香港中文大學（深圳）
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Introduction to Data Science

Lecture 17 Optimization (Cont.)

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Review: Convex Function

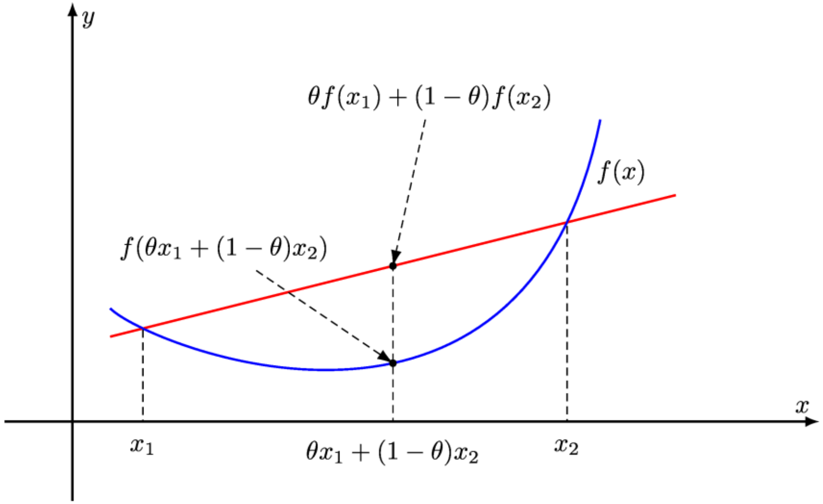
Convex Function

Definition: A function $f(\boldsymbol{x})$: $R^n \rightarrow R$ is **convex** if (1) its domain is a convex set, and

(2) for any $\boldsymbol{x_1}, \boldsymbol{x_2} \in dom(f)$ and any $0 \leq \lambda \leq 1$, we have

$$f(\boldsymbol{z}) \leq \lambda f(\boldsymbol{x_1}) + (1-\lambda) f(\boldsymbol{x_2})$$

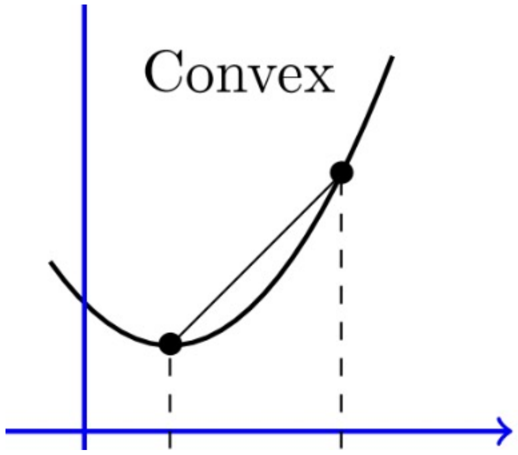
where $\boldsymbol{z}$ = $\lambda \boldsymbol{x_1} + (1-\lambda)\boldsymbol{x_2}$.

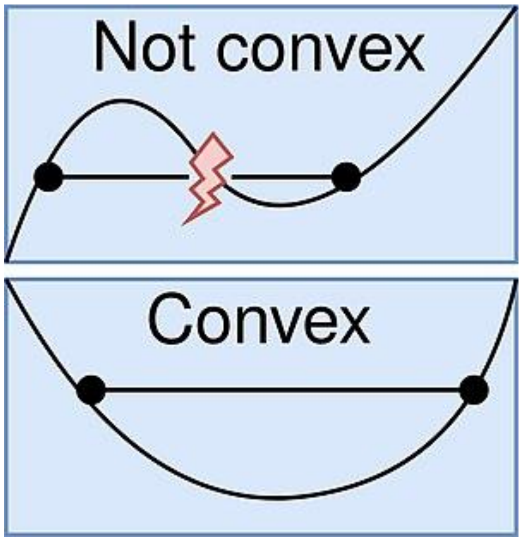


Function $\boldsymbol{f}$ evaluated at the combination of two points $\boldsymbol{x_1}, \boldsymbol{x_2}$ is no larger than the **same combination of $\boldsymbol{f(x_1)}$** and $\boldsymbol{f(x_2)}$

Other Definition

- Pick any two points on the function curve.
- The line segment between these two points is above the function curve.



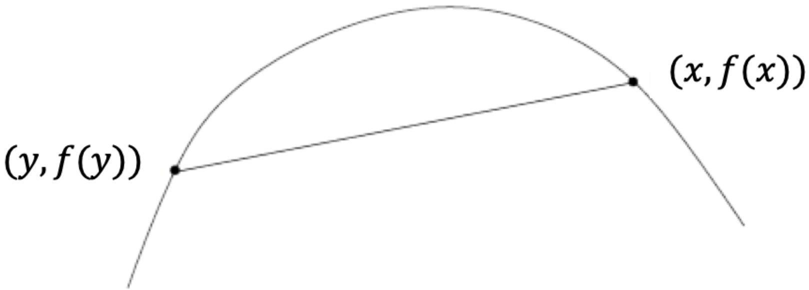


Concave Function

Definition: A function $f(\boldsymbol{x}): R^n \rightarrow R$ is **concave** if (1) the domain of f is a convex set, and (2) for any $\boldsymbol{x}, \boldsymbol{y} \in dom(f)$ and any $0 \leq \lambda \leq 1$, we have

$$f(\boldsymbol{z}) \geq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})$$

where $\boldsymbol{z} = \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}$.



If f is concave, then -f is convex!

If f is convex, then -f is concave!

Second Order Condition (SOC)

Suppose f is a twice continuously differentiable function. Then f is convex **if and only if**

(1) dom(f) is a convex set

(2) for any $\boldsymbol{x} \in$ dom(f), any unit vector $\boldsymbol{e}$,

$$\frac{d^2 f(\boldsymbol{x} + \theta \boldsymbol{e})}{d\theta^2}(0) \geq 0$$

One dimension: $\boldsymbol{f''(x)} \geq \mathbf{0}$ **for all** $\boldsymbol{x} \in \boldsymbol{dom(f)}$.

The Blind Box Problem (Newsvendor Model)

$$\text{minimize} \quad \overbrace{-(pE[\min(Q,D)] + sE[(Q-D)^+] - cQ)}^{\text{convex function of Q?}}$$

$$\text{subject to} \quad \underbrace{Q \geq 0}_{\text{Convex set}}$$

In order to show the convexity of the objective function, it suffices to show that $pE[\min(Q,D)] + sE[(Q-D)^+] - cQ$ is concave in Q.

Order Quantity Demand

$$\pi(Q) \stackrel{\text{def}}{=} pE[\min(Q,D)] + sE[(Q-D)^+] - cQ$$

Let the pdf of D be h(x) on [0, ∞)

$$\pi(Q) = \mathrm{p}\int_0^Q x\,h(x)dx + \mathrm{p}\int_Q^\infty Q\,h(x)dx + s\int_0^Q (Q-x)\,h(x)dx - cQ.$$

$$\begin{aligned} pE[\min(Q,D)] &= p\int_0^\infty \min(Q,x)\,h(x)dx \\ &= p\int_0^Q \min(Q,x)\,h(x)dx + p\int_Q^\infty \min(Q,x)\,h(x)dx \\ &= p\int_0^Q xh(x)dx + p\int_Q^\infty Qh(x)dx \end{aligned}$$

$$\pi(Q) \stackrel{\text{def}}{=} pE[\min(Q,D)] + sE[(Q-D)^+] - cQ$$

Leibniz Integral Rule

$$\frac{d}{dx}\int_u^v f(x,y)\,dy = -f(x,u)\frac{du}{dx} + f(x,v)\frac{dv}{dx} + \int_u^v \left[\frac{\partial}{\partial x} f(x,y)\right] dy.$$

Let the pdf of D be h(x) on [0, ∞)

$$\pi(Q) = \mathrm{p}\int_0^Q x\,h(x)dx + \mathrm{p}\int_Q^\infty Q\,h(x)dx + s\int_0^Q (Q-x)\,h(x)dx - cQ.$$

$$\pi'(Q) = \underbrace{\mathrm{p}\,Q\,h(Q)}_{\frac{d}{dQ}p\int_0^Q xh(x)dx = -0 + pQh(Q) + 0} - \mathrm{p}\,Q\,h(Q) + \mathrm{p}\int_Q^\infty h(x)dx + (Q-Q)h(Q) + s\int_0^Q h(x)dx - c$$

$$\pi(Q) \stackrel{\text{def}}{=} pE[\min(Q, D)] + sE[(Q-D)^+] - cQ$$

Let the pdf of D be h(x) on [0, ∞)

$$\pi(Q) = \mathrm{p}\int_0^Q x\, h(x)dx + \mathrm{p}\int_Q^\infty Q\, h(x)dx + s\int_0^Q (Q-x)\, h(x)dx - cQ.$$

$$\pi'(Q) = \mathrm{p}\, Q\, h(Q) - \mathrm{p}\, Q\, h(Q) + \mathrm{p}\int_Q^\infty h(x)dx + (Q-Q)h(Q) + s\int_0^Q h(x)dx - c$$

$$=\mathrm{p}\int_Q^\infty h(x)dx + s\int_0^Q h(x)dx - c.$$

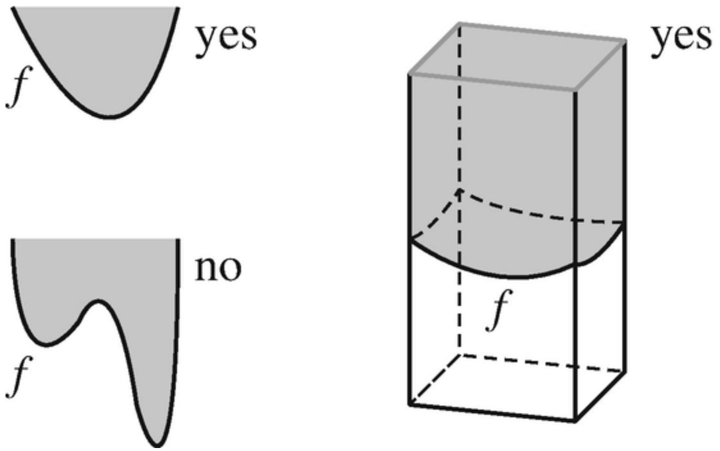
$$\pi''(Q) = -\mathrm{p}\, h(Q) + \mathrm{s}\;\, h(Q) = (s-p)h(Q).$$

As the salvage value is no larger than the price, $\pi''(Q) \leq 0$

Convex function *vs.* Convex set

- $C = \{\boldsymbol{x}: f(\boldsymbol{x}) \leq r\}$ is a convex set if $f(\boldsymbol{x})$ is a convex function

- $C = \{(\boldsymbol{x}, y): y \geq f(\boldsymbol{x})\}$ is a convex set if $f(\boldsymbol{x})$ is a convex function.



Convex Optimization Problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0, i = 1, \dots, m \\ & h_i(x) = 0, i = 1, \dots, n \end{array}$$

A convex optimization problem needs to satisfy the following two conditions:

- Its feasible set is a **convex set**.
- Its objective function is a **convex function**.

Theorem 1. *Consider an optimization problem*

$$\begin{aligned} &min.\ f(x) \\ &s.t.\ \ x \in \Omega, \end{aligned}$$

where f is a convex function and Ω is a convex set. Then, any local minimum is also a global minimum.

Remark

Definition. The graph of a differentiable function f is said to be

1. *concave up* on an open interval I if f' is increasing on I;
2. *concave down* on an open interval I if f' is decreasing on I.

- This is from **Calculus I**
- Concave up is convex in our class and optimization course
- Concave down is concave in our class and optimization course

Blind Box Problem

Optimality Condition

Theorem: If f is twice differentiable and convex, given $\boldsymbol{x}^*$ as an interior global minimizer, then

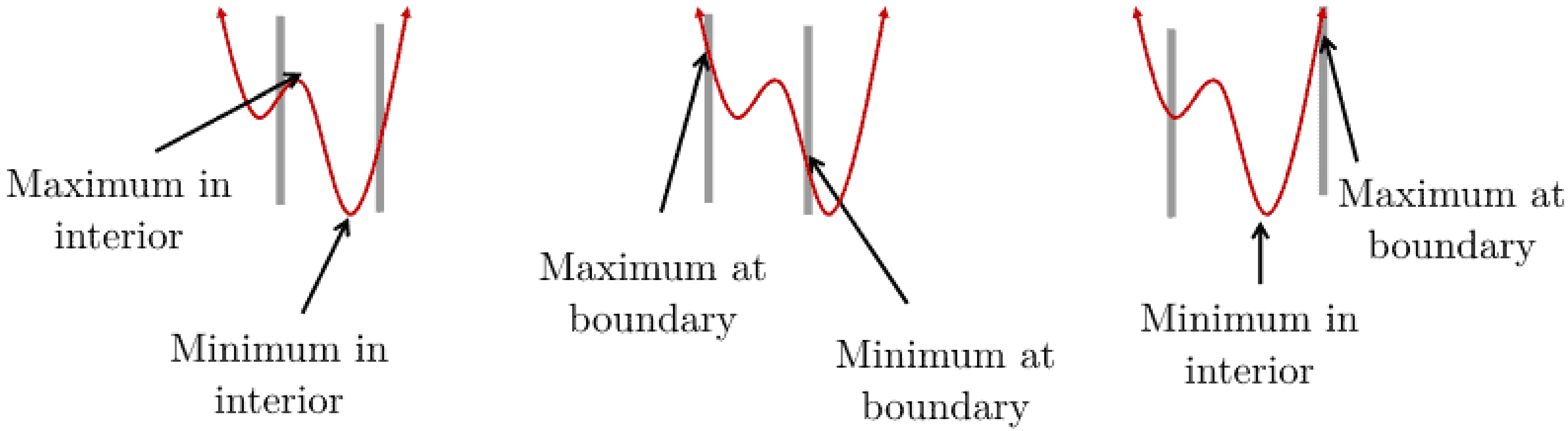
$$\nabla f(\boldsymbol{x}^*) = 0$$

Theorem: If f is twice differentiable and concave, given $\boldsymbol{x}^*$ as an interior global maximizer, then

$$\nabla f(\boldsymbol{x}^*) = 0$$

$$\nabla f(p) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(p) \\ \vdots \\ \frac{\partial f}{\partial x_n}(p) \end{bmatrix}$$

Proof not required.


Maximum in interior
Minimum in interior
Maximum at boundary
Minimum at boundary
Minimum in interior
Maximum at boundary

Blind Box Problem

- Suppose you want to start your own blind box business.
- Let D denote the one season (three months) random demand, with CDF $F(\cdot)$, and mean $\mu = E[D]$.
- At the beginning of each season, you place an order Q to Pop Mart, with a cost c for each blind box.
- Each blind box can be sold at a price of $p > c$.
- At the end of each season, unsold blind boxes are salvaged, and you get $s < c$ for each salvaged box.

Blind Box Problem

- In standard form, we have the following convex optimization problem

$$\begin{aligned} &\text{minimize} \quad && -(pE[\min(Q, D)] + sE[(Q-D)^+] - cQ) \\ &\text{subject to} \quad && -Q \le 0 \end{aligned}$$

Blind Box Problem

$$\pi(Q) \stackrel{\text{def}}{=} pE[\min(Q,D)] + sE[(Q-D)^+] - cQ$$

$$\pi(Q) = \mathrm{p}\int_0^Q x\, h(x)dx + \mathrm{p}\int_Q^\infty Q\, h(x)dx + s\int_0^Q (Q-x)\, h(x)dx - cQ.$$

$$\pi'(Q) = \mathrm{p}\int_Q^\infty h(x)dx + s\int_0^Q h(x)dx - c.$$

$$\pi''(Q) = -\mathrm{p}\, h(Q) + \mathrm{s}\ \ h(Q) = (s-p)h(Q) \le 0.$$

By Theorem 1, it suffices to solve $\pi'(Q) = 0$ (the first order condition you learned from Calculus I)

$$\mathrm{p}\int_Q^\infty h(x)dx + s\int_0^Q h(x)dx - c = 0 \Rightarrow F(Q) = \frac{p-c}{p-s} \Rightarrow Q^* = F^{-1}\left(\frac{p-c}{p-s}\right)$$

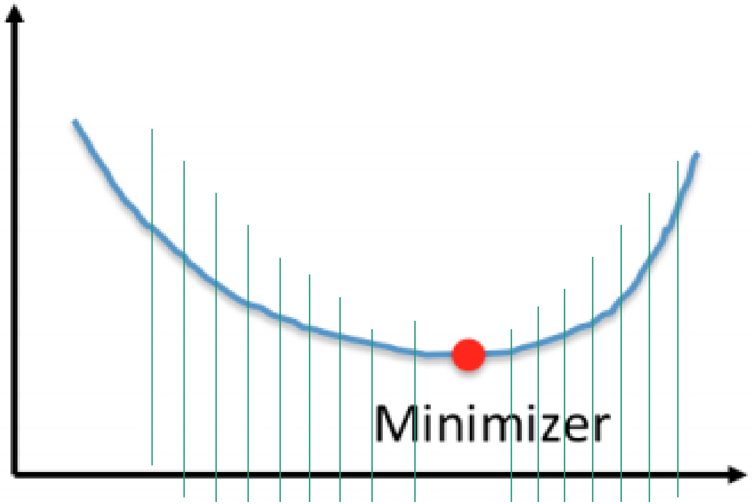
Blind Box Problem

- Suppose you want to start your own blind box business.
- Let D denote the one season (three months) random demand, which follows a uniform distribution in $[10,100]$.
- At the beginning of each season, you place an order Q to Pop Mart, with a cost 10 Yuan for each blind box.
- Each blind box can be sold at a price of 20 Yuan.
- At the end of each season, unsold blind boxes are salvaged, and you get 3 Yuan for each salvaged box.

- How many blind boxes should you order to maximize your expected profit?

$$Q^* = F^{-1}\left(\frac{p-c}{p-s}\right) \approx 62.9$$

Discrete Decision Variables



Sometimes, our decision variable is discrete, (1) we can first relax this assumption, (2) then find the optimal solution as if the decision variable is continuous (3) evaluate the performance for points near the derived solution.

More Examples

Regression Analysis

- Objective: we need to minimize

$$\Sigma_i (Y_i - \beta_1 X_i - \beta_0)^2$$

- Using the Second Order Condition, we can show that the above function is convex in (β_0, β_1). More on this later

1. Choose any $\mathbf{y}$, any unit vector $\boldsymbol{e}$
2. Define g(θ) = $f(\boldsymbol{y} + \theta\boldsymbol{e})$
3. Prove g'' (0) ≥ 0

MLE for Bernoulli RV

- Objective: we need to maximize

$$\mathbf{M} * \log \mathrm{p} + (\mathbf{N}-\mathbf{M}) * \log (1-\mathrm{p})$$

$$\frac{d^2}{dp^2}(\mathbf{M} * \log \mathrm{p} + (\mathbf{N}-\mathbf{M}) * \log (1-\mathrm{p})) = -\frac{M}{p^2} - \frac{N-M}{(1-p)^2} < 0$$

MLE for Exponential RV

- Suppose the lifetime of each bulb follows an exponential distribution.
- Given parameter λ, the pdf is $\mathrm{f(x)} = \lambda\, e^{-\lambda x}$
- Samples of lifetimes: $X_1, X_2 \ldots,\ X_n$
- $l(\lambda)$ = n $\log \lambda\ - \lambda\, \Sigma_i X_i$
- $l''(\lambda)\ = -\frac{n}{\lambda^2}$<0

Remark

From now on, whenever you are asked to find the optimal solution of a problem or to determine the MLE, you need to show the convexity/concavity of the problem before using the first order condition.

More on Convex Functions

Motivation for the remaining contents

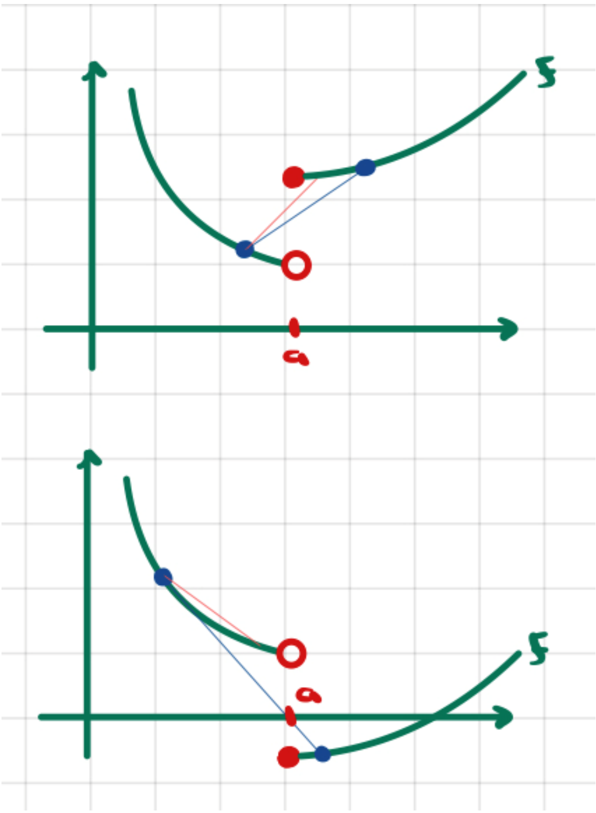
- Before this slide, we focus on convex functions that are twice **differentiable**

- There are a lot of convex functions which are not twice differentiable.

1. Discontinuity

- Consider a one-dimension convex function f defined on [a,b].

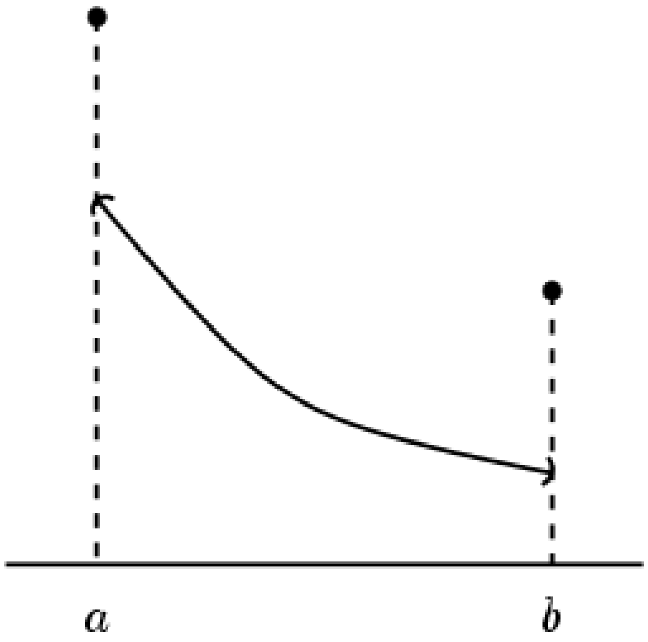
- Can f be discontinuous?



1. Discontinuity

- Consider a one-dimension convex function f defined on [a,b].
- It can be discontinuous at the boundary with

$f(a) > f_{x\downarrow a}(x)$ and $f(b) > f_{x\uparrow b}(x)$



2. Maximum of a set of Convex Functions

If $f_1, \dots, f_m$ are convex functions, then $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\}$ is also convex.

Proof: Pick any $x, y \in dom(f), \lambda \in [0, 1]$. Then,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f_j(\lambda x + (1-\lambda)y) \text{ (for some } j \in \{1, \dots, m\}) \\ &\leq \lambda f_j(x) + (1-\lambda) f_j(y) \\ &\leq \lambda \max\{f_1(x), \dots, f_m(x)\} + (1-\lambda) \max\{f_1(y), \dots, f_m(y)\} \\ &= \lambda f(x) + (1-\lambda) f(y). \ \square \end{aligned}$$

- Recall that $C = \{(\boldsymbol{x}, y): y \geq f(\boldsymbol{x})\}$ is a convex set if $f(\boldsymbol{x})$ is a convex function.

- If $C = \{(\boldsymbol{x}, y): y \geq f(\boldsymbol{x})\}$ is a convex set, is $f(\boldsymbol{x})$ necessarily a convex function?

- Recall that $C = \{(\boldsymbol{x}, y): y \geq f(\boldsymbol{x})\}$ is a convex set if $f(\boldsymbol{x})$ is a convex function.

- If $C = \{(\boldsymbol{x}, y): y \geq f(\boldsymbol{x})\}$ is a convex set, is $f(\boldsymbol{x})$ necessarily a convex function? This statement is TRUE

1. Choose any two points $(\boldsymbol{x_1}, f(\boldsymbol{x_1}))$ and $(\boldsymbol{x_2}, f(\boldsymbol{x_2}))$
2. These two points are in C
3. As C is a convex set, then $\lambda(\boldsymbol{x_1}, f(\boldsymbol{x_1}))+ (1-\lambda)(\boldsymbol{x_2}, f(\boldsymbol{x_2}))$ is in C. That is,

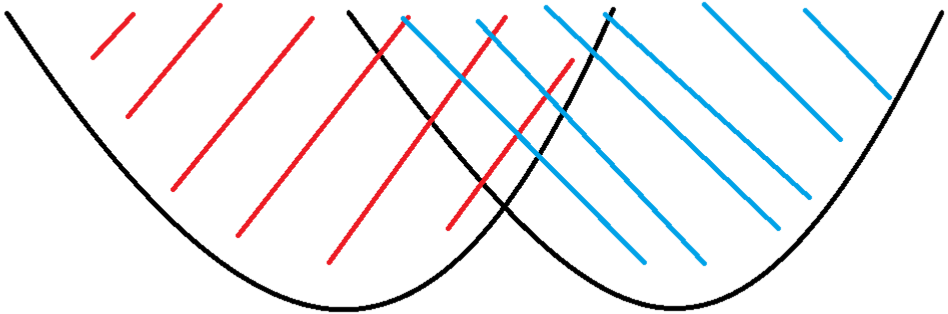
$$\lambda f(\boldsymbol{x_1})+(1-\lambda)f(\boldsymbol{x_2}) \geq \lambda \boldsymbol{x_1}+ (1-\lambda)\, \boldsymbol{x_2}$$

2. Maximum of a set of Convex Functions

If $f_1, \dots, f_m$ are convex functions, then $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})\}$ is also convex.

$C = \{(\boldsymbol{x}, y): y \geq f(\boldsymbol{x})\}$ is called the epigraph of function f, often denoted as **epi** f.

An alternative proof : **epi** f = $\cap_{i=1}^{m}$ $(\mathbf{epi}\ f_i)$. As $\mathbf{epi}\ f_i$ is a convex set, so is $\mathbf{epi}\ f$. Therefore, f is a convex function.



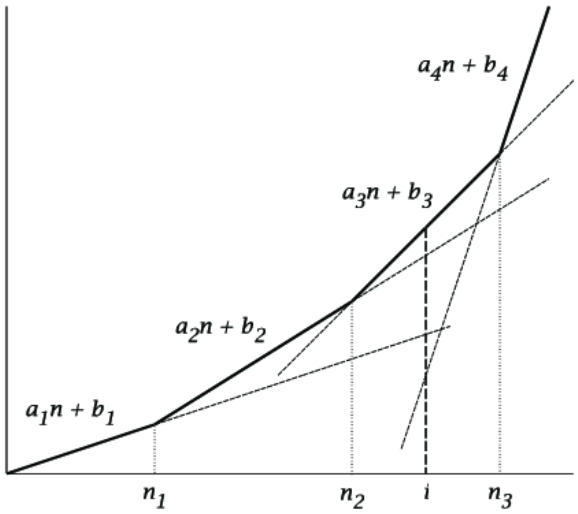
Examples

Piecewise-linear functions

- The maximum of L linear functions.

$$f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$$

where $a_i^T \boldsymbol{x} = \sum_m (a_i)_m x_m$



3. Nonnegative Weighted Sums

If $f_1, f_2, \dots, f_n$ are convex functions, and $w_1, w_2, \dots, w_n \geq 0$, then $f = w_1 f_1 + w_2 f_2 + \cdots + w_n f_n$ is also a convex function.

Proof: (also for high dimension)

- For any x and y, $0 \leq \lambda \leq 1$ and m

$$f_m(\lambda x + (1-\lambda)y) \leq \lambda fm(x) + (1-\lambda) f_m(y)$$

- Multiplying each side of the above inequality by w_m and summing over m, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Examples: Regression Analysis

- Objective: we need to minimize

$$\Sigma_i (Y_i - \beta_1 X_i - \beta_0)^2$$

- It is easy to verify that each component in the summation is a convex function. Therefore, the sum of them is still a convex function.

3. Nonnegative Weighted Sums

If $f_1, f_2, \ldots, f_n$ are convex functions, and $w_1, w_2, \ldots, w_n \geq 0$, then $f = w_1 f_1 + w_2 f_2 + \cdots + w_n f_n$ is also a convex function.

These properties extend to infinite sums and integrals. For example if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, and $w(y) \geq 0$ for each $y \in \mathcal{A}$, then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y) \, dy$$

is convex in x (provided the integral exists).

Proof not required.

Examples: blind box problem (newsvendor)

minimize $\boxed{-(pE[\min(Q,D)] + sE[(Q-D)^+] - cQ)}$

subject to $-Q \leq 0$

$\min(Q,D) = \mathrm{Q} - (\mathrm{Q} - \mathrm{D})^+$

Do not require differentiability of the cdf of D

$$\boxed{\underbrace{\overbrace{(p-c)(-Q) + (p-s)E[\max(Q-D,0)]}^{\text{nonnegative weighted sums of convex functions (in Q)}}}_{}}$$

$$(p-s)E\underbrace{[\max(Q-D,0)]}_{\text{maximum of convex functions (in Q)}}$$

Convex!