Truth Tables

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1 Self-Assessment Quiz

1. Write down the characteristic truth tables for each of our connectives.

Construct truth tables to determine whether or not the following sentences are tautologies:

2. $\neg P \to Q$
3. $\neg P \lor \neg Q \leftrightarrow \neg(P \land Q)$

2 Review

A **tautology** is a logical truth in sentential logic. Examples are:

4. $P \lor \neg P$
5. $\neg(P \land \neg P)$
6. $P \to P$
7. $P \to (Q \to P)$
8. $\neg(P \to Q) \to P \land \neg Q$

Every tautology is also a **theorem**, that is, a sentence that can be derived from no premises.

We show that a sentence is a tautology by showing that it must be true no matter what:

- A sentence is a **tautology** if and only if it is assigned T on every row of its truth table.

3 Truth Tables and Validity

Validity An argument is valid if and only if it is impossible for its premises to all be true but its conclusion false.

When we constructed our system of derivation, we were careful to choose rules and methods that were obviously valid. But we can also **prove** that each of our rules is valid using a truth table. For example, here is a truth table for MTP:

P	Q	$P \lor Q$	$\neg P$	$\vdash$	Q
T	T				
T	F				
F	T				
F	F				

As before, the left columns list each sentence letter, and, beneath the sentence letters, we list all possible assignments of T and F. But now, on the righthand side of the table, we have several columns: a column for each premise of the argument, a column for the conclusion of the argument, and a column under the "therefore" symbol.

To show that MTP is valid, we need to calculate the truth value of each premise on each row, and the truth value of the conclusion on each row. We are looking for rows on which the premises are all true, but the conclusion is false. If there are any such rows, that shows that the argument is not valid. If there aren't any such rows, that shows that the argument is valid.

We mark whether or not a row is an "offending row"—a counterexample—by filling in the column under the "therefore" ($\vdash$) sign. In the textbook, it is set up so that you enter "T" if the row is not offending, and "F" if it is offending. I think that is conceptually confused. You are not assessing the truth or falsehood of the therefore claim. You are assessing whether or not it is a counterexample to the validity of the argument. So, in the assignments that I create, you will mark it instead with a checkmark (if the row does not offend) or an x (if the row is a counterexample).

Use truthtables to determine whether or not the following arguments are valid:

9. $P \to Q, Q \to R \vdash R \to P$
10. $P, \neg P \vdash Q$
11. $P \vee Q, Q \vee R \vdash \neg P \to R$
12. $P \vee Q, \neg Q \vee R \vdash \neg R \to P$
13. $(P \wedge Q) \leftrightarrow R, \neg(R \leftrightarrow P) \vdash \neg Q$

4 Truth Tables, Derivations, and Metalogic

Truth tables have a couple of advantages over derivations:

- Truth tables are mechanical; derivations require insight
- Truth tables can be used to show that an argument is not valid; derivations can only show that an argument is valid.

Derivations have a couple of advantages over truth tables:

- Derivations allow us to represent and understand natural patterns of human reasoning;
- Truth tables grow in size exponentially; derivations do not;
- Truth tables only work for sentential logic; our system of derivation can be extended to work for more complex logics (as we will do after the next exam.)

More fundamentally, derivations and truth tables give us two different ways of thinking about validity:

- Truth tables are *semantic*: we assign a meaning (truth function) to each connective, and then theorize about what follows from what in terms of those meanings;
- Derivations are *syntactic*: we identify a few obviously valid *patterns* of reasoning, and then theorize about what follows from what in terms of those patterns.

But here is a worry: how can we be sure that our two ways of thinking about validity always deliver the same results? This is a worry about the truth of two claims (let Γ be any set of premises):

Soundness	If there is a derivation of $\Gamma \vdash \square$, then $\Gamma \vdash \square$ is valid according to its truth table.
Completeness	If $\Gamma \vdash \square$ is valid according to its truth table, then there is a derivation of $\Gamma \vdash \square$.

In *metalogic* we set out to *prove* these and other claims *about* our logic, and we explore alternative logics.