MAT1002 Midterm Examination

Saturday, March 25, 2023

Time: 9:30 - 11:30 AM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The total score of this examination is ***112****.*

3. There are ***eleven*** *questions (with parts) in total.*

4. The symbol [***N***] *at the beginning of a question indicates that the question is worth N points.*

5. Answer all questions on the ***answer book****.*

6. Show your intermediate steps ***except Question 1*** *— answers without intermediate steps will receive minimal (or even no) marks.*

MAT1002 Midterm Questions

1. [**18**] Short questions: no intermediate step is required.

(i) The Taylor series of the function $f(x)$ centered at $x = 2$ is given by $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+5}{n!}(x-2)^n$. What is the value of the third derivative $f'''(2)$?

A. 5

B. 9/2

C. 9

D. 14/6

E. 14

(ii) Find the third degree Taylor polynomial of $f(x) = e^x \sin(5x)$ centered at $x = 0$.

(iii) True (T) or False (F)? Let $\sum_{k=0}^{\infty} a_k x^k$ be the Taylor series of a given function f centered at $x = 0$. Suppose $\sum_{k=0}^{\infty} a_k x^k$ converges on $\mathbb{R}$. Then $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $\mathbb{R}$.

(iv) True (T) or False (F)? If for non-zero vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in 3D space, $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{c}$, then $\boldsymbol{b} = \boldsymbol{c}$.

(v) For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, if $|\boldsymbol{a}| = 3$, $|\boldsymbol{b}| = 8$, $|\boldsymbol{a}+\boldsymbol{b}| = 9$, then $|\boldsymbol{a}-\boldsymbol{b}| = (\quad)$.

(vi) Sketch the polar curve $r = \cos(2\theta)$, $\theta \in [0, \pi]$, on the xy-plane.

2. [**12**] If a series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms ($a_n \geq 0$) converges, which of the following series must always converge? If yes, give a reason. If no, provide an example.

(i) $\displaystyle\sum_{n=1}^{\infty} \cos(a_n)$

(ii) $\displaystyle\sum_{n=1}^{\infty} a_n^2$

(iii) $\displaystyle\sum_{n=1}^{\infty} \sqrt{a_n}$

3. [**16**] For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges.

(i) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(ii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)}\right)^{\frac{1}{n}}$

(iii) $\displaystyle\sum_{n=48}^{\infty} \frac{(-1)^n}{n \ln n}$

(iv) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(1 \cdot 3 \cdot \ldots \cdot (2n-1))}{4^n 2^n n!}$

4. [**6**] Evaluate the limit $\displaystyle\lim_{x \to 0^+} \frac{e^{2x^9} - 1 - 2x^8 \sin x}{\sqrt{1 + x^{11}} - 1}$.

5. [**8**] Consider the function $f(x) = \dfrac{3x^2}{\sqrt{4 - x^2}}$.

(i) Find the Taylor series of $f(x)$ centered at $x = 0$, and express your answer in sigma notation.

(ii) Find the radius of convergence for the series in (i).

6. [**6**] Find an optimal (smallest) n to ensure that the n-th degree Taylor polynomial of $\cos x$ centered at $x = 0$ gives an approximation of $\cos x$ on the interval $[-1, 1]$ with error magnitude no bigger than 6×10^{-3}.

7. [**4+6**] Consider the plane that passes the origin $(0, 0, 0)$ and is parallel to the following two lines:

$$l_1 = \begin{cases} x = 1 \\ y = -1 + t \\ z = 2 + t \end{cases} \qquad l_2 = \begin{cases} x = -1 + s \\ y = -2 + 2s \\ z = 1 + s \end{cases}.$$

(i) Find an equation of the plane.

(ii) Compute the **shortest** distances d_1 and d_2 from the plane to the lines l_1 and l_2, respectively.

8. [**6**] (Music Recommendation System.) Suppose in a famous music app, each song is coded as a 5-dimensional vector, say pop, rap, etc.. The app also keeps a record of your listening history and compute the average of the songs you listen, resulting into a 5-dimensional vector as your taste. The app will measure the angle of the vector of one song and the vector of your taste. The smaller the angle is, the better the song suits you.

Suppose your taste vector is $\langle 3, 1, 2, 0, 5\rangle$, and there are three musical candidates with their vectors

$$\boldsymbol{r}_1 = \langle 1, 0, 0, 6, 5\rangle, \quad \boldsymbol{r}_2 = \langle 2, 2, 6, 0, 3\rangle, \quad \boldsymbol{r}_3 = \langle 2, 1, 1, 0, 6\rangle.$$

Which of the three candidates above will be recommended to you by the app?

9. [**12**] Consider the cardioid $r = 2(1 + \cos\theta)$.

(i) Compute the arc length of the cardioid.

(ii) Find the area enclosed by the cardioid.

(iii) A particle moves along the cardioid counterclockwise. Find the angle between the velocity vector and the position vector at the point $P\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$.

10. [**8**] Consider a cycloid that is traced by a point P on a wheel of radius a rolling along the y-axis.

(i) Assume the cycloid intersects with x-axis at the point $P_1(2a, 0)$. Write a parametric equation for this cycloid.

(ii) Find the curvature of the cycloid at the point $P_2\left(a, a\left(1 + \frac{\pi}{2}\right)\right)$.

11. [**6+4**] (Ideal projectile motion.) A cannonball is fired with initial speed v_0 from the point (x_0, y_0) with an angle α made with the positive x-axis (which represents the ground). Here, $0 < \alpha < \pi/2$ and $y_0 > 0$. Let g be the acceleration due to gravity and let $t = 0$ be the initial time.

(i) **Deduce** the position $(x(t), y(t))$ of the cannonball at time t.

(ii) Find the time t_1 at which the cannonball hits the ground.