) - \sum_{j \in S} p_j \right) \geq \frac{v_i(U)}{\alpha}$$

which upon multiplying through and rearranging can be written as:

$$\sum_{S \subseteq U} \delta_S \sum_{j \in S} p_j - \sum_{j \in T} p_j \leq \sum_{S \subseteq U} \delta_S v_i(S \setminus T) - \frac{v_i(U)}{\alpha} \tag{4.1}$$

Now, we write the following LP for fixed δ with variables p_j for all $j \in U$ and slack variables ℓ_+ and ℓ_-:

$$\begin{aligned}
\max \quad & \ell_+ - \ell_- \\
\text{s.t.} \quad & \sum_{S \subseteq U} \delta_S \sum_{j \in S} p_j - \sum_{j \in T} p_j + \ell_+ - \ell_- \leq \sum_{S \subseteq U} \delta_S v_i(S \setminus T) - \frac{v_i(U)}{\alpha} && \text{for all } T \subseteq U \\
& p_j \geq 0 && \text{for all } j \in U \\
& \ell_+ \geq 0 \\
& \ell_- \geq 0
\end{aligned}$$

We can claim that there exist prices p_j for all $j \in U$ satisfying Equation 4.1 if and only if the optimal solution to the above LP has non-negative objective value, since a non-negative slack implies that Equation 4.1 holds. We show this via the dual program:

$$\begin{aligned}
\min \quad & \sum_{T \subseteq U} \mu_T \left(\sum_{S \subseteq U} \delta_S v_i(S \setminus T) - \frac{v_i(U)}{\alpha} \right) \\
\text{s.t.} \quad & \sum_{T \subseteq U} \sum_{S : j \in S} \mu_T \delta_S - \sum_{T : j \in T} \mu_T \geq 0 && \text{for all } j \in U \\
& \sum_{T \subseteq U} \mu_T = 1 \\
& \mu_T \geq 0 && \text{for all } T \subseteq U
\end{aligned}$$

Next, note the constraints in the dual LP can be rewritten as:

$$\begin{aligned}
0 &\leq \sum_{T \subseteq U} \sum_{S : j \in S} \mu_T \delta_S - \sum_{T : j \in T} \mu_T \\
&= \sum_{S : j \in S} \delta_S \sum_{T \subseteq U} \mu_T - \sum_{T : j \in T} \mu_T \\
&= \sum_{S : j \in S} \delta_S - \sum_{T : j \in T} \mu_T
\end{aligned}$$

The first line holds by rearranging the order of summation and the second by using that $\sum_T \mu_T = 1$. Notice that μ and δ can be thought of as distributions over subsets of U. By strong duality, we know that if for every feasible solution $\mu = (\mu_T)_{T\subseteq U}$ to the dual the corresponding objective value is non-negative, then the optimal objective value to the primal must be non-negative as well, as desired. Thus, the primal has optimal non-negative objective value if and only if for every μ, where $\sum_{T:j\in T} \mu_T \leq \sum_{S:j\in S} \delta_S$ we have:

$$\sum_{T\subseteq U} \sum_{S\subseteq U} \mu_T \delta_S v_i(S \setminus T) \geq \frac{v_i(U)}{\alpha} \tag{4.2}$$

Here, the LHS can be thought of as the reward (to the max player) of a zero sum game, where the max player plays mixed strategy δ and the min player plays mixed strategy μ, each being distributions over subsets of U. Thus, it would suffice to show that there exists a mixed strategy (resp. distribution) δ such that for all mixed strategies μ that put at most as much probability mass on each item $j \in U$ as δ we have reward at least $v_i(U)/\alpha$.

We let $\mathbf{q} = (q_1, \ldots, q_{|U|}) \in [0,1]^{|U|}$ and notate $\Delta(\mathbf{q})$ to be set of probability distributions ξ over subsets $S \subseteq U$ such that $\sum_{T:j\in T} \xi_T \leq q_j$ for all $j \in U$. Define the following:

$$g(\mathbf{q}) = \max_{\delta\in\Delta(\mathbf{q})} \min_{\mu\in\Delta(\mathbf{q})} \sum_{T\subseteq U} \sum_{S\subseteq U} \mu_T \delta_S v_i(S \setminus T)$$

Proposition 4.1.2. *If there exists a* $\mathbf{q}$*, such that* $g(\mathbf{q}) \geq v_i(U)/O(\log\log m)$ *then Lemma 4.1.1 must hold true.*

Proof. Suppose there exists a $\mathbf{q}$ such that $g(\mathbf{q}) \geq v_i(U)/O(\log\log m)$. This implies the existence of a $\delta^* \in \Delta(\mathbf{q})$ such that:

$$\min_{\mu\in\Delta(\mathbf{q})} \sum_{T\subseteq U} \sum_{S\subseteq U} \mu_T \delta^*_S v_i(S \setminus T) \geq v_i(U)/O(\log\log m)$$

Therefore, for all $\mu \in \Delta(\mathbf{q})$, we have:

$$\sum_{T \subseteq U} \sum_{S \subseteq U} \mu_T \delta_S^* v_i(S \setminus T) \geq v_i(U)/O(\log \log m) \tag{4.3}$$

Now suppose $\sum_T \mu_T = 1$ (condition 1) and $\sum_{T:j \in T} \mu_T \leq \sum_{S:j \in S} \delta_S^*$ (condition 2) for some μ. Then since $\delta^* \in \Delta(\mathbf{q})$ we know $\sum_{S:j \in S} \delta_S^* \leq q_j$. Thus, $\sum_{T:j \in T} \mu_T \leq q_j$ and $\mu \in \Delta(\mathbf{q})$. This implies from Equation 4.3 that:

$$\sum_{T \subseteq U} \sum_{S \subseteq U} \mu_T \delta_S^* v_i(S \setminus T) \geq v_i(U)/O(\log \log m)$$

for all μ such that condition 1 and 2 hold.

As a result, we claim that the existence of a $\mathbf{q}$, where $g(\mathbf{q}) \geq v_i(U)/O(\log \log m)$, implies the existence of a δ^* such that for every μ, where $\sum_{T:j \in T} \mu_T \leq \sum_{S:j \in S} \delta_S^*$, we have that Equation 4.2 holds for $\alpha = O(\log \log m)$. This directly implies that there exist prices p_j for all $j \in U$ satisfying Equation 4.1 thus proving Lemma 4.1.1.

$\square$

The authors show that there always exists a uniform vector $\mathbf{q} = (q, \ldots, q) \in [0,1]^{|U|}$ such that $g(\mathbf{q}) \geq v_i(U)/O(\log \log m)$, the proof of which is omitted so we defer the interested reader to [KL20].

$\square$

Theorem 4.1.3 ([KL20]). *For subadditive combinatorial auctions, there is an $O(\log \log m)$-competitive posted price mechanism that uses static, anonymous, item prices.*

4.2 Why is the Pointwise Lemma Insufficient?

As mentioned earlier, we show that the powerful proof technique used in the balanced price analysis, by converting a pointwise lemma into a global guarantee, is impossible while using the prices defined in Lemma 4.1.1.[1]

Proposition 4.2.1. *The pointwise technical ingredient, Lemma 4.1.1, is not sufficient to prove Theorem 4.1.3 in the Bayesian setting using the technique presented in [PDTKL17].*

A full proof of the above result is included in Appendix A.1. Due to Proposition 4.2.1, we understand the need for the Bayesian Lemma from [KL20] in order to prove the $O(\log \log m)$ guarantee in expectation. We provide the Bayesian Lemma below for completeness:

Lemma 4.2.2 ([KL20])**.** *For every probability distribution $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ over subadditive valuation profiles, there exist prices p_j for $j \in M$ and probability distributions $\delta^{i,\mathbf{v}}$ over $S \subseteq M$ for all i and $\mathbf{v}$ such that for all $T \subseteq M$:*

$$\sum_{j \in T} p_j + \mathop{\mathbb{E}}_{\mathbf{v}} \left[\sum_{i=1}^{n} \sum_{S \subseteq M} \delta_S^{i,\mathbf{v}} \left(v_i(S \setminus T) - \sum_{j \in S} p_j \right) \right] \geq \frac{1}{\alpha} \cdot \mathop{\mathbb{E}}_{\mathbf{v}}[\mathbf{v}(\mathit{OPT}(\mathbf{v}))]$$

where $\alpha \in O(\log \log m)$.[2]

It turns out this lemma allows for a simple proof of Theorem 4.1.3, as shown below:

Proof of Theorem 4.1.3. Consider posting the prices p_j for all $j \in M$, whose existence are guaranteed by Lemma 4.2.2. We approach this proof by using our usual utility and revenue bound structure. Before getting started, we notate $\mathbf{x}(\mathbf{v})$ as the allocation returned by our mechanism on the valuation profile $\mathbf{v}$.

Utility Bound: Note that for any two valuation profiles $\mathbf{v}$ and $\mathbf{v}'$, agent i considers drawing a set from the distribution $\delta^{i,(v_i,\mathbf{v}'_{-i})}$ whose existence was established

[1] It is possible to use Lemma 4.1.1 to prove a complete information guarantee corresponding to the valuation profile $\mathbf{v}$ that the prices are pointwise to by using a similar technique as in Chapter 3 for balanced prices.

[2] The proof of this Lemma is much more involved than that of Lemma 4.1.1, which is why defer the interested reader to Appendix A of [KL20].

in Lemma 4.2.2 and buying whatever is available from it. Also, note that whatever is sold prior to agent i's arrival on $\mathbf{v}$ is a subset of $\mathbf{x}(v_i', \mathbf{v}_{-i})$. Since this holds for any $\mathbf{v}'$, it also holds in expectation when $\mathbf{v}'$ is drawn from $\mathcal{D}$, fixing $\mathbf{v}$. Seeing that v_i' and $\mathbf{v}'_{-i}$ are independent, we can write:

$$\begin{aligned} u_i(\mathbf{v}) &\geq \mathop{\mathbb{E}}_{\mathbf{v}'}\left[\sum_{S\subseteq M} \delta_S^{i,(v_i,\mathbf{v}'_{-i})}\left(v_i\left(S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})\right) - \sum_{j\in S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})} p_j\right)\right] \\ &\geq \mathop{\mathbb{E}}_{\mathbf{v}'}\left[\sum_{S\subseteq M} \delta_S^{i,(v_i,\mathbf{v}'_{-i})}\left(v_i\left(S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})\right) - \sum_{j\in S} p_j\right)\right] \end{aligned}$$

The above line holds trivially, since we subtract a larger value on the RHS. Now we take an expectation over $\mathbf{v}$:

$$\begin{aligned} \mathop{\mathbb{E}}_{\mathbf{v}}\left[u_i(\mathbf{v})\right] &\geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{S\subseteq M} \delta_S^{i,(v_i,\mathbf{v}'_{-i})}\left(v_i\left(S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})\right) - \sum_{j\in S} p_j\right)\right] \\ &\geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{S\subseteq M} \delta_S^{i,(v_i',\mathbf{v}'_{-i})}\left(v_i'\left(S\setminus \mathbf{x}(v_i,\mathbf{v}_{-i})\right) - \sum_{j\in S} p_j\right)\right] \end{aligned}$$

The last line applies the notation defined earlier along with the fact that $(v_i, \mathbf{v}'_{-i})$ is i.i.d to $\mathbf{v}'$ and $(v_i', \mathbf{v}_{-i})$ is i.i.d to $\mathbf{v}$ therefore equivalent in expectation. Summing across all agents, we obtain:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{i=1}^{n}\sum_{S\subseteq M} \delta_S^{\mathbf{v}'}\left(v_i'\left(S\setminus \mathbf{x}(\mathbf{v})\right) - \sum_{j\in S} p_j\right)\right]$$

which holds by linearity of expectation. Since Lemma 4.2.2 applies pointwise for any $T \subseteq M$ as long as it is the same across buyers, we know that it will also hold in expectation over $\mathbf{v} \sim \mathcal{D}$, and considering $T = \mathbf{x}(\mathbf{v})$. Applying Lemma 4.2.2 yields:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq \frac{1}{\alpha}\cdot \mathop{\mathbb{E}}_{\mathbf{v}'}[\mathsf{v}'\mathsf{OPT}(\mathbf{v}')] - \mathop{\mathbb{E}}_{\mathbf{v}}\left[\sum_{j\in \mathbf{x}(\mathbf{v})} p_j\right]$$

for $\alpha \in O(\log \log m)$. Since both $\mathbf{v}$ and $\mathbf{v}'$ are i.i.d, they are equivalent in expectation, allowing us to write:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq \frac{1}{\alpha} \cdot \mathop{\mathbb{E}}_{\mathbf{v}}[\mathsf{vOPT}(\mathbf{v})] - \mathop{\mathbb{E}}_{\mathbf{v}}\left[\sum_{j \in \mathbf{x}(\mathbf{v})} p_j\right] \tag{4.4}$$

This gives us the desired utility bound.

Revenue Bound: This bound is simpler than the previous one. Note that the revenue generated by the mechanism is simply the expectation of the sum of prices of items sold over the valuation profile $\mathbf{v} \sim \mathcal{D}$. Mathematically:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\mathbf{v}}[r_i(\mathbf{v})] = \mathop{\mathbb{E}}_{\mathbf{v}}\left[\sum_{j \in \mathbf{x}(\mathbf{v})} p_j\right] \tag{4.5}$$

Summing Equations 4.4 and 4.5 completes the proof. □

The proof resembles one we have seen previously of bounding the revenue and utility in order to bound overall welfare as presented in Chapter 3. However, it is *vital* to note that Lemma 4.2.2 is a much stronger claim than the one in [PDTKL17] due to it being in expectation rather than pointwise to a specific valuation profile.

Overall, this provides a constructive posted price mechanism, proving an $O(\log \log m)$ approximation for general subadditive valuations. In fact, the paper provides a computationally efficient algorithm to compute the prices in Lemma 4.2.2 which solely establishes their existence. Doing this requires the assumption of having access to demand oracles – we defer the interested reader to Section 4 of [KL20]. Next, we will see why this technique cannot provide a guarantee of the form $o(\log \log m)$, motivating the need for a new mechanism altogether.

4.3 Why can't we beat $O(\log \log m)$?

At the beginning of this section, we mentioned the authors target some q fraction of the items. Recall that at the end of the proof of Lemma 4.1.1, we defined the corresponding min-max game whose value we lower bounded. In doing so, we considered some vector $\mathbf{q} = (q_1, \ldots, q_{|U|}) \in [0,1]^{|U|}$, where q_j represents an upper bound on the amount of probability mass that can be put on item j for each player's strategies. In other words, item j can be selected with probability at most q_j by each player. Most importantly, remember that the authors considered uniform vectors $\mathbf{q} = (q, \ldots, q)$.

The authors showed that for $U = M$, and, considering only uniform vectors $\mathbf{q} = (q, \ldots, q)$ for $q \in [0,1]$, that for the value of q which maximizes the value of the game, there exists a "bad" (to maintain the same nomenclature as used in Chapter 3) subadditive valuation v_i. Specifically, the maximal value of the game is at most $\frac{1}{\Omega(\log \log m)} \cdot v_i(M)$. The proof of existence of the above valuation is provided in Section 6 of [KL20], and is similar to that of the valuation provided in [BR11]. Overall, the existence of such a valuation implies that for uniform $\mathbf{q}$, this posted price algorithm cannot do better than a $O(\log \log m)$ prophet inequality.

At this point, we reach 3 main conclusions. The first is that balanced item prices fail to do well on general subadditive valuations; specifically, they cannot provide a guarantee of the form $o(\log m)$ in expectation. We also know that the powerful proof technique of the balanced price guarantee is not one that can be extended to the prices in [KL20]. Lastly, we now know that the prices in [KL20] cannot provide a guarantee of $o(\log \log m)$ while considering only uniform $\mathbf{q}$. In the next chapter, we relax the restriction of item prices by introducing *bundled prices*.

Chapter 5

Bundled Prices

Until now, we have specifically considered prices that are static, anonymous, and item. However, after relaxing the third constraint, there is now a posted price algorithm that posts price per bundle (posed by Feldman et al. in [FGL13]) and admits a 2-approximation for general subadditive valuation profiles in the complete information case. It turns out that they are also closely related to balanced prices as defined in Definition 3.3.3, a connection we will explore later.

5.1 $O(1)$-Bundled Price Mechanism for Complete Information Setting

The exact pricing scheme we show below is actually a special case of the work in [FGL13] and was presented in [CC22].

Theorem 5.1.1 ([CC22]). *For any general subadditive valuation profile $\mathbf{v} = (v_1, \dots, v_n)$, posting the following prices for any $S \subseteq M$ and $i \in N$:*

$$p_i^{\mathbf{v}}(S) = \frac{1}{2} \sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap S)$$

gives a 2-approximation to the optimal welfare.

Proof. We use our usual technique of a utility and revenue bound which we combine via quasi-linearity of utilities to achieve our final result.

Utility Bound: In order to do this, we first observe that the utility of any agent i is lower bounded by allowing agent i to purchase their optimal set on the remaining items such that their utility is maximized. Notating the set of remaining items as $R := M \setminus \cup_{i=1}^{n} \mathsf{ALG}_i(\mathbf{v})$, we can now write:

$$\begin{aligned}
\mathbf{u}(\mathbf{v}) &\geq \sum_{i=1}^{n} \max_{S \subseteq R} \{v_i(S) - p_i^{\mathbf{v}}(S)\} \\
&\geq \sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap R) - p_i^{\mathbf{v}}(\mathsf{OPT}_i(\mathbf{v}) \cap R) \\
&= \sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap R) - \frac{1}{2} \sum_{j=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v}) \cap (\mathsf{OPT}_i(\mathbf{v}) \cap R)) \\
&= \sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap R) - \frac{1}{2} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap R) \\
&= \frac{1}{2} \sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap R)
\end{aligned}$$

The first line holds as explained above. The second step uses the fact that $\mathsf{OPT}_i(\mathbf{v}) \cap R \subseteq R$. The third line simply inserts the definition of prices posted. To obtain the fourth line, we know that for all $i \neq j$, we have that $\mathsf{OPT}_i(\mathbf{v}) \cap \mathsf{OPT}_j(\mathbf{v}) = \varnothing$. We combine this with our assumption that valuations are normalized, i.e. $v_i(\varnothing) = 0$ then the inner sum only has one non-zero value when $i = j$. The remainder of the proof follows from basic algebra.

Revenue Bound: This bound is once again much simpler as we see below:

$$\begin{aligned}
\mathbf{r}(\mathbf{v}) &= \sum_{i=1}^{n} p_i^{\mathbf{v}}(\mathsf{ALG}_i(\mathbf{v})) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} v_j(\mathsf{OPT}_j(\mathbf{v}) \cap \mathsf{ALG}_i(\mathbf{v}))
\end{aligned}$$

$$\geq \frac{1}{2}\sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap (\cup_{j=1}^{n}\mathsf{ALG}_j(\mathbf{v})))$$

The first two equations hold by definition of revenue and the prices posted. The last line holds by interchanging the order of summation, then applying subadditivity, and subsequently reindexing. At this point, we finish the proof by adding the two respective bounds:

$$\begin{aligned}
\mathbf{v}(\mathsf{ALG}(\mathbf{v})) &\geq \mathbf{u}(\mathbf{v}) + \mathbf{r}(\mathbf{v}) \\
&\geq \frac{1}{2}\sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap R) + \frac{1}{2}\sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap (\cup_{j=1}^{n}\mathsf{ALG}_j(\mathbf{v}))) \\
&\geq \frac{1}{2}\sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap M) = \frac{1}{2}\mathbf{v}(\mathsf{OPT}(\mathbf{v}))
\end{aligned}$$

The final line holds by subadditivity since $R \cup (\cup_{i=1}^{n}\mathsf{ALG}_i(\mathbf{v})) = M$, which is the set of all items, thus allowing us to drop M from the expression and proving the claim. □

This provides a very clean proof and mechanism that results in a 2-approximation. However, it is not easy to extend its guarantee into a global guarantee or one in expectation. We attempt to find ways to extend this idea via the notion of balanced prices.

5.2 Connection between Balanced and Bundled Prices

Naturally, with the guarantee we found in Theorem 5.1.1, it would be ideal to have a theorem that allows us to immediately claim that the pointwise guarantee implies one in expectation. In fact, this would hold true if the defined bundled pricing rule satisfied balanced prices as defined in Definition 3.3.3 with $\alpha\beta = O(1)$. However, below, we show that this is not true.

Proposition 5.2.1. *The bundled prices as defined in Theorem 5.1.1 satisfy property*

(1) *of balanced prices as defined in Definition 3.3.3 with* $\alpha = 2$.

Proof. We first prove that the bundled prices satisfy property (a) directly. To recap, let us post for all $S \subseteq M$ and $i \in N$ the following prices: $p_i^{\mathbf{v}}(S) = \frac{1}{2}\sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v}) \cap S)$. Next, we consider any feasible allocation $\mathbf{x} \in \mathcal{F}$ where $\mathbf{x} = (x_1, \ldots, x_n)$. Thus, we can write:

$$\begin{aligned}
\sum_{i=1}^{n} p_i^{\mathbf{v}}(x_i) &= \sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v}) \cap x_i) \\
&= \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v}) \cap x_i) \\
&\geq \frac{1}{2} \sum_{j=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v}) \cap (\cup_{i=1}^{n} x_i)) \\
&\geq \frac{1}{2} \sum_{j=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v})) - v_j(\mathsf{OPT}_j(\mathbf{v}) \setminus (\cup_{i=1}^{n} x_i)) \\
&\geq \frac{1}{2} (\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x})))
\end{aligned}$$

The first line holds by definition, and the second by switching the order of summation. The next line holds by subadditivity. The fourth inequality also holds by subadditivity, but uses the fact that for any $S, T \subseteq M$ we have $v(S) \geq v(S \cup T) - v(T)$. The last line holds since $\sum_{j=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v}) \setminus (\cup_{i=1}^{n} x_i)) \leq \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))$ by definition, and the other term holds with equality. □

Proposition 5.2.2. *The bundled prices as defined in Theorem 5.1.1 fail to satisfy property* (2) *of balanced prices as defined in Definition 3.3.3 for any* $\beta < n/4$ *or* $\beta = o(n)$.

Proof. We show this via counter example. To this end, let us assume we have n agents and n^2 items both labelled numerically for convenience. Furthermore, consider $\mathbf{x}$ to be the empty allocation $\mathbf{x} = (\varnothing, \ldots, \varnothing)$, which is indeed feasible. Let us consider the following valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ such that for all $i \in [n]$ and any $S \subseteq [n^2]$, we have that $v_i(S) = \mathbb{I}[S \cap \mathsf{OPT}_i(\mathbf{v}) \neq \varnothing] \cdot \left(1 + \frac{|S \cap \mathsf{OPT}_i(\mathbf{v})|-1}{n}\right)$.[1] Furthermore, let us assume that for all $i \in [n]$, we have $\mathsf{OPT}_i(\mathbf{v}) = \{i, n+i, \ldots, (n-1)n+i\}$. Now consider the following allocation $\mathbf{x}'_i = \{n(i-1)+1, n(i-1)+2, \ldots, n(i-1)+n\}$ for all agents $i \in [n]$. One can see the intuition behind this example, where we exploit the fact that allowing for any arbitrary $\mathbf{x}'$ is simply too general for our prices to fulfill. We see this rigorously below:

$$\begin{aligned}\sum_{i=1}^{n} p_i^{\mathbf{v}}(x'_i) &= \sum_{i=1}^{n} \frac{1}{2} \sum_{j=1}^{n} v_j(\mathsf{OPT}_j(\mathbf{v}) \cap x'_i) \\ &= \frac{1}{2} \sum_{i=1}^{n} n = \frac{n^2}{2}\end{aligned}$$

The first line is by definition of bundled prices. The second line holds because for all $j \in [n]$, $\mathsf{OPT}_j(\mathbf{v}) \cap x'_i = \{n(i-1)+j\}$, and thus $v_j(\{n(i-1)+j\}) = 1$ since it is one of agent j's desired items. The remainder is just algebra.

Next, note that since $\mathbf{x}$ is the empty allocation, we know that $\mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x})) = \mathbf{v}(\mathsf{OPT}(\mathbf{v}))$ by definition. This leads to the following welfare:

$$\begin{aligned}\mathbf{v}(\mathsf{OPT}(\mathbf{v})) &= \sum_{i=1}^{n} v_i(\mathsf{OPT}_i(\mathbf{v})) \\ &= \sum_{i=1}^{n} v_i(\{i, n+i, \ldots, (n-1)n+i\}) \\ &= n\left(1 + \frac{n-1}{n}\right) = 2n-1\end{aligned}$$

Note that $\sum_{i=1}^{n} p_i^{\mathbf{v}}(x'_i) \leq \beta(\mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x})))$ does not hold for $\beta \leq n/4$, completing the proof. □

Therefore, we find that while our bundled prices satisfy some properties of bal-

[1] It is easy to see that such a definition is a form of subadditive valuation, and is left as an exercise to the reader.

anced prices, they do not satisfy them all for $\alpha\beta = O(1)$ implying that we cannot immediately claim the existence of a constant factor approximation. Nonetheless, since it does in fact satisfy property (1) of balanced prices, a natural extension is to consider whether loosening constraint (2) to weakly balanced prices as defined in [PDTKL17] translates to a constant factor approximation. Furthermore, we can also consider other bundled pricing rules that may potentially provide good guarantees or attempt to show an impossibility claim regarding bundled prices as a whole. The next chapter attempts to answer the latter open question.

Chapter 6

Balanced Bundled Prices

In Proposition 5.2.2, it was shown that the 2-approximation attained by Combinatorial Walrasian Equilibria (bundled) prices from [FGL13] were in fact not balanced for $\alpha\beta = O(1)$. However, this did not rule out the existence of appropriate bundled prices in general. In this chapter, we explore the connection between the two notions further and make progress in showing that, in some cases, bundled balanced prices in fact do not exist and we attempt to characterize a general relationship between the two. To begin, we reformulate Definition 3.3.3 to construct our varying relaxations.[1]

Definition 6.0.1 (Balanced Prices)**.** *Let $\alpha, \beta > 0$. A pricing rule profile $\mathbf{p}^{\mathbf{v}} = (p_1^{\mathbf{v}}, \ldots, p_n^{\mathbf{v}})$ defined by pricing rules (functions) $p_i^{\mathbf{v}} : 2^M \to \mathbb{R}_{\geq 0}$ is (α, β) balanced with respect to the valuation profile $\mathbf{v}$ if for all $\mathbf{x} \in \mathcal{F}$,*

1. $\sum_i p_i^{\mathbf{v}}(x_i) \geq \frac{1}{\alpha}(\mathbf{v}(\mathit{OPT}(\mathbf{v})) - \mathbf{v}(\mathit{OPT}(\mathbf{v}|\mathbf{x})))$,
2. *for all $\mathbf{x}' \in \mathcal{F}$ with $\mathbf{x}, \mathbf{x}'$ disjoint and $\mathbf{x} \cup \mathbf{x}' \in \mathcal{F}$,* $\sum_i p_i^{\mathbf{v}}(x_i) \leq \beta(\mathbf{v}(\mathit{OPT}(\mathbf{v}|\mathbf{x}')))$.

Note that there is no change in condition (1). However, we are allowed to rewrite the second constraint since we are still enumerating all the constraints for the pricing

[1] We use the special case definition, but all our results can be generalized to the general framework as well.

rule profile to be (α, β) balanced. Such a definition will be ideal in order to formulate the search for balanced set prices via a linear programming approach.

Before we proceed, we quickly justify why the computation of balanced set prices can be done via a linear program. This is because we can define the price for each set as a variable for the linear program and the constraints as is, which are indeed all linear. Thus, a feasibility problem would suffice. To solve this linear program, we consider several relaxations as a first pass.

6.1 Relaxation via Welfare and Subadditivity

We begin by defining value maximizing function or welfare function, which will allow us to tighten some of the balanced prices constraints thereby helping us construct our series of relaxations.

Definition 6.1.1 (Welfare Function)**.** *For any valuation profile* $\mathbf{v} = (v_1, \ldots, v_n)$*, we have a set function* $v : 2^M \to \mathbb{R}_{\geq 0}$ *such that for any* $S \subseteq M$*,* $v(S) = \max_{\mathbf{x} \in \mathcal{F}^S} \mathbf{v}(\mathbf{x})$*. Here we define* $\mathcal{F}^S$ *to be the set of feasible allocations such that for any* $\mathbf{x} \in \mathcal{F}^S$ *we have* $\cup_i x_i \subseteq S$*, or alternatively the restriction of* $\mathcal{F}$ *to* S*.*

The interpretation is that v searches over all allocations consisting of only items within the set its evaluated upon and returns welfare associated with value maximizing allocations amongst them. One might wonder whether v satisfies similar properties if $\mathbf{v}$ is a subadditive valuation profile. It turns out that it is in fact subadditive.

Proposition 6.1.2. *The function* v *in Definition 6.1.1 is a subadditive function for any general subadditive valuation profile support* $\mathbf{v} = (v_1, \ldots, v_n)$*, i.e., for any* $S, T \subseteq M$ *we have* $v(S \cup T) \leq v(S) + v(T)$*.*

Proof. By definition, $v(S \cup T) = \sum_i v_i(x_i^*)$, where we define the optimal allocation $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ to be a partition of $S \cup T$. Here, we have subadditivity of the functions v_i for all $i \in N$. Therefore,

$$\sum_i v_i(x_i^*) \leq \sum_i v_i(x_i^* \cap S) + v_i(x_i^* \cap T). \tag{6.1}$$

Observe that the sets $\{x_i^* \cap S\}_{i=1}^n$ partition S, and $\{x_i^* \cap T\}_{i=1}^n$ partition T. Thus, by optimality, we can write:

$$\sum_i v_i(x_i^* \cap S) + v_i(x_i^* \cap T) \leq v(S) + v(T) \tag{6.2}$$

Combining Equation 6.1 and 6.2 gives the desired result. □

We will utilize this function in order to construct our first linear programming relaxation.

Proposition 6.1.3. *The below optimization problem is a valid linear programming relaxation to the original problem:*

$$\begin{aligned}
\min_{\mathbf{p} \in \mathbb{R}^{2^m}} \quad & 0 && \qquad (1)\\
s.t. \quad & \sum_{i=1}^n p_{x_i} \leq C \cdot v(\cup_i x_i) && \textit{for all } \mathbf{x} \in \mathcal{F}\\
& \sum_{i=1}^n p_{x_i} \geq \frac{1}{C} \cdot v(\cup_i x_i) && \textit{for all } \mathbf{x} \in \mathcal{F}\\
& p_S \geq 0 && \textit{for all } S \subseteq M
\end{aligned}$$

Proof. We proceed to show why Problem 1 is indeed a valid relaxation. To do so, we determine that a solution $\mathbf{p}$ satisfies all constraints in Definition 6.0.1. We consider the first constraint (or constraints of type (1)) where for each $\mathbf{x} \in \mathcal{F}$ we have we must have $1/\alpha(\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x})))$. Note that $\mathbf{v}(\mathsf{OPT}(\mathbf{v}))$ is exactly $v(M)$ and $\mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))$ is $v(M \setminus \cup_i x_i)$ for v as defined in Definition 6.1.1. Next, Proposition 6.1.2 allows us to claim $v(\cdot)$ is subadditive, thereby giving $v(M) - v(M \setminus \cup_i x_i) \leq v(\cup_i x_i)$.

Therefore, enforcing the condition for all $\mathbf{x} \in \mathcal{F}$ that

$$\sum_{i=1}^{n} p_{x_i} \geq v(\cup_i x_i)/\alpha$$

satisfies all constraints of type (1).

Secondly, note that for each $\mathbf{x} \in \mathcal{F}$, we have several constraints of type (2). However, for all $\mathbf{x}'$ that are disjoint of $\mathbf{x}$, we know $\mathbf{x}$ is a valid candidate for $\mathsf{OPT}(\mathbf{v}|\mathbf{x}')$. Therefore, enforcing

$$\sum_{i=1}^{n} p_{x_i} \leq \beta \cdot v(\cup_i x_i)$$

for each $\mathbf{x} \in \mathcal{F}$ satisfies all constraints of type (2). Since, we require that $\alpha\beta = O(1)$, it is sufficient to have $\sum_{i=1}^{n} p_{x_i} \in [1/C \cdot v(\cup_i x_i), C \cdot v(\cup_i x_i)]$ for each $\mathbf{x} \in \mathcal{F}$ with some absolute constant $C \geq 1$. □

Proposition 6.1.4. *For any subadditive valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$, there do not exist prices p_S for all $S \subseteq M$ that satisfy every constraint in Problem 1 with $C = O(1)$, i.e., the problem is infeasible.*

Proof. In order to show this, consider setting prices to be a low as possible, specifically for any set $S \subseteq M$, we set $p_S = v(S)/C$. Then, for all $\mathbf{x} \in \mathcal{F}$, we have the following

$$\sum_{i=1}^{n} p_{x_i} = \sum_{i=1}^{n} \frac{v(x_i)}{C} \geq \frac{1}{C} \cdot v(\cup_i x_i)$$

where the last inequality follows by subadditivity. Note this is as low as we can set prices without violating the lower bound constraints. Therefore, if we can provide a subadditive valuation profile for which setting such prices violate the upper bound for $C = O(1)$ then we know that Problem 1 is infeasible.

To that end, consider n bidders and n items where bidder 1 has unit demand valuation, i.e., we have for any $S \subseteq M$

$$v_1(S) = \begin{cases} 0 & \text{if } S = \varnothing \\ 1 & \text{otherwise} \end{cases}$$

and all other bidders have value 0 for all items. It is evident that this is indeed a valid subadditive valuation profile. Consider the allocation $\mathbf{x}$ where each agent i is allocated item i and we see trivially that $\mathbf{x} \in \mathcal{F}$. Further, we can write

$$\sum_{i=1}^{n} p_{x_i} = \sum_{i=1}^{n} \frac{v(x_i)}{C} = \frac{n}{C}.$$

This is, since for all $i \in [n]$, we have that x_i is a singular item, which the welfare function will always award to agent 1 since they have value 1 for it. Furthermore, note that $Cv(\cup_i x_i) = Cv(M) = C$ since only agent 1 has positive value since they get value 1 upon receiving at least 1 item and 0 otherwise. To conclude, we know that to satisfy the required upper bound for this allocation $\mathbf{x}$, we need $n/C \leq C$ which holds only for $C \geq \sqrt{n}$. $\square$

An alternative approach to prove that the relaxation presented in Problem 1 is infeasible is via the dual, which was in fact the first approach. Despite having already shown infeasibility, we still characterize the properties of the dual optimization problem which may prove useful later.

Proposition 6.1.5. *The dual to Problem 1 is as follows:*

$$\begin{aligned} \max_{\mathbf{y},\mathbf{z} \in \mathbb{R}^{(n+1)^m}} \quad & \sum_{\mathbf{x} \in \mathcal{F}} \left(\frac{1}{C} z_{\mathbf{x}} v(\cup_i x_i) - C y_{\mathbf{x}} v(\cup_i x_i) \right) & & (2) \\ s.t. \quad & \sum_{\mathbf{x}: S \in \mathbf{x}} (y_{\mathbf{x}} - z_{\mathbf{x}}) \geq 0 & & \textit{for all } S \subseteq M \\ & y_{\mathbf{x}}, z_{\mathbf{x}} \geq 0 & & \textit{for all } \mathbf{x} \in \mathcal{F} \end{aligned}$$

Proof. The dual follows from introducing multipliers $y_{\mathbf{x}}, z_{\mathbf{x}}$ for each constraints and then following standard machinery. $\square$

Proposition 6.1.6. *If for every feasible solution* $(\mathbf{y}, \mathbf{z})$ *Problem 2 has a non-positive objective value, there exist balanced set prices for* $\alpha\beta = O(1)$ *from Definition 6.0.1.*

Proof. First, note that setting $(\mathbf{y}, \mathbf{z}) = (\mathbf{0}, \mathbf{0})$ gives objective value 0. Thus, by strong duality, it suffices to show that every dual feasible solution gives non-positive objective value to claim that the optimal dual value is 0, which coincides with the optimal primal value. That would imply that the primal is feasible and there exist balanced set prices. $\square$

Proposition 6.1.7. *Either there exists a feasible solution* $(\mathbf{y}, \mathbf{z})$ *to Problem 2 which gives positive objective value or Problem 2 is infeasible.*

Proof. By Proposition 6.1.4, we know that the primal, i.e. Problem 1 is infeasible. Therefore, by strong duality we know that either the dual is unbounded or infeasible proving the desired statement. $\square$

At this point, since we have shown Problem 1 to be infeasible, we know that our relaxation was still too tight. Thus, our search for constant factor balanced and bundled prices would require a more relaxed problem. To construct such a problem, we retrace our steps and note that our usage of the subadditivity of the welfare function to relax constraints of type (1) in Balanced Prices Definition 6.0.1 is the technique that failed. In the next section, we consider a "looser" relaxation in order to find appropriate prices.

6.2 Relaxation without Subadditivity

To construct this relaxation, we retrace our steps in constructing our first relaxation Problem 1 in order to determine which constraint led to infeasibility. As it turns out, infeasibility arises from our use of the subadditivity of the welfare function v in order to tighten constraints of type (1) from Definition 6.0.1. After reverting to this step, we find a new linear program, the feasibility of which would still yield a solution which were bundled prices balanced for $\alpha\beta = O(1)$.

Proposition 6.2.1. *The below optimization problem is a valid linear programming relaxation to the original problem:*[2]

$$\begin{aligned}
\min_{\mathbf{p}\in\mathbb{R}^{2^m}} \quad & 0 && && (3)\\
s.t. \quad & \sum_{i=1}^{n} p_{x_i} \leq \beta v(\cup_i x_i) && \text{for all } \mathbf{x} \in \mathcal{F} \\
& \sum_{i=1}^{n} p_{x_i} \geq \frac{1}{\alpha}(v(M) - v(M \setminus \cup_i x_i)) && \text{for all } \mathbf{x} \in \mathcal{F} \\
& p_S \geq 0 && \text{for all } S \subseteq M
\end{aligned}$$

Proof. Specifically, for each $\mathbf{x}$, we can tighten all respective constraints of type (2) to be $\sum_i p_i^{\mathbf{v}}(x_i) \leq \beta v(\cup_i x_i)$. This is because $v(\cup_i x_i) \leq \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}'))$ for all feasible allocations $\mathbf{x}'$ that are disjoint of $\mathbf{x}$ as the optimal allocation of the items $\cup_i x_i$ is always a candidate for $\mathsf{OPT}(\mathbf{v}|\mathbf{x}')$ and thus the optimum can potentially be better. So our problem can be reduced to finding prices p_S for all $S \subseteq M$ such that for all $\mathbf{x} \in \mathcal{F}$ we have

$$\frac{1}{\alpha}(\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))) = \frac{1}{\alpha}(v(M) - v(M \setminus \cup_i x_i)) \leq \sum_i p_{x_i} \leq \beta v(\cup_i x_i).$$

We can formulate the above problem as a linear program feasiblity problem instead with 2^m variables, $2(n+1)^m$ constraints and fixed α, β such that $\alpha\beta = O(1)$. $\square$

We know that if the above LP is feasible then any feasible solution proves **ex-**

[2] There exist slightly different forms of the linear program that might be useful based off the context. For further detail, see Appendix A.2.

istence of (α,β)-balanced set prices with $\alpha\beta = O(1)$. To determine feasibility one might hope to use similar techniques to those in Proposition 6.1.4 by setting prices as low possible and then setting a valuation profile that violated the upper bound. However, it is evident that the simple unit demand and null player valuation profile no longer work. One could also look at the dual problem for feasibility.

Proposition 6.2.2. *The dual to Problem 3 is as follows:*

$$\begin{aligned}
\max_{\mathbf{y},\mathbf{z}\in\mathbb{R}^{(n+1)^m}} \quad & \sum_{\mathbf{x}\in\mathcal{F}} \left(\frac{1}{\alpha} z_{\mathbf{x}}(v(M) - v(M \setminus \cup_i x_i)) - \beta y_{\mathbf{x}} v(\cup_i x_i) \right) && (4) \\
s.t. \quad & \sum_{\mathbf{x}:S\in\mathbf{x}} (y_{\mathbf{x}} - z_{\mathbf{x}}) \geq 0 && \textit{for all } S \subseteq M \\
& y_{\mathbf{x}}, z_{\mathbf{x}} \geq 0 && \textit{for all } \mathbf{x} \in \mathcal{F}
\end{aligned}$$

Proof. The dual follows from introducing multipliers $y_{\mathbf{x}}, z_{\mathbf{x}}$ for each type of constraint and then following standard machinery. □

Proposition 6.2.3. *If every feasible solution $(\mathbf{y},\mathbf{z})$ to Problem 4 has a non-positive objective value, there exists balanced set prices for $\alpha\beta = O(1)$ from Definition 6.0.1.*

Proof. First note that setting $(\mathbf{y},\mathbf{z}) = (\mathbf{0},\mathbf{0})$ gives objective value 0. Thus, by strong duality, it suffices to show that every dual feasible solution gives non-positive objective value to claim that the optimal dual value is 0. This coincides with the optimal primal value and would imply that the primal was indeed feasible thus there exist balanced set prices. □

Here, it is not obvious to show the existence of feasible solution with positive objective value or to determine that every feasible solution has non-positive objective value. Therefore, we use toward other techniques to establish the existence of constant factor balanced and bundled prices that are feasible to Problem 3.

6.3 Reduction to Item Pricing

It was demonstrated in Example 3.4.1 that there exists a subadditive valuation profile for which there does not exist balanced *item* prices for $\alpha\beta = o(\log m)$. We show difficulty in devising appropriate balanced prices by exploiting this fact.

Proposition 6.3.1. *For any subadditive valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ and $m \leq n$, there do not exist (α, β)-balanced bundled prices $\{p_S\}_{S \subseteq M}$ for $\alpha\beta = o(\log m)$ as defined in Definition 6.0.1.*

Proof. Assume for the sake of contradiction that there exist (α, β)-balanced bundled prices $\mathbf{p} = \{p_S\}_{S \subseteq M}$ for $\alpha\beta = o(\log m)$. Consider the following item pricing menu $I^{\mathbf{p}} = \{p_S : S \subseteq M, |S| = 1\}$ which is indeed a valid item pricing profile on M. We claim that in fact $I^{\mathbf{p}}$ defines an (α, β)-balanced item pricing scheme for $\alpha\beta = o(\log m)$. Note that for this to be true, we need the following constraints from Problem 3 to hold for all $\mathbf{x} \in \mathcal{F}$:

1. $\sum_{i=1}^{n} p_{x_i} = \sum_{i=1}^{n} \sum_{j \in x_i} I_j^{\mathbf{p}} \leq \beta v(\cup_i x_i)$,
2. $\sum_{i=1}^{n} p_{x_i} = \sum_{i=1}^{n} \sum_{j \in x_i} I_j^{\mathbf{p}} \geq \frac{1}{\alpha}(v(M) - v(M \setminus \cup_i x_i))$,

for $\alpha\beta = o(\log m)$. Now, since $m \leq n$, for any $\mathbf{x} \in \mathcal{F}$ we have that $|\cup_i x_i| \leq n$. Therefore, there exists an allocation or partition of item $\cup_i x_i$ such that each agent receives at most 1 item. To conclude, note that $\sum_{i=1}^{n} \sum_{j \in x_i} I_j^{\mathbf{p}} = \sum_{j \in \cup_i x_i} p_j$. Since $\mathbf{p}$ is balanced, the required inequalities

1. $\sum_{j \in \cup_i x_i} p_j \leq \beta v(\cup_i x_i)$
2. $\sum_{j \in \cup_i x_i} p_j \geq \frac{1}{\alpha}(v(M) - v(M \setminus \cup_i x_i))$

already exist as the RHS of both remain the same regardless of the allocation of the items $\cup_i x_i$. Therefore, $I^{\mathbf{p}}$ defines an (α, β)-balanced item pricing scheme for $\alpha\beta = o(\log m)$ which poses a contradiction to Example 3.4.1. $\square$

This effectively proves that appropriate bundled balanced prices do not exist for $m \leq n$. The question remains as to if balanced bundled pricing is also hard for $m > n$. Prior to analyzing the general case, we first gain intuition of what occurs within this regime of more items than agents. Specifically, we consider the case of 1 agent and m items.

Proposition 6.3.2. *Given any subadditive valuation profile* $\mathbf{v} = (v_1, \ldots, v_n)$ *with general* m *and* $n = 1$*, there exist* $(1, 1)$*-balanced bundled prices* $\{p_S\}_{S \subseteq M}$*, specifically* $p_S = v(S)$ *for any* $S \subseteq M$ *where* v *is welfare function from Definition 6.1.1*

Proof. Given the prices set, we immediately know that for any $\mathbf{x} = (x) \in \mathcal{F}$,

- $p_x \leq v(x)$
- $p_x \geq v(x)$.

However, by Lemma 6.1.2, we know that v is suabdditive and we can claim that

$$p_x \geq v(M) - v(M \setminus x),$$

from which we know that $\{p_S\}_{S \subseteq M}$ is a $(1, 1)$ balanced bundled pricing rule. $\square$

In the next section, we explore the more general setting with m items and n agents for $n \in \{2, \ldots, m-1\}$. We choose n in this range as Proposition 6.3.1 demonstrates the non-existence of $(1, 1)$-balanced bundled prices for general subadditive valuation profiles when $n \geq m$. Moreover, Proposition 6.3.2 establishes the existence for such when $n = 1$. It is an open question as to if $(1, 1)$-balanced prices exist for general subadditive valuation profiles for this range of n and general m. Moving forward to determine existence results in our range of agents, we introduce the notion of q-partitioning valuations. As we will see, subadditivity implies 2-partitioning valuations and, in the next section, we attempt to find appropriate prices as n increases by instead using progressively less complex models (i.e. as q increases).

6.4 q-partitioning and Interpolation

In particular, all we need for existence of $(1,1)$-balanced bundled prices in Proposition 6.3.2 is $\mathbf{v}$ being a subadditive valuation profile which implies the subadditivity of v by Proposition 6.1.2. We also know from Proposition 6.3.1 that as n gets larger (or specifically when $n \geq m$), there do not exist $(1,1)$-balanced bundled prices.

Nevertheless, from Theorem 3.3.2, we know the existence of $(1,1)$-balanced *item* prices for XOS or fractionally subadditive valuations for general m and n. In fact, existence of item prices always implies existence of bundled prices. Hence, one might wonder if we can characterize the relationship between the number of agents n and the "complexity" of the underlying valuation function class for which there exist $(1,1)$-balanced bundled prices. Here, we did not define what metric of complexity we are using, but note that XOS functions are a subset of subadditive ones and thereby less complex. To characterize this more formally, we introduce a notion developed by Bangachev and Weinberg in [BW23].

Definition 6.4.1 ([BW23])**.** *Let $q \in [2, m]$ be an integer. A valuation $v : 2^{[m]} \to \mathbb{R}_{\geq 0}$ satisfies the q-partitioning property if for any $S \subseteq [m]$ and any partition $(S_1, S_2, \ldots, S_q)$ of S into q (possibly empty) disjoint parts, and any fractional covering of $[q]$ (that is, any non-negative $\alpha(\cdot)$ such that for all $j \in [q], \sum_{T \ni j} \alpha(T) \geq 1$):*

$$v(S) \leq \sum_{T \subseteq [q]} \alpha(T) \cdot v\left(\bigcup_{j \in T} S_j\right).$$

We refer to the class of q-partitioning valuations over $[m]$ as $\mathcal{Q}(q, [m])$.

This definition may seem slightly abstract at first, however, after closer examination, one realizes that it provides a form of interpolating valuations between subadditive and XOS.

Proposition 6.4.2. *For all m, the following relations between classes of q-partitioning* *valuations hold:*

$$XOS(m) = \mathcal{Q}(m, [m]) \subseteq \mathcal{Q}(m-1, [m]) \subseteq \cdots \subseteq \mathcal{Q}(2, [m]) = Subadditive([m]).$$

Proof. For $q = m$, this definition is exactly that of XOS functions when written in their fractional cover form. Similarly, for $q = 2$, note that we only consider covers of $S \cup T$ with S and T as well as weights of 1 on each. The inclusions follow by considering an additional empty partition.[3] □

Given this new framework, we first establish a few baseline results that help construct more relevant questions. Specifically, we show that when a valuation profile $\mathbf{v}$ consists of valuation functions from a class $\mathcal{P}$, then the welfare function v supported on $\mathbf{v}$ is also of class $\mathcal{P}$ for only certain classes $\mathcal{P}$. Recall that Proposition 6.1.2 gave us exactly this result for $\mathcal{P}$ being the class of subadditive functions. We illustrate several such results below.

Proposition 6.4.3. *For any additive valuation profile* $\mathbf{v} = (v_1, \ldots, v_n)$*, the welfare function* v *supported on* $\mathbf{v}$ *is additive.*

Proof. Consider any $S \subseteq M$ and welfare function v supported on any additive valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$. First, we rewrite $\sum_{j \in S} v(\{j\})$ using its definition,

$$\sum_{j \in S} v(\{j\}) = \sum_{j \in S} \max_{1 \le i \le n} v_i(\{j\}).$$

Denote $i_j^* := \arg\max_{1 \le i \le n} v_i(\{j\})$ for all $j \in [m]$. The right hand side is exactly

$$\begin{aligned} &= \sum_{j \in S} \sum_{i=1}^{n} v_i(\{j\}) \mathbb{I}[i = i_j^*] \\ &= \sum_{i=1}^{n} \sum_{j \in S} v_i(\{j\}) \mathbb{I}[i = i_j^*] \end{aligned}$$

[3] In fact, it was demonstrated in [BW23] that these valuations have strict inclusion of function classes and satisfy a closeness property which was originally introduced in [BR11].

Now, define the allocation $\mathbf{y} = (y_1, \ldots, y_n)$, a partition of S where $y_i = \{j \in S : i = i_j^*\}$.[4] Thus we may write

$$\sum_{i=1}^{n} \sum_{j \in S} v_i(\{j\}) \mathbb{I}[i = i_j^*] = \sum_{i=1}^{n} \sum_{j \in y_i} v_i(\{j\}) \to \sum_{j \in S} v(\{j\}) = \sum_{i=1}^{n} \sum_{j \in y_i} v_i(\{j\}).$$

Hence, we have the following inequality where we define the optimal allocation $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ to be a partition of S, i.e., $v(S) = \sum_{i=1} v_i(x_i^*)$

$$\sum_{i=1}^{n} \sum_{j \in y_i} v_i(\{j\}) \leq \sum_{i=1}^{n} \sum_{j \in x_i^*} v_i(\{j\}) = \sum_{i=1}^{n} v_i(x_i^*) = v(S)$$

where the first inequality follows by optimality and the second by additivity of v_i for all $i \in [n]$. This effectively proves

$$\sum_{j \in S} v(\{j\}) \leq v(S). \tag{6.3}$$

To show the reverse inequality, we use Proposition 6.1.2, which is valid since an additive valuation profile is also subadditive. Therefore, we have that v is subadditive. Recursively applying the subadditivity property, we may claim for any $S \subseteq M$ that

$$\sum_{j \in S} v(\{j\}) \geq v(S). \tag{6.4}$$

Combining inequalities 6.3 and 6.4, we know that $v(S) = \sum_{j \in S} v(\{j\})$ proving that v is additive as desired. □

The above result is not particularly compelling as the class of additive valuation functions is notably restrictive. However, recall that XOS valuations are supported on a collection of additive valuations thereby providing reason to investigate such a

[4] It is easy to check that this defines a valid partition, i.e., $y_i \cap y_j = \varnothing$ for $i \neq j$ and $\cup_i y_i = S$. We can tie-break arbitrarily, for example, lexicographically.

claim. Furthermore, we will see that we can show Proposition 6.4.3 for $\mathcal{P}$ being the class of q-partitioning functions. These functions allow us to prove a general result for valuation profiles ranging from XOS to subadditive, but not for additive functions, making the previous result novel in some regard.

Proposition 6.4.4. *Let $q \in [2, m]$ be an integer. For any q-partitioning valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$, the welfare function v supported on $\mathbf{v}$ is q-partitioning.*

Proof. Consider any $S \subseteq M$ and welfare function v supported on any q-partitioning valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$. Let $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ be the allocation for which the welfare function v attains its maximum. That is,

$$v(S) = \sum_{i=1}^{n} v_i(x_i^*).$$

We need to show that for any partition of S into q disjoint parts $(S_1, S_2, \ldots, S_q)$, and for any fractional covering α of $[q]$, the q-partitioning property holds for v:

$$v(S) \leq \sum_{T \subseteq [q]} \alpha(T) \cdot v\left(\bigcup_{j \in T} S_j\right).$$

Furthermore, note that $(x_i^* \cap S_1, \ldots, x_i^* \cap S_q)$ is a partition of x_i^* for any partition $(S_1, \ldots, S_q)$ of S since $x_i^* \subseteq S$. Recall that each v_i is q-partitioning thus we can write for any fractional covering α of $[q]$,

$$v_i(x_i^*) \leq \sum_{T \subseteq [q]} \alpha(T) \cdot v_i\left(\bigcup_{j \in T} (x_i^* \cap S_j)\right) = \sum_{T \subseteq [q]} \alpha(T) \cdot v_i\left(x_i^* \cap \bigcup_{j \in T} S_j\right).$$

Summing over all agents $i \in [n]$, we get:

$$v(s) = \sum_{i=1}^{n} v_i(x_i^*)$$

$$\leq \sum_{i=1}^{n} \sum_{T \subseteq [q]} \alpha(T) \cdot v_i \left(x_i^* \cap \bigcup_{j \in T} S_j \right)$$
$$= \sum_{T \subseteq [q]} \alpha(T) \sum_{i=1}^{n} v_i \left(x_i^* \cap \bigcup_{j \in T} S_j \right).$$

Lastly, notice the allocation $\mathbf{x} = (x_1^* \cap (\cup_{j \in T} S_j), \ldots, x_n^* \cap (\cup_{j \in T} S_j))$ is such that $\mathbf{x} \in \mathcal{F}^{\cup_{j \in T} S_j}$ – the restriction of $\mathcal{F}$ to $\cup_{j \in T} S_j$. This is since $\mathbf{x}^*$ represents an allocation thus each entry in $\mathbf{x}$ is disjoint then we take the intersection with $\cup_{j \in T} S_j$ to ensure that $\mathbf{x} \in \mathcal{F}^{\cup_{j \in T} S_j}$. Now, by optimality of the right hand side:

$$\sum_{i=1}^{n} v_i \left(x_i^* \cap \bigcup_{j \in T} S_j \right) \leq v \left(\bigcup_{j \in T} S_j \right).$$

Plugging this into our previous result,

$$v(S) \leq \sum_{T \subseteq [q]} \alpha(T) \cdot v \left(\bigcup_{j \in T} S_j \right).$$

Hence, the welfare function v is q-partitioning. □

However, the converse does not hold for $q = 2$ or subadditive valuation profiles and welfare functions.

Proposition 6.4.5. *If the welfare function v supported on $\mathbf{v}$ is subadditive or 2-partitioning, the corresponding valuation profile $\mathbf{v} = (v_1, \ldots, v_n)$ need not be a subadditive valuation profile.*

Proof. We prove this by constructing a simple counterexample. Consider the case of two agents, agents 1 and 2, and two items a and b. We set agent 1's valuations to be $v_1(\{a\}) = 1$, $v_1(\{b\}) = 6$, and $v_1(\{a, b\}) = 8$. Clearly since $v_1(\{a\}) + v_1(\{b\}) \leq v_1(\{a, b\})$, this valuation function is not subadditive and is therefore not 2-partitioning. Now let agent 2's valuation function be $v_2(\{a\}) = 6$, $v_2(\{b\}) = 5$, and $v_2(\{a, b\}) = 11$, which is additive and thus subadditive.

Clearly, for welfare function v, $v(\{a,b\}) = 12$, $v(\{a\}) = 6$, and $v(\{b\}) = 6$. Hence, we see the following:

$$v(\{a,b\}) \leq v(\{a\}) + v(\{b\}).$$

Since this inequality is satisfied, v is 2-partitioning whereas its support is not. $\square$

We see that Proposition 6.4.4 gives a mechanism to bridge between the function class of agent valuations and the welfare function which is supported on them. Specifically, note that for $q = 2$, Proposition 6.4.4 aligns with our earlier findings regarding subadditive functions (Proposition 6.1.2). Similarly, for $q = m$, we have the result for XOS functions. Using this relationship, we can now make several claims given our earlier work in order to motivate the major question being explored within this section.

Utilizing this notion of q-partitioning valuations, we can now claim that following two statements:

1. By Proposition 6.3.2, for general m and $n = 1$, if $\mathbf{v} = (v_1)$ is a subadditive or 2-partitioning valuation profile, then there exist $(1,1)$-balanced bundled prices.

2. By Theorem 3.3.2, for general m and $n \geq m-1$, if $\mathbf{v} = (v_1, \ldots, v_n)$ is an XOS or m-partitioning valuation profile, then there exist $(1,1)$-balanced bundled prices.

Thus, we pose the following question as a potential future research direction.

Question 2. *For general m and $n \in \{1, 2, \ldots, m-1\}$, if $\mathbf{v} = (v_1, \ldots, v_n)$ is a $n+1$-partitioning valuation profile, then do we have existence of $(1,1)$-balanced bundled prices?*

If we assume $\mathbf{v} = (v_1, \ldots, v_n)$ is an $n+1$-partitioning valuation profile, then by Proposition 6.4.4, it follows that the welfare function v supported on $\mathbf{v}$ is also $n+1$-partitioning. Furthermore, Problem 3 presents the corresponding feasibility linear program. Its solution for $(\alpha, \beta) = (1,1)$ would give $(1,1)$-balanced bundled prices.

Recall that this optimization problem is defined with the welfare function known to be $(n+1)$-partitioning. Exploring this avenue could potentially address the conjecture posed in Question 2.

Chapter 7

Conclusion

In this paper, we attempt to design a posted price mechanism that achieves an $O(1)$ prophet inequality for general subadditive valuation profiles. Achieving this goal would make tremendous headway in the field, yielding a mechanism that can come arbitrarily close to the optimum in a multidimensional setting with only distributional knowledge a priori. We attempt to do so by setting prices, namely *static, anonymous,* and most importantly *item*; however, this is something that persists as a very difficult open problem.

In answering this question, we made several assumptions throughout the paper of which some may be relaxed to possibly achieve substantial further results. First, recall the work by Dütting, Kesselheim, and Lucier where they develop a new pricing mechanism, achieving an $O(\log \log m)$ bound. In this paper, we mention that their technique cannot provide a guarantee of the form $o(\log \log m)$, the proof of which was presented in [KL20] and uses a very similar construction of a bad valuation as our impossibility proof for balanced prices. As explained in Chapter 4, the tightness of the $\log \log m$ bound stems from the authors' main assumption in considering only uniform vectors $\mathbf{q} = (q, \ldots, q)$. This induces a symmetry where every item is selected with at most probability q. It is an open question if this bound can be improved by considering non-uniform $\mathbf{q}$, and one that was posed by the authors to be a worthwhile avenue to pursue for further research.

Moreover, on a different note, the assumption of static, anonymous, and item prices may be too restrictive. Naturally, one may wonder if relaxing any of these constraints allows for improvement in this model. We address this in the paper discussing bundled prices, which relaxes the item prices assumption. This notion was originally introduced by Correa and Cristi in [CC22] where they provide a bundled pricing rule that achieves a 2-approximation in the complete information setting. Naturally, a question that arises is if these prices maintain their approximation guarantee in expectation or, in other words, provide a constant factor prophet inequality. Recall that if these prices were (α, β) balanced for $\alpha\beta = O(1)$, then we would have exactly that result. However, we showed an explicit example where these are not (α, β) balanced for $\alpha\beta = O(1)$. Specifically, recall that the bundled pricing rule from [CC22] fails property (2) of balanced prices for $\beta \leq n/4$ (see Definition 3.3.3). The original balanced prices paper [PDTKL17] also provided a framework known as weakly balanced prices in Section 3 where they loosen property (2). Therefore, it may be interesting to use this framework instead with bundled prices, or even other prices, and see if it leads to desirable results.

Furthermore, it was an open question as to whether there exists a bundled price mechanism that is (α, β) balanced for $\alpha\beta = O(1)$, which thus achieves the desired bound. We answered this question using a linear programming approach in showing that for general subadditive valuations and $m \leq n$, i.e., when there are at most as many items as people that there do not exist (α, β)-balanced bundled prices for $\alpha\beta = o(\log m)$. Notice the similarity in impossibility result to balanced *item* prices. This is no coincidence and a result of the proof technique reducing the existence of appropriate balanced prices implying the existence of appropriate item prices which we had already shown to be hard. For the case of $m > n$, we show that for $n = 1$

general m and any subadditive valuation that there do exist $(1,1)$-balanced bundled prices. In fact, all you need is subadditivity of the agent's valuation function.

We recall work by [BW23] which introduces q-partitioning valuations as a construct to interpolate valuations between XOS and subadditive. It is an open question as to if this function class complexity (parameterized by q) is one that "scales" with the number of agents in order to retain nice balanced properties. This is stated precisely at the end of Chapter 6. Altogether, this paper makes significant progress in understanding which approaches are valid for finding a constant factor prophet inequality and analyzes their limitations. We motivate why this is a useful problem and leave several open questions to further explore.

Appendix A

Omitted Proofs

A.1 Proof of Proposition 4.2.1

Proof. We begin by drawing $\widetilde{\mathbf{v}} \sim \mathcal{D}$. Furthermore, notate the optimal allocation with respect to $\widetilde{\mathbf{v}}$ as $\mathsf{OPT}(\widetilde{\mathbf{v}}) = (U_1, \ldots, U_n)$. Next, for each $j \in U_i$, consider the price $p_j^{i,\widetilde{\mathbf{v}},U_i}$ from Lemma 4.1.1 and post the price $p_j = \mathbb{E}_{\widetilde{\mathbf{v}}}[p_j^{i,\widetilde{\mathbf{v}},U_i}]$ for all $i \in N$.[1] The remainder of the proof will use the same idea as earlier of exploiting the quasi-linearity of utility and approximating welfare by summing utility and revenue. Before getting started, we notate $\mathbf{x}(\mathbf{v})$ as the allocation returned by our mechanism on the valuation profile $\mathbf{v}$.

Utility Bound: Note that for any two valuation profiles $\mathbf{v}$ and $\mathbf{v}'$, agent i can consider drawing a set S from the distribution $\delta^{i,(v_i,\mathbf{v}'_{-i}),U_i}$, whose existence was established in Lemma 4.1.1, and buying whatever is available from it. Also, note that whatever is sold prior to agent i's arrival on $\mathbf{v}$ is a subset of $\mathbf{x}(v_i', \mathbf{v}_{-i})$. Since this holds for any $\mathbf{v}'$, it also holds in expectation when $\mathbf{v}'$ is drawn from $\mathcal{D}$, fixing $\mathbf{v}$.

[1] Posting prices as such fails to post prices for the set of items $M \setminus \cup_{i=1}^n U_i$. This is not a problem since valuations are monotone and the remaining items can be awarded to some arbitrary agent.

Seeing that v_i' and $\mathbf{v}'_{-i}$ are independent, we can write:

$$u_i(\mathbf{v}) \geq \mathop{\mathbb{E}}_{\mathbf{v}'}\left[\sum_{S\subseteq U_i} \delta_S^{i,(v_i,\mathbf{v}'_{-i}),U_i}\left(v_i\left(S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})\right) - \sum_{j\in S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})} p_j\right)\right]$$

$$\geq \mathop{\mathbb{E}}_{\mathbf{v}'}\left[\sum_{S\subseteq U_i} \delta_S^{i,(v_i,\mathbf{v}'_{-i}),U_i}\left(v_i\left(S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})\right) - \sum_{j\in S} p_j\right)\right]$$

The above line holds trivially since we subtract a larger value on the RHS. Now we take an expectation over $\mathbf{v}$:

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathbf{v}}\left[u_i(\mathbf{v})\right] &\geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{S\subseteq U_i} \delta_S^{i,(v_i,\mathbf{v}'_{-i}),U_i}\left(v_i\left(S\setminus \mathbf{x}(v_i',\mathbf{v}_{-i})\right) - \sum_{j\in S} p_j\right)\right]\\ &\geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{S\subseteq U_i} \delta_S^{i,(v_i',\mathbf{v}'_{-i}),U_i}\left(v_i'\left(S\setminus \mathbf{x}(v_i,\mathbf{v}_{-i})\right) - \sum_{j\in S} p_j\right)\right]\end{aligned}$$

The last line applies the notation defined earlier along with the fact that $(v_i, \mathbf{v}'_{-i})$ is i.i.d to $\mathbf{v}'$ and $(v_i', \mathbf{v}_{-i})$ is i.i.d to $\mathbf{v}$ therefore equivalent in expectation. Now, we substitute the definition of the prices we posted which gives:

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathbf{v}}\left[u_i(\mathbf{v})\right] &\geq \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\sum_{S\subseteq U_i} \delta_S^{i,\mathbf{v}',U_i}\left(v_i'\left(S\setminus \mathbf{x}(\mathbf{v})\right) - \sum_{j\in S} \mathbb{E}_{\widetilde{\mathbf{v}}}[p_j^{i,\widetilde{\mathbf{v}},U_i}]\right)\right]\\ &= \mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}',\widetilde{\mathbf{v}}}\left[\sum_{S\subseteq U_i} \delta_S^{i,\mathbf{v}',U_i}\left(v_i'\left(S\setminus \mathbf{x}(\mathbf{v})\right) - \sum_{j\in S} p_j^{i,\widetilde{\mathbf{v}},U_i}\right)\right]\\ &= \sum_{S\subseteq U_i}\left(\mathop{\mathbb{E}}_{\mathbf{v},\mathbf{v}'}\left[\delta_S^{i,\mathbf{v}',U_i}\cdot v_i'\left(S\setminus \mathbf{x}(\mathbf{v})\right)\right] - \mathop{\mathbb{E}}_{\mathbf{v}',\widetilde{\mathbf{v}}}\left[\delta_S^{i,\mathbf{v}',U_i}\cdot \sum_{j\in S} p_j^{i,\widetilde{\mathbf{v}},U_i}\right]\right)\end{aligned}$$

The last equality holds by linearity of expectation and by removing the expectation over $\widetilde{\mathbf{v}}$ on the first term and $\mathbf{v}$ on the second term due to a lack of terms dependent on those random vectors. Now, since $\mathbf{v}'$ is i.i.d to $\mathbf{v}$ and $\widetilde{\mathbf{v}}$, we can interchange them within each expectation:[2]

$$\mathop{\mathbb{E}}_{\mathbf{v}}[u_i(\mathbf{v})] \geq \mathop{\mathbb{E}}_{\mathbf{v},\widetilde{\mathbf{v}}}\left[\sum_{S\subseteq U_i} \delta_S^{i,\widetilde{\mathbf{v}},U_i} \cdot \widetilde{v}_i\left(S \setminus \mathbf{x}(\mathbf{v})\right) - \delta_S^{i,\mathbf{v},U_i} \cdot \sum_{j\in S} p_j^{i,\widetilde{\mathbf{v}},U_i}\right] \tag{A.1}$$

To complete the utility bound, we need to apply Lemma 4.1.1 to bound the inner terms of the expectation in Equation A.1. However, to utilize Lemma 4.1.1, we need the probability distribution and prices to be pointwise to the same valuation profile and set. Furthermore, the value must be determined by the i'th valuation function in the aforementioned valuation profile. Nevertheless, in the second term of Equation A.1, the distribution and prices are pointwise to different valuation profiles. Consequently, there is no way to apply the expectation interchanging property in order to use Lemma 4.1.1. This, is turn, means that we cannot actually achieve the desired utility bound thus highlighting why this pointwise technical ingredient is not sufficient to prove Theorem 4.1.3 using the technique presented in [PDTKL17]. The technical roadblock in this proof technique is the valuation dependent multiplier δ within the expectation. □

A.2 Alternative LP Construction in Relaxation 2

We could proceed by constructing a different feasibility LP.

$$\begin{aligned}
\max_{\mathbf{p}\in\mathbb{R}^{2^m}} \quad & \ell_+ - \ell_- & \\
\text{s.t.} \quad & \sum_{i=1}^{n} p_{x_i} + (\ell_+ - \ell_-) \leq \beta(\mathbf{v}(\mathbf{x})) & \text{for all } \mathbf{x} \in \mathcal{F} \\
& \sum_{i=1}^{n} p_{x_i} \geq \frac{1}{\alpha}(\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x}))) & \text{for all } \mathbf{x} \in \mathcal{F} \\
& p_S \geq 0 & \text{for all } S \subseteq M
\end{aligned}$$

[2]Note that such interchanging must be done properly - i.e. if one replaces a random variable with an i.i.d copy it must be done throughout the expectation.

$$\ell_+ \geq 0$$

$$\ell_- \geq 0$$

We know that if above LP with $\alpha\beta = O(1)$ has non-negative optimal value, the optimal solution proves the **existence** of (α, β)-balanced bundled prices with $\alpha\beta = O(1)$. To determine feasibility we proceed by computation of the dual problem:

$$\begin{aligned}
\min_{\mathbf{y},\mathbf{z}\in\mathbb{R}^{(n+1)^m}} \quad & \sum_{\mathbf{x}\in\mathcal{F}} \left(\beta y_{\mathbf{x}} \mathbf{v}(\mathbf{x}) - \frac{1}{\alpha} z_{\mathbf{x}}(\mathbf{v}(\mathsf{OPT}(\mathbf{v})) - \mathbf{v}(\mathsf{OPT}(\mathbf{v}|\mathbf{x})))\right) & \\
\text{s.t.} \quad & \sum_{\mathbf{x}:S\in\mathbf{x}} (y_{\mathbf{x}} - z_{\mathbf{x}}) \geq 0 & \text{for all } S \subseteq M \\
& \sum_{\mathbf{x}\in\mathcal{F}} y_{\mathbf{x}} = 1 & \\
& y_{\mathbf{x}}, z_{\mathbf{x}} \geq 0 & \text{for all } \mathbf{x} \in \mathcal{F}
\end{aligned}$$

The difference now is that we just need to show that for any feasible solution $(\mathbf{y}, \mathbf{z})$, we have non-negative objective value, hence, the optimum is also non-negative. Then, by strong duality, we can claim that the optimal value to the primal is also non-negative as desired. Furthermore, now $\mathbf{y}$ can be thought of as a probability distribution which is similar to the approach from [KL20] and perhaps tools from that paper of using a zero sum game might help.[3]

[3]However, note that $\mathbf{z}$ is not a probability distribution since it sums to some non-negative value strictly smaller than 1 which may complicate things.

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