PHY1001: Mechanics (Week 5, 6)

1 Rotation

Bodies that have perfectly definite and unchanging shape and size are called rigid bodies. In real world, rigid bodies do not exist, but they are the idealized model objects for us to study physics related to rotation. This leads to an important conclusion: the distance between any two points inside a rigid body never changes.

In this chapter, we are going to learn how to describe rotation kinetics and how to compute the moment of inertia, or rotational inertia.

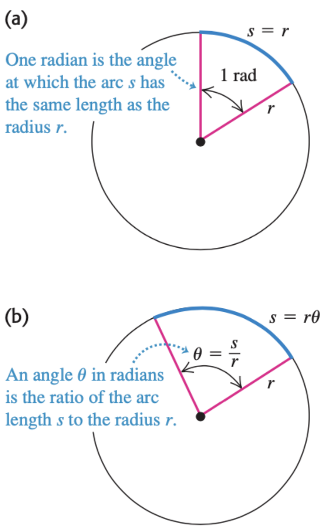
1.1 Angular displacement

In general, angular displacements (unless they are very small) are not vectors. Why not? Try this at home and think about it: Hold your right arm downward, palm toward your thigh. Keeping your wrist rigid, (1) lift the arm forward until it is horizontal, (2) move it horizontally until it points toward the right, and (3) then bring it down to your side. Your palm faces forward. If you start over, but reverse the steps, which way does your palm end up facing? From this, we conclude that the addition of two angular displacements depends on their order and they cannot be vectors since if you add two vectors, the order in which you add them does not matter. Rule of addition: $\vec{A}+\vec{B}=\vec{B}+\vec{A}$.

Nevertheless, one can show that infinitesimal angular displacements satisfy rule of addition and therefore small angular displacement together with angular velocity and acceleration can be treated as vectors (axial vectors).

1.2 Angular velocity and acceleration

In analyzing rotation, let us think about a simple example first. A rigid body rotates about a fixed axis - an axis that is at rest in some inertial frame of reference and does not change direction in that frame.

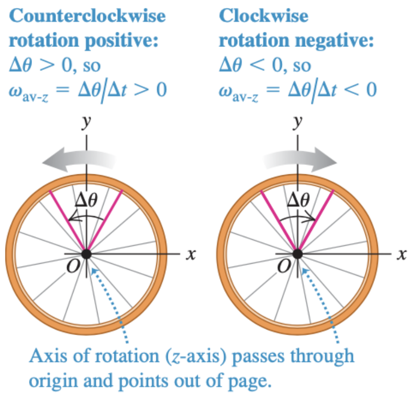


In this case, since the distance between any point and the axis is always a constant, it is better to use the angle θ as the coordinate for rotation. We use radians as the measure of the angle, which is defined as

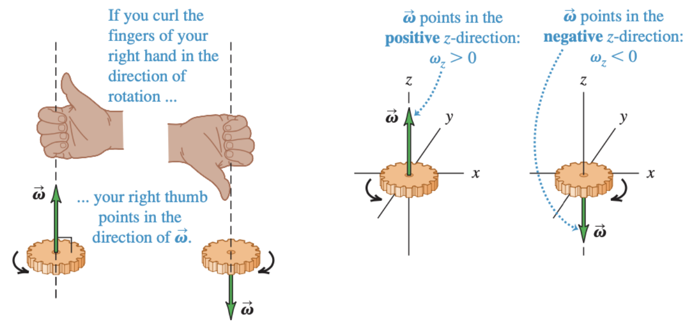
$$\theta \equiv \frac{s}{r}, \tag{1}$$

with s the length of the arc and 1 radian = 57.5° = $360/(2\pi)$. This definition is more useful and simpler when we discuss rotation. Remember 1 revolution $= 2\pi = 360°$.

Let us now define the the angular velocity in a rigid body. Angular velocity ω is the angular displacement per unit time. At any instant, every part of a rigid body has the same angular velocity. Think about what if this is NOT the case? (Hint: contradict with the definition of the rigid body.)



Conventionally, we use the right-hand rule to assign the direction of the angular velocity!



If you curl your fingers of right hand in the direction of the rotation, your right thumb points to the direction of $\vec{\omega}$. As shown above, in the right-handed coordinate system, $+z$ direction indicates $\omega > 0$ if the rotation is happening in the $x-y$ plane counterclockwise. For convenience, we usually choose the plane of the rotation to be the $x-y$ plane, then we know that the direction of $\vec{\omega}$ is along z direction and $\omega = \omega_z$. In general, $\vec{\omega}$ is along the rotation axis if the axis is fixed.

Mathematical definitions for ω.

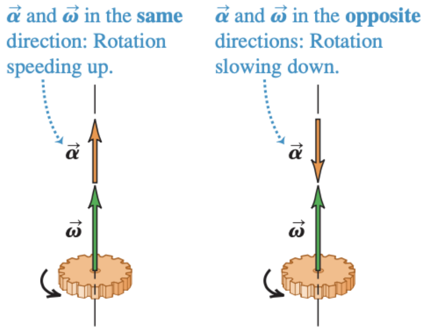
$$\text{scalar form: } \omega \equiv \frac{d\theta}{dt} \tag{2}$$

$$\text{vector form: } \vec{\omega}_z \equiv \frac{d\theta}{dt}\hat{k}. \tag{3}$$

Angular acceleration can be defined as follows

$$\alpha \equiv \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{4}$$

If the axis is fixed, then $\vec{\alpha}$ has the same direction as $\vec{\omega}$ if ω is increasing; $\vec{\alpha}$ has the opposite direction as $\vec{\omega}$ if ω is decreasing. Choose z as the direction of ω, then $\alpha_z \equiv \frac{d\omega_z}{dt}$.



Comparison of linear and angular motions:

$$v = \frac{dx}{dt}, \qquad \omega = \frac{d\theta}{dt}, \tag{5}$$

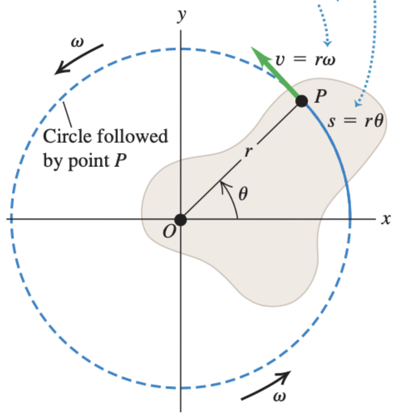
$$a = \frac{d^2x}{dt^2}, \qquad \alpha = \frac{d^2\theta}{dt^2}, \tag{6}$$

$$\text{constant } a/\alpha: \qquad v_x = v_0 + at, \quad \omega_z = \omega_{0z} + \alpha t,$$

$$x = x_0 + v_0 t + \frac{a}{2}t^2, \qquad \theta = \theta_0 + \omega_{0z} t + \frac{\alpha}{2}t^2. \tag{7}$$

They are very similar with notable difference in terms of dimension and directions.

1.3 Relating linear and Angular Kinetics



We need to find the linear speed in rigid body rotation in order to compute the kinetic energy of the rotating body. To proceed, let us consider the case a rigid body rotates about a fixed axis through point O. In this case: 1. there is only circular motion; 2. no radial motion (r is fixed in rigid body).

Therefore, the arc length $s = r\theta$ that the point P travels determines the linear speed as follows

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega. \tag{8}$$

In terms of vector form, note that radial velocity is zero ($v_{||} = v_{rad} = 0$) and the tangential velocity (perpendicular to $\vec{r}$) is given by

$$\vec{v} = \vec{v}_{tan} = \vec{v}_{\perp} = \vec{\omega} \times \vec{r}. \tag{9}$$

The direction of the velocity $\vec{v}$ is tangent to its circular path, and perpendicular to $\vec{r}$. Here we use the cross product to define the direction of the velocity. Note that you can also write $\vec{\omega} = \frac{1}{r^2}\vec{r} \times \vec{v}$. (Here r must be the perpendicular distance to the axis. In fact, $\vec{\omega}$ is known as the axial vector, or pseudovector, since it is defined through the cross product of two normal vector, $\vec{r}$ and $\vec{v}$.)

As to the acceleration, the situation is a bit different. There are both radial (due to centripetal acceleration) and tangential components.

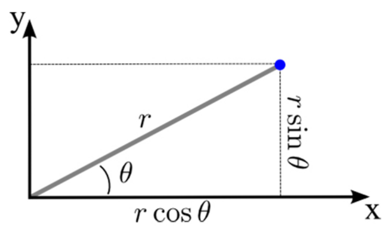
$$\text{Tangential: } a_{tan} = a_{||} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha, \tag{10}$$

$$\vec{a}_{tan} = \vec{\alpha} \times \vec{r}, \tag{11}$$

$$\text{Radial: } a_{rad} = a_{\perp} = \frac{v^2}{r} = \omega^2 r, \quad \text{inward}, \tag{12}$$

$$\vec{a}_{rad} = -\omega^2 \vec{r}. \tag{13}$$

1.4 Polar Coordinates



This part is a mathematical description of the circular motion a single particle with the use of polar coordinates. Let us consider a particle moving in a circular path with fixed radius= r in the $x-y$ plane with the following time-dependent cartesian components

$$x(t) = r\cos\theta, \quad y(t) = r\sin\theta, \tag{14}$$

where $\theta = \theta(t)$ is the time-dependent angular displacement. Therefore, we can express the vector $\vec{r}$ and its infinitesimal change as follows (note that r is fixed but $\vec{r}$ is changing with time)

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} = r\cos\theta\hat{i} + r\sin\theta\hat{j} \tag{15}$$

$$d\vec{r}(t) = dx\hat{i} + dy\hat{j} = r\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)d\theta. \tag{16}$$

Furthermore, dividing both sides by dt directly gives

$$\vec{v} = \frac{d\vec{r}(t)}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = \omega r\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right). \tag{17}$$

Use the right-hand rule, we set

$$d\vec{\theta} = d\theta\hat{k}, \tag{18}$$

which allows us to write

$$d\vec{r}(t) = d\vec{\theta} \times \vec{r}. \tag{19}$$

Now dividing both sides by dt, one obtains linear velocity (rate of change of $\vec{r}$) in terms of angular velocity $\vec{\omega}$

$$\vec{v} = \frac{d\vec{r}(t)}{dt} = \vec{\omega} \times \vec{r}, \tag{20}$$

where $\vec{\omega} \equiv \frac{d\vec{\theta}}{dt} = \frac{d\theta}{dt}\hat{k} = \omega\hat{k}$.

Now let us introduce the polar coordinate system. It is clear that the three vectors $\vec{r}$, $\vec{v}$ and $\vec{\omega}$ form a perpendicular system in the three respective directions $\hat{r}$, $\hat{\theta}$, and $\hat{k}$. Here we define the new unit vectors

$$\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}, \tag{21}$$
$$\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}. \tag{22}$$

These three directions satisfy the cyclic relation ($\hat{r} \to \hat{\theta} \to \hat{k} \to \hat{r}$)

$$\hat{r} \times \hat{\theta} = \hat{k}, \quad \hat{k} \times \hat{r} = \hat{\theta}, \quad \hat{\theta} \times \hat{k} = \hat{r}. \tag{23}$$

In the case of rotation, the polar coordinate is moving as follows according to Eqs (21) and (22)

$$\frac{d\hat{r}}{dt} = \left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)\frac{d\theta}{dt} = \omega\hat{\theta}, \tag{24}$$
$$\frac{d\hat{\theta}}{dt} = -\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)\frac{d\theta}{dt} = -\omega\hat{r}, \tag{25}$$
$$\frac{d\hat{k}}{dt} = 0, \tag{26}$$

where the last identity comes from our assumption that the axis is fixed.

It is very convenient to use the polar coordinate to describe circular motion. The velocity is then given by

$$\vec{v} = \frac{d\vec{r}(t)}{dt} = \vec{\omega} \times \vec{r} = \omega r\hat{\theta}. \tag{27}$$

Differentiating the velocity w.r.t. time gives the acceleration as follows

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left(\omega r\hat{\theta}\right) && (28)\\ &= \frac{d\omega}{dt}r\hat{\theta} + \omega r\frac{d\hat{\theta}}{dt}, && (29)\\ &= \alpha r\hat{\theta} - \omega^2 r\hat{r} = \alpha r\hat{\theta} - \omega^2\vec{r}. && (30) \end{aligned}$$

where Eq (25) is used in the last step. The above results is in agreement with what we discussed in the previous subsection.

1.5 Energy in rotational motion

Suppose a body is made up of a large number of particles, with mass $m_1, m_2, \cdots$ at distance $r_1, r_2, \cdots$ from the axis of rotation. Note that r_i is the perpendicular distance between the particle and the axis and the i-th particle. Therefore, the total kinetic energy due to rotation is

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i r_i^2\omega^2 = \frac{1}{2}I\omega^2, \tag{31}$$

where $I \equiv \sum_i m_i r_i^2$ is the moment of inertia, or rotational inertia, of the body for a given rotational axis. Several comments are in order.

1. The word "moment" means that I depends on how the body's mass is distributed in space. It is similar to moments of a distribution defined in statistics. In fact, in mathematics, the moments of a function are quantitative measures related its shape. If the function represents mass distribution, then the first moment is the center of the mass, and the second moment is the rotational inertia.

2. To understand the physical meaning of the moment of inertia. Let us compare it with mass M. For linear motion, the larger the mass M is, the harder it is to change is velocity. For rotation, the larger the rotational inertia I is, the harder it is to change its angular velocity. For this reason, I is also called the rotational inertia.

3. Since the rotational kinetic energy K increases with I for a given ω, it takes more work W to accelerate a rigid body with larger I from rest.

1.6 Gravitational Potential Energy for an Extended Body

Let us consider a body which is made up of a collection of infinitesimal mass elements. Th potential for element m_i is $m_i g y_i$, so the total gravitational potential energy is then

$$U = \sum_i m_i g y_i = g\sum_i m_i y_i = gMy_{cm}, \tag{32}$$

where we have assumed that g is a constant and employed the definition of y_{cm}.

Therefore, if g is the same at all points on the extended body, the potential is the same as if all the mass were concentrated at the center of the mass of the body.

1.7 Parallel-Axis Theorem

A rigid body's moment of inertia depends on where the axis is located. Nevertheless, there is a simple relation between I_{cm} (about the axis through its center of mass) and the moment of inertia I_P about any other axis parallel to the original one but displaced from the center of mass by a distance d. This relation is called the parallel-axis theorem, which states

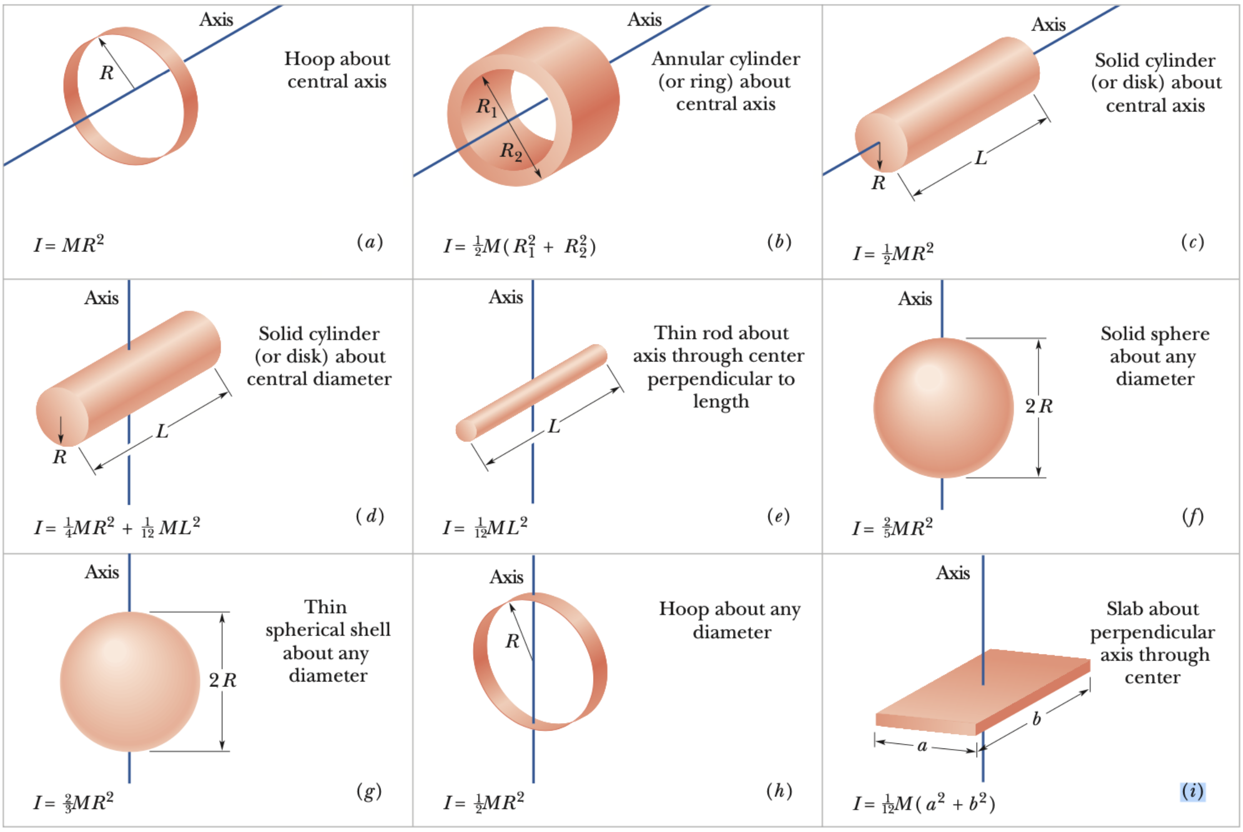
$$I_P = I_{cm} + Md^2. \tag{33}$$

This theorem also indicates that the moment of inertia is lower about the axis through the center of mass $I_P > I_{cm}$, and it is easier and more natural for a rotating body about the CM axis.

1.8 Several example calculations for rotational inertia

In the following figure, you can find several moments of inertia for some common shapes in terms of their masses and dimensions. We normally assume that the mass is uniformly distributed (with constant mass density) in the rigid body unless a different distribution is provided.

For continuous distributions of matter, the sum in $I = \sum_i m_i r_i^2$ becomes an integral. We will discuss how to compute some moments of inertia in class, and you will have the chance to show your tour de force in calculus and compute the rest in several homework problems.



Here are a few example calculations for moment of inertia of various shapes of uniform objects. The rest of derivations are included in the homework 5 and 6.

(a) Hoop about the center.

All the mass is equally distributed on a circle with equal distance R from the axis.

$$I = \int dmR^2 = MR^2. \tag{34}$$

(b) Hollow cylinder (or ring) about the center axis

Assume the mass density is ρ and then compute the mass and moment of inertia simultaneously as follows

$$M = \int dm = \int_{R_1}^{R_2} \rho L 2\pi r dr = \rho L\pi\left(R_2^2 - R_1^2\right), \tag{35}$$

$$I = \int dmr^2 = \int_{R_1}^{R_2} \rho L 2\pi r^3 dr = \frac{1}{2}\rho L\pi\left(R_2^4 - R_1^4\right). \tag{36}$$

Taking the ratio of the above results gives

$$I = \frac{1}{2}M\left(R_1^2 + R_2^2\right). \tag{37}$$

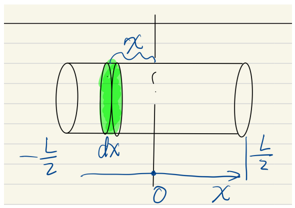
(c) Solid cylinder (or disk); (f), and (i) see Homework.

(d) Solid cylinder about central diameter. * * *

Consider thin slices of the cylinder parallel to its cross-section and first compute the moment of inertia through its diameter

$$dm = \rho\pi R^2 dx, \quad M = \rho\pi R^2 L \tag{38}$$

$$dI_{slice} = \rho dx \int_0^R rdr \int_0^{2\pi} d\theta r^2 \sin^2\theta = \frac{\pi}{4}\rho R^4 dx. \tag{39}$$



Employ the parallel axis theorem, one gets the moment of inertia of the disk colored in green in the above figure

$$dI(x) = \frac{\pi}{4}\rho R^4 dx + dmx^2 = \frac{\pi}{4}\rho R^4 dx + \rho\pi R^2 x^2 dx. \tag{40}$$

At last, let us integrate along its longitudinal length x from $-L/2$ to $L/2$ as follows

$$I = \int_{-L/2}^{+L/2} dx dI(x) = \frac{1}{4}MR^2 + \frac{1}{12}ML^2. \tag{41}$$

(e) Thin rod about center. Set $R = 0$ above.

(g) Thin walled hollow sphere. * *

Let σ be the mass density per unit area, then

$$M = \int dm = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\sigma R^2 = 4\pi R^2\sigma, \tag{42}$$

$$I = \sigma R^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta R^2 \sin^2\theta = \frac{2}{3}MR^2. \tag{43}$$

(h) Hoop about any diameter.

Let λ be the mass density per unit length, then

$$M = \int dm = 2\pi R\lambda, \tag{44}$$

$$I = \int_0^{2\pi} d\theta\lambda RR^2 \sin^2\theta = \pi R^3\lambda = \frac{1}{2}MR^2. \tag{45}$$