7.4

Exponential Change and Separable Differential Equations

Exponential Change

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Examples of such quantities include the size of a population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium. Such quantities are said to undergo **exponential change**.

If the amount present at time $t = 0$ is called y_0, then we can find y as a function of t by solving the following initial value problem:

$$\text{Differential equation:} \qquad \frac{dy}{dt} = ky \tag{1a}$$

$$\text{Initial condition:} \qquad y = y_0 \quad \text{when} \quad t = 0. \tag{1b}$$

If y is positive and increasing, then k is positive, and we use Equation (1a) to say that the rate of growth is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative, and we use Equation (1a) to say that the rate of decay is proportional to the amount still left.

We see right away that the constant function $y = 0$ is a solution of Equation (1a) if $y_0 = 0$. To find the nonzero solutions, we divide Equation (1a) by y:

$$
\begin{aligned}
\frac{1}{y} \cdot \frac{dy}{dt} &= k && y \neq 0 \\
\int \frac{1}{y}\frac{dy}{dt}dt &= \int k\,dt && \text{Integrate with respect to } t; \\
\ln|y| &= kt + C && \textstyle\int (1/u)\,du = \ln|u| + C. \\
|y| &= e^{kt+C} && \text{Exponentiate.} \\
|y| &= e^{C} \cdot e^{kt} && e^{a+b} = e^{a} \cdot e^{b} \\
y &= \pm e^{C}e^{kt} && \text{If } |y| = r, \text{ then } y = \pm r. \\
y &= Ae^{kt}. && A \text{ is a shorter name for } \pm e^{C}.
\end{aligned}
$$

By allowing A to take on the value 0 in addition to all possible values $\pm e^{C}$, we can include the solution $y = 0$ in the formula.

We find the value of A for the initial value problem by solving for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k \cdot 0} = A.$$

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is

$$y = y_0 e^{kt}. \tag{2}$$

Quantities changing in this way are said to undergo **exponential growth** if $k > 0$ and **exponential decay** if $k < 0$. The number k is called the **rate constant** of the change. (See Figure 7.15.)

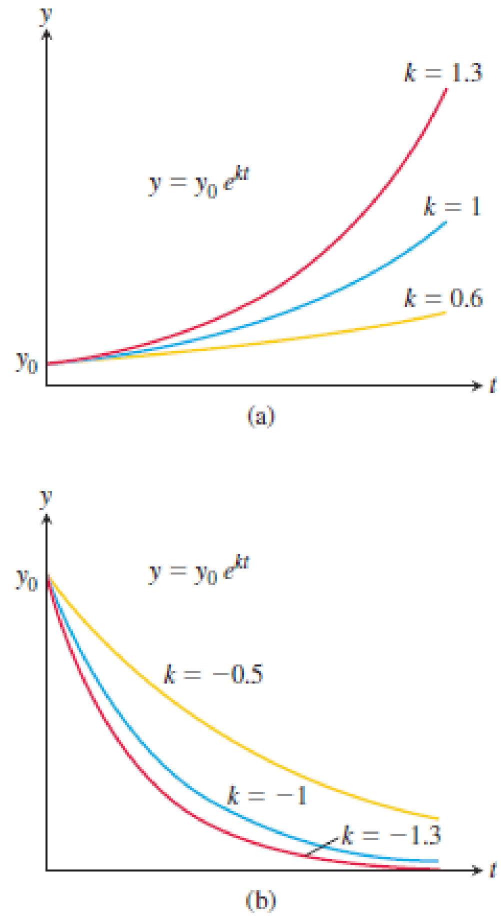


FIGURE 7.15 Graphs of (a) exponential growth and (b) exponential decay. As $|k|$ increases, the growth ($k > 0$) or decay ($k < 0$) intensifies.

Separable Differential Equations

Exponential change is modeled by a differential equation of the form $dy/dx = ky$ for some nonzero constant k. More generally, suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y), \tag{3}$$

where f is a function of *both* the independent and dependent variables.

Equation (3) is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

g is a function of x;
H is a function of y.

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

$H(y) = \frac{1}{h(y)}$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{4}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

EXAMPLE 1 Solve the differential equation

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

Solution Since $1 + y$ is never zero for $y > -1$, we can solve the equation by separating the variables.

$$\frac{dy}{dx} = (1 + y)e^x$$

$$dy = (1 + y)e^x \, dx$$

Treat dy/dx as a quotient of differentials and multiply both sides by dx.

$$\frac{dy}{1 + y} = e^x \, dx$$

Divide by $(1 + y)$.

$$\int \frac{dy}{1 + y} = \int e^x \, dx$$

Integrate both sides.

$$\ln(1 + y) = e^x + C$$

C represents the combined constants of integration.

The last equation gives y as an implicit function of x. ■

EXAMPLE 2 Solve the equation $y(x + 1)\dfrac{dy}{dx} = x(y^2 + 1)$.

Solution We change to differential form, separate the variables, and integrate:

$$y(x + 1)\,dy = x(y^2 + 1)\,dx$$

$$\frac{y\,dy}{y^2 + 1} = \frac{x\,dx}{x + 1} \qquad x \neq -1$$

$$\int \frac{y\,dy}{1 + y^2} = \int \left(1 - \frac{1}{x + 1}\right) dx \qquad \text{Divide } x \text{ by } x + 1.$$

$$\frac{1}{2}\ln(1 + y^2) = x - \ln|x + 1| + C.$$

The last equation gives the solution y as an implicit function of x.

Logistic Population Growth

To be realistic, the population growth is checked due to limited resources, and there is a limiting population or **carrying capacity** M beyond which the growth rate decreases. This gives the logistic differential equation for the population P, with initial population P_0:

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)}$$

$$\int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

separation of variables

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

partial fraction

$$\int \left(\frac{1}{P} + \frac{1}{M - P}\right) dP = \int k\,dt$$

$$\ln|P| - \ln|M - P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\frac{M-P}{P} = Ae^{-kt}$$

$$\frac{M-P_0}{P_0} = Ae^{0} = A$$ set $t = 0$

$$\frac{M}{P} - 1 = Ae^{-kt} \quad \Rightarrow \quad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

$$P = \frac{M}{1 + Ae^{-kt}}$$

$$\boxed{P(t) = \frac{M}{1 + Ae^{-kt}} \quad \text{where } A = \frac{M - P_0}{P_0}}$$

Heat Transfer: Newton's Law of Cooling

If H is the temperature of the object at time t and H_S is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_S). \tag{8}$$

If we substitute y for $(H - H_S)$, then

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}(H - H_S) = \frac{dH}{dt} - \frac{d}{dt}(H_S) \\
&= \frac{dH}{dt} - 0 && H_S \text{ is a constant.} \\
&= \frac{dH}{dt} \\
&= -k(H - H_S) && \text{Eq. (8)} \\
&= -ky. && H - H_S = y
\end{aligned}$$

Now we know that the solution of $dy/dt = -ky$ is $y = y_0 e^{-kt}$, where $y(0) = y_0$. Substituting $(H - H_S)$ for y, this says that

$$H - H_S = (H_0 - H_S)e^{-kt}, \tag{9}$$

where H_0 is the temperature at $t = 0$. This equation is the solution to Newton's Law of Cooling.

EXAMPLE 6 A hard-boiled egg at 98°C is put in a sink of 18°C water. After 5 min, the egg's temperature is 38°C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach 20°C?

Solution We find how long it would take the egg to cool from 98°C to 20°C and subtract the 5 min that have already elapsed. Using Equation (9) with $H_S = 18$ and $H_0 = 98$, the egg's temperature t min after it is put in the sink is

$$H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}.$$

To find k, we use the information that $H = 38$ when $t = 5$:

$$\begin{aligned} 38 &= 18 + 80e^{-5k} \\ e^{-5k} &= \frac{1}{4} \\ -5k &= \ln\frac{1}{4} = -\ln 4 \\ k &= \frac{1}{5}\ln 4 = 0.2\ln 4 \qquad \text{(about 0.28).} \end{aligned}$$

The egg's temperature at time t is $H = 18 + 80e^{-(0.2\ln 4)t}$. Now find the time t when $H = 20$:

$$\begin{aligned} 20 &= 18 + 80e^{-(0.2\ln 4)t} \\ 80e^{-(0.2\ln 4)t} &= 2 \\ e^{-(0.2\ln 4)t} &= \frac{1}{40} \\ -(0.2\ln 4)t &= \ln\frac{1}{40} = -\ln 40 \\ t &= \frac{\ln 40}{0.2\ln 4} \approx 13 \text{ min.} \end{aligned}$$

The egg's temperature will reach 20°C about 13 min after it is put in the water to cool. Since it took 5 min to reach 38°C, it will take about 8 min more to reach 20°C. ■

9.1

Solutions, Slope Fields, and Euler's Method

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \qquad \text{and} \qquad \frac{d}{dx}y = f(x, y)$$

are equivalent.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx} y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

EXAMPLE 1 Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

We need to show that the differential equation is satisfied when we substitute into it the expressions $(C/x) + 2$ for y, and $-C/x^2$ for dy/dx. That is, we need to verify that for all $x \in (0, \infty)$,

$$-\frac{C}{x^2} = \frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right].$$

This last equation follows immediately by expanding the expression on the right-hand side:

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

Therefore, for every value of C, the function $y = C/x + 2$ is a solution of the differential equation. ■

EXAMPLE 2 Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left side of the equation:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right side of the equation:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 9.1.

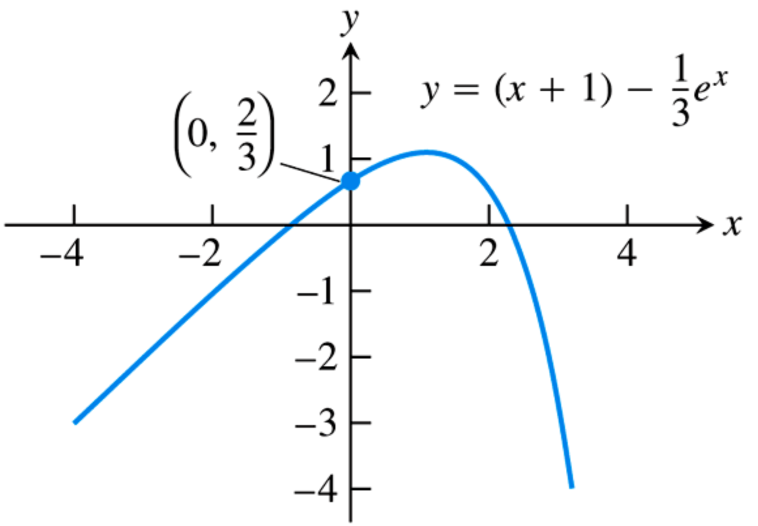


FIGURE 9.1 Graph of the solution to the initial value problem in Example 2.

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

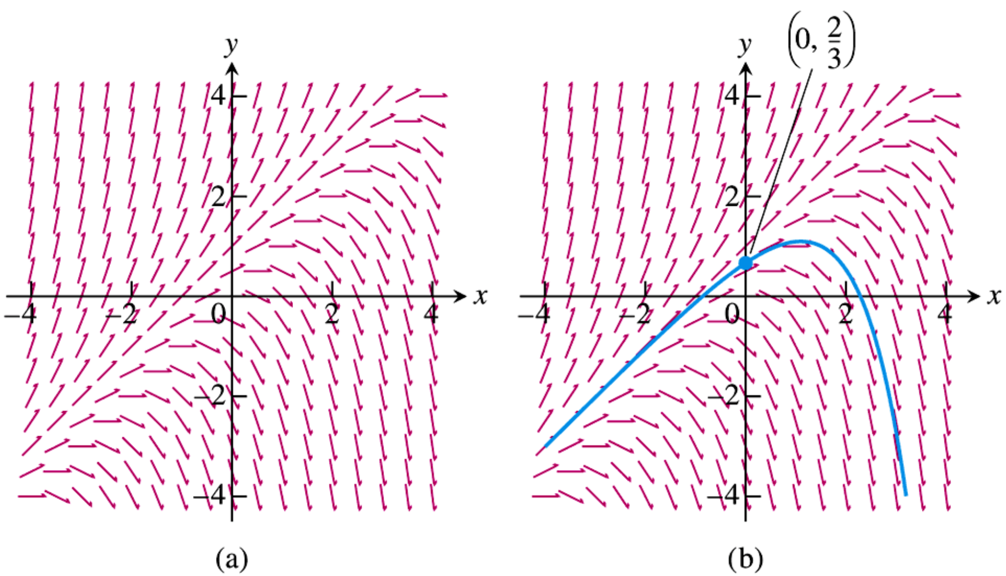


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

EXAMPLE 3

(a) Sketch the direction field for the differential equation $y' = x^2 + y^2 - 1$.
(b) Use part (a) to sketch the solution curve that passes through the origin.

SOLUTION

(a) We start by computing the slope at several points in the following chart:

x	-2	-1	0	1	2	-2	-1	0	1	2	. . .
y	0	0	0	0	0	1	1	1	1	1	. . .
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	. . .

Now we draw short line segments with these slopes at these points. The result is the direction field shown in Figure 5.

(b) We start at the origin and move to the right in the direction of the line segment (which has slope -1). We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 6. Returning to the origin, we draw the solution curve to the left as well. ■

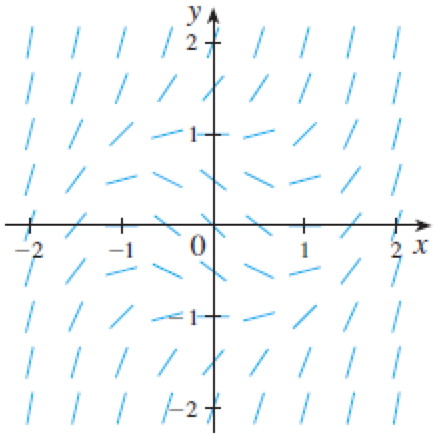


FIGURE 5

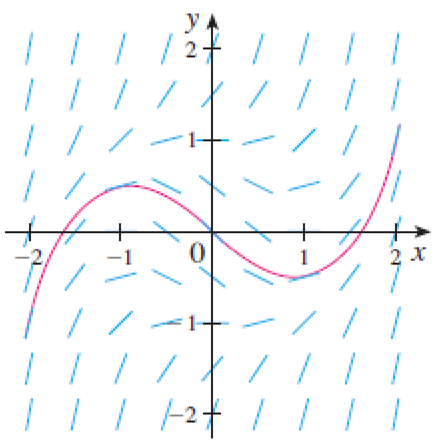


FIGURE 6

Optional

Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. We illustrate the method on the initial-value problem that we used to introduce direction fields:

$$y' = x + y \qquad y(0) = 1$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point (0, 1). As a first approximation to the solution we could use the linear approximation $L(x) = x + 1$. In other words, we could use the tangent line at (0, 1) as a rough approximation to the solution curve (see Figure 11).

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a midcourse correction by changing direction as indicated by the direction field. Figure 12 shows what happens if we start out along the tangent line but stop when $x = 0.5$. (This horizontal distance traveled is called the *step size*.) Since $L(0.5) = 1.5$, we have $y(0.5) \approx 1.5$ and we take (0.5, 1.5) as the starting point for a new line segment. The differential equation tells us that $y'(0.5) = 0.5 + 1.5 = 2$, so we use the linear function

$$y = 1.5 + 2(x - 0.5) = 2x + 0.5$$

as an approximation to the solution for $x > 0.5$ (the green segment in Figure 12). If we decrease the step size from 0.5 to 0.25, we get the better Euler approximation shown in Figure 13.

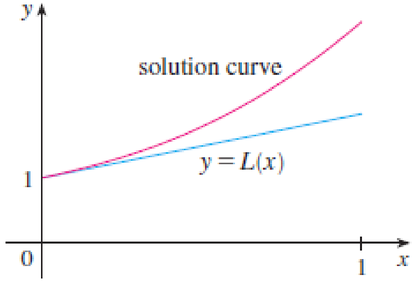


FIGURE 11
First Euler approximation

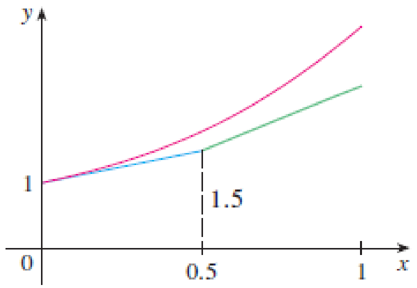


FIGURE 12
Euler approximation with step size 0.5

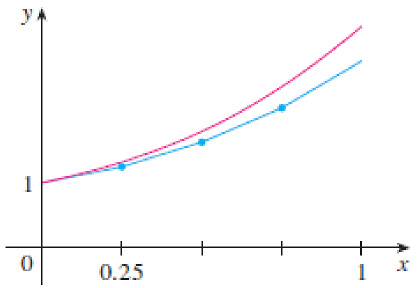


FIGURE 13
Euler approximation with step size 0.25

9.2

First-Order Linear Equations

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**. Since the exponential growth/decay equation $dy/dx = ky$ (Section 7.2) can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, they are not multiplied together, nor do they appear as the argument of a function $\left(\text{such as } \sin y,\ e^y, \text{ or } \sqrt{dy/dx}\right)$.

EXAMPLE 1 Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x.$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ■

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Original equation is in standard form.

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x)$$

Multiply by positive $v(x)$.

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x)$$

$v(x)$ is chosen to make $v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y)$.

$$v(x) \cdot y = \int v(x)Q(x)\,dx$$

Integrate with respect to x.

$$\boxed{y = \frac{1}{v(x)}\int v(x)Q(x)\,dx} \qquad (2)$$

Equation (2) expresses the solution of Equation (1) in terms of the functions $v(x)$ and $Q(x)$. We call $v(x)$ an **integrating factor** for Equation (1) because its presence makes the equation integrable.

We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy \qquad \text{Condition imposed on } v$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy \qquad \text{Derivative Product Rule}$$

$$y\frac{dv}{dx} = Pvy \qquad \text{The terms } v\frac{dy}{dx} \text{ cancel.}$$

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx \qquad \text{Variables separated, } v > 0$$

$$\int \frac{dv}{v} = \int P\,dx \qquad \text{Integrate both sides.}$$

$$\ln v = \int P\,dx \qquad \text{Since } v > 0 \text{, we do not need absolute value signs in } \ln v.$$

$$e^{\ln v} = e^{\int P\,dx} \qquad \text{Exponentiate both sides to solve for } v.$$

$$v = e^{\int P\,dx} \tag{3}$$

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int(-3/x)\,dx} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln|x|} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\frac{1}{x^3} \cdot \left(\frac{dy}{dx} - \frac{3}{x} y \right) = \frac{1}{x^3} \cdot x$$

$$\frac{1}{x^3} \frac{dy}{dx} - \frac{3}{x^4} y = \frac{1}{x^2}$$

$$\frac{d}{dx} \left(\frac{1}{x^3} y \right) = \frac{1}{x^2}$$ Left-hand side is $\frac{d}{dx}(v \cdot y)$.

$$\frac{1}{x^3} y = \int \frac{1}{x^2}\, dx$$ Integrate both sides.

$$\frac{1}{x^3} y = -\frac{1}{x} + C.$$

Solving this last equation for y gives the general solution:

$$y = x^3 \left(-\frac{1}{x} + C \right) = -x^2 + Cx^3, \qquad x > 0.$$

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx.$$

Left-hand side is vy.

Integration by parts of the right-hand side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4.$$

9.3

Applications

Motion with Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. Picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

If the resisting force is proportional to velocity, we have

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.4)

$$v = v_0 e^{-(k/m)t}. \tag{1}$$

Suppose that an object is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k} e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned} \lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) \\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}. \end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

EXAMPLE 1 For a 90-kg ice skater, the k in Equation (1) is about 5 kg/s. How long will it take the skater to coast from 3.3 m/s (11.88 km/h) to 0.3 m/s? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Equation (1) for t:

$$
\begin{aligned}
3.3e^{-t/18} &= 0.3 \qquad \text{Eq. (1) with } k = 5,\ m = 90,\ v_0 = 3.3,\ v = 0.3 \\
e^{-t/18} &= 1/11 \\
-t/18 &= \ln(1/11) = -\ln 11 \\
t &= 18 \ln 11 \approx 43\,\text{s}.
\end{aligned}
$$

We answer the second question with Equation (3):

$$
\begin{aligned}
\text{Distance coasted} &= \frac{v_0 m}{k} = \frac{3.3 \cdot 90}{5} \\
&= 59.4\ \text{m}.
\end{aligned}
$$

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 9.11).

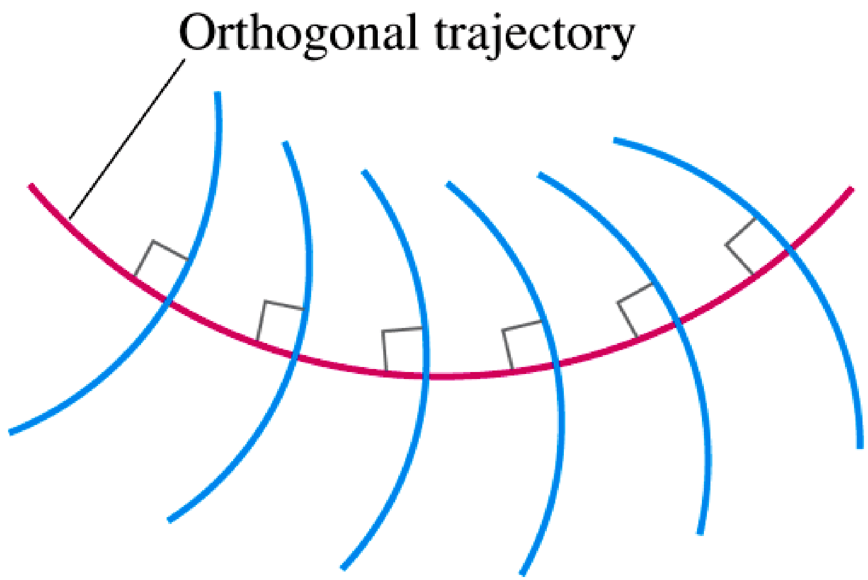


FIGURE 9.11 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

EXAMPLE 2 Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas having the coordinate axes as asymptotes. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

This differential equation is separable and we solve it as in Section 7.4:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{5}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (5) and sketched in Figure 9.13.

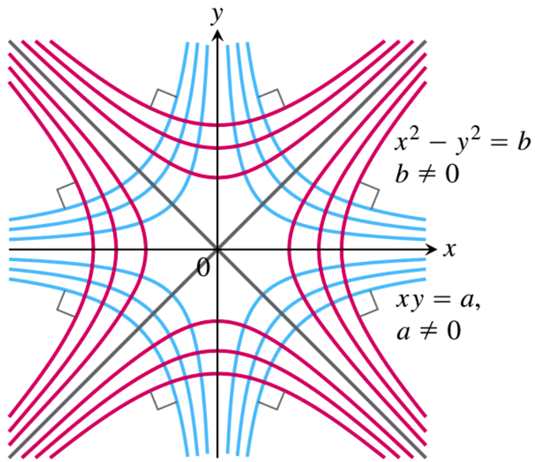


FIGURE 9.13 Each curve is orthogonal to every curve it meets in the other family (Example 2).

9.4

Graphical Solutions of Autonomous Equations

A differential equation for which dy/dx is a function of y only is called an **autonomous** differential equation.

DEFINITION If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

Thus, equilibrium values are those at which no change occurs in the dependent variable, so y is at *rest*.

Note that if r is any root of $g(y)$, i.e., $g(r) = 0$, then the constant function $y \equiv r$ is a solution to the autonomous differential equation.

To construct a graphical solution, we make a **phase line** for the equation, a plot on the y-axis that shows the equation's equilibrium values along with the intervals where y' and y'' are positive or negative. Then we know where the solutions are increasing or decreasing and the concavity of the solution curves, and we can thus determine the shapes of the curves without having to find formulae for them.

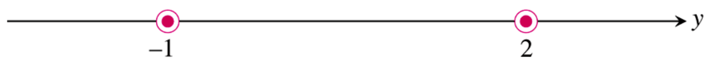
EXAMPLE 1 Draw a phase line for the equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

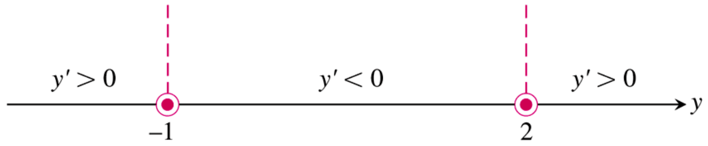
and use it to sketch solutions to the equation.

Solution

1. *Draw a number line for y and mark the equilibrium values $y = -1$ and $y = 2$, where $dy/dx = 0$.*



2. *Identify and label the intervals where $y' > 0$ and $y' < 0$.* This step resembles what we did in Section 4.3, only now we are marking the y-axis instead of the x-axis.



We can encapsulate the information about the sign of y' on the phase line itself. Since $y' > 0$ on the interval to the left of $y = -1$, a solution of the differential equation with a y-value less than -1 will increase from there toward $y = -1$. We display this information by drawing an arrow on the interval pointing to -1.

Similarly, $y' < 0$ between $y = -1$ and $y = 2$, so any solution with a value in this interval will decrease toward $y = -1$.

For $y > 2$, we have $y' > 0$, so a solution with a y-value greater than 2 will increase from there without bound.

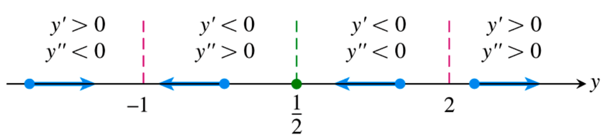
In short, solution curves below the horizontal line $y = -1$ in the xy-plane rise toward $y = -1$. Solution curves between the lines $y = -1$ and $y = 2$ fall away from $y = 2$ toward $y = -1$. Solution curves above $y = 2$ rise away from $y = 2$ and keep going up.

3. *Calculate y'' and mark the intervals where $y'' > 0$ and $y'' < 0$.* To find y'', we differentiate y' *with respect to* x, using implicit differentiation.

$$y' = (y + 1)(y - 2) = y^2 - y - 2 \qquad \text{Formula for } y' \ldots$$

$$\begin{aligned} y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - y - 2) \\ &= 2yy' - y' \qquad \text{differentiated implicitly with respect to } x \\ &= (2y - 1)y' \\ &= (2y - 1)(y + 1)(y - 2). \end{aligned}$$

From this formula, we see that y'' changes sign at $y = -1$, $y = 1/2$, and $y = 2$. We add the sign information to the phase line.



4. *Sketch an assortment of solution curves in the xy-plane.* The horizontal lines $y = -1$, $y = 1/2$, and $y = 2$ partition the plane into horizontal bands in which we know the signs of y' and y''. In each band, this information tells us whether the solution curves rise or fall and how they bend as x increases (Figure 9.15).

The "equilibrium lines" $y = -1$ and $y = 2$ are also solution curves. (The constant functions $y = -1$ and $y = 2$ satisfy the differential equation.) Solution curves that cross the line $y = 1/2$ have an inflection point there. The concavity changes from concave down (above the line) to concave up (below the line).

As predicted in Step 2, solutions in the middle and lower bands approach the equilibrium value $y = -1$ as x increases. Solutions in the upper band rise steadily away from the value $y = 2$.

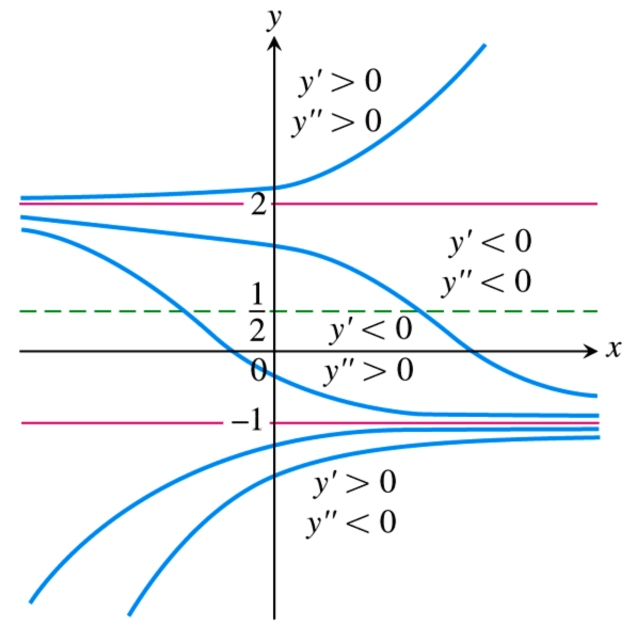
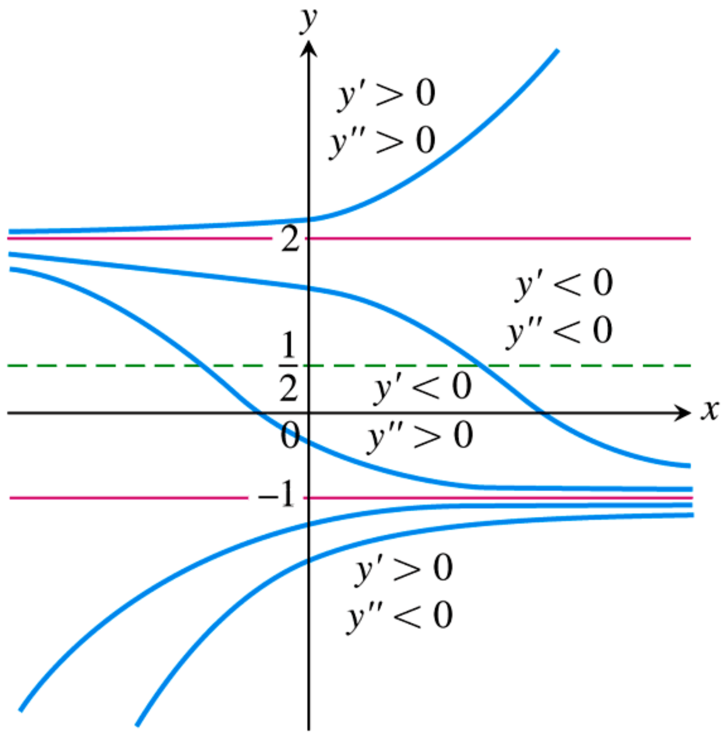


FIGURE 9.15 Graphical solutions from Example 1 include the horizontal lines $y = -1$ and $y = 2$ through the equilibrium values. No two solution curves can ever cross or touch each other.



Stable and Unstable Equilibria

Look at Figure 9.15 once more, in particular at the behavior of the solution curves near the equilibrium values. Once a solution curve has a value near $y = -1$, it tends steadily toward that value; $y = -1$ is a **stable equilibrium**. The behavior near $y = 2$ is just the opposite: All solutions except the equilibrium solution $y = 2$ itself move *away* from it as x increases. We call $y = 2$ an **unstable equilibrium**. If the solution is *at* that value, it stays, but if it is off by any amount, no matter how small, it moves away. (Sometimes an equilibrium value is unstable because a solution moves away from it only on one side of the point.)

Newton's Law of Cooling

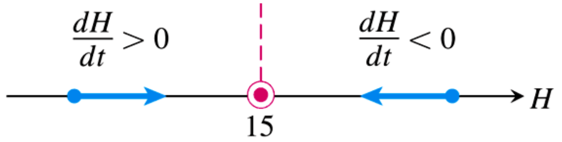


FIGURE 9.16 First step in constructing the phase line for Newton's law of cooling. The temperature tends towards the equilibrium (surrounding-medium) value in the long run.

The differential equation

$$\frac{dH}{dt} = -k(H - H_S), \qquad k > 0$$

models Newton's Law of Cooling, where H is the temperature of an object at time t and H_S is the constant temperature of the surrounding medium.

Suppose that the surrounding medium (say, a room in a house) has a constant Celsius temperature of 15°C. We can then express the difference in temperature as $H(t) - 15$. Assuming H is a differentiable function of time t, by Newton's Law of Cooling, there is a constant of proportionality $k > 0$ such that

$$\frac{dH}{dt} = -k(H - 15) \tag{1}$$

(*minus* k to give a negative derivative when $H > 15$).

Since $dH/dt = 0$ at $H = 15$, the temperature 15°C is an equilibrium value. If $H > 15$, Equation (1) tells us that $(H - 15) > 0$ and $dH/dt < 0$. If the object is hotter than the room, it will get cooler. Similarly, if $H < 15$, then $(H - 15) < 0$ and $dH/dt > 0$. An object cooler than the room will warm up. Thus, the behavior described by Equation (1) agrees with our intuition of how temperature should behave. These observations are captured in the initial phase line diagram in Figure 9.16. The value $H = 15$ is a stable equilibrium.

We determine the concavity of the solution curves by differentiating both sides of Equation (1) with respect to t:

$$\frac{d}{dt}\left(\frac{dH}{dt}\right) = \frac{d}{dt}(-k(H - 15))$$

$$\frac{d^2H}{dt^2} = -k\frac{dH}{dt}.$$

Since $-k$ is negative, we see that d^2H/dt^2 is positive when $dH/dt < 0$ and negative when $dH/dt > 0$. Figure 9.17 adds this information to the phase line.

The completed phase line shows that if the temperature of the object is above the equilibrium value of 15°C, the graph of $H(t)$ will be decreasing and concave upward. If the temperature is below 15°C (the temperature of the surrounding medium), the graph of $H(t)$ will be increasing and concave downward. We use this information to sketch typical solution curves (Figure 9.18).

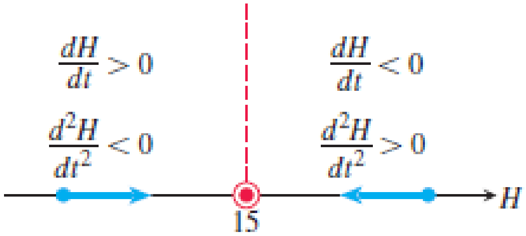


FIGURE 9.17 The complete phase line for Newton's Law of Cooling.

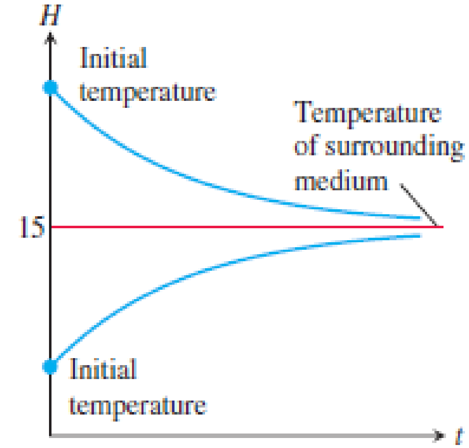


FIGURE 9.18 Temperature versus time. Regardless of initial temperature, the object's temperature $H(t)$ tends toward 15°C, the temperature of the surrounding medium.

Week 13

Assignment 13*

9.1: #1-6,11,14 (try 11 and 14 after learning 7.4 and 9.2)
7.4: #12,13,14,15,30,34,43
9.2: # 2,4,13,16,17
9.3: # 13,15
9.4: #1,7,11,12,14,18

*The questions above are for practice only --- no submission is needed (due to the forthcoming final exam).

Required Reading (Textbook)

- Sections 7.4, 9.1 to 9.4