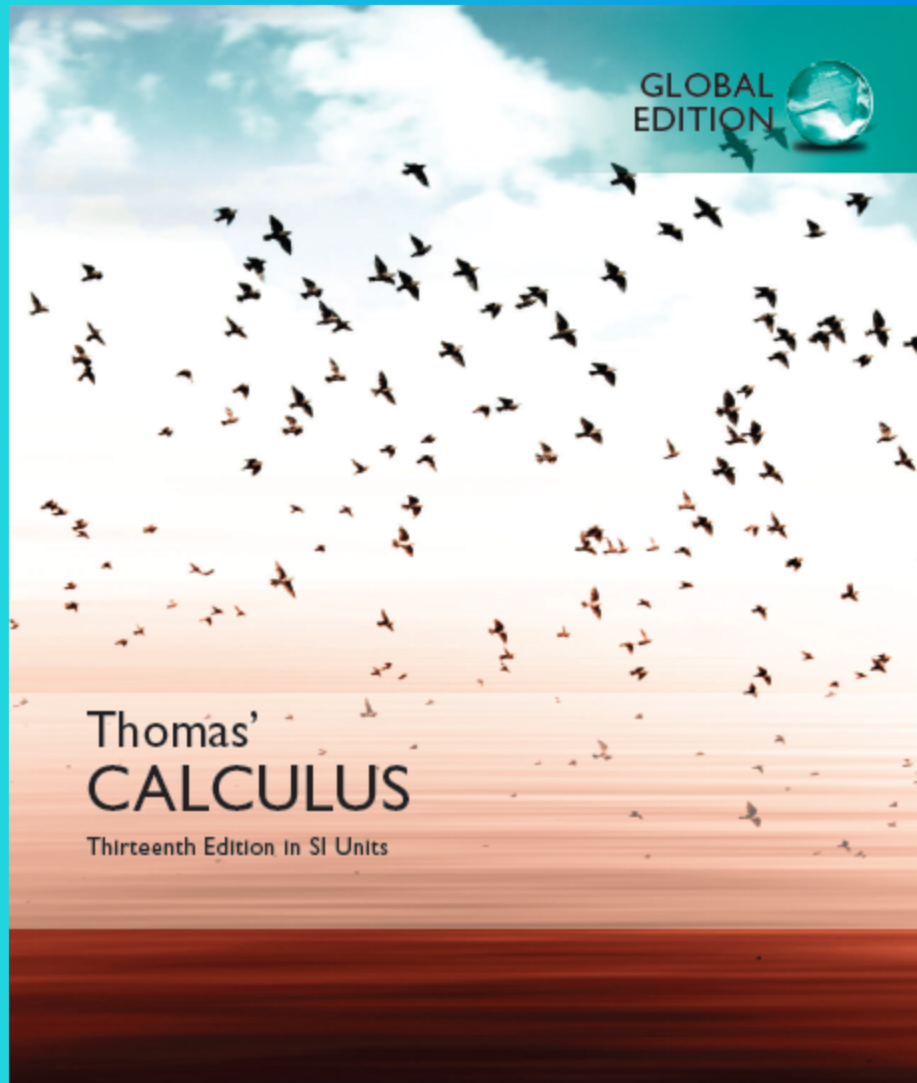

GLOBAL EDITION
Thomas'
CALCULUS
Thirteenth Edition in SI Units

Review of Some Basic Concepts

(mostly from Chapter 1)

DEFINITION A **function** f from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

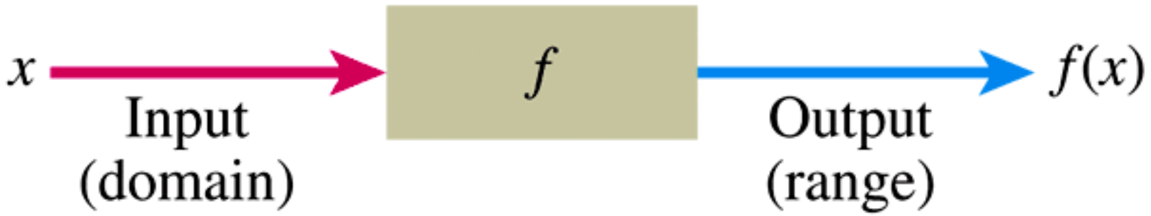


FIGURE 1.1 A diagram showing a function as a kind of machine.

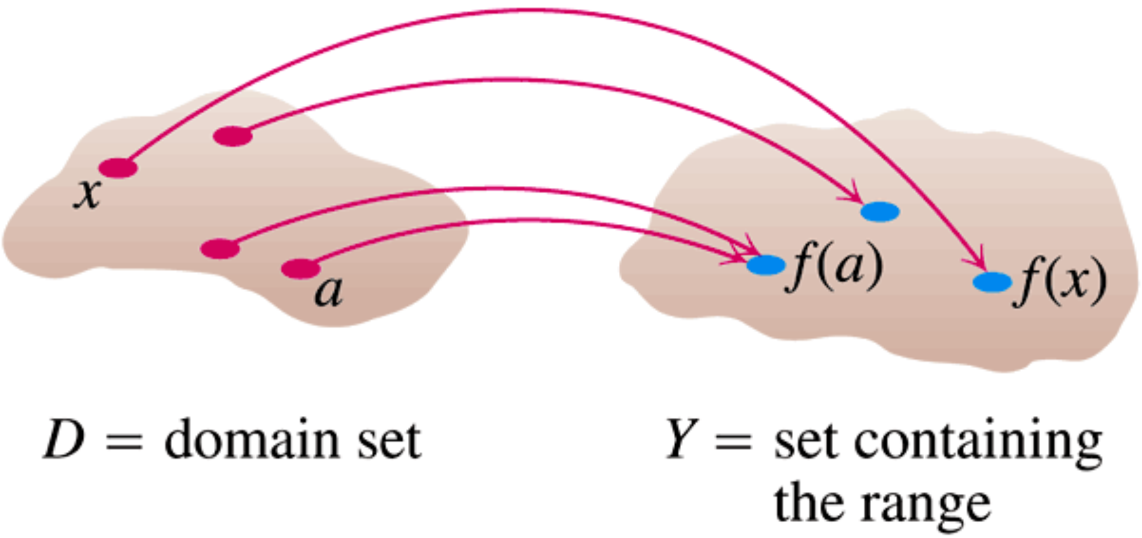


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

A set of real numbers x such that $a \leq x \leq b$ is called a **closed interval** and is denoted by $[a, b]$. The set $a < x < b$ is called an **open interval** and is denoted by (a, b). The sets $a < x \leq b$ and $a \leq x < b$, denoted respectively by $(a, b]$ and $[a, b)$, are called **half open** or **half closed intervals**.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

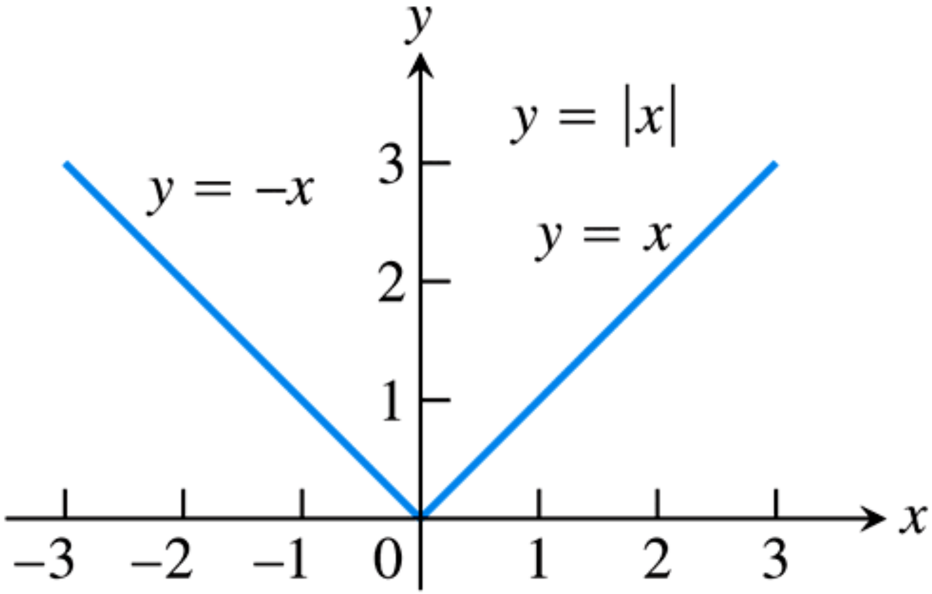


FIGURE 1.8 The absolute value function has domain $(-\infty, \infty)$ and range $[0, \infty)$.

DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

DEFINITIONS A function $y = f(x)$ is an

$$\textbf{even function of } \boldsymbol{x} \quad \text{if } f(-x) = f(x),$$
$$\textbf{odd function of } \boldsymbol{x} \quad \text{if } f(-x) = -f(x),$$

for every x in the function's domain.

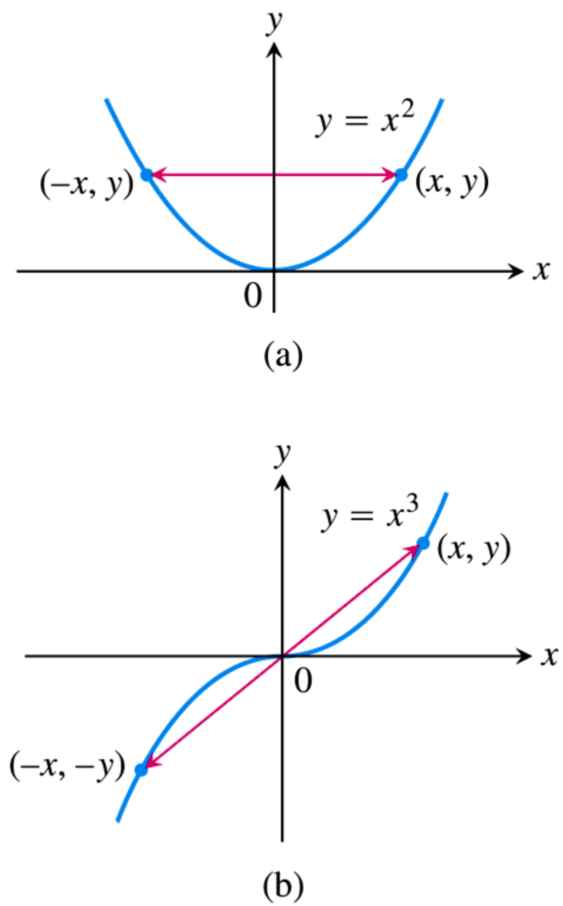


FIGURE 1.12 (a) The graph of $y = x^2$ (an even function) is symmetric about the y-axis. (b) The graph of $y = x^3$ (an odd function) is symmetric about the origin.

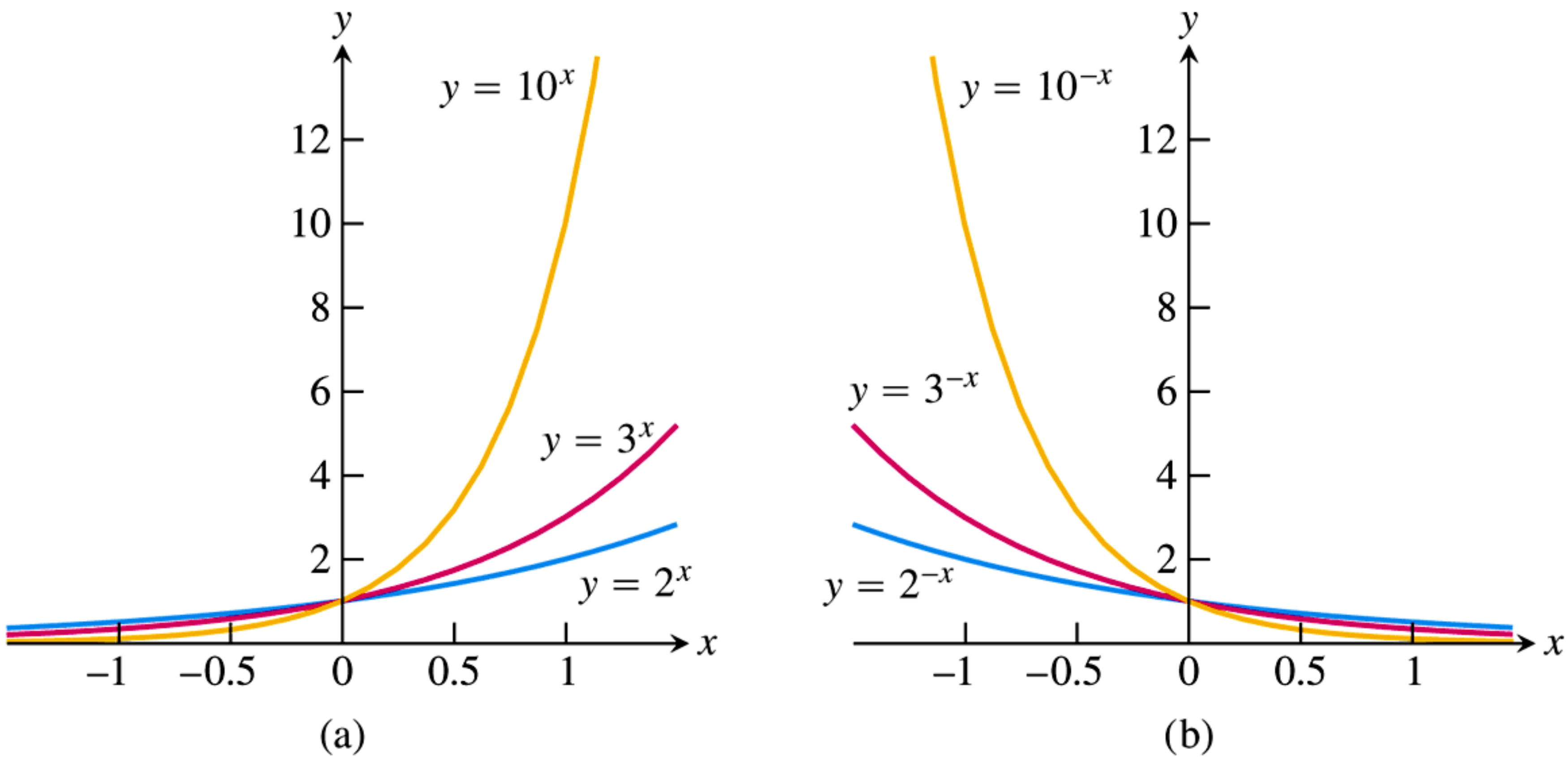


FIGURE 1.22 Graphs of exponential functions.

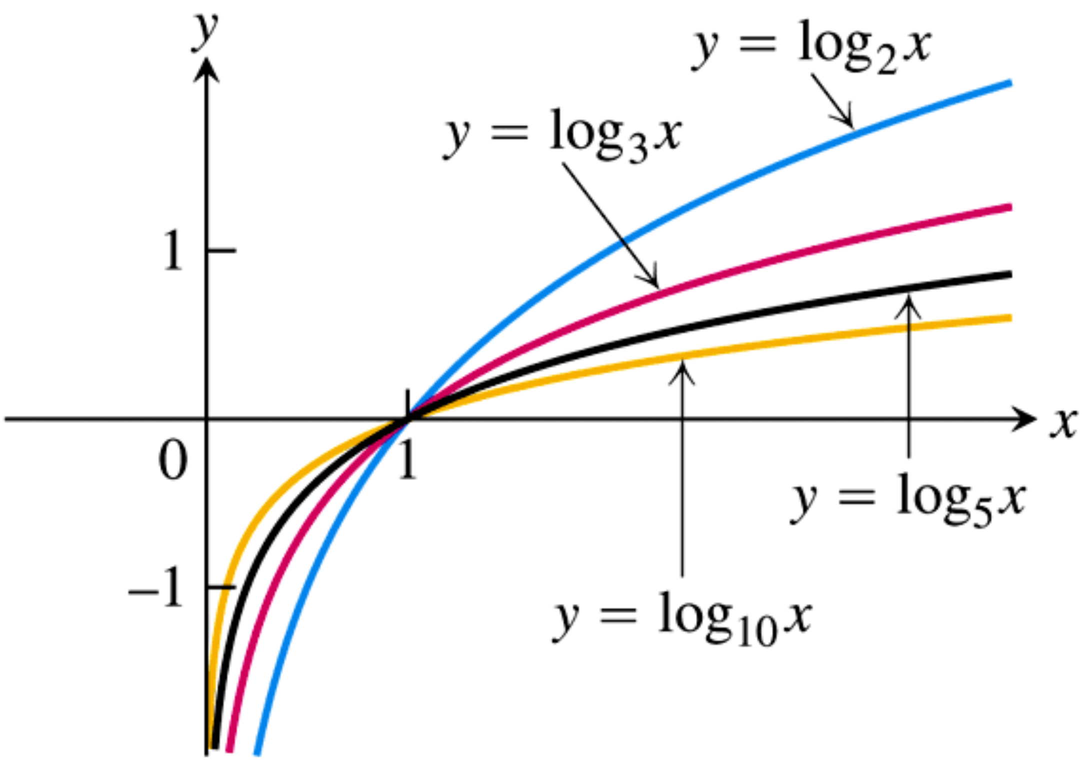


FIGURE 1.23 Graphs of four logarithmic functions.

DEFINITION If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

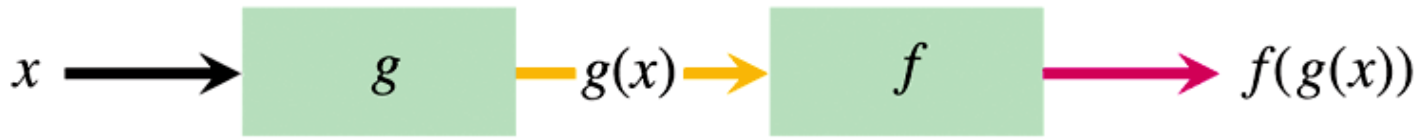


FIGURE 1.27 Two functions can be composed at x whenever the value of one function at x lies in the domain of the other. The composite is denoted by $f \circ g$.

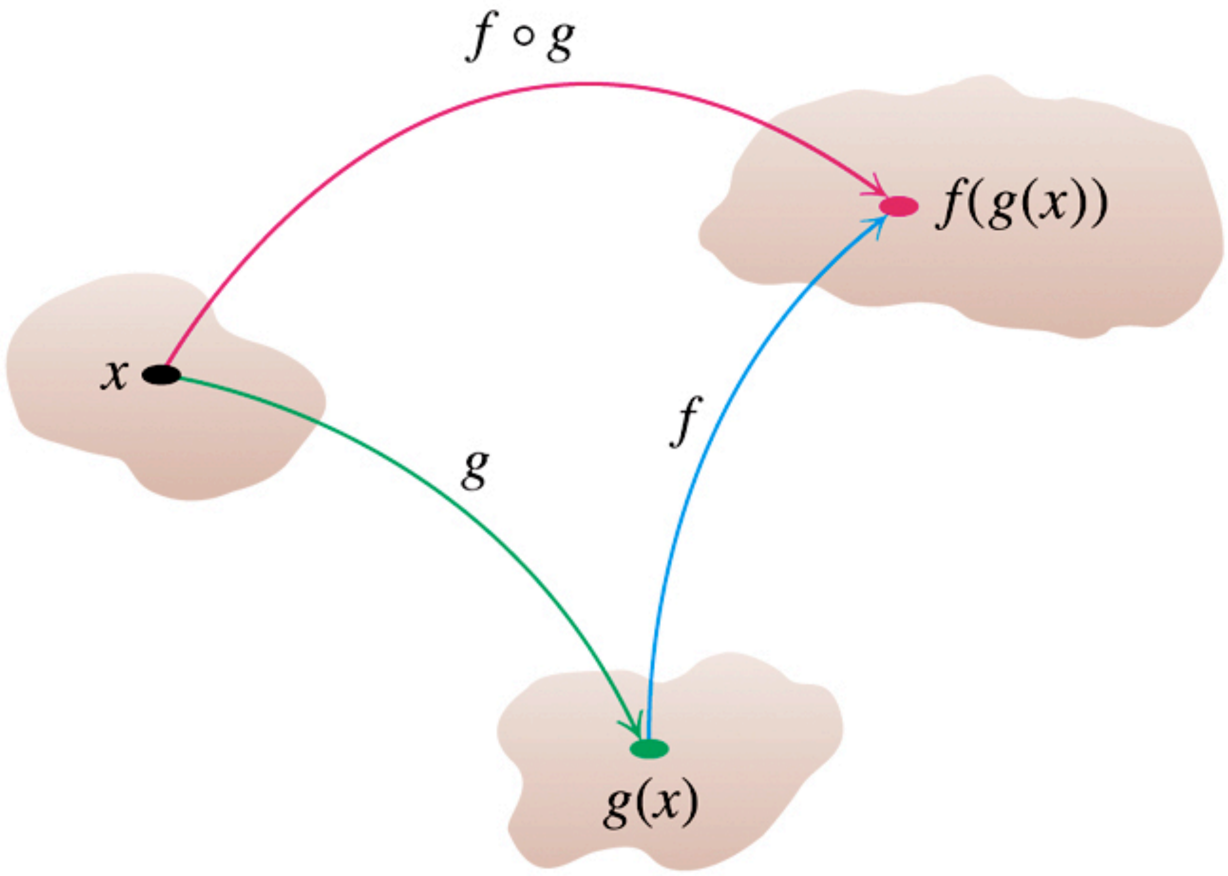


FIGURE 1.28 Arrow diagram for $f \circ g$.

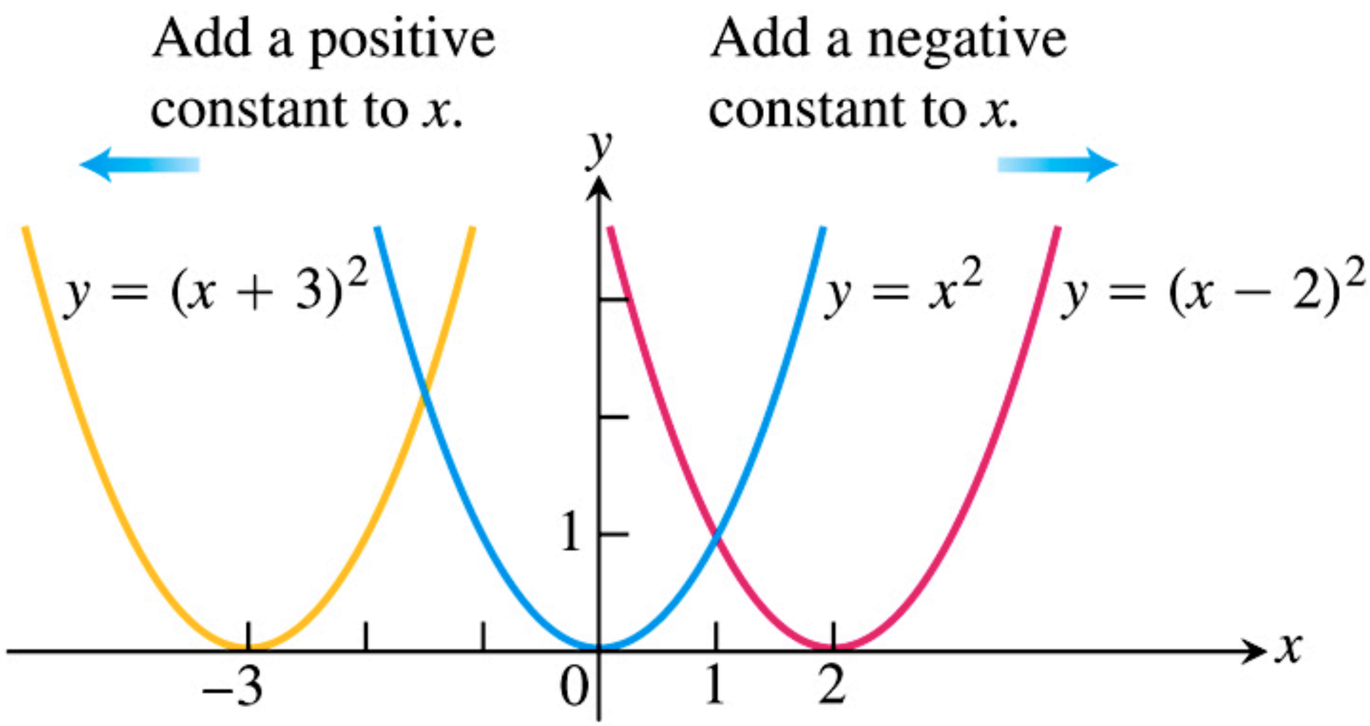


FIGURE 1.30 To shift the graph of $y = x^2$ to the left, we add a positive constant to x (Example 3c). To shift the graph to the right, we add a negative constant to x.

Chapter 2

Limits and Continuity

2.1

Rates of Change and Tangents to Curves

Galileo's Law: $y = 4.9t^2$, y is the distance a rock fallen in meters after t seconds. The following table suggests that the rock is falling at a speed of 9.8 m/s at $t_0 = 1$ s.

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{4.9(t_0 + h)^2 - 4.9t_0^2}{h}$$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	14.7	24.5
0.1	10.29	20.09
0.01	9.849	19.649
0.001	9.8049	19.6049
0.0001	9.80049	19.60049

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

A line joining two points of a curve is a **secant** to the curve

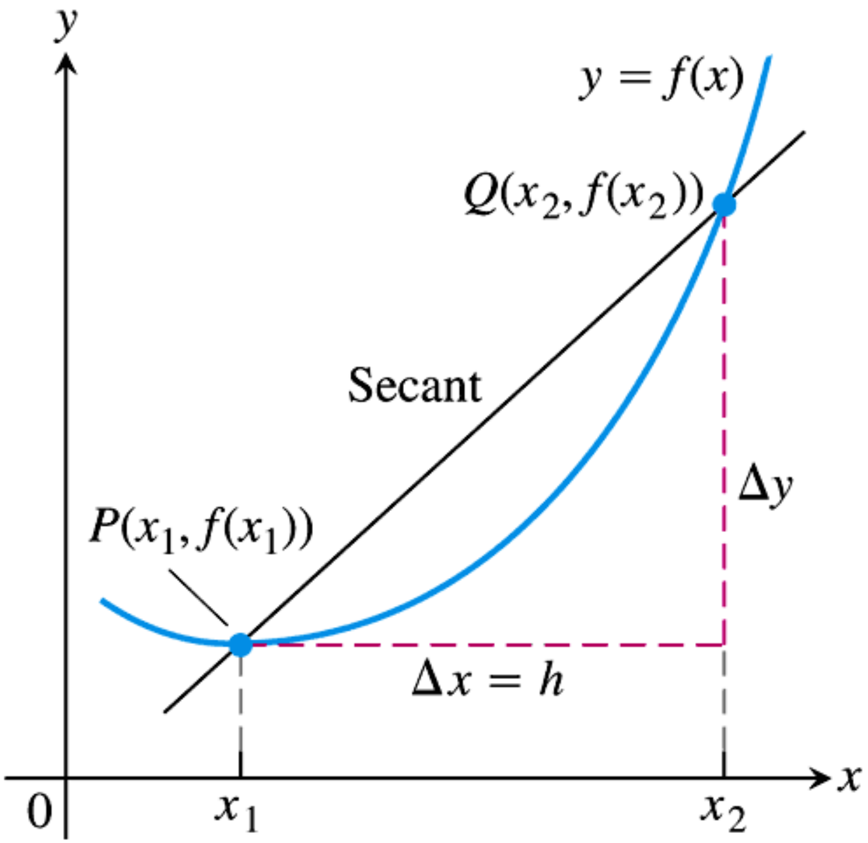


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y / \Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

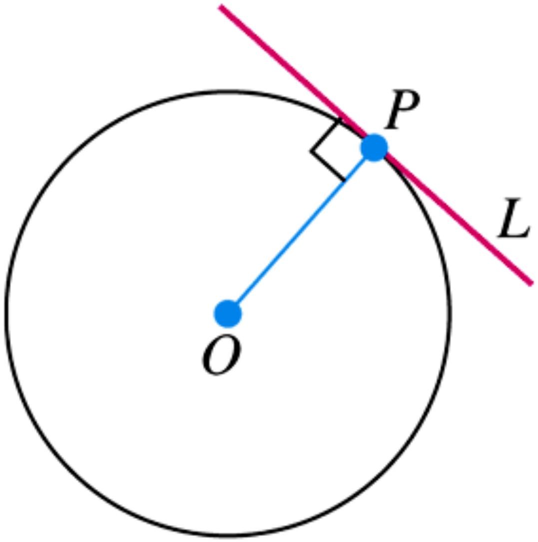


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

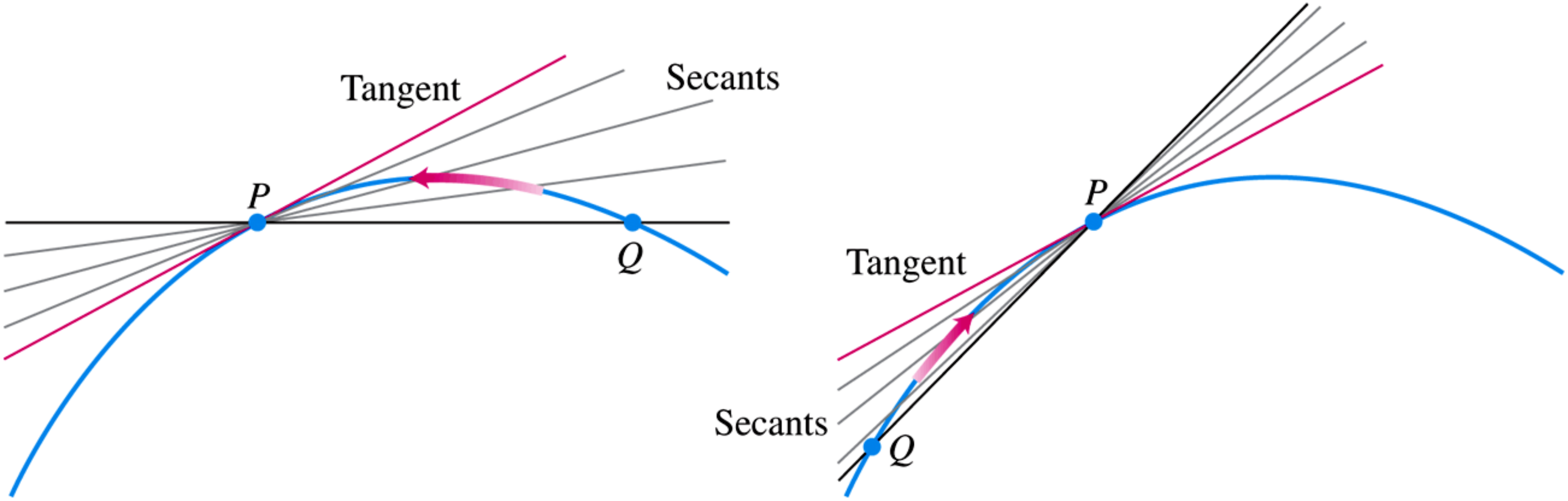


FIGURE 2.3 The tangent to the curve at P is the line through P whose slope is the limit of the secant slopes as $Q \rightarrow P$ from either side.

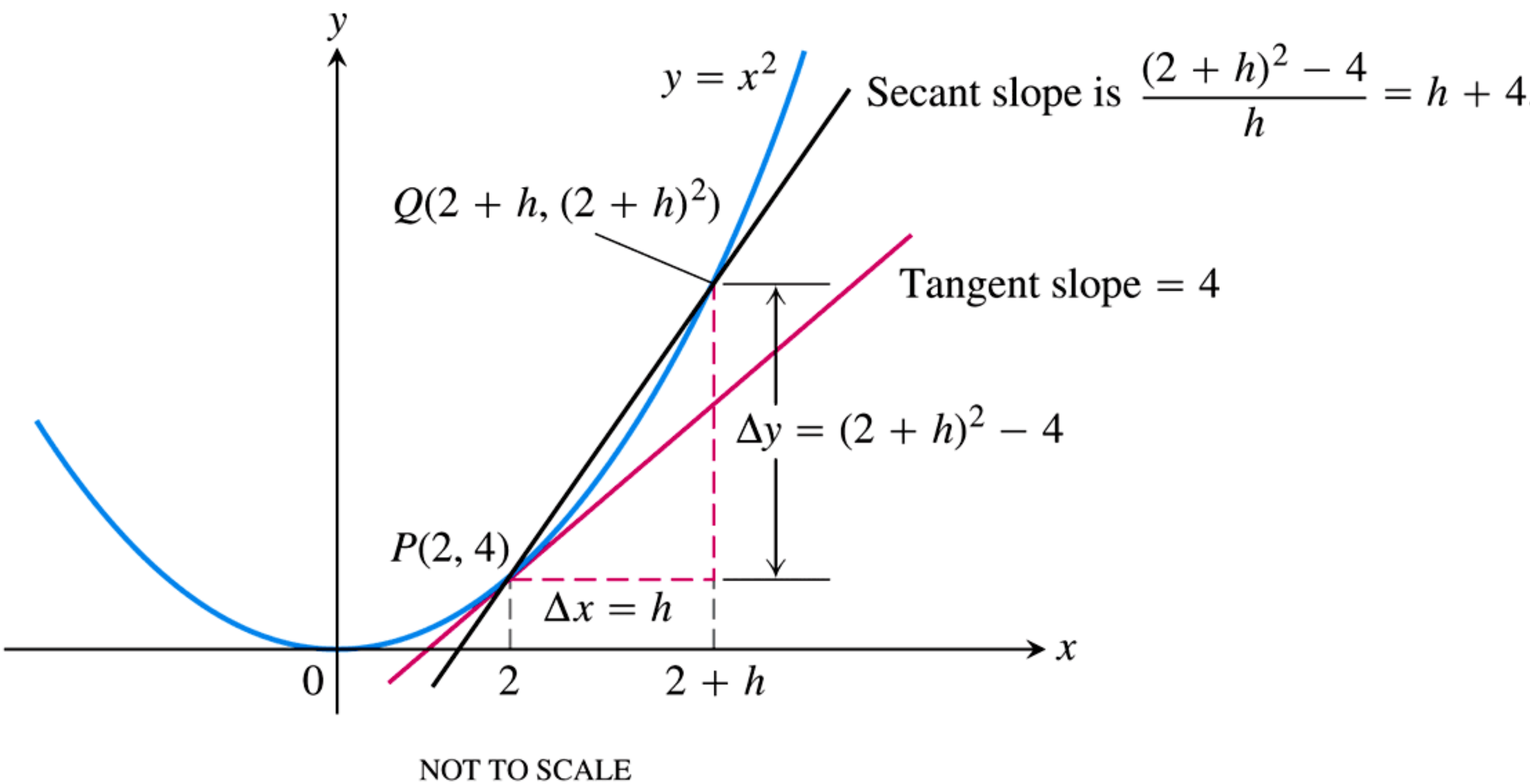


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

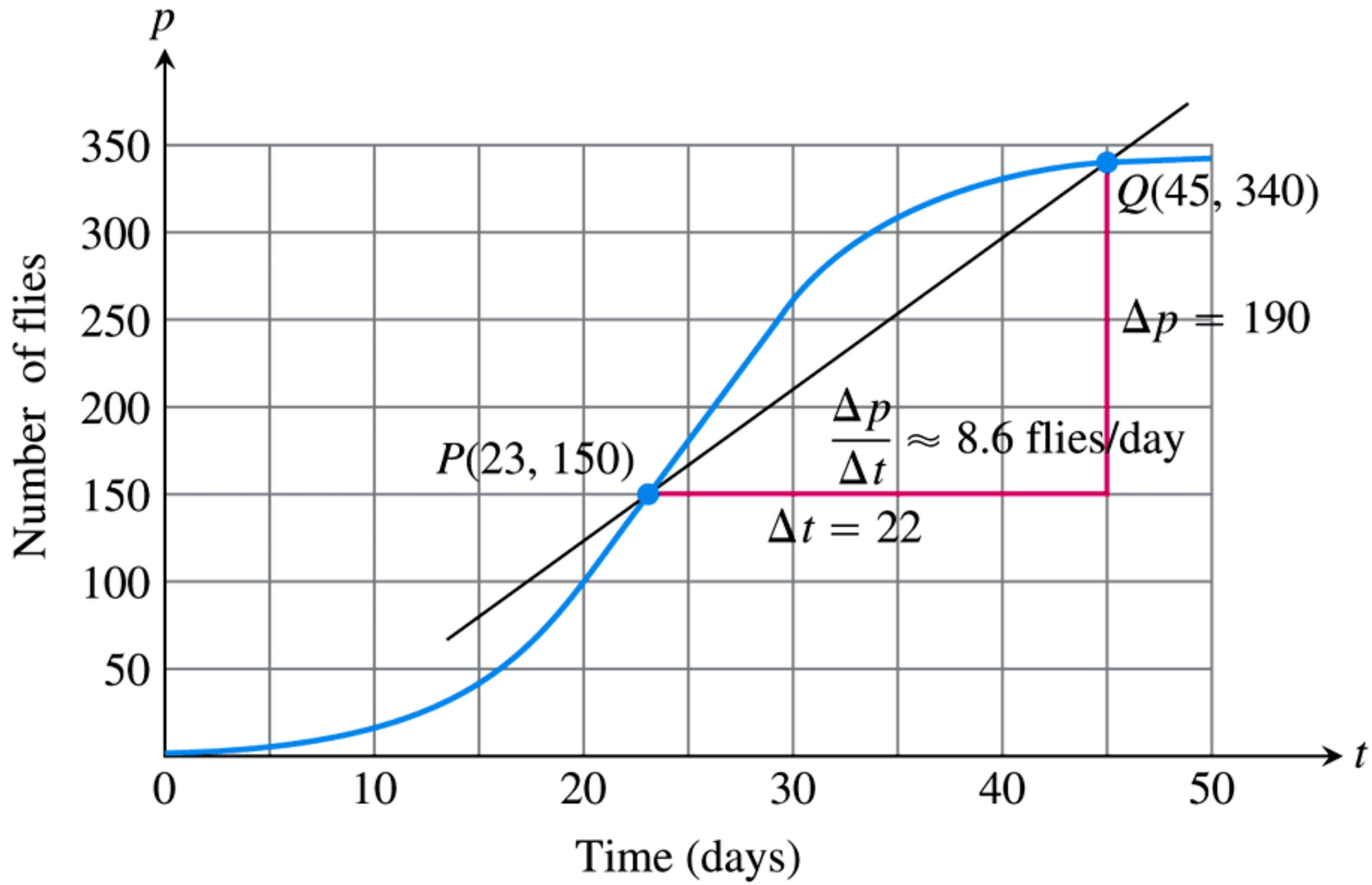


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

Example 5

How fast was the number of flies in the population growing on day 23?

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

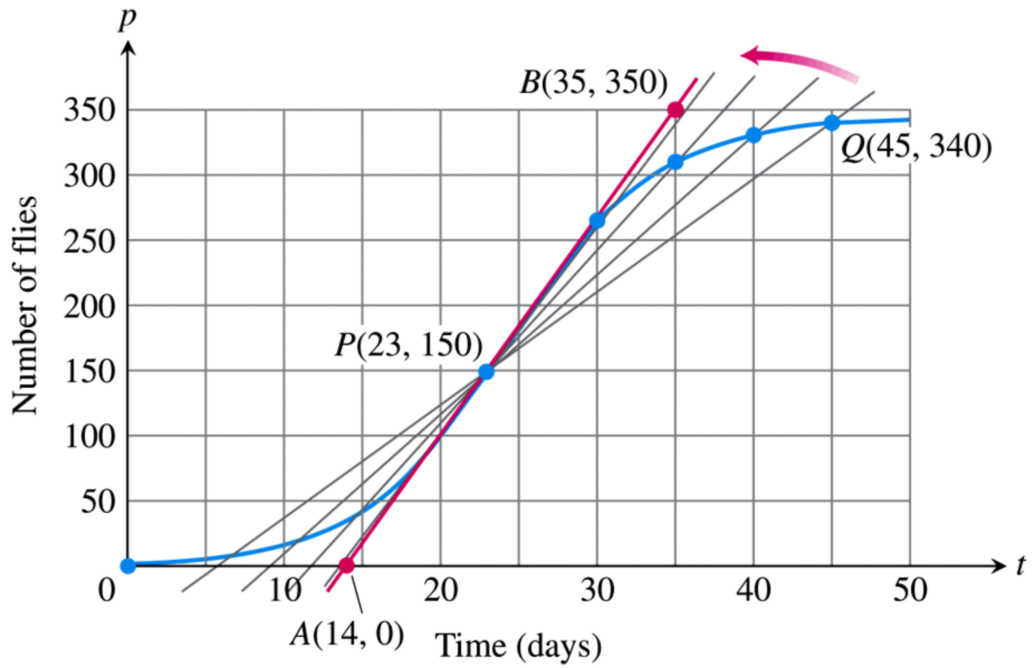
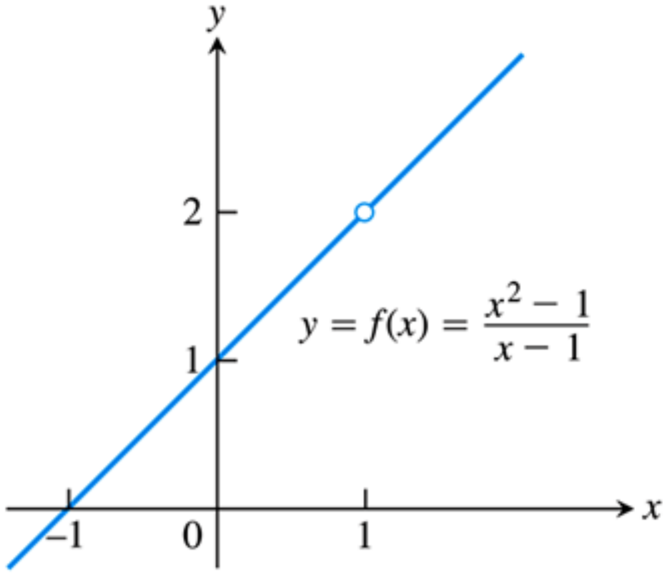


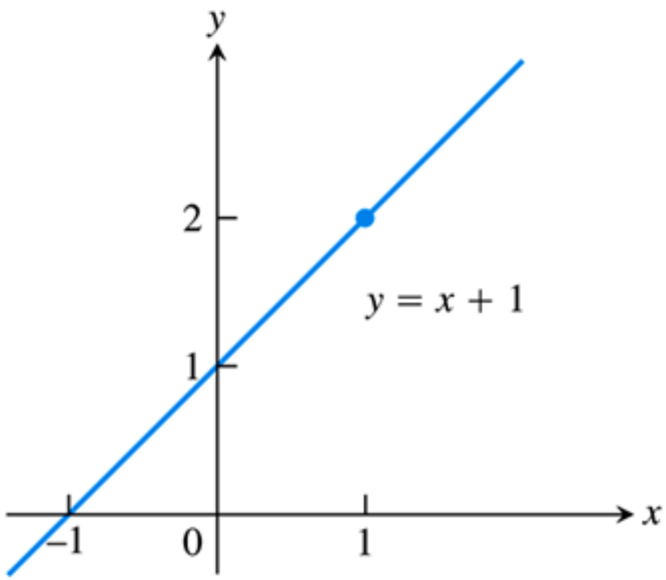
FIGURE 2.6 The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

The red tangent line to P appears to pass through the points (14,0) and (35,350) which has slope (350-0)/(35-14) = 16.7 flies/day

2.2

Limit of a Function and Limit Laws





Example 1

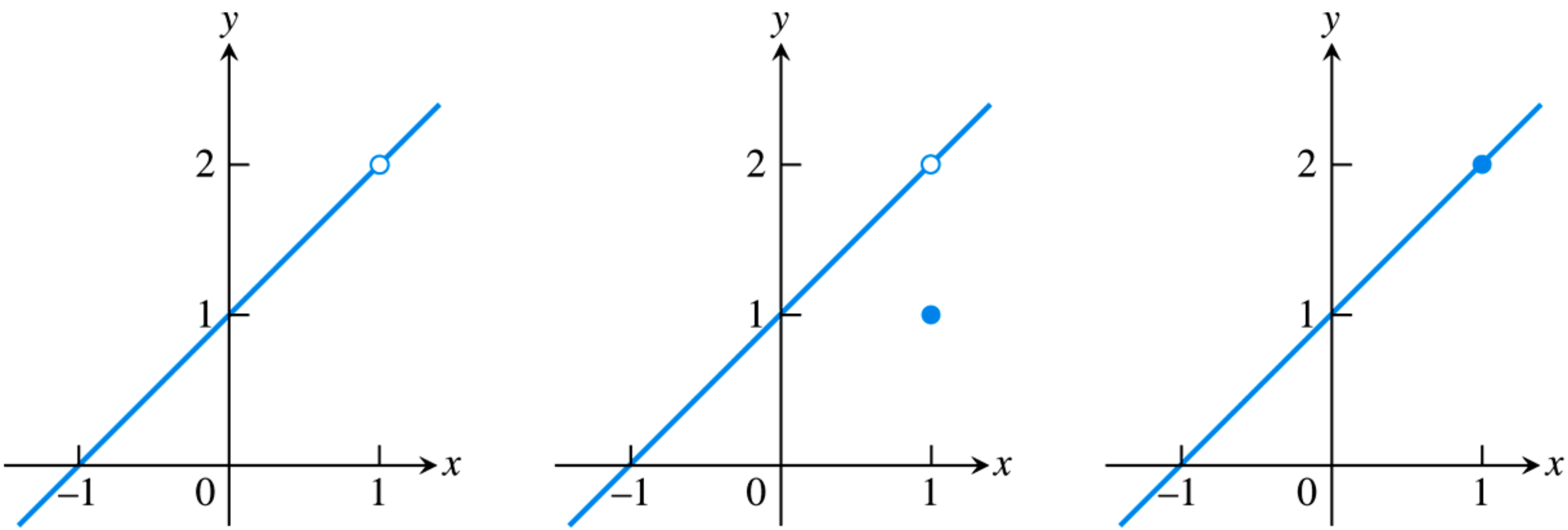
How does the given function $f(x)$ behave near $x = 1$? (Divide by 0 is not allowed)

FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \dfrac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Example 2



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

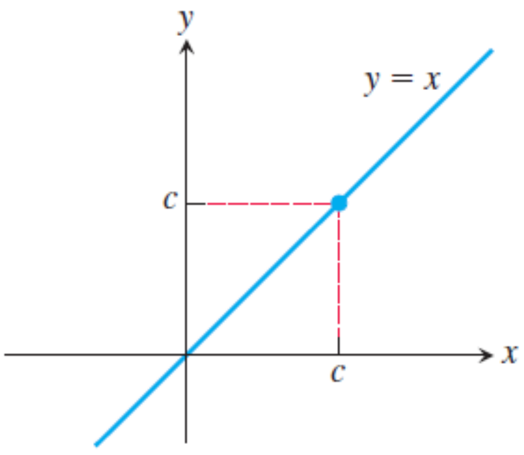
EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

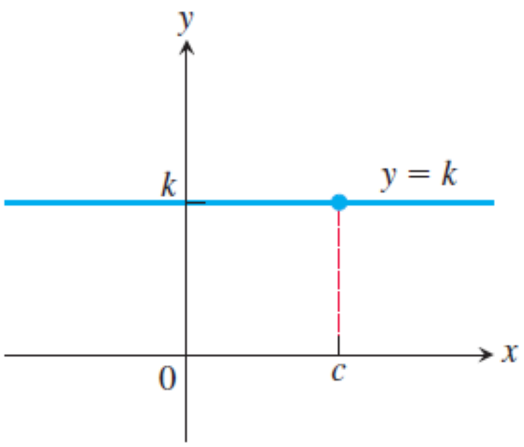
$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$



(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$

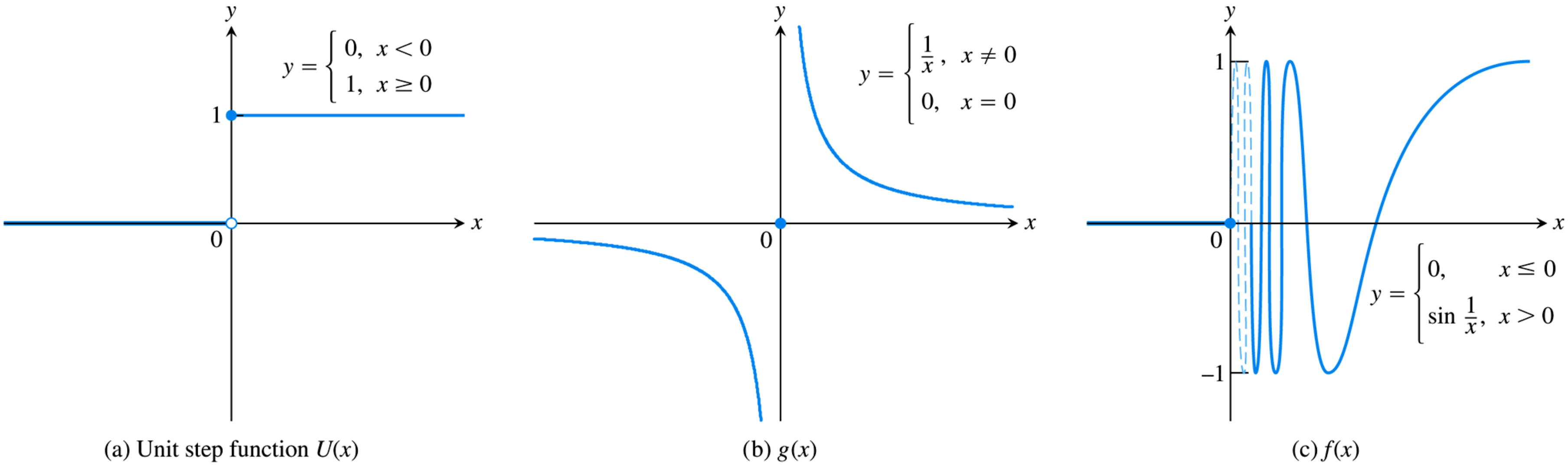


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Solution

(a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number (Figure 2.10b). We say the function is *not bounded*.

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$ (Figure 2.10c). ■

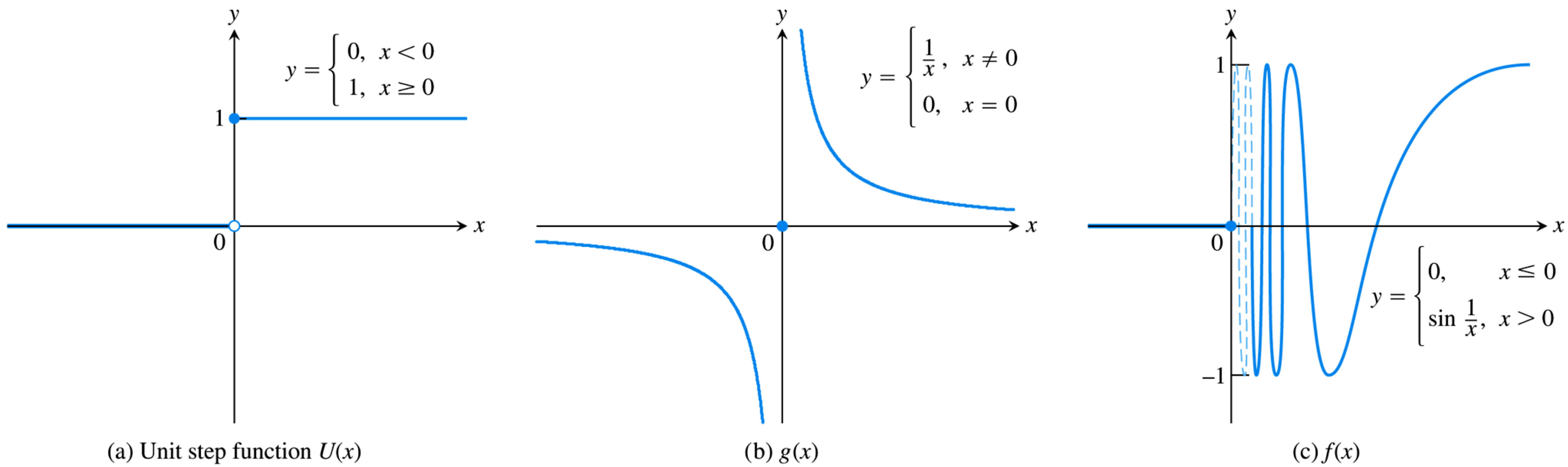


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

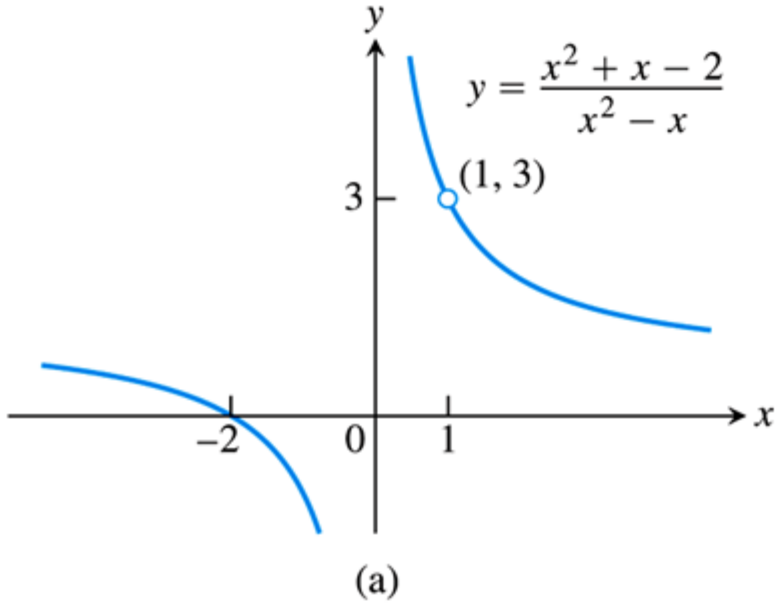
THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Identifying Common Factors

It can be shown that if $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.



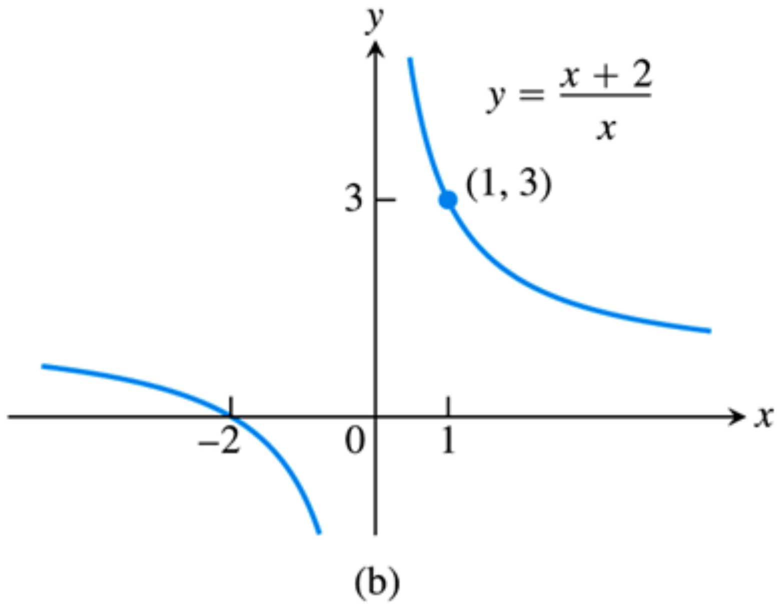


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 7).

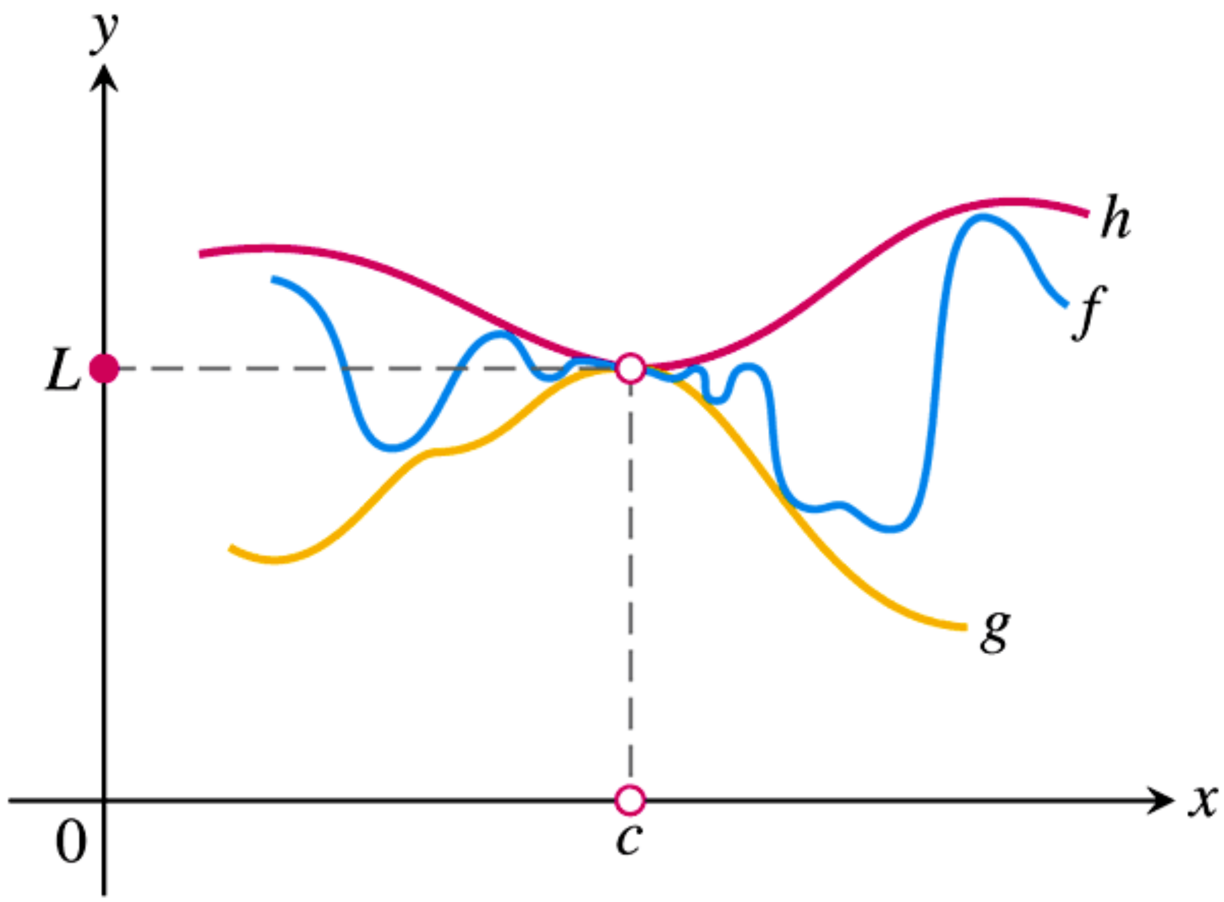


FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

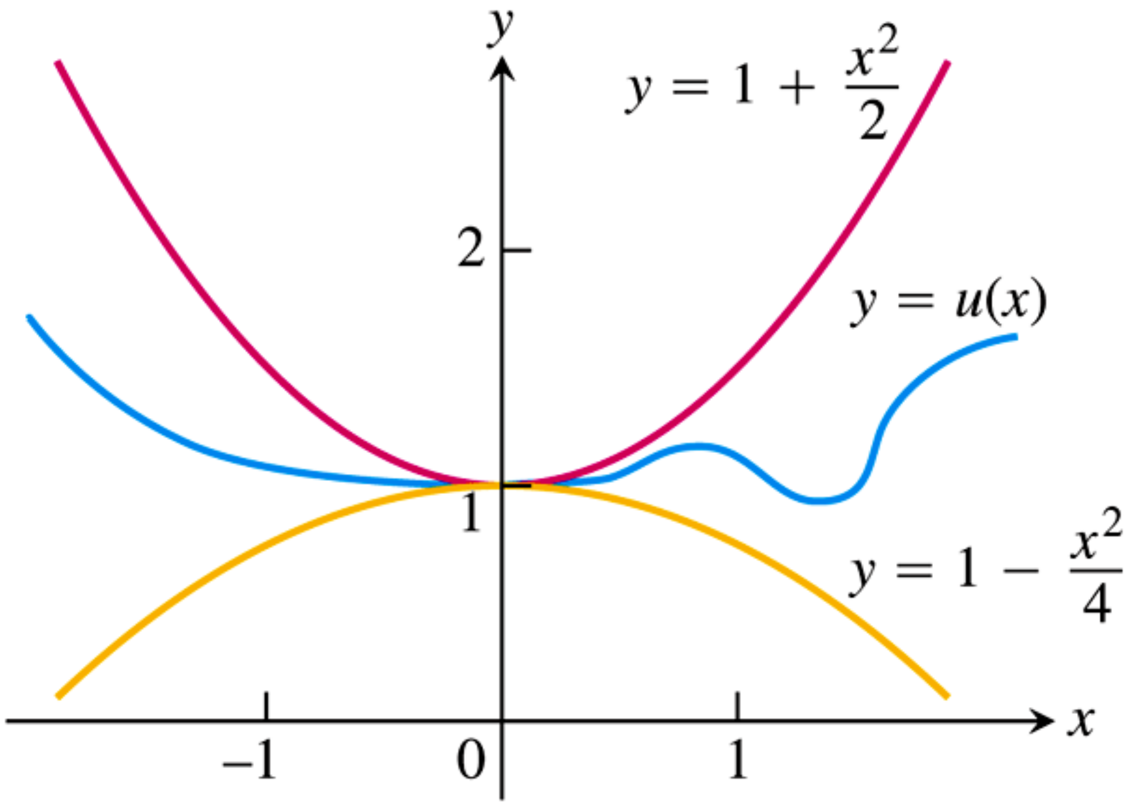


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

EXAMPLE 11 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta \to 0} \sin \theta = 0$ (b) $\lim_{\theta \to 0} \cos \theta = 1$

(c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution

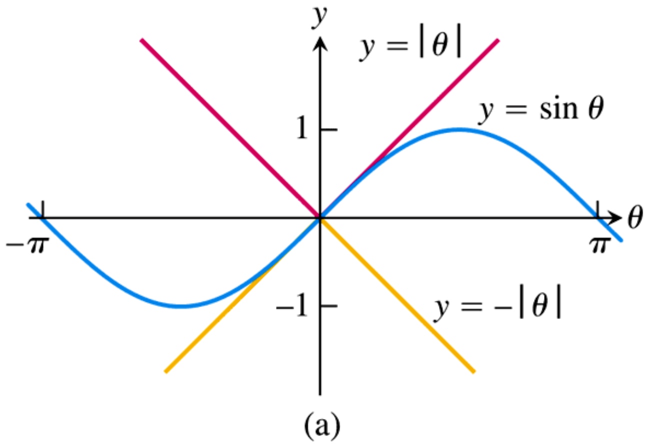
(a) In Section 1.3 we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$, we have

$$\lim_{\theta \to 0} \sin \theta = 0.$$

(b) From Section 1.3, $0 \le 1 - \cos\theta \le |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ or

$$\lim_{\theta \to 0} \cos \theta = 1.$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x\to c} f(x) = 0$. ■



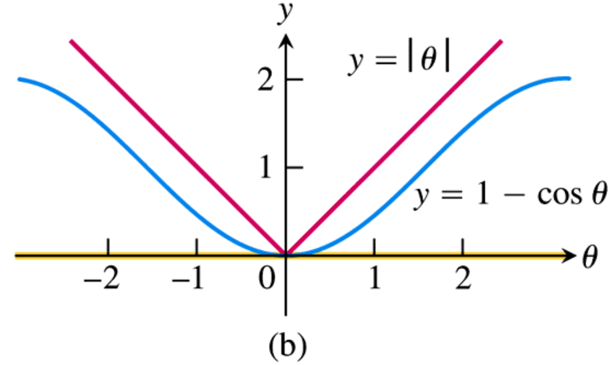


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 11.

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Week 1

Assignment 1

2.1: #1,7,8,15
2.2: #1,10,18,19,23,28,32,40,45,54,57,65(a),77,78

The questions above need to be submitted; you are encouraged to attempt other questions in Chapters 2.1 and 2.2 if you need more exercises.
Deadline: 10 PM, Friday, Sept 15 --- submitted online on Blackboard.

Required Reading (Textbook)

- Section 2.1
- Section 2.2

2.3

The Precise Definition of a Limit

(**Note**: this section is optional and will not be in ANY assignments/quizzes/exams)

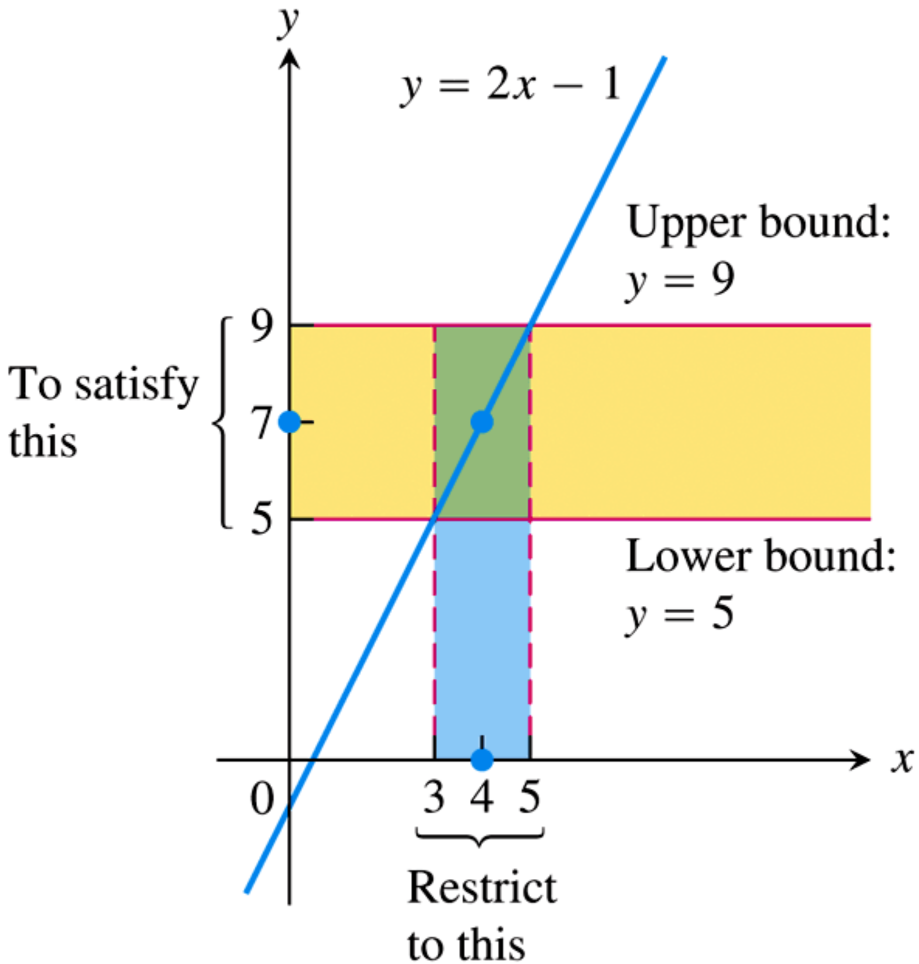


FIGURE 2.15 Keeping x within 1 unit of $x_0 = 4$ will keep y within 2 units of $y_0 = 7$ (Example 1).

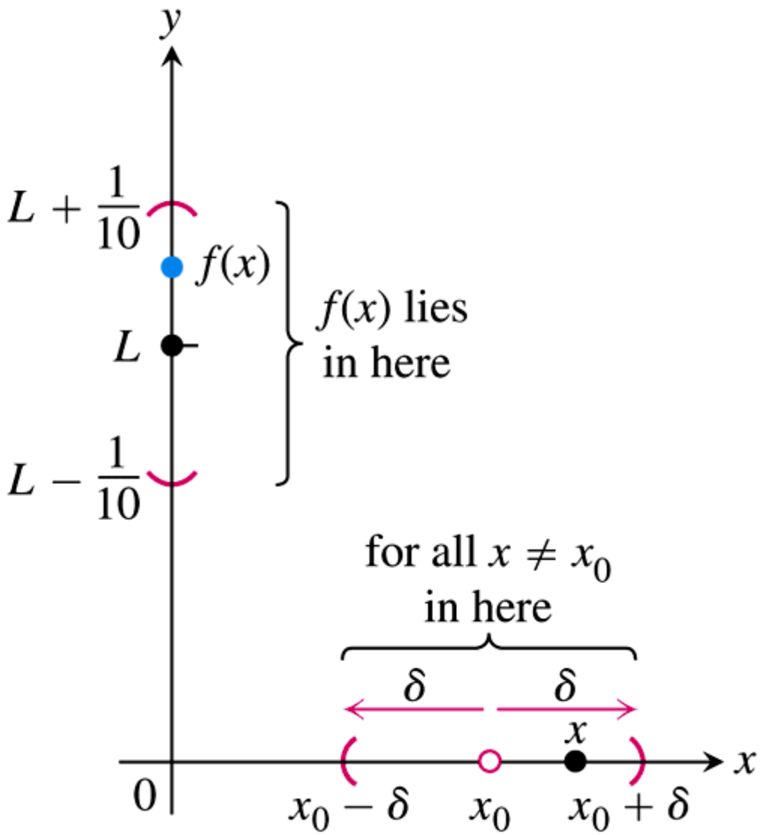


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(x_0 - \delta, x_0 + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

The gap between $f(x)$ and L can be made as small as we wish if x is kept close enough to x_0.

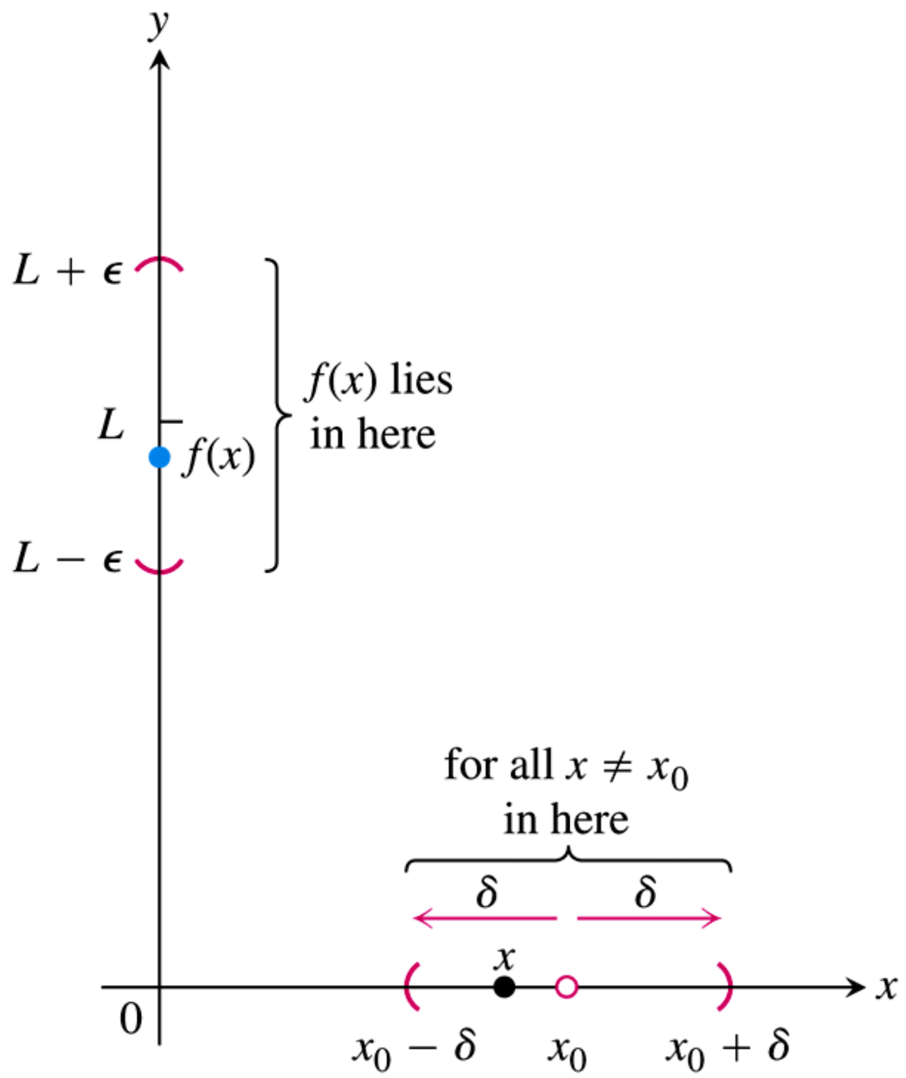


FIGURE 2.17 The relation of δ and ϵ in the definition of limit.

EXAMPLE 2 Show that

$$\lim_{x \to 1} (5x - 3) = 2.$$

Solution Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

We find δ by working backward from the ϵ-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5\,|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x \to 1}(5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ■

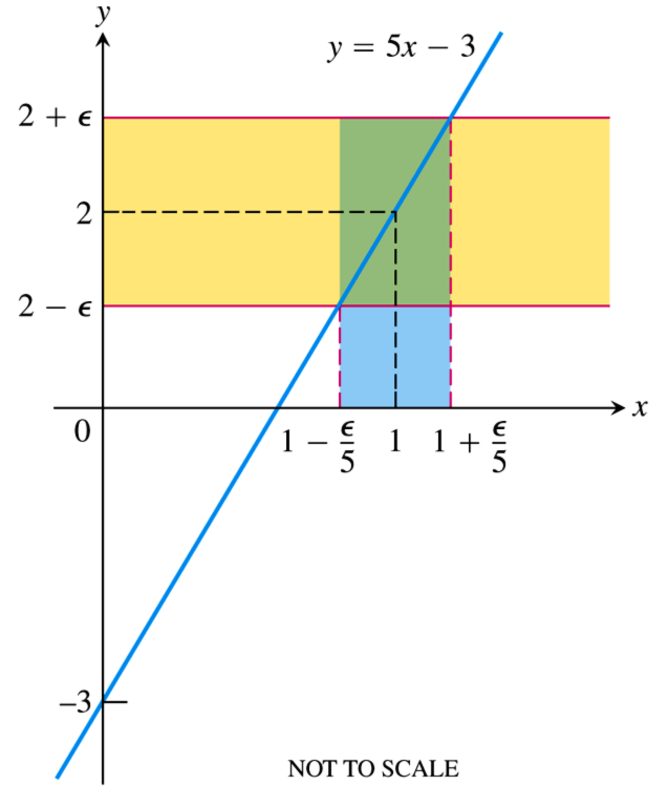


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).