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31265

Problem Chosen

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2014 Mathematical Contest in Modeling (MCM) Summary Sheet
(Attach a copy of this page to your solution paper.)

Type a summary of your results on this page. Do not include
the name of your school, advisor, or team members on this page.

Summary

Everywhere we go, every transport we take, we always look for minimizing the duration of the trip. Particularly, in a highway, people normally think that for his own purposes the better option is to overtake the slower vehicles that are ahead of them. Is it worthy to overtake all the slower vehicles? Or might this constant lane-changing provoke a lower traffic flow, increasing the trip time? Understanding overtaking and analyzing its traffic rules is an essential study since it lets the whole system progress.

We were asked to build and analyze a model in order to describe the overtaking phenomenon, according to the keep-on-the-right-except-to-pass rule. Passing another vehicle is an action which only affects few cars, for that reason we have built a microscopic model. We have considered that this kind of representation is a highly significant first step in order to study, in further work with more time, the possible macroscopic effects of overtaking via butterfly effect.

Specifically, we have trust in the power of cellular automata, which makes both space and time discrete, so further facilities a specific description of the rule and the extrapolation for more than two lanes. This decision naturally leads to a computational approach, so that we did. As relevant results we have found that the formation of traffic jams or phantom jams depends on initial conditions whereas in the stochastic one they can be formed spontaneously. Moreover, we found that for a two-lane case there exists a maximum value for flow at a specific density, called critical density.

The Automata Driver

ICM or MCM Contest Question A B C

Team # 31265

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Contents

1 Introduction

The main goal of our work is to build some model with which we are going to analyze the performance of the **Keep-In-Right-Except-To-Pass Rule** rule, by building models that represent it from a discrete point of view. Under several assumptions, we'll also argue the strengths and weaknesses of these models.

We approach this problem by considering 3 main steps and some computational algorithms.

- Firstly, we study the one lane road, and the distribution of cars on it, known as "Follow the leader".
- Secondly, we extend the previous model to interactions between to lanes, and we discuss its veracity and correction.
- Finally, we discuss the obtained results from the flow calculus in a C++ program.

2 First things first

As it was said in the introduction, our work is based on a discrete perspective, and both in a spacial (we are going to see an interpretation of this soon) and a temporal (i.e. if we look at the state of the system then we only check it out again after a certain interval Δt of time) sense. For this reason, we start announcing the fundamental concepts on which we will base our models.

2.1 Definitions

Since our main goal is to build a model that simulates the performance of a traffic rule, we should first define what a **road** and a **vehicle** is, shouldn't we?

Definition 1 (Road) *A **road** is the subset $R \subset \mathbb{N}^2$ defined by* $R = [0, N] \times [0, L]$.

Definition 2 (Lane) *A **lane** is a subset of R defined by* $l_i = [i-1, i] \times [0, L]$ *for* $i \in \{1, ..., N\}$.

Definition 3 (Height) *A **height** is a subset of R defined by* $h_i = [0, N] \times [i-1, i]$ *for* $i \in \{1, ..., L\}$.

Definition 4 (Box) *A **box** is a subset of R defined by* $b_{ij} = h_i \cap l_j$.

From this point of view, we can consider a road as a matrix, where heights are the rows, lanes are the columns and boxes are individual positions of the matrix, but since we consider that vehicles move upwards, the rows' usual indexes are inverted. Besides this, we will return to this concept of road as a matrix when we talk about our computational models.

Definition 5 (Vehicle) *We say that a vector* $v = (v_1, v_2, v_3, v_4) \in \mathbb{N}^4$ *is a **vehicle** over the road R if its two first components* $v_1 = j$ *and* $v_2 = i$ *indicate a certain box* b_{ij} *of the road. The two last components are known as the **speed** and the **maximum speed** of v.*

The box occupied by a vehicle in a certain instant of time is known as its **position**. The speed of the vehicle determines the number of boxes that it will move forward in the next step of time. For the forth component of the vector to have sense, we wait for the assumptions.

There are certain parameters that we need to define in order to give a more complete description of traffic.

Definition 6 (Flow and density) *We define **traffic flow** $q(b_{ij}, t)$ in a certain position at a certain time as the amount of vehicles that go through that position per unit of time.*

*We define **traffic density** $\rho(b_{ij}, t)$ in a certain position at a certain time as*

$$\rho(b_{ij}, t) = \frac{1}{d+1}$$

where d is the separation between the vehicles nearby.

2.2 Assumptions

1. **All kinds of vehicle have the same size.** For simplicity, we assume that the length of every vehicle is one box. Also, two cars neither can be in the same box nor pass over each other.

2. **Distance between vehicles.** Vehicles tend to keep a distance between them, that is the security distance. We define this parameter as the distance needed for a car to stop and not collide when is moving at a certain speed. In such a way, this makes density a function of velocity [1].

3. **Speed limit.** We assume that every sort of road has a concrete speed limit, which any kind of vehicle can overpass.

4. **Types of vehicles.** In our model, there are different types of vehicle depending on their own maximum speed, conditioned by their technical characteristics. This maximum speed is always slower or equal to the road speed limit.

5. **Driving attitude.** Drivers always will to go at their own maximum speed in order to arrive sooner at their destination.

3 First approach: Follow the leader

In this section, we consider that cars are distributed along one unique lane. Because vehicles cannot surpass each other due to assumption (1), and they want to keep a certain gap between them through assumption (2), it seems they are likely to distribute themselves in as many groups of vehicles as different maximum velocities there are. Each group with the same separation between automobiles, the first vehicle is the "leader", and the rest of automobiles follow him.

Since there is only one lane, we can ignore the first component of each car (the one that tells us the lane). This means that we can determinate the position of each car by only considering its height. It is necessary to introduce the notation in this section. If v^i is the i-th vehicle of a set V, let $x_i(t)$, $v_i(t)$ and $v_{i_{max}}(t)$ be v_2^i, v_3^i, v_4^i respectively.

The position of the vehicle at time t depends on its position at $t - \Delta t$ and on the speed associated in that moment, $v_i(t - \Delta t)$. We can express that in terms of the well-known expression of uniform linear motion relation

$$x_i(t) = x_i(t - \Delta t) + v_i(t - \Delta t)\Delta t \tag{1}$$

But note that $v_i(t - \Delta t)$ is not necessary the speed that the automobile would have in a real situation. It can be a good approximation of it when the system is almost-stable, or when it has already reached equilibrium (in those cases, speed is almost constant). In other situations, when accelerations in a continuum model are acting, $v_i(t - \Delta t)$ is a speed such that the distance covered with it is the same that the one covered considering accelerations in a interval of time Δt.

Now, we would like to know how speed of one vehicle changes by the influence of the car in front. First, we assume that there are only two cars, and the first one is moving slower than the second one. The second car have eventually to reduce its speed to reach the first's one speed. Besides, it will stay at a certain distance from the first car.

The speed of the second car is going to go down if its too near to the first one. But it could be greater if there was a large separation between them. All this explanation always taking as a reference the first car speed. In a linear approximation, this is described as

$$v_i(t) = v_{i+1}(t - \Delta t) + \frac{x_{i+1}(t - \Delta t) - x_i(t - \Delta t) - d}{\Delta t}$$

If we want to avoid collisions, d has to be a function of speed of the car. Actually, this is one of the given assumptions (2). The second car's position is determined by (1), which means that if he has to stop to avoid a collision, d equals to

$$d = v_i t - \Delta t \Delta t \tag{2}$$

Thus,

$$v_i(t) = v_{i+1}(t - \Delta t) - v_i(t - \Delta t) + \frac{x_{i+1}(t - \Delta t) - x_i(t - \Delta t)}{\Delta t} \tag{3}$$

Notice that if the second vehicle is not nearby the first one, the speed calculated at (3) could be greater than $v_{i_{max}}$. If that is the case, the vehicle will not assume (3), and he will go as fast as he can.

4 Second approach: Two lanes

Once we have completed the model for the case of one lane, we should pass to analyze the two-lane situation. The model should represent the rule that the statement proposes but also the motivation of the drivers, and according to the assumptions exposed in 2.2., we get to the following results.

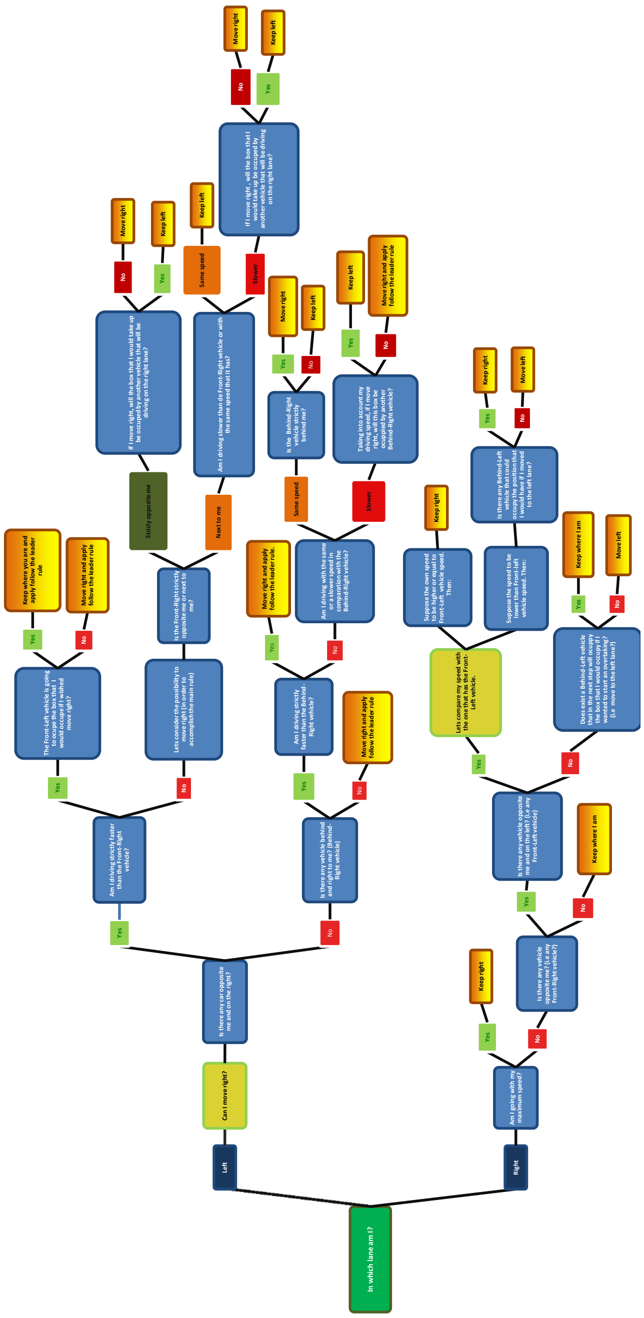
Lemma 4.1 (Priority criterion for the application of the algorithm rules) *For each moment (i.e. for each interval of time considered) every car follows, if possible, the following actions with decreasing priority order:*

1. *Overtaking. If is not possible then:*
2. *Move right. If is not possible then:*
3. *Follow the leader.*

Notice that if the considered automobile is not in the right lane, then first Lemmas? step is not possible. Thus, we should consider if it could do the second action, and if that wasn't possible, then it would end with the last option. The built model for the two lane road looks like a well-known model called "Cellular Automaton".

In general terms, we can define it as a deterministic model, and because of it, it is coherent if we consider, among other factors, that the vehicles are a kind of automata (i.e. they work by they own according to a set of rules already programmed in them, not tied to any type of stochastic or random processes).

According to what we want to study, our double-lane model will consists in vehicles acting as automata (following an algorithm). This algorithm is graphically described in the next page.



With this algorithm, what we achieve is that given any car in the road and a certain instant of time, we can know the next position where the car is going to be, after a time Δt has passed. Consequently, the final result, after using the algorithm with every car, is the new distribution of the vehicles in the road.

After trying to compute this algorithm, and making some adjustments, we conclude that it is only correct when the amount of cars in relation to the space they have (density) is not excessively elevated. We would understand that density corresponds to **light traffic**. This is due to the fact that in its deepest processes, the algorithm only checks, given a vehicle, the information given from the closest automobiles.

In order to specify a little more what we meant to say with light traffic, and make clear in which situations this model still works, consider the next case: if we were in a highway, where trucks can usually move at 90 km/h (25 m/s), and a private cars can speed up to 120 km/h (33,3 m/s), then the highest difference between velocities in that highway is about 8 m/s. It is not that bad to assume that 1 $m/s = 1\ box/\Delta t$, so our modelled speed would be 8 $box/\Delta t$. Then, for speeds higher than that, we would have to consider more vehicles around each one, so the algorithm is no longer useful.

With new rules, and basing a new code in a cellular automata algorithm [3][4], which consider more vehicles around each one, depending on the difference between the maximum and the minimum speed among all velocities of all cars (figure 1), we programed a simulation for a random distribution of cars and its velocities in a road of length L, representing the road as a matrix inside the program. The input of our program was an array containing different possible velocities and the number of cars generated.

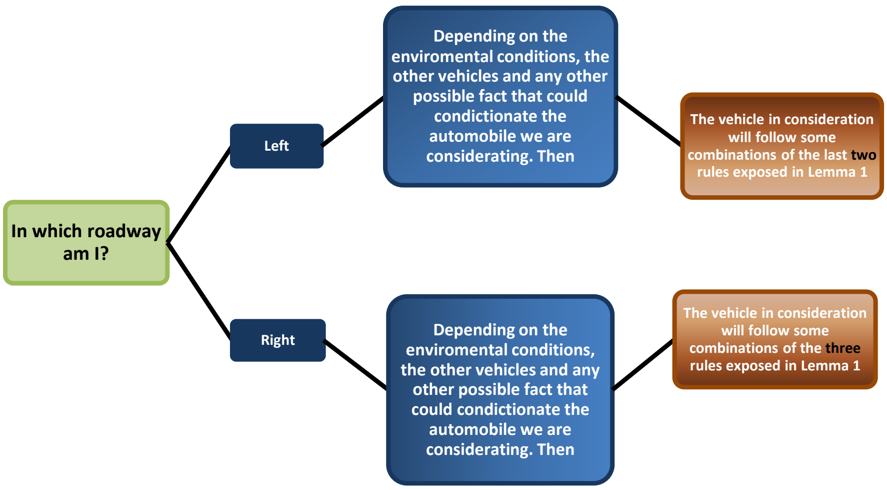


Figure 1: Resumed new diagram for distribution prediction

The main aspect to analyze about this two-lane model is to find the density of cars which provokes the maximum flow,the density varied between 0 and 1 $vehicles/box$. The main aspect to analyze about this two-lane model is to find the density of cars which provokes the maximum flow.

We repeatedly iterate our program with an initial random distribution of position and speed for the vehicles on the road. The density varied between 0 and 100 $cars/box$, with a speed of 8 $box/\Delta t$ for cars and 5 $box/\Delta t$ for another kind of vehicle, lorries for instance. The length of the road was taken to be 100 $boxes$.

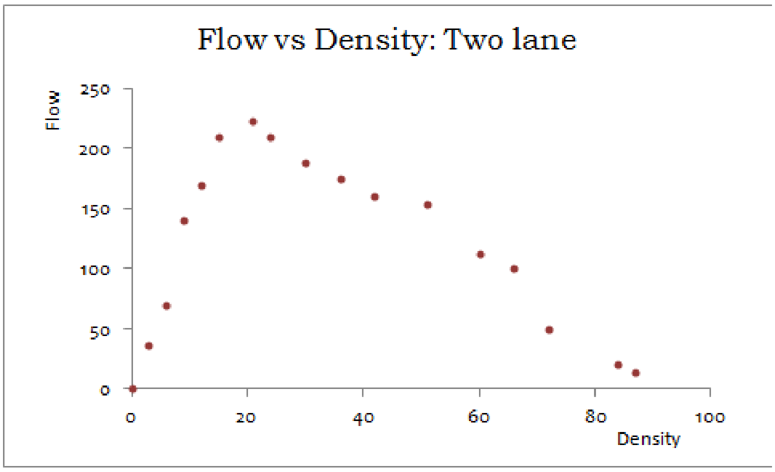


Figure 2: Density vs flow, considering two different speeds for the vehicles.

We can observe in the figure that there exists a concrete density that maximizes the flow, which is desirable because it means that the equilibrium between average speed and number of cars on the road is the best-balanced possible. The shape of the graphic has been well-studied and it is the fundamental diagram of traffic flow. [6]

5 A further step: Approximation to consider the driver's randomness.

The previous analyzed models were all deterministic in the sense that the formation of a traffic jam only depends on the initial conditions of the vehicles. If slow drivers were picked to the beginning of the road o at the beginning of a group of cars, for sure the average speed will tend to decrease. That is not the case of the stochastic models.

We are adding to the deterministic model a random parameter, which affects the acceleration or the braking of a vehicle, varying the speed in an interval of one unit. So that, spontaneous traffic jams can be observed [2]. That is a stochastic traffic cellular automata (**Nagel-Schreckenberg TCA**). It is true that this randomization has no logical background, it is a kind of experimental correction [7]. This mini-jams can evolution to generalized jams if the vehicle density is greater than the critical density (the density where the flow is maximum).
So in our simulation program, it is possible to represent this model by only adding a function of randomness at acting at some iterations upon the speed of some cars. If we had more time we would have analyzed the graphic of flow vs density in order to compare it with the deterministic one. Another interesting project could be to try to approximate the relation between the formation of a jam depending on the length of the interval of randomness applied to the drivers. We considered this idea because we wanted to determine if this randomness parameter is related with anger, the hour of the day or other human aspects which clearly determine real traffic pathologies.

6 Conclusion

Along the project we have achieve come important conclusions, we have proof by simulation that slow flow traffic only happens when a slow driver is on the head of a grupos of vehicles, and that situation remaind constant in time once it has happened. However, if we introduce a random paramether we get a more realistic model. Furthermore, we have computationally proved that there exists a situation where is possible to maximize the trafic flow. Moreover, unfortunate manouvers can lead to a phantom jam, which is likely to vanish if the density is less or equal to the critical denisity, for that reason is important to understand overtaking in detail. An intersesting further work could be to make a countinous model in time to study the process of acceleration and driver time reaction. We would like to have a little bit more time to think about a macroscopic representation of the effects of microscopi ovetaking based on the butterfly effect. The problem proposed by the contest had so many details to answer, for our team it was not possible to deal with all of them but he have discussed some points that we would want to explain.

Intuitively, we think that increasing the number of lanes in a road has a positive effect on the traffic flow above all in countries with a relaxed legislation about keeping on the right lane except to pass. But in countries where the law is more restrictive this ampliation of lanes is not as important as in the other case. With relation to the possibility of automathic driving control, we shoulde make a difference between automata and artificial intelligent robots. If all the cars in a highway were controlloled with computer dispositives it would be easier to increase the global flow since all of them would be able to coordinate themselves if another car wants to increase its speed.

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