Slide 3: Linear Systems and Matrices III

MAT2040 Linear Algebra

Three Possible Cases

$$[A|\mathbf{b}] \xrightarrow{\text{row operations}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

Unique solution.

$$[A|\mathbf{b}] \xrightarrow{\text{row operations}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinitely many solutions.

$$[A|\mathbf{b}] \xrightarrow{\text{row operations}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

No solution.

Definition 3.1 (Consistence) A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

Fact 3.2

Let $A\mathbf{x} = \mathbf{b}$ be a linear system with n variables, Suppose $[A|\mathbf{b}]$ is row equivalent to a matrix $[B|\mathbf{c}]$ in reduced row-echelon form. Then by the structure of the the reduced row echelon form, the following conclusion is valid.

(1) $A\mathbf{x} = \mathbf{b}$ is inconsistent

$\Leftrightarrow$ the last nonzero row of $[B|\mathbf{c}]$ is $(0, \cdots, 0, 1)$
$\Leftrightarrow$column $n+1$ is a pivot column.

(2) $A\mathbf{x} = \mathbf{b}$ is consistent

$\Leftrightarrow$ column $n+1$ is not a pivot columm.

$$[A|\mathbf{b}] \xrightarrow{\text{elementary row operations}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

The corresponding linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

$$[A|\mathbf{b}] \xrightarrow{\text{elementary row operations}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

$$[A|\mathbf{b}] \xrightarrow{\text{elementary row operations}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

The corresponding linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

Fact 3.3 (**Solution set for consistent linear systems**)
Assume the linear system $A\mathbf{x} = \mathbf{b}$ with n variables is consistent. Suppose $[A|\mathbf{b}] \xrightarrow{\text{elementary row operations}} [B|\mathbf{c}](RREF)$ and B has r nonzero rows (B has r pivot columns). Then $r \leq n$.

(1) $r = n$, the system has a unique solution.

(2) $r < n$, the system has infinitely many solutions and the solution set can be described by $n - r$ **free/independent variables** (**corresponding to the nonpivot columns in** B).

Remark: For consistent linear system, $r =$number of pivot columns=number of nonzero rows in the EERF=number of leading 1s.
Note: r is the number of "true equations" of the linear system, and there are $n - r$ redundant equations.

Example: for the case:

$$[A|\mathbf{b}] \xrightarrow{\text{elementary row operations}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right]$$

$r = n = 3$, the system has a unique solution.

Example: for the case:

$$[A|\mathbf{b}] \xrightarrow{\text{elementary row operations}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution can be described by

$$x_1 = 3 - x_3,$$
$$x_2 = 2 + x_3.$$

where x_3 is the **free variable** (**independent variable**) corresponding to nonpivot column in B, while x_1, x_2 are **dependent variables** corresponding to pivot columns in B.

Example 3.4 Find the solution of the following system:

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 8 \\ -3 & 4 & -5 & -6 & -12 \\ 1 & 1 & 4 & -5 & 4 \end{array}\right] \xrightarrow{\text{elemental row operations}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 4 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last column is not a pivot column, so it is a consistent system. Columns 1 and 2 are the pivot columns while columns 3 and 4 are non-pivot columns.

Thus, x_1, x_2 are dependent variables while x_3, x_4 are independent variables. In fact, $x_1 = -3x_3 + 2x_4 + 4, \quad x_2 = -x_3 + 3x_4$

Combining the above two facts, one has

Fact 3.5 (**Possible Solution Sets for Linear Systems**) For a system of linear equations $A\mathbf{x} = \mathbf{b}$, it can can have (1) a unique solution or (2) infinitely many solutions or (3) no solution.

Definition 3.6 (**Homogeneous System**) A system of linear equations $A\mathbf{x} = \mathbf{b}$ is called **homogeneous** if $\mathbf{b} = \mathbf{0}$ (the zero vector).

A homogeneous system looks like this:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0,\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0.
\end{aligned}$$

$\mathbf{0}$ is a solution to such a system, i.e., all variables equal to zero $(x_1 = x_2 = \cdots = x_n = 0)$ is a solution. This solution is called the **trivial** solution.

Property 3.7 (**Homogeneous systems are always consistent**)
Any homogeneous linear system is consistent.

Example 3.8 Determine the solutions for the following linear systems.

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= 0, \\ 5x_1 + 5x_2 + 7x_3 &= 0, \\ x_1 + 4x_3 &= 0. \end{aligned}$$

The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Hence it has only one solution, i.e., the trivial solution.

Example 3.9 Determine the solutions for the following linear systems.

$$
\begin{aligned}
x_1 - x_2 + 2x_3 &= 0, \\
2x_1 + x_2 + x_3 &= 0, \\
x_1 + x_2 &= 0
\end{aligned}
$$

The reduced row echelon form of the augmented matrix is

$$
\left[\begin{array}{ccc|c}
\boxed{1} & 0 & 1 & 0 \\
0 & \boxed{1} & -1 & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

The system is consistent. It has $n - r = 3 - 2 = 1$ free variable. The solution set is

$$
S = \left\{ \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \middle| t \in R \right\}.
$$

Theorem 3.10 (**Underdetermined homogeneous systems have infinite solutions**)
An **underdetermined homogeneous** linear system has **infinite** solutions.

For underdetermined homogeneous linear system ($m < n$):

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0,\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0,\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0.
\end{aligned}$$

Suppose $[A_{m\times n}|\mathbf{0}] \xrightarrow{\text{elementary row operations}} [B_{m\times n}|\mathbf{0}](RREF)$ and B has r nonzero rows, also B both have r pivot columns.
$\#$ of pivot columns in $B = \#$ of nonzero rows in $B{=}r \leq m < n$.

Example 3.11 Find the solution for the following homogeneous system.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0 \end{aligned}$$

$m = 3 < n = 4$, thus the above homogeneous linear system must have infinitely many solutions. In fact

$$\begin{bmatrix} 2 & 1 & 7 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix} \xrightarrow{\text{elementary row operations}} \begin{bmatrix} \boxed{1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus

$$x_1 = -3x_3 + 2x_4, \quad x_2 = -x_3 + 3x_4$$