Slide 8-Finding Matrix Inverse by using elementary matrices/row operations

MAT2040 Linear Algebra

Definition 8.1 (**Row Equivalent Matrices**) Let $A \in \mathbb{R}^{m \times n}$ and suppose we apply a series of elementary row operations $\mathrm{op_1}, \mathrm{op_2}, \cdots, \mathrm{op_k}$ on A and obtain the matrix B. Then, matrix A is said to be **row equivalent** to matrix B.

Moreover, suppose the corresponding elementary matrix for elementary row operation $\mathrm{op_i}$ $(i = 1, \cdots, k)(i = 1, \cdots, k)$, then

$$EA = B,$$

where $E = E_k E_{k-1} \cdots E_1$.
Illustration:

$$A \xrightarrow{op_1} A_1 \xrightarrow{op_2} A_2 \xrightarrow{op_3} \cdots \xrightarrow{op_k} A_k = B.$$

Thus, $B = E_k A_{k-1} = E_k E_{k-1} A_{k-2} = \cdots = E_k E_{k-1} \cdots E_2 A_1 = E_k E_{k-1} \cdots E_1 A = EA$.

Theorem 8.2 (**Equivalent conditions for invertible matrix**)
$A \in \mathbb{R}^{n \times n}$, the following are equivalent:
(1) A is invertible,
(2) the linear system $A\mathbf{x} = \mathbf{0}$ has only a trivial solution,
(3) matrix A is row equivalent to I_n,
(4) A is a product of elementary matrices,
(5) there exists an invertible matrix $E \in \mathbb{R}^{n \times n}$ such that $EA = I_n$.
(6) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$.

Proof.
(1)⇒(2)
Since A is invertible, then $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

(2)⇒(3)
Suppose

$$[A|\mathbf{0}] \xrightarrow{\text{elementary row operations}} [B|\mathbf{0}](reduced\ row - echelon\ form)$$

Since the linear system $A\mathbf{x} = \mathbf{0}$ has only a trivial solution, thus each row of B must has a leading 1.
Thus, $B = I_n$.

(3)$\Rightarrow$ (4)
By theorem 8.2, there are elementary matrices $E_1, \cdots, E_k$, such that $E_k \cdots E_1 A = I_n$. Thus $A = E_1^{-1} \cdots E_k^{-1}$ is a product of elementary matrices since $E_1^{-1}, \cdots, E_k^{-1}$ are also elementary matrices.

(4)$\Rightarrow$ (5)
Suppose $A = E_1 \cdots E_k$ ($E_1, \cdots, E_k$ are elementary matrices), then $E_k^{-1} \cdots E_1^{-1} A = I_n$. Let $E = E_k^{-1} \cdots E_1^{-1}$, then E is invertible and $EA = I_n$.

(5)$\Rightarrow$ (1)
Since $EA = I_n$ and E is invertible, then $A = E^{-1}$ is also invertible, and $A^{-1} = E$.

(1)$\Rightarrow$ (6)
$A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution for any $\mathbf{b}$.
(6)$\Rightarrow$ (1)
If A is singular, then $A\mathbf{x} = \mathbf{0}$ has infinity many solutions from (2) by the contrapositive statement. Now taking $\mathbf{z} \neq \mathbf{0}$ is the solution of $A\mathbf{x} = \mathbf{0}$, and suppose that $\mathbf{y}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$, then $A(\mathbf{y} + \mathbf{z}) = A\mathbf{y} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$, thus $\mathbf{y} + \mathbf{z}$ is also the solution of $A\mathbf{x} = \mathbf{b}$. But $\mathbf{y} \neq \mathbf{y} + \mathbf{z}$. This is a contradiction.

Remark.

If A is invertible, then A is row equivalent to I, i.e., $A \xrightarrow{\mathrm{row\ \ op_1, \cdots, op_k}} I$, suppose the corresponding elementary matrices for the row operations $\mathrm{op_1}, \mathrm{op_2}, \cdots, \mathrm{op_k}$ are $E_1, E_2, \cdots, E_k$, then $I = E_k \cdots E_1 A$. Thus, $A^{-1} = E_k \cdots E_1 I$. Thus, for the same series of elementary row operations, it will transforms a nonsingular matrix A to I and transform I to A^{-1}. This suggests a method to find A^{-1} by performing row operations for augmented matrix $[A|I]$.

Method to find A^{-1} (A is invertible)

$[A|I] \xrightarrow{\text{Gauss Jordan elimination}} [I|P]$, then $P = A^{-1}$.

Illustration: Suppose elementary row operations $\mathrm{op_1}, \mathrm{op_2}, \cdots, \mathrm{op_k}$ (the corresponding elementary matrices are $E_1, E_2, \cdots, E_k$) are used in the Gauss-Jordan elimination for $[A|I]$ to obtain the reduced row echelon form $[I|P]$. Then, $E_k \cdots E_1[A|I] = [I|P] \Rightarrow E_k \cdots E_1 A = I, E_k \cdots E_1 I = P \Rightarrow P = E_k \cdots E_1 = A^{-1} \Rightarrow P = A^{-1}$.

Example 8.3 Find the inverse of the following matrix

$$(1) \quad A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

Perform Gauss-Jordan elimination:

$$[A|\mathrm{I}] = \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

$$(2) \quad A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

$$[A|\mathrm{I}] = \left[\begin{array}{ccc|ccc} -7 & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & -12 & -9 \\ 0 & 1 & 0 & \frac{13}{2} & 8 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{5}{2} & 3 & \frac{5}{2} \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

Method to find X such that $AX = B$ (A is invertible)

$[A|B] \xrightarrow{\text{Gauss Jordan elimination}} [I|X]$, then $X = A^{-1}B$.

Illustration: Suppose elementary row operations $\mathrm{op}_1, \mathrm{op}_2, \cdots, \mathrm{op}_\mathrm{k}$ (the corresponding elementary matrices are $E_1, E_2, \cdots, E_k$) are used in the Gauss-Jordan elimination for $[A|B]$ to obtain the reduced row echelon form $[I|P]$. Then, $E_k \cdots E_1[A|B] = [I|P] \Rightarrow E_k \cdots E_1 A = I, E_k \cdots E_1 B = X \Rightarrow E_k \cdots E_1 = A^{-1} \Rightarrow X = A^{-1}B$.

Example 8.4 Find the solution X such that $AX = B$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -3 \\ -4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 15 & -15 & -30 \\ 15 & 30 & -15 \\ 5 & -10 & -5 \end{bmatrix}$$

Set

$$[A|\mathrm{B}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 15 & -15 & -30 \\ -2 & -1 & -3 & 15 & 30 & -15 \\ -4 & 5 & 6 & 5 & -10 & -5 \end{array}\right]$$

and perform Gauss-Jordan elimination to reduce it into

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -1 & -20 \\ 0 & 1 & 0 & 41 & 14 & -65 \\ 0 & 0 & 1 & -26 & -14 & 40 \end{array}\right]$$

Thus

$$X = \begin{bmatrix} 11 & -1 & -20 \\ 41 & 14 & -65 \\ -26 & -14 & 40 \end{bmatrix}$$

Theorem 8.5 (**Nonsingular Product has Nonsingular Terms**)
Suppose that A and B are square matrices with the same size. The product AB is nonsingular if and only if A and B are both nonsingular.

Proof.
$\Leftarrow$ Since $(AB)^{-1} = B^{-1}A^{-1}$, $(AB)^{-1}$ exists.

$\Rightarrow$ For this portion of the proof we will form the logically-equivalent contrapositive statement: "AB is nonsingular implies A and B are both nonsingular" is equivalent to "A or B is singular implies AB is singular".

Proof for Theorem 8.5

Case (1). Suppose B is singular, then there is z nonzero vector $\mathbf{z}$ s.t. $B\mathbf{z} = \mathbf{0}$, thus

$$AB\mathbf{z} = A(B\mathbf{z}) = A\mathbf{0} = \mathbf{0}$$

Therefore $AB\mathbf{x} = \mathbf{0}$ has a nonzero solution, this means that AB is singular.

Case (2). Suppose B is invertible and A is singular, then if AB is invertible, then there is a invertible matrix C, s.t. $ABC = I$, thus $A = C^{-1}B^{-1}$ which is invertible. This is a contradiction with the assumption A is singular.

Theorem 8.6 (**One-Sided Inverse Verification is Sufficient**) Suppose $A, B \in \mathbb{R}^{n \times n}$. If $BA = I_n$, then $AB = I_n$.

Proof. By theorem 8.5, both A and B are invertible. By theorem 8.2, there exists $E \in \mathbb{R}^{n \times n}$, s.t. $EB = I$. Thus

$$AB = IAB = (EB)AB = E(BA)B = EIB = EB = I.$$