MAT1002: Calculus II

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Review

Midterm Review

§10.1 Infinite Sequences

- An **infinite sequence**, or **sequence** is a list of numbers

$$\{a_1, a_2, \cdots, a_n, \cdots\}$$

- Convergence/divergence; divergence to ∞ or $-\infty$
- Calculating rules (sum, difference, constant multiple, product, quotient);
- Solve for the limit of a recursively defined sequence if it exists $x_{n+1} = x_n^2 - 4$
- Sandwich theorem for sequences
- Continuous function theorem for sequences e^{x_n}
- Consider the limit $\lim_{x\to\infty} f(x)$ given $f(n) = a_n$
- Common limits
  - $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
  - $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$
  - $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$
  - $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
  - $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x)
  - $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)
- Bounded and monotone sequences

§10.2 Infinite Series

- An **infinite series**, or **series**, is the sum of an sequence

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- Convergence of a series $\Leftrightarrow$ convergence of its nth partial sum.
- Geometric series $(|r| < 1)$

$$\sum_{n=5}^{\infty} ar^n = \frac{ar^5}{1-r}$$

$$\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$$

- The nth-term test for a Divergent Series, $a_n \not\to 0$
- Calculating rules (sum, difference, constant multiple)

§10.3-10.6 Convergence Tests for Series

- Integral test and approximation
  - positive, decreasing $f(n) = a_n$, check $\int_N^\infty f(x)dx$

$$\sum \frac{1}{n^p}, \quad \sum \frac{1}{n^2+1}, \quad \sum \frac{1}{n \ln n}$$

$p>1$ converges

$\ln(\ln n)$ diverges.

- (Limit) comparison test
  - nonnegative

$$d_n \le a_n \le c_n$$

  - nonnegative

$$\lim_{n\to\infty} \frac{a_n}{b_n} \begin{cases} = 0 & a_n < b_n \text{ for } n \text{ large enough} \\ = k & \frac{k}{2} b_n < a_n < (k+1) b_n \\ = \infty & a_n > b_n \text{ for } n \text{ large} \end{cases}$$

- Absolute convergent
- Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \begin{cases} > 1 \\ < 1 \\ = 1 \end{cases}$$

- Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} \begin{cases} > 1 \\ < 1 \\ = 1 \end{cases}$$

- Alternating series: $\sum(-1)^n u_n$ with $u_n > 0$
  - $u_n \ge u_{n+1} > 0$, $u_n \to 0$
  - Use derivative to check monotonicity
  - Approximation
- Conditional convergent, rearrangement theorem

§10.7 Power Series

- **A power series about** $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

  - converges absolutely for all x
  - converges at $x = a$ and diverges elsewhere
  - converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$
- Ratio or root test to find R, use other tests to check the convergence at both endpoints.
- Operations
  - Product of two power series
  - Substitute a function in a power series
  - Derivative of a power series
  - Integration of a power series

§10.8-10.9 Taylor Series

- Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then **its Taylor series at** $x = a$ is

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}}{n!}(x-a)^n+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

The **Taylor polynomial of order** n generated by f at $x = a$ is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- Taylor's Theorem

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

- Estimation error

§10.10 Binomial Series and Applications of Taylor Series

- Binomial Series ($|x| < 1$)

$$(1+x)^m$$

- Nonelementary integral

$$\int \sin x^2 dx$$

- Evaluating indeterminate forms

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

- Frequently used Taylor series

$$\frac{1}{1-x}, \frac{1}{1+x}, e^x, \sin x, \cos x, \ln(1+x), \tan^{-1} x$$

- Find Taylor series with these frequently used ones using operations of power series

$$e^x \sin x \cos x, (2-x^2)^{1/2}$$

§11.1-11.2 Parametrizations of plane curves

- $(x, y) = (f(t), g(t))$: parameter t is not unique
- Cycloids
- Tangent line
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
- Area
$$\int_a^b y(x)dx = \int_a^b y(t)\frac{dx}{dt}dt$$
- Smooth curve
$$|f'(t)|^2 + |g'(t)|^2 > 0 \quad \text{for all } t$$
- Length
$$\int_a^b \sqrt{|f'(t)|^2 + |g'(t)|^2}dt$$
$$y = f(x): \quad \int_a^b \sqrt{1 + |f'(x)|^2}dx$$
- Arc length function
$$ds = \sqrt{|f'(t)|^2 + |g'(t)|^2}dt$$

§11.3-11.4 Polar Coordinates

- $P(r, \theta) = (r\cos\theta, r\sin\theta)$
  - r can be negative,
- Symmetry test
  - about x-axis: $(r, -\theta)$
  - about the origin: $(-r, \theta)$
  - about y-axis: $(-r, -\theta)$

$r = f(\theta)$

- Sketch the polar curve (pick some points and connect them)
- Slope ($r = f(\theta)$)

$x = f(\theta)\cos\theta$

$y = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} =$$

- Area

$$\int_\alpha^\beta \frac{1}{2} r(\theta)^2 d\theta$$

- Length (parameter θ)

$$\int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2} d\theta$$

§12.1-12.4 3D Coordinate Systems

- Three-dimensional coordinates (right-hand rule)
- Distance, geometric interpretations
- Vector/directed line segment (length/magnitude, direction)

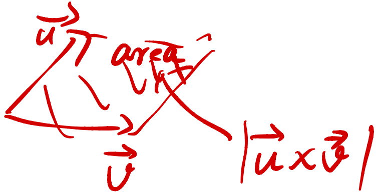
$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

- Vector operations: addition, scalar multiplication
- Unit vector/direction: $\vec{i}$, $\vec{j}$, $\vec{k}$
- Dot product: $\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cos\theta$; projecting one vector onto another
- Cross product (area of the parallelogram): $\vec{u} \times \vec{v} = (|\vec{u}| \cdot |\vec{v}| \sin\theta)\vec{n}$, $\vec{i} \times \vec{j} = \vec{k}$.

$\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$

-

$$|\vec{u} - \vec{v}| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} = \sqrt{|\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2}$$



§12.5-12.6 Line, Plane, Surface

- Line: one point and a direction

$$\vec{r}(t) - \vec{r}_0 = t\vec{a}$$

- Plane: one point and a normal vector

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

- Intersection of two planes (use cross product)
- Cylinders: a generating curve and a straight line
- Quadric surfaces

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

- Intersection of one line and one plane

§13.1-13.2 Derivative and Integral of Vector functions

Position vector

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

- Limit and continuity

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

- Derivatives

$$\vec{r}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

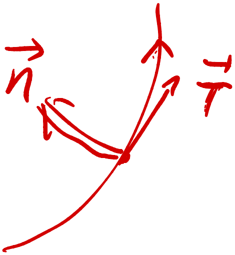
- Velocity vector, speed, direction of motion, acceleration vector
- Differentiation rules: constant, scalar multiple, sum, difference, dot product, cross product, chain
- Vector functions of constant length
- Integral of vector functions
- Example: ideal projectile motion

$$|\vec{r}(t)| = c$$

$$\vec{r}(t) \cdot \frac{d\vec{r}(t)}{dt} = 0$$

§13.3-13.4 Arc length, Curvature, and Normal Vectors

- Arc length
$$\int_a^b |\vec{r}'(t)|dt = \int_a^b |\vec{v}|dt$$
- Arc length parameter
$$ds = |\vec{v}|dt$$
- Unit tangent vector
$$\vec{T} = \boxed{\frac{\vec{v}}{|\vec{v}|}} = \frac{d\vec{r}}{ds}$$
- Curvature
$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$
- Principal unit normal
$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$
- ~~Circle of~~ curvature: point P, same $d\vec{T}/ds$



End of Midterm Review

§14.1-14.2 Functions of more variables

- $$f(x, y, z)$$
- Level curve (on the plane), level surface (in the space)
- Limit; rules (sum, difference, constant multiple, product, quotient, power, root)
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$
- Two-path test for the nonexistence of a limit
- Continuous
$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$$
- Interior and boundary of a set; open/closed; bounded/unbounded
- Extreme values on closed, bounded sets

§14.3-14.4 Partial Derivatives

- $$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)} = \frac{d}{dx}f(x,y_0)\Big|_{x=x_0} = f_x(x_0,y_0)$$
- Implicit differentiation
- High-order partial derivatives (order)
- Mixed derivative theorem $f_{xy} = f_{yx}$: continuity is required
- A function $z = f(x,y)$ is **differentiable** at (x_0,y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and Δz satisfies an equation of the form

  $$\begin{aligned}\Delta z \equiv& f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)\\ =& f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y\end{aligned}$$

  in which both $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.
- If partial derivatives are continuous, then the function is differentiable, and thus continuous
- Chain rule: $w = f(x(t), y(t))$

  $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$
- Implicit partial derivatives
- General chain rules

§14.5-14.6 Directional Derivatives, Tangent Planes

- $$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = (D_{\vec{u}}f)_{P_0} = (\nabla f)_{P_0} \cdot \vec{u} = |(\nabla f)_{P_0}| \cos\theta$$
- Increase rapidly; decrease rapidly; zero changes
- Gradient and tangents to level curves (2D)
- Algebra rules for gradient (sum, difference, constant multiple, product, quotient)
- Tangent lines (parametrized, implicit (2D)), tangent plane, and normal line (estimate the changes of f in direction $\vec{u}$.)
- Linearization and error estimation

§14.7-14.9 Exterem values, Lagrange Multiplier, Taylor

- Local maximum/minimum (first-derivative test), critical point
- Saddle point (second-derivative test)
- Absolute maxima/minima on closed bounded regions
- One constraint $\nabla f = \lambda \nabla g$ for $g(x, y, z) = 0$
- Two constraints
- Error estimation for Taylor's formular

§15.1-15.3 Double integral

- Rectangle

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy$$

- Change order, sketch the region
- General region

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

- Find the volume
- Properties (Constant Multiple, Sum and Difference, Domination, Additivity)
- Find the area ($f = 1$)
- Average value

§15.4-15.5, 15.7 Double integral in Polar, Triple Integral

- Polar form

$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)r\,dr\,d\theta$$

- Triple integral

$$\iiint_D f(x,y,z)dV = \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=h_1(x,y)}^{z=h_2(x,y)} f(x,y,z)dz\,dy\,dx$$

- Triple integral in cylindrical coordinates

$$\iiint_D f(r,\theta,z)r\,dz\,dr\,d\theta$$

- Spherical coordinates

$$\iiint_D f(\rho,\phi,\theta)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

§15.8 Substitutions

- Jacobian determinant for $x = g(u,v)$ and $y = h(u,v)$

$$\iint_R f(x,y)dxdy = \iint_G f(g(u,v),h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

§16.1-16.2 Line integral

- $$\int_a^b f(\vec{r}(t))|\vec{v}(t)|dt$$
- Mass, center of mass, moments of inertia
- Work done by a force over a curve

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t)dt$$

- Different ways

$$\int_C \vec{F} \cdot \vec{T}ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{v}dt = \int_a^b Mdx + Ndy + Pdz$$

- Flow integrals, circulation (closed loop)
- Flux across a simple closed plane curve

$$\int_C \vec{F} \cdot \vec{n}ds = \oint_C Mdy - Ndx = \int_a^b \left(M\frac{dy}{dt} - N\frac{dx}{dt}\right) dt$$

§16.3-16.4: path independence, Green's theorem

- Path independence, conservative, potential function
- $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$
- $\vec{T}$ is conservative $\Leftrightarrow \vec{T} = \nabla f \Leftrightarrow \oint_C \vec{F} \cdot \vec{r} = 0 \Leftrightarrow$ exact differential form.
- Find potential function; check conservative;
- Green's theorem:
  circulation:

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

  flux:

$$\oint_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

§16.5-16.6: Surfaces Integral

- Parametrization of surface

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$$

- Smooth surface: $\vec{r}_u \times \vec{r}_v \neq \vec{0}$.
- Surface area: $d\sigma = |\vec{r}_u \times \vec{r}_v| dudv$
- Implicit surface: $F(x,y,z) = c$, $d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dxdy$
- Explicit surface: $z = f(x,y)$: $d\sigma = \sqrt{f_x^2 + f_y^2 + 1} dxdy$
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$$\iint_S G(x,y,z) d\sigma$$

- Coordinates of the center of mass, Moments of inertia about coordinate axes
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$$\iint_S \vec{F} \cdot \vec{n} d\sigma$$

§16.7-16.8: Stokes' Theorem, Divergence Theorem

- Stokes' Theorem

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{counterclockwise}} = \underbrace{\iint_S \nabla \times \vec{F} \cdot \vec{n} d\sigma}_{\text{curl integral, flux}}$$

  - compute curl of a vector field
  - note the counterclockwise and the normal vector
  - surface independency
  - surfaces with holes
  - $\nabla \times \nabla f = \vec{0}$
  - connection to the conservative vector field
- Divergence theorem

$$\underbrace{\iint_S \vec{F} \cdot \vec{n} d\sigma}_{\text{outward flux}} = \underbrace{\iiint_D \nabla \cdot \vec{F} dV.}_{\text{divergence integral}}$$

  - compute the divergence
  - $\nabla \cdot (\nabla \times \vec{F}) = 0$
  - regions with bubbles