香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

DDA2001: Introduction to Data Science

Lecture 6: Common Distribution (Discrete)

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Recap: 1 - Some Useful formulas

Formula 1

Linearity: $E\left[\sum_i X_i\right] = \sum_i E[X_i]$

$\downarrow$

$\boxed{\text{No assumption on } X_i}$

A More General Formula

$$E[\sum_i C_i X_i] = \sum_i C_i E[X_i]$$

C_i is a constant

How to use the linearity of expectation?

Want to calculate $E[X]$, where X is a ‘complicated’ random variable

- Step 1: Figure out a suitable decomposition for X such that $X = \sum_i X_i$
- Step 2: Compute $E[X_i]$
- Step 3: Use the linearity of expectation ($E[X] = \sum_i E[X_i]$)

Hat Check

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Let's build a probability model

- Hats are labeled {1,2,3,…, n}, where hat m belongs to person m.
- People leave the party one by one in **a random order**.
- Everyone will take away the hat with the smallest label among the untaken hats.
- With same probability, **the order** is any of the n! permutations of {1,2,…, n}.

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- An outcome: people are ordered by (2,1,3,4,…,n).

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- What's the value of P(X=n)?
- How about P(X=n-2)?
- P(X=n-3)….

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Too hard to calculate f(x)!!!
Use linearity.

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For $i = 1, \ldots, n$, let $X_i = \begin{cases} 1, & \text{if person i got hat back} \\ 0, & \text{otherwise} \end{cases}$. Then $X = \sum_{i=1}^{n} X_i$.

E[X] = E[Σ_i X_i]= Σ_i E[X_i]

What's the value of E[X_i]?

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$X_i = 1$ if and only if person i is the i-th person who takes away a hat.

How many orders may contribute to the above events?

Fix person i at i-th position of the order, there are still (n-1)! possible orders.

Thus $P(X_i = 1) = \frac{1}{n}$ and $E[X_i]=1/n$, implying $E[X] = 1$.

Formula 2

$$\mathrm{E}[g(X)] = \Sigma_x\, g(x)P(X = x) = \Sigma_x\, g(x)f(x)$$

- $g(X)$ is also a random variable
- By definition of expectation, you need to figure out the pmf for $g(X)$ first

- Note: **$\mathrm{E}[g(X)] \neq g(\mathrm{E}[X])$**

- Toss a coin: head as 1, tail as -1.
- Then $\mathrm{E}[X^2] = 1^2 \times \frac{1}{2} + (-1)^2 \times \frac{1}{2} = 1$

$g(x) = x^2$

- But $(\mathrm{E}[X])^2 = 0$

Variance

$$E[\underbrace{(X - E[X])^2}_{X^2 - 2XE[X] + (E[X])^2}]$$

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Variance

$$\underbrace{E[(X - E[X])^2]}$$

$$X^2 - 2XE[X] + (E[X])^2$$

- $E[C] = C$ for constant C
- $E[X]$ is a constant
- $E[E[X]] = E[X]$

$$E[(X - E[X])^2] = E[X^2] - 2E[X]E[X] + (E[X])^2$$

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$$E[(X - E[X])^2] = \underbrace{E[X^2] - 2E[X]E[X] + (E[X])^2}_{-(E[X])^2}$$

More useful

$$Var(X) = E[(X - E[X])^2] \boxed{= E[X^2] - (E[X])^2}$$

Variance (Example)

Let X be the outcome of a fair 6-sided die roll. What is $Var(X)$?

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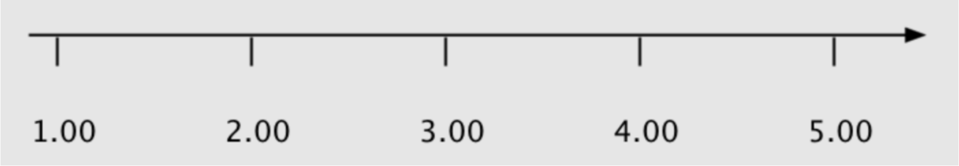
$$Var(X) = E[X^2] - E[X]^2 = \frac{91}{6} - (3.5)^2 = \frac{35}{12}$$

2. Common Distribution (Discrete)

Example 1

- You are the owner of a blind box store.
- Suppose that during each minute of the day, there is a probability p that a customer shows up in your store, and a probability $1 - p$ that no one shows up.



- Let X_i be the number of customers arrive in the i-th minute. What is the distribution of X_i?
- Let $Y_N = X_1 + \cdots + X_N$ be the number of customers arrive in the first N minutes of the day. What is the distribution of Y_N?
- Let Z be the minutes taken for the first customer to show up. What is the distribution of Z?

During each minute of the day, there is a probability p that a customer shows up in your store, and a probability $1-p$ that no one shows up.

?	?	?
Whether a customer arrives during a specific minute.	The number of customers arrive in N minutes	At which minute does the first customer arrive

1.Bernoulli distribution

- Take value 1 with probability p and value 0 with probability 1 − p.

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Other Examples

- Toss a coin
- Win or lose in a game

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Exercise: What is $E[X_i]$ and Var(X_i)?

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Other Examples

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Exercise: What is $E[X_i]$ and Var(X_i)?

- Mean=p
- Variance=p(1-p)

During each minute of the day, there is a probability p that a customer shows up in your store, and a probability $1 - p$ that no one shows up.

Bernoulli Distribution	**?**	**?**
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2.Binomial distribution

- One Bernoulli trial: take value 1 with probability p and value 0 with probability 1 – p.
- N Bernoulli trials??

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- With parameters N and p
- N independent experiments
- Each experiment: success (with probability p) or failure (with probability 1 – p).
- X: the number of success.

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$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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- Mean=np
- Variance=np(1-p)

During each minute of the day, there is a probability p that a customer shows up in your store, and a probability $1 - p$ that no one shows up.

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2.Binomial distribution: Application

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$$R(N,p) = \sum_{k=1}^{N} \Pr(X = k)\, R_k$$

2.Binomial distribution: Application

- Is increasing production yield rate always beneficial for a factory?

- Production Yield Rate: the percentage of non-defective items of all produced items

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- How much to invest?

- Choose p such that R(N,p) + C(p) is minimized

3.Geometric distribution

- Continuously draw a Bernoulli R.V.
- The X-th draw is the first success.
- X follows a geometric distribution.

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- Mean=1/p
- Variance=(1-p)/p^2

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Example

- Toss a coin until there is a head.
- Buy a blind box until you get a Harry Potter

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- At each day, a machine breaks down with a probability p.

- What’s the expected duration before the machine breaks down?

Connections

- A machine breaks down w.p. p on each date.

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