Slide 12-Basis and dimension

MAT2040 Linear Algebra

Definition 12.1 (Basis) A subset $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ from the vector space V is a **basis** of V if
(1) $\mathcal{U}$ is linearly independent.
(2) $\mathbf{Span}(\mathcal{U}) = V$, that is $\mathcal{U}$ spans V.

Remark 1. The basis $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_m\}$ has the maximum number of linearly independent vectors from V but has the smallest number of vectors that spans V. $\#$ of vectors in the basis cannot be too large and cannot be too small.

Remark 2. If $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is a basis of $\mathbb{R}^m$, we can show that $n = m$ (you will see it soon).

Example 12.2: $V = \mathbb{R}^2$. Are the following sets form a basis for V?

(a) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$? No, $\mathcal{U}$ is linearly dependent. Too many vectors that cannot form a basis.

(b) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$? No, **Span**$(\mathcal{U}) \subset V$ and $V \neq$ **Span**$(\mathcal{U})$. Too few vectors that cannot form a basis.

(c) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$?

Yes. $\mathcal{U}$ is linearly independent, each vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, thus **Span** $(\mathcal{U}) = \mathbb{R}^2$.

(d) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$?

Yes. $\mathcal{U}$ is linearly independent, each vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ can be written as $\mathbf{x} = (x_1 - x_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, thus $\mathbf{Span}\ (\mathcal{U}) = \mathbb{R}^2$.

Remark For a given vector space V, the basis is not unique.

Example 12.3 **(Standard basis for $V = \mathbb{R}^n$)** Let

$$\mathbf{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ i\text{th row}, \\ \\ \\ \\ \end{array}$$

then $\mathcal{E} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$. In particular, $\mathcal{E}$ is called the standard basis.

Example 12.4 (**Example of Basis for $V = \mathbb{R}^{2\times 2}$**) Given a vector space $\mathbb{R}^{2\times 2}$, the set B consists of

$$B_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is a basis of $\mathbb{R}^{2\times 2}$. B is linearly independent. Also notice that $\forall A \in \mathbb{R}^{2\times 2}$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aB_{11} + bB_{12} + cB_{21} + dB_{22}.$$

Example 12.5(Basis for P_n)

1. $V = P_1$ (polynomials of degree at most 1).

(1) $\mathbf{p}_1(x) = 1$, $\mathbf{p}_2(x) = x$, $\mathbf{p}_3(x) = 2 - 3x$. Is $\mathcal{U} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ a basis for P_1?

No! $\mathcal{U}$ is linearly dependent since $\mathbf{p}_3 = 2\mathbf{p}_1 - 3\mathbf{p}_2$.

(2) $\mathcal{U} = \{1, x\}$ is a basis for P_1.

2. For $V = P_n$ (polynomials of degree at most n),

$\mathcal{U} = \{1, x, x^2, \cdots, x^n\}$ is a basis for P_n.

Lemma 12.6 **(A vector set is linearly dependent if # of vectors in the set is larger than # of vectors in the basis)** Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be the basis of vector space V, then $T = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq V$ is linearly dependent if $m > n$.

Proof. Since $T = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\} \subseteq V = \textbf{Span}(\mathcal{U})$ with $m > n$, then

$$\mathbf{v}_j = a_{1j}\mathbf{u}_1 + \cdots + a_{nj}\mathbf{u}_n = \sum_{i=1}^{n} a_{ij}\mathbf{u}_i, \quad j = 1, \cdots, m.$$

Suppose that $c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$, then

$$c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \sum_{j=1}^{m} c_j\mathbf{v}_j = \sum_{j=1}^{m} c_j(\sum_{i=1}^{n} a_{ij}\mathbf{u}_i) = \sum_{i=1}^{n}(\sum_{j=1}^{m} a_{ij}c_j)\mathbf{u}_i = \mathbf{0}.$$

Since $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are linearly independent, we have

$$\sum_{j=1}^{m} a_{ij} c_j = 0, \quad i = 1, \cdots, n.$$

This is a linear system for $c_1, \cdots, c_m$ with m unknowns and n equations. Since $m > n$, the system has infinity many solutions. Thus, T is linearly dependent.

Theorem 12.7. If both $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ are bases for a vector space V, then $m = n$.

Proof. Since $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ is basis of V and $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ are linearly independent, from above lemma $m \leq n$. On the other hand, $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is basis of V and $\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ are linearly independent, then from above lemma $n \leq m$. Thus, $m = n$.

Remark. For $\mathbb{R}^n$, the standard basis is $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, if $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is another basis of $\mathbb{R}^n$. From above theorem, we have $m = n$. For a given vector space, the number of vectors in different bases must be the same.

Definition 12.8 (Dimension) Let V be a vector space and let $\mathcal{U}$ be a basis of V, then the number of vectors in $\mathcal{U}$ is called the dimension of V. Denoted by $\dim(V)$.

Example 12.9 (0) $\dim(\{\mathbf{0}\}) = 0$, since $\{\mathbf{0}\}$ has no basis vector.

Example 12.9 (1) $\dim(\mathbb{R}^n) = n$, since $\{\mathbf{e}_i, i = 1, \cdots, n\}$ is the standard basis.

Example 12.9 (2) $\dim(\mathbb{R}^{2\times 3}) = 6$, since

$$B_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
$$B_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ B_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a basis of $\mathbb{R}^{2\times 3}$.

Example 12.9 (3) $\dim(\mathbb{R}^{m\times n}) = mn$

Example 12.9 Let the set of all polynomials of degree $\leq n$ is $P_n = \left\{ \sum_{i=0}^{n} a_i x^i \middle| a_i \in \mathbb{R} \right\}$. Since $\{1, x, x^2, \cdots, x^n\}$ is a basis of P_n, thus $\dim(P_n) = n + 1$.

We know the standard basis for $\mathbb{R}^m$ has m vectors, so **every** basis for $\mathbb{R}^m$ has m vectors.

Theorem 12.10 Suppose that $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_m\} \subseteq \mathbb{R}^m$, then the following are equivalent:
(1) $\mathbf{u}_1, \cdots, \mathbf{u}_m$ is a basis of $\mathbb{R}^m$.
(2) $\mathbf{u}_1, \cdots, \mathbf{u}_m$ is linearly independent
Proof. It is not hard to prove it. It can be omitted in lecture.
(1)$\Rightarrow$ (2) by using the definition of basis.
(2)$\Rightarrow$ (1), take any vector $\mathbf{v} \in \mathbb{R}^m$, then $\mathbf{u}_1, \cdots, \mathbf{u}_m, \mathbf{v} \in \mathbb{R}^m$ is linearly dependent since $[\mathbf{u}_1, \cdots, \mathbf{u}_m, \mathbf{v}]\mathbf{x} = \mathbf{0}$ has nonzero solutions (it is the undetermined homogeneous linear system). But $\mathbf{u}_1, \cdots, \mathbf{u}_m$ are linearly independent, therefore, $\mathbf{v}$ can be written as a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_m$. Therefore $\mathbf{v} \in \mathbf{Span}(\mathbf{u}_1, \cdots, \mathbf{u}_m)$ and $\mathbb{R}^m = \mathbf{Span}(\mathbf{u}_1, \cdots, \mathbf{u}_m)$.

Example 12.11: $V = \mathbb{R}^2$. Are the following sets form a basis for V or not?

(a) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$?

Yes, $\mathcal{U}$ is linearly independent and since $\left[\begin{array}{cc} 1 & 2 \\ 1 & 3 \end{array} \right]$ is invertible.

(b) $\mathcal{U} = \left\{ \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$?

No, $\mathcal{U}$ is linearly dependent and $\left[\begin{array}{cc} 3 & 1 \\ 6 & 2 \end{array} \right]$ is singular.

Theorem 12.12 (**Any vector can be uniquely expressed as the linear combination of the basis vectors**)
Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis of vector space V, then each $\mathbf{v} \in V$ can be written uniquely as a linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_n$.

Proof. Since $\mathcal{U}$ is a basis of V, we have $\mathbf{Span}(\mathcal{U}) = V$, then for any $\mathbf{x} \in V$, we have $\mathbf{x} \in \mathbf{Span}(\mathcal{U})$. For any vector $\mathbf{x} \in V$ and suppose

$$\mathbf{x} = h_1\mathbf{u}_1 + \cdots + h_n\mathbf{u}_n, \quad \mathbf{x} = k_1\mathbf{u}_1 + \cdots + k_n\mathbf{u}_n$$

then

$$h_1\mathbf{u}_1 + \cdots + h_n\mathbf{u}_n = k_1\mathbf{u}_1 + \cdots + k_n\mathbf{u}_n,$$

thus

$$(h_1 - k_1)\mathbf{u}_1 + \cdots + (h_n - k_n)\mathbf{u}_n = \mathbf{0}.$$

Since $\mathbf{u}_1, \cdots, \mathbf{u}_n$ are linearly independent, thus,

$$h_1 - k_1 = \cdots = h_n - k_n = 0.$$

Definition 12.13 (**Coordinates in a general vector space**) For a vector space V, let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis for V. For any $\mathbf{x} \in V$, $\mathbf{x}$ can be written uniquely as

$$\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$$

where $c_1, \cdots, c_n$ are scalars. We denote the coordinates of $\mathbf{x}$ with respect to (relative to) the basis $\mathcal{U}$ by $[\mathbf{x}]_{\mathcal{U}}$:

$$[\mathbf{x}]_{\mathcal{U}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Question: Given two bases $\mathcal{U}$ and $\mathcal{V}$ of vector space V, let $\mathbf{x} \in V$, if we know $[\mathbf{x}]_{\mathcal{U}}$, then how can we find $[\mathbf{x}]_{\mathcal{V}}$?

Lemma 12.14: (The operator to take coordinate is the linear transformation) Let V be a vector space with a basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, and let $\mathbf{x}, \mathbf{y} \in V$. For any $\alpha, \beta \in \mathbb{R}$, one has

$$[\alpha\mathbf{x} + \beta\mathbf{y}]_{\mathcal{U}} = \alpha[\mathbf{x}]_{\mathcal{U}} + \beta[\mathbf{y}]_{\mathcal{U}}$$

Proof. suppose $[\mathbf{x}]_{\mathcal{U}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad [\mathbf{y}]_{\mathcal{U}} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$, then

$\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n, \quad \mathbf{y} = d_1\mathbf{u}_1 + \cdots + d_n\mathbf{u}_n$ and

$\alpha\mathbf{x} + \beta\mathbf{y} = (\alpha c_1 + \beta d_1)\mathbf{u}_1 + \cdots + (\alpha c_n + \beta d_n)\mathbf{u}_n$.

Thus, $[\alpha\mathbf{x} + \beta\mathbf{y}]_{\mathcal{U}} = \begin{bmatrix} \alpha c_1 + \beta d_1 \\ \alpha c_2 + \beta d_2 \\ \vdots \\ \alpha c_n + \beta d_n \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \beta \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \alpha[\mathbf{x}]_{\mathcal{U}} + \beta[\mathbf{y}]_{\mathcal{U}}$

Coordinate transformation in general vector spaces

Theorem 12.15 (**Transition Matrix between two bases**) Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ and $\mathcal{V} = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be two bases of vector space V. Then

$$[\mathbf{x}]_{\mathcal{V}} = A[\mathbf{x}]_{\mathcal{U}}$$

where the jth column of A is $[\mathbf{u}_j]_{\mathcal{V}}$.

Proof.

Let $[\mathbf{x}]_{\mathcal{U}} = (d_1, d_2, \cdots, d_n)^T$, then $\mathbf{x} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_n\mathbf{u}_n$.
Suppose $[\mathbf{u}_j]_{\mathcal{V}} = \mathbf{a}_j (j = \cdots, n)$. According to the above lemma 12.17,

$$\begin{aligned}
[\mathbf{x}]_{\mathcal{V}} &= d_1[\mathbf{u}_1]_{\mathcal{V}} + d_2[\mathbf{u}_2]_{\mathcal{V}} + \cdots + d_n[\mathbf{u}_n]_{\mathcal{V}} \\
&= d_1\mathbf{a}_1 + d_2\mathbf{a}_2 + \cdots + d_n\mathbf{a}_n \\
&= [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n](d_1, d_2, \cdots, d_n)^T \\
&= A[\mathbf{x}]_{\mathcal{U}}
\end{aligned}$$

A is the transition matrix corresponding to the change of basis from $\mathcal{U}$ to $\mathcal{V}$.

Fact: A is invertible.

Proof. Since $A = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n]$, assume $c_1\mathbf{a}_1 + c_2\mathbf{a}_2, \cdots + c_n\mathbf{a}_n = \mathbf{0}$, then by using lemma 12. 14, one has

$$\begin{aligned}\mathbf{0} =& c_1\mathbf{a}_1 + c_2\mathbf{a}_2, \cdots + c_n\mathbf{a}_n \\ =& c_1[\mathbf{u}_1]_{\mathcal{V}} + c_2[\mathbf{u}_2]_{\mathcal{V}}, \cdots + c_n[\mathbf{u}_n]_{\mathcal{V}} \\ =& [c_1\mathbf{u}_1 + c_2\mathbf{u}_2, \cdots + c_n\mathbf{u}_n]_{\mathcal{V}}\end{aligned}$$

Thus, $c_1\mathbf{u}_1 + c_2\mathbf{u}_2, \cdots + c_n\mathbf{u}_n = \mathbf{0}$. $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent. Therefore, A is invertible.

$$[\mathbf{x}]_{\mathcal{U}} = A^{-1}[\mathbf{x}]_{\mathcal{V}}$$

A^{-1} is the transition matrix corresponding to the change of basis from $\mathcal{V}$ to $\mathcal{U}$.

Example 12.16

Find the transition matrix corresponding to the change of basis from $\mathcal{U} = \left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right\}$ to $\mathcal{V} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Since

$$\mathbf{u}_1 = 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $A = [\mathbf{a}_1, \mathbf{a}_2]$, then

$$\mathbf{a}_1 = [\mathbf{u}_1]_{\mathcal{V}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}, \quad \mathbf{a}_2 = [\mathbf{u}_2]_{\mathcal{V}} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Thus

$$A = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$$

Example 12.17

Let $\mathcal{E} = \{1, x, x^2\}$ and $\mathcal{U} = \{1, 2x, 4x^2 - 2\}$ be two bases of P_3. Now find the transition matrix corresponding to the change from the basis $\mathcal{U}$ to the basis $\mathcal{E}$. Since

$$1 = 1 * 1 + 0 * x + 0 * x^2, \quad [1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$2x = 0 * 1 + 2 * x + 0 * x^2, \quad [2x]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$4x^2 - 2 = (-2) * 1 + 0 * x + 4 * x^2, \quad [4x^2 - 2]_{\mathcal{E}} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

Example 12.17

The transition matrix from $\mathcal{U}$ to $\mathcal{E}$ is

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$