PHY1001: Mechanics (Week 1)

The study of physics is like an adventure. You will find it challenging, frustrating, maybe occasionally painful, and yet often satisfying and exciting. Here are some useful tips in the process of study: 1. learn through thinking, reflection and practice; 2. discuss and work with your fellow classmates and learn from each other; 3. master math and other tools.

We begin with some important preliminaries of physics study. We will discuss the nature of physics, and idealized models. We will also introduce the system of units and order-of-magnitude estimates. In the end, we will study the basic aspect of vectors. We are going to learn motion in one, two and three dimensions and introduce the physical quantities displacement, velocity and acceleration. In this lecture, we can also learn that how derivatives and integrations are applied in physics to solve problems.

1 The Nature of Physics

Physics is the study of natural phenomena, and it is an experimental science. In physics, we build theories based on observation of nature and accepted fundamental principles. These theories can evolve into simple and well-established physics laws or principles.

The development of physics theories is always a two-way process that starts or ends with observations and experiments. On one hand, we develop theories based on experimental findings. On the other hand, theories must be able to withstand numerous tests of experiments.

No theory is ever regarded as the ultimate or final truth. Every theory has its own range of validity: it only applies within the range. Outside this range, this theory should be modified or revised, e.g., the development of Einstein theory of relativity from Newtonian Mechanics.

1.1 Idealized Models

Physics laws, such as Newton's laws, often are simple and elegant. When we study physics, we often seek simplicity: we build models which are simplified version of reality that allow us to gain insight into a physical process. To make an idealized model, we have to overlook (ignore) some minor effects to concentrate on the most important feature of the physical system. When we construct models, we need to simplify the system enough to make it doable and manageable, yet keep the essential features.

A Thrown Ball
Suppose we want to analyse the motion of a thrown baseball. If we are only interested in the trajectory of this ball, we often neglect the size and shape of the ball, as well as the air resistance. Therefore, it is treated as a point-like object moving in a vacuum.

Suppose we want to study the rotation of the ball or the physics of the curveball, then the size of the ball and air resistance can no longer be ignored.

2 Units

Any number that is used to describe a physical phenomenon quantitatively is called a *physical quantity*. A physical quantity is a measurable quantity and it includes two parts: a <u>number</u> and its <u>unit</u>. The number alone is not meaningful without the unit.

In science, we always use the International System of Units or SI (the abbreviation for its French name, **Systèm International d'unités**). One should be very careful if other unit system is used. In 1999, US lost the Mars Climate Orbiter because some part of the control system used the British **Metric system** instead of SI.

Now let us introduce three *fundamental* SI units used in mechanics.

- <u>Time</u>: the unit is second (abbreviated s). One second is the time taken by $9, 192, 631, 770$ oscillations of light emitted at the transition between the two hyperfine levels of the ground state of a cesium-133 atom. Normally it was defined as the time required for a one-meter-long pendulum to swing from one side to the other.
- <u>Length</u>: the unit of length is meter (m). The meter is the length of the path traveled by light in vacuum during a time interval of 1/299 792 458 of a second. Meter was defined as $1/10^7$ of the distance between the north pole and the equator with 0.02% error.
- <u>Mass</u>: the unit is kilogram (kg). The standard kilogram is a cylinder of platinum and iridium alloy kept in France. The atomic mass unit (u) is 1/12 of the mass of a carbon-12 atom, approximately $1.6605402 \times 10^{-27} kg$.

2.1 Unit prefixes (multiples of 10, or $1/10$)

We often need to introduce larger or smaller units for the same quantity. For example, the prefix kilo, abbreviated k, means a factor of 1000 . The important prefixes are:

- n =nano- (Greek "dwarf")=10^{-9} (thousand millionth)
- μ=micro- (Greek "small") = 10^{-6} (millionth)
- m = milli- (Latin "thousand") = 10^{-3} (thousandth)
- k = kilo- (Greek "thousand") = 10^3 (thousand)
- M = mega- (Greek "big") = 10^6 (million)
- G = giga- (Greek "giant") = 10^9 (billion)
- T = Tera- (Greek "monster") = 10^{12} (trillion)

Different physical scales:

- Distance: 1 kilometer= 1km= 10^3 m (ten to the third meters)
- Size of atoms: 1 nanometer= 1nm $=10^{-9}$m (ten to the minus nine meter)
- Size of nuclei: 1 femtometer = 1fm = 1fermi $=10^{-15}$m (ten to the minus fifteen meter) (Enrico Fermi)

x^n is called x to the power of n, or x to the n-th, or x to the n. Avogadro constant 6.02×10^{23} is called six point zero two times ten to the twenty-three.

2.2 Unit Consistency and Conversions

An equation must always be consistent in units. For a physical quantity, the unit tells the standard system used for the measurement and the number gives the quantity in that standard. To tell a quantity that we are measuring, we need to state the *dimension* of the physical quantity. Time, length and mass are all dimensions. (In terms of dimensions, we can write $[d] = L$, $[m] = M$ and $[t] = T$.)

For example, we can not add an area A to a speed v to obtain a meaningful sum. For the equation, $A = B + C$, all the quantities must have the same units and dimensions.

It is very useful to use dimension to check your result. For example, if you do a calculation and get $d = vt$ which implies $L = L/T \times T = L$. This is dimensionally consistent. On the other hand, if you get $d = vt^2$, you should check your algebra again.

Unit Conversion

$$
\begin{aligned}
90\,\mathrm{km/h} &= 90 \times \frac{10^3\,\mathrm{m}}{3600\,\mathrm{s}} = 25\,\mathrm{m/s} \\
&= 90\,\frac{\mathrm{km}}{\mathrm{h}} \times \frac{1\mathrm{mi}}{1.6\mathrm{km}} \simeq 56\,\mathrm{mi/h}.
\end{aligned}
$$

3 Uncertainty and Significant Figures

Measurements always have uncertainties/errors. The uncertainty of a measured value depends on the measurement technique used. We often indicate the accuracy of a measured value – that is, how close it is likely to be the true value – by writing the number, the symbol ±, and a second number indicating the uncertainty of the measurement. For example, suppose we measure the length of a table by using two different methods, and find two results written as follows.

Method 1	2.50 ± 0.01 m	3 sig. figures
Method 2	2.503 ± 0.001 m	4 sig. figures

It is easy to see that the second method is more accurate since it has smaller uncertainty. In many cases, the uncertainty is not written explicitly. If we drop the uncertainty and simply write 2.50 m or 2.503 m. We can still use the the significant figure to understand the accuracy of these two measurements.

- Significant figure is defined as the meaningful digits, or the number of reliably known digits in a measured quantity, excluding all the zeros used to locate the decimal points. For example, the result from the second measurement 2.503 m tells us that the first 3 digits are accurate, while the last digit 3 is uncertain. Therefore, it has 4 significant figures in total.
- When multiplying or dividing quantities, the number of significant figures in the final answer is no greater than that in the quantity with the fewest significant figures.
  Example: $1.58 \times 0.03 = 0.0474 = 0.05$. Do not forget to round up to 0.05 since we only have 1 significant figure here.
- When adding or subtracting quantities, the number of decimal places in the answer should match that of the term with the smallest number of decimal places.
  Example: $1.21342 - 1.040 = 0.17342 = 0.173$
- The number of significant figures in numbers with trailing zeros and no decimal point is ambiguous. We should use the scientific notation to indicate where the uncertainty is. In the scientific notation, the number is written as a product of a number between 1 and 10 and a power of 10.
  For example, the speed of light is $c = 299,792,458\,\mathrm{m/s}$, which is then written as $2.99792458 \times 10^8\,\mathrm{m/s}$. It has 9 significant figures. If we have a measured distance $x = 2300$ m, but the second digit 3 is already uncertain, then we should write it as $x = 2.3 \times 10^3$ m.

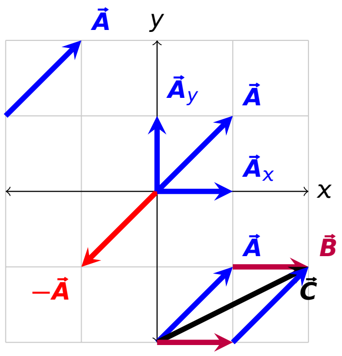
4 Estimates and Order-of-Magnitude

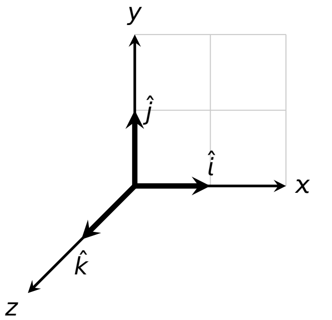
We have discussed the importance of knowing the accuracy of number that representing physical quantities. But sometimes even a very crude estimate of a quantity often gives us useful information. Such calculations are often called order-of-magnitude estimates. The great Italian-American nuclear physicist Enrico Fermi called them "back-of-envelop calculations".

The world's largest ball of string is about 2 m in radius. Estimate the total length of the string. Diameters of the string is around 0.004 m = 4mm. **Answer:** 3×10^6m.

5 Vectors and Vector Addition

When a physical quantity is described by a single number, we call it a scalar quantity, such as time, mass, density and temperature. In contrast, a vector quantity, such as the displacement, has both a magnitude and a direction. We draw a vector as a line with an arrowhead at its tip. The length of the line shows its magnitude and the direction of this line indicates the vector's direction.





As illustrated above, vectors have the following properties

1. Direction: If two vectors are parallel to each other, they have the same direction. If two vectors are anti-parallel to each other, they have the opposite direction.

2. Equality: Vectors are equal if their magnitudes and directions are the same. We call $\vec{V} = -\vec{A}$, if $\vec{V}$ and $\vec{A}$ have the same magnitude but opposite direction.

3. Addition: In vector addition, $\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$, we place the tail of the second vector at the head, or tip, of the first vector. Alternatively, we can add vectors by constructing a parallelogram.

4. Subtraction: We can subtract vectors as well as add them, because $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$. Thus, we first reverse $\vec{B}$ to obtain $-\vec{B}$, then we add $\vec{A}$ and $-\vec{B}$ together.

5. Components of Vectors: In 2-dimensional Cartesian coordinates, we can write $\vec{A} = \vec{A}_x + \vec{A}_y$ as the sum of the x and y components of vector $\vec{A}$, where the magnitudes
   $A_x = A\cos\theta$ and $A_y = A\sin\theta$,
   with θ defined as the angle between $\vec{A}$ and the positive x axis. This can be easily generalized to three-dimensional by adding a z component.

6. Unit Vectors: A unit vector is a vector that has a magnitude of 1, with no physical units. For example, we can use $\hat{\imath}$ and $\hat{\jmath}$ to describe (represent) the x and y direction, respectively. Then, we can write the vector in terms of its components as follows

The addition of two vectors in 2-dimension

$$\begin{aligned} \vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} \\ \text{Pythagorean theorem}: A &= \sqrt{A_x^2 + A_y^2}. \end{aligned}$$

$$\tan\theta = \frac{A_y}{A_x} \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x} \tag{1}$$

In terms of unit vectors, the addition of two vectors can be carried out very easily.

The addition of two vectors in 2-dimension

$$\begin{aligned} \vec{C} &= \vec{A} + \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\ &= \underbrace{(A_x + B_x)}_{C_x}\hat{\imath} + \underbrace{(A_y + B_y)}_{C_y}\hat{\jmath}. \\ C &= |\vec{C}| = \sqrt{C_x^2 + C_y^2}. \end{aligned} \tag{2}$$

For vectors in 3-dimensional space, we add $\hat{k}$ to represent the z direction.

The addition of two vectors in 3-dimension

$$\begin{aligned} \vec{C} &= \vec{A} + \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}), \\ &= \underbrace{(A_x + B_x)}_{C_x}\hat{\imath} + \underbrace{(A_y + B_y)}_{C_y}\hat{\jmath} + \underbrace{(A_z + B_z)}_{C_z}\hat{k}, \\ C &= |\vec{C}| = \sqrt{C_x^2 + C_y^2 + C_z^2}. \end{aligned} \tag{3}$$

7. Multiplying a vector by a scalar: If we multiply a vector $\vec{A}$ by a scalar s, each component of the product $\vec{B} = s\vec{A}$ is just the product of s and the corresponding component.

$$B_x = sA_x \quad \text{and} \quad B_y = sA_y. \tag{4}$$

$\vec{B} = s\vec{A}$ has the same direction as $\vec{A}$ if $s > 0$, while $\vec{B} = s\vec{A}$ has the same direction of $-\vec{A}$ if $s < 0$.

6 Products of Vectors

There are two types of products for vectors, namely the scalar product, which results in a scalar, and the vector product, which yields a vector. The scalar product is used in computing work, while the vector product describes torque and angular momentum.

6.1 Scalar product

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \cdot \vec{B}$. Because of this notation, the scalar product is also called the dot product. Although $\vec{A}$ and $\vec{B}$ are vectors, the quantity $\vec{A} \cdot \vec{B}$ is a scalar.

Scalar Product (Dot Product)

$$S = \vec{A}\cdot\vec{B} \equiv AB\cos\phi \tag{5}$$

where ϕ is the angle between $\vec{A}$ and $\vec{B}$ ranges from 0° to 180°. A and B are the magnitudes of $\vec{A}$ and $\vec{B}$, respectively.

The scalar product can be positive ($\phi < 90°$), negative ($\phi > 90°$) or zero ($\phi = 90°$). The scalar product of two perpendicular vectors is always zero.

If we express vectors in terms of unit vectors, we have $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$. By definition, we find $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$. We can also compute the scalar product using components and unit vectors, and find

$$S = \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z. \tag{6}$$

Now we have two ways to compute $\vec{A}\cdot\vec{B}$. It is always useful to try different methods to do calculations in general. This allows us to obtain an important relation

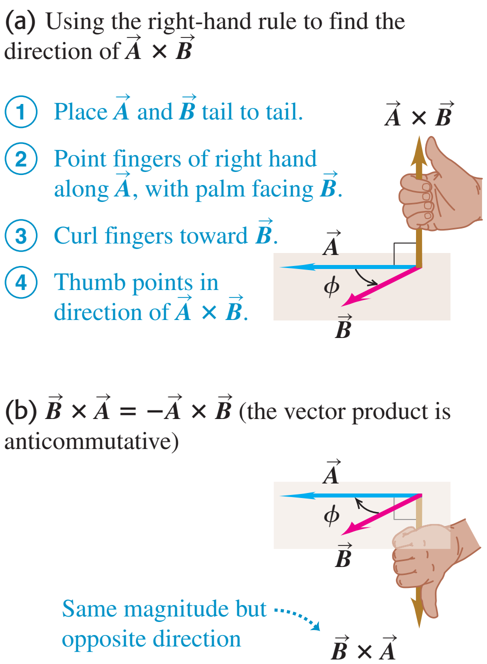
$$\cos\phi = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}. \tag{7}$$

Question: Can you use vector components and trigonometry to show the above relation?

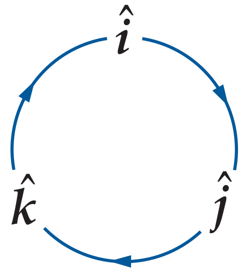
6.2 Vector product

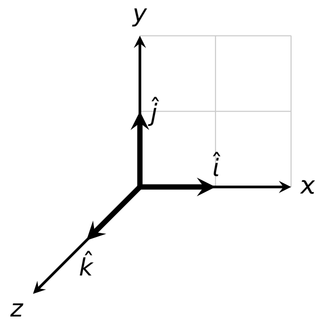
The vector product of two vectors $\vec{A}$ and $\vec{B}$, also called the cross product, is denoted by $\vec{A}\times\vec{B}$. The quantity $\vec{V} = \vec{A}\times\vec{B}$ is also a vector.

Definition: The magnitude of $\vec{A}\times\vec{B}$ is $V = AB\sin\phi$ and the direction of $\vec{V}$ is determined by the right-hand rule. The vector $\vec{V}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ and it is along the direction of the thumb of your right hand if you curl your fingers from the direction of $\vec{A}$ towards the direction of $\vec{B}$ as shown in the figure below.



It follows from the definition that $\vec{A}\times\vec{A} = 0$ and $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$ for any vectors $\vec{A}$ and $\vec{B}$.





By definition, the vector products of unit vectors are

$$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0 \tag{8}$$
$$\hat{\imath}\times\hat{\jmath} = \hat{k},\ \hat{\jmath}\times\hat{k} = \hat{\imath},\ \hat{k}\times\hat{\imath} = \hat{\jmath}. \tag{9}$$

Switching the order of product gives $\hat{\jmath}\times\hat{\imath} = -\hat{k}$, etc. The above cyclic figure is a very useful tool for remembering vector products of unit vectors. The sign is positive if one takes the product by going around the circle in the direction of the arrows, while the sign is negative when one goes against the arrows. Any coordinate system for which the above equations hold is called a right-handed coordinate system. We will always use a right-handed coordinate system in physics.

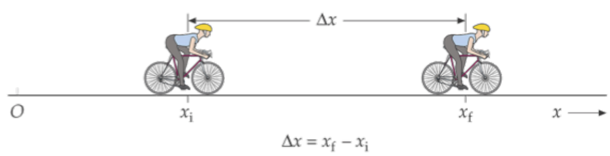
Using components, we can also find

Vector Product (Cross Product)

$$\begin{aligned}\vec{V} &= \vec{A}\times\vec{B} = -\vec{B}\times\vec{A} \\ &= (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}.\end{aligned} \tag{10}$$

The cyclic rule mentioned above is also very useful for us to remember the vector product equation, which can be written as the sum of all cyclic permutations (e.g., $A_xB_y\hat{k}$) and anti-cyclic permutations (e.g., $-A_yB_x\hat{k}$) of components of $\vec{A}$ and $\vec{B}$ together with unit vectors.

7 Displacement, Velocity and Speed



As shown above, shows a student on a bicycle at position x_i at time t_i. At a later time t_f, the student is at position x_f. Therefore, in the time interval $\Delta t \equiv t_f - t_i$, the change in student's position is defined as the displacement.

Definition of Displacement and Average Velocity

The change in the position: $\Delta x \equiv x_f - x_i$. (11)

Δx and Δt are symbols that mean “the change" in x and t, respectively. They should never be viewed as product of Δ and x (or t). In addition, we can also define the average velocity for the time interval Δt as

The average rate of change in the position $v_{av\,x} \equiv \dfrac{\Delta x}{\Delta t}$.

In contrast, we can also define the distance and average speed

Definition of Distance and Average Speed

The total distance traveled depends on the path Δs. (12)

and the average speed for the time interval Δt as

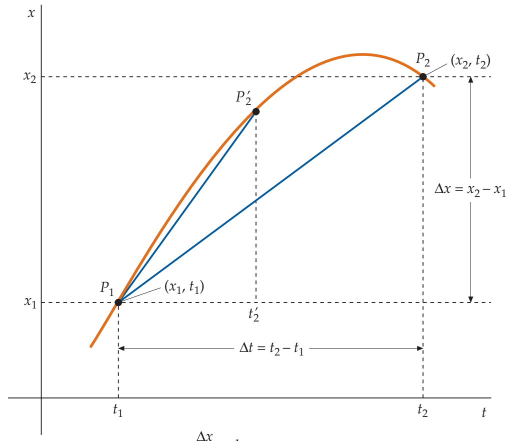
$$v \equiv \frac{\Delta s}{\Delta t}. \tag{13}$$

- Displacement only depends on the initial and final positions, while distance counts the length of the full path.

- We need to keep in mind that Δt is always positive by definition, but Δx can be positive or negative depending on our choice of positive x direction. When Δx is positive, it means that the displacement is along the positive x direction. When Δx is negative, it means that the displacement is along the negative x direction.

- The geometric interpretation of average velocity: the average velocity for the time interval $\Delta t = t_2 - t_1$ is the slope of the straight line connecting the points (t_1, x_1) and (t_2, x_2) on an x versus t graph. In geometry, the slope is a measure of the steepness of the straight line.

- The difference between velocity and speed: When a world-class swimmer swims a two-length in a 50-m pool in 50 seconds, his swimming speed is the distance (100 m) divided by the time interval (50 seconds) 100m/50s = 2m/s. But since he starts and ends in the same point, therefore his displacement and average velocity is zero. In addition, velocity is a vector as we will see later while speed is a scalar.



$\dfrac{\Delta x}{\Delta t}$ = slope = $v_{av\,x}$

8 Instantaneous Velocity and Speed

As we can see clearly, the average velocity sometimes can not tell us useful information about how fast a swimmer swims if the time interval is too large. Therefore, we need to use instantaneous velocity to denote the velocity at given time t (or specific coordinate x). It is defined as the limit of the ratio $\dfrac{\Delta x}{\Delta t}$ as Δt approaches zero.

Instantaneous Velocity $$v_x \equiv \lim_{\Delta t\to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{14}$$

In the language of calculus, $v_x = \dfrac{dx}{dt}$ is called the derivative of x with respect to t. Since Δx can be positive or negative, the instantaneous velocity v_x can be positive (object is moving in the $+x$ direction) or negative (object is moving in the $+x$ direction). Since $\Delta S = |\Delta x|$ when $\Delta t \to 0$, we can see that the instantaneous speed $v = \lim_{\Delta t\to 0} \dfrac{\Delta s}{\Delta t}$ is just the magnitude of v_x and it is always positive.

If a moving body has a constant velocity v_x, then its position x changes linearly with time, namely $x = x_0 + v_x t$ according to the definition of the velocity as the first derivative of x with respect to time.

9 Acceleration

Just as velocity describes the rate of change in position with time, acceleration describes the rate of change of velocity with time.

Definition of Average and Instantaneous Acceleration

Average Acceleration $$a_{av-x} \equiv \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t}. \tag{15}$$

Instantaneous Acceleration $$a \equiv \lim_{\Delta t\to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \tag{16}$$

When the v_x and a_x of a moving body have the same sign, the moving body is speeding up, which means the speed (the magnitude of the velocity) is increasing. When v_x and a_x have the opposite sign, the body is slowing down. Also the acceleration can be written as the second order derivative of x with respect to time, namely, $a_x = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$.

10 Motion with Constant Acceleration

The simplest example of motion with non-zero acceleration is the motion with constant acceleration. If a moving body has a constant acceleration a_x, then its velocity v_x changes linearly with time, namely $v_x = v_0 + a_x t$ according to the definition of the acceleration as the first derivative of v_x with respect to time.

Let us view this type of motion from the perspective of integration, which can be considered as the reverse operation of differentiation (derivative). Suppose we know $a_x = \frac{dv_x}{dt}$ as the function of t, we can ask ourselves what is the corresponding velocity at any give time. In order to answer this question by computing the change of velocity in a finite interval starting from the initial velocity v_0. This procedure is known as the integration. (If we know $v_x(t)$ to begin with, we can obtain a_x immediately by taking the derivative $\frac{dv_x}{dt}$).

According to the definition of the acceleration, we can write $dv_x = a_x dt$ and then find that the change in velocity for some time interval can be interpreted as the area under the a_x-vs-t curve for that interval

$$v_x(t_2) - v_x(t_1) = \Delta v_x = \lim_{\Delta t \to 0}\left(\sum_i a_{ix}\Delta t\right) = \int_{t_1}^{t_2} a_x dt. \quad (17)$$

Let us set $t_1 = 0$ and $t_2 = t$, then we can arrive at

$$v_x(t) = v_{0x} + \int_0^t a_x dt. \quad (18)$$

For constant acceleration, we have $v_x(t) = v_{0x} + a_x t$. Similarly, the position $x(t)$ can be derived as well

$$\frac{dx}{dt} = v_x \quad \Rightarrow \quad x = x_0 + \int_0^t v_x dt, \quad (19)$$

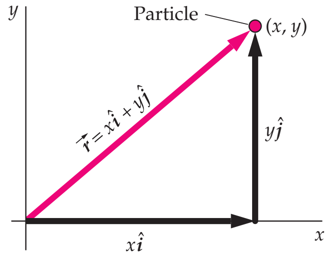
where $x(t = 0) = x_0$. For constant acceleration, we can arrive at

$$x = x_0 + \int_0^t v_x dt = x_0 + v_{0x}t + \frac{1}{2}a_x t^2. \quad (20)$$

11 Two and Three dim Kinematics

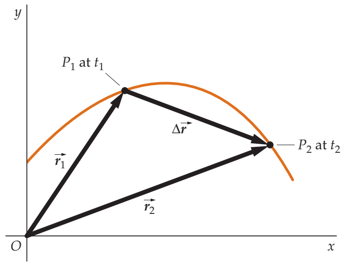
We have already seen the 1-D kinematics equations:

$$x = x(t), \quad v_x = \frac{dx}{dt}, \quad a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}.$$



Let us consider two dimensional motion in two-dimensional Cartesian coordinates. For a particle in the x, y plane at the point with coordinates (x, y), the position vector $\vec r$ is

$$\vec r = x\hat i + y\hat j. \quad (21)$$



Let us consider the displacement as shown in the above figure. At time t_1, the particle is at P_1 with position vector $\vec r_1$; by time t_2, the particle is at P_2 with position vector $\vec r_2$. Therefore, the displacement vector can be written as

$$\Delta\vec r = \vec r_2 - \vec r_1 \quad (22)$$
$$= \underbrace{(x_2 - x_1)}_{\Delta x}\hat i + \underbrace{(y_2 - y_1)}_{\Delta y}\hat j. \quad (23)$$

Then, in terms of components of unit vectors, the velocity is

$$\vec v = \lim_{\Delta t \to 0}\frac{\Delta\vec r}{\Delta t} = \lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t}\hat i + \lim_{\Delta t \to 0}\frac{\Delta y}{\Delta t}\hat j, \quad (24)$$
$$= \frac{dx}{dt}\hat i + \frac{dy}{dt}\hat j = v_x\hat i + v_y\hat j, \quad (25)$$

where v_x and v_y are the x and y components of the velocity. Th magnitude of the velocity is given by $v = \sqrt{v_x^2 + v_y^2}$ and the direction of the velocity is also determined $\tan\theta = \frac{v_y}{v_x}$.

Similarly, one can add a z direction, thus the position, velocity, and acceleration of a particle in 3 dimensions can be expressed as:

$$\vec r(t) = x(t)\hat i + y(t)\hat j + z(t)\hat k, \quad (26)$$
$$\vec v(t) = \lim_{\Delta t \to 0}\frac{\Delta\vec r}{\Delta t} = \frac{dx}{dt}\hat i + \frac{dy}{dt}\hat j + \frac{dz}{dt}\hat k,$$
$$= v_x\hat i + v_y\hat j + v_z\hat k, \quad (27)$$
$$\vec a(t) = \frac{d^2x}{dt^2}\hat i + \frac{d^2y}{dt^2}\hat j + \frac{d^2z}{dt^2}\hat k,$$
$$= a_x\hat i + a_y\hat j + a_z\hat k. \quad (28)$$

Parallel and Perpendicular components of acceleration can have different impact on the velocity. The parallel component changes the magnitude of the velocity, while the perpendicular component of the acceleration changes the direction of the velocity. This becomes very useful when we discuss the circular motion.

11.1 Projectile Motion

A projectile is any body that is given an initial velocity and then follows a path determined entirely by the effects of gravitational acceleration and air resistance (e.g., a thrown ball). The path followed by a projectile is called trajectory. Usually we neglect the effects of air resistance and the curvature and rotation of the earth.

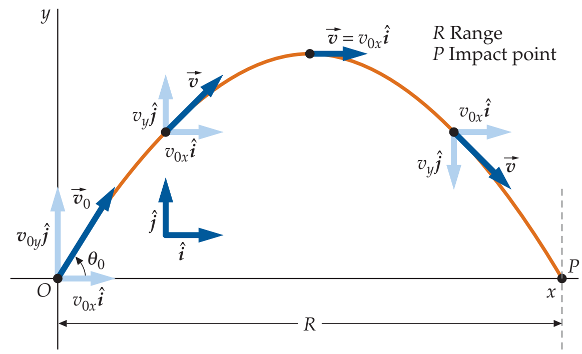
In general, the projectile motion is always confined to a plane determined by $\vec{v}_0$ and gravitational acceleration $-\vec{g}$, since we can always find a vertical plane which these two vectors $\vec{v}_0$ and $-\vec{g}$ are lying on. Therefore, in this plane, the projectile motion is a two dimensional motion. We usually call the plane of motion the xy-coordinate plane, with the x-axis horizontal and the y-axis vertically upward. The projectile motion is a combination of horizontal motion with constant velocity and vertical motion with constant acceleration as follows

$$a_x = 0, \qquad a_y = -g, \tag{29}$$
$$v_x = v_{0x}, \qquad v_y = v_{0y} - gt, \tag{30}$$
$$x = x_0 + v_{0x}t, \qquad y = y_0 + v_{0y}t - \frac{1}{2}gt^2. \tag{31}$$

The horizontal and vertical components of projectile motion are independent, which means we can consider the projectile motion in x and y axises separately. Choosing $x_0 = 0$ and $y_0 = 0$ in the xy-coordinate, we can obtain

$$y(x) = \frac{v_{0y}}{v_{0x}}x - \frac{g}{2v_{0x}^2}x^2, \tag{32}$$
$$= (\tan\theta_0)x - \frac{g}{2v_0^2\cos^2\theta_0}x^2, \tag{33}$$

where in the last step we have substituted for the velocity components using $v_{0x} = v_0\cos\theta_0$ and $v_{0y} = v_0\sin\theta_0$. This equation is of the form $y = ax + bx^2$, which is the equation for a parabola passing through the origin as shown below.



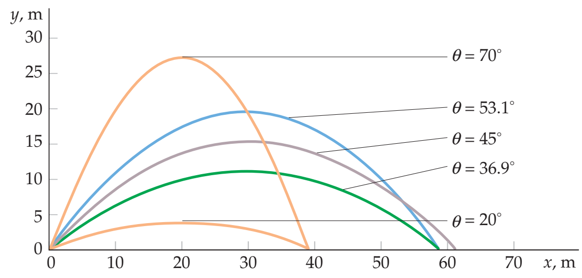
As shown above, suppose the projectile impacts the ground at P, which has the same height (elevation) as the position where it is launched. The horizontal distance Δx between the launch and impact position is called the horizontal range R.

By setting $\frac{dy(x)}{dx} = 0$, we can obtain the maximum height of the trajectory

$$y_{\max} = h = \frac{v_0^2\sin^2\theta_0}{2g}. \tag{34}$$

The horizontal range R can also computed by setting $y = 0$ and solve for x. There are two solutions, namely $x = 0$ and $x = \frac{2v_0^2\sin\theta_0\cos\theta_0}{g} = \frac{v_0^2\sin 2\theta_0}{g}$. Therefore the horizontal range R is

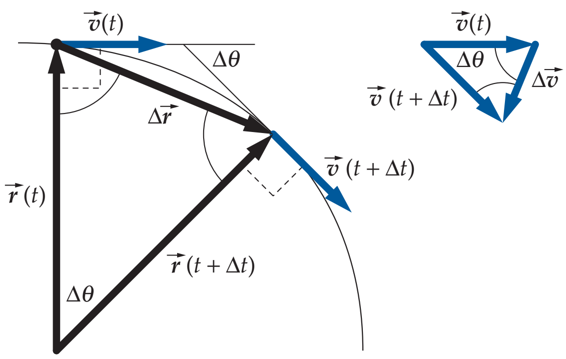
$$R = \Delta x = \frac{v_0^2\sin 2\theta_0}{g}. \tag{35}$$



As shown above, the range R becomes the largest when $\sin 2\theta_0 = 1$, that is to say $\theta_0 = \frac{\pi}{4}$.

11.2 Uniform Circular Motion

Motion along a circular path, or a segment of a circular path is called circular motion. As a particle moves along a circular arc, the direction from the particle toward the center of the circle is called the radial (centripetal) direction and the direction of the velocity vector is called the tangential direction.



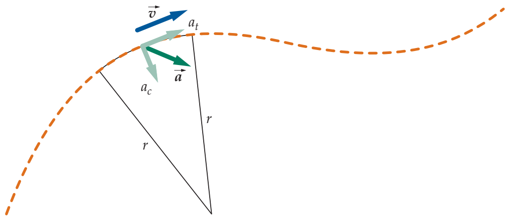
Motion in a circle at constant speed is called uniform circular motion. The position and velocity vectors for a particle moving in a circle at constant speed are shown in above figure. The two triangles formed by $\vec{r}$ and $\vec{v}$ are similar, and corresponding lengths of similar geometric figures are proportional. Thus,

$$\frac{|\Delta\vec{v}|}{|\Delta\vec{r}|} = \frac{v}{r} \quad \Rightarrow \quad \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v}{r}\frac{|\Delta\vec{r}|}{\Delta t} = \frac{v^2}{r}. \tag{36}$$

In circular motion, the direction of the moving object's velocity is changing all the time due to the Centripetal acceleration

$$\text{Centripetal acceleration} \quad a_c = \frac{v^2}{r}. \tag{37}$$

Centripetal acceleration is the component of the acceleration vector in the centripetal direction, and it only changes the direction of the velocity without changing the magnitude. The motion of a particle moving in a circle with constant speed is often described in terms of the time $T = \dfrac{2\pi r}{v}$ required for one complete revolution, called the period.



In contrast, as shown above, the parallel component of the acceleration, which is also known as the tangential acceleration $a_t = \dfrac{dv}{dt}$, changes the magnitude of the velocity $v = |\vec{v}|$.

11.3 Relative Velocity

Velocity is defined relative to a frame of reference. A frame of reference (or reference frame) is a coordinate system plus a time scale. In general, when two observer measure the velocity of a moving body, they get different results if one observer is moving relative to the other. The velocity seen by a particular observer is called the velocity relative to that observer, or simply relative velocity.

Now let us consider the velocity of an object (p) in frame A and B. To start, we first write down their relationship in the coordinate system as follows

$$\vec{r}_{\text{p/B}}(t) = \vec{r}_{\text{p/A}}(t) + \vec{r}_{\text{A/B}}(t), \tag{38}$$

where $\vec{r}_{\text{p/B}}$ ($\vec{r}_{\text{p/A}}$) represents the position of object (p) in frame B (A), while $\vec{r}_{\text{A/B}}$ stands for the position of the origin of frame A with respect to the origin of frame B.

Second, we differentiate both side of the above equation with respect to t, and obtain the so-called the Galilean velocity transformation by assuming that the time scale in frame A and B are the same

$$\vec{v}_{\text{p/B}}(t) = \vec{v}_{\text{p/A}}(t) + \vec{v}_{\text{A/B}}(t), \tag{39}$$

It is important to note that the right hand side of the above formula is a vector sum.

In addition, by definition, the vector $\vec{r}_{\text{A/B}}$ is the vector from the origin of B to the origin of A, then $\vec{r}_{\text{B/A}}$ is in the opposition direction of $\vec{r}_{\text{A/B}}$. Therefore,

$$\vec{r}_{\text{A/B}} = -\vec{r}_{\text{B/A}} \quad \Rightarrow \quad \vec{v}_{\text{A/B}} = -\vec{v}_{\text{B/A}}. \tag{40}$$

12 Summary of Calculus

12.1 Derivatives

Derivatives give the rate of change. In terms of Newton's notation, the derivative of $f(x)$ with respect to x is $f'(x)$, while it is written as $\dfrac{df(x)}{dx}$ or $\dfrac{d}{dx}f(x)$ in Leibniz's notation. Here are some useful formulas with a a constant and $e \simeq 2.718$ the Euler's number.

$$\begin{aligned}
&\frac{d}{dx}x^n = nx^{n-1}, && \frac{d}{dx}e^{ax} = ae^{ax}, \\
&\frac{d}{dx}\cos ax = -a\sin ax, && \frac{d}{dx}\sin ax = a\cos ax, \\
&\frac{d}{dx}\ln ax = \frac{1}{x}, && \frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v.
\end{aligned}$$

12.2 Integrals

Integrals provide the inverse operation of derivatives. Here are some useful integral formulas.

$$\begin{aligned}
&\int x^n dx = \frac{x^{n+1}}{n+1}\ (n \neq -1), && \int \frac{dx}{x} = \ln x, \\
&\int \sin ax dx = -\frac{1}{a}\cos ax, && \int \cos ax dx = \frac{1}{a}\sin ax, \\
&\int e^{ax} dx = \frac{1}{a}e^{ax}, && \int u\,dv = uv - \int v\,du.
\end{aligned}$$

12.3 Power Series

It is also useful to know some power series as follows with the radius of convergence shown at the end of each formula.

$$\begin{aligned}
(1+x)^k &= 1 + kx + \frac{k(k-1)x^2}{2!} + \cdots\ (|x|<1) \\
\frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots\ (|x|<1) \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots\ (|x|<1) \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\ (\text{all } x) \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots\ (\text{all } x) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots\ (\text{all } x)
\end{aligned}$$