PHY1001: Mechanics (Week 8)

1 Chapter 12 Equilibrium and Elasticity

In the following, we will study the concept of equilibrium and learn the conditions that must be satisfied for a body or structure to be in equilibrium, and use them to help us solve physics problems. Later, we will also study how to analyze situations in which a body is deformed by tension, compression, pressure, or shear.

1.1 Equilibrium

Equilibrium, in physics, is the condition of a system when neither its state of motion nor its internal energy state tends to change with time. A simple mechanical body is said to be in equilibrium if it experiences neither linear acceleration nor angular acceleration; unless it is disturbed by an outside force, it will continue in that condition indefinitely. In short, equilibrium means everything (force and torque) is well-balanced in such a way that the motion of a body does not change.

Types of Equilibrium:

1. If a rigid body is at rest, it is said to be in static equilibrium.
2. Consider an airplane in flight with constant speed, direction, and altitude, we say such a body is in equilibrium but is not static. (Dynamical)

Equilibrium and Stability

1. Stable Equilibrium: Minimum in the energy diagram. Low center of gravity.
2. Unstable Equilibrium: Maximum in the energy diagram. high center of gravity.

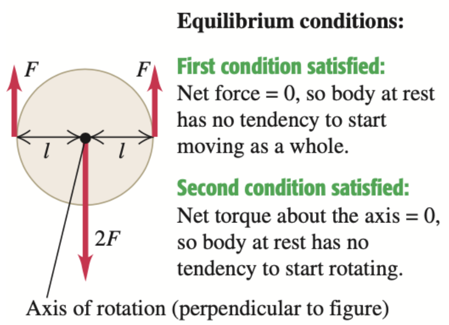
Thus, let us first write down the Conditions for Equilibrium

$$\text{First condition: } \sum \vec{F} = 0\,, \tag{1}$$

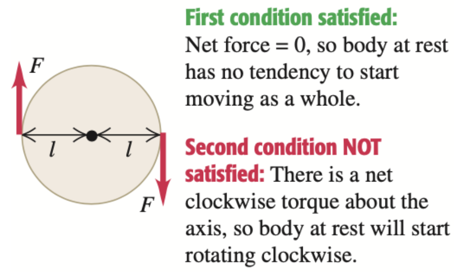
$$\text{Second condition: } \sum \vec{\tau} = 0 \quad \text{about any point.} \tag{2}$$

These conditions for an extended body make sure that the body has no tendency to move and rotate.

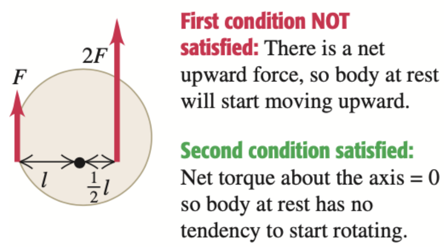
(a) This body is in static equilibrium.



(b) This body has no tendency to accelerate as a whole, but it has a tendency to start rotating.



(c) This body has a tendency to accelerate as a whole but no tendency to start rotating.



The consequence of the above conditions for a body in equilibrium is that

1. The linear momentum $\vec{P}$ of its center of mass is constant.
2. Its angular momentum $\vec{L}$ about its center of mass, or about any other point, is also constant.

1.2 Center of Gravity

In most equilibrium problems, one of the forces acting on the body is its weight. We need to be able to calculate the torque of this force. If $\vec{g}$ has the same value at all points on a body, its center of gravity is identical to its center of mass. But we can always calculate the torque due to the body's weight by assuming that the entire force of gravity (weight) is concentrated at a point called the center of gravity (abbreviated "cg").

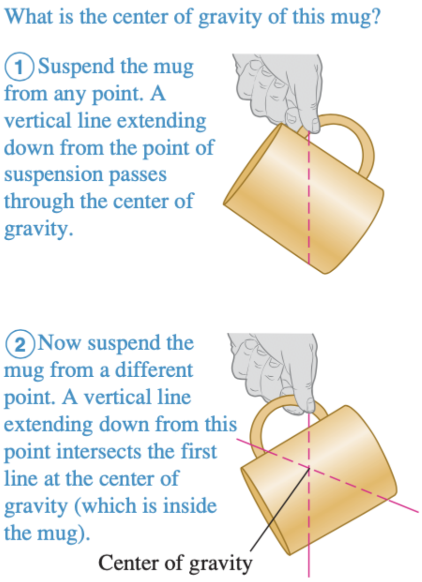
Now consider the gravitational torque on a body of arbitrary shape and assume $\vec{g}$ is a constant, thus we can write the total torque due to gravity as

$$\vec{\tau} = \sum_i \vec{r}_i \times m_i \vec{g} = \sum_i m_i \vec{r}_i \times \vec{g} \tag{3}$$

$$\vec{\tau} = M\vec{r}_{cm} \times \vec{g} = \vec{r}_{cm} \times (M\vec{g}). \tag{4}$$

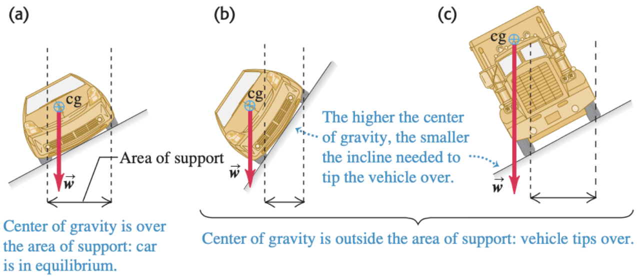
The above equation implies that the total gravitational torque is the same as though the total weight $w = Mg$ were acting on the position $\vec{r}_{cm}$ of the center of mass, which we also call the center of gravity.

By definition, we can often use symmetry considerations to locate the center of gravity of a body. The center of gravity of a homogeneous sphere, cube, circular sheet, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.



When a body acted on by gravity is suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the body could not be in rotational equilibrium. Above figure shows how to use this fact to determine experimentally the location of the center of gravity of an irregular body.

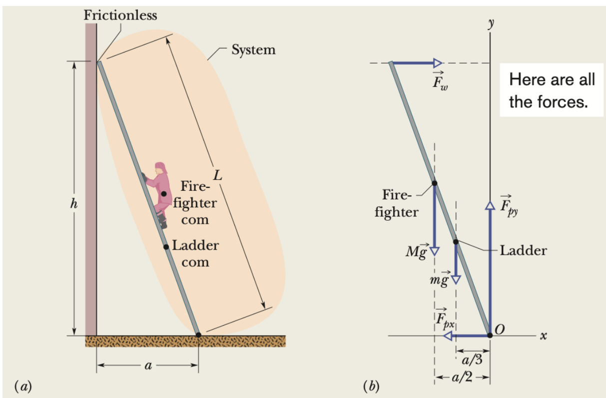
The lower the center of gravity and the larger the area of support, the more difficult it is to overturn a body.



In (a) the center of gravity is within the area bounded by the supports, and the car is in equilibrium. The car in (b) and the truck in (c) will tip over because their centers of gravity lie outside the area of support.

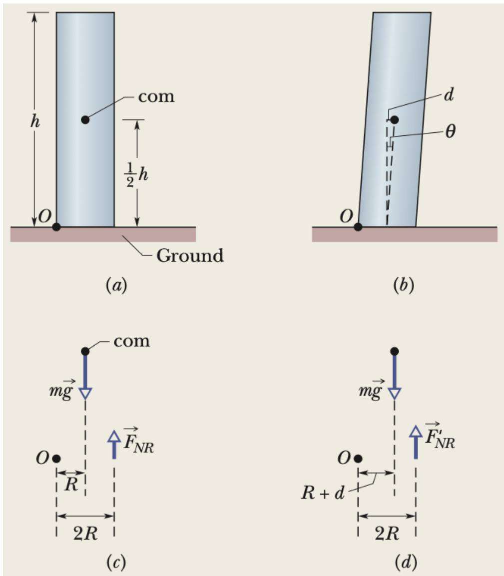
1.3 Solving Rigid-Body Equilibrium Problems

Example 1: Balancing the leaning ladder



Example 2: Balancing the leaning Tower of Pisa

Let's assume that the Tower of Pisa is a uniform hollow cylinder of radius R = 9.8 m and height h = 60 m. The center of mass is located at height $h/2$, along the cylinder's central axis. In Fig.a, the cylinder is upright. In Fig. b, it leans rightward (toward the tower's southern wall) by $\theta = 5.5°$, which shifts the com by a distance d. Let's assume that the ground exerts only two forces on the tower. A normal force F_{NL} acts on the left (northern) wall, and a normal force F_{NR} acts on the right (southern) wall. By what percent does the magnitude FNR increase because of the leaning?



Answer: 29%

1.4 Elasticity

When we talked about the elastic potential energy, we mentioned the Hooke's law $F_x = -kx$ for the elastic spring with spring constant k. F_x is the force generated by the spring when it is stretched by a length of x from its equilibrium position. This law can be also understood as follows: let us try to use hand to pull the spring, and then the magnitude of the force exerted to the spring is $F_{\text{hand}} = kx$ to reach the elongation x.

The rigid body is a useful idealized model, but the stretching (tensile), squeezing (bulk) and twisting (shear) of real bodies when forces are applied are often too important to ignore. Here we systematically develop the physical description of these deformation and the corresponding forces. For real bodies with finite extent, the proportionality of stress and strain in general is called Hooke's law.

For each kind of deformation we will introduce a quantity called stress that characterises the strength of the forces causing the deformation, on a 'force per unit area' basis. Another quantity, strain, describes the resulting deformation. When the stress and strain are small enough, we often find that the two are directly proportional, and we call the proportionality constant an elastic modulus. Thus, Hooke's law can be expressed as the equation in general

$$\frac{\text{stress (force per unit area)}}{\text{strain (fractional deformation)}} = \text{elastic modulus}. \quad (5)$$

Intuitively, the harder you pull on something, the more it stretches; the more you squeeze it, the more it compresses. The elastic modulus of a material defines an object's or substance's resistance to being deformed elastically. It is roughly a constant for given materials.

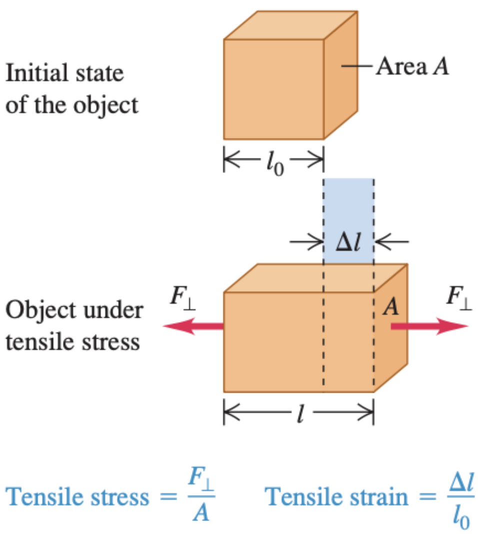
In the above language, the elongation of an ideal spring (x akin to strain) is proportional to the stretching force (F akin to stress). The spring constant k can be viewed as the corresponding elastic modulus.

Remember that Hooke's law is not really a general law but an experimental finding that is valid over only a limited range (under certain conditions).

For real (not rigid) bodies, there are several types deformations that can occur when forces are applied. Let us discuss the corresponding Hooke's law as follows.

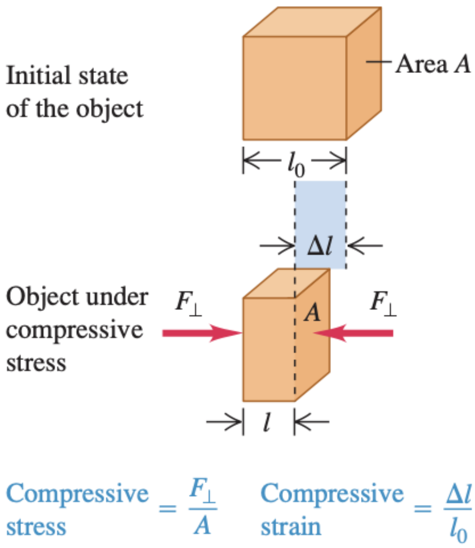
1. Tensile stress is tensile force per unit area, $F_\perp/A$. Tensile strain is fractional change in length, $\Delta l/l_0$. The elastic modulus is called Young's modulus Y, which is given by

$$Y = \frac{\text{Tensile stress}}{\text{Tensile strain}} = \frac{F_\perp/A}{\Delta l/l_0}. \quad (6)$$



Typical values of Y for metals are of the order of $10^{10} - 10^{11}$ Pa. In comparison, the tendon, which connects your foot to the large muscle, has a Young's modulus of 1.2×10^9 Pa, much less than for the metal materials. Thus, tendon stretches substantially (up to 2.5% of its length) in response to the stresses experienced in walking and running.

2. Compressive stress and strain are defined in the same way as the tensile ones.

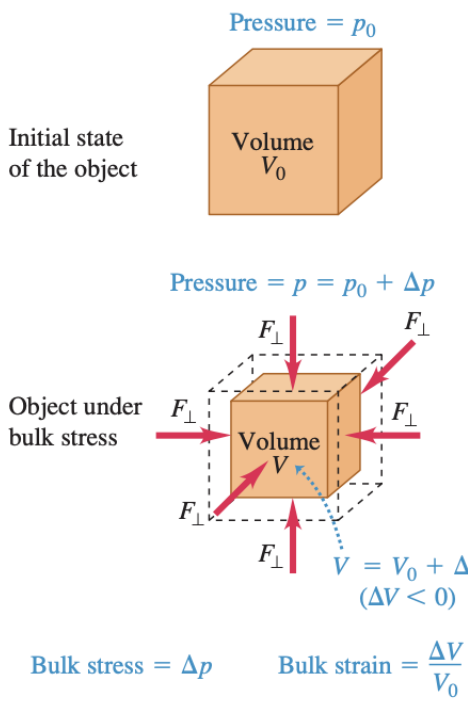


For many materials, Young's modulus has the same value for both tensile and compressive stresses. Composite materials such as concrete and stone are an exception; they can withstand compressive stresses but fail under comparable tensile stresses. These two properties of materials have a lot of applications in engineering and constructions.

3. For fluid, we usually use bulk modulus to describe its properties under force. Pressure in a fluid is force per unit area. Bulk stress is pressure change, Δp, and bulk strain is fractional volume change, $\Delta V/V0$. The elastic modulus is called the bulk modulus, B is defined by

$$B = \frac{\text{Bulk stress}}{\text{Bulk strain}} = -\frac{\Delta p}{\Delta V/V_0}. \quad (7)$$

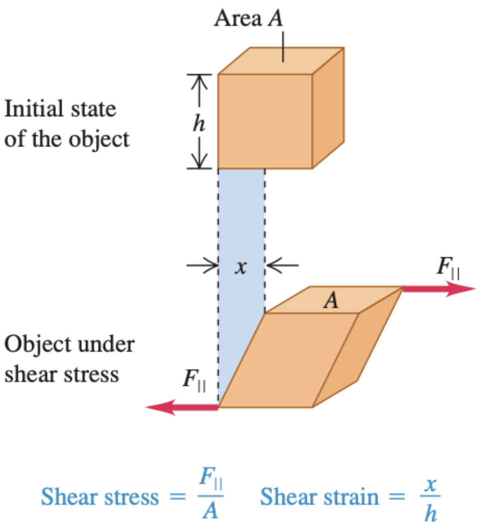
where a minus sign is included because an increase of pressure always causes a decrease in volume. In other words, if Δp is positive,ΔV is negative. The bulk modulus B itself is a positive quantity.



For small pressure changes in a solid or a liquid, we consider B to be constant. Typical vaule of B for metal and liquid is of the order of $10^{10} \sim 10^{9}$ Pa. The bulk modulus of a gas, however, depends on the initial pressure p_0 (see homework).

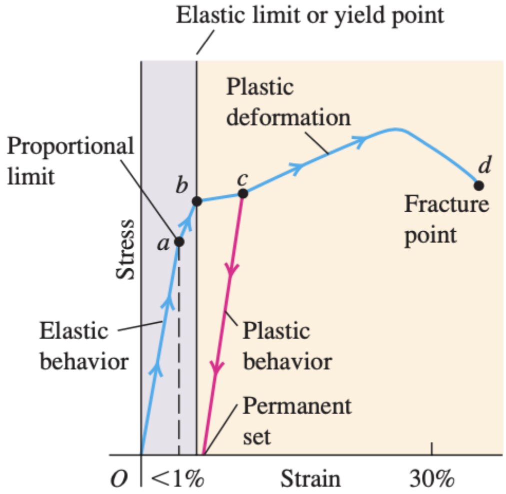
4. Shear stress is force per unit area, $F_{||}/A$, for a force applied tangent to a surface. Shear strain is the displacement x of one side divided by the transverse dimension h. The elastic modulus is called the shear modulus, denoted by S:

$$S = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{F_{||}/A}{x/h}. \tag{8}$$



For a given material, S is usually one-third to one-half as large as Young's modulus Y for tensile stress. Keep in mind that the concepts of shear stress, shear strain, and shear modulus apply to solid materials only. The reason is that shear refers to deforming an object that has a definite shape. This concept does not apply to gases and liquids, which do not have definite shapes.

In the preceding discussion, we briefly mentioned that Hooke's law—the proportionality of stress and strain in elastic deformations— has a limited range of validity. We know that if you pull, squeeze, or twist anything hard enough, it will bend or break. Let us provide a more precise description. Below we show a typical stress-strain diagram for a ductile metal under tension, which indicates the elasticity and plasticity of the metal.



1. The first portion is a straight line, indicating Hooke's law behavior with stress directly proportional to strain. This straight-line portion ends at point a; the stress at this point (the maximum stress for which stress and strain are proportional) is called the proportional limit. Beyond the proportional limit, Hooke's law is not valid.

2. From a to b, stress and strain are no longer proportional, and Hooke's law is no longer valid. The deformation is reversible, and the forces are conservative; the energy put into the material to cause the deformation is recovered when the stress is removed. The elastic limit is the stress beyond which irreversible deformation occurs.

3. When we increase the stress beyond point b, the strain continues to increase. But now when we remove the load at some point beyond b, say c, the material does not come back to its original length. The length at zero stress is now greater than the original length; the material has undergone an irreversible deformation and has acquired what we call a permanent set. Further increase of load beyond c produces a large increase in strain for a relatively small increase in stress, until a point d is reached at which fracture takes place. Plasticity: The behavior of the material from b to d is called plastic flow or plastic deformation. A plastic deformation is irreversible; when the stress is removed, the material does not return to its original state.

4. The stress required to cause actual fracture of a material is called the breaking stress, the ultimate strength, or (for tensile stress) the tensile strength. Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses.