Probability and Statistics I

Mid-term Examination
SDS, CUHK(SZ)

June 29, 2024

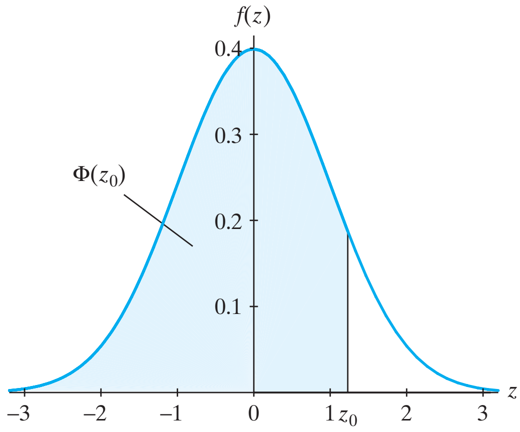
Name: ____________________ Student ID: ____________________

Answer the multiple choice questions in the boxes below. Answers outside the boxes below will NOT be graded.

Answers:

1.	2.	3.	4.	5.
6.	7.	8.	9.	10.
11.	12.	13.	14.	15.
16.	17.	18.	19.	20.
21.	22.	23.	24.	25.

Table Va The Standard Normal Distribution Function



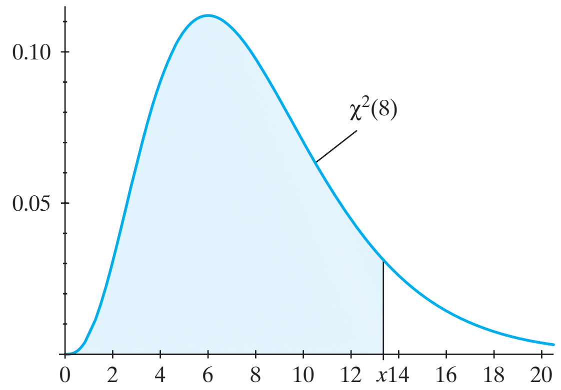
$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

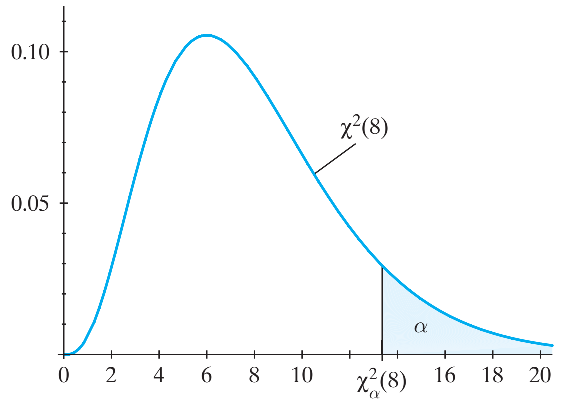
$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.240	2.326	2.576	2.807	3.291

Table IV The Chi-Square Distribution



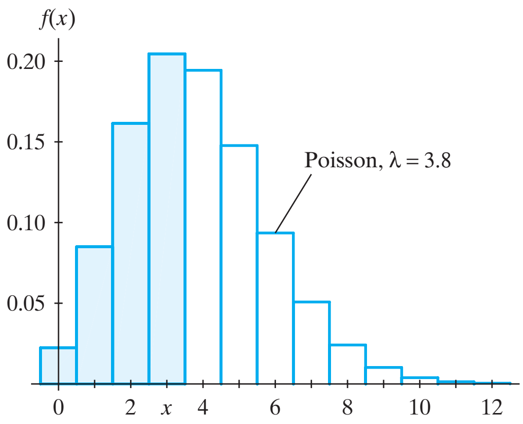


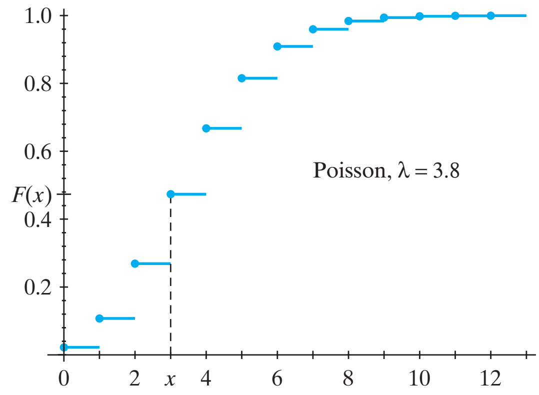
$$P(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2} dw$$

	$P(X \le x)$							
	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
r	$\chi^2_{0.99}(r)$	$\chi^2_{0.975}(r)$	$\chi^2_{0.95}(r)$	$\chi^2_{0.90}(r)$	$\chi^2_{0.10}(r)$	$\chi^2_{0.05}(r)$	$\chi^2_{0.025}(r)$	$\chi^2_{0.01}(r)$
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.34
4	0.297	0.484	0.711	1.064	7.779	9.488	11.14	13.28
5	0.554	0.831	1.145	1.610	9.236	11.07	12.83	15.09
6	0.872	1.237	1.635	2.204	10.64	12.59	14.45	16.81
7	1.239	1.690	2.167	2.833	12.02	14.07	16.01	18.48
8	1.646	2.180	2.733	3.490	13.36	15.51	17.54	20.09
9	2.088	2.700	3.325	4.168	14.68	16.92	19.02	21.67
10	2.558	3.247	3.940	4.865	15.99	18.31	20.48	23.21
11	3.053	3.816	4.575	5.578	17.28	19.68	21.92	24.72
12	3.571	4.404	5.226	6.304	18.55	21.03	23.34	26.22
13	4.107	5.009	5.892	7.042	19.81	22.36	24.74	27.69
14	4.660	5.629	6.571	7.790	21.06	23.68	26.12	29.14
15	5.229	6.262	7.261	8.547	22.31	25.00	27.49	30.58
16	5.812	6.908	7.962	9.312	23.54	26.30	28.84	32.00
17	6.408	7.564	8.672	10.08	24.77	27.59	30.19	33.41
18	7.015	8.231	9.390	10.86	25.99	28.87	31.53	34.80
19	7.633	8.907	10.12	11.65	27.20	30.14	32.85	36.19
20	8.260	9.591	10.85	12.44	28.41	31.41	34.17	37.57
21	8.897	10.28	11.59	13.24	29.62	32.67	35.48	38.93
22	9.542	10.98	12.34	14.04	30.81	33.92	36.78	40.29
23	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64
24	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98
25	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31
26	12.20	13.84	15.38	17.29	35.56	38.88	41.92	45.64
27	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96
28	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28
29	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59
30	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89
40	22.16	24.43	26.51	29.05	51.80	55.76	59.34	63.69
50	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15
60	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38
70	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4
80	53.34	57.15	60.39	64.28	96.58	101.9	106.6	112.3

This table is abridged and adapted from Table III in *Biometrika Tables for Statisticians*, edited by E.S.Pearson and H.O.Hartley.

Table III The Poisson Distribution





$$F(x) = P(X \le x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

	$\lambda = E(X)$									
x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.905	0.819	0.741	0.670	0.607	0.549	0.497	0.449	0.407	0.368
1	0.995	0.982	0.963	0.938	0.910	0.878	0.844	0.809	0.772	0.736
2	1.000	0.999	0.996	0.992	0.986	0.977	0.966	0.953	0.937	0.920
3	1.000	1.000	1.000	0.999	0.998	0.997	0.994	0.991	0.987	0.981
4	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.996
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	0.333	0.301	0.273	0.247	0.223	0.202	0.183	0.165	0.150	0.135
1	0.699	0.663	0.627	0.592	0.558	0.525	0.493	0.463	0.434	0.406
2	0.900	0.879	0.857	0.833	0.809	0.783	0.757	0.731	0.704	0.677
3	0.974	0.966	0.957	0.946	0.934	0.921	0.907	0.891	0.875	0.857
4	0.995	0.992	0.989	0.986	0.981	0.976	0.970	0.964	0.956	0.947
5	0.999	0.998	0.998	0.997	0.996	0.994	0.992	0.990	0.987	0.983
6	1.000	1.000	1.000	0.999	0.999	0.999	0.998	0.997	0.997	0.995
7	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.999
8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
x	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
0	0.111	0.091	0.074	0.061	0.050	0.041	0.033	0.027	0.022	0.018
1	0.355	0.308	0.267	0.231	0.199	0.171	0.147	0.126	0.107	0.092
2	0.623	0.570	0.518	0.469	0.423	0.380	0.340	0.303	0.269	0.238
3	0.819	0.779	0.736	0.692	0.647	0.603	0.558	0.515	0.473	0.433
4	0.928	0.904	0.877	0.848	0.815	0.781	0.744	0.706	0.668	0.629
5	0.975	0.964	0.951	0.935	0.916	0.895	0.871	0.844	0.816	0.785
6	0.993	0.988	0.983	0.976	0.966	0.955	0.942	0.927	0.909	0.889
7	0.998	0.997	0.995	0.992	0.988	0.983	0.977	0.969	0.960	0.949
8	1.000	0.999	0.999	0.998	0.996	0.994	0.992	0.988	0.984	0.979
9	1.000	1.000	1.000	0.999	0.999	0.998	0.997	0.996	0.994	0.992
10	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.998	0.997
11	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999
12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Multiple Choices (100 points)

- **4 points for each correct answer; -1.5 point for each incorrect answer; 0 points for no answer.**
- **For each question, only choose (at most) one out of four given choices (A,B,C and D). If you choose more than one choice in one question, your answer will be incorrect and 1.5 point will be deduced.**

1. If $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$, find (a) $P(A \cup B)$, (b) $P(A' \cap B)$.

   A. (a) 0.7, (b) 0.3. B. (a) 0.6, (b) 0.2.
   C. (a) 0.6, (b) 0.3. D. (a) 0.7, (b) 0.2.

   Solution: (C)
   (a) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$.
   (b) $P(A' \cap B) = P(B) - P(A \cap B) = 0.3$.

2. Count the number of distinct ways of putting 5 balls into 4 boxes (each box can hold multiple balls) when:
   (a) all boxes and balls are distinguishable;
   (b) the boxes are different but the balls are identical.

   A. (a) 4^5, (b) 35. B. (a) 5^4, (b) 35.
   C. (a) 4^5, (b) 56. D. (a) 5^4, (b) 56.

   Solution: (C)
   (a) The number is 4^5.
   (b) The number is $\binom{5+4-1}{5} = 56$ (arrangements of 5 balls and 3 fences).

3. Suppose that 35 people are divided in a random manner into two teams in such a way that one team contains 10 people and the other team contains 25 people. What is the probability that two particular people A and B will be on the same team?

A. 0.6592. B. 0.5798. C. 0.4227. D. 0.5926.

Solution: (B)
The probability is

$$\frac{\binom{33}{8}+\binom{33}{10}}{\binom{35}{10}} = \frac{10\times 9+25\times 24}{35\times 34} = 0.5798.$$

4. A man can go from his home at point h to one of three points c, d and e via one of two intermediate points a and b. The probabilities of taking the various routes are as follows:

$$P(\text{ from } h \text{ to } a) = 0.6,\quad P(\text{ from } h \text{ to } b) = 0.4$$

$$P(\text{ from } a \text{ to } c) = 0.2,\quad P(\text{ from } a \text{ to } d) = 0.4,\quad P(\text{ from } a \text{ to } e) = 0.4$$

$$P(\text{ from } b \text{ to } c) = 0.4,\quad P(\text{ from } b \text{ to } d) = 0.3,\quad P(\text{ from } b \text{ to } e) = 0.3$$

Given that he arrived at point c, what is the conditional probability that he has passed through intermediate point a?

A. $\frac{3}{7}$. B. $\frac{4}{7}$. C. $\frac{3}{8}$. D. $\frac{5}{8}$.

Solution: (A)

$$\begin{aligned} P(\text{from } a \mid \text{arrived at } c) &= \frac{P(\text{from } a \cap \text{arrived at } c)}{P(\text{arrived at } c)} \\ &= \frac{P(\text{arrived at } c \mid \text{from } a)P(\text{from } a)}{P(\text{arrived at } c)} \end{aligned}$$

$P(\text{arrived at } c)$
$=P(\text{arrived at } c \mid \text{from } a)P(\text{from } a) + P(\text{ arrived at } c \mid \text{from } b)P(\text{from } b)$
$=(0.2)(0.6) + (0.4)(0.4) = 0.28$

$$P(\text{from } a \mid \text{arrived at } c) = \frac{(0.2)(0.6)}{0.28} = \frac{3}{7}.$$

5. Consider 3 urns. Urn A contains 4 white and 2 red balls; urn B contains 7 white and 5 red balls; and urn C contains 3 white and 5 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn B is white, given that exactly 2 white balls are selected?

   A. $\frac{91}{121}$. B. $\frac{90}{121}$. C. $\frac{101}{121}$. D. $\frac{100}{121}$.

Solution: (A)

The probability in the question should be

$$
\begin{aligned}
& P(\text{ Ball from B white } \mid 2 \text{ white balls selected }) \\
= & \frac{P(\text{ Ball from B white } \cap 2 \text{ white balls selected })}{P(2 \text{ white balls selected })}
\end{aligned}
$$

First, consider P (Ball from B white $\cap$ 2 white balls selected). This means that the ball chosen from B must be white and either the ball from A or C is white and the other one is not. The probability of drawing a white ball from B is $\frac{7}{12}$. Likewise, the probability of drawing a white ball from A, C is $\frac{4}{6}$ and $\frac{3}{8}$ respectively. The probability of not drawing the white ball from A, C is $\frac{2}{6}$ and $\frac{5}{8}$ respectively. So, P(Ball from B white $\cap$ 2 white balls selected)$=$ $\frac{7}{12}\left(\frac{4}{6} \cdot \frac{5}{8} + \frac{3}{8} \cdot \frac{2}{6}\right)$ Now consider P (2 white balls selected). There are 3 ways we can choose the two white balls: choose a white ball from A and B and a red ball from C, choose a white ball from B and C and a red ball from A and choose a white ball from A and C and a red ball from B. So the denominator is

$$
\frac{4}{6} \cdot \frac{7}{12} \cdot \frac{5}{8} + \frac{4}{6} \cdot \frac{5}{12} \cdot \frac{3}{8} + \frac{2}{6} \cdot \frac{7}{12} \cdot \frac{3}{8}
$$

Therefore the probability is $\frac{91}{121}$.

6. A small plane went down and was missing, and the search was organized into three regions. Starting with the likeliest, they are:

Region	Initial chance the plane is there	Chance of being overlooked in the search
Mountains	0.4	0.4
Prairie	0.35	0.3
Sea	0.25	0.95

The second column gives the chance that if the plane is there, it will not be found. For example, if it went down at sea, there is 95% chance it will have disappeared, or otherwise not be found. Since the pilot is not equipped to long survive a crash in the mountains, it is particularly important to determine the chance that the plane went down in the mountains. (a) Before any search is started, what is the chance that the plane is in the mountains? (b) The initial search was in the mountains, and the plane was not found. Now what is the chance the plane is nevertheless in the mountains? (c) The search was continued over the other two regions, and unfortunately the plane was not found anywhere. Finally now what is the chance that the plane is in the mountains?

A. (a) 0.40, (b) 0.2105, (c) 0.3816.
B. (a) 0.40, (b) 0.2314, (c) 0.3184.
C. (a) 0.35, (b) 0.2314, (c) 0.3816.
D. (a) 0.40, (b) 0.2105, (c) 0.3184.

Solution: (D)
Let M, P, S be the events that the plane went down in the mountains, prairie, and sea, respectively. Let OM, OP, OS be the events that the plane is not found in mountains, prairie, sea, respectively. Then we have:

$$\mathrm{P}(M) = 0.4, \qquad \mathrm{P}(P) = 0.35, \quad \mathrm{P}(S) = 0.25,$$

$$\mathrm{P}(OM \mid M) = 0.4, \ \mathrm{P}(OP \mid M) = \mathrm{P}(OS \mid M) = 1,$$

$$\mathrm{P}(OP \mid P) = 0.3, \quad \mathrm{P}(OM \mid P) = \mathrm{P}(OS \mid P) = 1,$$

$$\mathrm{P}(OS \mid S) = 0.95, \quad \mathrm{P}(OM \mid S) = \mathrm{P}(OP \mid S) = 1.$$

(a) $\mathrm{P}(M) = 0.4$.
(b)

$$\begin{aligned} &\mathrm{P}(M \mid OM) \\ &= \frac{\mathrm{P}(OM \mid M)P(M)}{\mathrm{P}(OM \mid M)\,\mathrm{P}(M) + \mathrm{P}(OM \mid P)\,\mathrm{P}(P) + \mathrm{P}(OM \mid S)\,\mathrm{P}(S)} \\ &= \frac{(0.4)(0.4)}{(0.4)(0.4) + 0.35 + 0.25} = 0.2105. \end{aligned}$$

(c) Let $O = OM \cap OP \cap OS$. Note that

$$\begin{aligned} &\mathrm{P}(O) = \mathrm{P}(O \mid M)\,\mathrm{P}(M) + \mathrm{P}(O \mid P)\,\mathrm{P}(P) + \mathrm{P}(O \mid S)\,\mathrm{P}(S) \\ =&\,\mathrm{P}(OM \mid M)\,\mathrm{P}(M) + \mathrm{P}(OP \mid P)\,\mathrm{P}(P) + \mathrm{P}(OS \mid S)\,\mathrm{P}(S) \\ =&(0.4)(0.4) + (0.3)(0.35) + (0.95)(0.25) = 0.5025. \end{aligned}$$

So, $$\mathrm{P}(M \mid O) = \frac{\mathrm{P}(OM \mid O)P(M)}{\mathrm{P}(O)} = \frac{(0.4)(0.4)}{0.5025} = 0.3184.$$

7. Three students A, B, and C are enrolled in the same class. Suppose that A attends class 30 percent of the time, B attends class 50 percent of the time, and C attends class 80 percent of the time. If these students attend class independently of each other, what is (a) the probability that at least one of them will be in class on a particular day and (b) the probability that exactly one of them will be in class on a particular day?

A. (a) 0.93, (b) 0.45. B. (a) 0.87, (b) 0.38.
C. (a) 0.93, (b) 0.38. D. (a) 0.87, (b) 0.45.

Solution: (C)
Let A, B, and C stand for the events that each of the students is in class on a particular day.
(a) We want $P(A \cup B \cup C)$. We can use Theorem 1.1-6. Independence makes it easy to compute the probabilities of the various intersections.

$$\begin{aligned} P(A \cup B \cup C) = 0.3 + 0.5 + 0.8 - [0.3 \times 0.5 + 0.3 \times 0.8 + 0.5 \times 0.8] \\ + 0.3 \times 0.5 \times 0.8 = 0.93. \end{aligned}$$

(b) Once again, use independence to calculate probabilities of intersections.

$$\begin{aligned} &P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) \\ &\quad = (0.3)(0.5)(0.2) + (0.7)(0.5)(0.2) + (0.7)(0.5)(0.8) = 0.38. \end{aligned}$$

8. Suppose we roll two fair six-sided dice. Let A denote the event that the two dices show the same value. Let B be the event that the sum of the two dices is equal to 7. Let C be the event that the first die shows 5. Let D be the event that the second die shows 5. Which of the following 5 statements are **true**?
   1. B and C are independent.
   2. A and C are independent.
   3. A and D are independent.
   4. C and D are independent.
   5. A, C, D are mutually independent.

   A. All of them.
   B. Only 1, 2, 3 and 4.
   C. Only 2, 3 and 4.
   D. Only 4.

Solution: (B)
Observe that

$$P(A) = P(B) = P(C) = P(D) = 1/6,$$
$$P(B \cap C) = P(A \cap C) = P(A \cap D) = P(C \cap D) = 1/36,$$
$$P(A \cap C \cap D) = 1/36.$$

9. How many of the following three statements are **true**?

   (a) A random variable is a mapping (function) from the original sample space to the set of real numbers (or a subset of the real numbers).
   (b) A discrete random variable (distribution) must have finite support.
   (c) If f is the probability density function of a continuous random variable X, then $f(x) \in [0,1], \forall x \in \bar{S}$.

   A. 0. B. 1. C. 2. D. 3.

Solution: (B)
(a) True.

(b) False, the support of a discrete random variable (distribution) can be countably infinite.
(c) False. Consider the continuous uniform distribution on (0, 1/2).

10. Let X follow a discrete uniform distribution on $\{1, 2 \ldots, 10\}$. Find (a) $E(X)$, (b) the moment generating function of X for $t = 0$, and (c) the moment generating function of X for $t = 1$,

A. (a) $\frac{11}{2}$, (b) 0, (c) $\frac{e\left(1-e^{10}\right)}{10(1-e)}$.
B. (a) $\frac{11}{2}$, (b) 1, (c) $\frac{e\left(1-e^{9}\right)}{10(1-e)}$.
C. (a) $\frac{11}{2}$, (b) 1, (c) $\frac{e\left(1-e^{10}\right)}{10(1-e)}$.
D. (a) $\frac{10}{2}$, (b) 0, (c) $\frac{e\left(1-e^{9}\right)}{10(1-e)}$.

Solution: (C)
(a) $\mathrm{E}(X) = \sum_{x=1}^{10} \frac{x}{10} = \frac{(1+10)(10-1+1)}{2 \cdot 10} = \frac{11}{2}$.

(b)
$$M_X(t) = \mathrm{E}\left(e^{tX}\right) = \sum_{x=1}^{10} \frac{e^{tx}}{10} = \frac{e^t}{10}\left[1 + e^t + \cdots + e^{9t}\right]$$
$$= \begin{cases} \frac{e^t\left(1-e^{10t}\right)}{10(1-e^t)}, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0 \end{cases}$$

11. Consider two random variables X and Y. The distribution of X is $b(3, p)$ and the distribution of Y is $b(4, p)$, where $0 \leq p \leq 1$. If $P(X \geq 1) = \frac{37}{64}$, then what is the value of $P(Y \geq 1)$?

A. $\frac{173}{256}$.
B. $\frac{175}{256}$.
C. $\frac{83}{256}$.
D. $\frac{81}{256}$.

Solution: (B)
Since $P(X \geq 1) = 1 - P(X < 1) = 1 - \binom{3}{0} p^0 (1-p)^3 = 1 - (1-p)^3 = \frac{37}{64}$, we have $1 - p = \frac{3}{4}$. Hence,
$$P(Y \geq 1) = 1 - P(Y < 1) = 1 - \binom{4}{0} p^0 (1-p)^4 = 1 - \left(\frac{3}{4}\right)^4 = \frac{175}{256}.$$

12. Suppose that the random variables $X_1, \ldots, X_n$ form n Bernoulli trials with parameter p. Determine the conditional probability that $X_1 = 1$ given that $\sum_{i=1}^{n} X_i = k \in \{1, 2, \ldots, n\}$.

    A. $\dfrac{k(k-1)}{n(n-1)}$. B. $\dfrac{k^2}{n^2}$. C. $\dfrac{k}{n}$. D. $\dfrac{k-1}{n}$.

Solution: (C)

$$P(X_1 = 1 | \sum_{i=1}^{n} X_i = k) = \frac{P(X_1 = 1 \text{ and } \sum_{i=2}^{n} X_i = k-1)}{P(\sum_{i=1}^{n} X_i = k)}.$$

Since X_1 and $\sum_{i=2}^{n} X_i$ are independent, $\sum_{i=1}^{n} X_i \sim b(n,p)$, and $\sum_{i=2}^{n} X_i \sim b(n-1,p)$, we get

$$P(X_1 = 1 | \sum_{i=1}^{n} X_i = k) = \frac{p \cdot \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{k}{n}.$$

13. Find $P(X = 5)$ if X has a Poisson distribution such that $P(X = 2) = P(X = 3)$.

    A. 0.101. B. 0.156. C. 0.114. D. 0.087.

Solution: (A)
Note that $P(X = 2) = P(X = 3) \implies \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{\lambda^3 e^{-\lambda}}{3!} \implies \lambda = 3$.
So, from the Poisson table, $P(X = 5) = P(X \leq 5) - P(X \leq 4) = 0.916 - 0.815 = 0.101$.

14. For a class of students, consider the probability that at least two of them have their birthdays on the same day. What's the minimum size of class, i.e., the minimum number of students in the class, such that the probability is larger than 0.75? (For simplicity, suppose each year has 365 days.) [**Hint**: You can assume that birthday matches across different pairs of students are mutually independent. The probability of a binomial distribution $b(n,p)$ can be approximated by the probability of a Poisson distribution with $\lambda = np$ for $n \geq 20$ and $p \leq 0.05$, and moreover $\ln 4 = 1.386$.]

A. 31. B. 32. C. 33. D. 34.

Solution: (C)
Let N be the number of students in a class, there are $n = \binom{N}{2} = \frac{N(N-1)}{2}$ pairs of students sharing birthdays in this class. For each pair, both students are born on the same day with probability $p = 1/365$. Each pair is a Bernoulli trial because the two birthdays either match or not match. Besides, all matches are mutually independent. Therefore, X, the number of pairs sharing birthdays, follows a binomial distribution.

P(there are two students sharing birthday)$=1 - P$(no matches)
$=1 - P(X = 0) = 1 - (1 - \frac{1}{365})^n$.

We note that the probability of a binomial distribution $b(n, p)$ can be approximated by that of a Poisson distribution with $\lambda = np$ for $n \geq 20, p \leq 0.05$. Then, we can use Poisson approximation with $\lambda = np = N(N-1)/730$, so we have $1 - P(X = 0) \approx 1 - e^{-\lambda} = 1 - e^{-N(N-1)/730} > 0.75$. This implies $N(N-1) > 730 \ln 4 \approx 1011.995$. Therefore, the minimum class size should be 33.

15. Products produced by a machine has a 5% (independent) defective rate. The first 20 inspections have been found to be free of defectives. What is the probability that the first defective will occur on the 24th inspection.

A. $0.95^5 \times 0.05$. B. $0.95^4 \times 0.05$.
C. $0.95^3 \times 0.05$. D. 1 - $0.95^4 \times 0.05$.

Solution: (C)
Let X be the number of products needed to detect the first defective product. Then, $X \sim \text{geometric}(p)$, where $p = 0.05$. Let $q = 1-p = 0.95$. Then,

$$P(X = 24|X > 20) = \frac{P(X = 24)}{P(X > 20)} = \frac{q^{23}p}{\sum_{k=21}^{\infty} q^{k-1}p} = q^3 p.$$

(Or, we can directly apply the memoryless property.)

16. Suppose that two players A and B are trying to throw a basketball through a hoop. The probability that player A will succeed on any given throw is p and he throws until he has succeeded r times. The probability that player B will succeed on any given throw in mp, where $m \geq 2$ is a given integer such that $mp < 1$, and she throws until she has succeeded mr times.
(a) For which player is the expected number of throws smaller?
(b) For which player is the variance of the number of throws smaller?

A. (a) player A, (b) player B.
B. (a) player B, (b) player A.
C. (a) They have the same expected throws, (b) player A.
D. (a) They have the same expected throws, (b) player B.

Solution: (D)
(a) Let X_A and X_B denote the number of throws for player A and B, respectively. Since $X_A \sim$ negative binomial(r, p) and $X_B \sim$ negative binomial(mr, mp), we have

$$E(X_A) = \frac{r}{p} = \frac{mr}{mp} = E(X_B).$$

(b)

$$\text{Var}(X_A) = \frac{r(1-p)}{p^2}, \text{Var}(X_B) = \frac{mr(1-mp)}{(mp)^2} = \frac{r(1-p)}{p^2}\frac{1-mp}{(1-p)m}.$$

Since $\frac{1-mp}{(1-p)m} < 1$, we have $\text{Var}(X_B) < \text{Var}(X_A)$.

17. Buses arrive at the bus stop as a Poisson process at the rate of μ buses per hour. A man arrives at the bus stop and finds that there is no bus. What is the probability that he has to wait for more than α minutes for the next bus?

A. $1 - \exp\left[-\frac{\alpha}{60/\mu}\right]$.
B. $\exp\left[-\frac{\alpha}{60/\mu}\right]$.
C. $1 - (60/\mu)\exp\left[-\frac{\alpha}{60/\mu}\right]$.

D. $(\mu/60)\exp\left[-\frac{\alpha}{60/\mu}\right]$.

Solution: (B)

Poisson arrivals imply exponential interarrival times

μ buses per hour implies $\mu/60$ arrivals per minute

Mean interarrival time is $60/\mu$ minutes

W = waiting time in minutes

$P(W > \alpha) = \exp\left[-\frac{\alpha}{60/\mu}\right]$

18. Consider the nonnegative random variable X with the pdf

$$f_X(x) = \alpha x e^{-x^2} + \beta I(0,1)$$

Here, α and β are constants to be determined. $I(0,1)$ is the indicator function given by

$$I(0,1) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Find α and β such that the 80^{th} percentile is at the point $\pi_{0.8} = 1$.

A. $\alpha = 0.4e, \beta = 1 - 0.2e$.
B. $\alpha = 1 - 0.2e, \beta = 0.4e$.
C. $\alpha = 1 - 0.4e, \beta = 1 - 0.2e$.
D. $\alpha = 0.4e, \beta = 0.2e$.

Solution: (A)

Normalization:

$$1 = \int_0^\infty \left(\alpha x e^{-x^2} + \beta I(0,1)\right) dx = \frac{\alpha}{2}\int_0^\infty e^{-x^2} d\left(x^2\right) + \beta = \frac{\alpha}{2} + \beta$$

80^{th} percentile implies

$$\begin{aligned}
0.8 &= \int_0^1 \left(\alpha x e^{-x^2} + \beta I(0,1)\right) dx \\
&= \frac{\alpha}{2} \int_0^{x=1} e^{-x^2} d\left(x^2\right) + \beta \\
&= \frac{\alpha}{2}\left(1 - e^{-1}\right) + \beta
\end{aligned}$$

Solving the two equations gives $\frac{\alpha}{2}e^{-1} = 0.2$, or $\alpha = 0.4e$ while $\beta = (1 - 0.2e) > 0$.

19. The number of cars passing a speed camera follows a Poisson distribution, and the average number of cars passing the camera in 1 minute is 2. Suppose current time is $t = 0$ and the unit is minute, and let the random variable Y be the time that the 4th car passes the speed camera, and let the random variable Z be the time that the 1st car passes the speed camera. Consider the following statements:

    i. The probability of at least 3 cars passing the speed camera in the first 2 minute is 0.7619 approximately.

    ii. $E(Y) = 2$, $Var(Y) = 1$, $E(Y^4) = 52.5$.

    iii. For any $t, s > 0$, $P(Z > s + t | Z > t) = P(Z > s)$.

    Find which one of the following statements is correct.

    A. Only (i) is true.
    B. Only (i) and (ii) are true.
    C. Only (ii) and (iii) are true.
    D. All statements are true.

Solution: (D)

i. The mean number of car passing the speed camera in 2 minute is 4. Let X be the number of car passing the speed camera in the first 2 minute, so $X \sim Poisson(4)$, and then

$$P(X \geq 3) = 1 - \frac{e^{-4}(4)^0}{0!} - \frac{e^{-4}(4)^1}{1!} - \frac{e^{-4}(4)^2}{2!} = 0.76189.$$

ii. Note that Y follows a Gamma distribution with $\alpha = 4$ and $\theta = 1/2 = 0.5$.

$$\begin{aligned} &E(Y) = \alpha\theta = 2,\ Var(Y) = \alpha\theta^2 = 1 \\ &E(Y^4) = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\theta^4 = 52.5. \end{aligned}$$

iii. This is the memoryless property of the exponential distribution as shown in the assignment.

20. Suppose that X is a continuous random variable with pdf

$$f(x) = \begin{cases} 1+x, & -1 \leq x < 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Define a random variable $Y = X^2 + 1$. What is the value of $P(\frac{5}{4} < Y \leq \frac{7}{4})$?

A. $\sqrt{3} - \frac{3}{2}$. B. $2\sqrt{3} - 3$. C. $2 - \sqrt{3}$. D. $3 - 2\sqrt{2}$.

Solution: (A) We have $F_Y(y) = P(Y \leq y) = P(X^2 + 1 \leq y)$. Apparently, when $y \leq 1$, $F_Y(y) = 0$; when $y \geq 2$, $F_Y(y) = 1$. When $1 < y < 2$,

$$\begin{aligned} F_Y(y) =& P(-\sqrt{y-1} \leq X \leq \sqrt{y-1}) \\ =& \int_{-\sqrt{y-1}}^{\sqrt{y-1}} f_X(x)dx \\ =& \int_{-\sqrt{y-1}}^{0} (1+x)dx + \int_{0}^{\sqrt{y-1}} (1-x)dx \\ =& 2\sqrt{y-1} - y + 1. \end{aligned}$$

Hence $P(\frac{5}{4} < Y \leq \frac{4}{7}) = F_Y(\frac{7}{4}) - F_Y(\frac{5}{4}) = \sqrt{3} - \frac{3}{2}$.

21. Suppose that $X \sim N(3,4)$, find the largest d below such that $P(X > d) \geq 0.8$.
A. 5.56. B. 4.68. C. 2.08. D. 1.31.

Solution: (D)

$$\begin{aligned} P(X>d) =& 1-P(X\leq d) = 1-P(\frac{X-3}{2}\leq\frac{d-3}{2}) \\ =& 1-\Phi(\frac{d-3}{2}) \geq 0.8. \end{aligned}$$

$$\Phi(\frac{d-3}{2}) \leq 0.2.$$

Check that table we know that $\Phi(0.84) = 0.8$, and $\Phi(-0.84) = 0.2$. Thus $\frac{d-3}{2} \leq -0.84$, $d \leq 1.32$.

22. Suppose that the moment-generating function $M_X(t)$ of the continuous random variable X has the property $M_X(t) = e^t M_X(-t)$ for all t. What is $E(X)$?
A. $\frac{1}{4}$. B. $\frac{1}{2}$. C. 1. D. 2.

Solution: (B)
$E(e^{tX}) = e^t E(e^{-tX}) = E(e^{t(1-X)})$ and thus $M_X(t) = M_{1-X}(t)$ for all t. Since the moment generating function determines uniquely the probability distribution, it follows that the random variable X has the same distribution as the random variable $1-X$. Hence $E(X) = E(1-X)$ and so $E(X) = \frac{1}{2}$.

23. Let X be lifetime (measured in hours) of a cetain type of electronic device, and its probabiltiy density function given by

$$f(x) = \begin{cases} \frac{10}{x^2}, & x > 10 \\ 0, & x \leq 10. \end{cases}$$

What is the probability that at least 1 of 4 such types of devices will function for at least 15 hours?

A. $\frac{80}{81}$. B. $\frac{72}{81}$. C. $\frac{66}{81}$. D. $\frac{54}{81}$.

Solution: (A)

Let X be the lifetime (measured in hours) of a certain type of device. Then,

$$P(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2} dx = \frac{2}{3}.$$

Let Y be the number of devices that function at least 15 hours. Then

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - (\frac{1}{3})^4 = \frac{80}{81}.$$

24. Suppose that $f_1(x)$ is the pdf of the standard normal distribution, and $f_2(x)$ is the pdf of the uniform distribution over $[-2, 4]$. Let $f(x)$ be a function defined as

$$f(x) = \begin{cases} af_1(x), & x \leq 0 \\ bf_2(x), & x > 0 \end{cases}$$

where $a > 0$ and $b > 0$. If $f(x)$ is a pdf, then which one of the following equalities must hold?

A. $3a+4b = 6$. B. $3a+2b = 4$. C. $a+b = 1$. D. $a+b = 2$.

Solution: (A)

According to definition of probability density function: $\int_{-\infty}^{\infty} f(x)dx = 1$. Since $f_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$, and

$$f_2(x) = \begin{cases} \frac{1}{6}, & -2 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases} .$$

We have

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} af_1(x)dx + \int_{0}^{\infty} bf_2(x)dx$$
$$= a\frac{1}{2} + b\frac{2}{3} = 1.$$

25. Consider the following statements:

   i. Let $Z \sim N(0,1)$, then the probability that the quadratic equation $0.1x^2 + Zx + 0.04 = 0$ has real roots is 0.9.

   ii. If $X \sim N(0,3)$, then $E(X^6) = 3E(X^4)$.

   Find which one of the following statements is correct.

   A. Only (i) is true.
   B. Only (ii) is true.
   C. Both (i) and (ii) are true.
   D. Both (i) and (ii) are not true.

Solution: (A)
To have real roots, the coefficients of the equations must satisfy $b^2 - 4ac \geq 0$, which is equivalent to $Z^2 - 4 \times 0.1 \times 0.04 \geq 0$ in this question. As $Z \sim N(0,1)$, using the mgf technique we may show that $Z^2 \sim \chi^2(1)$. Hence, we obtain $P(Z^2 \geq 0.016) = 1 - P(\chi^2(1) < 0.016) = 0.9$, where we obtain the result by checking the chi-square table.

Using the mgf of a normal random variable, we can show that $EX^4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$, $EX^6 = \mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$. Plug in the numbers, then we have $EX^6 = 15EX^4$.