Slide 13: Null Space, Column Space

MAT2040 Linear Algebra

Definition 13.1 (**Null Space**) Let $A \in \mathbb{R}^{m \times n}$, then the null space of A is the set Null(A)=$\{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$. The Null Space of A is also the solution set for $A\mathbf{x} = \mathbf{0}$.

Theorem 13.2 (**Null(A) is subspace**)
Let $A \in \mathbb{R}^{m\times n}$, then Null($A$) is a subspace of $\mathbb{R}^n$, hence is a vector space.

Proof.
(1). $\mathbf{0} \in$ Null(A), thus Null(A) is not $\emptyset$.
(2). Let $\mathbf{x}, \mathbf{y} \in$ Null(A), then $A\mathbf{x} = \mathbf{0} = A\mathbf{y}$. Thus
$A(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}) + A\mathbf{y} = A\mathbf{x} + A\mathbf{y} = \mathbf{0}+\mathbf{0} = \mathbf{0}$, therefore $\mathbf{x}+\mathbf{y} \in$ Null(A).
(3) Let $\mathbf{x} \in$ Null(A), $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \mathbf{0}$, thus $\alpha\mathbf{x} \in$ Null(A), for any α.

Theorem 13.3 (**Nonsingular matrix has only zero null space**)
Let $A \in \mathbb{R}^{n\times n}$, then A is invertible if and only if Null(A)$= \{\mathbf{0}\}$.

Proof.
"$\Rightarrow$" If $A\mathbf{x} = \mathbf{0}$, $\mathbf{x} = A^{-1}A\mathbf{0} = \mathbf{0}$ since A is invertible.

"$\Leftarrow$" Assume $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$ where $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are columns vectors from A, then $A\mathbf{c} = \mathbf{0}$, where $\mathbf{c} = [c_1, \cdots, c_n]^T$ and $A = [\mathbf{u}_1, \cdots, \mathbf{u}_n]$. Since Null($A$)=$\{\mathbf{0}\}$, then $\mathbf{c} = \mathbf{0}$, thus, $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are linearly independent. Thus $A\mathbf{x} = \mathbf{0}$ has only zero solution, and A is invertible by using Theorem 10.6.

Definition 13.4 Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$, the column space of A is defined as the vector space

$$\operatorname{Col}(A) = \mathbf{Span}(\mathbf{a}_1, \cdots, \mathbf{a}_n) \subseteq \mathbb{R}^m$$

Some book uses $C(A)$ to denote the column space of A.

Fact: Let A be an $m \times n$ matrix. $\operatorname{Col}(A)$ is a subspace of $\mathbb{R}^m$.

Question: Given $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$, how to find a basis of $\operatorname{Col}(A)$?

If $\mathbf{a}_1, \cdots, \mathbf{a}_n$ (the column vectors of A) are linearly dependent, one can keep deleting some vectors until reach a smallest subset which keeps the same span. The smallest subset has maximum number of linearly independent vectors and form a basis for $\operatorname{Col}(A) = \mathbf{Span}(\mathbf{a}_1, \cdots, \mathbf{a}_n)$.

In the following, we provide a systematic way to find the basis for $\operatorname{Col}(A) = \mathbf{Span}(\mathbf{a}_1, \cdots, \mathbf{a}_n)$.

Theorem 13.5 (**Row operations preserve the linear dependence relation between column vectors**)
Suppose B is the matrix of A obtained by applying row operations, then the linear dependence relation between column vectors of A are the same as the linear dependence relation between column vectors of B.

Proof. Suppose $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$, there is a finite number of elementary matrices, $E_1, \cdots, E_k$, such that
$B = [\mathbf{b}_1, \cdots, \mathbf{b}_n] = E_k \cdots E_1 A = EA = [E\mathbf{a}_1, \cdots, E\mathbf{a}_n]$, where
$E = E_k \cdots E_1$. Take any subset $\{i_1, \cdots, i_s\}$ of $\{1, \cdots, n\}$, obviously, $\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s}$ and $\mathbf{b}_{i_1}, \cdots, \mathbf{b}_{i_s}$ have the same linear dependence relation.
That is:
$\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s}$ is linearly independent $\iff$ $\mathbf{b}_{i_1}, \cdots, \mathbf{b}_{i_s}$ is linearly independent.
$\mathbf{a}_{i_1}, \cdots, \mathbf{a}_{i_s}$ is linearly dependent $\iff$ $\mathbf{b}_{i_1}, \cdots, \mathbf{b}_{i_s}$ is linearly dependent.

Remark

Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \xrightarrow{Elementary\ \ Row\ \ operations} B = [\mathbf{b}_1, \cdots, \mathbf{b}_n]$, then take any subset $\{i_1, \cdots, i_s\}$ of $\{1, \cdots, n\}$, one has

$$\begin{aligned} & k_1, \cdots, k_s \quad \text{satisfies} \quad \mathrm{k}_1\mathbf{a}_{\mathrm{i}_1} + \cdots + \mathrm{k}_\mathrm{s}\mathbf{a}_{\mathrm{i}_\mathrm{s}} = \mathbf{0} \\ \Longleftrightarrow & k_1, \cdots, k_s \quad \text{satisfies} \quad \mathrm{k}_1\mathbf{b}_{\mathrm{i}_1} + \cdots + \mathrm{k}_\mathrm{s}\mathbf{b}_{\mathrm{i}_\mathrm{s}} = \mathbf{0} \end{aligned}$$

where $k_i \in \mathbb{R}, i = 1, \cdots, s$.

Example 13.6 Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 \\ 2 & 8 & -1 & 3 & 9 \\ 0 & 0 & 2 & -3 & -4 \\ -1 & -4 & 2 & 4 & 8 \end{bmatrix}$$

Row reduce A into the echelon form:

$$B = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1,3,4 in B are linear independent, the corresponding columns 1,3,4 in A are also linear independent.
Columns 1,2,3,4 in B are linear dependent, the corresponding columns 1,2,3,4 in A are also linear dependent.

Moreover, it is easy to check that $\mathbf{b}_5 = 2\mathbf{b}_1 + \mathbf{b}_3 + 2\mathbf{b}_4$, then one has $\mathbf{a}_5 = 2\mathbf{a}_1 + \mathbf{a}_3 + 2\mathbf{a}_4$.

Theorem 13.7
(The pivot columns of A form a basis for Col (A))

Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$ be row equivalent to reduced row-echelon form B with pivot columns indices $d_1, \cdots, d_r$. Let $T = \{\mathbf{a}_{d_1}, \cdots, \mathbf{a}_{d_r}\}$, then

(1) T is linearly independent.
(2) $\mathbf{Span}(T) = \mathbf{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} = \mathrm{Col}\,(A)$.

Remark 1: $\mathbf{a}_{d_1}, \cdots, \mathbf{a}_{d_r}$ are pivot columns of A, the rest of columns are nonpivot columns of A. The nonpivot columns of B are linear combinations of pivot columns of B since B is in reduced row-echelon form. Thus, nonpivot columns of A are also linear combinations of pivot columns of A.

Remark 2: $T = \{\mathbf{a}_{d_1}, \cdots, \mathbf{a}_{d_r}\}$ (pivot columns of A) form a basis for $\mathrm{Col}\,(A)$. Thus, the dimension of $\mathrm{Col}\,(A)$=the number of pivot columns of A.

Recall: $\dim(\mathrm{Null(A)})$=the number of independent variables for the linear system $A\mathbf{x} = \mathbf{0}$=the number of nonpivot columns of A.

Together with $\dim(\mathrm{Col}\,(A))$=the number of nonpivot columns of A. We have the following corollary:

Corollary: Let A be an $m \times n$ matix.

$$\dim(\mathrm{Col}\,(A)) + \dim(\mathrm{Null}\,(A)) = n.$$

Example 13.8 Let

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 \\ 2 & 8 & -1 & 3 \\ 0 & 0 & 2 & -3 \\ -1 & -4 & 2 & 4 \end{bmatrix}$$

Thus, in order to find $\mathrm{Col}\,(A)$, we reduce A into the row-echelon form:

$$B = \begin{bmatrix} \boxed{1} & 4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the pivot columns of A are columns 1,3,4. (Pivot column indices are 1,3,4)

Thus,

$$\mathrm{Col}\,(A) = \textbf{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}\right)$$

But

$$\mathrm{Col}\,(B) = \textbf{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) \neq \mathrm{Col}\,(A)$$

Remark. The pivot of column indices of reduced row echelon form B only tells which columns of A form a basis of $\mathrm{Col}\,(A)$, but in general, $\mathrm{Col}\,(A) \neq \mathrm{Col}\,(B)$. **Row operations does not preserve the column space.**

Theorem 13.9 Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \in \mathbb{R}^{n \times n}$ $(\mathbf{a}_i \in \mathbb{R}^n (i = 1, \cdots, n))$, then A is invertible $\Leftrightarrow$ $\mathrm{Col}\,(A) = \mathbb{R}^n$.

Proof.

$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ is invertible
$\Leftrightarrow$ $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} \subseteq \mathbb{R}^n$ is linearly independent (Theorem 10.5)
$\Leftrightarrow$ $\mathbf{a}_1, \cdots, \mathbf{a}_n$ is a basis of $\mathbb{R}^n$ (Theorem 12.16)
$\Leftrightarrow$ $\mathrm{Col}\,(A) = \mathbf{Span}(\mathbf{a}_1, \cdots, \mathbf{a}_n) = \mathbb{R}^n$. (Definition of the basis)

Example 13.10 Observe the columns of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

are linearly independent, and A is invertible by Theorem 10.5. Thus $\mathrm{Col}\,(A) = \mathbb{R}^2$.

Theorem 13.11[Invertible Matrix Theorem] Given a $n \times n$ matrix $A \in \mathbb{R}^{n \times n}$ (square!). The following are equivalent:

1. A is invertible
2. the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution
3. A is row equivalent to I
4. A is a product of elementary matrices
5. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$
6. the columns of A are linearly independent
7. the columns of A form a basis of $\mathbb{R}^n$, i.e. $\mathrm{Col}\,(A) = \mathbb{R}^n$
8. the columns of A span $\mathbb{R}^n$
9. $\mathrm{Null}\,(A) = \{\mathbf{0}\}$
10. $\mathrm{Col}\,(A) = \mathbb{R}^n$
11. $\dim(\mathrm{Null}\,(A)) = 0$.
12. $\dim(\mathrm{Col}\,(A)) = n$

The equivalence between 1-6 is given in Theorem 10.5.
The equivalence between 1,6,7 is given in Theorem 12.10.
The equivalence between 1 and 9 is given in Theorem 13.3.
The equivalence between 1 and 10 is given in Theorem 13.11.
The equivalence between 1 and 8 is given in Theorem 13.11. Indeed, 8 and 10 are the same.
The equivalence between 9 and 11 is obvious since $\dim(\text{Null}(A)){=}0 \Leftrightarrow \text{Null}(A){=}\{\mathbf{0}\}$. In fact, $\dim(V) = 0 \Leftrightarrow V = \{\mathbf{0}\}$ for any vector space V.
The equivalence between 11 and 12 is due to that $\dim(\text{Col}\,(A)) + \dim(\text{Null}\,(A)) = n$.