STA2001 Probability and Statistics (I)

Lecture 4

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Review

- Conditional probability of an event A, given that event B has occurred, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$. Note: conditional probability is a probability function.

- Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

The occurrence of one of them does not change the probability of the occurrence of the other.

Review

(Mutually) Independent Events:

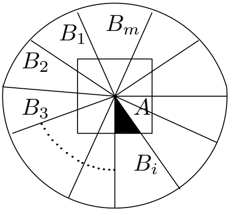
- A and B are independent, if and only if any pair of the following events are independent
  (a) A and B'
  (b) A' and B
  (c) A' and B'

- A, B, C are independent, if
  1. pairwise independent
  2. $P(A \cap B \cap C) = P(A)P(B)P(C)$

  Many properties hold.

Review

Bayes' Theorem



Assume

1. $S = B_1 \cup B_2 \cup \cdots B_m, \quad B_i \cap B_j = \Omega$
2. $P(B_i) > 0$

Then

$$P(A) = \sum_{k=1}^{m} P(A \cap B_i) = \sum_{k=1}^{m} P(B_i) P(A|B_i)$$

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{P(A)}, \text{ provided } P(A) > 0$$

Chapter 2 Discrete Distribution

Section 2.1 Random Variable of the Discrete Type

Motivations

1. Flip a coin.
2. Select a color from 256 colors.

original sample space		new and numeric sample space
S	$\leftrightarrow$	$\overline{S}$
$S = \{H, T\}$	$\leftrightarrow$	$\{1, 0\}$
$S = \{R, G, \cdots, B\}$	$\leftrightarrow$	$\{1, 2, \cdots, 256\}$
nonnumeric		numeric

There are other motivations ...

Random Variable (RV)

Definition[Random Variable]

Given a random experiment with sample space S, a function $X : S \rightarrow \overline{S} \subseteq R$ that assign one real number $X(s) = x$ to each $s \in S$ is called a Random Variable (RV).

- $\overline{S}$ denote the range of X: $\overline{S} = \{x | X(s) = x, s \in S\}$.

Understand a RV

Question

What's the relation between S and X? What's the relation between S and $\overline{S}$?

$$X : S \to \overline{S}$$

- RV defines a new random experiment with a numeric sample space $\overline{S}$
- If X is one to one, then old random experiment with S

  $\Leftrightarrow$ new random experiment with $\overline{S}$
- If X is not one to one, then old random experiment with S

  $\not\Leftrightarrow$ new random experiment with $\overline{S}$ (example will be given later)
- repeat the new random experiment is to generate a number randomly from $\overline{S}$

Example 1

The old random experiment is to roll a die with a, b, c, d, e, f on each side of the die with sample space $S = \{a, b, c, d, e, f\}$

1. define a RV: $X(a) = 1, \cdots, X(f) = 6$

$$X : S = \{a, b, c, d, e, f\} \to \overline{S} = \{1, 2, 3, 4, 5, 6\}$$

   the new random experiment is to roll a die with $1, 2, 3, 4, 5, 6$ on each side of the die
2. the old random experiment with sample space $S \Longleftrightarrow$ the new random experiment with numeric sample space $\overline{S}$
3. repeat the new random experiment is to generate a number randomly from $\overline{S} = \{1, 2, 3, 4, 5, 6\}$

Some Conventions

- uppercase letters, e.g. $X, Y, Z \to$ RVs
- lowercase letters, e.g. $x, y, z \to$ the numeric values that RV X, Y, Z can take, respectively

For a given random experiment, two probability functions are involved through $X : S \to \overline{S}$,

- $P_S(\cdot)$ is the probability function associated with S
- $P(\cdot)$ is the probability function associated with $\overline{S}$

$$P(X = x) \triangleq P(\{X = x\}) = P_S(\{s|X(s) = x, s \in S\})$$
$$P(X \in A) \triangleq P(\{X \in A\}) = P_S(\{s|X(s) \in A, s \in S\})$$

Example 1, continued

The old random experiment is to roll a die with a, b, c, d, e, f on each side of the die with sample space $S = \{a, b, c, d, e, f\}$

1. define a RV: $X(a) = 1, \cdots, X(f) = 6$

$$X : S = \{a, b, c, d, e, f\} \to \overline{S} = \{1, 2, 3, 4, 5, 6\}$$

   the new random experiment is to roll a die with $1, 2, 3, 4, 5, 6$ on each side of the die
2. the old random experiment with sample space $S \iff$ the new random experiment with numeric sample space $\overline{S}$
3. repeat the new random experiment is to generate a number randomly from $\overline{S} = \{1, 2, 3, 4, 5, 6\}$
4. Let $x = 1$ and $A = \{1, 2\}$

$$P(X = x) \triangleq P(\{X = x\}) = P_S(\{s | X(s) = 1, s \in S\})$$
$$P(X \in A) \triangleq P(\{X \in A\}) = P_S(\{s | X(s) \in A, s \in S\})$$

Discrete Random Variable

Definition

Recall that $\overline{S}$ denote the range of X: $\overline{S} = \{x|X(s) = x, s \in S\}$.

A RV X is said to be discrete if its range $\overline{S}$ is finite or countably infinite.

Example 1, continued

The old random experiment is to roll a die with a, b, c, d, e, f on each side of the die with sample space $S = \{a, b, c, d, e, f\}$

1. define a RV: $X(a) = 1, \cdots, X(f) = 6$

$$X : S = \{a, b, c, d, e, f\} \to \overline{S} = \{1, 2, 3, 4, 5, 6\}$$

2. X is discrete, because $\overline{S}$ is finite, i.e., it contains a finite number of outcomes

Probability Mass Function (pmf)

Definition

Suppose that X is a RV with range $\overline{S}$. Then a function $f(x) : \overline{S} \to (0, 1]$ is called pmf, if

1. $f(x) > 0, \quad x \in \overline{S}$.
2. $\sum_{x \in \overline{S}} f(x) = 1$.
3. $P(X \in A) = \sum_{x \in A} f(x), \quad A \subseteq \overline{S}$,

   which defines the probability function for an event A. In particular, taking $A = \{x\}$ yields the probability of $X = x$, i.e.,
   $$P(X = x) = f(x)$$

Probability Mass Function (pmf)

We often extend the domain of $f(x)$ from $\overline{S}$ to R and let

$f(x) = 0$, $x \notin \overline{S}$. In this case, $\overline{S}$ is called the support of $f(x)$.

Definition

Suppose that X is a RV with range $\overline{S}$. Then a function $f(x) : R \to [0, 1]$ is called pmf, if

1. $f(x) \geq 0, \quad x \in R.$
2. $\sum_{x \in \overline{S}} f(x) = 1.$
3. $P(X \in A) = \sum_{x \in A} f(x), \quad A \subseteq \overline{S}.$

Example 1, continued

The old random experiment is to roll a die with a, b, c, d, e, f on each side of the die with sample space $S = \{a, b, c, d, e, f\}$

1. define a RV: $X(a) = 1, \cdots, X(f) = 6$

$$X : S = \{a, b, c, d, e, f\} \to \overline{S} = \{1, 2, 3, 4, 5, 6\}$$

2. X is discrete, because $\overline{S}$ is finite, i.e., it contains a finite number of outcomes

3. pmf $f(x) = \frac{1}{6}, \quad x \in \overline{S}$, and $f(x) = 0, \quad x \notin \overline{S}$

Uniform Distribution

Definition[uniform distribution]

A RV X is said to have a uniform distribution if

$$f(x) = \text{constant for} \quad x \in \overline{S}$$

Example 2

Question: Roll a fair four-sided die twice and let X be the maximum of the two outcomes. Find the pmf of X, $f(x)$.

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Question: Roll a fair four-sided die twice and let X be the maximum of the two outcomes. Find the pmf of X, $f(x)$.

1. The sample space S for rolling a fair four-sided die twice is

$$S = \{(d_1, d_2) | d_1 = 1, 2, 3, 4; d_2 = 1, 2, 3, 4\}$$

2. For any $s = (d_1, d_2) \in S$, $X(s) = \max\{d_1, d_2\}$. Clearly, this RV is not one-to-one! and the range of X, i.e., $\overline{S} = \{1, 2, 3, 4\}$
3. To find $f(x)$, the pmf of X, is to find the value of $f(x) = P(X = x)$ for $x \in \overline{S}$, i.e., $x = 1, 2, 3, 4$:

$$\begin{aligned}
f(1) &= P(X = 1) = P_S(\{(1, 1)\}) = 1/16, \\
f(2) &= P(X = 2) = P_S(\{(1, 2), (2, 1), (2, 2)\}) = 3/16, \\
f(3) &= P(X = 3) = P_S(\{(1, 3), (3, 1), (2, 3), (3, 2), (3, 3)\}) = 5/16, \\
f(4) &= P(X = 4) = P_S(\{(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), \\
&\qquad (4, 3), (4, 4)\}) = 7/16,
\end{aligned}$$

Line Graph and Probability Histogram

Definition[Line Graph]

A line graph of the pmf $f(x) : \overline{S} \to (0,1]$ of a RV X is a graph having a vertical line segment drawn from $(x, 0)$ to $(x, f(x))$ at each $x \in \overline{S}$

Definition[Probability Histogram]

If a RV X with range $\overline{S}$ that only contains integers, then a probability histogram of the pmf $f(x) : \overline{S} \to (0, 1]$ is a graph having a rectangle of height $f(x)$ and a base of length 1, centered at x, for each $x \in \overline{S}$.

Example 2, continued

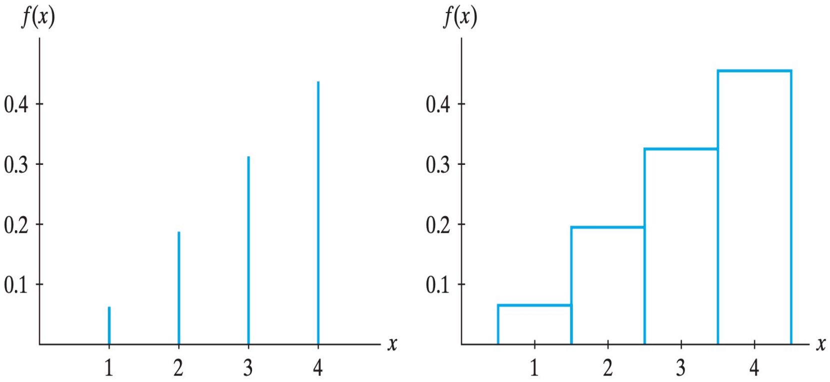

Figure 2.1-1 Line graph and probability histogram

Cumulative Distribution Function (cdf)

Definition[cdf]

The function $F(x) : R \to [0\ 1]$:

$$F(x) = P(X \leq x)$$

is called the cumulative distribution function (cdf).

1. $F(x)$ is nondecreasing and moreover,

$$P(X \leq x) = \sum_{x' \leq x, x' \in \overline{S}} f(x').$$

2. relation between the probability function and the cdf

$$P(a < X \leq b) = F(b) - F(a)$$

Example 1, continued

The old random experiment is to roll a die with a, b, c, d, e, f on each side of the die with sample space $S = \{a, b, c, d, e, f\}$

1. define a RV: $X(a) = 1, \cdots, X(f) = 6$

$$X : S = \{a, b, c, d, e, f\} \to \overline{S} = \{1, 2, 3, 4, 5, 6\}$$

2. X is discrete, because $\overline{S}$ is finite, i.e., it contains a finite number of outcomes

3. pmf $f(x) = \frac{1}{6}, \quad x \in \overline{S}$, and $f(x) = 0, \quad x \notin \overline{S}$

4. cdf

$$F(x) = P(X \leq x) = \sum_{x' \leq x, x' \in \overline{S}} f(x')$$

$$= \begin{cases} 0, & x < 1 \\ \frac{k}{6}, & k \leq x < k+1, k = 1, 2, 3, 4, 5 \\ 1, & x \geq 6 \end{cases}$$

Section 2.2 Mathematical Expectation

Motivation

We will learn many probability distributions, it is important to introduce concepts to summarize their key characteristics.

- Mean
- Variance
- Moments
- Moment generating function

Motivation Example

An enterprising man proposes a game: let the player throw a die and then the player receives payment as follows:

$$A = \{1, 2, 3\} \to 1 \text{ dollar}$$

$$B = \{4, 5\} \to 2 \text{ dollars}$$

$$C = \{6\} \to 3 \text{ dollars}$$

Motivation Example

1. This defines explicitly a RV $X : S \to \overline{S}$, where $S = \{1, 2, 3, 4, 5, 6\}$ and $\overline{S} = \{1, 2, 3\}$.

$$\text{for } s \in A = \{1, 2, 3\}, \qquad X(s) = 1$$

$$\text{for } s \in B = \{4, 5\}, \qquad X(s) = 2$$

$$\text{for } s \in C = \{6\}, \qquad X(s) = 3$$

The RV X represents the payment the player receives and is NOT one-to-one!

Motivation Example, continued

2. The RV X is discrete.
3. pmf of X:

$$f : \overline{S} \to (0, 1] \quad \overline{S} = \{1, 2, 3\}$$

$$f(x) = \frac{4 - x}{6}, \quad x = 1, 2, 3.$$

Motivation Example, continued

Question

The man charges the player 2 dollars for each play. Can the man make profit if the game is repeated for a large number of times?

Motivation Example, continued

4. payment of $\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$ occur $\begin{Bmatrix} \frac{3}{6} \\ \\ \frac{2}{6} \\ \\ \frac{1}{6} \end{Bmatrix}$ of the times.
5. average payment is

$$1 \cdot \frac{3}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{1}{6} = \frac{10}{6} = \frac{5}{3}$$

so the man can earn $2 - \frac{5}{3} = \frac{1}{3}$ per play on average

Mathematical Expectation

More generally, we are interested in the average value of a function of X, say $g(X)$.

Definition[Mathematical Expectation]

Assume X is a discrete RV with range $\overline{S}$ and $f(x)$ is its pmf. If $\sum_{x\in\overline{S}} g(x)f(x)$ exists, then it's called the mathematical expectation of $g(X)$ and is denoted by

$$E[g(X)] = \sum_{x\in\overline{S}} g(x)f(x)$$

Example 1, page 59

Question

Let X be a RV with $\overline{S} = \{-1, 0, 1\}$ and its pmf is $f(x) = \frac{1}{3}$ for $x \in \overline{S}$. What's $E[X^2]$?

Example 1, page 59

Question

Let X be a RV with $\overline{S} = \{-1, 0, 1\}$ and its pmf is $f(x) = \frac{1}{3}$ for $x \in \overline{S}$. What's $E[X^2]$?

$$E[X^2] = \sum_{x \in \overline{S}} x^2 f(x) = (-1)^2 \frac{1}{3} + 0^2 \frac{1}{3} + 1^2 \frac{1}{3} = \frac{2}{3}$$

Theorem 2.2-1, page 60 (Properties of mathematical expectation)

Theorem 2.2-1

Assume that X is a discrete RV with range $\overline{S}$ and $f(x)$ is its pmf. When the involved mathematical expectations exist, the following properties hold:

(a) If c is a constant, $E[c] = c$.

(b) If c is a constant and $g(X)$ is a function.

$$E[cg(X)] = cE[g(X)]$$

(c) If c_1 amd c_2 are constants, $g_1(X)$ and $g_2(X)$ are functions;

$$E[c_1 g_1(X) + c_2 g_2(X)] = c_1 E[g_1(X)] + c_2 E[g_2(X)]$$

Mathematical expectation is a linear operator.

Example 2, page 61

Let $g(X) = (X - b)^2$ where b is a constant to be chosen and suppose $E[(X - b)^2]$ exists. Find the value of b for which $E[(X - b)^2]$ is minimized.

Example 2, page 61

Let $g(X) = (X - b)^2$ where b is a constant to be chosen and suppose $E[(X - b)^2]$ exists. Find the value of b for which $E[(X - b)^2]$ is minimized.

$$\begin{aligned}
E[(X-b)^2] &= E[X^2 - 2bX + b^2] \\
&= E[X^2] - 2bE[X] + b^2 \triangleq h(b) \\
\frac{dh(b)}{db} &= -2E[X] + 2b = 0 \qquad \Rightarrow \qquad b = E[X]
\end{aligned}$$

Motivation Example, revisited

4. payment of $\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$ occur $\begin{Bmatrix} \frac{3}{6} \\ \\ \frac{2}{6} \\ \\ \frac{1}{6} \end{Bmatrix}$ of the times.

5. average payment is

$$1 \cdot \frac{3}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{1}{6} = \frac{10}{6} = \frac{5}{3}$$

   so the man can earn $2 - \frac{5}{3} = \frac{1}{3}$ per play on average

6. Formally, the average payment is given by

$$E(X) = \sum_{x \in \overline{S}} x f(x) = 1 \cdot \frac{3}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{1}{6} = \frac{10}{6} = \frac{5}{3}$$