Indirect Derivations

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1 First Midterm

The first midterm is on Thursday, Feb 13. Next Tuesday will be devoted to practice and review.

The midterm will contain a few very short answer questions testing your understanding of some key terms and ideas. For example,

1. Circle the main connective: $\mathrm{P} \land \mathrm{Q} \land \mathrm{R} \to \mathrm{S}$
2. Circle the antecedent: $\neg(\mathrm{P} \to \mathrm{Q}) \to \mathrm{R}$

It will contain some questions that ask you to translate a symbolic sentence into English, and some questions that ask you to translate an English sentence into symbols. In both cases, a scheme of abbreviation will be supplied.

And it will contain some derivations, including some simple direct derivations, some conditional and indirect derivations, and some nested derivations.

2 Self-Assessment Quiz

Do you know the basic rules yet? For each of the following, what other premise would you need to apply MP? to apply MT? and what would be allowed to infer in each case?

3. $\mathrm{Q} \to \mathrm{P}$
4. $\neg\mathrm{P} \to \neg\mathrm{Q}$
5. $(\mathrm{P} \to \mathrm{Q}) - (\mathrm{R} \to \mathrm{S})$

Construct derivations for:

6. $\mathrm{P}, \mathrm{P} \to \mathrm{Q}, \mathrm{Q} \to \mathrm{R} \vdash \mathrm{R}$
7. $\mathrm{P} \to \mathrm{Q}, \mathrm{Q} \to \mathrm{R} \vdash \mathrm{P} \to \mathrm{R}$
8. $\top \vdash \mathrm{R} \to (\mathrm{Q} \to \mathrm{R})$

3 Indirect Derivations

An indirect derivation establishes the truth of a negation, $\not\phi$, by showing that, if the unnegation were true, a contradiction would follow. Here is an informal example:

Show. There is no smallest positive rational number.
Proof. Suppose there is a smallest positive rational number, r. Note that $r/2$ is also a positive rational number, and $r/2 < r$. So r is not the smallest positive rational number.

Here we manage to show that it is impossible for there to be a smallest positive rational number, because no matter what number it might be, we can always find a rational number that is yet smaller. So, *there is a smallest possible rational number* is a sentence that *can't be true*. So its negation, *there is no smallest possible rational number*, is a sentence that must be true. This is how indirect derivations work.

Here is another (more controversial) informal example:

Show. God does not exist.
Proof. Suppose God does exist. If God exists, then God—being all-knowing and all-powerful—could prevent all evil. And God—being perfectly good—would prevent all evil that he could. But evil occurs. So God does not exist.

Here we argue that God does not exist. Our argument proceeds by showing that, from the assumption that God does exist (and some other premises) we can derive a contradiction.

Let's now work through some examples to demonstrate how this looks in our formal system.

4 Exercises

9. $P \to Q, Q \to \neg P \vdash \neg P$
10. $P \to Q, P \to \neg Q \vdash \neg P$
11. $\neg(P \to Q) \vdash \neg Q$
12. $\neg(P \to Q) \vdash \neg\neg P$
13. $\neg(P \to Q) \vdash P$