STA2001 Probability and Statistics (I)

Lecture 6

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Review

Definition[Special mathematical expectation]

$$E[g(X)] = \sum_{x \in \overline{S}} g(x) f(x)$$

$$g(X) = \begin{cases} X \to \text{ Mean} \\ (X - E[X])^2 \to \text{ Variance} \\ X^r \to \text{ Moment} \\ e^{tX}, \text{ for } |t| < h, \to \text{ Mgf:} M(t) = \begin{cases} M(0) = 1 \\ M'(0) = E[X] \\ M''(0) = E[X^2] \end{cases} \end{cases}$$

Review

We are interested in the number of successes in n Bernoulli trials.

Definition[Binomial distribution]

A RV X is said to have a binomial distribution with n Bernoulli trials and the probability of success p, if the range space $\overline{S} = \{0, 1, \cdots, n\}$ and the pmf $f(x)$ is in the form of

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots, n.$$

We can simply denote it by $X \sim b(n, p)$.

Mgf of Binomial Distribution

Let $X \sim b(n, p)$. Then by definition,

$$\begin{aligned} M(t) = E[e^{tX}] &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= [(1-p) + pe^t]^n \qquad -\infty < t < \infty \end{aligned}$$

From the expansion of

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} \text{ with } a = pe^t, \quad b = 1-p$$

Mgf of Binomial Distribution

Question

What is the use of mgf?

Mgf of Binomial Distribution

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What is the use of mgf?

$$M'(t) = n[(1-p)+pe^t]^{n-1}pe^t \Rightarrow M'(0) = E[X] = np$$

$$M''(t) = n(n-1)[(1-p)+pe^t]^{n-2}p^2e^{2t} + n[(1-p)+pe^t]^{n-1}pe^t$$

$$M''(0) = E[X^2] = n(n-1)p^2 + np$$

$$Var[X] = E[X^2] - (E[X])^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p)$$

By the way, when $n = 1$ in $b(n, p)$, the binomial distribution reduces to Bernoulli distribution denoted by $b(1, p)$.

cdf of Binomial Distribution

$$F(x) = P(X \le x) = \sum_{y \in \{X \le x\}} f(y) = \sum_{y=0}^{[x]} \binom{n}{y} p^y (1-p)^{n-y},$$

where $x \in (-\infty, \infty)$ and $[x]$ is the largest integer $\le x$.

Example 3

A kind of chicken are raised for laying eggs. Let $p = 0.5$ be the probability that the newly hacked chick is a female. Assuming independence, let X be the number of female chicken out of 10 newly hatched chicks selected at random.

$$P(X \leq 5)?$$

$$P(X = 6)?$$

$$P(X \geq 6)?$$

Example 3

Then $X \sim b(10, 0.5)$

$$P(X \leq 5) = \sum_{x=0}^{5} \binom{10}{x} 0.5^x 0.5^{5-x}$$

$$P(X = 6) = \binom{10}{6} 0.5^6 0.5^4 = P(X \leq 6) - P(X \leq 5)$$

$$P(X \geq 6) = 1 - P(X \leq 5)$$

Section 2.5 Negative Binomial Distribution

Negative Binomial Distribution

Description: We are interested in the number of Bernoulli trials

until exactly r successes occur, where r is a fixed positive integer.

Negative Binomial Distribution

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until exactly r successes occur, where r is a fixed positive integer.

Define a RV X to denote the trial number at which the rth success

is observed. Then X has the range $\overline{S} = \{r, r+1, \cdots\}$.

Let $f(x)$ denote the pmf of X. Then recall $f(x) = P(X = x)$

Negative Binomial Distribution

$$
\begin{aligned}
f(x) &= P(\{\text{at the } x\text{th trial, the } r\text{th success is observed}\}) \\
&= P(\{\underbrace{\text{for the first } x-1 \text{ trials, } r-1 \text{ success have been observed}}_{A}\} \\
&\cap \{\underbrace{\text{at the } x\text{th trial, the outcome is a success}}_{B}\}) \\
&= P(A \cap B) = P(A)P(B)(\text{because } A \text{ and } B \text{ are independent})
\end{aligned}
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\end{aligned}
$$

$$
P(A) = \binom{x-1}{r-1} p^{r-1}(1-p)^{x-r}, \quad P(B) = p
$$

Therefore

$$
f(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r}, \quad x = r, r+1, \cdots
$$

Negative Binomial Distribution

Definition[Negative Binomial Distribution]

A RV X is said to have a negative binomial distribution with the probability of success p and the number of successes r we are interested in, if the range $\overline{S} = \{r, r+1, \cdots\}$ and the pmf $f(x)$ is in the form of

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \cdots .$$

This distribution get its name due to the negative binomial series

$$(1-w)^{-r} = \sum_{x=r}^{\infty} \left(\begin{array}{c} x-1 \\ r-1 \end{array} \right) w^{x-r}$$

Geometric Distribution

Definition[Geometric Distribution]

A RV X is said to have a geometric distribution with the probability of success p, if the range $\overline{S} = \{1, 2, \cdots\}$ and the pmf $f(x)$ is in the form of

$$f(x) = p(1-p)^{x-1}, \quad x = 1, 2, \cdots.$$

For a positive integer k,

$$P(X > k) = \sum_{x=k+1}^{\infty} p(1-p)^{x-1} = \frac{(1-p)^k p}{1-(1-p)} = (1-p)^k$$

$$P(X \leq k) = \sum_{x=1}^{k} p(1-p)^{x-1} = 1 - P(X > k) = 1 - (1-p)^k$$

Example 1, page 83

Biology students are checking eye color of fruit flies. For each fly,

$$P(\text{white}) = \frac{1}{4}, \quad P(\text{red}) = \frac{3}{4}.$$

Assume the observations are independent Bernoulli trials.

To observe 1 white fly, what's the probability one has to check

at least 4 flies?

at most 4 flies?

4 flies?

Example 1, page 83

We define X to be the number of fruit flies one has to check until the first white-eye fly is observed.

Then X has the geometric distribution with probability of success $1/4$. So the probability one has to check

at least 4 flies? $\longrightarrow P(X \geq 4) = P(X > 3) = (1 - \frac{1}{4})^3 = (\frac{3}{4})^3$

at most 4 flies? $\longrightarrow P(X \leq 4) = 1 - (1 - \frac{1}{4})^4$

4 flies? $\longrightarrow P(X = 4) = \frac{1}{4} \cdot (\frac{3}{4})^3$

Mathematical Expectations of Negative Binomial Distribution

Mean and Variance

$$\text{Mean}: E[X] = \frac{r}{p}$$

$$\text{Variance}: Var[X] = E[X^2] - (E[X])^2 = \frac{r(1-p)}{p^2}$$

can be calculated by using the mgf

$$\text{Mgf}: \ M(t) = E[e^{tX}] = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \ \text{for } (1-p)e^t < 1$$

which can be obtained by using the negative binomial series

$$(1-w)^{-r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} w^{x-r}$$

Section 2.6 Poisson Distribution

Motivation

Description: There are experiments that result in counting the number of times that particular events occur within a given period or for a given physical object:

- the number of flaws in a 100 feet long wire.
- the number of customers that arrive at a ticket window between 7:00-8:00 pm.

Counting such events can be seen as observations of a RV associated with an approximate Poisson process (APP).

Approximate Poisson Process (APP)

Definition[Approximate Poisson Process (APP)]

Let the number of occurrences of some event in a given continuous interval be counted. Then we have an APP with parameter $\lambda > 0$ if

(a) The number of occurrences in non-overlapping subintervals are <u>independent</u>.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately $\underline{\lambda h}$.

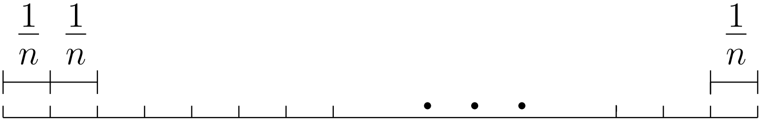
(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially <u>0</u>.

Poisson Distribution

Consider a random experiment described by APP. Let X denote the number of occurrences in **an interval with length 1**. We aim to find an approximation for $f(x) = P(X = x)$ with $x = 0, 1, 2, \cdots$.

To this goal,

1. Partition the unit interval into n equally spaced subintervals.



2. If n is sufficiently large ($n >> x$), $P(X = x)$ can be approximated by the probability that exactly x of these n subintervals each has one occurrence.

Poisson Distribution

2.1 By condition (c), the probability of two or more occurrences in any sufficiently short subinterval is 0. [n Bernoulli experiments.]

2.2 By condition (b), the probability of one occurrence in any subinterval (with length $\frac{1}{n}$) is approximately $\lambda\frac{1}{n}$. [Same probability of success $\lambda\frac{1}{n}$.]

2.3 By condition (a), the n Bernoulli experiments are independent. [n Bernoulli trials with probability of success $\lambda\frac{1}{n}$.]

Therefore occurrence and nonoccurrence in the n

subintervals are n Bernoulli trials with probability of success $\frac{\lambda}{n}$

Poisson Distribution

3. Therefore, $P(X = x)$ can be approximated by the probability of x successes for $b(n, p = \frac{\lambda}{n})$

$$\frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x}$$

4. Let $n \to \infty$. Then

$$\lim_{n\to\infty}\frac{n!}{x!(n-x)!}(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x}$$

$$= \lim_{n\to\infty}\frac{n!}{(n-x)!n^x}\cdot\frac{\lambda^x}{x!}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x}$$

Poisson Distribution

Noting

$$\lim_{n\to\infty} \frac{n!}{(n-x)!n^x} = 1$$

$$\lim_{n\to\infty} (1 - \frac{\lambda}{n})^n = e^{-\lambda},$$

$$\lim_{n\to\infty} (1 - \frac{\lambda}{n})^{-x} = 1$$

We have

$$P(X = x) = \lim_{n\to\infty} \frac{n!}{(n-x)!n^x} \cdot \frac{\lambda^x}{x!}(1 - \frac{\lambda}{n})^n(1 - \frac{\lambda}{n})^{-x} = \frac{\lambda^x e^{-\lambda}}{x!}$$

Poisson Distribution

It can be verified

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, , \cdots$$

is a well-defined pmf.

Definition[Poisson Distribution]

A RV X is said to have a Poisson distribution with the parameter λ, if the range $\overline{S} = \{0, 1, \cdots, \}$ and the pmf $f(x)$ is in the form of

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, , \cdots .$$

We can simply denote it by $X \sim \text{Poisson}(\lambda)$.

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Question

What's the implication of λ?

Mean and Variance

The mgf of a Poisson distributed RV X is

$$
\begin{aligned}
M(t) &= E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)} \\
M'(t) &= \lambda e^t e^{\lambda(e^t-1)} \Rightarrow M'(0) = \lambda \\
M''(t) &= \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)} \Rightarrow M''(0) = \lambda + \lambda^2 = E[X^2] \\
E[X] &= M'(0) = \lambda \\
Var[X] &= E[X^2] - (E[X])^2 = \lambda + \lambda^2 - \lambda^2 = \lambda
\end{aligned}
$$

λ is the mean and variance of $X \sim \text{Poisson}(\lambda)$: the average number of occurrences in **the unit interval!**

Example 1, page 91

Question

In SZ, telephone calls to 110 come on the average of 2 calls every 3 minutes. If one models with APP, what's the probability of 5 or more calls arrive in a 9-minute period?

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Let X denote the number of calls in a 9-minute period, which is **the unit interval** for the problem. Then $X \sim \text{Poisson}(\lambda)$.

We need to determine λ.

$$E[X] = 6 = \lambda \Longrightarrow f(x) = \frac{6^x e^{-6}}{x!}$$

Therefore,

$$P(X \geq 5) = 1 - P(X \leq 4) = 1 - \sum_{x=0}^{4} \frac{6^x e^{-6}}{x!}$$