Sample Midterm

DURATION OF EXAMINATION: 2 hours

Your exam sheet shall include 6 **problems**. If not, notify the instructors.

1. **(26 pt) Solving a Linear System of Equations**
   Consider the linear system:

$$
\begin{aligned}
2x_1 - 3x_2 + x_3 + x_4 &= 1 && (1)\\
-x_2 + x_3 + x_4 &= 0 && (2)\\
x_1 + 2x_2 + 4x_3 - 2x_4 &= -2. && (3)
\end{aligned}
$$

   (a) (3 pt) Write out the *coefficient matrix* $\boldsymbol{A}$ of this system.

   (b) (10 pt) Write out the *augmented matrix* for this system and calculate its *row-reduced echelon form.*

   (c) (7 pt) Write out the *complete set* of solutions. Note: You shall write in the form of $\mathbf{x}_p + \alpha_1\mathbf{v}_1 + \ldots \alpha_k\mathbf{v}_k$, where $\mathbf{x}_p, \mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors that you need to find out.

   (d) (3 pt) What is the *rank* of the coefficient matrix $\boldsymbol{A}$? Justify your answer.

   (e) (3 pt) Find a *basis* of the *null space* of matrix $\boldsymbol{A}$.

2. **(12 pt) Question Answering**

   (a) (3 pt) A 6th grade student Xiao Ming told you that she/he has learned from the school about how to get a specific solution of a 2×2 linear system of equations by substituting variables. Can you give an example of a 2×2 system that Xiao Ming may not be able to solve? Note: Your example should be a specific 2×2 linear system.

   (b) (4 pt) If a matrix A has full row rank, then what can we say about the solution set of $Ax = b$? If A has full column rank, what can we say about the solution set of $Ax = b$?

   (c) (3 pt) Write one 2×2 invertible matrix that is both upper triangular and symmetric.

   (d) (2 pt) What theorem makes the definition of "dimension" valid (i.e. well defined)? Choose one from the following choices.

   (A) The column space and row space of a matrix have the same dimension.
   (B) Any two bases of a linear space has the same number of elements.
   (C) A square matrix has full rank iff the columns are linearly independent.
   (D) The sum of the rank and the nullity of a matrix equals the number of columns.

3. **(16 pt) Matrix Transpose, Inverse and Multiplication**

   (a) (6 pt) Find the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and justify what you found is indeed its inverse.

   (b) (4 pt) Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

   Compute $(AB)^\top$ and $B^\top A^\top$.

(c) (6 pt) Suppose A and B are 2×2 matrices. Suppose $A = [\mathbf{a}_1, \mathbf{a}_2] = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \end{bmatrix}$, $B = [\mathbf{b}_1, \mathbf{b}_2] = \begin{bmatrix} \mathbf{v}_1^\top \\ \mathbf{v}_2^\top \end{bmatrix}$. Express $B^\top A^\top$ at least in two forms using the vectors $\mathbf{a}_i, \mathbf{u}_i, \mathbf{b}_i, \mathbf{v}_i, i = 1, 2$.

Remark: Your expressions can NOT contain scalars like a_{ij} or b_{ij}. Your expressions can NOT contain matrices like A or B. Your expressions can ONLY contain vectors $\mathbf{a}_i, \mathbf{u}_i, \mathbf{b}_i, \mathbf{v}_i, i = 1, 2$.

4. **(16 pt) True or False**

For each question, you do NOT need to justify. (Only writing T or F is enough)

(a) Any invertible matrix can be written as the product of elementary matrices.

(b) If A and AB are invertible where A and B are both square matrices with the same size, then B is also invertible.

(c) Any square matrix can be written as the product of a lower triangular matrix and an upper triangular matrix.

(d) The time complexity of conducting Gaussian elimination for an $n \times n$ matrix is $O(n^2)$.

(e) If the columns of a **square** matrix A are linearly independent, then the columns of A^2 are also linearly independent.

(f) The three vectors $[0, 0, 0]$, $[1, 4, 5]$ and $[1, 2, 3]$ are linearly dependent.

(g) Suppose $A \in \mathbb{R}^{n \times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. If $A\mathbf{x} = A\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$.

(h) $\text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) = \text{span}(\{\mathbf{u}, \mathbf{u} - \mathbf{v} + \mathbf{w}, 2\mathbf{u} - \mathbf{v} + \mathbf{w}\})$ for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

5. **(15 pt) Linear Space, Column Space and Null Space**

(a) (8 pt) Suppose A is a matrix. Prove: the null space of A is a linear space.

Remark: We talked about this result in class. You cannot directly quote the theorem in class. Instead, you need to provide a proof of this result.

(b) (7 pt) Show that if the product of two matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$ is a zero matrix $0 \in \mathbb{R}^{m \times k}$, i.e., $AB = 0$, then the column space of A is contained in the null space of B, i.e., $C(B) \subseteq N(A)$.

6. **(15 pt) Linear Dependence and Rank**

In all subproblems, I_k is the $k \times k$ identity matrix.

(a) (5 pt) Prove: All rows of the matrix $[A, I_k, U] \in \mathbb{R}^{k \times (m+k+n)}$ are linearly independent, where $A \in \mathbb{R}^{k \times m}$ and $U \in \mathbb{R}^{k \times n}$ are two arbitrary matrices.

(b) (5 pt) Suppose $A \in \mathbb{R}^{k \times k}$, and define the matrix $M = \begin{bmatrix} I_k & 0 \\ 0 & A \end{bmatrix} \in \mathbb{R}^{(2k) \times (2k)}$. Prove that $\text{rank}(M) = k + \text{rank}(A)$.

(c) (5 pt) Suppose $A, B, C \in \mathbb{R}^{k \times k}$, and the matrix $Q = \begin{bmatrix} I_k & B \\ C & A \end{bmatrix} \in \mathbb{R}^{(2k) \times (2k)}$. Does $\text{rank}(Q) = k + \text{rank}(A)$ always hold? If yes, provide a proof. If not, provide an alternative expression of $\text{rank}(Q)$ in terms of k and $\text{rank}(f(A, B, C))$ where f is a certain function of A, B, C that you need to figure out, and justify the expression.

The following are some additional sample questions for the midterm.

7. Given an m by n matrix A with rank s, take s columns from A and s rows from A. Their intersection forms an $s \times s$ matrix. Show that the intersection is non-singular if these selected rows and columns are independent.

8. Let $A = CR$, where C is an m by r matrix and R is r by n. Prove that if the rank of A is r, then C has full column rank and R has full row rank

9. Let $u, v \in \mathbb{R}^n$, assume $u^T v = 0$, prove that $(I + uv^T)^{-1} = I - uv^T$.

10. Consider block elimination matrices, show that

$$\left[\begin{array}{c|cc} I_k & & \\ \hline & 1 & \\ & l & I_{n-k-1} \end{array}\right] \left[\begin{array}{c|c} I_k & u \\ \hline & I_{n-k} \end{array}\right] = \left[\begin{array}{c|cc} I_k & & u \\ \hline & 1 & \\ & l & I_{n-k-1} \end{array}\right] \triangleq J.$$

Also, for $A = \left[\begin{array}{c|cc} u_1 & u & * \\ \hline & \alpha & \\ & l & * \end{array}\right]$, find J of the above form s.t. $JA = \left[\begin{array}{c|cc} u_1 & 0 & * \\ \hline & \alpha & \\ & 0 & * \end{array}\right]$

11. Compute $PA = LU$ for $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$