**MAT2041**

# Final Exam

Time: Dec 20, 2023 Wed 1:30pm - 4:00pm

DURATION OF EXAMINATION: 2.5 hours

Your exam sheet shall include 6 **problems** plus one more bonus problem. If not, notify the instructors.

1. **(18 pt) Linear System and Determinant**

(a) (10 pt) Solve the following linear system

$$\begin{bmatrix} 0 & 2 & 6 \\ 1 & 1 & 3 \\ 0 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ -4 \end{bmatrix}.$$

(b) (3 pt) A student claims that the determinant of the coefficient matrix in the above linear system equals to the *negative* of the determinant of

$$B = \begin{bmatrix} 2 & 6 \\ 2 & -6 \end{bmatrix}.$$

Without explicit computation of the values of the determinant(s), can you justify this claim?

(c) (5 pt) Find LU decomposition of the above matrix $B$.

***Solution:*** (a) $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, 10 marks for correctly finding the solution (inluding those solving equations); partial marks will be granted if the steps are correct but the final answer is wrong.

(b) $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 1 & 3 \\ 0 & 2 & -6 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 2 & -6 \end{vmatrix} = -\begin{vmatrix} 2 & 6 \\ 2 & -6 \end{vmatrix}$ Students may have multiple ways to show the answer. 3 marks if the procedure makes sense. 0 if they directly calculate the determinant.

(c) $B = \begin{bmatrix} 2 & 6 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 12 \end{bmatrix} = LU.$

2. **(20 pt) Computation of Eigenvalues and Eigenvectors**

Consider the following $3 \times 3$ matrix $A$:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a) (2pt) Without computing the eigenvalues explicitly, explain why all the eigenvalues of $A$ must be real.

(b) (10pt) Find all eigenvalues of matrix $A$.

(c) (4pt) For the smallest eigenvalue of $A$ *only*, determine a corresponding eigenvector.

(d) (2pt) Compute the trace of $A$ and the determinant of $A$ using the definitions.

(e) (2pt) Compute the trace of $A$ and the determinant of $A$ using the properties of eigenvalues.

***Solution:*** (a) Because it is a real symmetric matrix. (b)$det(A - \lambda I) = -(\lambda - 1)(\lambda - 2)^2 = 0$ The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 4$.

(c)

$$\lambda = 1, u_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; \lambda = 2, u_2 = b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \lambda = 4, u_3 = b = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix};$$

(d) Trace is the sum of diagonal elements. Calculate determinant using cofactors.

(e) **Sum of Eigenvalues**:(1pt)

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 4 = \text{trace}(A)$$

**Product of Eigenvalues**:(1pt)

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1 \cdot 2 \cdot 4 = \det(A)$$

3. **(16 pt) True or False** You do NOT need to justify. (Only writing T or F is enough)

   (a) Performing elementary row operations on any square matrix does not change its determinant.

   ***Solution:*** **F**alse

   (b) The eigenvalues of a real square matrix must be real.

   ***Solution:*** **False**

   (c) For any matrix $A \in \mathbb{R}^{m \times n}$, the sum of the dimension of its row space and the dimension of its null space equals $n$.

   ***Solution:*** **T**rue

   (d) Let $A \in \mathbb{R}^{n \times n}$ and $\text{rank}(\lambda I_n - A) < n$ for some $\lambda$, then $\lambda$ is an eigenvalue of $A$.

   ***Solution:*** **T**rue

   (e) All eigenvectors of a matrix are orthogonal to each other.

   ***Solution:*** **F**alse

   (f) The number of non-zero singular values of a matrix is equal to the rank of the matrix.

   ***Solution:*** **T**rue

   (g) For any square matrix $A$, the absolute value of its eigenvalue is a singular value of $A$.

   ***Solution:*** **F**alse

   (h) Suppose $\lambda$ is an eigenvalue of $A$, $\mu$ is an eigenvalue of $B$. If $C = A + B$, then $\lambda + \mu$ must be an eigenvalue of $C$.

   ***Solution:*** **F**alse

4. **(10 pt) Linear Transformation and Linear Space**
   For the following linear transformation $T : R^n \to R^m$, find a matrix $A$ such that $T(x) = Ax$ for all $x \in R^n$.

   (a) (4 pt) $T : R^2 \to R^3$ and

   $$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - y \\ -x \\ 2x + 3y \end{bmatrix}$$

   (b) (6 pt) $T : R^2 \to R^2$, and

   $$T \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \quad T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

***Solution:***

(a)

$$A = \begin{bmatrix} 4 & -1 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$$

(b) Solve $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then we can obtain $c_1 = -0.2$, $c_2 = 0.6$, $c_3 = 0.4$, and $c_4 = -0.2$.
Thus, $T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -0.2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + 0.6 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2.8 \\ 0 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.4 \begin{bmatrix} 5 \\ 6 \end{bmatrix} - 0.2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2 \end{bmatrix}$.
We have $A = \begin{bmatrix} -2.8 & 2.6 \\ 0 & 2 \end{bmatrix}$.

5. **(16 pt) SVD and a Least Squares Problem**

(a) (10 pt) Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 2 \end{bmatrix}$. Find the SVD $U\Sigma V^\top$ of $A$, where $U \in R^{3\times3}$ and $V \in R^{2\times2}$ are orthogonal matrices. You **must** place the singular values in descending order diagonally in $\Sigma$, otherwise you lose all points.

(b) (2 pt) Based on the SVD of the above matrix $A$, find an orthonormal basis for each of the four subspaces associated with $A$, i.e., $C(A), N(A^\top), C(A^\top)$, and $N(A)$, where $C(\cdot)$ and $N(\cdot)$ denote the column space and nullspace of a matrix, respectively.

(c) (4 pt) Consider the above matrix $A$, and $b = [\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}]^\top$. Does the least squares problem $\min_{x\in\mathbb{R}^{2\times1}} \|Ax - b\|$ have a unique solution? Provide a reason for your answer and write down the solution set of the least squares problem.

***Solution:***

(a) $U = \begin{bmatrix} \frac{5\sqrt{66}}{66} & \frac{\sqrt{2}}{2} & \frac{2\sqrt{33}}{33} \\ \frac{5\sqrt{66}}{66} & -\frac{\sqrt{2}}{2} & \frac{2\sqrt{33}}{33} \\ \frac{2\sqrt{66}}{33} & 0 & -\frac{5\sqrt{33}}{33} \end{bmatrix}$ (3pts), $\Sigma = \begin{bmatrix} \sqrt{33} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$ (4pts), $V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ (3pts).
There are different correct answers for $U, V$

(b) As the question says "based on the SVD", the basis you list must correspond to your SVD.
$C(A) = span\{[\frac{5\sqrt{66}}{66}, \frac{5\sqrt{66}}{66}, \frac{2\sqrt{66}}{33}]^T, [\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0]^T\}$
$N(A^T) = span\{[\frac{2\sqrt{33}}{33}, \frac{2\sqrt{33}}{33}, \frac{-5\sqrt{33}}{33}]^T\}$
$C(A^T) = span\{[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T, [\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}]^T\}$
$N(A) = \{[0, 0]\}$. (it is not 0) Its basis is $\varnothing$. (1,2 correct gets 1 pt, 3,4 correct gets 2 pts).

(c) Yes (1pt), because $A$ is of full column rank(1pt).
$\arg\min_x ||Ax - b|| = [-\sqrt{2}/11, 10\sqrt{2}/11]^T$(2pts).

6. **(20 pt) Proofs**

(a) (5 pt) Suppose $A^2 = 0, A \in R^{n\times n}$. Show all eigenvalues of $A$ are zero.

(b) (5 pt) Let $A \in \mathbb{R}^{3\times3}$, $\mathbf{x}$ be an eigenvector of $A$, and

$$B = [\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}] \in \mathbb{R}^{3\times3}.$$

(i) Prove: $B$ is not invertible. (ii) What is the rank of $B$, and why?

(c) (5 pt) Let $A = [a_1, a_2] \in \mathbb{R}^{n\times2}$ with the vectors $a_1, a_2 \in \mathbb{R}^n$. If $a_1$ is orthogonal to $a_2$, show $\|a_1\|$ and $\|a_2\|$ are two singular values of $A$.
[Hint: Consider $A^\top A$.]

(d) (5 pt) Let $Q = [q_1, \ldots, q_k] \in R^{n\times k}$ be orthonormal, i.e., $q_i^T q_j = 0$ for $i \neq j$, and $\|q_i\| = 1, i = 1, \ldots, k$, where $q_i$ is the $i$-column of $Q$. For any $a \in R^n$, show: (i) $a - Q(Q^T a)$ is orthogonal to $q_i, i = 1, \ldots, k$; (2) if $a \in C(Q)$, the column space of $Q$, then $a = Q(Q^T a)$.

***Solution:***

(a) Since $A^2x = A(Ax) = \lambda^2 x = 0$, the eigenvalues of $A$ are zero.

(b) (i) Since $\mathbf{x}$ is the eigenvector of the matrix $A$, we have the following logic chain

$$A\mathbf{x} = \lambda\mathbf{x},$$

$$\Rightarrow A^2\mathbf{x} = A(A\mathbf{x}) = A\lambda\mathbf{x} = \lambda^2\mathbf{x}$$

Hence, $B$ can be written as the following form

$$B = [\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}] = [\mathbf{x}, \lambda\mathbf{x}, \lambda^2\mathbf{x}]$$

Now $\mathbf{x} \neq 0$, and the second column of $B$ is multiple of its first column, $(B)$ must be singular.
(ii) The second and third columns of $B$ are multiple of its first column, therefore, $\text{rank}(B) = 1$

(c) Consider $A^T A$, and because $a_1$ is orthogonal to $a_2$,

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 \\ a_2^T a_1 & a_2^T a_2 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2 & 0 \\ 0 & \|a_2\|^2 \end{bmatrix}$$

Therefore, $\|a_1\|^2$ and $\|a_2\|^2$ are the eigenvalues of $A^T A$, therefore they are two singular values of $A$.

(d) (i)

$$\begin{aligned} q_i^T Q(Q^T a) &= (e_i Q^T)a \\ &= q_i^T a. \end{aligned}$$

Therefore, $q_i^T(a - Q(Q^T a)) = 0$, they are orthogonal.
(ii) Let $a = \sum_{i=1}^k c_i q_i$. $Q^T a = [c_1, ..., c_k]^T$. Therefore, $Q(Q^T a) = \sum_{i=1}^k c_i q_i = a$.

**Bonus Question (5 bonus pt)** Let $P$ and $Q$ be two orthogonal matrices. If $\det(P) = 1$ and $\det(Q) = -1$, show $P + Q$ is singular.

***Solution:***

$$P + Q = P(I + P^T Q)$$

We just need to show $I + P^T Q$ is singular. Now

$$\det(I + P^T Q) = \det(I + P^T Q) = \det(I + Q^T P) = \det[Q^T P(I + P^T Q)] =$$

$$= \det Q^T \det P \det(I + P^T Q) = -\det(I + P^T Q)$$

Therefore, $\det(I + P^T Q) = 0$