4.7

Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ (b) $g(x) = \cos x$ (c) $h(x) = \sec^2 x + \dfrac{1}{2\sqrt{x}}$

Solution We need to think backward here: What function do we know has a derivative equal to the given function?

(a) $F(x) = x^2$ (b) $G(x) = \sin x$ (c) $H(x) = \tan x + \sqrt{x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$, and the derivative of $H(x) = \tan x + \sqrt{x}$ is $\sec^2 x + \left(1/2\sqrt{x}\right)$. ■

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

EXAMPLE 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C. Substituting $x = 1$ into $F(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$. So

$$F(x) = x^3 - 2$$

is the antiderivative satisfying $F(1) = -1$. Notice that this assignment for C selects the particular curve from the family of curves $y = x^3 + C$ that passes through the point $(1, -1)$ in the plane (Figure 4.51). ∎

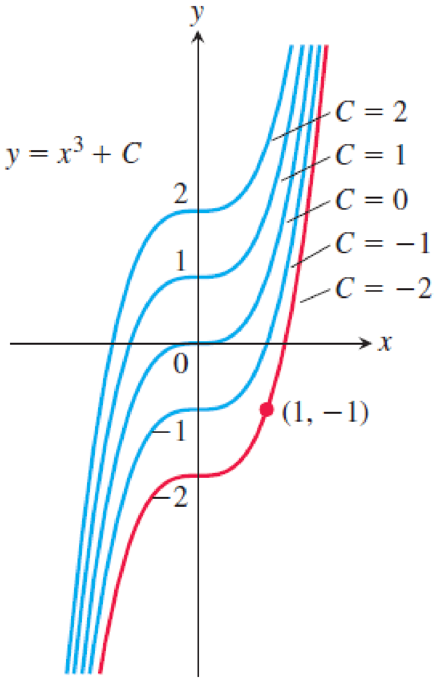


FIGURE 4.51 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

TABLE 4.2 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

EXAMPLE 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \sqrt{x}$

(c) $h(x) = \sin 2x$ **(d)** $i(x) = \cos \frac{x}{2}$

Solution In each case, we can use one of the formulas listed in Table 4.2.

(a) $F(x) = \frac{x^6}{6} + C$ Formula 1 with $n = 5$

(b) $g(x) = x^{1/2}$, so

$$G(x) = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$$ Formula 1 with $n = 1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$ Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin\frac{x}{2} + C$ Formula 3 with $k = 1/2$ ■

EXAMPLE 4 Find the general antiderivative of

$$f(x) = 3\sqrt{x} + \sin 2x.$$

Solution We have that $f(x) = 3g(x) + h(x)$ for the functions g and h in Example 3. Since $G(x) = 2x^{3/2}/3$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3(2x^{3/2}/3) = 2x^{3/2}$ is an antiderivative of $3g(x) = 3\sqrt{x}$. Likewise, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 2x^{3/2} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where C is an arbitrary constant. ■

Initial Value Problems and Differential Equations

Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We can fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**.

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

EXAMPLE 6 Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

EXAMPLE 7 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

SOLUTION Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$ ■

5.1

Area and Estimating with Finite Sums

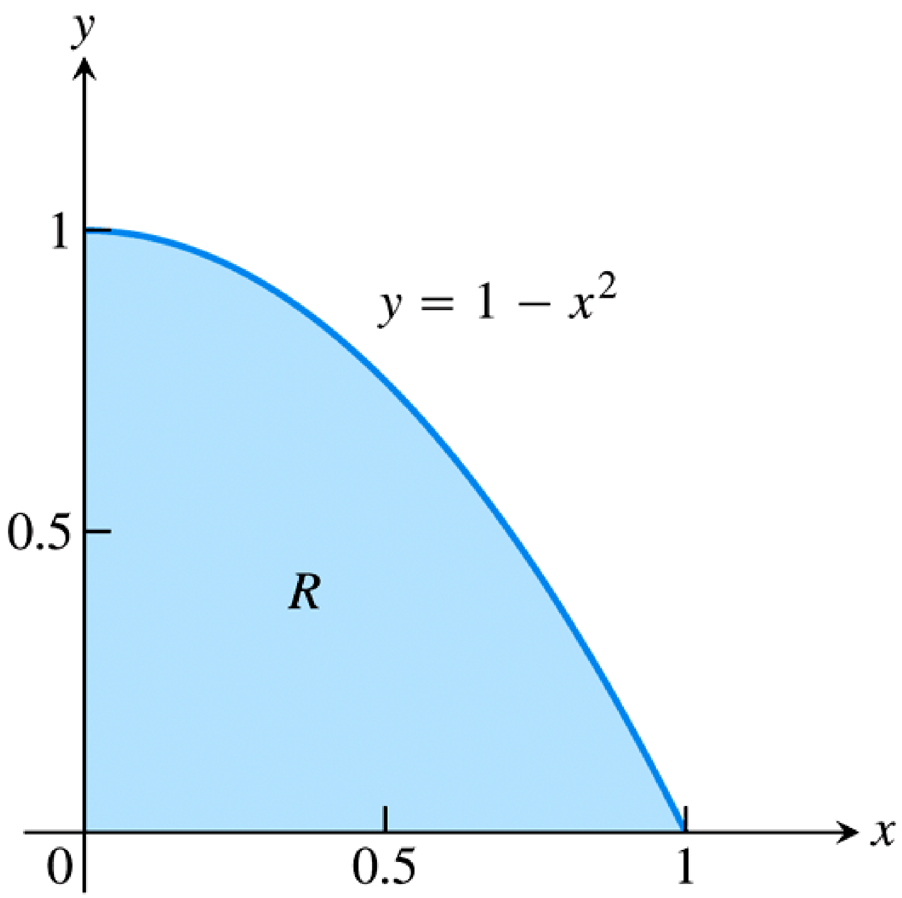


FIGURE 5.1 The area of the region R cannot be found by a simple formula.

Physical Interpretation of the Area

Suppose the curve represents the velocity over time (i.e. y is the velocity and x is the time), then the area of a rectangle is (velocity $\times$ time) which gives the approximate distance traversed in in that time interval.

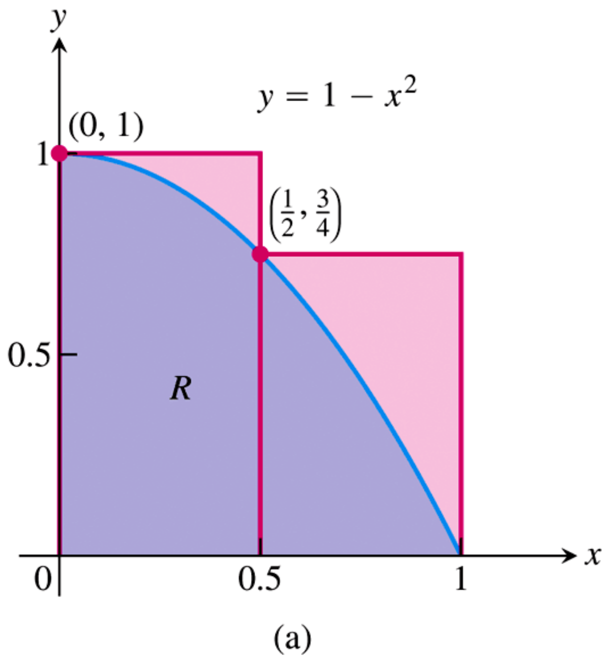


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

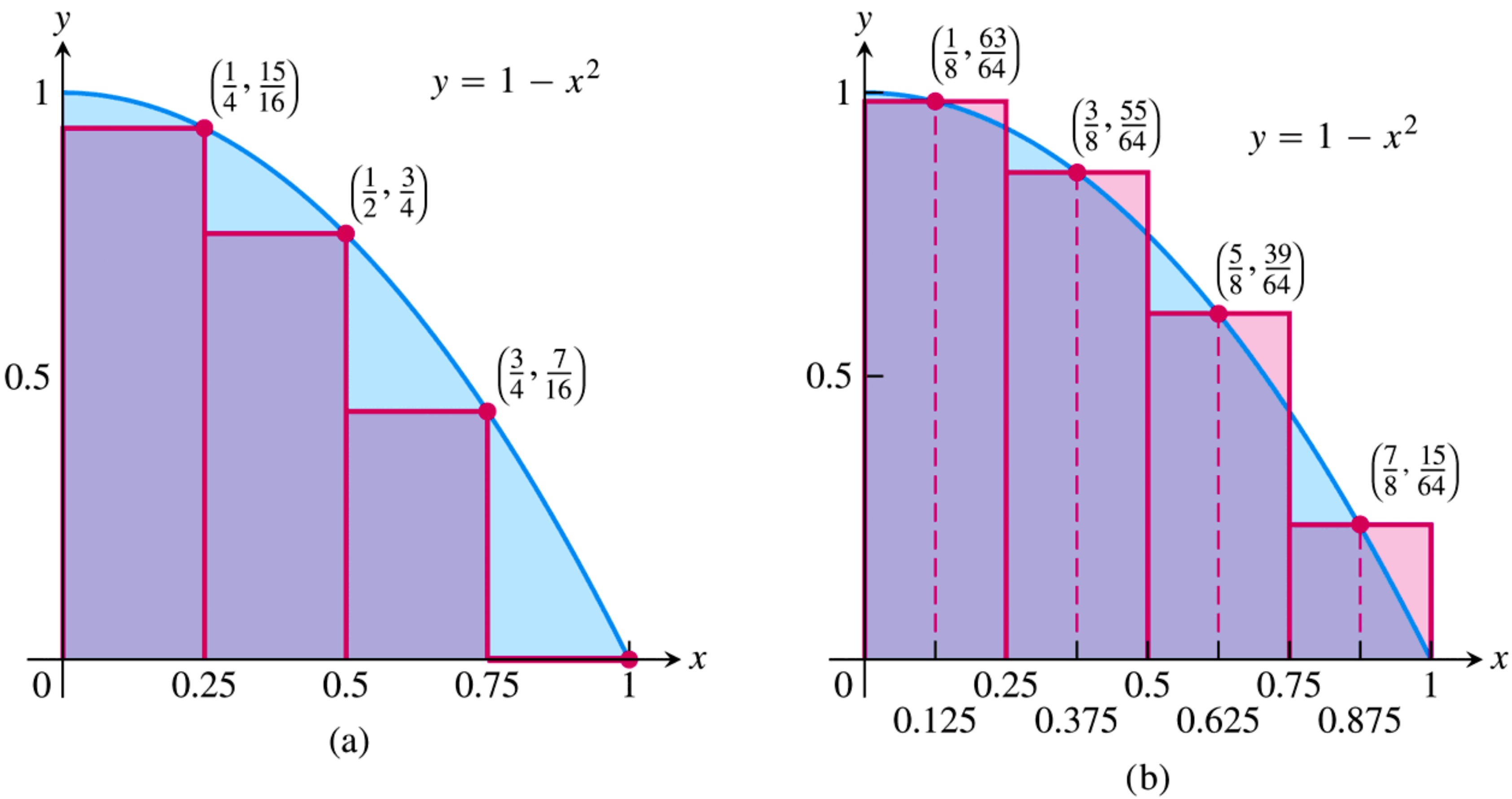
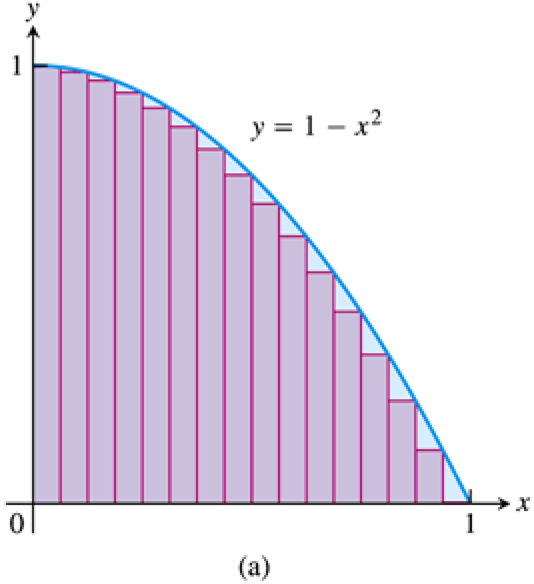


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value by the amount shaded in light blue. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases. The estimate appears closer to the true value of the area because the light red overshoot areas roughly balance the light blue undershoot areas.



Number of subintervals	**Lower sum**	**Midpoint rule**	**Upper sum**
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

TABLE 5.1 Finite approximations for the area of R

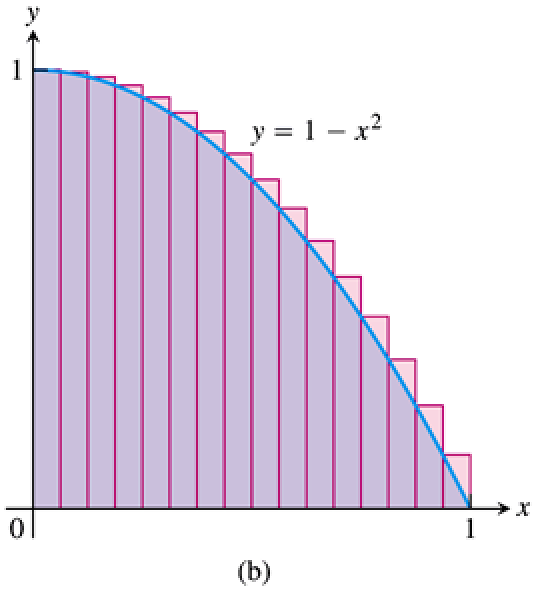


FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

5.2

Sigma Notation and Limits of Finite Sums

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) — $\displaystyle\sum_{k=1}^{n} a_k$ — a_k is a formula for the kth term.

The index k starts at $k = 1$.

The sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

Algebra Rules for Finite Sums

1. *Sum Rule*: $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule*: $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule*: $$\sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k$$ (Any number c)

4. *Constant Value Rule*: $$\sum_{k=1}^{n} c = n \cdot c$$ (c is any constant value.)

The first n squares: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes: $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

The set

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

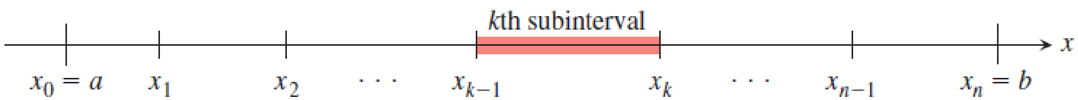
Riemann Sum

is called a **partition** of $[a, b]$.

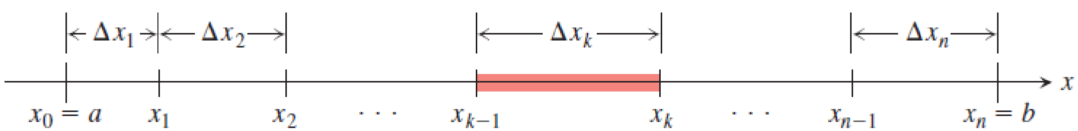
The partition P divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the **kth subinterval** of P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$ (Figure 5.9).

On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

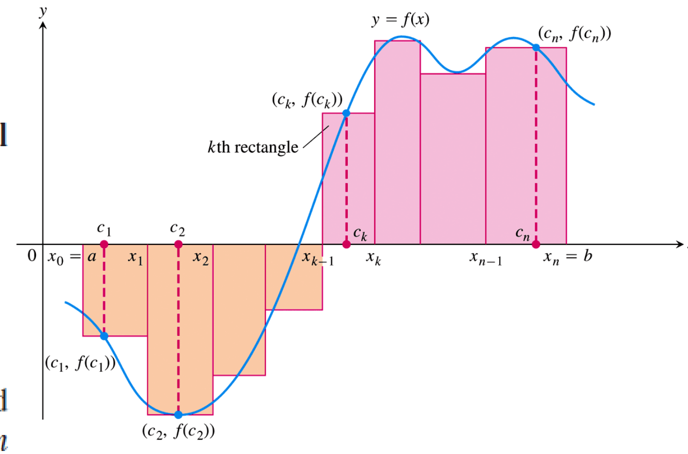


FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis. Figure 5.8 has been enlarged to enhance the partition of $[a, b]$ and selection of points c_k that produce the rectangles.

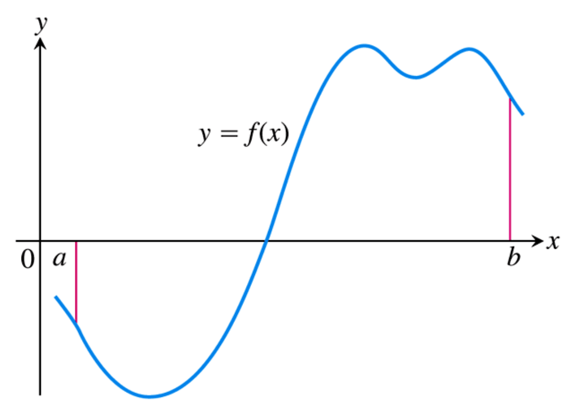


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

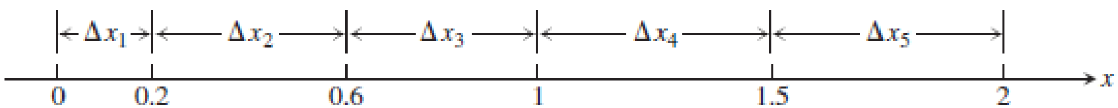
The sum S_P is called a **Riemann sum for f on the interval $[a, b]$**. There are many such sums, depending on the partition P we choose, and the choices of the points c_k in the subintervals. For instance, we could choose n subintervals all having equal width $\Delta x = (b - a)/n$ to partition $[a, b]$, and then choose the point c_k to be the right-hand endpoint of each subinterval when forming the Riemann sum (as we did in Example 5). This choice leads to the Riemann sum formula

$$S_n = \sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\cdot\left(\frac{b-a}{n}\right).$$

Similar formulas can be obtained if instead we choose c_k to be the left-hand endpoint, or the midpoint, of each subinterval.

In the cases in which the subintervals all have equal width $\Delta x = (b - a)/n$, we can make them thinner by simply increasing their number n. When a partition has subintervals of varying widths, we can ensure they are all thin by controlling the width of a widest (longest) subinterval. We define the **norm** of a partition P, written $\|P\|$, to be the largest of all the subinterval widths. If $\|P\|$ is a small number, then all of the subintervals in the partition P have a small width. Let's look at an example of these ideas.

EXAMPLE 6 The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

5.3

The Definite Integral

DEFINITION Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over $[a, b]$** and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ if the following condition is satisfied:

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $\|P\| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left| \sum_{k=1}^{n} f(c_k)\,\Delta x_k - J \right| < \epsilon .$$

We have many choices for a partition P with norm going to zero, and many choices of points c_k for each partition. The definite integral exists when we always get the same limit J, no matter what choices are made. When the limit exists we write it as the definite integral

$$J = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k .$$

And we say that f is **integrable** over $[a, b]$.

Upper limit of integration

The function is the integrand.

Integral sign

x is the variable of integration.

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

Briefly, when f is continuous we can choose each c_k so that $f(c_k)$ gives the maximum value of f on the subinterval $[x_{k-1}, x_k]$, resulting in an upper sum. Likewise, we can choose c_k to give the minimum value of f on $[x_{k-1}, x_k]$ to obtain a lower sum. The upper and lower sums can be shown to converge to the same limiting value as the norm of the partition P tends to zero. Moreover, every Riemann sum is trapped between the values of the upper and lower sums, so every Riemann sum converges to the same limit as well. Therefore, the number J in the definition of the definite integral exists, and the continuous function f is integrable over $[a, b]$.

For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectangles. Our first example shows a function that is not integrable over a closed interval.

EXAMPLE 1 The function

This is the Dirichlet Function over [0, 1], and it is nowhere continuous

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

has no Riemann integral over $[0, 1]$. Underlying this is the fact that between any two numbers there is both a rational number and an irrational number. Thus the function jumps up and down too erratically over $[0, 1]$ to allow the region beneath its graph and above the x-axis to be approximated by rectangles, no matter how thin they are. We show, in fact, that upper sum approximations and lower sum approximations converge to different limiting values.

If we pick a partition P of $[0, 1]$ and choose c_k to be the point giving the maximum value for f on $[x_{k-1}, x_k]$ then the corresponding Riemann sum is

$$U = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (1)\,\Delta x_k = 1,$$

since each subinterval $[x_{k-1}, x_k]$ contains a rational number where $f(c_k) = 1$. Note that the lengths of the intervals in the partition sum to 1, $\sum_{k=1}^{n} \Delta x_k = 1$. So each such Riemann sum equals 1, and a limit of Riemann sums using these choices equals 1.

On the other hand, if we pick c_k to be the point giving the minimum value for f on $[x_{k-1}, x_k]$, then the Riemann sum is

$$L = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} (0)\,\Delta x_k = 0,$$

since each subinterval $[x_{k-1}, x_k]$ contains an irrational number c_k where $f(c_k) = 0$. The limit of Riemann sums using these choices equals zero. Since the limit depends on the choices of c_k, the function f is not integrable. ■

Advanced Note (Optional): *Lebesgue Integral*

The ***Lebesgue Integral*** is able to integrate the Dirichlet function above and evaluates the integral to 0.

More generally, while the Riemann integral partitions the x-axis, the Lebesgue Integral partitions the y-axis, and multiplies the given y-values by the lengths of the associated x-intervals (which may consist of a fragmented collection of x-intervals).

The Lebesgue integral may be regarded as a generalization of the Riemann integral:
Every bounded Riemann integrable function defined on [a, b] is Lebesgue integrable, and the two integrals are the same.

"*I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral*."
Henri Lebesgue

THEOREM 2 When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules in Table 5.6.

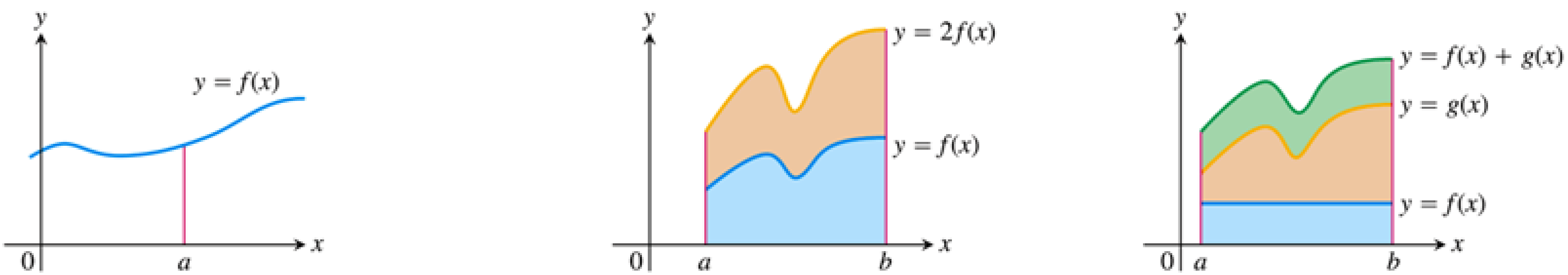
TABLE 5.6 Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special case)



(a) *Zero Width Interval:*

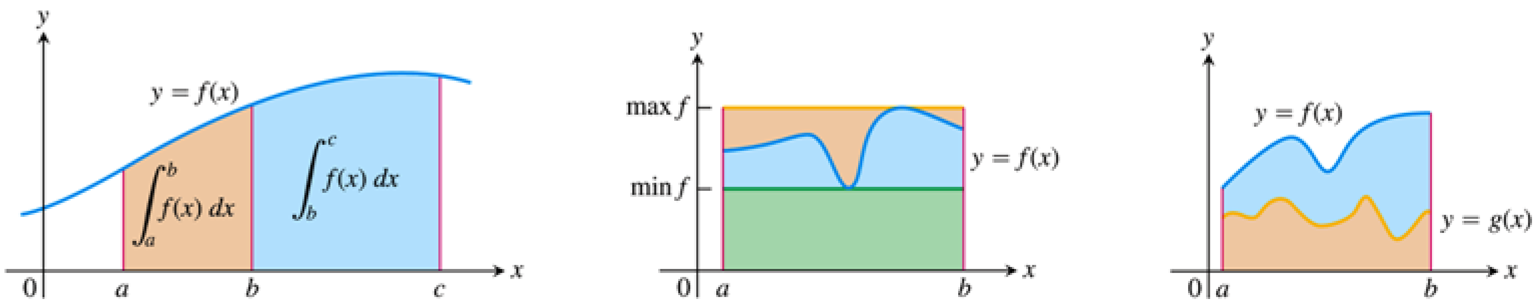
$$\int_a^a f(x)\,dx = 0$$

(b) *Constant Multiple:* $(k = 2)$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(c) *Sum: (areas add)*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$



(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$

(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11 Geometric interpretations of Rules 2–7 in Table 5.6.

EXAMPLE 2 To illustrate some of the rules, we suppose that

$$\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$$

Then

1. $$\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$$ Rule 1

2. $$\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$$ Rules 3 and 4
$$= 2(5) + 3(7) = 31$$

3. $$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$$ Rule 5

EXAMPLE 3 Show that the value of $\int_0^1 \sqrt{1 + \cos x}\,dx$ is less than or equal to $\sqrt{2}$.

Solution The Max-Min Inequality for definite integrals (Rule 6) says that $\min f \cdot (b - a)$ is a *lower bound* for the value of $\int_a^b f(x)\,dx$ and that $\max f \cdot (b - a)$ is an *upper bound*. The maximum value of $\sqrt{1 + \cos x}$ on $[0, 1]$ is $\sqrt{1 + 1} = \sqrt{2}$, so

$$\int_0^1 \sqrt{1 + \cos x}\,dx \le \sqrt{2} \cdot (1 - 0) = \sqrt{2}.$$

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

This is evident from the formation of the Riemann Sum

EXAMPLE 4 Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.

Since the area equals the definite integral for a nonnegative function, we can quickly derive the definite integral by using the formula for the area of a triangle having base length b and height $y = b$. The area is $A = (1/2)\,b \cdot b = b^2/2$.

$$\Rightarrow \quad \int_0^b x\,dx = \frac{b^2}{2}.$$

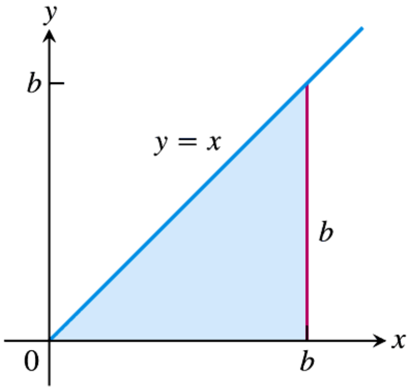


FIGURE 5.12 The region in Example 4 is a triangle.

Example 4 can be generalized to integrate $f(x) = x$ over any closed interval $[a, b]$, $0 < a < b$.

$$\begin{aligned} \int_a^b x\,dx &= \int_a^0 x\,dx + \int_0^b x\,dx && \text{Rule 5} \\ &= -\int_0^a x\,dx + \int_0^b x\,dx && \text{Rule 1} \\ &= -\frac{a^2}{2} + \frac{b^2}{2}. && \text{Example 4} \end{aligned}$$

In conclusion, we have the following rule for integrating $f(x) = x$:

$$\boxed{\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \qquad (1)}$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \tag{1}$$

This remains true even when both *a* and *b* are negative or when *a* is negative

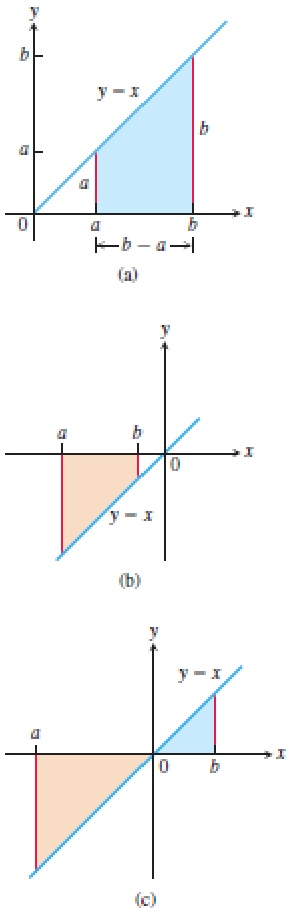


FIGURE 5.13 (a) The area of this trapezoidal region is $A = (b^2 - a^2)/2$. (b) The definite integral in Equation (2) gives the negative of the area of this trapezoidal region. (c) The definite integral in Equation (2) gives the area of the blue triangular region added to the negative of the area of the tan triangular region.

DEFINITION If f is integrable on $[a, b]$, then its **average value on $[a, b]$**, also called its **mean**, is

$$\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Notice that av(f) is the height of the rectangle with base $(b - a)$ which has the same area as the integral.

EXAMPLE 5 Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution We recognize $f(x) = \sqrt{4 - x^2}$ as a function whose graph is the upper semicircle of radius 2 centered at the origin (Figure 5.15).

Since we know the area inside a circle, we do not need to take the limit of Riemann sums. The area between the semicircle and the x-axis from -2 to 2 can be computed using the geometry formula

$$\text{Area} = \frac{1}{2} \cdot \pi r^2 = \frac{1}{2} \cdot \pi(2)^2 = 2\pi.$$

Because f is nonnegative, the area is also the value of the integral of f from -2 to 2,

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi.$$

Therefore, the average value of f is

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Notice that the average value of f over $[-2, 2]$ is the same as the height of a rectangle over $[-2, 2]$ whose area equals the area of the upper semicircle (see Figure 5.15). ■

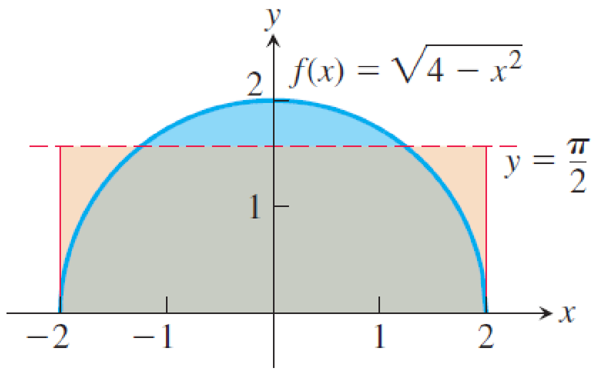


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

Week 6

Assignment 6

4.7: #28,33,46,55,85,95,98,100,103
5.1: #4,10
5.2: #28,30,32,35,38,42,46
5.3: #5,6,10,22,69,72,76,81,82,86

The questions above need to be submitted.
Deadline: 10 PM, Friday, Oct 27 --- solutions should be submitted online on Blackboard.

Required Reading (Textbook)

- Section 4.7
- Sections 5.1 to 5.3