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Problem Chosen

A

2016
MCM/ICM
Summary Sheet
(Your team's summary should be included as the first page of your electronic submission.)

Type a summary of your results on this page. Do not include
the name of your school, advisor, or team members on this page.

Abstract

It had been a very long day, but Alice was finally home. She was feeling a little bit hungry, but what she really needed was a moment of relax. After leaving everything in her room, she went to the bathroom and started to fill the bathtub with hot water.

"This is fantastic!" she thought, as she was getting into the water. But Alice was a mathematician, and soon, a huge set of thoughts started to trouble her mind. "How long will I be able to stay warm?", "How fast is the water going to cold?", "What if I leave the faucet flowing? Would this delay the cooling process?", "But, how much water would I waste if I did that?", "Is there something else I could do?",...

In this report, we have wanted to give our dear Alice a simple answer to her problem (check the user guide "How to keep your bathtub as hot as possible", in the next page). Specifically, we have dealt with two models to accomplish that.

The first one, and simplest, allows us to study the heating and cooling processes of the water in general terms, assuming an instantaneous propagation of heat throughout the water. With that, we have been able to determine the amount of water required to keep the bathtub hot, according to the way that Alice manage the faucet.

The second one explores the actual propagation of heat due to the finite conductivity of water and its velocity with the use of convection-diffusion and Navier-Stokes equations. With this, we have been able to realize about the difficulties of having the temperature of water even throughout the bathtub when the water remains still. We have also accomplished to give some notions about what would happen if Alice decided to use her shower handle to make hot water fall in different locations, or if she moved the water in a particular direction.

User guide

How to keep your bathtub as hot as possible

Dear customer,

We are thankful that you decided to acquire our newest bathtub model. It is clear that one of the greatest advantages of bathtubs among shower plates is its functionality, because, apart from the regular use you will make of it, you are now able to take a hot bath whenever you want. For that, we hope you enjoy all the relaxing experiences that our bathtub can provide you.

However, since this is not an auto-heating bathtub, you may feel, when taking a long bath, that the water gets gradually colder, because heat will tend to escape into the air. Therefore, we give you the following tips to reduce this sensation:

- Add a constant trickle of water from the faucet flowing at the hottest temperature it can provide. This will make the temperature of the water remain more or less constant, because it will counter its natural cooling.
- If you are worried about the amount of water you spend, just turn the faucet on at maximum flow for a while, and then turn it off when you feel comfortable again. This will make you spend less water than in the previous case.

Even if you do this, you might feel one part of your body warmer than others (in particular, in the place where the water from the faucet falls). That is because water does not mix itself quick enough, so it will not have the same temperature everywhere. In that case, we suggest you two other solutions:

- Redirect the water from the faucet to your shower handle and make it fall in a different place. Please have in mind that this will only concentrate heat in another point.
- Push with your hands the water that gets from the faucet to the opposite side of the bathtub, so you keep hot water as far as possible from the overflow drain. By doing this, you will not achieve to completely even the temperature of the water, but at least, you will get water at temperatures that will result comfortable to you.

If you keep having troubles keeping your water warm, please consider buying one of our spa-bathtubs in order to completely eliminate this problem.

Team # 53449
Problem A

Relaxing maths

February 1, 2016

Contents

1 Introduction and problem approach

Taking a hot bath for a long time in a bathtub without a heating system (such a spa bathtub) can result in a undesirable, chilling experience, since the water will gradually lose its heat and reach room temperature. Our subject, Alice, wants to avoid this when taking a bath, so she will try to keep the temperature more or less the same throughout the bathtub, and close enough to a temperature where she is comfortable. We want to explore some of the options she has to accomplish this.

In general terms, we consider the following situation. Alice's bathtub is defined as an entity of the three dimensional space that can contain a volume of water closed by the following region $\mathcal{R}$:

$$\mathcal{R} = \{\vec{r} = (x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 50,\ 0 \leq y \leq 150,\ 0 \leq z \leq 35\}^1$$

The values that define the region are measured in *cm*, and they have been chosen intuitively, according to the general dimensions of a bathtub. This definition given will be necessary in the calculations done later.

Alice has filled the bathtub with hot water at a temperature T_{max}, which we will call *maximum temperature*. This temperature corresponds to the highest one in which Alice feels well inside the water. Since the human body is not particularly sensitive to small temperature changes, we define another two temperatures, T_{min} and T_d, named *minimum temperature* and *desired temperature*, which correspond to the lowest and the optimal temperatures in which she still feels the water ideally hot. Therefore, $T_{min} < T_d < T_{max}$, and we will naively relate them with the following expression

$$T_d = \frac{T_{max} + T_{min}}{2} \tag{1}$$

After doing a little research, we have concluded that we may set $T_{max} = 311\ K$ and $T_{min} = 305\ K$, so $T_d = 308\ K$.

After filling the bathtub, she gets into the water, and the excess of volume escapes through an overflow drain. We will assume, due to general bathtub designs, and according to our first definition, that this drain is located on the point $(25, 0, 35) \in \mathcal{R}$. Also, the overflow drain will be supposed big enough to provide an exit flow of water as large, at least, as the maximum flow of water that the faucet of the bathtub can give. Thus, the volume of water inside the bathtub will remain constant. This final assumption will be necessary in the following discussions.

[1] From now on, we may understand $\mathcal{R}$ as the bathtub itself, since we will not consider any other of its characteristics.

We will also neglect every "abrupt" movement that Alice would make while resting inside the water. Actually, this assumption will only be necessary in section 3; before that, the model given ignores the movement of the water.

Finally, in order to keep the water warm, Alice will have turn on or off the faucet. The way she does that will determine the amount of water she needs to stay comfortable inside the water, and we will see through our models that this depends on the temperature at which she wants to keep the water or the time of bathing, among other factors. It is assumed that in this moment, the hottest water that the faucet can provide will be flowing, which is $T_{fau} = 323\ K$ for the model presented in section 2 and $T_{max} = 311\ K$ for the one (more realistic) presented in section 3. We will also comment other strategies that she can follow to keep the temperature of the water, such as gently move the water with her hands, or change the position where the water from the faucet falls.

In order to examine the whole situation, two different models have been proposed. In the following pages, we will present and discuss this models and extract different conclusions from them. We will start with a simple model that describes water as whole entity and examines its cooling and heating times. The second one deals with the nature of water as a fluid, and focus on the actual distribution of temperatures that can be given in the bathtub.

2 Simple model

In such a complex problem like this, simplifications and restatements have become an essential part of the work. In this section we explain our first and more simplified approach. This model gives a partial answer to the question: *Is it better to keep the faucet flowing so as to achieve a steady state at temperature T_d or keeping it off until water colds to temperature T_{min} and turning it on until it achieves T_{max} again?.*

2.1 Assumptions

- The water temperature is instantaneously homogenized (infinite conductivity).
- The heat loss by the water-air interaction follows the Newton's cooling law (convection).
- The heat loss due to Alice and the bathtub is negligible. This means that the heat transfer by conduction between the water and the system Alice-bathtub is negligible.
- We will assume a couple of numbers that should be experimental results: the cooling time of the water from T_{max} to T_{min} and the maximum volume flux in the faucet.

2.2 Heat loss by convection with air

We will first consider that the only heat loss is produced by the convection due to air. Here we decided to take the famous Newton's law of cooling:

$$\frac{dQ}{dt} = -\lambda A(T - T_a)$$

Here Q represents the heat. A is the surface in contact with the air and T_a the ambient temperature. The coefficient λ has to be determined experimentally and we will find it later. As the magnitude we are interested in is the temperature T, we will make it appear by using the first law of thermodynamics: $dQ = \rho V c dT$. Here ρ is the density of water, V the total volume of water in the bathtub and c the specific heat capacity of water. Thus, the equation we will use to describe the convection with air is:

$$\frac{dT}{dt} = -\frac{\lambda A}{\rho V c}(T - T_a) \tag{2}$$

To determine the value of λ the best option would have been to carry out an experiment with the real bathtub. Here we will just suppose the value of the cooling time within the range of temperatures we are going to be working with. This way, integrating the equation above we can obtain the value of λ:

$$\int_{T_{max}}^{T_{min}} \frac{dT}{T - T_a} = -\frac{\lambda A}{\rho V c_V} \int_0^{t_c} dt$$

implies that:

$$\lambda = \frac{\rho V c}{A t_c} \ln\left(\frac{T_{min} - T_a}{T_{max} - T_a}\right) \tag{3}$$

where t_c is the cooling time of the water from T_{max} to T_{min}.

2.3 Faucet heat contribution

This term has been derived by the following arguments. First we consider a standard fluid mixing problem where the only heat transfer is due to the different temperature of the fluids and one is equal to the other:

$$Q_1 = m_1 c_1 (T - T_1) = -m_2 c_2 (T - T_2) = -Q_2$$

where m_1 and m_2 are the masses of the fluids that are mixed, c_1 and c_2 are their specific heat capacity and T is the final temperature after the mixing whereas T_1 and T_2 are the initial temperatures. Now the fluids are both water so $c_1 = c_2 = c$. The first fluid will be an infinitesimal piece of water flowing out of the faucet at temperature $T_1 = T_{fau}$ and mass $dm_1 = \rho dV = \rho \phi dt$ where ϕ is the volume flux coming out the faucet per unit of time and dt an infinitesimal of time. Thus, $dQ_1 = \rho \phi c (T - T_{fau})$. The second one will be the water in the bathtub, with $m_2 = \rho V$ (the volume is considered to be constant because we suppose that the bathtub is filled so that the same quantity of water coming out of the faucet goes out in the overflow drain). The change in temperature will be dT. Considering $dQ_1 = -dQ_2$ one obtains the following equation:

$$\frac{dT}{dt} = \frac{\phi}{V}(T_{fau} - T) \tag{4}$$

what gives us a positive change in temperature when $T < T_{fau}$.

Now we have already found the only two contributions to the temperature changes in time that we were looking for.

2.4 Temperature in time equation

The equation we obtain by summing the contributions of (2) and (4) is:

$$\frac{dT}{dt} = \frac{\phi}{V}(T_{fau} - T) - \frac{\lambda A}{\rho V c}(T - T_a) \tag{5}$$

Recall that:

- ϕ is the volume flux coming out the faucet per unit of time.
- V is the volume of water in the bathtub, which is considered to be constant. We shall subtract the volume of Alice underwater.

- λ is the coefficient that describes the cooling of water by convection with air.
- A is the surface of water in contact with air. Again a correction because of the presence of Alice shall be made.
- c is the specific heat capacity of water.
- T_a is the ambient temperature and T_{fau} the temperature of the water that flows out the faucet.

With this equation we can already make an approximate calculus of the quantity of water we need for the following possible strategies:

1. We can find the steady state in which for a certain volume flux per time ϕ the temperature in the bathtub will remain at T_d constant in time. This is going to be given by the condition $\frac{dT}{dt} = 0$. If we manipulate the equation (5), we find that it must be:

$$\phi = \frac{\lambda A}{\rho c} \frac{T_d - T_a}{T_{fau} - T_d} \tag{6}$$

And in this case the total volume of spent water is given by:

$$V_T = \phi t_T \tag{7}$$

where t_T is the total of time Alice is going to be in the bath.

2. Another option for Alice is to wait t_c seconds until the water reaches T_{min}, when she starts to feel cold, and then open the faucet at the maximum flux to warm the water up again to T_{max}. In this case what we find is the time she needs to reach again the maximum temperature by integrating (5) from T_{min} to T_{max} and from $t = 0$ to $t = t_w$. The value of t_w can be isolated after solving the differential equation, which is trivial.

To calculate the total volume of spent water we can consider that Alice won't start warming up if she is going to finish the bath before the temperature reaches again the maximum (because she wants to save water). In this case the value of spent water in function of the total bath time will be:

$$V_T = \left[\frac{t_T}{t_w + t_c} \right] \phi t_w \tag{8}$$

where the square brackets mean the integer part of the quantity inside.

2.5 Predictions

Here we shall calculate the total volume of additional water needed to follow each of the mentioned strategies. To begin with we have looked for real values of the parameters that the equation (5) requires. The first thing we need to calculate is the convection coefficient λ. Here is the list of the values we have considered in the later calculations:

ρ	993.73 (Kg/m^3)
V	$1.5 \times 0.35 \times 0.5 - 1.5 \times 0.18 \times 0.3 = 0.1815$ (m^3)
c	4178 (J/Kg K)
A	$1.5 \times 0.5 - \pi \times 0.06^2 = 0.74$ (m^2)
t_c	30×60 (s)
T_a	295.15 (K)
T_{fau}	323.15 (K)

Using (3) we have obtained the following convection coefficient:

$$\lambda = 266.37 \, \frac{J}{m^2 s}$$

Now with (6) we find the volume flow per time that will end up in a steady state at temperature T_d:

$$\phi = 2.46 \ (L/min)$$

With this flow, the volume of water used per bath would be:

$$V_T = 2.46 \ t_T \ (L)$$

Considering a maximum flow in the faucet of 12L/min we calculate the warming time t_w:

$$t_w = 7.87 \ (min)$$

To illustrate the results we have displayed in the next graphic both strategies:

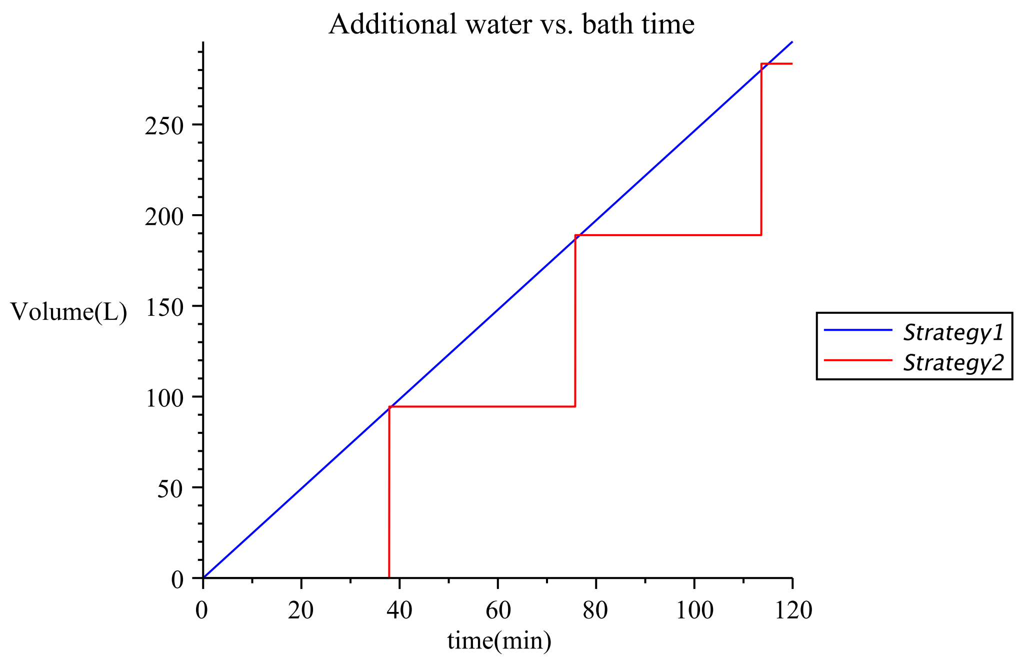

Figure 1: Additional volume of water in function of the bath time.

What we have found is that depending on the time that Alice is going to be relaxing it would be more appropriate to follow strategy 1 or 2. For example, if the bath is going to take less than 30 minutes, then it would be more appropriate to go for strategy 2 because this way there is no added water. But if the bath is going to last 38 minutes or 76 minutes approximately, then strategy 1 is better because she will waste the same water, but having the water of the bathtub at the constant desired temperature.

After all this we are already able to understand why a better optimization model should not contain the *temperature homogenizes instantaneously* hypothesis. What is happening with this model is that the water that escapes through the overflow drain is always as warmed as the rest of the water. If we think about it for a while we will realize that the best strategy would be that in which Alice puts the plug off so as to remove enough cold water in a way in which, when she puts it in again and refills the bathtub with the maximum flow and the maximum temperature of the faucet, it gets eventually at the beginning point. This way, any part of the heat that the faucet water is adding would be wasted.

3 Fluid dynamics approach

As a second, more realistic model, we have wanted to take into account the nature of the water as a fluid to obtain the distribution of temperatures. In particular, we have looked for a function $T : \mathcal{R} \longrightarrow \mathbb{R}^+$ that gives the temperature of the point (x, y, z) for each $(x, y, z) \in \mathcal{R}$ in the steady state (that is, when the value of the temperature in that point does not change in time), using differential equations that model the velocities field of a moving fluid and the propagation of heat throughout it.

These equations are the well-known Navier-Stokes equations (for the movement of the fluid) and the convection-diffusion equation, for the propagation of heat. We now explain why we have used them, how we have dealt with them, and what difficulties we have encountered in doing this.

3.1 Navier-Stokes equations

According to [4] (p. 1), given an incompressible fluid of density ρ and viscosity μ under a pressure field $\pi(\vec{r}, t)$ and an external force per unit of mass $\vec{F}(\vec{r}, t)$, its field of velocities $\vec{v}$ is given by the following Navier-Stokes equation, also known as the *momentum equation*

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = \mu \Delta \vec{v} - \nabla \pi - \rho \vec{F} \tag{9}$$

As we will see in the following section, solving this equation is mandatory for giving a realistic velocities field to use in equation (11), and model the propagation of temperatures in a proper way.

We will make some simplifications in order to get a equation which will be easier to solve. It has been already said that we are looking for the steady state solutions. Thus, we can set $\partial \vec{v}/\partial t = 0$. Now, we make the following assumptions:

- In our situation, since there are no external forces acting on the water (mainly, because we neglect the movements of Alice), we may take $\vec{F} = 0$.
- The main difficulty to numerically solve this equation, in general, is the unknown pressure field π, which depends on the velocities of each point of the fluid and the internal forces. Expressing these internal forces would have been a task out of the scope of this work. Thus, we have limited π to represent the resulting pressures from the Venturi effect.
- As boundary conditions for solving this equation, we have set $\vec{v}(\vec{r}) = 0$ for all $\vec{r} \in \partial\mathcal{R}$ (condition known as *no-slip* boundary condition, which is actually very reasonable according to [9] (p. 451)), where $\partial\mathcal{R}$ means the boundary of $\mathcal{R}$, except for the points where the water leaves the bathtub (point $(25, 0, 35)$, as said in the introduction) and the point where the water from the faucet enters, which we can consider to be $(25, 15, 35)$.

Both velocities have been considered to have equal modulus u, and according to the geometry of the situation, the corresponding vectors in each point should be

$$\vec{v}(25,0,35) = (0,-u,0)$$
$$\vec{v}(25,15,35) = (0,0,-u)$$

After these considerations, we have managed to write (9) as follows

$$(\vec{v}\cdot\nabla)\,\vec{v} = \nu\Delta\vec{v} + \sum_{i=1}^{3} v_i\,(\nabla v_i) \tag{10}$$

where $\nu = \mu/\rho$, named *kinematic viscosity*. Implementing a finite-difference method as taught in [1] in Python 2, we have numerically solved this expression for values such as $u = 50\ cm/s$ and $\nu = 8010\ cm^2/s$ (which is the value for the kinematic viscosity of water at 303 K, according to [10]). The following figure is a $3D$ representation of the vector field obtained near the overflow drain of the bathtub.

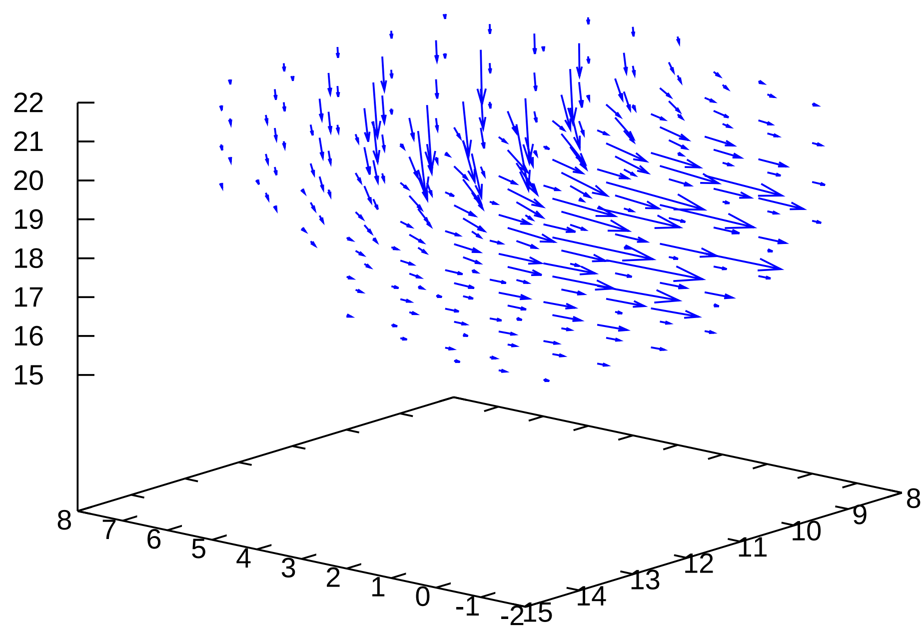

Figure 2: Velocities field near the overflow drain calculated with the discretization of (10)

Various calculations of the velocities field for different values of u have lead us to conclude that, actually, most of the water in the bathtub remains still. Also, the water that moves corresponds to the hot water from the faucet that immediately leaves the bathtub through the overflow drain. Assuming that, in that little time lapse, the temperature of that water does not change dramatically, it is possible to understand this situation as having a non-moving hot spot, always at the same temperature, substituting the region of moving water.

3.2 Convection-diffusion equation

The convection-diffusion equation describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection [2]. Using the variables involved in our problem, the convection-diffusion equation (11) can be written as follows .

$$\frac{\partial T}{\partial t} = \nabla \cdot (D \nabla T) - \nabla \cdot (T \vec{v}) + R \tag{11}$$

where $T = T(\vec{r}, t)$ is the temperature, D the thermal diffusivity [3], $\vec{v} = v(\vec{r}, t)$ is the average velocity of the water flow and $R = R(\vec{r}, t)$ is a function that describes sources and sinks of temperature; ∇ represents the gradient operator, which acts on scalar functions, and $\nabla\cdot$ the divergence operator, which acts on vector fields.

Now we give an interpretation of each term involved in equation (11), according to the situation we are studying:

- The first term, $\nabla \cdot (D\nabla T)$ describes diffusion of heat, and in our problem it can be thought as the contribution of heat conduction through water, as described by Fourier's law.
- The second term $-\nabla \cdot (T\vec{v})$ describes water convection. So as to understand it better, suppose that hot water from the tap is flowing into the bathtub, and there is also a velocity field associated to bathtub's water. For example, if the bathtub is full of water and we continue adding water continuously, excess water will escape through the overflow drain, and it will create a velocity field pointing to it. Moreover, the velocity field is associated to Alice movements done during bathing. Nevertheless, it is possible to understand the velocities field just as the one associate with Alice movements, because, as we have argued in the previous section, the natural movement of the water due to the escape through the overflow drain can be neglected.
- Finally, R in this case would represent the influence of Alice in the water, as a sink of temperatures. Nevertheless, in order to simplify things, we will take it as zero (considering that Alice is not immersed in water, or that she does not influence in temperature values).

Our approach considers some reasonable simplifications:

1) The diffusion coefficient D is assumed to be constant, since the range of temperatures we are working with is small. From [8], we have that $D = 0.143\cdot 10^{-6}\ m^2/s$.
2) Water is considered to be an incompressible fluid, what lead us to consider that the velocity field describes an incompressible flow. Mathematically, this implies that $\nabla\cdot\vec{v} = 0$.

[2]The general expression can be found in [2]
[3]See [8] for more details.

3) As we have said, for the function R, we set $R = 0$.

Using these simplifications in (11) we have that

$$\nabla \cdot (D\nabla T) - \nabla \cdot (T\vec{v}) = D \cdot \nabla^2 T - \vec{v} \cdot \nabla T - T\nabla \cdot \vec{v} = D \cdot \nabla^2 T - \vec{v} \cdot \nabla T$$

so that the final simplified convection-diffusion equation considered is

$$\frac{\partial T}{\partial t} = D \cdot \nabla^2 T - \vec{v} \cdot \nabla T \tag{12}$$

We are interested in solutions of (12) such that the temperature hardly varies, since what we want is to keep the temperature of the bathtub almost at the desired temperature. Therefore, we want to find solutions of the stationary equation derived from (12) considering $\frac{\partial T}{\partial t} = 0$.

3.3 The model: temperature distribution

We have finally found that the equation to solve is

$$D \cdot \nabla^2 T - \vec{v} \cdot \nabla T = 0$$

which can be written as

$$D\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - v_x\frac{\partial T}{\partial x} - v_y\frac{\partial T}{\partial y} - v_z\frac{\partial T}{\partial z} = 0 \tag{13}$$

where $\vec{v} = (v_x, v_y, v_z)$, and $v_i = v_i(x, y, z, t)$, $\forall i \in \{x, y, z\}$.

So as to solve equation (13) numerically, we have implemented a finite-difference method in Python 2, which finds the values of the function T for all the points $(x, y, z) \in \mathcal{R}$. For the simulations, $\mathcal{R}$ is treated as a discrete set, and it is obtained from $\mathcal{R}$ by dividing it in smaller volumes (which are the points where the temperatures are calculated).

The main initial condition to consider, as we said in the introduction, is that the bathtub is full of water at the temperature $T_{max} = 311$ K. On the other hand, here we study the situation in which hot water is constantly falling at temperature T_{max}.

Boundary conditions considered: let $\partial\mathcal{R}$ be boundary of the bathtub $\mathcal{R}$ now considered in a general form:

$$\mathcal{R} = \{\vec{r} = (x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le x_0,\ 0 \le y \le y_0,\ 0 \le z \le z_0\}$$

- The temperature at the regions $\{x = 0\} \cap \partial\mathcal{R}$, $\{x = x_0\} \cap \partial\mathcal{R}$, $\{y = 0\} \cap \partial\mathcal{R}$, $\{y = y_0\} \cap \partial\mathcal{R}$ is $T_0 = 300$ K.
- The temperature at the bottom of the bathtub is considered to be $T_{bottom} = 303$ K.
- To take into account the loss of temperature due to natural convection between the upper surface of the bathtub and the air, we set the constant temperature $T_{up} = 300$ K for all point in $\{z = z_0\} \cap \partial\mathcal{R}$.

- So as to model the hot water coming from the faucet, we assume that a certain region $\mathcal{R}_0 \subset \mathcal{R}$, which is located around the hot water, is always at the same temperature T_{max}. For example, if the faucet is near the point $(x, y, z) = \left(\frac{1}{2}x_0, 0, z_0\right)$ (on the left), a reasonable $\mathcal{R}_0$ will be:

$$\mathcal{R}_0 = \left\{(x, y, z) \in \mathcal{R} \mid 0 \leq x \leq x_0,\ 0 \leq y \leq \frac{1}{6}y_0,\ 0 \leq z \leq z_0\right\}$$

- Finally, we neglect the existent heat transfer between water and all bathtub walls (for example, assuming them to be adiabatic), as well as between water and Alice.

3.3.1 Predictions: Non-presence of velocity field

When free of any velocity field, so water is treated as a steady fluid, calculations are pretty straight-forward. In general, it can be observed that temperature decays exponentially with the distance from the faucet. As an example, we present the following figures, that represent the temperature for each coordinate (x, y) at $z = 15$ cm (half of the height of water).

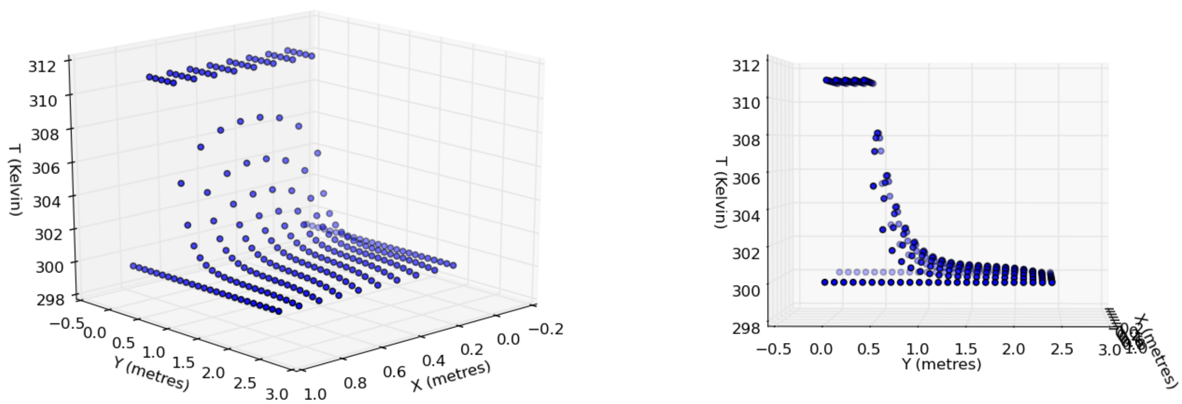

Figure 3: Temperature distribution in one layer of water without any velocity field

Due to the symmetry of the problem, it is easy to imagine that, if Alice would use the shower handle to make water fall in a different position, a similar function would have been obtained, with a peak wherever the shower handle is, and exponential decreases on both sides of that peak.

3.3.2 Predictions: Presence of a constant velocity field

Now, we take a look at the corresponding data when adding a velocity field, for the same layer of water as before. Velocity will be added in the y component (that is, as if Alice were moving the water with her hands from the faucet to the opposite side). Calculations have been made for two different velocities, which allow us to compare the obtained distributions in terms of the magnitude of the velocities.

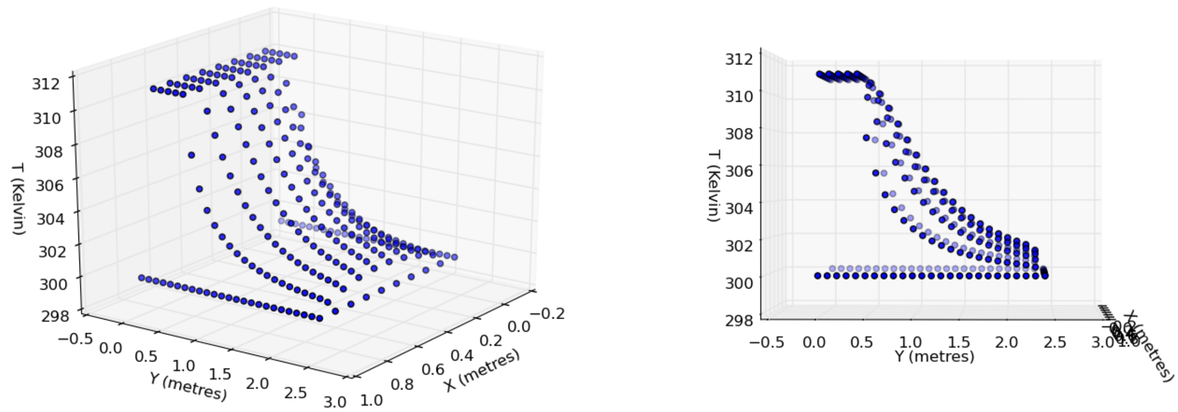

Figure 4: Temperature distribution in one layer of water with low speed in the y direction

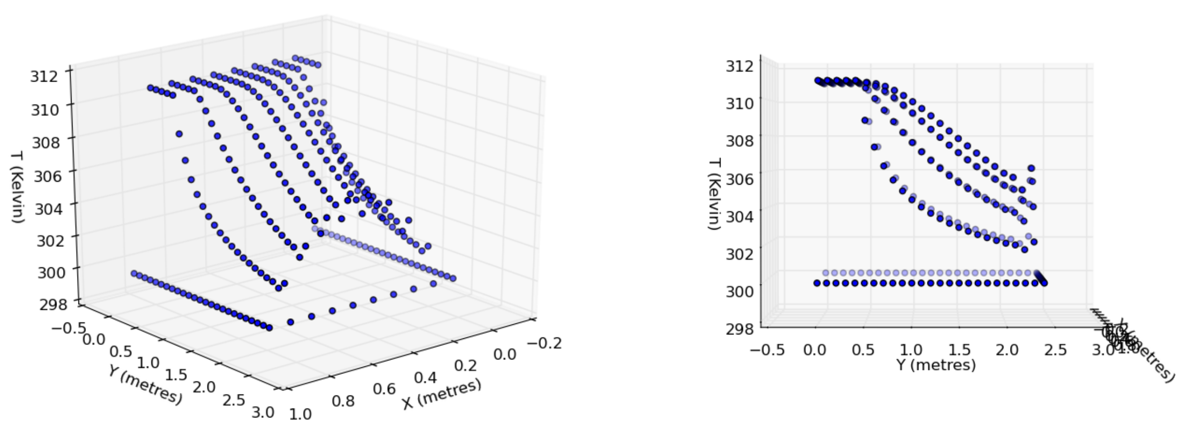

Figure 5: Temperature distribution in one layer of water with high speed in the y direction

We can observe significant differences in the distribution of temperatures between figures 3, 4, 5. To begin with, having no velocity in water makes temperature concentrate in a single region, while adding a velocities field helps to homogenize temperature throughout the bathtub. But also, the level of homogenization is also significant. Considering that Alice will feel comfortable when the temperature of the water is between T_{min} and T_{max} (that is, in the interval $[305, 311]$ K), and supposing that Alice's width would be around 30 cm (placed in the x axis) figure 5 let us infer that, with enough speed, the water around her would be at a suitable temperature for her.

3.4 Difficulties

Here we want to discuss about some of the main difficulties encountered during this work. It has to be mentioned that solving the Navier-Stokes and convection-diffusion equations is a really hard work. One of the biggest challenges when one tries to solve them numerically is the good choice and the correct implementation of the boundary conditions. The time we have invested trying to solve the equations has deprived us from drawing more conclusions. Maybe more numerical methods and PDE's mathematical knowledge would enable us to treat properly with this model.

4 Strenghts and weaknesses

4.1 First approach

Strengths

- It has enabled us to make estimations of the volume flow that would be desired to mantain the average temperature constant troughout the bath.
- It gives a qualitative view of the problem.
- It is a very simple model that says things about a very complex phenomenon.
- It can be solved analytically.
- It is flexible and can adapt well to empirical data.

Weaknesses

- It has not take conductivity or velocity into account.
- It does not give a model of temperature dependent on space.
- It is not sensible to the position of the entering warm water.
- Instantaneous homogenization hypothesis does not leave place for strategies based on optimizing temperature of the water that escapes through the overflow drain.

4.2 Fluid dynamics approach

Strengths

- It intends to describe very precisely the motion and thermodynamics of the studied system, which could be used to extract very specific conclusions.
- It is a model similar to the ones used in thermodynamics engineering and fluid dynamics for solving problems of the same complexity.
- On the one hand, equation (12) allows us to describe how temperature distribution will evolve as a function of time and, in the other hand (setting $\frac{\partial T}{\partial t} = 0$), which temperature distribution will result in the steady state as a function of space. In both cases only considering the presence of an arbitrary velocity field and the heat conduction through water, neglecting heat loss through the bathtub walls and the presence of Alice.

Weaknesses

- It cannot be solved analytically.
- There is a need of high level computation to solve the equations efficiently.
- The influence of Alice and the bathtub walls has been neglected in all the approaches made.
- Probably boundary conditions can be improved so as to reproduce temperature distributions more realistic.

5 Conclusions

The problem proposed was open to a lot of creative solutions. We have first developed a simple model that permits to obtain the flux in the faucet that should be set to achieve the desired temperature as steady state. It also gives two possible strategies to follow depending on the amount of time spent in the bathtub. Additionally, it shows that the real optimization of the water spent has to be found with a model that takes into account the difference of temperatures throughout the bathtub. Otherwise, there is little to do a part from avoiding additional convection (the water that sticks to the skin of Alice). As for the model that does improve this, it must consider conduction or diffusion (or both).

The second approach is an attempt to go in this way. Including the behavior described by the Navier-Stokes equation and the diffusion-convection equation, we have been able to treat water as something closer to what it really is: a fluid with a certain capacity of heat conduction and movement.

This has allowed us to conclude something that the first model could not see: when the water remains quiet, it is impossible to obtain a distribution of temperatures that could be considered as pleasant. Also, since the calculations performed with the Navier-Stokes equation have shown us that the natural movement of water throughout the bathtub (flowing in the direction of the overflow drain) is negligible, changing the position where the water falls would not help too.

Therefore, the best option to homogenize water temperatures as much as possible, according to this model, is to introduce a certain velocity in a particular direction (the one opposite to the drain). Alice could achieve this by moving her hands in that direction, and for certain speeds, she would be able to get pleasant water (that is, water at temperatures between T_{min} and T_{max}) all around her.

6 Further work

First of all it has to be mentioned the necessity of improving the first model we developed. The idea that one should follow in a further investigation based in what we have found is to force the appearance of a dependence with the position where the faucet flow is falling on. We leave place for an original idea in this way.

Anyway, the application of fluid dynamics and thermodynamics is required to make an intensive study of the phenomenon. We believe that once the convection-diffusion and Navier-Stokes equations are properly solved there must be a lot of opportunities and place for extracting specific conclusions.

Additionally, even with a better numerical solution of the considered equations, we think that there is still more place for further investigation as long as we have not studied the dependence of the model on the following aspects: shape of the bathtub, heat transfer between the water and the bathtub walls, effect of the velocity of water in convection with air, the heat transfer between Alice and the water and the effect of extra convection that results when the wet skin of Alice is in contact with the air.

We have not studied neither the presence of foam in the surface. We believe that this could help to reduce convection with the air, since it could produce a kind of *green house effect*. If this hypothesis was confirmed, using a bubble bath additive while initially filling the bathtub could be a very good idea if we want to keep the temperature of the water hotter enough.

Apart from this, experiments with real bathtubs should be carried out to test all the models exposed and find possible not considerated weaknesses.

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