STA2001 Probability and Statistics (I)

Lecture 3

Tianshi Chen

The Chinese University of Hong Kong, Shenzhen

Review

- Revisit the method of enumeration by multiplication principle
  - permutation
  - combination
  - distinguishable permutation

Section 1.4 Independent Events

Motivation

Motivation

For certain pair of events, the occurrence of one of them does not change the probability of the occurrence of the other.

Example 1

Experiment: flip a coin twice and observe the sequence of heads and tails.

Sample space: $S = \{HH, HT, TH, TT\}$

Assumption: the four outcomes are "equally likely"

Events:

$$
\begin{aligned}
A &= \{\text{heads on the first flip}\} = \{HH, HT\} \\
B &= \{\text{tails on the second flip}\} = \{HT, TT\} \\
C &= \{\text{tails on both flips}\} = \{TT\}
\end{aligned}
$$

Example 1

$$P(A) = \frac{2}{4}, \quad P(B) = \frac{2}{4}, \quad P(C) = \frac{1}{4}$$

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Given that C has occurred, then

$$P(B|C) = 1 \text{ because } C \subset B \text{ or } \frac{P(B \cap C)}{P(C)} = \frac{P(C)}{P(C)} = 1$$

Given that A has occurred, then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2} = P(B)$$

Given that B has occurred, then

$$P(A|B) = \frac{1}{2} = P(A)$$

Example 1

So we have

$$P(B|A) = P(B), \text{and } P(A|B) = P(A)$$

the occurrence of one of them does not affect the probability of the occurrence of the other. Leading to the definition of independent events.

Independent Events

Definition

Events A and B are independent if

$$P(A \cap B) = P(A)P(B).$$

Otherwise, events A and B are called dependent events

- When $P(A) \neq 0$ and $P(B) \neq 0$, we have

$$P(A|B) = P(A), \qquad P(B|A) = P(B)$$

Example 2, page 38

Question

A red die and a white die are rolled.

$$S = \{(1,1),(1,2),\cdots\}, \quad N(S) = 36$$

$$A = \{\text{4 on the red die}\}, \quad B = \{\text{sum of dice is odd}\}$$

Assuming the two dice are fair. Are A and B independent?

Example 2, page 38

A red die and a white die are rolled.

$$S = \{(1,1),(1,2),\cdots\}, \quad N(S) = 36$$

$$A = \{4 \text{ on the red die}\}, \quad B = \{\text{sum of dice is odd}\}$$

Assuming the two dice are fair. Are A and B independent?

$$P(A) = \frac{6}{36}, \quad P(B) = \frac{18}{36}, \quad P(A \cap B) = \frac{3}{36}$$
$$P(A \cap B) = \frac{3}{36} = P(A)P(B) = \frac{6}{36} \cdot \frac{18}{36}$$

$\Rightarrow$ A and B are independent.

Properties of Independent Events

Theorem 1.4-1

A and B are independent, if and only if any pair of the following events are independent

(a) A and B'

(b) A' and B

(c) A' and B'

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A and B are independent, if and only if any pair of the following events are independent

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(b) A' and B

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Proof:

$$\begin{aligned} P(A) &= P(A \cap (B \cup B')) = P((A \cap B) \cup (A \cap B')) \\ &= P(A \cap B) + P(A \cap B') = P(A)P(B) + P(A \cap B') \end{aligned}$$

$$P(A \cap B') = P(A)(1 - P(B)) = P(A)P(B')$$

Independent Events

Definition

Events A, B and C are mutually independent if

1. A, B, C are pairwise independent, i.e.,

$$\begin{cases} P(A \cap B) & = P(A)P(B) \\ P(A \cap C) & = P(A)P(C) \\ P(B \cap C) & = P(B)P(C) \end{cases}$$

2. $P(A \cap B \cap C) = P(A)P(B)P(C)$

   - multiplication rule for three independent events.

Example 3, page 39

An urn contains four balls number 1,2,3,4 and we draw one ball randomly from the urn.

$$A = \{1, 2\}, \quad B = \{1, 3\}, \quad C = \{1, 4\}$$

Then are A, B, C mutually independent?

Example 3, page 39

$$P(A) = P(B) = P(C) = \frac{1}{2}$$
$$P(A \cap B) = P(\{1\}) = \frac{1}{4} = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C) = \frac{1}{4}$$
$$P(B \cap C) = P(B)P(C) = \frac{1}{4}$$

$$P(A \cap B \cap C) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$$

So A, B, C are pairwise independent but not mutually independent.

Properties of Independent Events (Continued)

- Mutual independence can be extended to four or more events: Each pair, triple, quartet of the events are independent and moreover

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$$

- If A, B, C are mutually independent, then
  1. A and $(B \cap C)$ independent,
  2. A' and $(B \cap C')$ independent,
  3. A and $(B \cup C)$ independent,
  4. A', B', C' independent

Properties of Independent Events (Continued)

①A and $(B \cap C)$ independent

$$P(A \cap (B \cap C)) = P(A)P(B)P(C) = P(A)P(B \cap C)$$

②A' and $(B \cap C')$ independent,

By Theorem 1.4-1, ② $\Leftrightarrow$ A and $B \cap C'$ independent

$$\begin{aligned} P(A \cap B \cap C') &= P(A \cap B) - P(A \cap B \cap C) = P(A \cap B)P(C') \\ &= P(A)P(B)P(C') = P(A)P(B \cap C') \end{aligned}$$

Properties of Independent Events (Continued)

③A and $(B \cup C)$ independent

$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C)) = P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) = P(A)(P(B) + P(C) - P(B)P(C)) = P(A)P(B \cup C)$

④A', B', C' independent

The pairwise independence is obvious and then from ③ $\Leftrightarrow$ A' and $B' \cap C'$ independent

$$P(A' \cap (B' \cap C')) = P(A')P(B' \cap C') = P(A')P(B')P(C')$$

Properties of Independent Events (Continued)

- Many experiments consist of a sequence of n trials. If the outcomes of ith trial, in fact, does not have anything to do with the others, then events such that each is associated with a different trial should be independent in the probability sense. That is, if the event A_i is associated with the ith trial, $i = 1, 2, \cdots, n$, then $A_1, A_2, \cdots, A_n$ are mutually independent and in particular

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$$

Example 4, page 40

Question

A fair 6-sided die is rolled six independent times. Let $A_i = \{$a match on the ith roll, i.e., the side i is observed on the ith roll$\}$, $i = 1, 2, \cdots, 6$. Let $B = \{$at least one match occur$\}$, what is $P(B)$?

Example 4, page 40

Question

A fair 6-sided die is rolled six independent times. Let $A_i =$ {a match on the ith roll, i.e., the side i is observed on the ith roll}, $i = 1, 2, \cdots, 6$. Let $B =$ {at least one match occur}, what is $P(B)$?

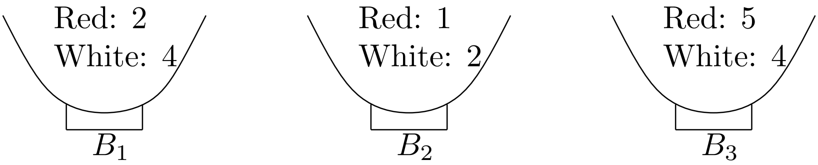
$$P(B) = 1 - P(B') \quad \text{where } B' = \{\text{no matches occur in 6 rolls}\}$$

$$= 1 - P(A_1' \cap A_2' \cdots \cap A_6') \quad \text{since } A_1' \cdots A_6' \text{ are independent}$$

$$= 1 - P(A_1')P(A_2') \cdots P(A_6') = 1 - \left(\frac{5}{6}\right)^6$$

Section 1.5 Bayes's Theorem

A Motivation Example



Experiment: Select a bowl first, and then draw a chip from the selected bowl.

Assumption: All chips are "equally likely" and moreover,

$$P(B_1) = \frac{1}{3}, \quad P(B_2) = \frac{1}{6}, \quad P(B_3) = \frac{1}{2}.$$

$P(B_i)$: the probability to select the ith bowl.

A Motivation Example

Question 1

Let R={draw a red chip}. What is $P(R)$?

A Motivation Example

Question 1

Let R={draw a red chip}. What is $P(R)$?

$$P(R) = P(S \cap R), \text{ where } S=\{\text{all chips}\}$$

$$= P((B_1 \cup B_2 \cup B_3) \cap R) = P((B_1 \cap R) \cup (B_2 \cap R) \cup (B_3 \cap R))$$

$$= P(B_1 \cap R) + P(B_2 \cap R) + P(B_3 \cap R)$$

$$= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) + P(B_3)P(R|B_3)$$

$$= \frac{1}{3} \cdot \frac{2}{6} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{5}{9} = \frac{4}{9}$$

A Motivation Example

Question 2

Suppose now that the outcome of the experiment is a red chip but we don't know from which bowl the chip was drawn. We are interested in

$$P(B_1|R), \quad P(B_2|R), \quad P(B_3|R)$$

A Motivation Example

Question 2

Suppose now that the outcome of the experiment is a red chip but we don't know from which bowl the chip was drawn. We are interested in

$$P(B_1|R), \quad P(B_2|R), \quad P(B_3|R)$$

From the definition of conditional probability, e.g., Consider

$$P(B_i|R) = \frac{P(B_i \cap R)}{P(R)} = \frac{P(B_i)P(R|B_i)}{P(R)}, \quad i = 1, 2, 3.$$

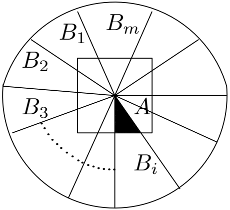
$$P(B_1|R) = \frac{1}{4}, \quad P(B_2|R) = \frac{1}{8}, \quad P(B_3|R) = \frac{5}{8}$$

Bayes' Theorem

Assume that

1. S is a sample space, and $B_1, B_2, \cdots, B_m$ are mutually exclusive and exhaustive w.r.t the sample space S.
2. the prior probabilities of B_i is positive, i.e., $P(B_i) > 0, i = 1, \cdots, m$. Then we have

Bayes' Theorem



(a) For any event A,

$$P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(B_i)P(A|B_i)$$

$\rightarrow$ total probability

(b) If $P(A) > 0$, then

$$P(B_k|A) = \frac{P(B_k \cap A)}{P(A)}, \quad k = 1, \cdots, m$$

$$P(B_k|A) = \frac{P(B_k)P(A|B_k)}{P(A) = \sum_{i=1}^{m} P(B_i)P(A|B_i)} \rightarrow \text{Bayes Theorem}$$

Bayes' Theorem

$$P(B_k) \rightarrow \text{ prior probability}$$

$$P(B_k|A) \rightarrow \text{ posterior probability}$$

$$P(A|B_k) \rightarrow \text{ likelihood of } B_k,\ A \text{ is called a data}$$

Thomas Bayes

Thomas Bayes is known for formulating a specific case of the theorem that bears his name: Bayes' theorem.

Figure: Thomas Bayes (1701 – 1761) was an English statistician, philosopher and Presbyterian minister.

Pierre-Simon Laplace

However, it was Pierre-Simon Laplace (1749–1827) who introduced what is now called Bayes' theorem, and the Bayesian was in fact pioneered and popularised by Pierre-Simon Laplace.

Figure: Pierre-Simon Laplace (1749–1827) was a French scholar and polymath whose work was important to the development of engineering, mathematics, statistics, physics, astronomy, and philosophy.