Team # 207
Problem B

Physics of a spinning golf hit

November 15, 2015

Abstract

A solution to a golf problem that involves the study and developing of a ball flight aerodynamics and a three dimensional collision has been found and explained. The main point is to find the most optimal solution to the problem of hitting a golf ball in way in which it achieves the necessary spin to make it go around a tree and end up in the green. The first approach is divided into two stages. Firstly, the equations that describe the ball flight trajectory are found assuming that the forces acting on the ball are: (i) gravity, (ii) drag force and (iii) the lift or 'Magnus' force. Secondly, the equations concerning the club-head/ball collision are developed to find the post-impact velocity and spin of the ball depending on the clubhead speed and its loft and spin angles at the moment of impact.

Once the equations to use have been stated the problem is coded in Maple as follows: (i) a first code is designed to find the trajectories of the ball in function of its post-impact conditions, (ii) a second code is made to find the flight time for each trajectory, (iii) a third code takes a set of post-impact conditions being within a reallistical and approximated range of values for the club-head impact and calculates whether the correspondent trajectories are succesful or not.

Although the goal is to present a solution to the problem, as all the research has been done in less than 48h, the model is afterwards re-analysed and some suggestions are made so that other researchers may go into some further extensions of the equations and the solution.

Contents

1 Introduction

1.1 Problem approach

A golfer hits an errant shot and finds the ball directly behind a tree. The ball, with radius r and mass m, is 120 meters from the green. The green can be supposed as a cercle of 10 meters in diameter with its center placed right behind the center of the tree, as seen from the location of the ball. The tree is too close and too tall to hit the ball over it so the golfer is forced to hit the ball either to the right or to the left. We can model the tree as a cylinder of $R = 5$ meters in radius because of the branches and with its center 10 meters from the ball.

What we want to determine is the parameters of the club-ball impact that would result in a shot that would go around the tree and end up in the green. Our purpose is, fixed a model, find values or, even better, a range of values of the following parameters that would accomplish the requirement of getting the ball to land on the green:

1. Direction and module of the clubhead's impact velocity.
2. Orientation of the clubhead at the moment of impact.

Regardless of the model considered to describe the physics of the problem we make the following interpretations:

- It is assumed that there is no wind.
- We assume the green to be flat and at the same height with respect to the height of the ball before the shot.
- To simplify the problem, we will consider a shot to be succesful when it ends up landing on the green. In other words, we won't take into account the bouncing and rolling phase of the ball trajectory. Although this is important to be reallistic, this is beyond our goal.

The coordinate system choosen, positively oriented, is uniquely determined by the orientation of the unitary vectors $\vec{i}$ (which define x axis) and $\vec{j}$ (which define y axis) in the usual notation. In figure 1 we have an schematic representation of the xy plane where we can see the circles with which we identify the ball, the tree and the green.

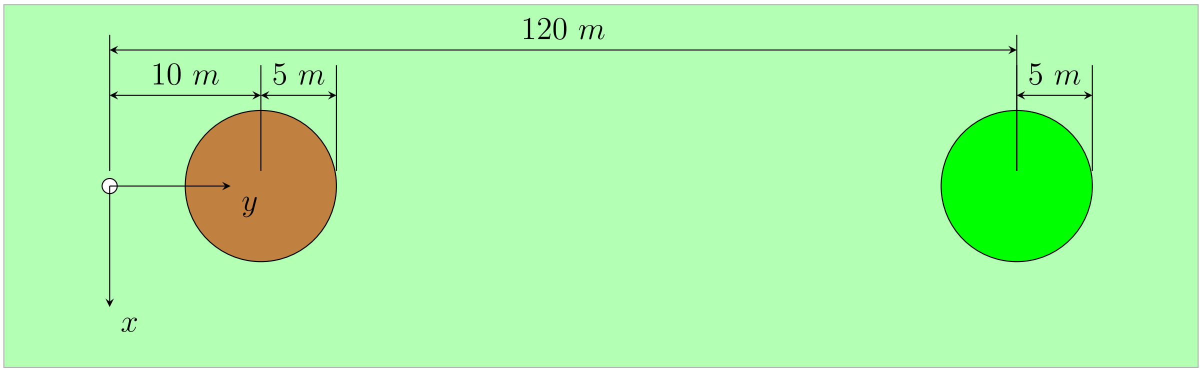

Figure 1: Scheme of the situation of the problem (not scaled). Also, x and y axis are represented. z axis is perpendicular to the figure and goes out of it.

To determinate if a given shot makes the ball ending up on the green and does not hit the tree, it will also be useful to know that the equations of de cercles associated to the elements of our sistem in the choosen coordinate system are, according to the statements of the problem:

- **The ball**: defined by the volume $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z-r)^2 \leq r^2\}$ in $\mathbb{R}^3$.
- **The tree**: defined by the volume $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + (y-10)^2 \leq R^2, z \geq 0\}$ in $\mathbb{R}^3$.
- **The green**: defined by the volume $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + (y-120)^2 \leq R^2\}$ in $\mathbb{R}^3$.

To obtain the equations for cercles mentioned we must set z to zero in all the above volumes. For the ball, we have considered that it is just over the origin so its position vector is $r_0 = (0, 0, r)$.

2 Ball flight

In order to describe de trajectory followed by a golf ball we have considered a model derived in [1] which gives a coupled differential equations system as from the second newton's law by taking into account the following forces acting on the ball: the gravitational force $\vec{F_g}$, the drag force due to the air $\vec{F_D}$ and the contribution of the Magnus effect $\vec{F_L}$. As we are dealing with short distances contributions of the centrifugal and the Coriolis forces are neglected.

Coming up next we expose the assumptions and considerations made with each force which will lead us to the motion equation used for our model. This equation, in its general form, describes the motion of a spherical projectile rotating about an arbitrary axis in presence of an arbitrary wind.

The parameters and vectors refered to our fixed coordinate system that will determine the dynamics and kinematics of the golf ball and which we haven't defined yet are:

- The position vector $\vec{r}(t) = (x(t), y(t), z(t))$, with module r,
- The linear velocity vector $\vec{v}(t) = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$, with module v,
- The angular velocity or spin vector $\vec{\omega}(t) = (\omega_x(t), \omega_y(t), \omega_z(t))$, with module ω,
- Cross-sectional area of the ball, denoted by A
- Air density ρ,
- The initial conditions will be $v_0 = v(0), \omega_0 = \omega(0)$.

From now on we work with the International System of Units.

2.1 Gravitational effect

The trajectory of the ball would be a simple projectile motion if the gravity was the only force to act on the ball. Regarding to the coordinates system that we mentioned in the introduction, our gravity force will be considered to be in the z axis:

$$\vec{F}_g = -m_b\vec{g} \tag{1}$$

where $\vec{g} = (0, 0, -9.81)\frac{m}{s^2}$.

The only thing that we will assume here is the gravitational field to be constant along the trajectory, because the variation of this magnitud in the range of heights within the ball will move can be neglected.

2.2 Drag force

Any object moving in a fluid experiments a drag force which opposes to motion. This effect is still being studied in the field of fluid dynamics. As long as it concerns to a spherical projectile moving through air it is widely accepted (see for example 1,ref2) that it is proportional to the square of the velocity module [1] and it satisfies the following equation:

$$\vec{F_D} = -\frac{1}{2}\rho A C_D v^2 \frac{\vec{v}}{|\vec{v}|} \tag{2}$$

where C_D is a dimensionless frictional term that in general can depend on ball's velocity v and also on its spin ω. The determination of the C_D parameter depends on the situation considered. It is usual to rely on very especific experiments, appropiate for the cases studied, in order to see the empiric dependance with the magnituds of v and ω. We will make our particular assumptions in section 4.

[1] Here the velocity must be the one relative to the air, but in our case as there is no wind considered, it coincide with the velocity with respect to the fixed coordinate system.

2.3 Magnus effect

This effect is observed when a projectile moving through air experiments a curving away force due to its spin or angular momentum. Let θ be the angle between $\vec{\omega}$ and $\vec{v}$ in a given time. According to experimental results we have that:

- On the one hand, for spins perpendicular to the projectile velocity (i.e. $\theta = \frac{\pi}{2}$), such as a golf ball with its backspin, it is reasonable to assume that the Magnus force is proportional to v^2 and act at right angles to both $\vec{v}$ and $\vec{\omega}$ (i.e. points to the $\vec{\omega} \times \vec{v}$). In this situation the module of the Magnus force is accepted to be

$$F_L = \frac{1}{2}\rho A C_L v^2 \tag{3}$$

  where C_L is a dimensionless lift coeficient which is in general a function of ω and other variables.

- On the other hand, for spins such that $\vec{\omega}$ is anti-parallel or parallel to $\vec{v}$ (i.e. for $\theta \in \{0, \pi\}$) it is also found empirically that $F_L = 0$.

Therefore it seems to be that the force of the magnus effect is maximum when $\vec{v}$ and $\vec{\omega}$ are perpendicular, while for the parallel and antiparallel case there is no force.

In general, for a given θ there is not a clear knowledge in the literature of how F_L behaves. For this reason, if we take into account the two above cases it is reasonable to assume that for θ arbitrary, the force is dependent on the perpendicular component of $\vec{\omega}$ with respect to $\vec{v}$ so that the expression for the Magnus effect force - which is also called *lift force* in the case of backspin because it causes a ball lift - in this model is assumed to satisfy:

(i) The direction of $\vec{F}_L$ is given by $\vec{\omega} \times \vec{v}$.

(ii) F_L varies smoothly as $\sin\theta$, since $\sin\theta$ gives the perpendicular component of $\vec{F}_L$.

with this assumptions, as shown in [1] we obtain an expression for $\vec{F}_L$:

$$\vec{F}_L = \frac{1}{2}\rho A C_L v \cdot \left(\frac{\vec{\omega} \times \vec{v}}{\omega}\right) \tag{4}$$

where C_L is a dimensionless term called lift coefficient that in general depends on v and ω. As in the case of the drag coefficient, there is a lot of discussion about how to determine this term. We will make our own simplifications in section 4.

2.4 Motion equations

If we gather the three forces exposed above, with expressions given by the equations (**??**), (2) and (4), and consider that there is no wind,[2] then the second Newton's law equation turns

[2] If there was wind, to obtain the Newton's second law equation it would be enough to make a frame of reference change to the coordinate system where the velocity of wind is zero.

out to be:

$$m\ddot{\vec{r}} = -\frac{1}{2}\rho A C_D v^2 \frac{\vec{v}}{|\vec{v}|} + \frac{1}{2}\rho A C_L v \cdot \left(\frac{\vec{\omega}\times\vec{v}}{\omega}\right) + m\vec{g} \tag{5}$$

Resolving equation (5) into components we find a system of three coupled second order differential equations:

$$m\ddot{x} = -\frac{1}{2}\rho A(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}\left[C_D\dot{x} - C_L\left(\frac{\omega_y\dot{z}-\omega_z\dot{y}}{\omega}\right)\right] \tag{6}$$

$$m\ddot{y} = -\frac{1}{2}\rho A(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}\left[C_D\dot{y} - C_L\left(\frac{\omega_z\dot{x}-\omega_x\dot{z}}{\omega}\right)\right] \tag{7}$$

$$m\ddot{z} = -\frac{1}{2}\rho A(\dot{x}^2+\dot{y}^2+\dot{z}^2)^{\frac{1}{2}}\left[C_D\dot{z} - C_L\left(\frac{\omega_x\dot{y}-\omega_y\dot{x}}{\omega}\right)\right] - mg \tag{8}$$

The initial conditions would be $\vec{r_0} = (x_0, y_0, z_0)$, $\vec{v_0} = (v_{0x}, v_{0y}, v_{0z})$ and $\vec{\omega_0} = (\omega_{0x}, \omega_{0y}, \omega_{0z})$ and it can be integrated with numerical methods. For our purposes we will assume that $\vec{r_0} = (0, 0, r)$ and $\vec{\omega_0} = (\omega_x, \omega_y, \omega_z) = \vec{\omega}$ will be constant as long as we will neglict the rotational speed deceleration.

3 Club-head/ball interaction

As we mentioned before, what we want to determine how the interaction between the club-head and the ball has to be in the moment of the impact in order to obtain a trajectory that will make the ball land on the green. In the previous section, we have described the equations that rule the behaviour of the trajectory, given certain initial conditions, such as the velocity of the ball and its spin. What we are going to expose here is which sets of initial condicions can be given to the ball, considering that it has to be thrown away with a golf club, according to the characteristics of the club and the way the swing is performed.

Specifically, we focus on four factors that affect the launch of the ball: the mass of the clubhead m_c, the velocity of the clubhead in the moment of the impact $\vec{u}$ with module u, the loft angle β, and angle between the xy diretion of the normal vector to the face of the clubhead and the direction of its velocity (as shown in figure 2), that will be refered, from now on, as *spin angle*, and will be denoted by θ.

It is shown in [9] the deduction of the equations for the post-impact velocity of the ball in the case where $\theta = 0$. In that situation, it is seen that the ball obtains the following velocities, from the laboratory frame of reference:

$$v_n = \frac{(1+e)m_c}{m_c+m} u\cos\beta \tag{9}$$

$$v_p = \frac{2}{7}\frac{m_c}{m_c+m} u\sin\beta \tag{10}$$

Where $\vec{v_n}$ is parallel to the normal vector to the face of the clubhead, and $\vec{v_p}$ is perpendicular to the normal vector and lives inside the plane defined by $\vec{v_n}$ and $\vec{v}$ (as seen in figure 3) and

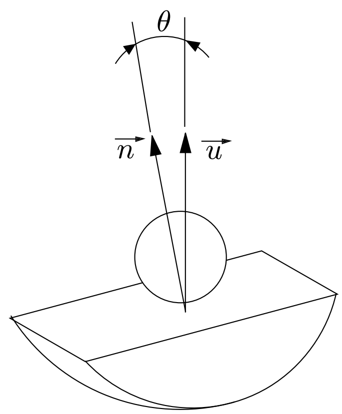

Figure 2: Representation of the spin angle

e is the *restitution coefficient*, which is related with the energy lost during the impact due to the deformation of the objects that collide[3].

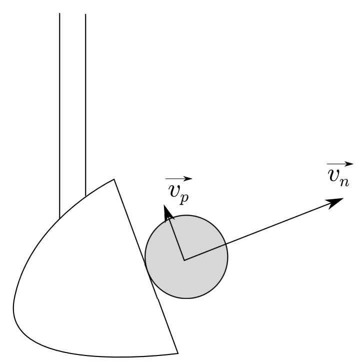

Figure 3: Clubhead seen from the side and vectors $\vec{v_n}$ and $\vec{v_p}$

The origin of the velocity v_p is an angular velocity ω_p that appears due to the loft angle of the club, and makes the ball rotate arround the axis that goes through the middle of the ball and its perpendicular to the directions of $\vec{v_n}$ and $\vec{v_p}$, and up along the face of the clubhead. ω_p satisfies the following equation:

$$\omega_p = \frac{5}{7}\frac{u \sin\beta}{r} = \frac{5}{2}\frac{(m_c + m)v_p}{m_c\ r} \tag{11}$$

where r is the radius of the ball.

Assuming that $\vec{u} = (0, u, 0)$, those expressions can be expressed according to our coordinate system applying the proper rotation matrix, so they result in:

[3]That is not negligible at all. As seen in [10], the ball can be significally compressed during the impact of the club head, recovering its round shape once it takes over.

$$
\begin{aligned}
v_y &= \frac{m_c}{m_c+m} u \left[(1+e)\cos^2\beta + \frac{2}{7}\sin^2\beta \right] \\
v_z &= \frac{m_c}{m_c+m} u \sin\beta\cos\beta \left(\frac{5}{7} + e \right) \\
\omega_p &= \frac{5}{7}\frac{u\sin\beta}{r}
\end{aligned}
$$

The validity of these equations rely on the following assumptions:

1. The clubhead moves completely horizontal at the moment of the impact with the ball (that is, $\vec{u} = (u_x, u_y, 0)$, according to our coordinate system).

2. The clubhead is free of any external force at the moment of the impact, so $\vec{u}$ arround that very moment is constant. That would mean that the golfer applies force to the club only during the first moments of the swing, and while this is not a very reallistic assumption, it is necessary to correctly deal with the linear and angular momenta in general terms, because it avoids considering the particular technique of the golfer and the forces that he or she applies.

We want to generalize the expressions above to obtain formulas for the case where the club head strikes the ball with a spin angle $\theta \neq 0$. In that case, there will appear another rotation ω_e on the ball that will make it spin arround the z axis and, according to the Magnus force, will curve the ball to the left or to the right (depending on the signum of θ). Also, the expressions for the velocity of the ball after the impact will vary. These formulas will be used in our model to describe the possible initial conditions that could be given to the ball.

We note that, since we base our deduction on the formulas from [9], we are forced to make the same assumptions that have been already exposed.

We start finding the initial velocities of the ball along our coordinate system. For that purpose, we consider an auxiliar coordinate system $(\vec{p_2}, \vec{n}, \vec{p_1})$ defined over the face of the clubhead, as seen in figure 4.

For vector $\vec{u_1}$, we may apply the expressions (9) and (10), so we get v_n and v_{p_1} along the directions of $\vec{n}$ and $\vec{p_1}$. Since $u_1 = u\cos\beta$, we have

$$v_n = \frac{(1+e)m_c}{m_c+m} u \cos\beta\cos\theta \tag{12}$$

$$v_{p_1} = \frac{2}{7}\frac{m_c}{m_c+m} u \sin\beta\cos\theta \tag{13}$$

Now we use vector u_2 to find v_{p_2} applying the expression (10). We have that

$$v_{p_2} = \frac{2}{7}\frac{m_c}{m_c+m} u \sin\theta \tag{14}$$

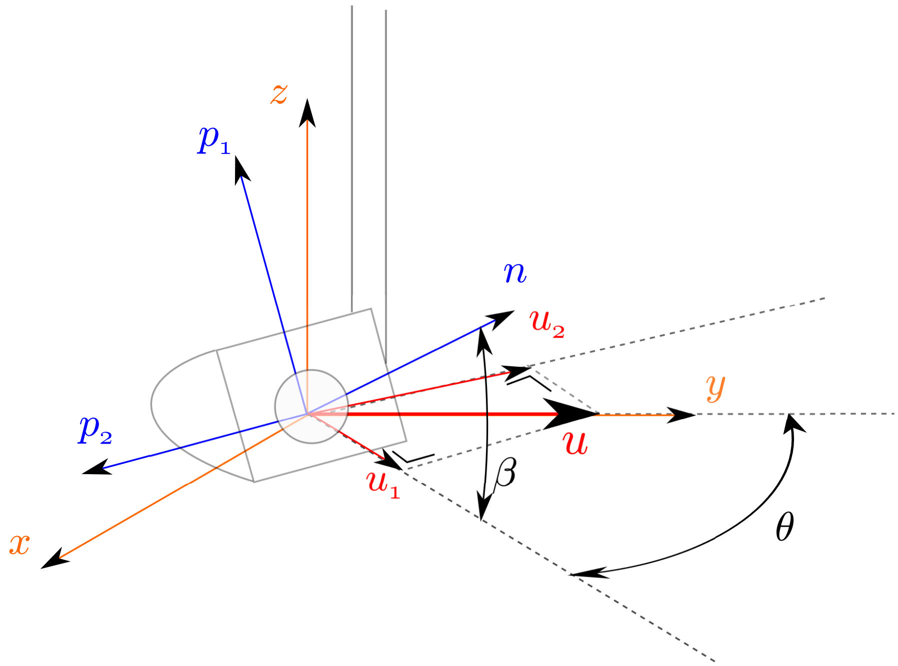

Figure 4: Auxiliar coordinate system and velocity of the clubhead

All together, we have that the velocity $\vec{v}$ in the coordinate system $(\vec{p_2}, \vec{n}, \vec{p_1})$ is

$$\vec{v} = \left(\frac{2}{7} \frac{m_c}{m_c + m} u \sin\theta \ , \ \frac{(1+e)m_c}{m_c + m} u \cos\beta \cos\theta \ , \ \frac{2}{7} \frac{m_c}{m_c + m} u \sin\beta \cos\theta \right)$$

Similarly to the previous case, v_{p_1} and v_{p_2} are consequences of certain angular velocities ω_p and ω_e induced in the ball. Extending (11) to v_{p_1} and v_{p_2}, we obtain

$$\omega_p = \frac{5}{7} \frac{u \sin\beta \cos\theta}{r} \tag{15}$$

$$\omega_e = \frac{5}{7} \frac{u \sin\theta}{r} \tag{16}$$

We note that the ball rolls up along the direction of p_1 with angular velocity ω_p, and rolls to the right along the direction of p_2 with angular velocity ω_e. That means that, in the coordinate system $(\vec{p_2}, \vec{n}, \vec{p_1})$, the angular velocity vector $\vec{\omega}$ can be written as

$$\vec{\omega} = \left(\frac{5}{7} \frac{u \sin\beta \cos\theta}{r} \ , \ 0 \ , \ \frac{5}{7} \frac{u \sin\theta}{r} \right)$$

It is clear from figure (4) too that, in order to go from $(\vec{p_2}, \vec{n}, \vec{p_1})$ to (x, y, z) coordinate systems, one applies the following rotation matrix

$$R = \begin{pmatrix} \cos\theta & \cos\beta \sin\theta & -\sin\beta \sin\theta \\ -\sin\theta & \cos\beta \cos\theta & -\sin\beta \cos\theta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}$$

so, to obtain the velocity vector and the angular velocity vector in our coordinate system, we multiply this matrix with the vectors $\vec{v}$ and $\vec{\omega}$ above. We do not show this expressions here due to their extension, but we remark that this is done in the successive computer calculations.

4 Proceeding and results

To give an answer to the problem, we are going to use the motion equations that we found in the section 2, but the initial conditions of the equations will be given by the expressions in section 3, that is, $\vec{v_0}$ and $\vec{\omega_0}$ are determined by (12), (13), (14), (15) and (16). The goal is to find which initial conditions bring to good shots. According to the expressions that we have just mentioned, that means that what we have to find is which initial velocities of the club head and spin angles, and which *orientation angles* α, that determine the direction of the initial velocity projected into the xy plane, make the ball arrive to the green without touching the tree.

We have, then, that the list (u, α, θ) determines a trajectory of the ball and it will be a valid shot if and only if its trajectory equation is such that the center of the ball is always at a distance from the center of the tree greater or equal than the sum of the ball radius and the tree radius. In mathematical language: $\|(x(t), y(t)) - (0, y_{tree})\| > r + R$ for all $t \geq 0$, where $y_{tree} = 10m$. If additionally it lands on the green, the shortlist will be an answer. The condition will be: $\|(x(t_{land}), y(t_{land})) - (0, y_{green})\| \leq r + R$ where $y_{green} = 10m$

4.1 Assumptions

- After a little research (see [3]) we have found that the typical club velocities at the moment of impact are between 40 and $60m/s$.
- As we want to reach approximately 120m with the shot, regarding to [2] the iron 8 is a good club choice.
- According to [4] the iron 8 has a loft angle of 39º and so we will consider. From the same reference we extract the value of the clubhead mass, which is $m_c = 0.274kg$
- For the drag coefficient we will be considered the value of $C_D = 0.25$. The choice has been done accordingly to [8].
- For the lift coefficient in our simulations we use the value of $C_L = 0.325$. In this case we will justify this choice with the results shown in the study made in [6]. To choose the number we took $v \approx 45\ m/s$ and $\omega \approx 600\ rad/s$ and calculated the S_p parameter, called spin ratio, which is defined by

  $$S_p = \frac{r\omega}{v}$$

  and so the correspondent C_L for a golf ball with the characeristics of the type A in [6] must be in the range $[0.30, 0.35]$.

- The mass m_b, diameter d and coefficient of restitution e of a standard ball is extracted as well from [4]. It will be set to $m_b = 0.459kg$, $d = 0.427m$ and $e = 0.68$.

4.2 Numerical simulations with Maple

The differential equations in (8) have been solved using numerical methods provided by the software Maple. A procedure has been made that, given the list (u, α, θ), calculates, in the first place, the initial velocity and angular velocity of the ball, then uses this information as initial conditions of the ball, and finally calculates its trajectory. Figure 5 shows an example of trajectory calculated with this program.

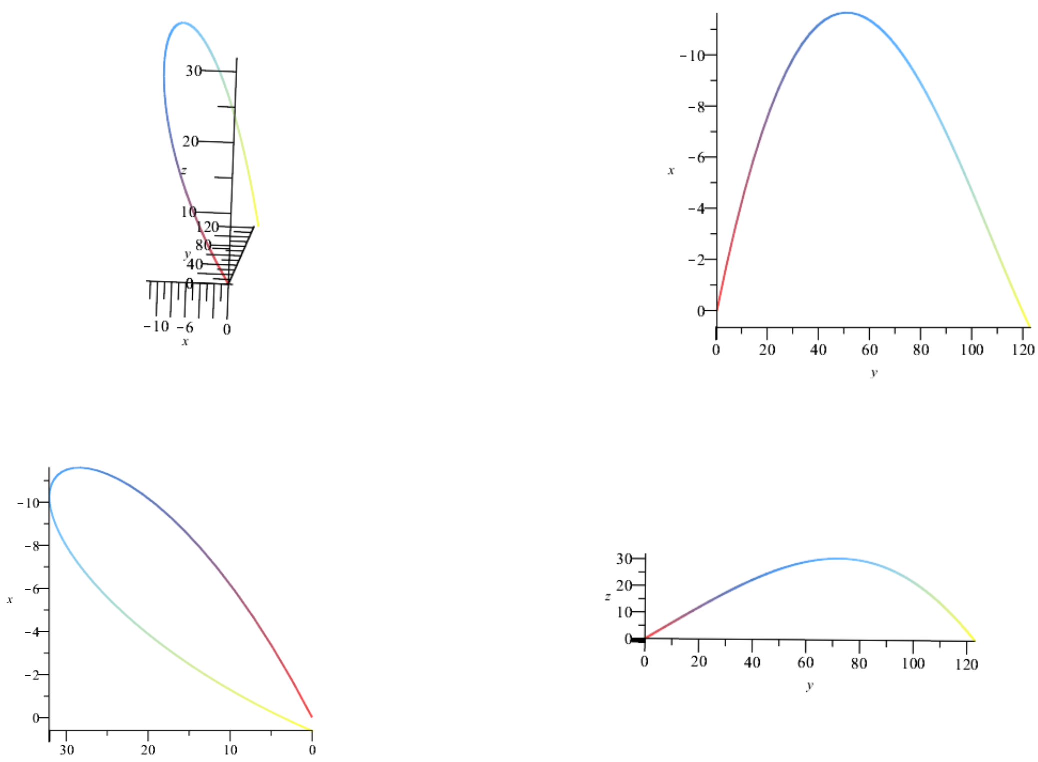

Figure 5: Exemple of trajectory followed by the ball with initial conditions of the shot $(u, \alpha, \theta) = (47\ m/s, 26^o, 39^o)$.

It is obvious that a lot of possible lists (u, α, θ) could be given in order to calculate a trajectory, but much of them will not fit the requierement of hitting the green without hitting the tree. Therefore, additional procedures are requiered to analyse these situations.

First of all, a piece of code has been written to calculate the time of flight of the ball, t_{land}, that is, the time when the ball first touches the ground. This has been done to limit the range of the solution given by the first program. Secondly, two extra codes have been written: one avaluates wether or not the ball is at a distance lower than 5 m from the center of the green when $t = t_{land}$, the other monitorizes the distance from the ball to the center of the tree and while the position y of the ball verifies that $y \leq 10$. Only when the two programs give positive results, the trajectory performed will be considered as successful.

A final piece of code has been written to avaluate possible trajectories faster. This program tries all possible lists (u, α, θ), considering u, α and β inside certain intervals, and executes the three programs descrived above for each list. u has been considered in the interval $[40, 60]$ according to the assumptions mentioned before. α has been considered to move between 26º and 35º, since a golf player would want to throw the ball as close to the tree as possible (in order avoid huge deviations of the landing position of the ball from the center of the green) and the mininum orientation angle α that can be used is given by the position of the tree and its radius, $\tan \alpha_{min} = 1/2$, $\alpha_m in \approx 26^{\circ}$. Finally, θ has been considered to be in the interval $[20, 50]$.

The following results have been obtained. Figure 6 gather four point graphics, where each point represent a valid set of initial conditions for a trajectory that would land inside the green without touching the tree.

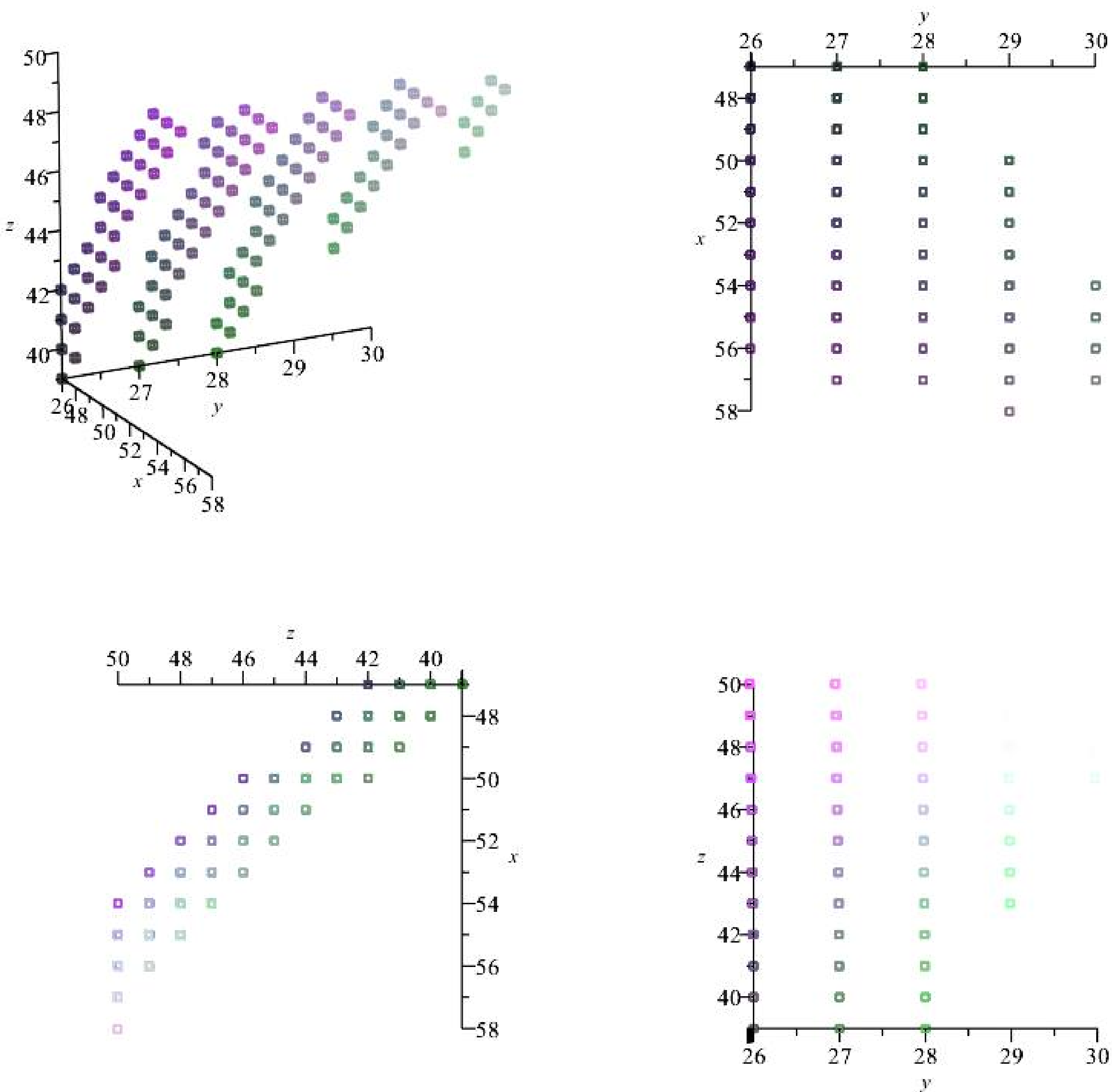

Figure 6: Possible sets of (u, α, θ) that perform valid trajectories. u, α and θ are represented in the x, y and z axis, respectively

5 Strenghts and weaknesses of the model

Here we will analise the pros and cons of the work done. We will focus first on the discussion that concerns the two models we have developed and then we make some comments about our general approach.

5.1 Ball flight model

Strengths

- The procedure that we have build to apply the model ables us to obtain results really fast. To obtain all the valid shots we only needed about 15 minutes of computing.
- The motion equation can be solved easily with numeric methods even if we change some of the variables. We could change for example the value of C_D by a function of ω and v if we had experimental evidence of this dependance. The model is flexible and can be improved as soon as we get more experimental data.
- We haven't showed empirical evidence for the validity of our approach but, if we have done right, it could be easily extrapolated to any other similar situation.
- Besides the particular assumptions that we took to calculate the solutions, the model is based on just three widely studied and accepted formulas.

Weaknesses

- The fact of being so flexible (we can choose what values of the variables to use) means also less confindence about a particular result. One has to be careful with the values of the variables that introduces because if some of them were wrong, the results could be not realistic.
- The model doesn't contain a direct dependance with the roughness or dimples number of the ball. Even though it could be considered implicit within the values of the drag and lift coefficient, there is no enough evidence to formulate a general relation between this variables, so the model should be applied only to a ball with empirically measured values of the drag and lift coefficient.
- The particular case in which we choosed a constant value of C_D doesn't take into account the drag crisis. This effect has been in fact quite observed with golf balls, so it would be interesting to introduce a empirical dependance formula to relate C_D with the instant values of v and with ω.
- C_L is taken constant or dependent on ω, but it can also depend on other things such as C_D (see e.g [5]). This problem is of the same nature than the prior point.
- We doesn't consider the rotational speed deceleration at any point. We are not pretty sure about how reallistic this assumption is. This opens up another way to continue with further investigations.

5.2 Club-Ball interaction model

Strengths

- The solution is analytical.
- It gives a relatively easy answer to an apparently very complex phenomenon.

Weaknesses

- In our model we consider that there is no external force acting on the clubhead during the collision. It remains to be analised how good this approximation is. Experiments could be done in order to approve or refuse this hypothesis.
- The deformation of the ball is only taken into account when calculating the energy loss, the geometric change that it implies in the collision is neglected. This is another assumption that should be constrasted with experiments.

5.3 The problem approach

Strengths

- We have got a considerable amount of presentable results.
- The nature of the problem has been well analised.

Weaknesses

- We haven't considered to be wind. This could be easily fixed by setting a frame reference change in the motion equation in the case that the wind velocity was constant along the trajectory and known.
- We have supposed that the ball will remain in the green if it lands on it, but it could be probably false. The study of the bouncing and roll phase is a part of the trajectory of the ball that we have left for further investigations.
- We haven't taken into account the case in which the height of the green is not the same as where the ball is. It could be studied easily if demanded.

6 Conclusions

Regarding to the results that we obtained in section 4 within the velocity range $[40, 60]\frac{m}{s}$ and spin angle range $[20, 50]\frac{rad}{s}$ that we considered we have been able to find that:

- The valid orientation angle range is $\alpha \in [26, 30]^{\text{o}}$.
- The valid spin angle range is $\theta \in [39, 50]^{\text{o}}$.
- The valid impact velocity range is $u \in [47, 60]\frac{m}{s}$

Notice that for a velocity below 47 $\frac{m}{s}$ there is no solution. As a particular example of succesful shot we have taken $(u, \alpha, \theta) = (47\frac{m}{s}, 26^{\text{o}}, 39^{\text{o}})$, which seems a realistic shot.

To sum up, we have got theoretical solutions. Nevertheless, the subject opens up a lot of further investigations that we comment in the next section.

7 Further Investigations

We have been mentioning several ways to go on with the research. We would like to sum up all this suggestions in this section.

- One of the first things than could be done to reach major results using our model is taking into account the bouncing and rolling phase of the ball trajectory. It is a possibility to investigate how it works and make estimations of where should the ball land so that it actually gets to end up on the green.
- Another interesting way of improving our results is to focus the investigation in gathering all the possible data to choose the best values or functions for the drag and lift coefficients. It is shown in many studies as in [6] that the Reynolds number and the number of the dimples and its characteristics play an important role. Wind tunnel experiments like the one in [7] could bring up interesting and useful results.
- Another possibility to go further is to investigate whether the equations developed in the section 3 match with the experimental data.

8 Appendix: Maple code

```
#CONSTANT VALUES IN THE SI UNITS

g:=9.81: # Gravity acceleration
m:= 4.59*10^(-2): # Ball mass
rho:=1.22 : # Air density
r:= 4.27*10^(-2)*0.5: #Ball radius
A:= Pi*r^2:  # Cross-sectional area of the ball
mc:= 0.274: # Clubhead
f:= 0.7: # Restitution coefficient
Cd:= 0.25: # Drag coefficient
Cl:=0.325: # Lift/Magnus coefficient
beta:=39*2*Pi/360: # Loft angle of the clubhead, iron 9

sgn:= -1; # It determines if the ball is hit to the left or to the right of the
   tree
          for sgn = -1 the ball is hit to the left. If sgn = +1 the ball is hit
             to the     right.
```

The *Trajectory()* procedure determinates the trajectory equation $\vec{r}(t)$ and its derivative $\dot{\vec{r}}(t)$ of the ball given the initial conditions (u, α, θ) of the clubhead

```
#Numerical solution for the trajectory that returns r(t) and d/dt r(t) for a
   given values of u, alpha and theta.

Trajectory:= proc(u, alpha, theta) # u, alpha and theta described in the report.

#Declaration of local variables
local var,v0, v, initc, eqnt0, w, rt, res0, xt, yt, zt, drt, dxt,dyt,dzt,wx, wy,
   wz,we,wb, vecvel:=Vector(3),
   MR:=Matrix(3,3),velxyz:=Vector(3),phi,MRfin:=Matrix(3,3),velxyzfin:=Vector(3),
   wfinal:=Vector(3);

#Declaration of global variables
global g,m,rho, Cd, Cl, r, A, sgn, beta,f, mc;

#Here we calculeta the initial velocity of the ball in the reference frame of the
   clubhead

vecvel:=<(2/7)*(mc/(mc+m))*u*sin(theta),
   (1+f)*mc/(mc+m)*u*cos(beta)*cos(theta),(2/7)*(mc/(m+mc))*u*sin(beta)*cos(theta)>;

#Here we make a reference frame change to obtain the velocity in the fixed
   reference fram xyz multiplying by a rotation matrix
```

```
MR:=Matrix(3, 3, {(1, 1) = cos(theta), (1, 2) = -cos(beta)*sin(theta), (1, 3) =
    -sin(beta)*sin(theta), (2, 1) = sin(theta), (2, 2) = cos(beta)*cos(theta),
    (2, 3) = sin(beta)*cos(theta), (3, 1) = 0, (3, 2) = -sin(beta), (3, 3) =
    cos(beta)});

velxyz:=map(abs,MR.vecvel);

#We want the ball to come out in the direction indicated by alpha. Therefore, the
    golfer must rotate an angle alpha = +90-phi to hit the ball, where phi is the
    angle between the x component and the component vector velxyz "

phi:= arctan(velxyz[2]/velxyz[1]);


#Another rotation matrix, which determines the xyz definitely have the ball
    speeds so pulling out in the direction of alpha

MRfin:= <<cos(alpha+Pi/2-phi),sin(alpha+Pi/2-phi),0>|
        <-sin(alpha+Pi/2-phi),cos(alpha+Pi/2-phi),0>|<0,0,1>>;
velxyzfin:= MRfin.velxyz;

#Lack determine the rotation of the ball To do this, we start with wb and the
    clubhead reference system.

wb:=evalf((5/7)*u/r*cos(theta)*sin(beta));
we:=evalf((5/7)*u/r*sin(theta));

#Now, we multiply by rotation matrices that we found before.

wfinal:= MRfin.MR.<wb,0,-we>; #Final spin vector of the ball
wx:=wfinal[1];
wy:=wfinal[2];
wz:=wfinal[3];

  var:={x(t), y(t), z(t), dx(t), dy(t), dz(t)}: #Motion variables
  v:=(dx^2 +dy^2 +dz^2 )^(1/2): #Definition of the velocity

  w:=(wx^2+wy^2+wz^2)^(1/2): # pendiente de si lo ponemos en
                                funcion de angulos o que
  initc:= x(0) = 0, y(0)=0, z(0) = r,
       dx(0) = velxyzfin[1], dy(0) = velxyzfin[2], dz(0)= velxyzfin[3]:

#Differential equations of motion
eqnt0:=
diff(x(t),t) = dx(t), diff(y(t),t) = dy(t), diff(z(t),t) = dz(t),

diff(dx(t), t) = 1/m*(-0.5*rho*A*v(t) *( Cd*dx(t) -Cl*(wy*dz(t)-wz*dy(t))/w )),
```

```
diff(dy(t), t) = 1/m*(-0.5*rho*A*v(t) *( Cd*dy(t) -Cl*(wz*dx(t)-wx*dz(t))/w )),
diff(dz(t), t) = 1/m*(-0.5*rho*A*v(t) *( Cd*dz(t) -Cl*(wx*dy(t)-wy*dx(t))/w ))-g:

#Numerical solution
res0:=dsolve({eqnt0, initc}, var, numeric, output = listprocedure);

xt:=subs( res0, x(t)):
yt:=subs( res0, y(t)):
zt:=subs( res0, z(t)):

dxt:=subs( res0, dx(t)):
dyt:=subs( res0, dy(t)):
dzt:=subs( res0, dz(t)):

rt:=[xt,yt,zt];
drt:=[dxt,dyt,dzt];
return rt, drt;

end proc;
```

The *zzero()* procedure finds the flight time (also called in the report as land time). This code have been partially extracted from [11].

```
zzero := proc(t0,z,dz, nmax) local tn, ts, up, counter;
     # find root of z(t) = 0
     # using Newton method
     # using diff(z,t) = dz(t)
    tn:= t0; ts:=0.0; counter:=0;
    while abs((tn-ts)/tn) > 10^(-4) do;
         ts:=tn;

         tn:= ts- z(ts)/dz(ts);
         counter:=counter+1;
         if counter > nmax then break;
         return "Error: maximum number of iterates reached";
         end if;
    od;
    tn;
end proc;
```

The *ValidGreenTrajectory()* procedure finds different initial conditions that leads to trajectories of the ball in which it ends up on the green. This function uses $Trajectory()$ and $zzero()$.

```
ValidGreenTrajectory:=proc()

#Declaration of local variables
```

```
local i,j,k, rt:=Vector(3), drt:=Vector(3), distgreen, tmax0, T;

#Declaration of global variables
global g,m,rho, Cd, Cl, r, A, sgn, beta,f, mc;

#We can choose a range for the velocities (i) of the club head, in our case
   between 40 and 60.
#We can choose a range for the angle alpha (j) such as from 26 to 35.
#We can choose a range for the angle theta (k) such as from 20 to 50.

for i from 40 to 60 do
for j from 26 to 35 do
for k from 20 to 50 do
   rt, drt:=trajectory(i, j*2*Pi/360, k*2*Pi/360):
   tmax0:= evalf(zzero(10, rt[3], drt[3],7)):
   T:=tmax0:
   distgreen:= rt[1](tmax0)^2 + (rt[2](tmax0)-120)^2:

    #Here is checked if the current values for i,j and k give a valid trajectory
    if distgreen < 25 then
         print("Valid trajectory");
         print("Velocity of the clubhead = "), print(i);
         print("Alpha = "), print(j);
         print("Theta = "), print(k);
         print(" ");
    end if;
end do;
end do;
end do;
end proc;
```

The following loop checks if in a given trajectory, the ball touches or not the tree.

```
#Given the vector solution for a trajectory, rt:=r(t)=(x(t), y(t), z(t)) this
   procedure check if the ball touches the tree.

t:=0;
counter:=1;
#T is the flight time
while t < T do

  pos:= x(t)^2+(y(t)-10)^2;
  t:= t+0.01;
  if pos < (R+r)^2 then
     print("In this trajectory the ball touch the tree");
     break;
```

```
  end if;

  if y(t)>10 then
      print("In this trajectory the ball doesn't touch the tree");
      break;
  end if;
end do;
```

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