

4.4

Concavity and Curve Sketching

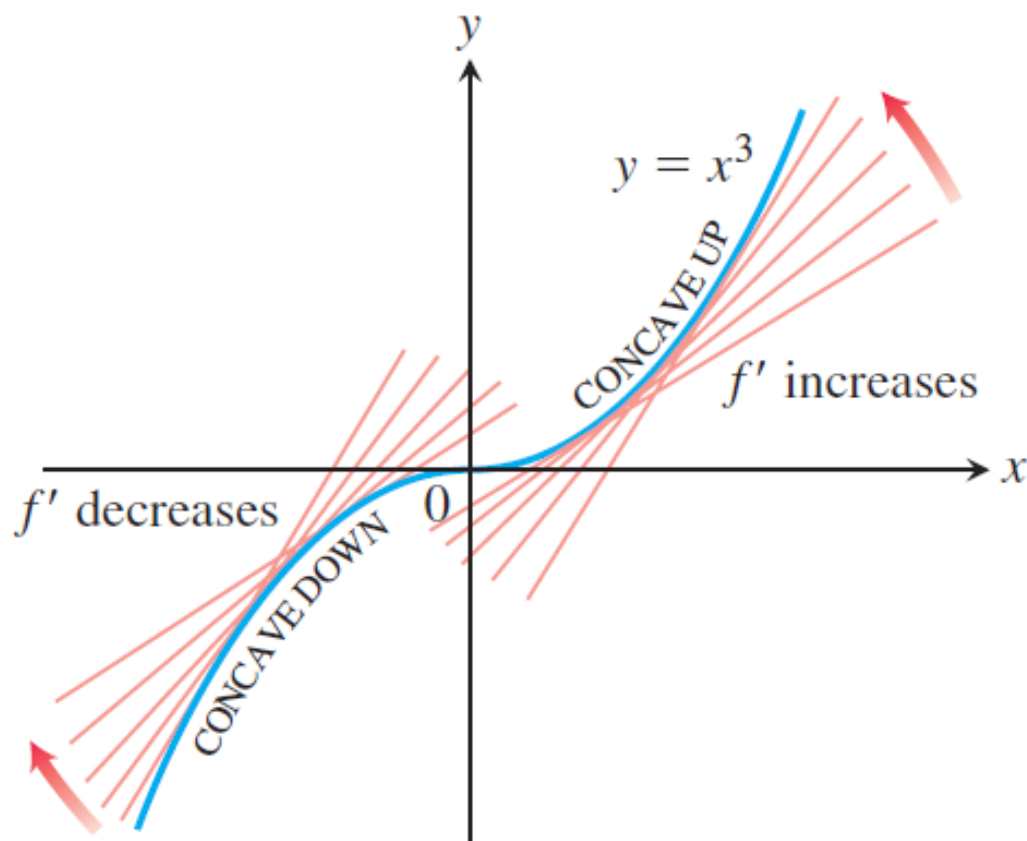


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

DEFINITION The graph of a differentiable function $y = f(x)$ is

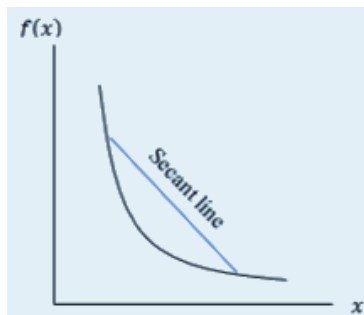
- (a) **concave up** on an open interval I if f' is increasing on I ;
- (b) **concave down** on an open interval I if f' is decreasing on I .

Concavity and Secant Lines

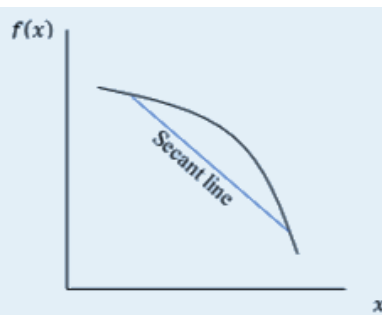
Let f be continuous on $[a, b]$, and differentiable on (a, b) .

- (i) If f is concave up on (a, b) , then the graph of f lies below the secant line joining $(a, f(a))$ and $(b, f(b))$ on (a, b) .
- (ii) If f is concave down on (a, b) , then the graph of f lies above the secant line joining $(a, f(a))$ and $(b, f(b))$ on (a, b) .

Concave Up



Concave Down

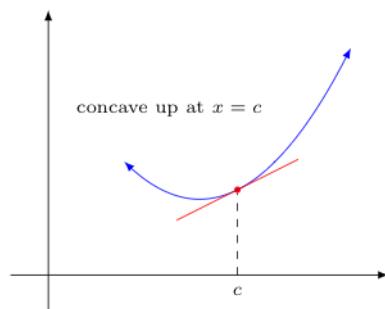


Concavity and Tangent Lines

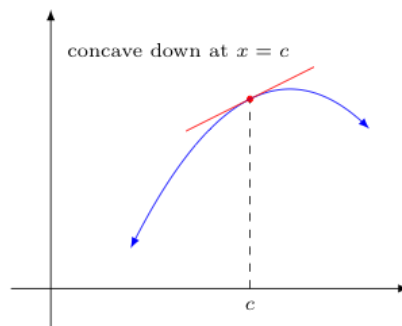
Let f be continuous on $[a, b]$, and differentiable on (a, b) .

- (i) If f is concave up on (a, b) , then for any $c \in (a, b)$, the tangent line to $y = f(x)$ at c lies below the graph of $y = f(x)$.
- (ii) If f is concave down on (a, b) , then for any $c \in (a, b)$, the tangent line to $y = f(x)$ at c lies above the graph of $y = f(x)$.

Concave Up



Concave Down



The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I .

1. If $f'' > 0$ on I , the graph of f over I is concave up.
2. If $f'' < 0$ on I , the graph of f over I is concave down.

Proof: Immediate from Corollary 3 earlier.

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b) .

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

EXAMPLE 1

- (a) The curve $y = x^3$ (Figure 4.24) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.
- (b) The curve $y = x^2$ (Figure 4.25) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ■

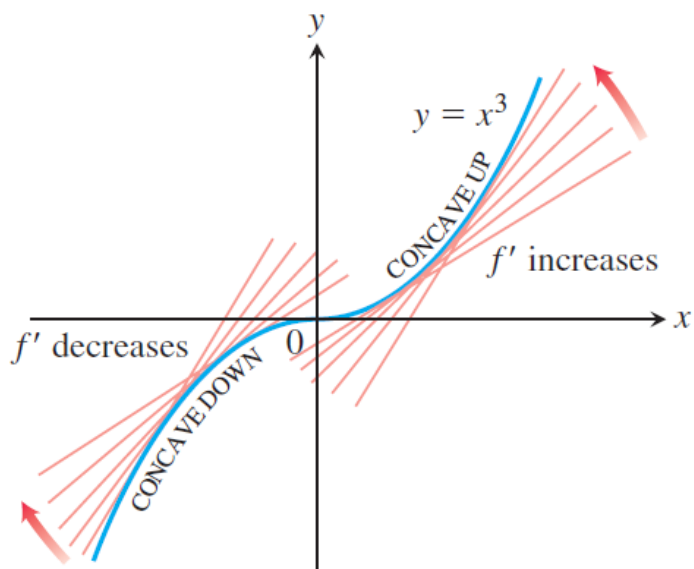


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

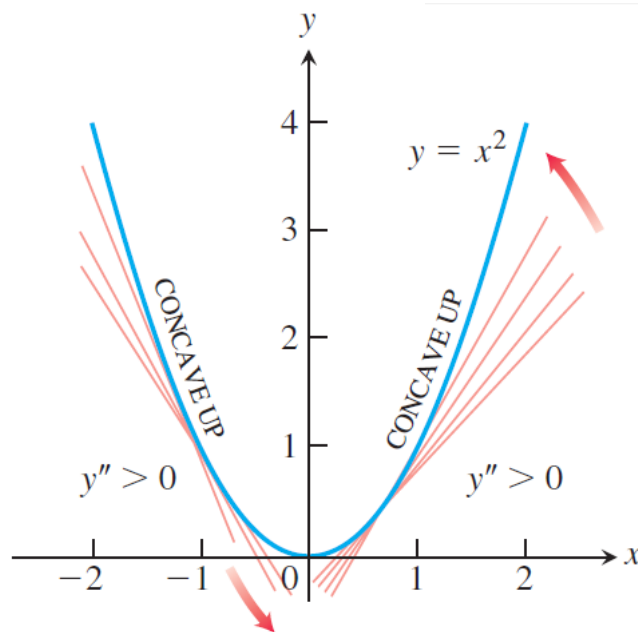


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

EXAMPLE 2 Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The first derivative of $y = 3 + \sin x$ is $y' = \cos x$, and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.26). ■

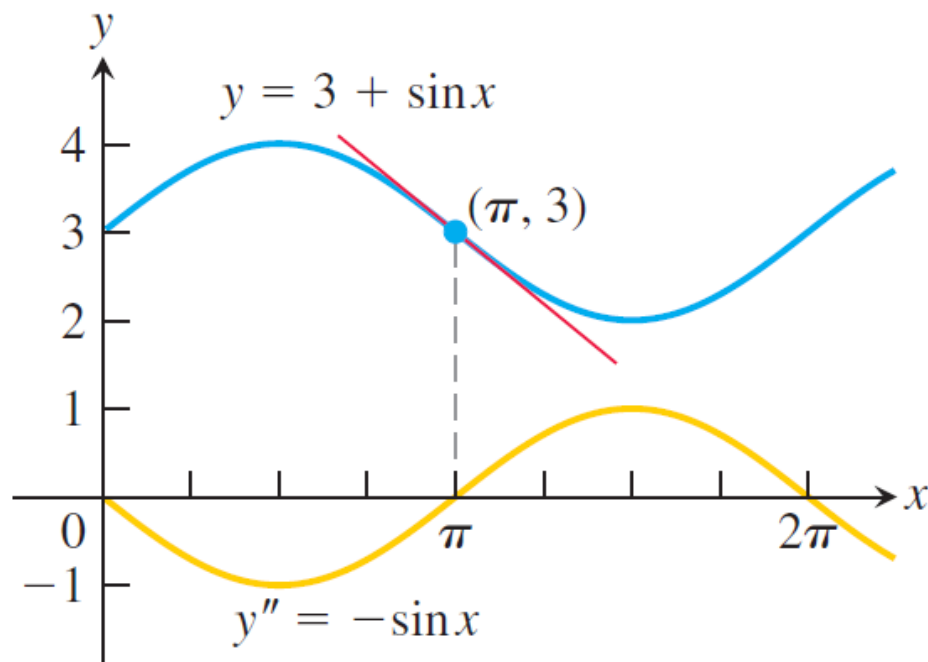


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

EXAMPLE 3 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx} \left(\frac{5}{3}x^{2/3} \right) = \frac{10}{9}x^{-1/3}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.27. ■

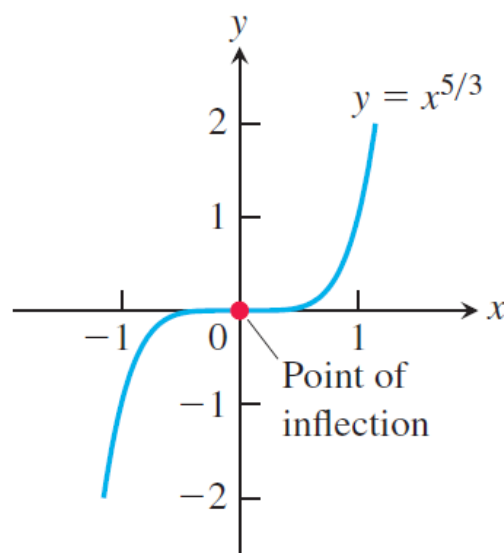


FIGURE 4.27 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 3).

EXAMPLE 4 The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.28). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign. ■

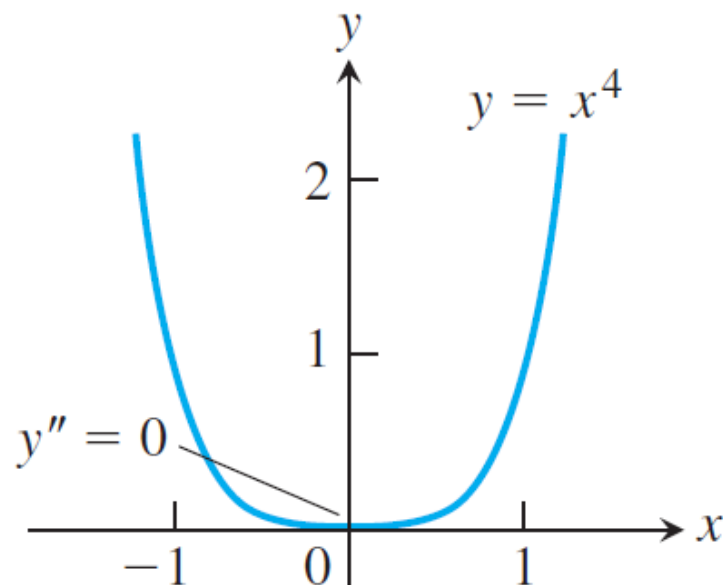


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

EXAMPLE 5 The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}(x^{1/3}) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = x^{-2/3}/3$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.29. ■

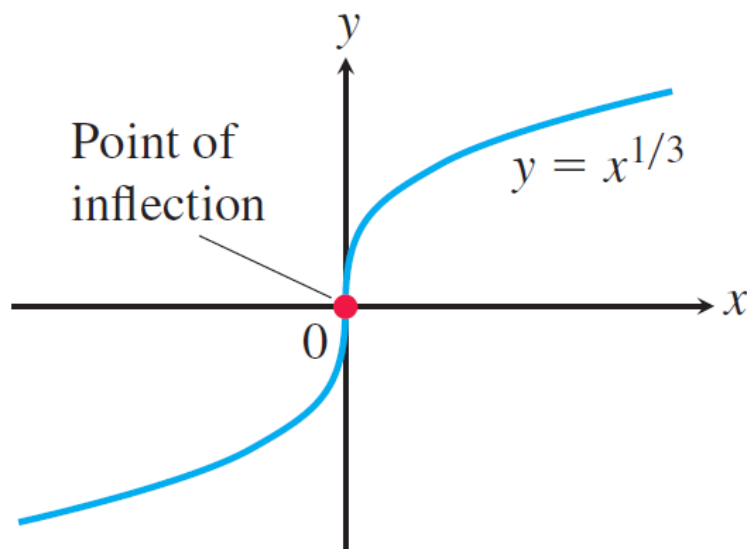


FIGURE 4.29 A point of inflection where y' and y'' fail to exist (Example 5).

THEOREM 5—Second Derivative Test for Local Extrema

on an open interval that contains $x = c$.

Suppose f'' is continuous

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c , since f'' is continuous. Therefore, f' is decreasing on I . Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

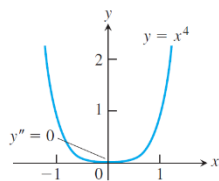


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 4).

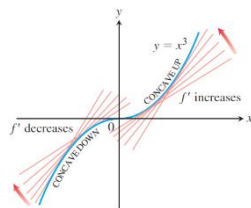
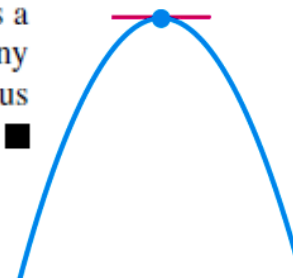
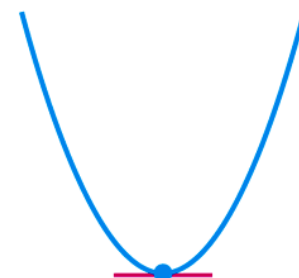


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).



$f' = 0, f'' < 0$
 \Rightarrow local max



$f' = 0, f'' > 0$
 \Rightarrow local min

EXAMPLE 7 Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

- (a) Identify where the extrema of f occur.
- (b) Find the intervals on which f is increasing and the intervals on which f is decreasing.
- (c) Find where the graph of f is concave up and where it is concave down.
- (d) Sketch the general shape of the graph for f .
- (e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f' . Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	–	–	+
Behavior of f	decreasing	decreasing	increasing

- (a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.
- (b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

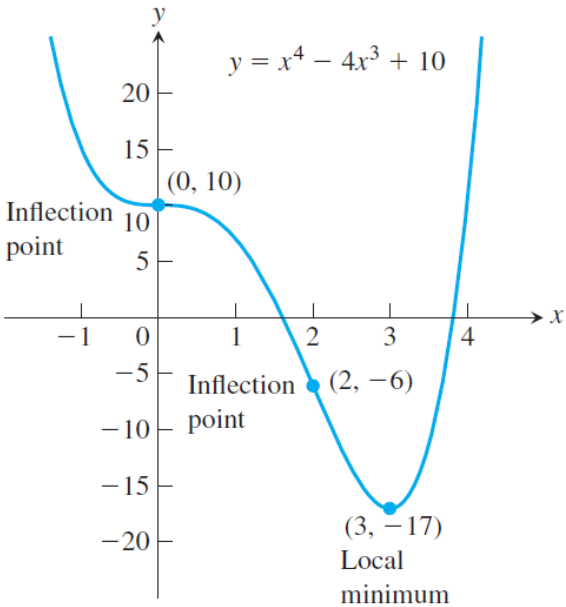


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	+	-	+
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

(d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing	decreasing	decreasing	increasing
concave up	concave down	concave up	concave up

The general shape of the curve is shown in the accompanying figure.

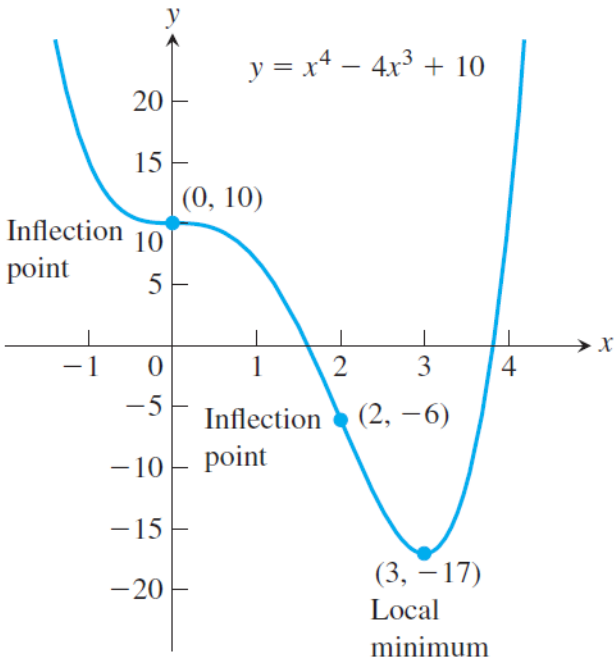
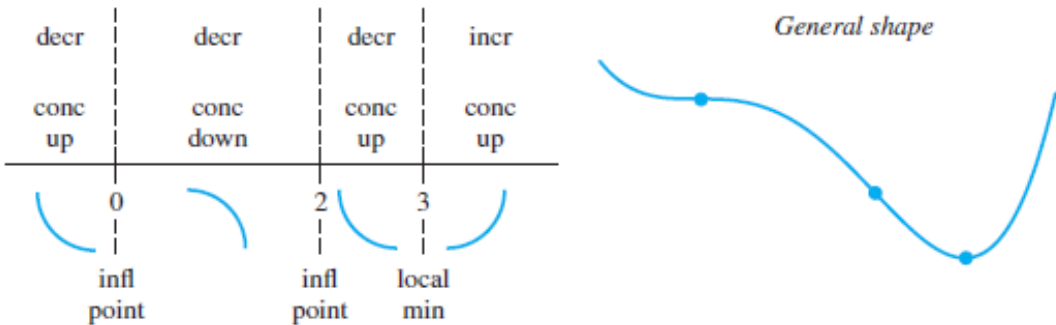


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 7).

(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of f . ■

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y'' .
3. Find the critical points of f , if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

EXAMPLE 8 Sketch the graph of $f(x) = \frac{(x + 1)^2}{1 + x^2}$.

Solution

- 1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.1).
- 2. Find f' and f'' .

$$f(x) = \frac{(x + 1)^2}{1 + x^2}$$

$$f'(x) = \frac{(1 + x^2) \cdot 2(x + 1) - (x + 1)^2 \cdot 2x}{(1 + x^2)^2}$$

$$= \frac{2(1 - x^2)}{(1 + x^2)^2}$$

$$f''(x) = \frac{(1 + x^2)^2 \cdot 2(-2x) - 2(1 - x^2)[2(1 + x^2) \cdot 2x]}{(1 + x^2)^4}$$

$$= \frac{4x(x^2 - 3)}{(1 + x^2)^3}$$

x -intercept at $x = -1$,
 y -intercept ($y = 1$) at
 $x = 0$

Critical points: $x = -1, x = 1$

After some algebra

- 3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f . At $x = -1$, $f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.
- 4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

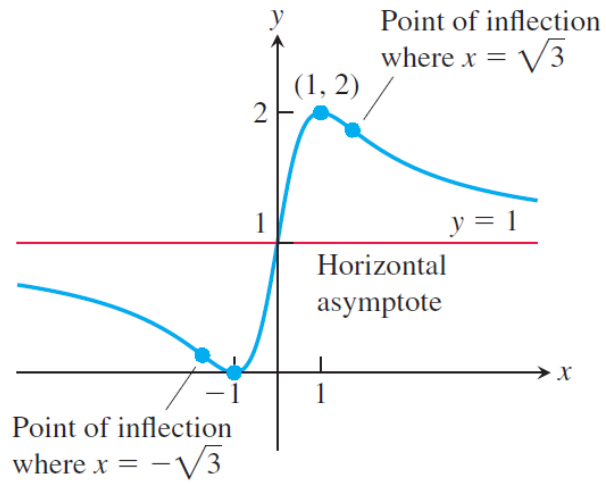


FIGURE 4.31 The graph of $y = \frac{(x + 1)^2}{1 + x^2}$ (Example 8).

5. **Inflection points.** Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

6. **Asymptotes.** Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$\begin{aligned} f(x) &= \frac{(x+1)^2}{1+x^2} = \frac{x^2 + 2x + 1}{1+x^2} && \text{Expanding numerator} \\ &= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. && \text{Dividing by } x^2 \end{aligned}$$

We see that $f(x) \rightarrow 1^+$ as $x \rightarrow \infty$ and that $f(x) \rightarrow 1^-$ as $x \rightarrow -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \leq y \leq 2$).

7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \rightarrow -\infty$, and concave up in its approach to $y = 1$ as $x \rightarrow \infty$. ■

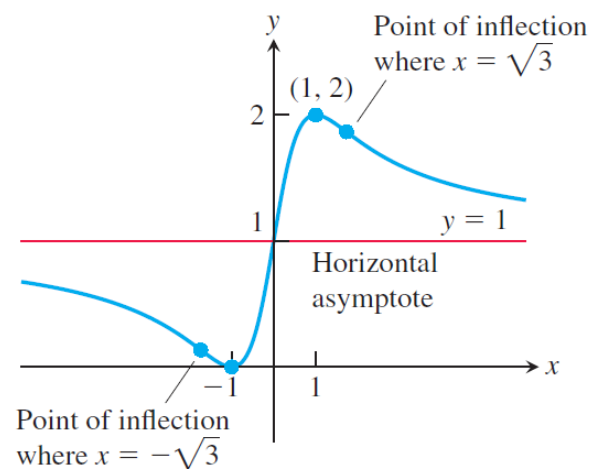


FIGURE 4.31 The graph of $y = \frac{(x+1)^2}{1+x^2}$ (Example 8).

EXAMPLE 9 Sketch the graph of $f(x) = \frac{x^2 + 4}{2x}$.

Solution

- 1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
- 2. We calculate the derivatives of the function, but first rewrite it in order to simplify our computations:

$$f(x) = \frac{x^2 + 4}{2x} = \frac{x}{2} + \frac{2}{x}$$

Function simplified for differentiation

$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$$

Combine fractions to solve easily $f'(x) = 0$.

$$f''(x) = \frac{4}{x^3}$$

Exists throughout the entire domain of f

- 3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.

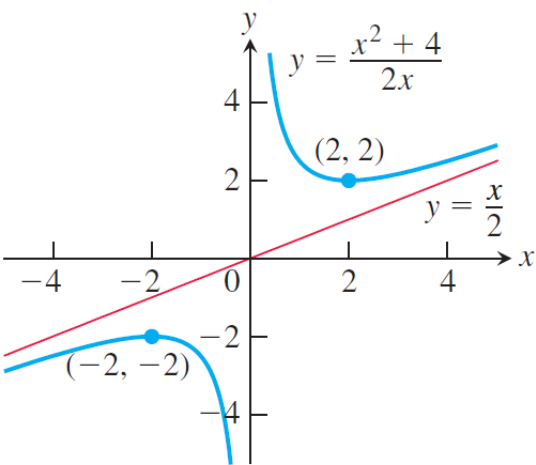


FIGURE 4.32 The graph of $y = \frac{x^2 + 4}{2x}$ (Example 9).

4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f . The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
6. From the rewritten formula for $f(x)$, we see that

$$\lim_{x \rightarrow 0^+} \left(\frac{x}{2} + \frac{2}{x} \right) = +\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \left(\frac{x}{2} + \frac{2}{x} \right) = -\infty,$$

so the y -axis is a vertical asymptote. Also, as $x \rightarrow \infty$ or as $x \rightarrow -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.

7. The graph of f is sketched in Figure 4.32.

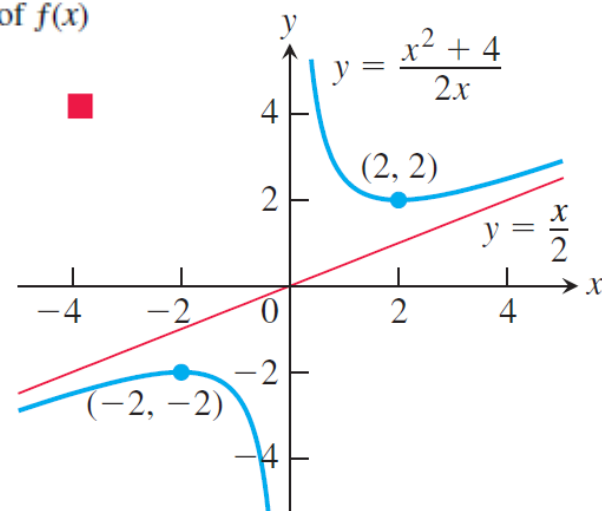


FIGURE 4.32 The graph of $y = \frac{x^2 + 4}{2x}$ (Example 9).

EXAMPLE 10 Sketch the graph of $f(x) = \cos x - \frac{\sqrt{2}}{2}x$ over $0 \leq x \leq 2\pi$.

Solution The derivatives of f are

$$f'(x) = -\sin x - \frac{\sqrt{2}}{2} \quad \text{and} \quad f''(x) = -\cos x.$$

Both derivatives exist everywhere over the interval $(0, 2\pi)$. Within that open interval, the first derivative is zero when $\sin x = -\sqrt{2}/2$, so the critical points are $x = 5\pi/4$ and $x = 7\pi/4$. Since $f''(5\pi/4) = -\cos(5\pi/4) = \sqrt{2}/2 > 0$, the function has a local minimum value of $f(5\pi/4) \approx -3.48$ (evaluated with a calculator) by The Second Derivative Test. Also, $f''(7\pi/4) = -\cos(7\pi/4) = -\sqrt{2}/2 < 0$, so the function has a local maximum value of $f(7\pi/4) \approx -3.18$.

Examining the second derivative, we find that $f'' = 0$ when $x = \pi/2$ or $x = 3\pi/2$. We conclude that $(\pi/2, f(\pi/2)) \approx (\pi/2, -1.11)$ and $(3\pi/2, f(3\pi/2)) \approx (3\pi/2, -3.33)$ are points of inflection.

Finally, we evaluate f at the endpoints of the interval to find $f(0) = 1$ and $f(2\pi) \approx -3.44$. Therefore, the values $f(0) = 1$ and $f(5\pi/4) \approx -3.48$ are the absolute maximum and absolute minimum values of f over the closed interval $[0, 2\pi]$. The graph of f is sketched in Figure 4.33.

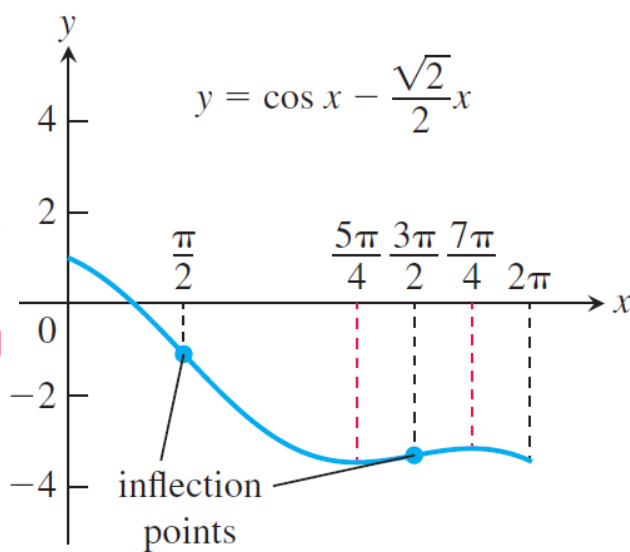
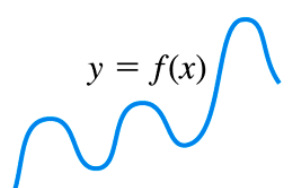
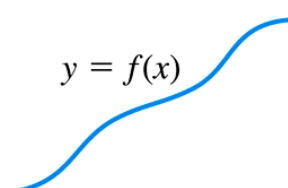
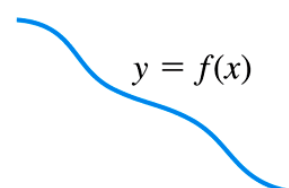
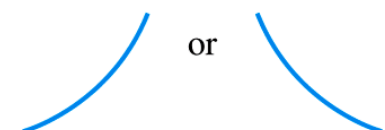
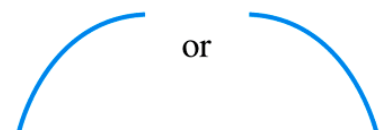

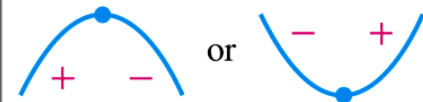
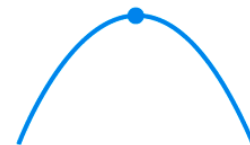



FIGURE 4.33 The graph of the function in Example 10.

Summary of information from derivatives

 <p>$y = f(x)$</p> <p>Differentiable \Rightarrow smooth, connected; graph may rise and fall</p>	 <p>$y = f(x)$</p> <p>$y' > 0 \Rightarrow$ rises from left to right; may be wavy</p>	 <p>$y = f(x)$</p> <p>$y' < 0 \Rightarrow$ falls from left to right; may be wavy</p>
 <p>or</p> <p>$y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall</p>	 <p>or</p> <p>$y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall</p>	 <p>y'' changes sign at an inflection point</p>
 <p>or</p> <p>y' changes sign \Rightarrow graph has local maximum or local minimum</p>	 <p>$y' = 0$ and $y'' < 0$ at a point; graph has local maximum</p>	 <p>$y' = 0$ and $y'' > 0$ at a point; graph has local minimum</p>

4.5

Applied Optimization

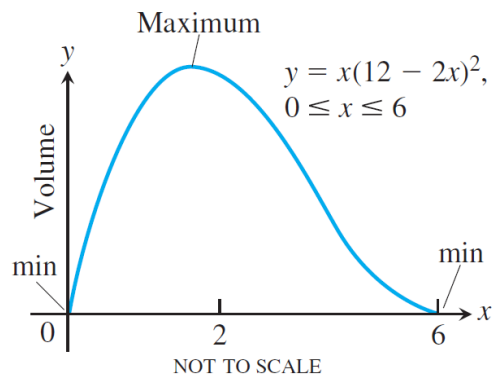
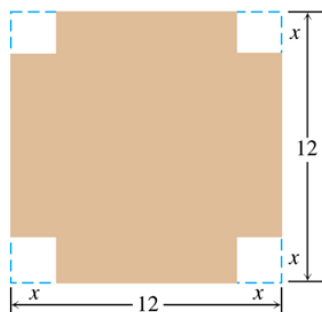
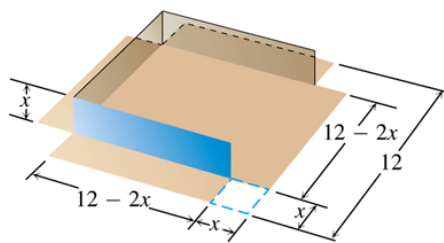


FIGURE 4.35 The volume of the box in Figure 4.34 graphed as a function of x .



(a)



(b)

EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-cm-by-12-cm sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Figure 4.34). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \quad V = htw$$

Since the sides of the sheet of tin are only 12 cm long, $x \leq 6$ and the domain of V is the interval $0 \leq x \leq 6$.

A graph of V (Figure 4.35) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x :

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

$$\text{Critical point value: } V(2) = 128$$

$$\text{Endpoint values: } V(0) = 0, \quad V(6) = 0.$$

The maximum volume is 128 cm^3 . The cutout squares should be 2 cm on a side. ■

FIGURE 4.34 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

EXAMPLE 2 You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.36). What dimensions will use the least material?

Solution *Volume of can:* If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \qquad 1 \text{ liter} = 1000 \text{ cm}^3$$

Surface area of can: $A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi rh}_{\text{cylindrical wall}}$

How can we interpret the phrase “least material”? For a first approximation we can ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000 \text{ cm}^3$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of A . Figure 4.37 suggests that such a value exists.

Notice from the graph that for small r (a tall, thin cylindrical container), the term $2000/r$ dominates (see Section 2.6) and A is large. For large r (a short, wide cylindrical container), the term $2\pi r^2$ dominates and A again is large.

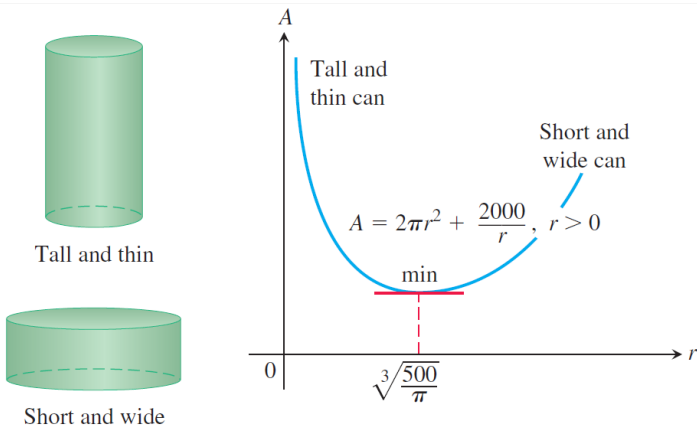


FIGURE 4.37 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2} \quad \text{Set } dA/dr = 0.$$

$$4\pi r^3 = 2000 \quad \text{Multiply by } r^2.$$

$$r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42 \quad \text{Solve for } r.$$

What happens at $r = \sqrt[3]{500/\pi}$?

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A . The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{500/\pi}$ is an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2 \sqrt[3]{\frac{500}{\pi}} = 2r.$$

The one-liter can that uses the least material has height equal to twice the radius, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm. ■

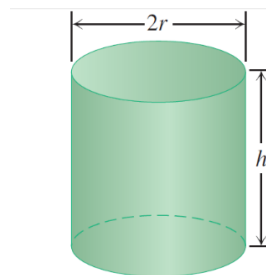


FIGURE 4.36 This one-liter can uses the least material when $h = 2r$ (Example 2).

EXAMPLE 3 A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solution Let $(x, \sqrt{4 - x^2})$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.38). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \quad \text{Height: } \sqrt{4 - x^2}, \quad \text{Area: } 2x\sqrt{4 - x^2}.$$

Notice that the values of x are to be found in the interval $0 \leq x \leq 2$, where the selected corner of the rectangle lies.

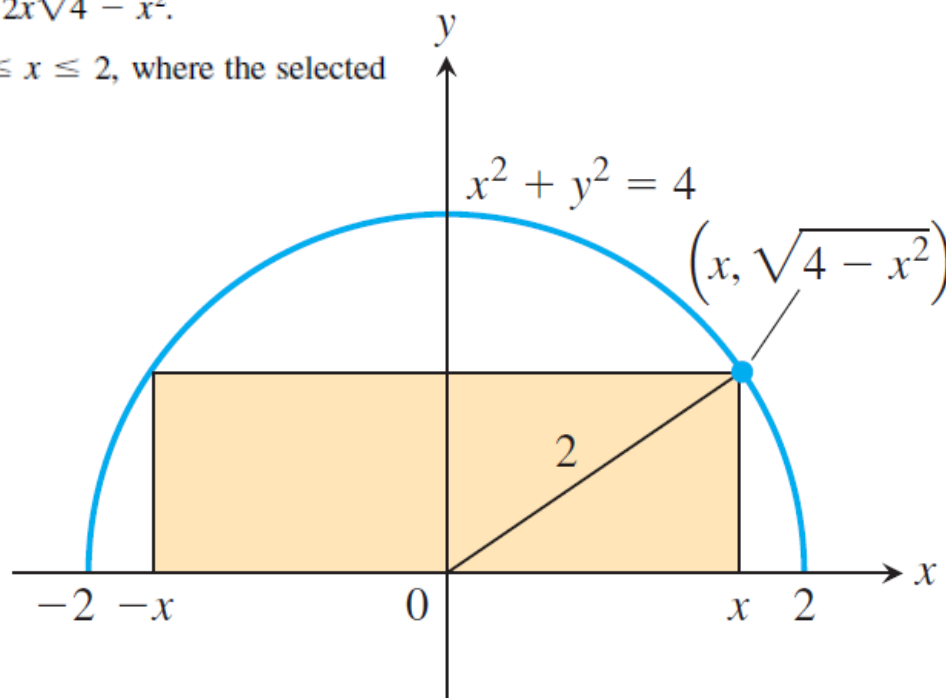


FIGURE 4.38 The rectangle inscribed in the semicircle in Example 3.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$


is not defined when $x = 2$ and is equal to zero when

$$\begin{aligned}\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} &= 0 \\ -2x^2 + 2(4 - x^2) &= 0 \\ 8 - 4x^2 &= 0 \\ x^2 &= 2 \\ x &= \pm \sqrt{2}.\end{aligned}$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A 's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\text{Critical point value: } A(\sqrt{2}) = 2\sqrt{2}\sqrt{4 - 2} = 4$$

$$\text{Endpoint values: } A(0) = 0, \quad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. 

EXAMPLE 4 The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Describe the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2 .

Solution Since light traveling from A to B follows the quickest route, we look for a path that will minimize the travel time. We assume that A and B lie in the xy -plane and that the line separating the two media is the x -axis (Figure 4.39).

In a uniform medium, where the speed of light remains constant, “shortest time” means “shortest path,” and the ray of light will follow a straight line. Thus the path from A to B will consist of a line segment from A to a boundary point P , followed by another line segment from P to B . Distance traveled equals rate times time, so

$$\text{Time} = \frac{\text{distance}}{\text{rate}}.$$

From Figure 4.39, the time required for light to travel from A to P is

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From P to B , the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

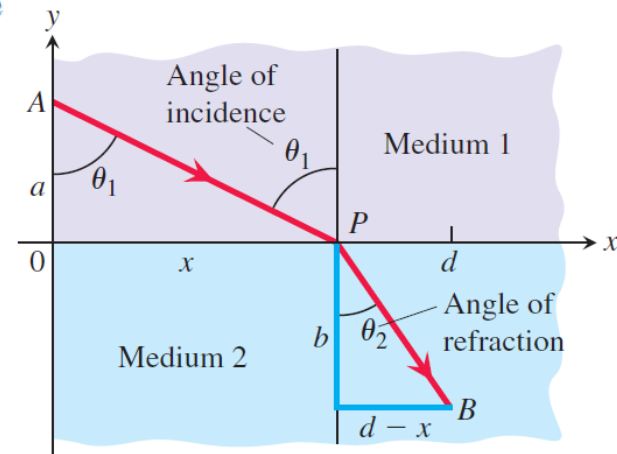


FIGURE 4.39 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

This equation expresses t as a differentiable function of x whose domain is $[0, d]$. We want to find the absolute minimum value of t on this closed interval. We find the derivative

$$\frac{dt}{dx} = \frac{x}{c_1 \sqrt{a^2 + x^2}} - \frac{d - x}{c_2 \sqrt{b^2 + (d - x)^2}}$$

and observe that it is continuous. In terms of the angles θ_1 and θ_2 in Figure 4.39,

$$\frac{dt}{dx} = \frac{\sin \theta_1}{c_1} - \frac{\sin \theta_2}{c_2}.$$

The function t has a negative derivative at $x = 0$ and a positive derivative at $x = d$. Since dt/dx is continuous over the interval $[0, d]$, by the Intermediate Value Theorem for continuous functions (Section 2.5), there is a point $x_0 \in [0, d]$ where $dt/dx = 0$ (Figure 4.40). There is only one such point because dt/dx is an increasing function of x (Exercise 62). At this unique point we then have

$$\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2}.$$

This equation is **Snell's Law** or the **Law of Refraction**, and is an important principle in the theory of optics. It describes the path the ray of light follows. ■

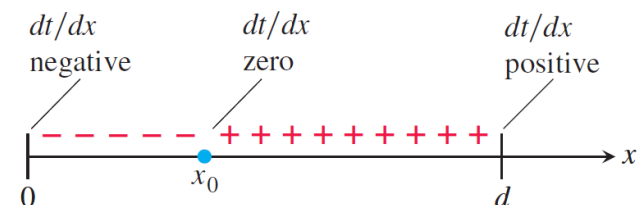


FIGURE 4.40 The sign pattern of dt/dx in Example 4.

Examples from Economics

Suppose that

$r(x)$ = the revenue from selling x items

$c(x)$ = the cost of producing the x items

$p(x) = r(x) - c(x)$ = the profit from producing and selling x items.

Although x is usually an integer in many applications, we can learn about the behavior of these functions by defining them for all nonzero real numbers and by assuming they are differentiable functions. Economists use the terms **marginal revenue**, **marginal cost**, and **marginal profit** to name the derivatives $r'(x)$, $c'(x)$, and $p'(x)$ of the revenue, cost, and profit functions. Let's consider the relationship of the profit p to these derivatives.

If $r(x)$ and $c(x)$ are differentiable for x in some interval of production possibilities, and if $p(x) = r(x) - c(x)$ has a maximum value there, it occurs at a critical point of $p(x)$ or at an endpoint of the interval. If it occurs at a critical point, then $p'(x) = r'(x) - c'(x) = 0$ and we see that $r'(x) = c'(x)$. In economic terms, this last equation means that

At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.41).

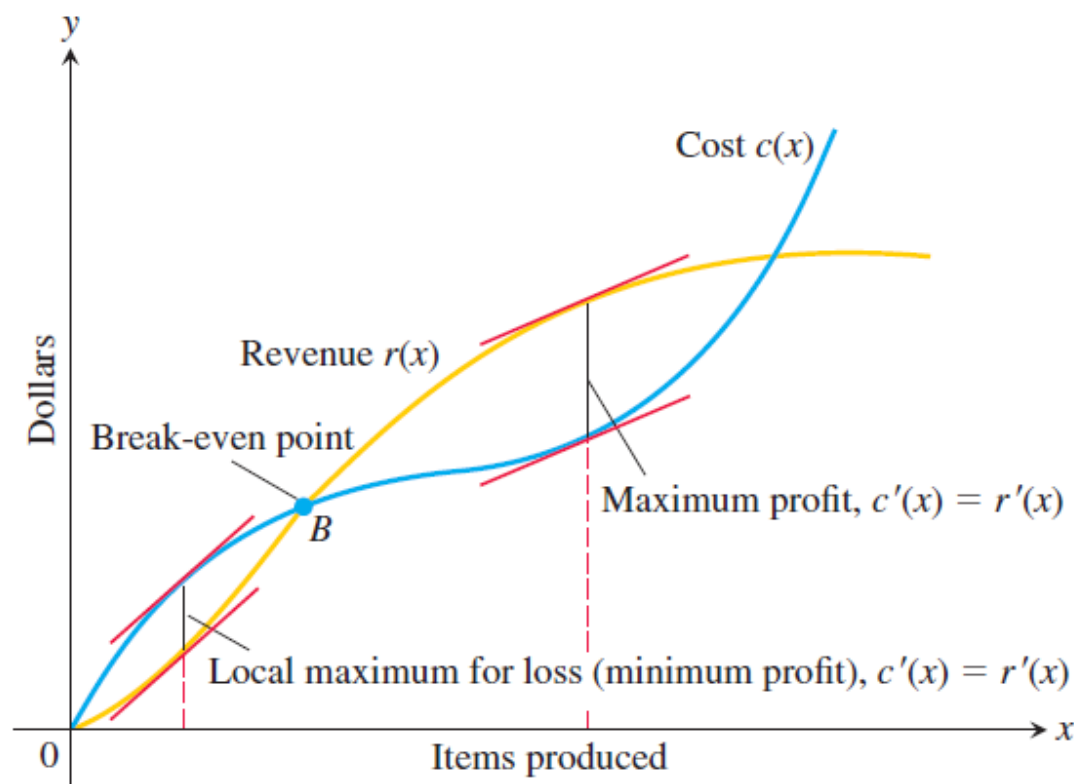


FIGURE 4.41 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B . To the left of B , the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

EXAMPLE 5 Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents millions of MP3 players produced. Is there a production level that maximizes profit? If so, what is it?

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \quad \text{Set } c'(x) = r'(x).$$

$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$

$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ million MP3 players or $x \approx 3.414$ million. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$, which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graphs of $r(x)$ and $c(x)$ are shown in Figure 4.42. ■

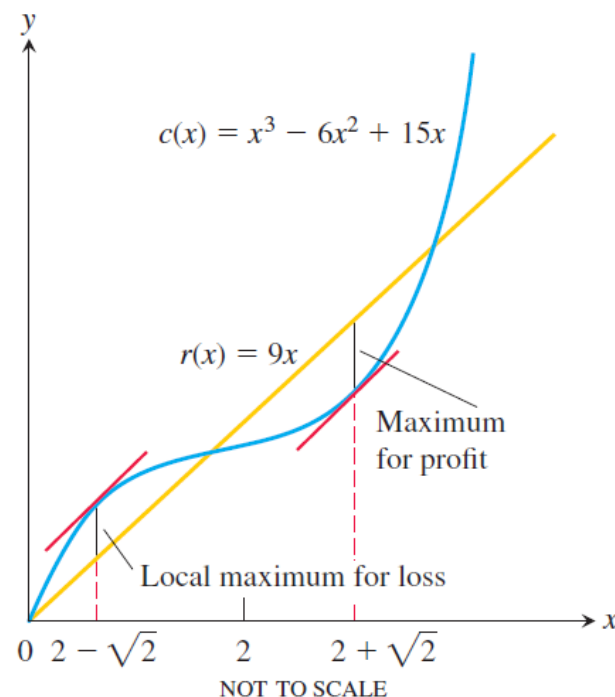


FIGURE 4.42 The cost and revenue curves for Example 5.

EXAMPLE 6 A cabinetmaker uses mahogany wood to produce 5 desks each day. Each delivery of one container of wood is \$5000, whereas the storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 desk. How much material should be ordered each time, and how often should the material be delivered, to minimize her average daily cost in the production cycle between deliveries?

Solution If she asks for a delivery every x days, then she must order $5x$ units to have enough material for that delivery cycle. The *average* amount in storage is approximately one-half of the delivery amount, or $5x/2$. Thus, the cost of delivery and storage for each cycle is approximately

Cost per cycle = delivery costs + storage costs

$$\text{Cost per cycle} = \underbrace{5000}_{\text{delivery cost}} + \underbrace{\left(\frac{5x}{2}\right)}_{\text{average amount stored}} \cdot \underbrace{x}_{\text{number of days stored}} \cdot \underbrace{10}_{\text{storage cost per day}}$$

We compute the *average daily cost* $c(x)$ by dividing the cost per cycle by the number of days x in the cycle (see Figure 4.43).

$$c(x) = \frac{5000}{x} + 25x, \quad x > 0.$$

As $x \rightarrow 0$ and as $x \rightarrow \infty$, the average daily cost becomes large. So we expect a minimum to exist, but where? Our goal is to determine the number of days x between deliveries that provides the absolute minimum cost.

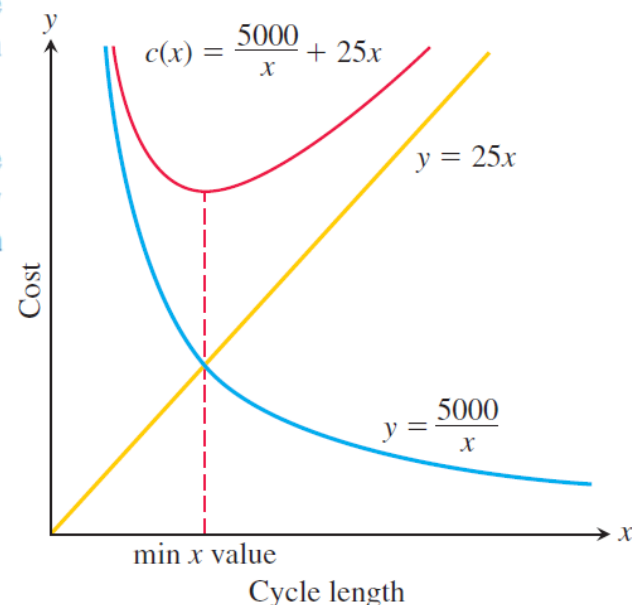


FIGURE 4.43 The average daily cost $c(x)$ is the sum of a hyperbola and a linear function (Example 6).

We find the critical points by determining where the derivative is equal to zero:

$$\begin{aligned}c'(x) &= -\frac{500}{x^2} + 25 = 0 \\x &= \pm \sqrt{200} \approx \pm 14.14.\end{aligned}$$

Of the two critical points, only $\sqrt{200}$ lies in the domain of $c(x)$. The critical point value of the average daily cost is

$$c(\sqrt{200}) = \frac{5000}{\sqrt{200}} + 25\sqrt{200} = 500\sqrt{2} \approx \$707.11.$$

We note that $c(x)$ is defined over the open interval $(0, \infty)$ with $c''(x) = 10000/x^3 > 0$. Thus, an absolute minimum exists at $x = \sqrt{200} \approx 14.14$ days.

The cabinetmaker should schedule a delivery of $5(14) = 70$ units of the mahogany wood every 14 days. ■

4.6

Newton's Method

Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of f crosses the x -axis (Figure 4.44). At each step the method approximates a zero of f with a zero of one of its linearizations. Here is how it works.

The initial estimate, x_0 , may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the x -axis (Figure 4.44). The number x_1 is usually a better approximation to the solution than is x_0 . The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x -axis is the next approximation in the sequence. We continue on, using each approximation to generate the next, until we are close enough to the root to stop.

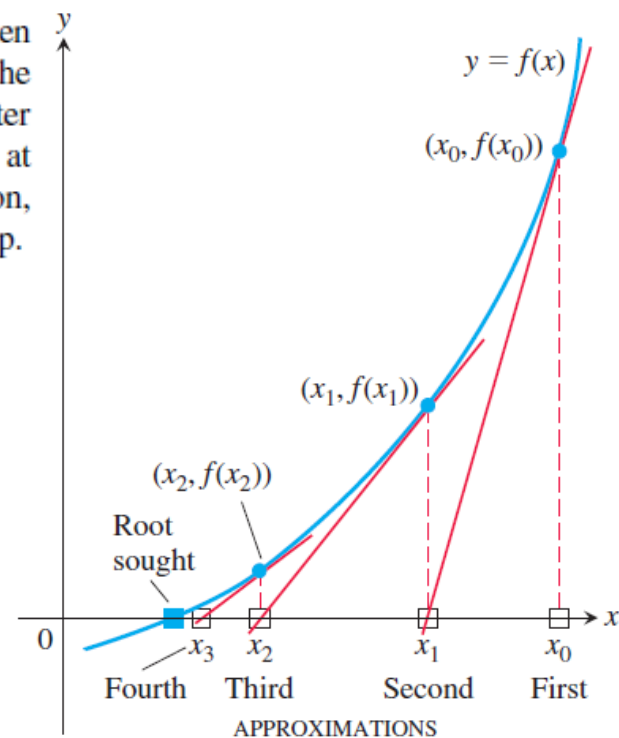


FIGURE 4.44 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n , the point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

We can find where it crosses the x -axis by setting $y = 0$ (Figure 4.45):

$$\begin{aligned} 0 &= f(x_n) + f'(x_n)(x - x_n) \\ -\frac{f(x_n)}{f'(x_n)} &= x - x_n \\ x &= x_n - \frac{f(x_n)}{f'(x_n)} \end{aligned} \quad \text{If } f'(x_n) \neq 0$$

This value of x is the next approximation x_{n+1} .

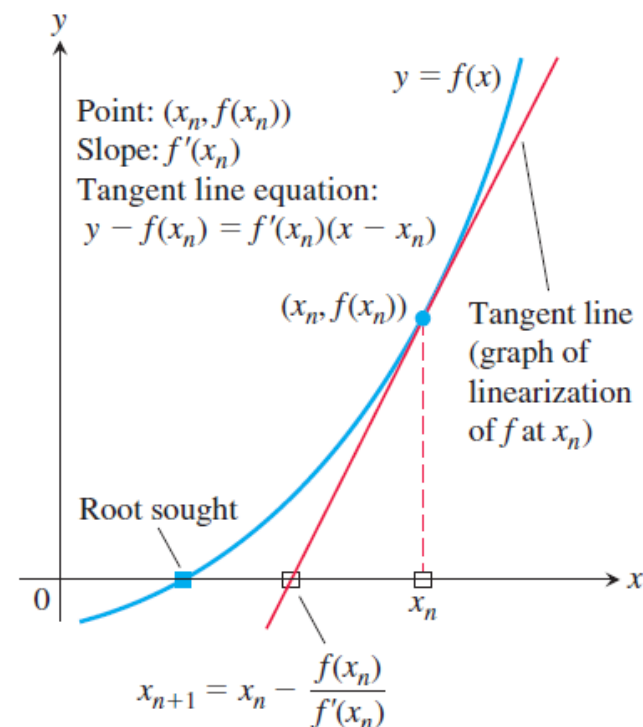


FIGURE 4.45 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1} .

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{if } f'(x_n) \neq 0. \quad (1)$$

EXAMPLE 2 Find the x -coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? Since $f(1) = -1$ and $f(2) = 5$, we know by the Intermediate Value Theorem there is a root in the interval $(1, 2)$ (Figure 4.46).

We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 4.1 and Figure 4.47.

At $n = 5$, we come to the result $x_6 = x_5 = 1.3247\,17957$. When $x_{n+1} = x_n$, Equation (1) shows that $f(x_n) = 0$. We have found a solution of $f(x) = 0$ to nine decimals. ■

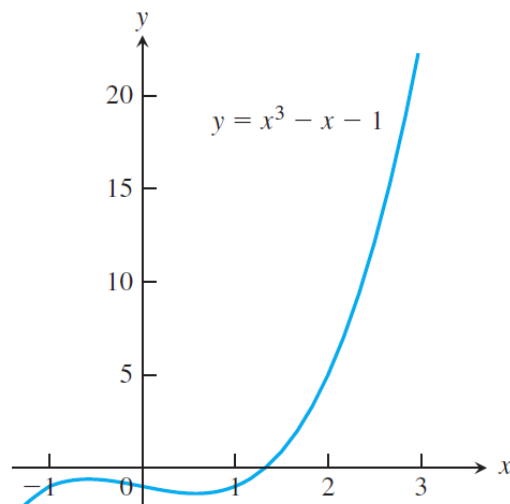


FIGURE 4.46 The graph of $f(x) = x^3 - x - 1$ crosses the x -axis once; this is the root we want to find (Example 2).

TABLE 4.1 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	-1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	-1.8672E-13	4.2646 32999	1.3247 17957

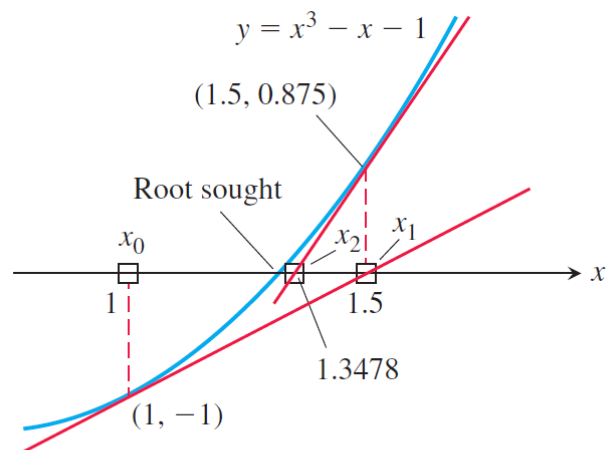


FIGURE 4.47 The first three x -values in Table 4.1 (four decimal places).

In Figure 4.48 we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x -axis, but the tangent at B_0 crosses the x -axis at about $(2.12, 0)$, so x_1 is still an improvement over x_0 . If we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we obtain the nine-place solution $x_7 = x_6 = 1.3247\ 17957$ in seven steps.

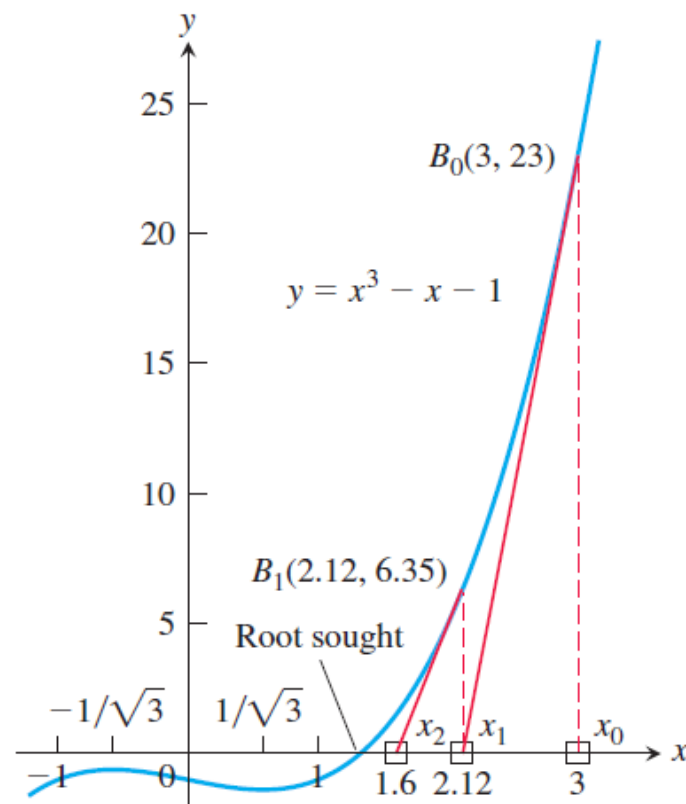


FIGURE 4.48 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r-x}, & x < r \\ \sqrt{x-r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.49. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

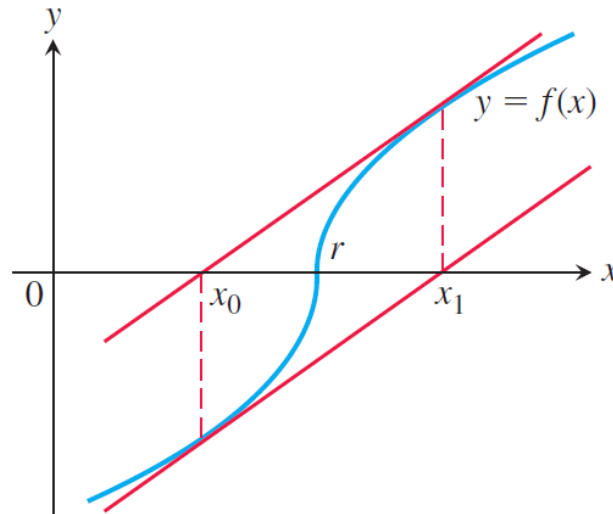


FIGURE 4.49 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0 , never getting any closer to r .

A Sufficient Condition for Convergence in Newton's Method

Let f have its derivative f' that is continuous on (a, b) which contains a root r of f . If $f'(r) \neq 0$, then there exists $\delta > 0$ such that with any starting point $x_0 \in (r - \delta, r + \delta)$, the sequence $\{x_n\}$ converges to r .

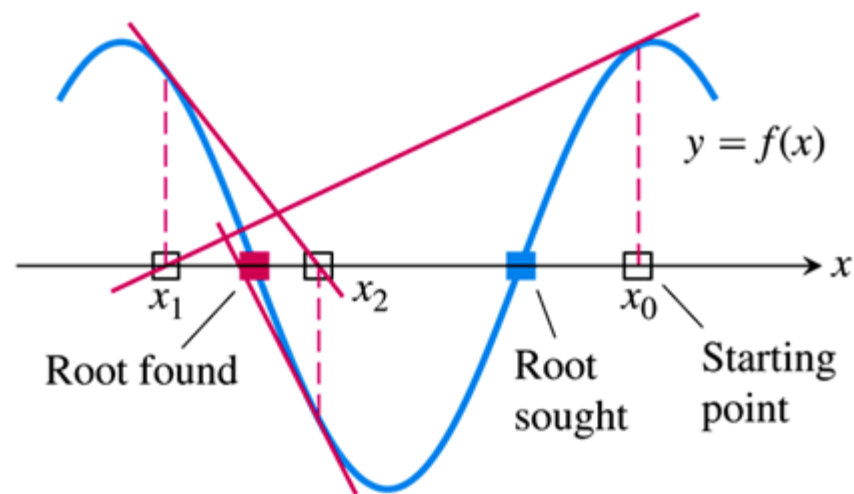
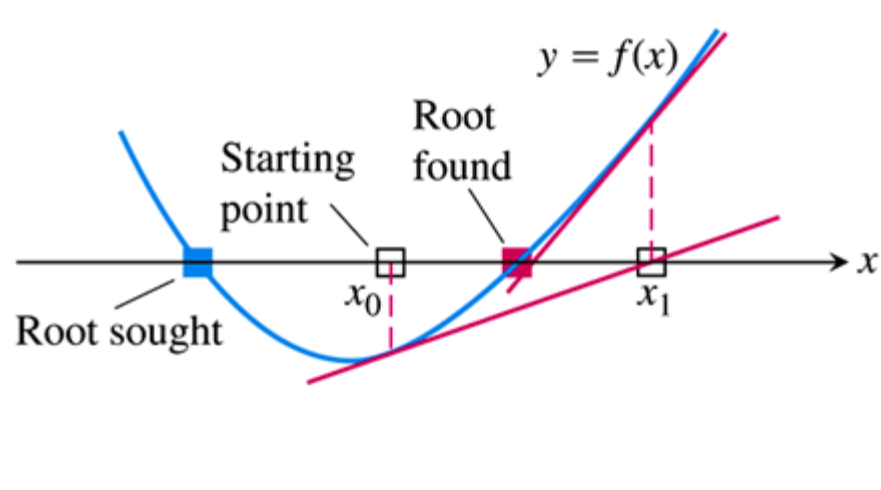


FIGURE 4.50 If you start too far away, Newton's method may miss the root you want.

Week 5

Assignment 5

4.4: #7,16,36,87,98,99,108,111

4.5: #12,15,20a,22,39,45,48,49,52,53,54,60a

4.6: #3,7,9,10

The questions above need to be submitted. Deadline: 10 PM, Friday, Oct 20 --- solutions should be submitted online on Blackboard

Required Reading (Textbook)

- Sections 4.4 to 4.6