PHY1001: Mechanics (Week 3)

1 Work and Kinetic Energy

In the following, we will study the idea of work and energy, and use them to help us solve physics problems. Sometimes, it is a very useful tool and is much easier than using Newton's laws directly.

Let us first use the motion with constant acceleration as an example to see the idea of kinetic energy and the work-energy theorem.

- In the case of constant acceleration
  1. Displacement as function of t:
  $$x - x_0 = v_0 t + \frac{1}{2}at^2$$
  2. Velocity as function of t:
  $$v = v_0 + at$$
  3. Use $t = \dfrac{v - v_0}{a}$ to eliminate t
  $$x - x_0 = v_0\frac{v - v_0}{a} + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 = \frac{1}{2a}\left(v^2 - v_0^2\right)$$
  4. Multiply both sides with $F = ma$:
  $$F(x - x_0) = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$
- If we say that $F(x - x_0)$ is the work done by the net force F, and $K = \frac{1}{2}mv^2$ is the kinetic energy, then we can conclude that the work done by the net force on an object equals the change in the object's kinetic energy, which known as the work-energy theorem.

In general, from Newton's second law of motion, we can derive the work-energy theorem for varying force F_{net} as follows

1. Use Newton's second law of motion:
$$F_{\text{net}} = ma = m\frac{dv}{dt}$$
2. Multiply both sides with $dx = vdt$:
$$F_{\text{net}}dx = m\frac{dv}{dt}vdt = mvdv$$
3. Use the famous trick $vdv = \frac{1}{2}d(v^2)$ and integrate on both sides
$$\int_{x_0}^{x} F_{\text{net}}dx = \int_{v_0}^{v} mvdv = \int_{v_0}^{v} \frac{m}{2}dv^2 = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$
4. For 3-dimensional case, we can generalize the above derivation by using the dot product and obtain:
$$\int_{P_1}^{P_2} \vec{F}_{\text{net}} \cdot \vec{v}dt = \int_{P_1}^{P_2} \vec{F}_{\text{net}} \cdot d\vec{r} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2,$$
where $\vec{v} \cdot \vec{v} = v^2 = v_x^2 + v_y^2 + v_z^2$. The aforementioned trick becomes $\vec{v} \cdot d\vec{v} = \frac{1}{2}d(\vec{v}^2)$. The work $W = \int_{P_1}^{P_2} \vec{F}_{\text{net}} \cdot d\vec{r}$ is then path dependent where P_1 and P_2 stand for the initial and final position of the curved path of the motion.

Some additional comments:

- The introduction of dot product. How does it arise? Kinetic energy only depends on $F_{||}$. $F_{\perp}$ changes the direction of $\vec{v}$, has no effect on the magnitude of $|v|$.
- $\vec{F}_{\text{net}}$ now does not have to be a constant.
- If force is perpendicular to the path, then it does not do any work. Consider the example of uniform circular motion.
- The W can be negative or positive depending on the dot product between $\vec{F}_{\text{net}}$ and the path as we know the dot product can be either negative or positive.
- When W is negative, the final kinetic energy is less than the initial kinetic energy. When W is positive, the kinetic energy increases.
- The unit of energy and work is Joule. 1 Joule = 1 kg·(m/s)^2.

1.1 Spring and work

Let us apply the idea of work to the stretched spring. First, suppose spring pulls an object and moves from x_1 to x_2. According to the Hooke's law $F_x = -kx$, the work done by a spring is then

$$W_{\text{spring}} = \int_{x_1}^{x_2} F_x dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2, \tag{1}$$

where the equilibrium position (at which the spring is neither stretched nor compressed) is set to be $x = 0$.

Second, let us try to use hand to pull the spring, and then the force exerted to the spring is kx. (action and reaction forces) Now your hand must do work if the spring is pulled from x_1 to x_2

$$W_{\text{hand}} = \int_{x_1}^{x_2} kxdx = -\frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2. \tag{2}$$

You always find that $W_{\text{hand}} = -W_{\text{spring}}$ due to Newton's third law. In practice, we need to be always clear about which object is doing the work and which one is receiving the work.

1.2 Power

Power is the time rate at which work is done.

$$P = \frac{dW}{dt}, \quad \text{unit: Watt, W.} \tag{3}$$

Like Joule (J=kg m^2/s^2), Watt (W=J/s) is a derived unit, which is a unit of measurement derived from the SI base units. Sometimes we use kw, Mw or horse power (1hp =746 W).

2 Energy Conservation

2.1 Potential Energy

The concept of potential energy: a measure of potential or possibility for work to be done. For example, a stone on the top of a high hill have the potential to do a lot of work when it rolls downhill.

- Gravitational Potential Energy: Consider an object is moved from y_1 to y_2, then the gravitational force $-mg$ does the work in this process $W_g = -mg(y_2 - y_1) = mg(y_1 - y_2)$. If $y_1 > y_2$ ($y_1 < y_2$), then gravity does positive (negative) work. Thus, we can define the gravitational potential energy $U_g(y) = mgy$ (by introducing the definition U_g)
$$W_g = -[U_g(y_2) - U_g(y_1)] = -\Delta U_g. \tag{4}$$
- Understanding the minus sign: the decrease of potential energy is used to do work to increase other energy, for example, the kinetic energy.
- Any constant can be added to the potential energy U, which has no physical significance. It is the change in U that matters.
- Elastic Potential Energy: A body is called elastic if it returns to its original shape and size after being deformed. For example, a spring or rubber band can be viewed as elastic, and they can store the so-called elastic potential energy. As we discussed earlier, the work can be done by a spring is
$$W_{\text{spring}} = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 = U_{el}(x_1) - U_{el}(x_2) = -\Delta U_{el}. \tag{5}$$

2.2 Energy Conservation

In general, we can define the total potential energy to be
$$U = U_g + U_{el}, \tag{6}$$
If all the work comes from the potential energy, then based on the work-energy theorem $\Delta K = W_{tot} = -\Delta U$, we can obtain $\Delta K + \Delta U = 0$ and
$$K_1 + U_1 = K_2 + U_2 = \text{Constant}, \tag{7}$$
which is known as the conservation of mechanical energy. The total mechanical energy of the system E can also defined as
$$E = K + U = \frac{1}{2}mv^2 + mgy + \frac{1}{2}kx^2. \tag{8}$$
The conservation of mechanical energy indicates that E always has the same value during a physical process, which is also called the conserved quantity. In this case, we often say that kinetic energy can be converted into potential energy and stored in terms of potential energy, and we always have in mind that later we can retrieve it (get it back) again as kinetic energy. For example: a ball is thrown up in the air. If there is no air friction, the total energy is conserved while the kinetic energy and the potential energy are converted into each others during this process.

If there are other sources which contribute to the total work W, then the mechanical energy is not conserved, we need to modify the above formula accordingly
$$W_{\text{other}} = \Delta E = \Delta K + \Delta U \tag{9}$$

More generally, we have the Law of conservation of energy:
The total energy of the universe is constant. Energy can be converted from one form to another, or transmitted from one region to another, but energy can never be created or destroyed.

- No exception to the law of conservation of energy has ever been found. Although we learned it based on Newton's laws, the law of conservation of energy appears to be more fundamental!
- In this law, we have banished (given up) the concept of work, since work can be always viewed in terms of the conservation of energy from one form to another.
- Conservative Forces vs Non-conservative forces. Conservative force: A force that offers the opportunity of two-way conversion between kinetic and potential energies is called a conservative force. Typically, the gravitational force, spring force and static electric force are conservative forces. Non-conservative force: In contrast, the kinetic friction and fluid resistance always do negative work, and the work they do depends on the path. They are irreversible and they always cause mechanical energy to be lost or dissipated, therefore they are also called dissipative force.
- Non-conservative forces often take away some mechanical energy (frictions) or inject some mechanical energy (the firing of a gun) into the system. The friction and resistance convert kinetic energies into the internal energies of the contacting bodies, causing the increase of temperatures of these bodies, especially near their contacting surface.
- Mathematical criteria of conservative force. There are three equivalent conditions for conservative forces.
1. The curl of F is zero: $\vec{\nabla} \times \vec{F} = 0$.
2. Loop work is zero: $\oint \vec{F} \cdot d\vec{s} = 0$.
3. F can be written as $\vec{F} = -\vec{\nabla} U$.
These three statements can be used to identify conservative force. Since they are equivalent, satisfying one is already sufficient.

  In terms of words, work done by a conservative force only depends on the initial and final positions (zero if initial and final are the same). It is independent of path and is reversible (no loss).