Slide 4-Matrices Algebra I

MAT2040 Linear Algebra

Definition 4.1 (Set of Matrices)

$$\mathbb{R}^{m\times n} = \{m \times n \text{ Matrix}|\text{ entries} \in \mathbb{R}\}$$

$$\mathbb{C}^{m\times n} = \{m \times n \text{ Matrix}|\text{ entries} \in \mathbb{C}\}$$

Given a $n \times n$ Matrix A, A is called a **square matrix**.

$$\mathbb{R}^{n\times n} = \{n \times n \text{ Matrix}|\text{ entries} \in \mathbb{R}\}$$

$$\mathbb{C}^{n\times n} = \{n \times n \text{ Matrix}|\text{ entries} \in \mathbb{C}\}$$

Definition 4.2 (Set of Column Vectors)

$$\mathbb{R}^{n} = \mathbb{R}^{n\times 1} = \{n \times 1 \text{ Matrix}|\text{ entries} \in \mathbb{R}\}$$

$$\mathbb{C}^{n} = \mathbb{C}^{n\times 1} = \{n \times 1 \text{ Matrix}|\text{ entries} \in \mathbb{C}\}$$

Matrix Operation Definition

Definition 4.3 (Matrix Equality) Let A and B be two $m \times n$-matrices. A and B are **equal** (written as "$A = B$") if $a_{ij} = b_{ij}$, for every $i = 1, \cdots, m$, $j = 1 \cdots, n$.

Definition 4.4 (Matrix addition) Let A and B be two $m \times n$-matrices. The **sum of A and B** (written as "$A + B$") is defined to be the $m \times n$-matrix C with entries $c_{ij} = a_{ij} + b_{ij}$, for every $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Definition 4.5 (Scalar Multiplication) Let A be an $m \times n$-matrix and α be any real number (α in $\mathbb{R}$). The **multiplication** of α and A (written as "αA") is defined to be the $m \times n$-matrix D with entries $d_{ij} = \alpha a_{ij}$, for every $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Example 4.6
(1)

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -5 & 4 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -5 \\ -2 & 4 \\ 3 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & x & 3 \\ -5 & 4 & y \end{bmatrix}$$

Then $A \neq B$ (A and B have different sizes).

$A = C \Rightarrow x = -2, y = -1.$

(2)

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} A + (-1)B &= \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + (-1) \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + \begin{bmatrix} -6 & -2 & 4 \\ -3 & -5 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -4 & -5 & 8 \\ -2 & -5 & -9 \end{bmatrix} \end{aligned}$$

$$A - B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & -5 & 8 \\ -2 & -5 & -9 \end{bmatrix}$$

Definition 4.7 (Zero Matrix) Let $A = (a_{ij})_{m\times n}$ such that $a_{ij} = 0, \forall i = 1, \cdots, m, j = 1, \cdots, n$, then A is a **zero matrix**, denoted by $O = O_{m\times n}$.

Theorem 4.8 (Properties of Matrices Operations)
Let $A, B, C \in \mathbb{R}^{m \times n}$, $\alpha, \beta \in \mathbb{R}$.

(1) $A + B = B + A$.

(2) $A + (B + C) = (A + B) + C$.

(We can therefore use the notation $A + B + C$.)

(3) $\alpha(\beta A) = (\alpha\beta)A$.

(4) $\alpha(A + B) = \alpha A + \alpha B$.

(5) $(\alpha + \beta)A = \alpha A + \beta A$.

Proof of (4) $\alpha(A+B) = \alpha A + \alpha B$ (others are left as exercises). Fix a row number i, and column number j.

$$\begin{aligned} (\alpha(A+B))_{ij} =& \alpha(A+B)_{ij} && \text{(definition of scalar multiplication)} \\ =& \alpha(a_{ij}+b_{ij}) && \text{(definition of matrix addition).} \end{aligned}$$

Similarly,

$$\begin{aligned} (\alpha A+\alpha B)_{ij} =& (\alpha A)_{ij} + (\alpha B)_{ij} && \text{(definition of matrix addition)} \\ =& \alpha a_{ij} + \alpha b_{ij} && \text{(definition of scalar multiplication).} \end{aligned}$$

Now we are just comparing real (or complex) numbers, and you know the rules for comparing these:

$$\alpha(a_{ij}+b_{ij}) = \alpha a_{ij} + \alpha b_{ij} \qquad \text{(distributive property).}$$

We see that the (i,j)th entry of $\alpha(A+B)$ equals the (i,j)th entry of $\alpha A + \alpha B$ for all i and j, which is exactly the definition of two matrices being equal.

(**Vectors from Matrix**) Let $A = (a_{ij})_{m \times n}$ be a matrix, then the **ith row vector** is given by

$$\vec{\mathbf{a}}_i = (\mathrm{a}_{i1}, \mathrm{a}_{i2}, \cdots, \mathrm{a}_{in}), \quad i = 1, \cdots, m$$

And the **jth column vector** is given by

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad j = 1, \cdots, n$$

Then

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

If $A, B \in \mathbb{R}^{m\times n}, \alpha \in \mathbb{R}$,

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}, B = [\mathbf{b}_1, \cdots, \mathbf{b}_n] = \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_m \end{bmatrix}$$

Then

$$A + B = [\mathbf{a}_1 + \mathbf{b}_1, \cdots, \mathbf{a}_n + \mathbf{b}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 + \vec{\mathbf{b}}_1 \\ \vec{\mathbf{a}}_2 + \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m + \vec{\mathbf{b}}_m \end{bmatrix}$$

$$\alpha A = [\alpha\mathbf{a}_1, \cdots, \alpha\mathbf{a}_n] = \begin{bmatrix} \alpha\vec{\mathbf{a}}_1 \\ \alpha\vec{\mathbf{a}}_2 \\ \vdots \\ \alpha\vec{\mathbf{a}}_m \end{bmatrix}$$

Matrix-Vector Multiplication

Definition I (From Steven's Book)

Let $A = (a_{ij})_{m\times n} = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} \in \mathbb{R}^{m\times n}$, where $\vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \cdots, \vec{\mathbf{a}}_m$ are row vectors and $\mathbf{u} = (u_i)_{n\times 1}$ is a column vector, then

$$A\mathbf{u} = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{u} \\ \vec{\mathbf{a}}_2\mathbf{u} \\ \vdots \\ \vec{\mathbf{a}}_m\mathbf{u} \end{bmatrix}$$

where $$\vec{\mathbf{a}}_i\mathbf{u} = a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{in}u_n$$

is the scalar product of $\vec{\mathbf{a}}_i$ and $\mathbf{u}$.

Matrix-Vector Multiplication: Second Definition

Definition II (From Beezer's Notes)
Let $A = (a_{ij})_{m\times n} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, where $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are column vectors and $\mathbf{u} = (u_i)_{n\times 1}$ is also a column vector, then the matrix-vector product $A\mathbf{u}$ is

$$u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n$$

which is a **linear Combination** of column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ with weights $u_1, \cdots, u_n$.

Remark: Two definitions produce the same results.

Example 4.9

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 5 \\ -2 & 1 & 3 & 0 & -1 \\ 0 & 7 & -1 & -2 & 4 \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ -1 \end{bmatrix},$$

By definition 1:

$$A\mathbf{u} = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{u} \\ \vec{\mathbf{a}}_2\mathbf{u} \\ \vec{\mathbf{a}}_3\mathbf{u} \end{bmatrix} = \begin{bmatrix} 1*1+4*(-2)+2*0+3*5+5*(-1) \\ (-2)*1+1*(-2)+3*0+0*5+(-1)*(-1) \\ 0*1+7*(-2)+(-1)*0+(-2)*5+4*(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix}$$

By definition 2:

$$
\begin{aligned}
A\mathbf{u} &= u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3 + u_4\mathbf{a}_4 + u_5\mathbf{a}_5 \\
&= 1 \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 5 \cdot \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -3 \\ -28 \end{bmatrix}
\end{aligned}
$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We can write $A\mathbf{x}$ as the linear combination of the column vectors of A with weights $x_1, x_2, \ldots, x_n$, so $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Now use the definition of scalar multiplication and the matrix addition, one has

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Thus, $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Theorem 4.10 (**Equivalent Condition for a Consistent Linear System**) The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the column vectors of A.

Proof. Suppose that $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

By the definition of matrix-vector multiplication, $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}.$$

This is also equivalent to that $\mathbf{b}$ is a linear combination of column vectors of A.

Definition 4.11 (**Matrix Product**) Let $A \in \mathbb{R}^{m\times n}$ and $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_l] \in \mathbb{R}^{n\times r}$, then the matrix product of A by B is a $m \times r$ matrix defined by

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

Remark

1. Matrix product is a natural generalization of the matrix-vector product.
2. AB exists only and if only the number of columns of A equal to the number of rows of B.

Theorem 4.12 (Matrix Product Alternative Definition) Let $A = (a_{ik})_{m\times n} = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$ and $B = (b_{kj})_{n\times l} = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r] \in \mathbb{R}^{n\times r}$, then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_l] = C \triangleq (c_{ij})_{m\times r}$$

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \vec{\mathbf{a}}_i\mathbf{b}_j$.

Note that: $\vec{\mathbf{a}}_i$ is a $1\times n$ matrix (row vector) while $\mathbf{b}_j$ is a $n\times 1$ matrix (column vector), the product $\vec{\mathbf{a}}_i\mathbf{b}_j$ will be a 1×1 matrix which is a scalar.

Proof.

$$c_{ij} = (A\mathbf{b}_j)_i = \begin{bmatrix} \vec{\mathbf{a}}_1\mathbf{b_j} \\ \vdots \\ \vec{\mathbf{a}}_m\mathbf{b_j} \end{bmatrix}_i ----------(\text{ith entry of } \mathrm{A}\mathbf{b}_{\mathrm{j}})$$

$$\begin{aligned} c_{ij} &= (A\mathbf{b}_j)_i \\ &= \vec{\mathbf{a}}_i\mathbf{b}_j - -(\text{by using matrix} - \text{vector multiplication definition 2}) \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \end{aligned}$$

Remark. Most of the book uses the second definition for the matrix product.

Example 4.13

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix},$$

Then by using matrix-matrix multiplication definition 1,

$$AB = \begin{bmatrix} A \begin{bmatrix} 1 \\ -1 \\ 1 \\ 6 \\ 1 \end{bmatrix}, A \begin{bmatrix} 6 \\ 4 \\ 1 \\ 4 \\ -2 \end{bmatrix}, A \begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \\ 3 \end{bmatrix}, A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 28 & 17 & 20 & 10 \\ 20 & -13 & -3 & -1 \\ -18 & -44 & 12 & -3 \end{bmatrix}$$

Alternatively by using matrix-matrix multiplication definition 2, let $AB = C = (c_{ij})_{3\times 4}$, all the entries of C can be figured out.
For example, $c_{12} = \vec{\mathbf{a}}_1 \mathbf{b}_2 = 1 * 6 + 2 * 4 + (-1) * 1 + 4 * 4 + 6 * (-2) = 17$, and other entries can also be calculated.

Note that BA does not exist, because the number of column of B is not equal to the number of rows of A.

Remark

1. AB exists if and only if the number of columns of A equals to the number of rows of B.

2. AB exists does not imply that BA exists.

3. Even if both AB and BA exists, they are generally not equal (A and B are generally not commutative).

Theorem 4.14 **(Properties of Matrix-vector multiplication)** Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, then
(1) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$,
(2) $A(\alpha\mathbf{x}) = (\alpha A)\mathbf{x} = \alpha(A\mathbf{x})$,

Proof. Only show (1). Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$, then

$$A(\mathbf{x}+\mathbf{y}) = (x_1+y_1)\mathbf{a}_1 + (x_2+y_2)\mathbf{a}_2 + \ldots + (x_n+y_n)\mathbf{a}_n$$

$$\begin{aligned} A\mathbf{x} + A\mathbf{y} &= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n + y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \ldots + y_n\mathbf{a}_n \\ &= (x_1+y_1)\mathbf{a}_1 + (x_2+y_2)\mathbf{a}_2 + \ldots + (x_n+y_n)\mathbf{a}_n. \end{aligned}$$

Corollary: If $A \in \mathbb{R}^{m\times n}$, $\mathbf{x}_i \in \mathbb{R}^n$, $\alpha_i \in \mathbb{R}$ $(i = 1, \cdots, s)$, then

$$A(\alpha_1\mathbf{x}_1 + \cdots + \alpha_s\mathbf{x}_s) = \alpha_1 A\mathbf{x}_1 + \cdots + \alpha_s A\mathbf{x}_s.$$

Theorem 4.15 (**Properties of Matrix Product I**) Let $A \in \mathbb{R}^{m\times n}$, $B, C \in \mathbb{R}^{n\times l}$, $\alpha \in \mathbb{R}$, then
(1) $A(B+C) = AB + AC$
(2) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.
If $A \in \mathbb{R}^{n\times l}$, $B, C \in \mathbb{R}^{m\times n}$, then
(3) $(B+C)A = BA + CA$

Proof. Only show (1). Others are excises.
Method 1: Suppose

$$B = [\mathbf{b}_1, \cdots, \mathbf{b}_l], C = [\mathbf{c}_1, \cdots, \mathbf{c}_l],$$

then

$$B + C = [(\mathbf{b}_1 + \mathbf{c}_1),\ (\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ (\mathbf{b}_l + \mathbf{c}_l)].$$

Therefore

$$A(B+C) = [A(\mathbf{b}_1 + \mathbf{c}_1),\ A(\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ A(\mathbf{b}_l + \mathbf{c}_l)].$$

On the other hand,

$$AB = [A\mathbf{b}_1,\ A\mathbf{b}_2,\ \cdots\ ,\ A\mathbf{b}_l], \quad AC = [A\mathbf{c}_1,\ A\mathbf{c}_2,\ \cdots\ ,\ A\mathbf{c}_l].$$

Thus

$$\begin{aligned} AB + AC &= [A\mathbf{b}_1,\ A\mathbf{b}_2,\ \cdots\ ,\ A\mathbf{b}_l] + [A\mathbf{c}_1,\ A\mathbf{c}_2,\ \cdots\ ,\ A\mathbf{c}_l] \\ &= [A(\mathbf{b}_1 + \mathbf{c}_1),\ A(\mathbf{b}_2 + \mathbf{c}_2),\ \ldots\ ,\ A(\mathbf{b}_l + \mathbf{c}_l)] \\ &= A(B + C) \end{aligned}$$

Method 2: The (i, k)-entry of A is a_{ik}, the (k, j)-entry of $B + C$ is $b_{kj} + c_{kj}$, thus the (i, j)-entry of $A(B + C)$ is

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}.$$

On the other hand, the (i, j)-entry of $AB + AC$ is

$$\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}.$$

Thus,

$$A(B + C) = AB + AC.$$

Lemma 4.16 Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $\mathbf{x} \in \mathbb{R}^p$, then

$$(AB)\mathbf{x} = A(B\mathbf{x}). \quad (Assocativity)$$

Proof. Let $B = [\mathbf{b}_1, \mathbf{b}_2, \ \ldots, \mathbf{b}_p]$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$ Then,

$$\begin{aligned} A(B\mathbf{x}) &= A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots + x_p\mathbf{b}_p) \\ &= x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \ldots + x_p A\mathbf{b}_p \\ &= [A\mathbf{b}_1, \ A\mathbf{b}_2, \ \cdots \ , \ A\mathbf{b}_p]\mathbf{x} \\ &= (AB)\mathbf{x} \end{aligned}$$

Where the corollary of Theorem 4.14 (linearity of $A\mathbf{x}$) is used.

Theorem 4.17 **(Property of Matrix Product II)** Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times \ell}$, then

$$(AB)C = A(BC). \quad (\textit{Assocativity})$$

Proof.
Method 1:
Let $C = [\mathbf{c}_1, \mathbf{c}_2, \ \ldots, \mathbf{c}_\ell]$, then

$$(AB)C = [(AB)\mathbf{c}_1, \ (AB)\mathbf{c}_2, \ \ldots, \ (AB)\mathbf{c}_\ell]$$

and because $BC = [B\mathbf{c}_1, \ B\mathbf{c}_2, \ \ldots, B\mathbf{c}_\ell]$, we have

$$A(BC) = [A(B\mathbf{c}_1), \ A(B\mathbf{c}_2), \ \ldots, \ A(B\mathbf{c}_\ell)].$$

Since $(AB)\mathbf{c}_i = A(B\mathbf{c}_i)$ $(i = 1, \cdots, \ell)$ by using **Lemma 4.16**, thus $(AB)C = A(BC)$.
Another proof is put into the appendix.

Therefore, we can write $(AB)C = A(BC) = ABC$.