

MAT1002 Midterm Reference Solution (2024)

1. T T F F F T

2. (i) $\frac{1+\sqrt{5}}{2}$

(ii) $\sqrt{90}$ (or $3\sqrt{10}$)

(iii) $\langle \frac{8}{11}, -\frac{8}{11}, \frac{24}{11} \rangle$ (or $\frac{8}{11}\vec{i} - \frac{8}{11}\vec{j} + \frac{24}{11}\vec{k}$)

(iv) $\langle 0, -1 \rangle$ (or $-\vec{j}$)

3. False. Consider $a_n = b_n = (-1)^n \frac{1}{\sqrt{n}}$. Then both $\sum a_n$ & $\sum b_n$ convs by the alternating series test. But

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

divgs as the harmonic series.

4. (i) Since

$$|a_n| = \left(1 - \frac{2024}{n}\right)^n = \left(1 + \frac{-2024}{n}\right)^n \rightarrow e^{-2024} \text{ as } n \rightarrow \infty,$$

we have $\lim_{n \rightarrow \infty} a_n \neq 0$, so series diverges by the n^{th} -term test.

(ii) Consider $S := \sum_{n=1}^{\infty} \frac{1}{(n+1)(\ln(1+n))^2}$ first. Since

$$\int_1^b \frac{dx}{(1+x)(\ln(1+x))^2} = \int_{\ln 2}^{\ln(1+b)} \frac{du}{u^2} \quad \begin{array}{l} u = \ln(1+x) \\ du = dx/(1+x) \end{array}$$

$$= \left. \frac{-1}{u} \right|_{u=\ln 2}^{\ln(1+b)} = \frac{1}{\ln 2} - \frac{1}{\ln(1+b)},$$

$$\int_1^{\infty} \frac{dx}{(1+x)(\ln(1+x))^2} = \lim_{b \rightarrow \infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln(1+b)} \right) = \frac{1}{\ln 2} \quad \text{Cvgs.}$$

By the integral test, S converges. Since $\forall n \geq 1$,

$$0 \leq \frac{1}{\underbrace{(20n+24)}_{20(n+\frac{24}{20})}(\ln(1+n))^2} < \frac{1}{(n+1)(\ln(1+n))^2},$$

$20(n+\frac{24}{20}) > (n+1)$

Series converges absolutely by direct comparison test.

(iii) Since $|\cos(y)| \leq 1, \forall y$, we have

$$0 \leq \left| \frac{\cos(\frac{1}{2}n^2\pi)}{n\sqrt{n}} \right| \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

By direct comparison with the convergent p -series $\sum \frac{1}{n^{3/2}}$ ($p=3/2$), series converges absolutely.

$$(iv) \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \frac{(2n)!}{n!n!} = \frac{(n+1)^2}{(2n+2)(2n+1)} \longrightarrow \frac{1}{4} (< 1).$$

as $n \rightarrow \infty$

By the ratio test, series converges absolutely.

5.

$$\begin{aligned}
 S &= \int_0^2 2\pi y \, ds \quad \text{could write } \sqrt{(x'(t))^2 + (y'(t))^2} \, dt \\
 &= \int_0^2 2\pi(4t)(4\sqrt{t^2 + 1}) \, dt \\
 &= 32\pi \int_0^2 t\sqrt{1 + t^2} \, dt \\
 &= \frac{32\pi}{3} (t^2 + 1)^{3/2} \Big|_0^2 \\
 &= \frac{32\pi}{3} (5^{3/2} - 1).
 \end{aligned}$$

6. A normal vector of the plane is given by

$$\begin{aligned}
 \vec{PQ} \times \vec{PR} &= \langle 2, 3, -4 \rangle \times \langle -3, 6, -3 \rangle \\
 &= \langle 15, 18, 21 \rangle.
 \end{aligned}$$

Hence the plane is

$$15(x-1) + 18(y+2) + 21(z-3) = 0,$$

which gives

$$\begin{aligned}
 15x + 18y + 21z &= 42 \\
 (\text{or } 5x + 6y + 7z &= 14) .
 \end{aligned}$$

7. Since $\vec{r}'(t) = 3(t-1)^2 \vec{i} - \pi \sin(\pi t) \vec{j}$, \vec{r}' is continuous for all t , but $\vec{r}'(t) = \vec{0} \Leftrightarrow t=1$, so the curve is not smooth, and the only cusp occurs at the point

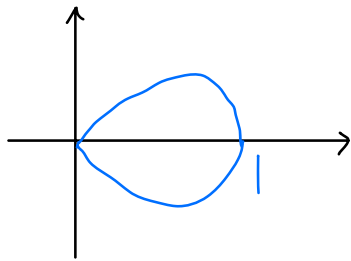
$$((t-1)^3, \cos(\pi t)) \Big|_{t=1} = (0, -1).$$

*: Technically, finding where $\vec{r}' = 0$ would only give you where the motion is non-smooth (and the curve may still be smooth there); e.g., $\vec{r}(t) = \langle t^3, t^3 \rangle$ would give you a smooth curve (a line) with no cusp, although $\vec{r}'(0) = \langle 0, 0 \rangle$ (this is only a candidate for a cusp). For this question, the point in the answer is indeed a cusp (try graphing it with Desmos), and we do not require you to justify any further beyond the reference solution. If you would like a bit of insight why it is a cusp, note that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\pi \sin(\pi t)}{3(t-1)^2}, \text{ so } \lim_{x \rightarrow 0^-} \frac{dy}{dx} = \lim_{t \rightarrow 1^-} \frac{dy}{dx} = -\infty$$

and similarly $\lim_{x \rightarrow 0^+} \frac{dy}{dx} = \infty$, giving a sharp "corner" at $x=0$.

8. (i)



(Since $\theta^2 = (-\theta)^2$,
curve is
symmetric about
the x-axis).

(ii)

$$L = \int_{-1}^1 \sqrt{r^2 + (r')^2} d\theta = 2 \int_0^1 (1 + \theta^2) d\theta = \frac{8}{3}.$$

(iii)

$$A = \int_{-1}^1 \frac{1}{2} r^2 d\theta = \int_{-1}^1 \frac{1}{2} (1 - \theta^2)^2 d\theta = \frac{8}{15}$$

9. (a) The curve intersects the plane when $2 + 3(2t) + 3(t^2) = 5$,
whence

$$3t^2 + 6t - 3 = 0,$$

which has roots $t = -1 \pm \sqrt{2}$. Since $t \geq 0$, we have $t_0 = \sqrt{2} - 1$.

(b) Substituting $t = \sqrt{2} - 1$ into $\vec{r}(t)$ gives the point of impact:

$$(1, 2\sqrt{2} - 2, 3 - 2\sqrt{2}).$$

(c) The plane has a normal $\vec{n} = \langle 2, 3, 3 \rangle$. Since

$$\vec{r}'(t) = \langle 0, 2, 2t \rangle,$$

its tangent at $t_0 = \sqrt{2} - 1$ is $\vec{r}'(t_0) = \langle 0, 2, 2\sqrt{2} - 2 \rangle$. Compute

$$\cos(\theta) = \frac{\vec{n} \cdot \vec{r}'(t_0)}{|\vec{n}| |\vec{r}'(t_0)|} = \frac{6\sqrt{2}}{\sqrt{22} \cdot 2\sqrt{4-2\sqrt{2}}} = \frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}.$$

$$\left(\text{or } \frac{3\sqrt{4-2\sqrt{2}}}{\sqrt{11}(4-2\sqrt{2})} \text{ or } \frac{3(2+\sqrt{2})\sqrt{4-2\sqrt{2}}}{4\sqrt{11}} \text{ or } \frac{3(2+\sqrt{2})\sqrt{4-2\sqrt{2}}\sqrt{11}}{44} \right)$$

Since this value is > 0 , the angle is acute and

$$\cos \theta_0 = \frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}, \text{ or } \theta_0 = \arccos\left(\frac{3}{\sqrt{11}\sqrt{4-2\sqrt{2}}}\right).$$

10. (a) $\vec{r}'(t) = \langle \sqrt{2} \sin t, \sqrt{1 + \cos(2t)}, 1 \rangle$

$$\begin{aligned} s(t) &= \int_0^t |\vec{r}'(u)| \, du = \int_0^t \sqrt{2 \sin^2 u + 1 + \cos(2u) + 1} \, du \\ &= \int_0^t \sqrt{2 \sin^2 u + 1 + 2 \cos^2 u} \, du \\ &= \int_0^t \sqrt{3} \, du = \sqrt{3} t \end{aligned}$$

Since $s_0 = s(T) = \sqrt{3} T$, we have $T = \frac{s_0}{\sqrt{3}}$.

Average speed is $\frac{s_0}{T} = \sqrt{3}$.

(b) Since

$$\begin{aligned} f(t) &= \int_0^t \sqrt{1 + \cos(2u)} \, du = \int_0^t \sqrt{2 \cos^2 u} \, du = \sqrt{2} \int_0^t \cos u \, du \\ &= \sqrt{2} \sin t, \end{aligned}$$

we have $c(t) = |\vec{r}(t)| = \sqrt{2 \cos^2 t + 2 \sin^2 t + t^2} = \sqrt{2 + t^2}$

and

$$c'(t) = \frac{1}{2} \frac{2t}{\sqrt{2+t^2}} = \frac{t}{\sqrt{2+t^2}}.$$

* Note that the integral above only works for $t \in [0, \frac{\pi}{2}]$: $\sqrt{\cos^2 u} = |\cos u|$ instead and the form of $f(t)$ will be more complicated beyond $t > \frac{\pi}{2}$. For this reason, if you get the concept correctly without figuring out the integral, it is still considered correct.

- (C) · $\left| \frac{d}{dt} \vec{r}(t) \right|$ measures speed along the curve C , i.e., rate of change of distance measured along the curve.
- $\frac{d}{dt} C(t)$ measures the rate of change of distance NOT measured along the curve (it is the distance from the origin).

$$11. (a) \cos(\sqrt{t}) = \sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{t})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(2n)!}$$

$$\Rightarrow \int_0^x \cos(\sqrt{t}) dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(n+1)} x^{n+1}$$

$$(b) F(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(n+1)} \text{ by (a).}$$

By alternating series approximation, we want $U_k < 0.001$,

$$\text{where } U_k = \frac{1}{(2k)!(k+1)}. \text{ Since } U_2 = \frac{1}{4! \cdot 3} = \frac{1}{72} > 0.001$$

but $\underbrace{U_3 = \frac{1}{(6!) \cdot 4}}_{\text{first unused term}} < 0.001$, we should take terms for $n=0, 1, 2$,

i.e., need $N=3$ terms.

12. (a) Method 1

Note that $f(x) = g'(x)$, where $g(x) = \frac{1}{1-x}$. Hence

$$f(x) = \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = \sum_{n=1}^{\infty} n x^{n-1} \quad (= \sum_{n=0}^{\infty} (n+1) x^n).$$

Method 2

$$\text{Note that } f(x) = (1+(-x))^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} (-x)^n.$$

$$\text{Since } \binom{-2}{n} = \frac{(-2)(-3)\dots(-(n+1))}{n!} = \frac{(-1)^n \cdot (n+1)!}{n!} = (-1)^n (n+1),$$

(also holds for $n=0$)

$$\text{We have } f(x) = \sum_{n=0}^{\infty} (n+1) x^n \quad (= \sum_{n=1}^{\infty} n x^{n-1}).$$

(b) Let $a_n := (n+1)x^n$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = (n+1)|x| \xrightarrow{n \rightarrow \infty} |x| \quad \begin{cases} \text{series convs for } |x| < 1 \\ \text{divgs for } |x| > 1 \end{cases}$$

\therefore Radius of convergence is $R=1$.

Check endpoints for $x=1$ & -1 , $|a_n| = n+1 \rightarrow \infty$

as $n \rightarrow \infty$, so $\lim_{n \rightarrow \infty} a_n \neq 0$; series divgs at $x=\pm 1$

by n^{th} -term test.

\therefore Series convs only for $x \in (-1, 1)$.

13. It converges.

$$\cdot S_1 := \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{4} \frac{1}{1-\frac{3}{4}} = 3.$$

• Since $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ by Q12, we have

$$\sum_{n=1}^{\infty} \frac{n}{4^{n-1}} = \sum_{n=1}^{\infty} n \left(\frac{1}{4}\right)^{n-1} = \frac{1}{(1-\frac{1}{4})^2} = \frac{16}{9},$$

So

$$S_2 := \sum_{n=1}^{\infty} \frac{2n}{4^n} = \frac{2}{4} \sum_{n=1}^{\infty} \frac{n}{4^{n-1}} = \frac{8}{9}.$$

Therefore, series = $S_1 + S_2 = 3 + \frac{8}{9} = \frac{35}{9}.$