# STA 2001 Final Exam

Full marks: 100 points

13:30-16:00, July 25,2023

1. (8 points) Let $Y$ have a uniform distribution $U(0,1)$, and let

$$W = a + (b-a)Y, \ a < b.$$

   (a) (4 point) Find the cdf of $W$.

   (b) (4 point) How is $W$ distributed?

2. (12 points) Let $X$ equal the weight in grams of a miniature candy bar. Assume that $\mu = E(X) = 24.43$ and $\sigma^2 = \text{Var}(X) = 2.20$. Let $\bar{X}$ be the sample mean of a random sample of $n = 30$ candy bars. Find

   (a) (4 point) $E(\bar{X})$.

   (b) (4 point) $\text{Var}(\bar{X})$.

   (c) (4 point) $P(24.17 \le \bar{X} \le 24.82)$, approximately.

3. (10 points) A car dealer sells $X$ cars each day and always tries to sell an extended warranty on each of these cars. (In our opinion, most of these warranties are not good deals.) Let $Y$ be the number of extended warranties sold; then $Y \le X$. The joint pmf of $X$ and $Y$ is given by

$$f(x,y) = c(x+1)(4-x)(y+1)(3-y),$$

   where $x = 0,1,2,3$, $y = 0,1,2$, with $y \le x$.

   (a) (2 point) Find the value of $c$.

   (b) (2 point) Compute $\mu_X$ and $\sigma_X^2$.

   (c) (2 point) Compute $\mu_Y$ and $\sigma_Y^2$.

   (d) (2 point) Compute $\text{Cov}(X,Y)$.

   (e) (2 point) Determine $\rho$, the correlation coefficient.

4. (10 points) If the distribution of $X$ is $N(\mu,\sigma^2)$, then $M(t) = E(e^{tX}) = \exp(\mu t + \sigma^2 t^2/2)$. We then say that $Y = e^X$ has a lognormal distribution because $X = \text{ln}Y$.

(a) (4 point) Show that the pdf of $Y$ is

$$g(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\big[ - (\ln y - \mu)^2/2\sigma^2 \big], \; 0 < y < \infty$$

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(b) (6 point) Find (i) $E(Y)$, (ii) $E(Y^2)$, and (iii) $\mathrm{Var}(Y)$.

5. (10 points) Let $Y$ be $\chi^2(n)$. Use the central limit theorem to demonstrate that $W = (Y - n)/\sqrt{2n}$ has a limiting cdf that is $N(0,1)$. Hint: Think of $Y$ as being the sum of a random sample from a certain distribution.

6. (10 points) Let $X_i, i = 1, \ldots, n$ be independent binary random variables with uniform distribution over $\{0,1\}$, where $n > 0$ is an integer. (You may consider $X_1, \ldots, X_n$ as a sequence of Bernoulli trials with success probability one half.) We also call $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ a random binary sequence.

   (a) (2 point) What is the probability that $\mathbf{X}$ is the all "1" sequence (i.e., $X_i = 1, i = 1, \ldots, n$)?

   (b) (2 point) What is the probability that $\mathbf{X}$ has only "1"s on its odd positions and "0"s on its even positions (i.e., $X_i = 1$ if $i$ is odd and $X_i = 0$ if $i$ is even)?

   (c) (6 points) The number of "1"s in $\mathbf{X}$ is $Y = \sum_{i=1}^{n} X_i$. Show that for any $\epsilon > 0$, the probability that $Y \in (n(0.5-\epsilon), \quad n(0.5+\epsilon))$ is at least $1-\epsilon$ when $n$ is sufficiently large.

7. (10 points) Let $X_1, X_2$, and $X_3$ be indepenent random variables with pdf $f(x) = e^{-x}, 0 < x < \infty$, zero elsewhere. Let $Y = \min(X_1, X_2, X_3)$, the minimum of $X_1$, $X_2$ and $X_3$.

   (a) (6 points) Find the pdf of $Y$.

   (b) (4 points) Find the value of $E(Y)$.

8. (8 points) A fair six-sided die is rolled 42 independent times. Let $X$ be the number of threes and $Y$ the number of fives.

   (a) (4 points) What is the conditional pmf of $X$, given $Y = y$?

   (b) (4 points) What is the joint pmf of $X$ and $Y$?

9. (12 points) Let $X_1$ and $X_2$ be independent Poisson random variables with mean $\lambda_1$ and $\lambda_2$ respectively, and let $Y = X_1 + X_2$.

   (a) (4 point) Prove that the distribution of $Y$ is also Poisson.

   (b) (4 points) Find the correlation coefficient of $X_1$ and $Y$.

(c) (4 points) Find the joint distribution of $X_1$ and $X_2$ conditioned on $Y = m$, where $m > 0$ is an integer. (Hint: consider $P(X_1 = k_1, X_2 = k_2 | Y = m)$, where $k_1, k_2 \geq 0$ are integers.)

10. (10 points) Let $Z_1$, $Z_2$ and $Z_3$ have independent standard normal distribution $N(0,1)$.

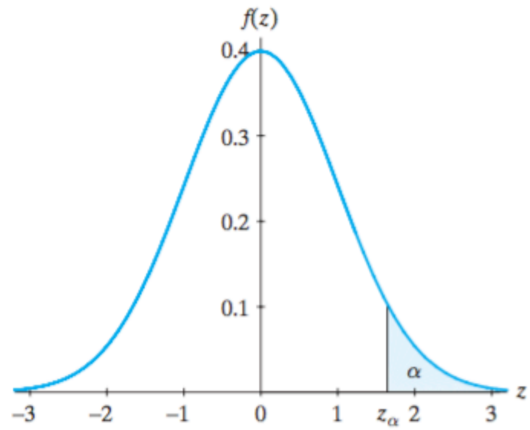
(a) (2 point) Find the distribution of

$$W = \frac{Z_1}{\sqrt{(Z_2^2 + Z_3^2)/2}}.$$

(b) (8 points) Show that

$$V = \frac{Z_1}{\sqrt{(Z_1^2 + Z_2^2)/2}}$$

has pdf $f(v) = \frac{1}{\pi\sqrt{2-v^2}}$, $-\sqrt{2} < v < \sqrt{2}$. You may use the fact that $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, where $\Gamma(\cdot)$ is the Gamma function in the given distribution table for $t$ distribution.

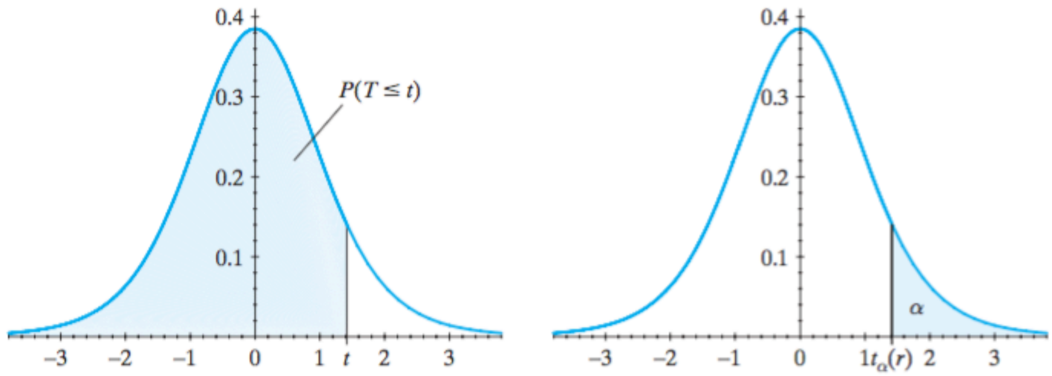



$$P(Z > z_\alpha) = \alpha$$

$$P(Z > z) = 1 - \Phi(z) = \Phi(-z)$$

| $z_\alpha$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |
| 0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| 0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| 0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| 0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| 0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| 0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| 0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| 0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| 0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| 1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| 1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| 1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| 1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| 1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| 1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| 1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| 1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| 1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| 1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| 2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| 2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| 2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| 2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| 2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| 2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| 2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| 2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| 2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| 2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| 3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| 3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| 3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| 3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| 3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |

Figure 1: Distribution table for normal distribution



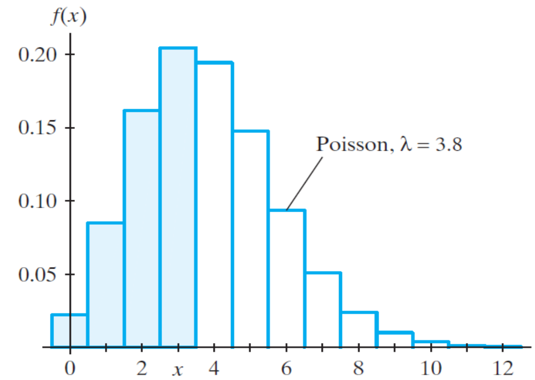

$$P(T \le t) = \int_{-\infty}^{t} \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)(1+w^2/r)^{(r+1)/2}}\,dw$$

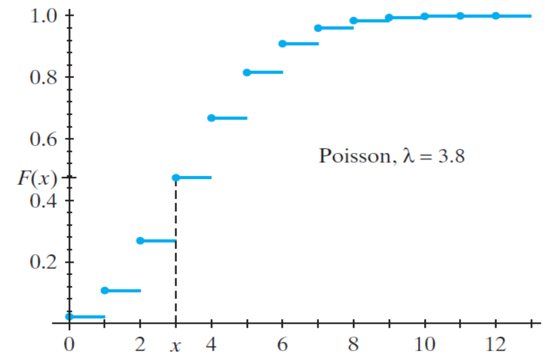
$$P(T \le -t) = 1 - P(T \le t)$$

| | $P(T \le t)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.60 | 0.75 | 0.90 | 0.95 | 0.975 | 0.99 | 0.995 |
| $r$ | $t_{0.40}(r)$ | $t_{0.25}(r)$ | $t_{0.10}(r)$ | $t_{0.05}(r)$ | $t_{0.025}(r)$ | $t_{0.01}(r)$ | $t_{0.005}(r)$ |
| 1 | 0.325 | 1.000 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| 2 | 0.289 | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 |
| 3 | 0.277 | 0.765 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 |
| 4 | 0.271 | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 |
| 5 | 0.267 | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 |
| 6 | 0.265 | 0.718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 |
| 7 | 0.263 | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 |
| 8 | 0.262 | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 |
| 9 | 0.261 | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 |
| 10 | 0.260 | 0.700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 |
| 11 | 0.260 | 0.697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 |
| 12 | 0.259 | 0.695 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 |
| 13 | 0.259 | 0.694 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 |
| 14 | 0.258 | 0.692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.997 |
| 15 | 0.258 | 0.691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 |
| 16 | 0.258 | 0.690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 |
| 17 | 0.257 | 0.689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 |
| 18 | 0.257 | 0.688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 |
| 19 | 0.257 | 0.688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 |
| 20 | 0.257 | 0.687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 |
| 21 | 0.257 | 0.686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 |
| 22 | 0.256 | 0.686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 |
| 23 | 0.256 | 0.685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 |
| 24 | 0.256 | 0.685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 |
| 25 | 0.256 | 0.684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 |
| 26 | 0.256 | 0.684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 |
| 27 | 0.256 | 0.684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 |
| 28 | 0.256 | 0.683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 |
| 29 | 0.256 | 0.683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 |
| 30 | 0.256 | 0.683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 |
| $\infty$ | 0.253 | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 |

Figure 2: Distribution table for t distribution

**Table III** The Poisson Distribution




$$F(x) = P(X \le x) = \sum_{k=0}^{x} \frac{\lambda^k e^{-\lambda}}{k!}$$

| | $\lambda = E(X)$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 4.2 | 4.4 | 4.6 | 4.8 | 5.0 | 5.2 | 5.4 | 5.6 | 5.8 | 6.0 |
| 0 | 0.015 | 0.012 | 0.010 | 0.008 | 0.007 | 0.006 | 0.005 | 0.004 | 0.003 | 0.002 |
| 1 | 0.078 | 0.066 | 0.056 | 0.048 | 0.040 | 0.034 | 0.029 | 0.024 | 0.021 | 0.017 |
| 2 | 0.210 | 0.185 | 0.163 | 0.143 | 0.125 | 0.109 | 0.095 | 0.082 | 0.072 | 0.062 |
| 3 | 0.395 | 0.359 | 0.326 | 0.294 | 0.265 | 0.238 | 0.213 | 0.191 | 0.170 | 0.151 |
| 4 | 0.590 | 0.551 | 0.513 | 0.476 | 0.440 | 0.406 | 0.373 | 0.342 | 0.313 | 0.285 |
| 5 | 0.753 | 0.720 | 0.686 | 0.651 | 0.616 | 0.581 | 0.546 | 0.512 | 0.478 | 0.446 |
| 6 | 0.867 | 0.844 | 0.818 | 0.791 | 0.762 | 0.732 | 0.702 | 0.670 | 0.638 | 0.606 |
| 7 | 0.936 | 0.921 | 0.905 | 0.887 | 0.867 | 0.845 | 0.822 | 0.797 | 0.771 | 0.744 |
| 8 | 0.972 | 0.964 | 0.955 | 0.944 | 0.932 | 0.918 | 0.903 | 0.886 | 0.867 | 0.847 |
| 9 | 0.989 | 0.985 | 0.980 | 0.975 | 0.968 | 0.960 | 0.951 | 0.941 | 0.929 | 0.916 |
| 10 | 0.996 | 0.994 | 0.992 | 0.990 | 0.986 | 0.982 | 0.977 | 0.972 | 0.965 | 0.957 |
| 11 | 0.999 | 0.998 | 0.997 | 0.996 | 0.995 | 0.993 | 0.990 | 0.988 | 0.984 | 0.980 |
| 12 | 1.000 | 0.999 | 0.999 | 0.999 | 0.998 | 0.997 | 0.996 | 0.995 | 0.993 | 0.991 |
| 13 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 0.999 | 0.999 | 0.998 | 0.997 | 0.996 |
| 14 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 0.999 | 0.999 | 0.999 |
| 15 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 |
| 16 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

| $x$ | 6.5 | 7.0 | 7.5 | 8.0 | 8.5 | 9.0 | 9.5 | 10.0 | 10.5 | 11.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.002 | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 1 | 0.011 | 0.007 | 0.005 | 0.003 | 0.002 | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 |
| 2 | 0.043 | 0.030 | 0.020 | 0.014 | 0.009 | 0.006 | 0.004 | 0.003 | 0.002 | 0.001 |
| 3 | 0.112 | 0.082 | 0.059 | 0.042 | 0.030 | 0.021 | 0.015 | 0.010 | 0.007 | 0.005 |
| 4 | 0.224 | 0.173 | 0.132 | 0.100 | 0.074 | 0.055 | 0.040 | 0.029 | 0.021 | 0.015 |
| 5 | 0.369 | 0.301 | 0.241 | 0.191 | 0.150 | 0.116 | 0.089 | 0.067 | 0.050 | 0.038 |
| 6 | 0.527 | 0.450 | 0.378 | 0.313 | 0.256 | 0.207 | 0.165 | 0.130 | 0.102 | 0.079 |
| 7 | 0.673 | 0.599 | 0.525 | 0.453 | 0.386 | 0.324 | 0.269 | 0.220 | 0.179 | 0.143 |
| 8 | 0.792 | 0.729 | 0.662 | 0.593 | 0.523 | 0.456 | 0.392 | 0.333 | 0.279 | 0.232 |
| 9 | 0.877 | 0.830 | 0.776 | 0.717 | 0.653 | 0.587 | 0.522 | 0.458 | 0.397 | 0.341 |
| 10 | 0.933 | 0.901 | 0.862 | 0.816 | 0.763 | 0.706 | 0.645 | 0.583 | 0.521 | 0.460 |
| 11 | 0.966 | 0.947 | 0.921 | 0.888 | 0.849 | 0.803 | 0.752 | 0.697 | 0.639 | 0.579 |
| 12 | 0.984 | 0.973 | 0.957 | 0.936 | 0.909 | 0.876 | 0.836 | 0.792 | 0.742 | 0.689 |
| 13 | 0.993 | 0.987 | 0.978 | 0.966 | 0.949 | 0.926 | 0.898 | 0.864 | 0.825 | 0.781 |
| 14 | 0.997 | 0.994 | 0.990 | 0.983 | 0.973 | 0.959 | 0.940 | 0.917 | 0.888 | 0.854 |

Figure 3: Distribution table for poisson distribution