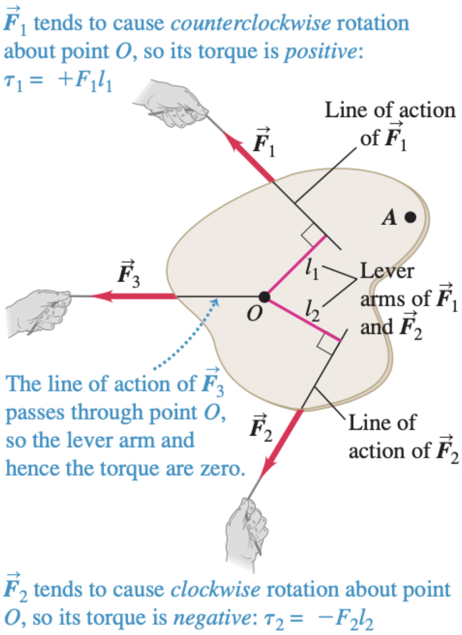
# PHY1001: Mechanics (Week 6, 7)

## 1 Rolling, Torque, Angular Momentum

In this chapter, we will learn about rolling motion, and the torque, which describes the twisting or turning effort of a force, and the conservation of angular momentum as well as the example of the gyroscope.

### 1.1 Torque




We have known the concept of torque ever since the time of the ancient Greek Philosopher Archimedes, who famously said, "Give me a place to stand, and I shall move the earth." Torque is the quantity which measures the tendency of a force cause or change a body's rotational motion. Usually, we use the Greek letter $\tau$ (tau) for torque and define the torque as

$$\text{Scalar form def:} \quad \tau = Fl \tag{1}$$

where $F$ is the magnitude of the force and $l$ is called the lever arm or the moment arm, which is the perpendicular distance of the line of action from the axis.

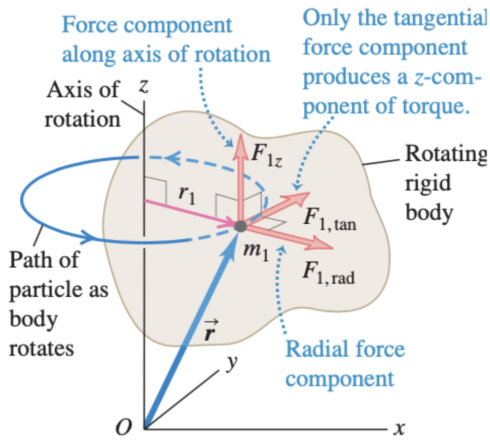
> **Torque is always measured about a point**
> For example, the torque defined above is defined with respect to point $O$. If you change the axis to point $A$, then the corresponding values of torque change as well. You must specify the axis point by saying "the torque of $\vec{F}$ w.r.t. point $X$", or "the torque of $\vec{F}$ about point $X$."

As shown above, the torque has two different directions, namely, counterclockwise ($\tau > 0$) and clockwise $\tau < 0$. Again, we can use cross product to define the direction of torque, similar to the case of angular velocity, as follows

$$\text{Vector form def:} \quad \vec{\tau} = \vec{r} \times \vec{F}. \tag{2}$$

Daily life application: Bring out a bottle water, and try to unscrew (open) the cap with your right hand. Is the torque positive or negative? Similar rule can be applied to screws and bolts.

### 1.2 Torque and Angular Acceleration




To develop the relation between the torque and the angular acceleration, we imagine the rotating body is made up of a number of particles with mass $m_1, m_2, \cdots$ at distance $r_1, r_2, \cdots$ from the axis of rotation.

Consider the net force $F_1$ acting on the particle 1. It has a component $F_{1,rad}$ along the radial direction, a component $F_{1,tan}$ that is tangent to the circle of radius $r_1$ in which the particle moves as the body rotates, and a component $F_{1z}$ along the axis of rotation. The only component responsible for the rotation about the $z$ axis through point $O$ is the tangential component. Newton's second law for the tangential component is

$$F_{1,tan} = m_1 a_{1,tan} \quad \text{with} \quad a_{1,tan} = r_1 \alpha_z\,, \tag{3}$$

$$\tau_1 = F_{1,tan} r_1 = m_1 a_{1,tan} r_1 = m_1 r_1^2 \alpha_z, \tag{4}$$

$$\tau = \sum_i \tau_i = \sum_i m_1 r_1^2 \alpha_z = I\alpha_z, \tag{5}$$

where sum over all the particles and used the definition of the moment of inertia. Here some comments are in order.

1. $I$ is the same moment of inertia defined in the last chapter for kinetic energy.

2. $\tau = \sum_i \tau_i$ only depends on the external forces, since the torque from internal forces always come in pairs, and they are equal in magnitude and opposite in direction, thus give zero.

3. For the gravitational force, since $g$ is the same at every point in the body, we can show that center of gravity = center of mass. The torque due to gravity

is the same as if all the mass were concentrated at the center of the mass of the body.

$$\vec{\tau}_g = \sum_i \vec{\tau}_i = \sum_i \vec{r}_i \times m_i \vec{g} \quad (6)$$

$$= \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \times M\vec{g} = \vec{r}_{cm} \times M\vec{g}, \quad (7)$$

where $M = \sum_i m_i$ is the total mass of the system.

## 1.3 Rigid Body Rotation about a Moving Axis

In this case, the motion of the body is combined translation and rotation. The key to understanding such situation is this: Every possible motion of a rigid body (at any moment) can be represented as a combination of translational motion (should be viewed as general motion) of the reference axis (point R) and rotation about an axis through the point R.

Translating the above statement into math gives the following expression for the velocity $v_i$ of a point $i$ in a rigid body

$$\vec{v}_i = \vec{v}_R + \vec{\omega} \times \vec{r}_i, \quad (8)$$

where $\vec{v}_R$ is the velocity of the axis point $R$ and $\vec{r}_i$ represents the vector from $R$ to the point $i$.

1. The above formula can be understood from the relative velocity formula $\vec{v}_A = \vec{v}_B + \vec{v}_{A/B}$ by choosing $B = R$ and $A = i$. Since the distance between two points in a rigid body never changes, $\vec{v}_{A/B}$ can only be tangential, therefore $\vec{v}_{i/R} = \vec{\omega} \times \vec{r}_i$. This formula is only true for the idealized model of rigid bodies. (Note that the magnitude of $\vec{r}_i$ (the position relative to the axis) can NOT change, only the direction can vary.)

2. In principle, $R$ can be any points on the rigid body and it can even be a point outside the body (in the "body extended"). Out of all possible choices for $R$, there are two special points which can greatly simplify the calculation. First one is the center of mass, while the second one is called instant center, or instantaneous axis with $v_R = 0$.

### 1.3.1 Choose COM as the axis

If we choose the center of mass as the rotational axis, then

$$\vec{v}_i = \vec{v}_{cm} + \vec{\omega} \times \vec{r}_i', \text{with} \quad \vec{r}_i' = \vec{r}_i - \vec{r}_{cm}. \quad (9)$$

In this case, $\vec{r}_i'$ represents the relative position vector w.r.t. the center of mass. We decompose the motion of point $i$ as the combination of center of mass motion + rotation about the COM.

Then, the total kinetic energy of the rigid body can be written as the sum of two parts

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2. \quad (10)$$

where $I_{cm}$ is the moment of inertia about COM. The first part of the kinetic energy is associated with the translational motion of the COM, while the second part is corresponding to the rotation about the COM.

To show the above result, one can write

$$K = \sum_i K_i = \frac{1}{2}\sum m_i v_i^2 \quad (11)$$

$$= \frac{1}{2}\sum m_i\left(\vec{v}_{cm} + \vec{\omega} \times \vec{r}_i'\right)\cdot\left(\vec{v}_{cm} + \vec{\omega} \times \vec{r}_i'\right) \quad (12)$$

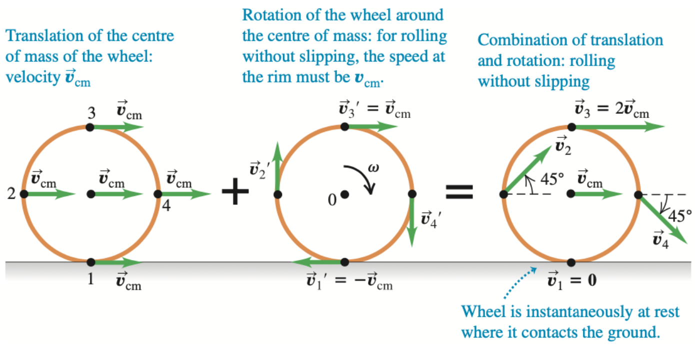
$$= \frac{1}{2}\sum m_i v_{cm}^2 + \frac{1}{2}\sum m_i r_i'^2\omega^2 + \vec{v}_{cm}\cdot\left(\vec{\omega} \times \sum m_i \vec{r}_i'\right)$$

$$= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2. \quad (13)$$

It is important to note that the cross term in the above derivations vanishes following from $\sum m_i \vec{r}_i' = \sum m_i(\vec{r}_i - \vec{r}_{cm}) = 0$ by definition.

### 1.3.2 Choose instant center as the axis

The instantaneous axis (instant center point) is the location which makes $\vec{v}_R = 0$ as shown in the figure below. In fact, at any given moment, we can find a location in the space (not necessary on/in the rotating body), where $\vec{v}_i = \vec{\omega} \times \vec{r}_i''$ with $\vec{r}_i''$ being the position vector w.r.t. the instant center. That is to say that at this instant/moment in time, all the points of this rigid body are rotating about this point of instantaneous axis, and this axis itself is in fact at rest.




Rolling without slipping, as shown above, is a typical example of combined translation and rotation and instant center. Suppose the surface on which the wheel rolls is at rest, and the point on the wheel that contacts the surface must be instantaneously at rest so that it does not slip . This leads to condition for rolling without slipping

$$v_{cm} = \omega R. \quad (14)$$

Thus point 1, the point of contact, is instantaneously at rest. At any instant we can think of the wheel as rotating about an "instantaneous axis" of rotation that passes through the point of contact with the ground. The angular velocity $\omega$ is the same for this axis as for an axis through the center of mass; an observer at the center. The whole wheel is rotating about it with $I_1 = I_{cm} + MR^2$ with no translational motion. Thus, the kinetic energy of the wheel is then

$$K = \frac{1}{2}I_1\omega^2 = \frac{1}{2}(I_{cm} + MR^2)\omega^2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2, \quad (15)$$

which is in agreement with the picture in the center of mass, as expected. There are further discussion in the homework set 6 for you to get familiar with the concept of instantaneous axis.

### 1.3.3 Combined Translation and Rotation: Dynamics and an Example

Let us now analyze the combined translational and rotational motions of a rigid body from the standpoint of dynamics. As we discussed in previous chapter, the body's acceleration is determined by the sum of external forces

$$M\vec{a}_{cm} = \sum \vec{F}_{ext}. \tag{16}$$

In the meantime, the rotational motion about the center of mass is described by the rotational analog of Newton's second law

$$\sum \vec{\tau}_z = I_{cm}\alpha_{cm}, \tag{17}$$

where the axis is chosen to be the center of mass.

One should note that usually the formula $\sum \vec{\tau}_z = I_{cm}\alpha_{cm}$ under the assumption that the axis of rotation was stationary. In fact, one can extend the validity of this equation and use it even when the axis of rotation moves, provided the following two conditions are met:

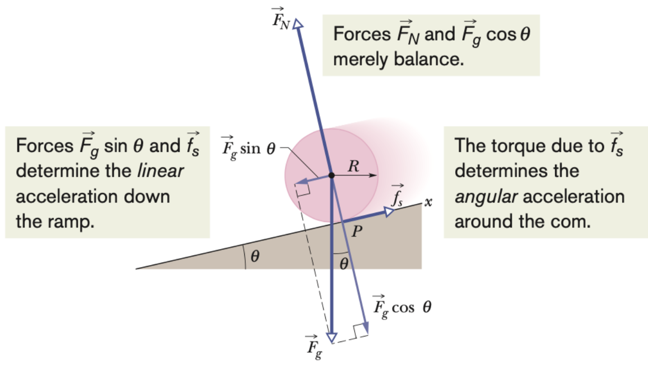
1. The axis through the center of mass must be an axis of symmetry. (Like spherical or cylindrical shapes)

2. The axis must not change direction. (See example of gyroscope)

Note that in general this moving axis of rotation is not at rest in an inertial frame of reference when $a_{cm} \neq 0$ in an inertial frame. In case of accelerating axis, one needs to take into account the ficticious "non-inertial force" since now the center of mass frame is a non-inertial frame. If the above two conditions are met, one can show that the "non-inertial force" does not affect the rotational motion about the CM axis.

In contrast, one can always use the instant center as the axis in an inertial frame. This is because the instant center is at rest by definition. The instantaneous axis is the stationary axis that we used to derive the rotational version of the Newton's 2nd law $\sum \vec{\tau} = I_P\alpha$. In this case, the location of the instant center needs to be determined first.

Example: ∗∗ Acceleration of a rolling ball
A solid ball rolls without slipping (also called rolling without sliding) down a ramp, which is inclined at an angle $\theta$ to the horizontal (Fig. below). What are the ball's acceleration and the magnitude of the friction force on the spherical ball?




This problem is a typical problem involving the torque and the physics of rolling without slipping. Let us try to use three difference methods to solve it. These three methods teach us different aspect of the related physics.

1. Energy Conservation Method
When the solid ball rolls without slipping ($v = \omega R$) down a ramp, the contact point has zero velocity w.r.t. the ramp. Thus, there is no kinetic friction since there is no relative motion between the contact point and the ramp. In this process, the mechanical energy is conserved. Therefore, it gives

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I_{cm}\frac{v^2}{R^2} + mgx\sin\theta = \text{constant}. \tag{18}$$

Differentiating the above equation w.r.t. time yields

$$\frac{7}{5}Mv\frac{dv}{dt} + mg\frac{dx}{dt}\sin\theta = 0 \tag{19}$$
$$a = \frac{dv}{dt} = -\frac{5}{7}g\sin\theta, \tag{20}$$

where $a$ is negative since its direction is parallel to the ramp and pointing downward. In this case, when we write $U = mgh = mgx\sin\theta$, we have already assumed that the direction of the x axis is pointing upward.

Or one can obtain $v^2 = -\frac{10}{7}g\Delta h$ from above energy conservation and simply use $v^2 = 2a\Delta x$ with $\Delta h = \Delta x\sin\theta$ to find $a = -\frac{5}{7}g\sin\theta$.

As to the friction, it is then easy to use Newton's second law and write

$$f - mg\sin\theta = ma, \quad \Rightarrow \tag{21}$$
$$f = mg\sin\theta + ma = \frac{2}{7}mg\sin\theta. \tag{22}$$

It is positive since it is upward.

2. Rotating about the center of mass:
The free-body diagram shows that only the friction force exerts a torque about the center of mass. To make sure the ball roll without sliding, the friction must be upward to produce a counterclockwise torque and balance the downward acceleration at the contact point, namely, $a_{cm} + \alpha_z R = 0$.

Consider the Newton's law for the linear and angular motions

$$f - mg\sin\theta = ma_{cm}, \tag{23}$$
$$fR = I_{cm}\alpha_z = \frac{2}{5}MR^2\alpha_z, \tag{24}$$

which gives

$$f = \frac{2}{7}mg\sin\theta, \quad a_{cm} = -\frac{5}{7}g\sin\theta. \tag{25}$$

3. Rotating about the instantaneous axis
Since the ball does not slip at the instantaneous point of contact with the ramp, this is a static friction force; it gives the ball its angular acceleration and prevents it from slipping. About the instant center, $I_P = I_{cm} + MR^2 = (7/5)MR^2$. If we reverse the positive $x$ direction, and one should write $\alpha_z R = a_{cm}$. In such case, we have

$$-f + mg\sin\theta = ma_{cm}, \tag{26}$$
$$mg\sin\theta R = I_P\alpha_z = \frac{7}{5}MR^2\alpha_z, \tag{27}$$

where the only non-zero torque about the instantaneous axis through point $P$ is provided by the gravity force. The result is the same as obtained above.

## 1.4 Work and Power in Rotational Motion

Similar to the linear motion, the work done by the torque can be written as

$$\begin{aligned} W &= \int_{\theta_1}^{\theta_2} d\theta\tau_z = \int dt\omega I\alpha = I\int \omega dt\frac{d\omega}{dt} \qquad (28)\\ &= I\int_{\omega_1}^{\omega_2}\omega d\omega = \frac{1}{2}I\left(\omega_2^2 - \omega_1^2\right). \qquad (29)\end{aligned}$$

The above equation is nothing but the rotational version of the work-energy theorem. Similar to the power defined in the linear motion $P = F\cdot v$, one can write $P = \tau_z\omega_z$.

## 1.5 Angular Momentum

So far, we have encountered a few rotational quantities, such as angular velocity and rotational kinetic energy. They are analogous to the corresponding physical quantities in the translational (linear) motion of an object. The analog of linear momentum of a particle is angular momentum, a vector quantity denoted as $\vec{L}$, which is defined similar to the torque as follows

$$\vec{L} \equiv \vec{r}\times\vec{P} = \vec{r}\times m\vec{v} \tag{30}$$

for a particle with mass $m$ and velocity $v$. Here $\vec{r}$ is the position vector relative to the origin $O$ of an inertial frame.

Let us follow the derivations which relates the change of momentum and external forces. We can show at the rate of change of angular momentum of a particle equals the torque of the net force acting on it, following the use Newton's second law.

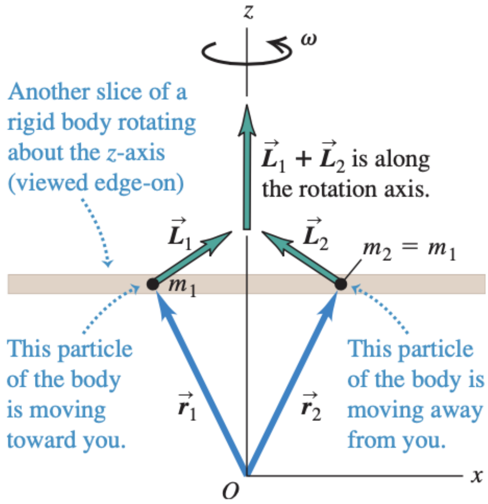
$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}\times\vec{P}}{dt} = \underbrace{\frac{d\vec{r}}{dt}\times\vec{P}}_{\vec{v}\times m\vec{v}=0} + \vec{r}\times\frac{d\vec{P}}{dt} = \vec{r}\times\vec{F} = \vec{\tau}. \tag{31}$$

This formula is more general than the one we derived before ($I\vec{\alpha} = \vec{\tau}$) since it does not assume the axis is fixed. As long as we choose the inertial frame in which Newton's 2nd law is valid, the above formula should be valid and it can be used to describe the rotational motion of moving axis.

For rigid bodies, its angular momentum can be written as the sum of the angular momentum about the center of mass and angular momentum of the center of mass as follows

$$\begin{aligned} \vec{L} &= \sum_i \vec{r}_i\times m_i\vec{v}_i \\ &= \sum_i(\vec{r}_i - \vec{r}_{cm})\times m_i\vec{v}_i + \vec{r}_{cm}\times\sum_i m_i\vec{v}_i \qquad (32)\\ &= \vec{L}_{cm} + \vec{r}_{cm}\times\vec{P}, \qquad (33)\end{aligned}$$

where $\vec{P} \equiv \sum_i m_i\vec{v}_i$ is the total momentum of the rigid body.




In general, $\vec{L}_{cm}$ may not be parallel to $\vec{\omega}$. One needs to pay extra attention to this issue. Suppose $\vec{\omega}$ is along the z axis. But if the z-axis is an axis of symmetry, the perpendicular components for particles on opposite sides of this axis add up to zero. Then $\vec{L}_{cm} = I_{cm}\vec{\omega}$. Again, this formula is only valid when the axis is also the axis of symmetry and it does not change direction. These two conditions are the same as those conditions for $\tau = I_{cm}\alpha$ mentioned previously.

## 1.6 Conservation of Angular Momentum

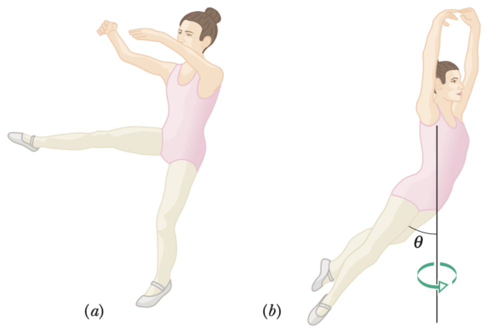
The principle of conservation of angular momentum, which is like conservation of energy and of linear momentum, is a universal conservation law, valid at all scales from atomic and nuclear systems to the motions of galaxies. When $\tau = 0$, then

$$\frac{d\vec{L}}{dt} = 0, \quad L = \text{constant}. \tag{34}$$

When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved). The torques of the internal forces can transfer angular momentum from one body to the other, but they can not change the total angular momentum of the system.

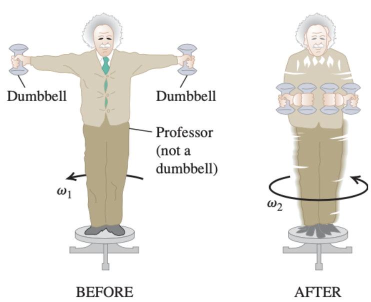
Several examples of angular momentum conservation.

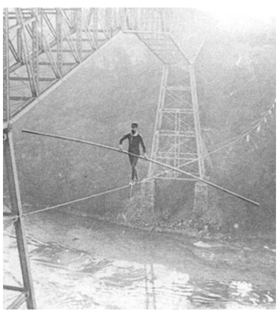
1. Tour jeté is a movement in gymnastics and ballet in which the ballerina (dancer) leaps from one foot, makes a half turn in the air, and lands on the other foot.



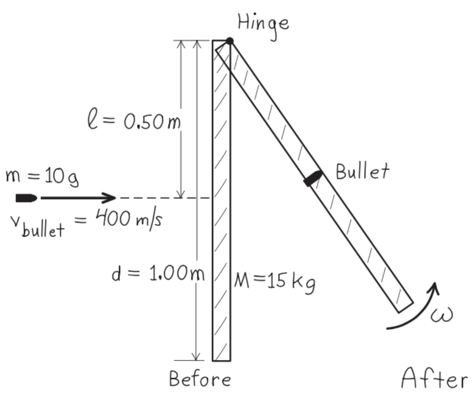

(a)Initial phase of a tour jeté: large rotational inertia and small angular speed. (b) Later phase: smaller rotational inertia and larger angular speed.

2. Anyone can be a ballerina (or a tightrope walker). A physics professor stands at the center of a frictionless turntable with arms outstretched and a dumbbell in each hand. When he pulls the dumbbells in to his stomach, his angular velocity increases a few times and he spins as fast as a ballerina. Similarly, the tightrope walker (see picture from 1890) use the moment of inertia of a long rod to increase the total moment of inertia and for better balance.



3. Angular momentum in a collision.




Find the door's angular speed. (Answer: $0.40\, rad/s$) According to angular momentum conservation about the Hinge of the door, one should find that the initial $L_i$ (before the bullet hits the door) equals the final $L_f$ (after the bullet hits the door). Thus

$$L_i = mvl = L_f = I\omega, \quad \Rightarrow \quad \omega = \frac{mvl}{I}, \tag{35}$$

$$I = I_{door} + I_{bullet} \simeq \frac{Md^2}{3} = 5\, kgm^2, \tag{36}$$

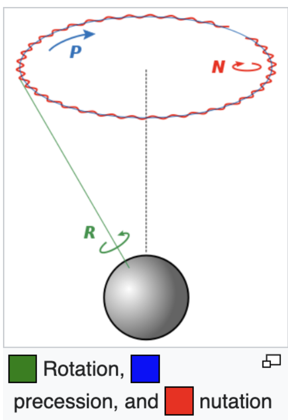
where one can neglect $I_{bullet}$ in $I$ since $I_{door} \gg I_{bullet}$.

## 1.7 Gyroscopes and Precession

A rigid body has six degrees of freedom (dof), which can be counted as follows. Pick up a point A on the rigid body, and we need three parameters $x_A, y_A, z_A$ to specify its location in the space. Then choose a point B on the rigid body, we have three parameters $x_B, y_B, z_B$ plus one constraint $d_{AB} = const$ for point B. The constraint arises from the consideration that the distance $d_{AB}$ between A and B can not change in a rigid body. Similarly, the third point C has three coordinates $x_C, y_C, z_C$ but they are subject to two constraints $d_{AC} = const$ and $d_{BC} = const$. Remember if you consider a fourth point D, you will find that it brings three coordinates together with three constraints, so there are no more new parameters needed. So the total number of degrees of freedom equals the number of parameters minus the number of constaints, which is six.

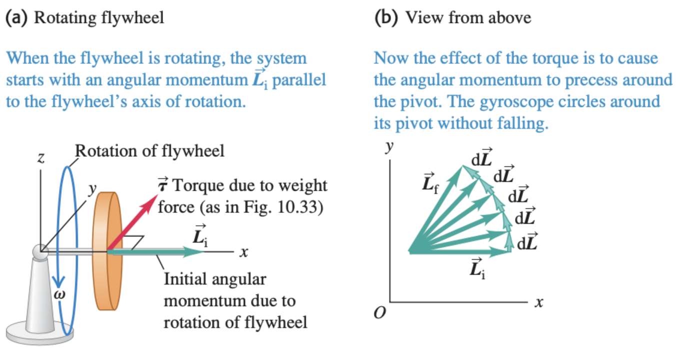
Question: How about the dofs for a two dimensional rigid body? How many of which are linear and angular dofs? (Answer: 3=2+1) How about the case in four dimensional space? (Answer: 10=4+6)

Among these six dofs, three of them are related to the translational motion which can be viewed as the linear motion of the center of mass. The other three dofs are associated with angles and thus angular motions, namely, rotation, precession, and nutation.




We have learned the intrinsic rotation (spin) which is the spinning about a fixed axis. The precession is the rotation of the spin axis about another axis, while the nutation is the small oscillation normal to the precession as shown in the above figure. In the following, let us study a very interesting example of precession without involving too much math.

Let us consider a gyroscope supported at one end. (It is a device used for measuring or maintaining orientation and angular velocity with a heavy flywheel. It has tremendous applications in navigation.) The horizontal circular motion of the flywheel and axis is called precession.




To study this strange phenomenon of precession, we must remember that angular velocity, angular momentum, and torque are all vector quantities. If the flywheel in the figure above is initially not spinning, its initial angular momentum is zero. Then if we place it on the pivot as shown above, it certainly will start to fall down due to the torque $\vec{\tau} = \vec{r} \times \vec{w}$ created by its weight around the pivot. During the fall, as the wheel gains more angular velocity, it develops an angular momentum pointing to the y direction.

Now let's see what happens if the wheel rotates very fast around its symmetry axis. Now the change in angular momentum $d\vec{L} = \vec{\tau}dt$ is perpendicular to the angular momentum $\vec{L}$ of the wheel. Recall the centripetal force in the case of uniform circular motion. Now we see that the effect of the torque is change $\vec{L}$ in such a way that it rotates in a circle, which is called precession. Replace $\vec{p}$ with $\vec{L}$ and the centripetal force $\vec{F} = mv^2/R$ with $\vec{\tau}$ in the physical picture of uniform circular motion, and you will see that precession is simply the rotational analog of uniform circular motion. As a result, the gyroscope can circle around without falling down. The angular velocity of the precession can be computed in analogous to the uniform circular motion a point particle as follows

$$\text{Rotation of } \vec{v}\text{: } \omega = \frac{|d\vec{v}/dt|}{v} = a/v = v/R, \tag{37}$$

$$\text{Rotation of } \vec{r}\text{: } \omega = \frac{|d\vec{r}/dt|}{r} = \frac{|r\omega\hat{\theta}|}{r} = \omega, \tag{38}$$

$$\text{“Rotation" of } \vec{L}\text{: } \Omega = \frac{d\phi}{dt} = \frac{|dL/dt|}{L} = \frac{\tau}{I\omega} = \frac{mgr}{I\omega}, \tag{39}$$

where the rotational analog of Newton's 2nd law ($d\vec{L}/dt = \vec{\tau}$) has been used and $r$ is the distance of the center of mass from the pivot. As shown in the above diagram, the flywheel axis of the gyroscope has turned through a small angle $d\phi = dL/L$. The rate at which the axis moves, denoted as $\Omega = d\phi/dt$, is called the precession angular speed. Thus the precession angular speed is inversely proportional to the angular speed of spin about the axis. A rapidly spinning gyroscope precesses slowly; if friction in its bearings causes the flywheel to slow down, the precession angular speed increases!

Furthermore, by analyzing the forces acting on the wheel, one can conclude that the pivot must provide the force to balance the weight of the wheel and a centripetal force $M\Omega^2 r$ associated with the precession.

One key assumption made in the above analysis is that the angular momentum vector $\vec{L}$ is purely horizontal and sufficiently large, and $\omega \gg \Omega$. When the precession is not slow, additional effects show up, including an up-and-down wobble or nutation of the flywheel axis that's superimposed on the precessional motion. You can see nutation occurring in a gyroscope as its spin slows down. In this case, the motion of the wheel becomes quite complicated and the vertical component of the angular momentum can no longer be neglected. (Of course, this is much beyond the scope of this course.)

This very interesting and non-intuitive motion of the axis is called precession. Precession is found in nature as well as in rotating machines such as gyroscopes. The earth itself is precessing; its spin axis (through the north and south poles) slowly changes direction, going through a complete cycle of precession every 26,000 years. This is due to the fact that the earth is slightly elliptic and thus the gravitational influences of the Sun and moon can generate small torques which results in the slow precession of the earth.

## 1.8 Summry of Rotation

In the study of last two chapters, we have introduced several important physical quantities related to the rotation as follows (Pay attention to the pattern $\vec{r} \times \cdots$)

$$\omega = \frac{\vec{r} \times \vec{v}}{r^2}, \tag{40}$$

$$\vec{\tau} = \vec{r} \times \vec{F}, \tag{41}$$

$$\vec{L} = \vec{r} \times \vec{P} = \vec{r} \times M\vec{v}. \tag{42}$$

These three rotational quantities are usually called axial vectors (pseudovectors) since they are defined as the cross product of the usual polar vectors such as $\vec{v}$, $\vec{r}$, $\vec{F}$ and $\vec{P}$. The difference between axial vectors and polar vectors is that polar vectors change sign when coordinates $x, y, z$ are changed to $-x, -y, -z$, while axial vectors do not.

When we consider rotational problem, we usually start by writing down the rotation theorem (analog of 2nd law)

$$\frac{d\vec{L}}{dt} = \vec{\tau}. \tag{43}$$

When the axis through the center of mass of a rigid body happens to also be the axis of symmetry. We can normally find that

$$\vec{L}_{cm} = I_{cm}\vec{\omega} \quad \Rightarrow \quad \frac{d(I\vec{\omega})}{dt} = \vec{\tau}. \tag{44}$$

This is to say $\vec{L} || \omega$. (Note that $\vec{L}$ and $\omega$ are not necessarily parallel to each other in general.) We have discussed two special cases.

1. $\vec{\tau} || \omega$: This can be rolling of balls or cylinders with stationary or moving axis. We can either choose the COM or the instant center to be the axis.

2. $\vec{\tau} \perp \omega$: This leads to the precession, which describes the rotation of $\vec{L}$ ($I\omega$). Similar to the uniform circular motion, the magnitude of $\vec{\omega}$ remains the same while the direction is changing by $\tau$ constantly.