Slide 10-Vectors II

MAT2040 Linear Algebra

Definition 10.1 (**Linearly independent**) Given a set of vectors $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$, we say that $\mathcal{U}$ is linearly independent if $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$ $(c_1, \cdots, c_n \in \mathbb{R})$ is valid only when $c_1 = c_2 = \cdots = c_n = 0$.

Definition 10.2 (**Linearly dependent**) Given a set of vectors $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$, then $\mathcal{U}$ is linearly dependent if there exists a set of real numbers $c_1, \cdots, c_n$ which are not all zeros $((c_1, \cdots, c_n) \neq (0, 0, \cdots, 0))$, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$.

Some simple examples

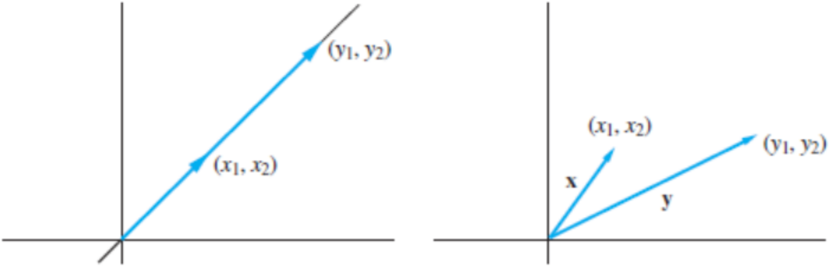
$$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ is linearly independent.}$$

$$\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ is linearly dependent.}$$

$\mathcal{U} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{0}\}$ is linearly dependent.

Geometrical Interpretation: 2D case

For $\mathbb{R}^2$, if $\mathbf{x}$ and $\mathbf{y}$ are linear dependent, there exists c_1, c_2 which are not all zeros, such that $c_1\mathbf{x} + c_2\mathbf{y} = \mathbf{0}$, suppose that $c_1 \neq 0$, then $\mathbf{x} = -\dfrac{c_2}{c_1}\mathbf{y}$, one vector can be written as a scalar multiple of the other. Geometrically, if both vectors are placed at the origin, they will lie along the same line.



(a) **x** and **y** linearly dependent (b) **x** and **y** linearly independent

In matrix notation, we can ask the following question: Does the following homogeneous system above have a unique solution?

$$\begin{bmatrix} \mathbf{u}_1, & \mathbf{u}_2, & \ldots, & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If it has the unique solution, then $\mathcal{U}$ is linearly independent. Otherwise the linear system will have infinite number of solutions, and $\mathcal{U}$ is linearly dependent.

Remark If $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$ is linearly dependent, then there is some vector $\mathbf{u}_s (s = 1, \cdots, n)$ which is the linear combination of other vectors in $\mathcal{U}$. And vice versa.

Brief illustration: $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$ is linearly dependent, then there exist a set of real numbers $c_1, \cdots, c_n$ which are not all zeros $((c_1, \cdots, c_n) \neq (0, 0, \cdots, 0))$, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n = \mathbf{0}$. Suppose that $c_s \neq 0$, then $\mathbf{u}_s = -\frac{c_1}{c_s}\mathbf{u}_1 - \cdots - \frac{c_{s-1}}{c_s}\mathbf{u}_{s-1} - \frac{c_{s+1}}{c_s}\mathbf{u}_{s+1} - \cdots - \frac{c_n}{c_s}\mathbf{u}_n$. Thus, $\mathbf{u}_s$ is the linear combination of other vectors.

Example 10.3

Given the set

$$\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To determine linear independence we first form a relation of linear dependence,

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This will give a linear system

$$\begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 1 & 2 & -6 & 0 \\ -1 & 2 & 1 & 7 & 0 \\ 3 & -1 & -3 & -1 & 0 \\ 1 & 5 & 6 & 1 & 0 \\ 2 & 2 & 1 & 1 & 0 \end{array}\right]$$

can be row reduce into the reduce row-echelon form as

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ is the unique solution. Thus, these four vectors are linearly independent.

Theorem 10.4 (**More Equivalent Conditions for Invertible Matrix**)
Suppose that A is a square matrix. The following are equivalent.

(1). A is invertible (nonsingular, nondegenerate).

(2). The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

(3). A row-equivalent to the identity matrix.

(4). The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(5). A is a product of elementary matrices.

(6). There exists invertible E, such that $EA = I$.

(7). The column vectors of A are linearly independent.

Proof. The equivalence of (1)-(6) can be found in Theorem 8.7. The equivalence of (7) and (4) can be found in the remark 1 below definition 10.2.

Definition 10.5 (**Span of a vector set**) Let $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$, then the span of $\mathcal{U}$ is the set of all linear combinations of $\mathbf{u}_1, \cdots, \mathbf{u}_n$, denoted by $\textbf{Span}(\mathcal{U}) = \{k_1\mathbf{u}_1 + \cdots + k_n\mathbf{u}_n | k_i \in \mathbb{R}\}$ (notation in Steven's book) or $\langle \mathcal{U} \rangle = \{k_1\mathbf{u}_1 + \cdots + k_n\mathbf{u}_n | k_i \in \mathbb{R}\}$ (notation in Beezer's note).

Exercises:

1. What is $\textbf{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$?
2. What is $\textbf{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$?

Observation:
If $\mathbf{u}, \mathbf{v}$ are linearly dependent, then $\textbf{Span}\{\mathbf{u}, \mathbf{v}\} = \textbf{Span}\{\mathbf{u}\}$.

3. What is $\textbf{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$?

$$\textbf{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$$

Geometrically, it means the whole two-dimensional plane.

The solution set of linear system by using Span

Recall: **Example 9.4**

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

The solution in parametric vector form is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}, x_3, x_4 \in \mathbb{R}$$

Using notation of Span, one can rewrite the solution as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \textbf{Span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

One can not simplify it further since $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.

Definition 10.6 (**Spanning set**) Suppose that $\mathcal{U} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$is a subset of $\mathbb{R}^m$, and **Span**$(\mathcal{U}) = \mathbb{R}^m$, then we say that $\mathcal{U}$ is a spanning set of $\mathbb{R}^m$, or $\mathcal{U}$ spans $\mathbb{R}^m$.

Example: Span $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$. Thus, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the spanning set of $\mathbb{R}^2$.