STA2001 Probability and Statistics (I)

Lecture 8

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Review

Definition

A RV X with $\overline{S}$ that is an interval or unions of intervals is said to be continuous RV. If there exists a function $f(x) : \overline{S} \to (0, \infty)$ such that

1. $f(x) > 0, \quad x \in \overline{S}$
2. $\int_{\overline{S}} f(x)dx = 1$
3. If $(a, b) \subseteq \overline{S}$

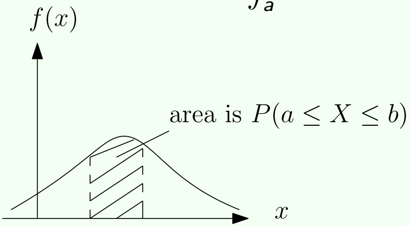
$$P(a \leq X \leq b) \triangleq \int_a^b f(x)dx$$

f is the so-called probability density function (pdf).

Review

Interpretation

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



Review

Definition

cdf $F(x) : R \to [0, 1]$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

1. relation between the pdf and the cdf

$$f(x) = F'(x)$$

   for those values of x at which $F(x)$ is differentiable
2. relation between the probability function and the cdf

$$P(a \leq X \leq b) = F(b) - F(a)$$

Review

Mathematical Expectations $E[g(X)] = \int_{\overline{S}} g(x)f(x)dx$

1. $[g(X) = X]$: Mean of X, $E[X] = \int_{\overline{S}} xf(x)dx$
2. $[g(X) = (X - E[X])^2]$: Variance of X,
$$Var[X] = E[(X - E[X])^2] = \int_{\overline{S}} (x - E[X])^2 f(x)dx$$
3. $[g(X) = X^r]$, Moments of X:
$$E[X^r] = \int_{\overline{S}} x^r f(x)dx$$
4. $[g(X) = e^{tX}]$: mgf, if there exists $h > 0$, such that
$$M(t) = E[e^{tX}] = \int_{\overline{S}} e^{tx} f(x)dx, \quad -h < t < h \text{ for some } h > 0$$
$$M^{(r)}(0) = E[X^r], E[X] = M'(0), \quad Var[X] = M''(0) - (M'(0))^2$$

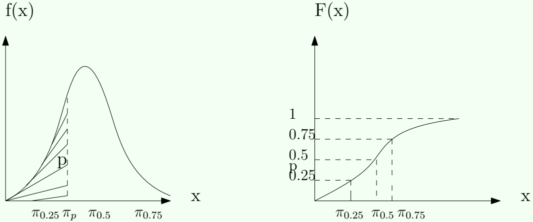
Review

Definition[(100p)th percentile]

It is a number π_p such that the area under $f(x)$ to the left of π_p is p. That is

$$p = \int_{-\infty}^{\pi_p} f(x)dx = F(\pi_p)$$

The 50th percentile is called the median. The 25th and 75th percentiles are called the first and third quantiles, respectively. The median is also called the 2nd quantile



Section 3.2 Exponential, Gamma and Chi-square Distribution

Poisson Distribution

Now consider the approximate Poisson process (APP) with average number of occurrence λ in a unit interval.

Poisson distribution: let X describe the number of occurrences of some events in the unit interval with

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \cdots .$$

$$E[X] = \lambda, \quad Var[X] = \lambda$$

Number of occurrences in an interval with length T

For an interval with length T, which should be treated as a new "unit interval", the number of occurrences Y has a Poisson distribution with $E[Y] = \lambda T$ and thus its pmf is

$$f(y) = \frac{(\lambda T)^y e^{-\lambda T}}{y!}, \quad y = 0, 1, \cdots$$

Therefore, we have for any APP with average number of occurrence λ in a unit interval, the probability of having no occurrence in an interval with length T is

$$P(Y = 0) = e^{-\lambda T} = P(\text{no occurrence in the interval with length T})$$

Exponential distribution

1. Description: Consider an APP. We are interested in the waiting time until the first occurrence.
2. Define the waiting time by W. Then our goal is to derive the pdf of W.

   Idea: $\begin{cases} 1.\text{derive cdf of W}, F(w) \\ 2. f(w) = F'(w) \end{cases}$

   $F(w) = P(W \leq w)$

   Assume that the waiting time is nonnegative. Then,

   $$F(w) = 0, \quad \text{for } w < 0.$$

   For $w \geq 0$,

   $$F(w) = P(W \leq w) = 1 - P(W > w)$$

Exponential distribution

where

$$P(W > w) = P(\text{no occurrences in [0,w]}) = e^{-\lambda w}$$

Therefore,

$$F(w) = 1 - e^{-\lambda w} \text{ for } w \geq 0$$

leading to

$$f(w) = F'(w) = \lambda e^{-\lambda w}, w \geq 0$$

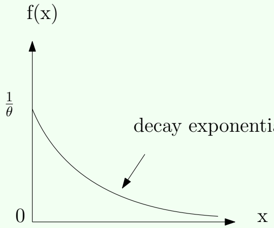
Exponential Distribution

Definition

A RV X has an exponential distribution if its pdf is

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x \geq 0, \theta > 0$$

Accordingly, the waiting time until the first occurrence for an APP has an exponential distribution with $\theta = \frac{1}{\lambda}$ [λ: the average number of occurrences per unit time]



Mathematical expectations

3. mgf, mean and variance

$$M(t) = E[e^{tX}] = \int_0^\infty e^{tx} \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$$

$$= \frac{1}{\theta} \frac{1}{(t - \frac{1}{\theta})} e^{(t-\frac{1}{\theta})x} \Big|_0^\infty = \frac{1}{1 - t\theta}, \quad t < \frac{1}{\theta}$$

$$\Rightarrow M'(t) = \frac{\theta}{(1-\theta t)^2}, \quad M''(t) = \frac{2\theta^2}{(1-\theta t)^3}$$

$$\Rightarrow M'(0) = \theta = E[X], \quad M''(0) = 2\theta^2 \Rightarrow Var[X] = \theta^2$$

Example 1, page 105

Question

Customers arrive in a shop according to APP at a mean rate of 20 per hour. What's the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Example 1, page 105

Question

Customers arrive in a shop according to APP at a mean rate of 20 per hour. What's the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Let X denote the waiting time in minute until the first customer arrives and note that $\lambda = \frac{1}{3}$ is the average number of customers per minute. Thus

$$\theta = \frac{1}{\lambda} = 3 \quad \text{and} \quad f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \quad x \geq 0$$

Hence

$$P(X > 5) = \int_5^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx = e^{-\frac{5}{3}} \approx 0.18$$

Poisson Distribution

Let X describe the number of occurrences of some events in a unit interval with

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \cdots, \quad E[X] = Var[X] = \lambda$$

For an interval with length T, which should be treated as a new "unit interval", the number of occurrences Y has $E[Y] = \lambda T$ and thus its pmf is

$$f(y) = \frac{(\lambda T)^y e^{-\lambda T}}{y!}, \quad y = 0, 1, \cdots$$

Poisson Distribution

Let X describe the number of occurrences of some events in a unit interval with

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, \cdots, \quad E[X] = Var[X] = \lambda$$

For an interval with length T, which should be treated as a new "unit interval", the number of occurrences Y has $E[Y] = \lambda T$ and thus its pmf is

$$f(y) = \frac{(\lambda T)^y e^{-\lambda T}}{y!}, \quad y = 0, 1, \cdots$$

Then for $\alpha = 1, 2, \ldots,$

$$\begin{aligned} P(Y < \alpha) &= \sum_{k=0}^{\alpha-1} \frac{(\lambda T)^k e^{-\lambda T}}{k!} \\ &= P(\{\text{the number of occurrence smaller than } \alpha \\ &\qquad \text{in the interval with length T}\}) \end{aligned}$$

Gamma distribution

1. Description: Consider an APP. We are interested in the waiting time until the αth occurrence, $\alpha = 1, 2, \ldots$.
2. Define the waiting time by W. Then our goal is to derive the pdf of W.

$$\text{Idea: } \begin{cases} 1.\text{derive cdf of W}, F(w) \\ 2. f(w) = F'(w) \end{cases}$$

$F(w) = P(W \leq w)$

Assume that the waiting time is nonnegative. Then,

$$F(w) = 0, \quad \text{for } w < 0.$$

For $w \geq 0$,

$$F(w) = P(W \leq w) = 1 - P(W > w)$$

$P(W > w) = P(\{\text{number of occurrences in } [0, w] \text{ smaller than } \alpha\})$

Gamma distribution

$$
\begin{aligned}
P(W > w) &= P(\{\text{number of occurrences in } [0, w] \text{ smaller than } \alpha\}) \\
&= \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!}
\end{aligned}
$$

$$
f(w) = F'(w) = \frac{\lambda^\alpha w^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w}, w > 0.
$$

pdf of this form is said to be of the Gamma type and W is said to have Gamma distribution.

The waiting time until the αth occurrence in the APP, has a Gamma distribution with parameters α and λ.

Gamma Function

With the so-called Gamma function, we can obtain a more general definition of Gamma distribution with $\alpha > 0$.

Definition[Gamma function(generalized factorial)]

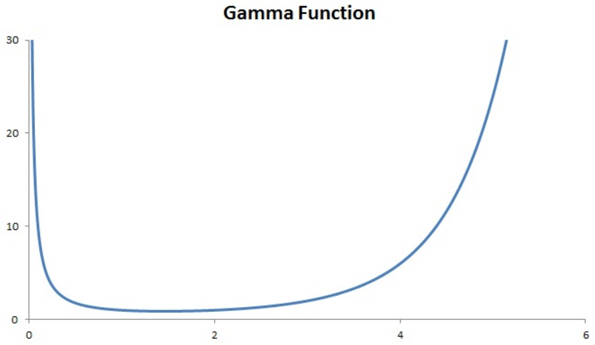
$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \quad t > 0$$

$$\Gamma(t) = -y^{t-1} e^{-y}\Big|_0^\infty + \int_0^\infty (t-1) y^{t-2} e^{-y} dy = (t-1)\Gamma(t-1)$$

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) \\ &= \cdots = (n-1)\cdots 2\Gamma(1) = (n-1)!(\Gamma(1) = 1)\end{aligned}$$

Gamma Function

With the so-called Gamma function, we can obtain a more general definition of Gamma distribution with $\alpha > 0$.



Gamma Distribution

Definition

A RV X has a Gamma distribution if its pdf is

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}}, \quad x \geq 0, \alpha > 0, \theta > 0,$$

where θ and α are the two parameters.

- Gamma pdf $f(x)$ is well-defined pdf (by the definition of $\Gamma(\alpha)$)
- mgf (exercise 3.2-7)

$$M(t) = \frac{1}{(1-\theta t)^{\alpha}}, \quad t < \frac{1}{\theta}$$

$$E[X] = \alpha\theta, \quad Var[X] = \alpha\theta^2$$

A special case: when $\alpha = 1$, Gamma distribution reduces to exponential distribution.

Chi-square Distribution

Definition

Let X have a Gamma distribution with $\theta = 2$, $\alpha = \frac{r}{2}$, r is an integer. The pdf of X is

$$f(x) = \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, \quad x > 0$$

Then X has chi-square distribution with degrees of freedom r, and denoted by $X \sim \chi^2(r)$

$$E[X] = \alpha\theta = \frac{r}{2} \cdot 2 = r \quad Var[X] = \alpha\theta^2 = \frac{r}{2} \cdot 2^2 = 2r$$

$$\text{Mgf}: M(t) = (1-2t)^{-\frac{r}{2}}, \quad t < \frac{1}{2}$$

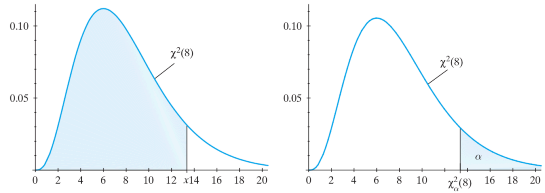
Note: The interpretation of Chi-square distribution is deferred.

Remark

Chi-square distribution plays an important role in statistics, the tables of cdf of chi-square distribution are given

$$F(x) = P(X \leq x) = \int_0^x f(t)dt.$$

Table IV The Chi-Square Distribution



$$P(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} dw$$

	$P(X \leq x)$							
	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
r	$\chi^2_{0.99}(r)$	$\chi^2_{0.975}(r)$	$\chi^2_{0.95}(r)$	$\chi^2_{0.90}(r)$	$\chi^2_{0.10}(r)$	$\chi^2_{0.05}(r)$	$\chi^2_{0.025}(r)$	$\chi^2_{0.01}(r)$
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.34
4	0.297	0.484	0.711	1.064	7.779	9.488	11.14	13.28
5	0.554	0.831	1.145	1.610	9.236	11.07	12.83	15.09

Example 2

Let X have a chi-square distribution with $r = 5$ degrees of freedom. Then using table IV in Appendix B on page 501, we have

$$P(1.145 \leq X \leq 12.83) = F(12.83) - F(1.145)$$

Example 2

Let X have a chi-square distribution with $r = 5$ degrees of freedom. Then using table IV in Appendix B on page 501, we have

$$\begin{aligned} P(1.145 \leq X \leq 12.83) &= F(12.83) - F(1.145) \\ &= 0.975 - 0.05 = 0.925. \end{aligned}$$