Slide 15-Determinants I

MAT2040 Linear Algebra

Recall: to determine a square matrix is invertible or not, there are many different ways, see Theorem 13.14 for details, as I mentioned, you need to know some of the most important criteria, such as using its reduced row echelon form, solving the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, the linear independence of its column vectors.

It is possible to define a number called determinant that tests the square matrix is invertible or not.

Let A be a $n \times n$ matrix and its associated determinant is a scalar, denoted by $\det(A)$ or $|A|$ ($|A|$ is not the absolute value but the determinant of the matrix A). **The number $\det(A)$ will tell you whether the matrix is invertible (nonsingular) or not.**

Examples:

Case 1. 1×1 **Matrix** If $A = [a_{11}]$ is a 1×1 matrix, then A is nonsingular if and only if $a_{11} \neq 0$, thus we define

$$\det(A) \triangleq |A| = |[a_{11}]| = a_{11}.$$

$\det(A) = a_{11} \neq 0$ means that A is invertible and $\det(A) = a_{11} = 0$ means that A is not invertible. Thus, $\det(A)$ determines the matrix A is invertible or not.

Case 2. 2×2 **Matrix** Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

From previous lecture (theorem 8.7 (Equivalent Conditions for the Non-singular Matrix)), A will be nonsingular if and only if it is row equivalent to the identity matrix.

(I) $\mathbf{a_{11} \neq 0}$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{R_2 \to -\frac{a_{21}}{a_{11}} R_1 + R_2} \begin{bmatrix} a_{11} & a_{12} \\ 0 & \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} \end{bmatrix}$$

A is nonsingular if and only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$

(II) $\mathbf{a_{11} = 0}$

$$\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{bmatrix}$$

It will row equivalent to the identity matrix if and only if $a_{12}a_{21} \neq 0$.

For both (I) and (II): A **is nonsingular if and only if** $\mathbf{a_{11}a_{22} - a_{12}a_{21} \neq 0}$.
Thus we can define

$$\begin{aligned} \det(\mathrm{A}) &\triangleq a_{11}a_{22} - a_{12}a_{21} \\ &= a_{11}\det([a_{22}]) - a_{12}\det([a_{21}]) \\ &= a_{11}\det(M_{11}) - a_{12}\det(M_{12}) \end{aligned}$$

where M_{11} is the **submatrix** obtained by deleting the first row and first column while M_{12} is the **submatrix** obtained by deleting the first row and second column.

Observation: Without considering the sign, the determinant of 2×2 matrix is a summation of 2 terms, each term is a multiplication of the element in the first row and the determinant of the corresponding submatrix obtained by deleting the row and the column containing the element.

Generalization to $n \times n$ matrix: The definition of the determinant for a $n \times n$ matrix can be defined **recursively** as the a summation of n terms, where, without considering the sign, each term is a multiplication of the element in the first row and the determinant of the submatrix obtained by deleting the row and the column containing the element.

Remark: actually, the sign of each term needs to be considered. It will be useful to introduce the **cofactor** for the entry in the matrix.

Definition 15.1 (**Minors and Cofactor**) Let $A = (a_{ij})_{n\times n}$ be a square matrix and let M_{ij} denotes the $(n-1)\times(n-1)$ matrix obtained by deleting the row and column containing a_{ij}. The $\det(M_{ij})$ is called the **minor** of a_{ij}, and the **cofactor** A_{ij} of a_{ij} is given by

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

Using the definition of cofactor, the determinant of 2×2 matrix can be written as

$$\det(A) = a_{11}A_{11} + a_{12}A_{12}$$

Remark. M_{ij} can be obtained just by deleting the ith row and jth column of A, which is a **submatrix** of A.

Definition 15.2 (**Determinant**) (**Recursive definition**) Let $A = (a_{ij})_{n\times n}$ be a square matrix, the determinant $\det(A)$ of A is a scalar defined recursively as

$$\det(A) = \begin{cases} a_{11}, & \text{if } n = 1, \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}, & \text{if } \quad n > 1, \end{cases} \tag{1}$$

where

$$A_{1j} = (-1)^{1+j}M_{1j}, \; j = 1, \cdots, n$$

This is the **cofactor expansion** formula or the **Laplace expansion formula** for the determinant.

In fact, the expansion is not limited to the first row, it can be done along any row.

Without proof, the following two theorem are provided. (If you want to know the detail proof, please refer to Beezer's notes P347-350).

Theorem 15.3 (**Expansion along any row/column**) Let $A = (a_{ij})_{n\times n}$ be a square matrix, the determinant $\det(A)$ can be expressed as a cofactor expansion along any row:

$$\begin{aligned}\det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}\\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}\end{aligned}$$

for any $i = 1, 2, \cdots, n, j = 1, 2, \cdots, n$.

Example

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1)^{1+1} a \det([d]) + (-1)^{1+2} b \det([c]) = ad - bc.$$

$$\begin{aligned} &\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \\ =&(-1)^{1+1} a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + (-1)^{1+2} b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (-1)^{1+3} c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ =&a(ei - hf) - b(di - gf) + c(dh - ge) \\ =&aei + bfg + cdh - cge - ahf - bdi \end{aligned}$$

Example 15.4 Compute the determinant of following matrix:

$$A = \begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix}$$

Expanding along the fourth row yields,

$$\begin{aligned}
\det(A) =&(4)(-1)^{4+1}\begin{vmatrix} 3 & 0 & 1 \\ -2 & 0 & 1 \\ 3 & -2 & -1 \end{vmatrix} + (1)(-1)^{4+2}\begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\
&+(2)(-1)^{4+3}\begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} + (6)(-1)^{4+4}\begin{vmatrix} -2 & 3 & 0 \\ 9 & -2 & 0 \\ 1 & 3 & -2 \end{vmatrix} \\
=&(4)(-1)(10) + (1)(1)(-22) + (2)(-1)(61) + (6)(1)(46) = 92.
\end{aligned}$$

Expanding along the third column yields,

$$\begin{aligned}\det(A) =&(0)(-1)^{1+3}\begin{vmatrix} 9 & -2 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} + (0)(-1)^{2+3}\begin{vmatrix} -2 & 3 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} \\ &+(-2)(-1)^{3+3}\begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 4 & 1 & 6 \end{vmatrix} + (2)(-1)^{4+3}\begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} \\ =&0 + 0 + (-2)(1)(-107) + (2)(-1)(61) = 92.\end{aligned}$$

Property 15.5 Suppose A is an upper triangular square matrix, i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Then

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

Proof. Expand the determinant along the first column recursively gives

$$
\begin{aligned}
\det(A) =& a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}\right) \\
=& a_{11}a_{22}\det\left(\begin{bmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}\right) \\
=& \cdots \\
=& a_{11}a_{22}\cdots a_{n-1n-1}\det([a_{nn}]) \\
=& a_{11}a_{22}\cdots a_{n-1n-1}a_{nn}
\end{aligned}
$$

Remark: $\det(I_n) = 1$.

Theorem 15.6 (**Determinant of Transpose**) If A is a $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Proof. By using mathematical induction.
(1) When $n = 1$, $A = (a_{11})$, A is symmetric. $\det(A^T) = \det(A)$.
(2) Now assume that the result holds for all $k \times k$ matrices and A is a $(k+1) \times (k+1)$ matrix. By the definition:

$$\det(A) = a_{11}(-1)^{1+1}\det(M_{11}) + \cdots + a_{1n}(-1)^{1+n}\det(M_{1n})$$

All $M_{11}, \cdots, M_{1n}$ are $k \times k$ matrices, by the induction assumption, one has $\det(M_{11}) = \det(M_{11}^T), \cdots, \det(M_{1n}) = \det(M_{1n}^T)$. Thus,

$$\det(A) = a_{11}(-1)^{1+1}\det(M_{11}^T) + \cdots + a_{1n}(-1)^{1+n}\det(M_{1n}^T)$$

The right-hand side is exactly the expansion of $\det(A^T)$ along the first column. ($M_{11}^T, \cdots, M_{1n}^T$ are submatrices of A^T corresponding to $a_{11}, \cdots, a_{1n}$ (the entries of first column of A^T))
Thus, $\det(A) = \det(A^T)$.

Property 15.7 (**Determinant with Zero Row or Column**) Suppose that A is a square matrix with a zero row/column (zero row/column means that the entries in the row/column are all zeros). Then $\det(A) = 0$.

Proof. Suppose the ith row of A are all zeros, then expand along this row. one will get that

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = 0A_{i1} + 0A_{i2} + \cdots + 0A_{in} = 0.$$

Similar result can be obtained if A has a zero column.

Example 15.8 (1) Compute

$$\begin{vmatrix} a & b & c & d \\ e & 0 & 0 & 0 \\ f & 0 & 0 & 0 \\ g & 0 & 0 & 0 \end{vmatrix} (Expand\ along\ 4th\ row)$$

$$=(-1)^{4+1}g \begin{vmatrix} b & c & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} (the\ 3rd\ row\ is\ entirely\ zeros)$$

$$=0$$

Example 15.8 (2) Compute

$$\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} (\textit{Expand along 1st column})$$

$$=(-1)^{1+4}2 \begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} (\textit{Expand along 3rd column})$$

$$=(-2)(-1)^{3+3}3 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= -6(2 * 5 - 3 * 4) = 12$$

Example 15.8 (3) always expanding along the first row

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = (-1)^{1+4} a_{14} \begin{vmatrix} 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

$$= -a_{14}(-1)^{1+3} a_{23} \begin{vmatrix} 0 & a_{32} \\ a_{41} & a_{42} \end{vmatrix}$$

$$= -a_{14} a_{23} (-1)^{1+2} a_{32} |a_{41}| = a_{14} a_{32} a_{23} a_{41}$$

Property 15.9 (**Determinant with Equal Rows or Columns**) Suppose that A is a square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$.

Proof. Using mathematical induction, this can be easily proved. I leave as an exercise. Also see exercise 9 in section 2.1 of Steven's book.