Derivations Derivations Derivations Derivations

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1 New Rules

Fill in the ?s with sentences, and name the rule.

1. $P \lor Q, \neg Q \vdash ?$
2. $Q \lor R, ? \vdash Q$
3. $P \land Q \vdash ?$
4. $?, ? \vdash (R \to P) \land (P \lor T)$
5. $? \vdash P \land P$
6. $? \vdash P \lor Z$
7. $R \leftrightarrow S \vdash (R \leftrightarrow S) \lor ?$
8. $P \leftrightarrow Q \vdash ?$
9. $?, ? \vdash (P \land Q) \leftrightarrow (R \lor S)$
10. $(P \leftrightarrow Q) \lor (R \to S), ? \vdash P \leftrightarrow Q$

For each of the following, what other premise would you need to apply MP, MT, or MTP? What would you get?

11. $\neg P \lor \neg Q$
12. $P \lor Q \lor R$
13. $(P \to Q) \lor R$
14. $P \to Q \lor R$
15. $Q \lor P \to R$
16. $\neg(P \lor Q) \lor (Q \leftrightarrow R)$

Fill in the ?s with sentences, and name the derived rule.

17. $R \to S \vdash ? \to (R \to S)$
18. $\neg(P \land Q) \vdash P \land Q \to ?$
19. $\neg(P \lor Q) \to \neg(R \land S) \vdash R \land S \to ?$
20. $\neg(P \lor Q), P \lor Q \vdash P \lor Q \leftrightarrow P \lor Q$

2 Derivations

21. $P, P \lor Q \to R \vdash R$
22. $P \lor Q, \neg Q \lor \neg R \vdash \neg P \to R$
23. $P \land Q, Q \to R, P \to S \vdash R \land S$
24. $P \leftrightarrow Q, Q \leftrightarrow R \vdash P \to R$
25. $P \leftrightarrow Q, Q \leftrightarrow R \vdash P \leftrightarrow R$

3 More Derived Rules

One of the big headaches is dealing with lines that have the form ¬(CRAP). So some of the most useful derived rules are rules that allow us to transform such lines into lines that are easier to use.

(D-NC) $\neg(\square \rightarrow \bigcirc) \vdash \square \wedge \neg\bigcirc$ "Negation of a Conditional"

This rule is related to the fact that our conditional is a material conditional.

(D-NB) $\neg(\square \leftrightarrow \bigcirc) \vdash \square \leftrightarrow \neg\bigcirc$ "Negation of a Biconditional"

Remember that the biconditional expresses agreement: $\square \leftrightarrow \bigcirc$ is true when $\square$ and $\bigcirc$ have the same truth value. Can you see, then, why this inference is valid?

There are a few different "distribution" laws that describe how our connectives interact. Perhaps the most important are DeMorgan's laws, which describe how negation distributes over conjunctions and disjunctions:

(D-DMA) $\neg(\square \wedge \bigcirc) \vdash \neg\square \vee \neg\bigcirc$
(D-DMO) $\neg(\square \vee \bigcirc) \vdash \neg\square \wedge \neg\bigcirc$

Try to tease out what each of these arguments is saying. $\neg(\square \wedge \bigcirc)$ means "not both $\square$ and $\bigcirc$." $\neg(\square \vee \bigcirc)$ means "neither $\square$ nor $\bigcirc$."

Here are some more derived rules that you might find useful or interesting:

(D-R) $\square \vdash \square$ "Repetition"

(D-R) is never useful, but it is valid!

(D-SC) $\square \vee \bigcirc, \square \rightarrow \triangle, \bigcirc \rightarrow \triangle \vdash \triangle$ "Separation of Cases"

(D-SC) suggests a different way of thinking about how we reason *from* disjunctive information. If you know that one or the other of $\square$ and $\bigcirc$ must be true, and you know that, in either case, $\triangle$ is true, then you know that $\triangle$ is true.

Algebra is built on facts about commutativity, associativity, and distributivity for arithmetic operations. It is interesting to think about our connectives in these terms as well. Three of our connectives are both commutative and associative:

(D-COMA) $\square \wedge \bigcirc \vdash \bigcirc \wedge \square$
(D-COMO) $\square \vee \bigcirc \vdash \bigcirc \vee \square$
(D-COMB) $\square \leftrightarrow \bigcirc \vdash \bigcirc \leftrightarrow \square$
(D-ASSOCA) $\square \wedge \bigcirc \wedge \triangle \vdash \square \wedge (\bigcirc \wedge \triangle)$
(D-ASSOCO) $\square \vee \bigcirc \vee \triangle \vdash \square \vee (\bigcirc \vee \triangle)$
(D-ASSOCB) $(\square \leftrightarrow \bigcirc) \leftrightarrow \triangle \vdash \square \leftrightarrow (\bigcirc \leftrightarrow \triangle)$

A lot is made, in algebra, of the fact that multiplication distributes over addition and subtraction. Negation does not distribute in a clean way over any of our connectives. But our conditional distributes over conjunctions and disjunctions:

(D-DISTCA) $\square \rightarrow \bigcirc \wedge \triangle \vdash (\square \rightarrow \triangle) \wedge (\square \rightarrow \bigcirc)$
(D-DISTCO) $\square \rightarrow \bigcirc \vee \triangle \vdash (\square \rightarrow \bigcirc) \vee (\square \rightarrow \triangle)$

And conjunctions distribute over disjunctions, and vice versa:

(D-DISTAO) $\square \wedge (\bigcirc \vee \triangle) \vdash (\square \wedge \bigcirc) \vee (\square \wedge \triangle)$
(D-DISTOA) $\square \vee (\bigcirc \wedge \triangle) \vdash (\square \vee \bigcirc) \wedge (\square \vee \triangle)$

These last several rules are interesting, but not that useful for constructing derivations. The rules that are most useful for constructing derivations are (D-NC), (D-NB), (D-DMA) and (D-DMB).