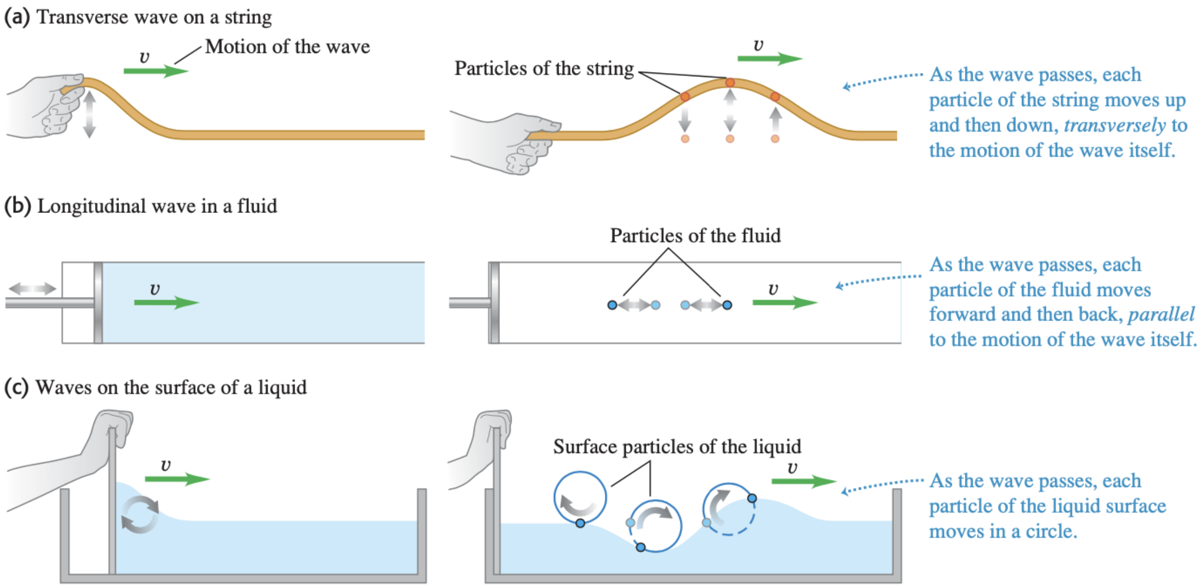
PHY1001: Mechanics (Week 12-14)

In this chapter, we will discuss waves, which is defined as the oscillation (in time) that travels in space and transfers energy from one place to another. The oscillation with time and wave propagation are described by the wave equation.

When the displacement of the medium are perpendicular or transverse to the propagation direction of the wave along the medium, we define this type of wave as the transverse wave. For example, the wave travels along a string and the electromagnetic wave (light wave) are transverse waves.[1]

When the displacement of the medium are along the same direction as the propagation direction of the wave, we name this type as the longitudinal wave. For example, the sound wave is a longitudinal wave. Earthquake waves have both longitudinal and transverse components.



Three ways to make a wave that moves to the right: (a) The hand moves the string up and then returns, producing a transverse wave. (b) The piston moves to the right, compressing the gas or liquid, and then returns, producing a longitudinal wave. (c) The board moves to the right and then returns, producing a combination of longitudinal and transverse waves.

1 Waves in general

Usually, there are three types of waves.

1. A mechanical wave is a disturbance that travels through some material or substance called the medium for the wave. Ripples on a pond, musical sounds, earthquakes (seismic tremors) are all wave phenomena. As the wave travels through the medium, particles which make up the medium undergo displacement and carry energy as the wave propagates along the direction of the wave.

2. Electromagnetic waves–including light, radio waves, infrared and ultraviolet radiation, and X rays–can propagate even in empty space (vacuum), where there is no medium. These waves satisfy the electromagnetic wave equation derives from Maxwell's equations and travel at speed of light in vacuum.

3. Matter waves. In quantum mechanics, according to the de Broglie hypothesis (1924), there are waves are associated with electrons, protons, and other fundamental particles, and even atoms and molecules. All matter exhibits properties of both particles and waves, and there is a relationship between the wavelength, λ and the momentum, p, through the Planck constant, $h = 6.63 \times 10^{-34}$ J.s:

$$\lambda = \frac{h}{p}. \tag{1}$$

Because we commonly think of these particles as constituting matter, such waves are called matter waves. The wave equation for the matter wave in QM is known as the the Schrödinger equation.

Wave-particle duality: every particle may be partly described in terms of not only particles, but also of waves. Neither particle nor wave can fully explain the physics, but together they do. This is the quantum nature of physical entities. They are like two sides of the same coin, or two faces of the same reality. We have to look at the particle (object) from different angles to get the full picture of the reality, because we are living in the same reality.

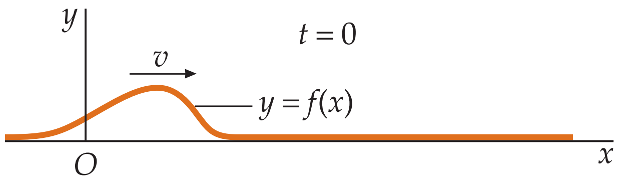
[1]Strictly speaking electromagnetic waves include light, radio waves, X rays, gamma rays, and microwaves, among others. The various types of electromagnetic waves differ only in wavelength and frequency.

In this course, we will only focus on the Mechanical waves which have three interesting properties:

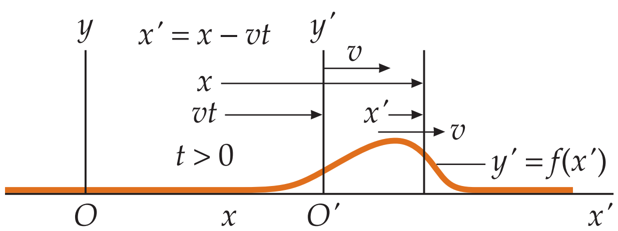
- The disturbance travels or propagates with a definite speed through the medium, which is called the speed of propagation, or simply the wave speed v.
- The medium itself does not travel through space; its individual points undergo transverse or longitudinal motions around their equilibrium positions. It is the overall pattern or shape of the wave disturbance is what travels.
- Last but not least, to create a disturbance, we have to put in energy by doing work on the system, and let the wave motion transport this energy as the wave travels. It is important to note that waves transport energy, but not matter, from one region of space to another.

Wave properties

Wave function: The displacement of the medium from the EP as the function of space and time (x and t).



(a)



(b)

Above figure (a) shows a pulse (disturbance) on a string at $t=0$. The shape of the string at this moment can be described by the function $y=f(x)$. At some later time, the pulse is further down the string which can be represented by the function $y=f(x-vt)$. The most convenient way to view the propagation of the wave pulse is to change coordinate system to O' in which $x'=x-vt$. The string is described in this frame by $f(x')$ for all times. Thus for fixed point on the pulse $x'=x-vt=$ constant, it travels at speed of v in the O frame as shown below

$$\frac{dx'}{dt}=0=\frac{dx}{dt}-v, \quad \Rightarrow \quad \frac{dx}{dt}=v. \tag{2}$$

Another way is to look at the peak of the pulse. Suppose the peak of $f(x)$ is at $x=x_m$, then the peak position moves to x_m+vt after time t. Therefore, the function $y=f(x-vt)$ describes a pulse moving to the right. Similarly, $y=f(x+vt)$ describes a pulse moving to the left. They both satisfies the differential wave equation

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y}{\partial x^2}. \tag{3}$$

A few comments are in order.

1. Traveling wave solutions
$$y=f(x-vt) \quad \text{moving in the } +x \text{ direction,} \tag{4}$$
$$y=f(x+vt) \quad \text{moving in the } -x \text{ direction.} \tag{5}$$
2. v is the speed of propagation of the wave. (Because v is a speed and not a velocity, it is always a positive quantity.) The function $y=f(x-vt)$ is called a wave function.
3. For waves on a string, the wave function represents the transverse displacement of the string.
4. For sound waves in air, the wave function can be the longitudinal displacement of the air molecules, or the pressure of the air.

Speed of waves

Interestingly, it turns out that for many different type of mechanical waves, the expression for wave speed has the general form

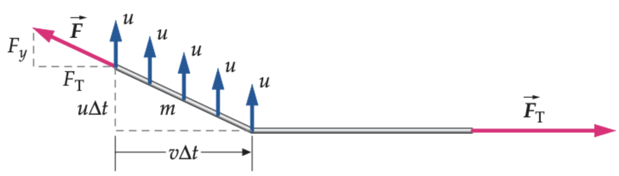
$$v=\sqrt{\frac{\text{Restoring Force}}{\text{Inertia}}} \tag{6}$$

1.0.1 Transverse wave

A general property of waves is that their speed relative to the medium depends on the properties of the medium, but is independent of the motion of the disturbance. For wave pulses on a taut rope, the speed of wave is only depending on the tension of the rope and linear mass density of the rope. If F_T is the tension (we use F_T rather than T for tension because we use F_T for the period) and μ is the linear mass density (mass per unit length), then the wave speed is

$$v=\sqrt{\frac{F_T}{\mu}}. \tag{7}$$

F_T plays the role of the restoring force, since it tends to bring the string back to its undisturbed, equilibrium configuration. The linear mass density μ provides the inertia.



There are at least three ways to derive the above wave speed.

1. Consider the propagation of the wave front movement as shown above

$$\underbrace{F_y \Delta t = F_T \Delta t \frac{u}{v} = \Delta m v = (\mu v \Delta t) u}_{\text{Propagation of the wave front}} \quad \Rightarrow v = \sqrt{\frac{F_T}{\mu}}. \quad (8)$$

   As the end of the string moves upward at constant speed u, the point where the string changes from horizontal to inclined moves to the right at the wave speed v.

2. Consider the centripetal acceleration of the wave crest (or trough) in the co-moving frame with speed v as illustrated in the textbook. $2F_T \frac{\Delta l}{2R} = (\mu \Delta l) \frac{v^2}{R}$.

3. Consider an arbitrary segment of the string as discussed below.

1.0.2 Longitudinal wave

Sound propagates due to the compression of the medium along the direction of the propagation. Similarly, the speed of a longitudinal sound wave in a fluid (including gas and liquid) is given by

$$v_s = \sqrt{\frac{B}{\rho}}, \quad (9)$$

where $B \equiv -V \frac{dP}{dV} = \frac{-\text{Pressure Change}}{\text{Fractional Volume Change}}$ is the bulk modulus. B is the measure of how resistant to compression that substance is. The bulk modulus tells us exactly how large the "restoring force" is against compression.

In addition, the expression for the bulk modulus of the ideal gas is $B = \gamma P$.

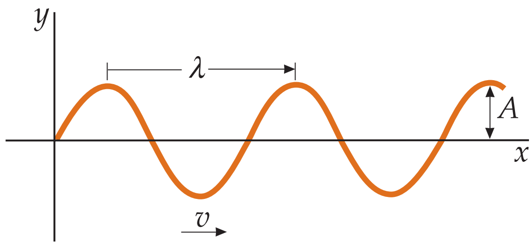
$$v_s = \sqrt{\frac{\gamma P}{\rho}} = \sqrt{\frac{\gamma R T}{M}}, \quad (10)$$

where P is the pressure of the gas, $R = 8.31 J.mol^{-1}.K^{-1}$ is the molar gas constant and T is the temperature. M is the average molar mass air is $29 \times 10^{-3} kg/mol$. Roughly speaking, the gas pressure provides the force while the inertia is given by the density of the gas. This ratio $\gamma = 5/3$ for an ideal monoatomic gas and $\gamma = 7/5$ for air, which is predominantly a diatomic gas.(The major constituents of Earth atmosphere are nitrogen and oxygen.) Sir Newton missed the γ factor which results in a speed off by about 15%, when he first studied this problem. Laplace rectified this error by correctly noticing that sound wave compression and expansion of air is an adiabatic process, not an isothermal process.

It is useful to use the above equation to estimate the speed of sound in air and find $v_s \simeq 350 m/s$ in air. Sound waves tend to travel faster in water. It is around $1500 m/s$, which is 4.3 times as fast as in air.

2 Harmonic (sinusoidal) waves

A more interesting situation develops when we give the free end of the string a repetitive, or periodic, motion. Then each particle in the string also undergoes periodic motion as the wave propagates, and we have a periodic wave. As shown below, if each point of the medium moves with simple harmonic motion, we call the corresponding wave the harmonic wave. Harmonic wave is one of the most common periodic waves.



The minimum distance after which the wave repeats itself is called the wavelength λ. Similarly, the periodicity in time is called the period T. So the speed is also given by

$$v = \frac{\lambda}{T} = f\lambda. \quad (11)$$

Since the above relations arises only from the definition of wavelength and frequency, it applies to all kinds of periodic waves. The displacement $y(x,t)$ of a harmonic wave traveling along $+x$ direction is described by

$$y(x,t) = A\sin(kx - \omega t) = A\sin\left[k\left(x - \frac{\omega}{k}t\right)\right] \quad (12)$$

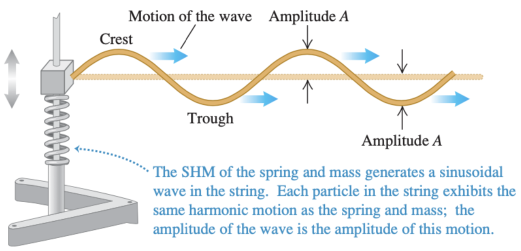
where $k = \frac{2\pi}{\lambda}$ is called the (angular) wave number which has the dimension of L^{-1} and $\frac{\omega}{k} = \frac{2\pi}{T}\frac{\lambda}{2\pi} = v$ with the angular frequency ω. This wave is propagating along the $+x$ direction, since the position $x = vt + x_0$ in the wave with the same y displacement $A\sin kx_0$ increases with t. In contrast, $y(x,t) = A\sin(kx + \omega t)$ describes the wave traveling in the $-x$ direction.

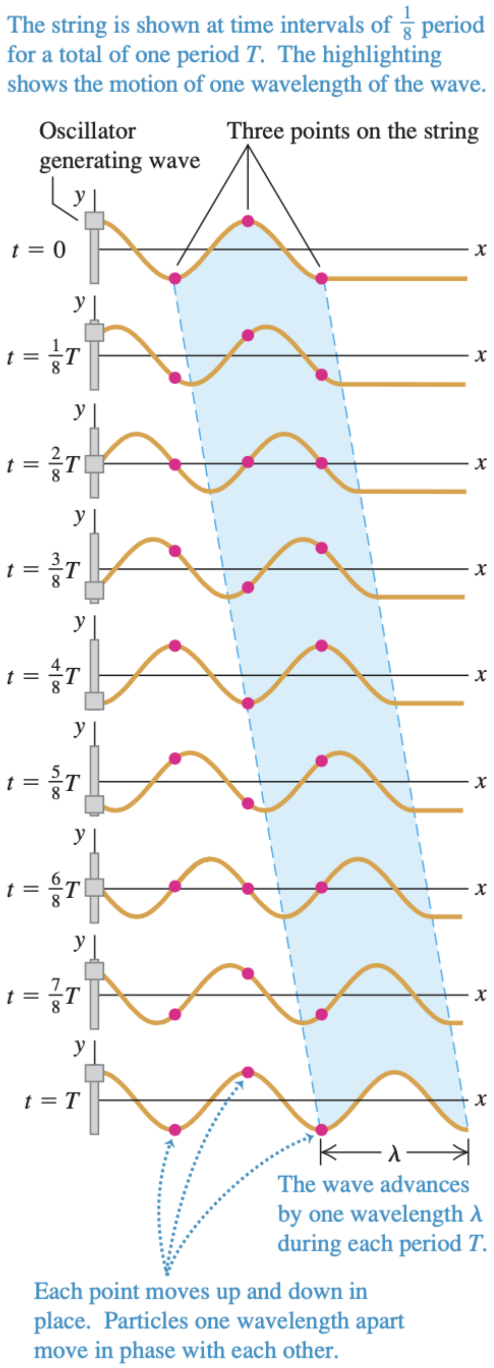
Harmonic waves are periodic in both space (x) and time (t).

- Periodic in space x:
  $y(x+\lambda,\, t) = A\sin(kx + 2\pi - \omega t) = y(x,\, t)$.

- Periodic in time t:
  $y(x,\, t+T) = A\sin(kx - \omega t - 2\pi) = y(x,\, t)$.

The sinusoidal wave generated by SHM advances steadily toward the right, as indicated by the highlighted area.





The transverse velocity and acceleration are then given by

$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = -\omega A\cos(kx-\omega t), \quad (13)$$

$$a_y(x,t) = \frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A\sin(kx-\omega t) \quad (14)$$

$$= -\omega^2 y(x,t), \quad (15)$$

respectively. Here ∂t is called partial derivative w.r.t. time t (when t varies), while the other variable x is constant because we are looking at a particular point on the string. The partial derivative is a simple extension of the derivative.

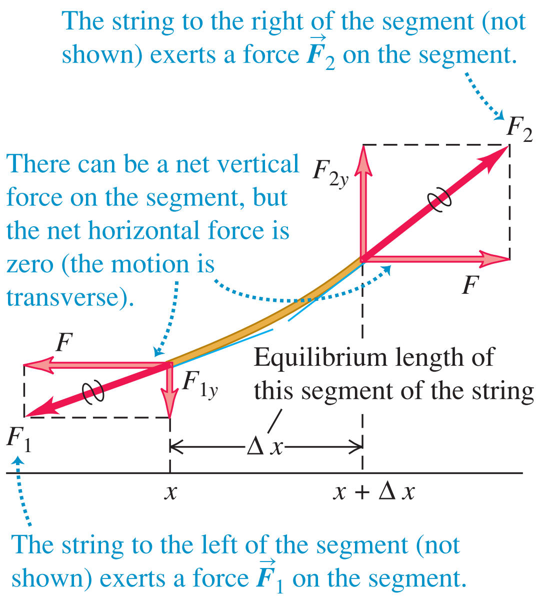
We can also compute partial derivatives of $y(x,t)$ with respect to x, holding t constant. The first derivative $\partial y(x,t)/\partial x$ is the slope of the string at point x and at time t. The second partial derivative with respect to x is the curvature of the string:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 A\sin(kx-\omega t) \quad (16)$$

$$= -k^2 y(x,t) = \frac{k^2}{\omega^2} a_y = \frac{a_y}{v^2}, \quad (17)$$

which means that a_y in a string is tightly related to the curvature of the wave!

Advanced topic: the wave equation



Consider the free-body diagram for a segment Δx of string, we can obtain the following equation of motion for this segment based on the Newton's second law in the y direction

$$F_y = F_{2y} - F_{1y} = ma_y, \quad (18)$$

where $m = \mu\Delta x$ (assuming the string can be stretched but there is no movement along the x direction), $a_y = \frac{\partial^2 y}{\partial t^2}$, and

$$F_{1y} = F_T\tan\theta_1 = F\frac{\partial y(x)}{\partial x}, \quad (19)$$

$$F_{2y} = F_T\tan\theta_2 = F\frac{\partial y(x+\Delta x)}{\partial x}, \quad (20)$$

$$F_{2y} - F_{1y} = F_T\left(\frac{\partial y(x+\Delta x)}{\partial x} - \frac{\partial y(x)}{\partial x}\right) = F_T\frac{\partial^2 y(x)}{\partial x^2}\Delta x. \quad (21)$$

We use partial derivatives since y is the function of both x and t. At the end of the day, by plugging the above results back to Eq. (18), we can obtain the wave equation

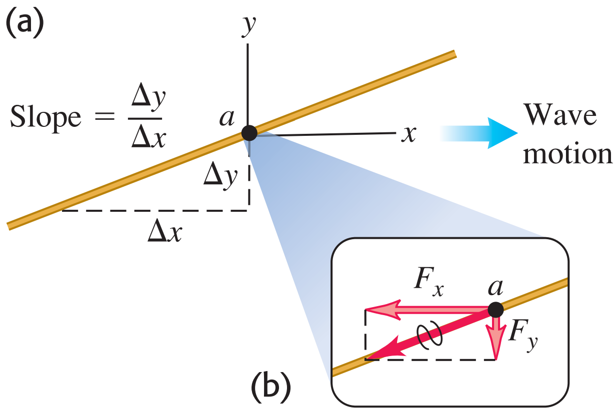
$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F_T}\frac{\partial^2 y}{\partial t^2}, \quad (22)$$

and it is straightforward to see that $y = f(x-vt)$, which describes a traveling wave with velocity v, is the solution to Eq. (22) as long as $v = \sqrt{\frac{F_T}{\mu}}$. Therefore, we can always write the wave equation as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}. \quad (23)$$

3 Energy in Wave Motion

Traveling waves carry energy, such as the energy we received from sunlight and the destructive energy released from an earth quake. To produce any wave motion, we have to apply a force to a portion of the wave medium. As the wave propagates, each portion of the medium exerts a force and does work on the adjacent portion. In this way wave can transport energy from one region of space to another.



Let us picture a wave $y(x,t) = A\cos(kx-\omega t)$ traveling from left to right (the $+x$ direction) on a string, and consider a particular point a on the string as shown above. Let us divide the string into two parts: the part to the left of point a and the string to the right of point a. We want to compute the energy per unit time transferred from the left part to the right part. That is to say that we are focusing on the work done by the left part of the string to the right part of the string. It is also important to notice that we are considering the transverse wave, which means that every point in the string has no longitudinal motion. Therefore, the corresponding power (rate of doing work) can be written as

$$P(x,t) = F_y(x,t)v_y(x,t), \tag{24}$$

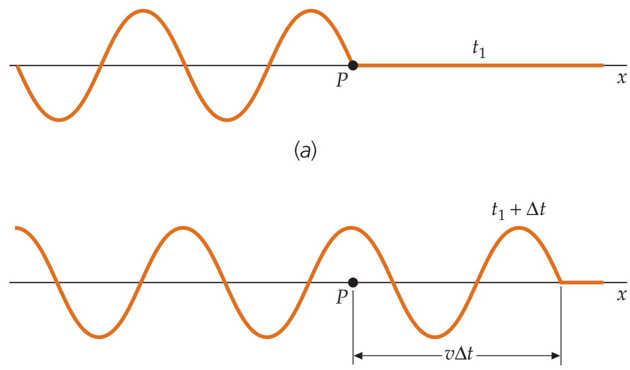
where $F_y(x,t)$ and $v_y(x,t)$ are the transverse force exerted on the right part of the string and velocity of the point a, respectively. In addition, $F_y = -\frac{\partial y}{\partial x}F_T = F_T kA\sin(kx-\omega t)$ with F_T the string tension along the x direction and $v_y(x,t) = \frac{\partial y}{\partial t} = \omega A\sin(kx-\omega t)$. As shown in the figure, we need to put a minus sign in F_y since it is negative when the slope is positive. Putting everything together, we have

$$P(x,t) = -\frac{\partial y}{\partial x}F_T\frac{\partial y}{\partial t} = F_T k\omega A^2\sin^2(kx-\omega t) \geq 0. \tag{25}$$

Using $v = \sqrt{\frac{F_T}{\mu}}$ and $v = \frac{\lambda}{T} = \frac{\omega}{k}$, we have

$$P(x,t) = \mu v\omega^2 A^2\sin^2(kx-\omega t), \tag{26}$$

and the average power $\langle P\rangle = \frac{1}{2}\mu v\omega^2 A^2$, because the time average of $\sin^2(kx-\omega t)$ in a period is 1/2.



(a)

(b)

As shown above, the energy travels along a taut (tightly pulled) string at an average speed equal to the wave speed v, thus the average energy $\langle\Delta E\rangle$ flowing past point P during a time interval Δt is

$$\langle\Delta E\rangle = \langle P\rangle\Delta t = \frac{1}{2}\mu\omega^2 A^2 v\Delta t = \frac{1}{2}\mu\omega^2 A^2\Delta x. \tag{27}$$

Note that $\mu\Delta x = \Delta m$ and therefore

$$\langle\Delta E\rangle = \frac{1}{2}\Delta m\omega^2 A^2. \tag{28}$$

Furthermore, from our last lecture, we have learnt that the average energy of a simple harmonic oscillator with mass m is $\frac{1}{2}m\omega^2 A^2$. Therefore, we can understand and interpret the effect of the energy flow in a string. The energy flowed into the right part of the string are used to provide the oscillation energy for the string, which is at rest before the energy flow arrives.

Kinetic Energy in the string

For the wave function $y(x,t) = A\cos(kx-\omega t)$, the kinetic energy dK associated with a string element of mass $dm = \mu dx$ is given by $dK = \frac{1}{2}dmv_y^2$, where v_y is the transverse speed of the oscillating string element, obtained by differentiating with respect to time while holding x constant:

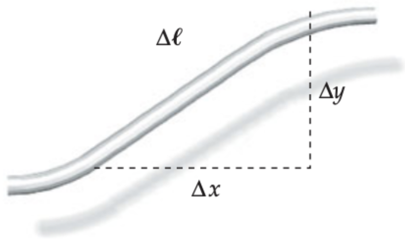
$$v_y(x,t) = \frac{\partial y(x,t)}{\partial t} = \omega A\sin(kx-\omega t) \tag{29}$$

Thus, dividing dK by dt gives the rate at which kinetic energy passes through a string element, and thus the rate at which kinetic energy is carried along by the wave. The dx/dt is the wave speed v, so the rate at which kinetic energy is transported is

$$\frac{dK}{dt} = \frac{1}{2}\mu v\omega^2 A^2\sin^2(kx-\omega t), \tag{30}$$

and its average over time is $\langle\frac{dK}{dt}\rangle_{av} = \frac{1}{4}\mu v\omega^2 A^2$. It is half of the total power while the other half is stored as the potential energy.

Potential Energy in the string



Let us derive an expression for the potential energy of a segment of a string carrying a traveling wave. The potential energy of a segment equals the work done by the tension in stretching the string, which is $\Delta U = F_T(\Delta l - \Delta x)$, where F_T is the tension, Δl is the length of the stretched segment, and Δx is its original length.

1. $\Delta l = (\Delta x^2 + \Delta y^2)^{1/2} = \Delta x(1 + (\Delta y/\Delta x)^2)^{1/2}$
   $= \Delta x[1 + \frac{1}{2}(\Delta y/\Delta x)^2 + \frac{1/2(1/2-1)}{1\times 2}(\Delta y/\Delta x)^4 + \cdots]$
   $\approx \Delta x[1 + \frac{1}{2}(\Delta y/\Delta x)^2]$ in the limit of $\Delta y/\Delta x \ll 1$.
   So, $\Delta l - \Delta x \approx \frac{1}{2}(\Delta y/\Delta x)^2 \Delta x \Rightarrow \Delta U \approx \frac{1}{2}F_T(\Delta y/\Delta x)^2\Delta x$
2. $y = A\cos(kx - \omega t) \Rightarrow$
   $\partial y/\partial x = -Ak\sin(kx - \omega t) \Rightarrow$
   $\Delta U = \frac{1}{2}F_T(\partial y/\partial x)^2\Delta x = \frac{1}{2}F_T A^2 k^2 \sin^2(kx - \omega t)\Delta x.$

Recall that the average kinetic energy and the average potential energy are equal, we expect that elastic potential energy is carried along with the wave, and at the same average rate as the kinetic energy.

Note that for a transverse wave, both potential and kinetic energies are maximum as the wave crosses the $y = 0$ point. The speed is maximum and the stretch is maximum at the $y = 0$. Both potential and kinetic energies are zero at $y = A$ point. (The speed is zero and the stretch is zero at the $y = A$ point.)

Wave Intensity

Waves on a string carry energy in just one dimension of space (along the direction of the string). But other types of waves, including sound waves in air , carry energy across all three dimensions of space. For waves that travel in three dimensions, we define the intensity (denoted as I) to be the power per unit area transported by the wave across a surface perpendicular to the direction of propagation. It is usually measured in watts per square meter (W/m^2).

If waves spread out equally in all directions from a centered source, then intensity through a sphere with radius r and surface area $4\pi r^2$ is

$$I = \frac{P}{4\pi r^2}, \tag{31}$$

which follows directly from energy conservation. This is called inverse-square law for intensity.

4 The Principle of Superposition

1. Wave superposition: The principle of superposition states that the total wave displacement at any point where two or more waves overlap is the sum of the displacement of the individual waves.

   $$y(x,t) = y_1(x,t) + y_2(x,t), \quad \text{superposition} \tag{32}$$

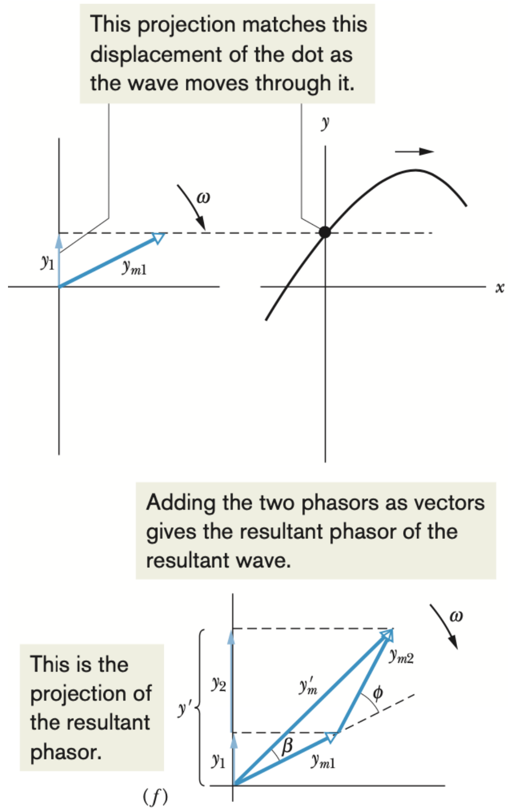
   The overlapping of waves is also called interference in physics.

2. Overlapping waves algebraically add to produce a resultant wave (or net wave).

3. Overlapping waves do not in any way alter the travel of each other.

4.1 Phasor

A phasor is a vector that rotates around its tail, which is pivoted at the origin of a coordinate system. The magnitude of the vector is equal to the amplitude of the wave that it represents. The angular speed of the rotation is equal to the angular frequency ω of the wave. For example, the wave $y = y_{m1}\sin(kx - \omega t)$ is represented by the phasor shown below. A second wave that lags behind by ϕ (delayed in time) can be written as $y = y_{m2}\sin(kx - \omega t + \phi)$.

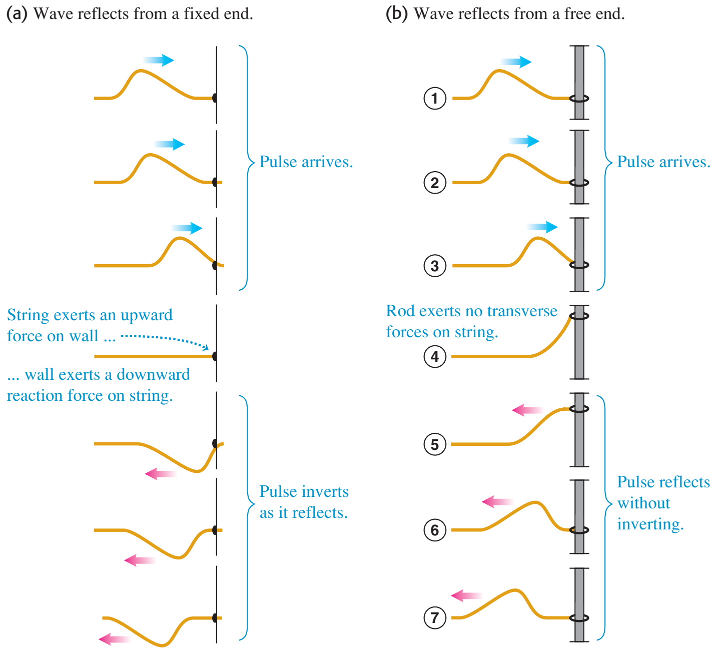


When two waves travel along the same string in the same direction, we can represent them and their resultant wave in a phasor diagram. We can use phasors to combine waves even if their amplitudes are different.

4.2 Standing Wave

Standing wave on a string: For a given string, there are certain frequencies for which superposition results in a stationary vibration pattern called a standing wave. (Standing wave is no longer traveling. To emphasize the difference, a wave that does move along the string is called a traveling wave.) Let us consider the superposition of two waves

$$
\begin{aligned}
y_1(x,t) &= A\cos(kx-\omega t), \quad \text{right-moving}\\
y_2(x,t) &= -A\cos(kx+\omega t), \quad \text{left-moving.}
\end{aligned}
$$



In fact, $y_2(x,t)$ can be viewed as the reflected wave after the incident right wave $y_1(x,t)$ reflects from a fixed end. Note also that the change in sign corresponds to a shift in phase of 180° or π radians due to the reaction force from the fixed end during the reflection. When the wave arrives at the fixed end, it exerts a force to the fixed end, and the reaction to this force from the fixed end (is opposite to the action based on Newton's third law), "kicks back" on the string and sets up a reflected wave in the reverse direction. In contrast, wave reflected from a free end has no such phase shift.

The resulting superposition of these two waves is then

$$
\begin{aligned}
y(x,t) &= A\left[\cos(kx-\omega t)-\cos(kx+\omega t)\right]\\
&= (2A\sin kx)\sin\omega t = (A_{SW}\sin kx)\sin\omega t, \qquad (33)
\end{aligned}
$$

where the amplitude of the standing wave $A_{SW} = 2A$. The above equation has two factors: a function of x and a function of t. The factor $A_{SW}\sin kx$ shows that at each moment the shape of the string is a sine curve. But unlike a wave traveling along a string, the wave shape stays in the same position, oscillating up and down as described the $\sin\omega t$ factor.

We can find that the amplitude of the oscillation $A_{SW}\sin kx$ is always zero as long as $kx = n\pi$ with n an integer. These places are the nodes. At a node the displacements of the two waves are always equal and opposite and cancel each other out, which is known as destructive interference. Midway between two adjacent nodes are the points of greatest amplitude given by $kx = n\pi + \dfrac{\pi}{2}$. They are called antinodes due to constructive interferences of two waves. The distance between successive nodes or between successive antinodes is one half-wavelength, or $\lambda/2$.

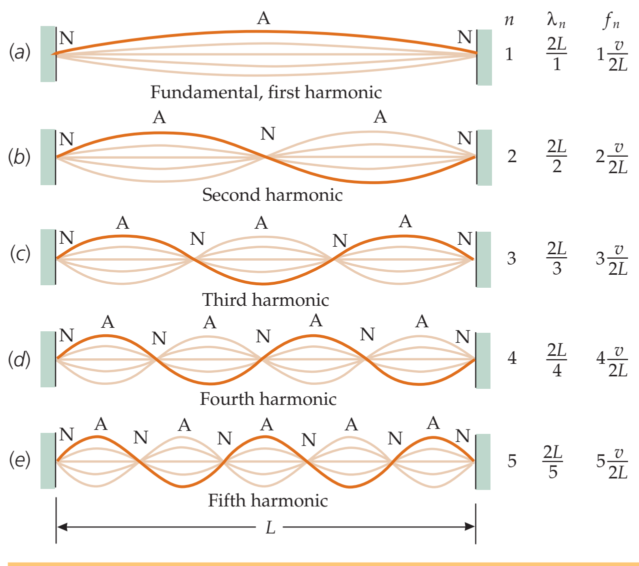


FIGURE 16-12 Standing waves on a string that is fixed at both ends. Antinodes are labeled A and nodes are labeled N. The nth harmonic has n antinodes, where $n = 1, 2, 3, \ldots$.

Let us consider a string of a definite length L, rigidly fixed at both ends. Such strings are found in many musical instruments. Only standing waves with certain wavelengths can form on this string. We can also see from above figure that the standing wave must have a node at both end of the string due to fixed boundary condition. Therefore we have the following standing wave condition

$$L = n\frac{\lambda_n}{2}, \quad n = 1, 2, 3, \cdots. \qquad (34)$$

The corresponding frequency then can be computed by

$$f_n = \frac{1}{T_n} = \frac{v}{\lambda_n} = n\frac{v}{2L}, \quad n = 1, 2, 3, \cdots, \qquad (35)$$

where $v = \sqrt{\dfrac{F_T}{\mu}}$ is the velocity of the wave. Each frequency with its associated vibration pattern is called a normal mode. The lowest frequency f_1 is called the fundamental frequency.

As opposed to the traveling wave, standing waves do not transfer energy from one end to another, since equal amount of energy flows carried by the incident wave and reflected wave cancel each other. (The mechanical energy flows back and forth from a node to an antinode as the string goes from maximum curvature to straight, and does not propagate along the string.)