STA2001 Probability and Statistics (I)

Lecture 13

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Review of Last Lecture

Key concepts and/or techniques:

Let X and Y be two continuous random variables and (X, Y) be a pair of RVs with their range denoted by $\overline{S} \subseteq R^2$. Then (X, Y) or X and Y is said to be a bivariate continuous RV.

[Road map]

To study the bivariate continuous random variable

discrete RV $\longrightarrow$ continuous RV		Mathematical expectations
pmf $\longrightarrow$ pdf		mean
joint pmf $\longrightarrow$ joint pdf		variance
marginal pmf $\longrightarrow$ marginal pdf		covariance
conditional pmf $\longrightarrow$ conditional pdf		correlation coefficient

Review of Last Lecture

[Marginal pdf]

The marginal pdf of X and Y are:

$$f_X(x) = \int_{\overline{S_Y}(x)} f(x,y)dy : \overline{S_X} \to (0,\infty)$$

$$\overline{S_Y}(x) = \{y|(x,y) \in \overline{S}\} \text{ for } x \in \overline{S_X}$$

$$f_Y(y) = \int_{\overline{S_X}(y)} f(x,y)dx : \overline{S_Y} \to (0,\infty)$$

$$\overline{S_X}(y) = \{x|(x,y) \in \overline{S}\} \text{ for } y \in \overline{S_Y}$$

Review of Last Lecture

[Covariance and Correlation Coefficient]

Let $g(X, Y)$ be a function of X and Y, whose joint pdf $f(x, y) : \overline{S} \to [0, \infty)$. Then

$$\boxed{E[g(X, Y)] = \iint_{\overline{S}} g(x, y) f(x, y) dxdy}$$

1. covariance $Cov(X, Y)$
2. correlation coefficient $\rho(X, Y)$

[Independent Variables]

Two continuous RVs X and Y are independent if

$$f(x, y) = f_X(x) f_Y(y), \qquad x \in \overline{S_X}, \quad y \in \overline{S_Y}$$

Review of Last Lecture

[Conditional pdf and Conditional Expectation]

The conditional pdf, mean, and variance of Y, given that $X = x$ are

$$h(y|x) = \frac{f(x,y)}{f_X(x)} \quad \text{for} \quad f_X(x) > 0, y \in \overline{S_Y}(x)$$

$$P(Y \in A|X = x) = \int_{y \in A} h(y|x)dy, A \subseteq \overline{S_Y}(x)$$

$$E(Y|X = x) = \int_{\overline{S_Y}(x)} yh(y|x)dy$$

$$\begin{aligned}
Var(Y|X = x) &= E\{[Y - E(Y|X = x)]^2|X = x\} \\
&= \int_{\overline{S_Y}(x)} [y - E(Y|X = x)]^2 h(y|x)dy \\
&= E[Y^2|X = x] - [E(Y|X = x)]^2
\end{aligned}$$

Section 4.5 Bivariate Normal distribution

Pdf of Bivariate Normal Distribution

Definition

Let X and Y be 2 continuous RVs and have the joint pdf

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp[-\frac{1}{2}q(x,y)], x \in \mathbb{R}, y \in \mathbb{R},$$

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \geq 0$$

where $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$ and $|\rho| < 1$. Then X and Y are said to be bivariate normally distributed.

Key components: Scaled exponential function with a quadratic and negative function as its exponent.

Properties of Bivariate Normal Distribution

1. Marginal pdf of X and Y are normal with

$$X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$$

Properties of Bivariate Normal Distribution

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$$X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2)$$

2. Conditional pdf of X given that $Y = y$ is normal with mean

$$\mu_X + \frac{\sigma_X}{\sigma_Y}\rho(y - \mu_Y)$$

and variance

$$(1 - \rho^2)\sigma_X^2$$

i.e.,

$$X|Y = y \sim N\left(\mu_X + \frac{\sigma_X}{\sigma_Y}\rho(y - \mu_Y), (1 - \rho^2)\sigma_X^2\right)$$

Moreover,

$$Y|X = x \sim N\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1 - \rho^2)\sigma_Y^2\right)$$

Proof of the Two Properties

First, recall that

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Clearly, it is equivalent to prove that

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right).$$

To this goal, we arrange $f(x, y)$ as follows:

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \times \\ &\quad \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2 - \frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right) \times h(y|x) \end{aligned}$$

Proof of the Two Properties

$$h(y|x) = \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{(1-\rho^2)\sigma_Y^2}} \exp\left(-\frac{1}{2}\left[\frac{y-[\mu_Y+\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)]}{\sqrt{(1-\rho^2)\sigma_Y^2}}\right]^2\right)$$

Clearly, to complete the proof, we only need to show that

$$\int_{-\infty}^{\infty} h(y|x)dy = 1$$

which is true because $h(y|x)$ is the probability density function of $N(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x-\mu_X), (1-\rho^2)\sigma_Y^2)$.

Actually, $h(y|x)$ is the conditional probability density function of Y given $X = x$ because

$$h(y|x) = \frac{f(x,y)}{f_X(x)}.$$

Properties of Bivariate Normal Distribution

3. Independence $\Longleftrightarrow$ Uncorrelation

- prove that ρ is indeed the correlation coefficient of X and Y
  - $\mathrm{Cov}(X, Y) = \rho\sqrt{Var(X)}\sqrt{Var(Y)} = \rho\sigma_X\sigma_Y$
  - $f(x, y) = h(y|x)f_X(x)$
- Uncorrelation means $\rho = 0 \Longrightarrow$

$$f(x, y) = f_X(x)f_Y(y) \quad \text{means independence}$$

The proof is left as a question in the assignment.

Example 1, page 165

Question

Observe a group of college students. Let X and Y denote their grades in high school and in the 1st year in college, respectively, have a bivariate normal distribution with

$$\mu_X = 2.9, \quad \mu_Y = 2.4, \quad \sigma_X = 0.4, \quad \sigma_Y = 0.5 \quad \text{and} \quad \rho = 0.8$$

Find $P(2.1 < Y < 3.3)$ and $P(2.1 < Y < 3.3 | X = 3.2)$.

Example 1, page 165

Question

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Find $P(2.1 < Y < 3.3)$ and $P(2.1 < Y < 3.3 | X = 3.2)$.

Since $Y \sim N(\mu_Y, \sigma_Y^2) = N(2.4, 0.5^2)$, then

$$P(2.1 < Y < 3.3) = P\left(\frac{2.1 - 2.4}{0.5} < \frac{Y - 2.4}{0.5} < \frac{3.3 - 2.4}{0.5}\right)$$
$$= \Phi(1.8) - \Phi(-0.6) = 0.69$$

Example 1, page 165

Note that

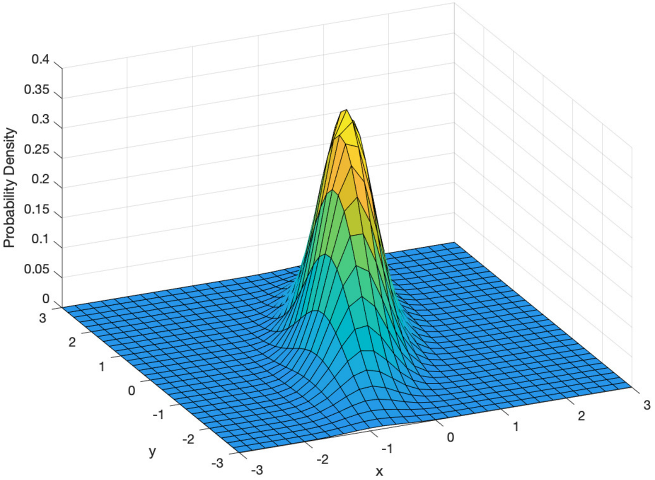
$$Y|X = x \sim N\left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1 - \rho^2)\sigma_Y^2\right)$$

Let $X = 3.2$. Then $Y|X = 3.2 \sim N(2.7, 0.3^2)$

$$\begin{aligned}
&P(2.1 < y < 3.3|X = 3.2) \\
&= P\Big(\frac{2.1 - 2.7}{0.3} < \frac{Y - 2.7}{0.3} < \frac{3.3 - 2.7}{0.3}\Big|X = 3.2\Big) \\
&= \Phi(2) - \Phi(-2) = 0.95
\end{aligned}$$

Geometrical interpretation [P.166]

Let $z = f(x, y)$ and draw it in $x - y - z$ 3-dimensional space



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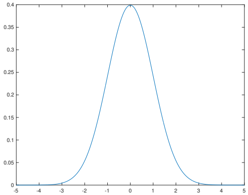

1. Consider $z = f(x_0, y) = f_X(x_0)h(y|x_0)$

- the intersection of the surface $z = f(x, y)$ with the plane $x = x_0$, which is parallel to the yz-plane
- a bell-shaped curve and has the shape of a normal pdf

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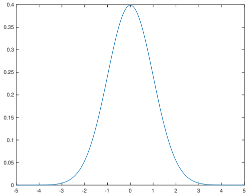


2. Consider $z = f(x, y_0) = f_Y(y_0)g(x|y_0)$

- the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$, which is parallel to the xz-plane
- a bell-shaped curve and has the shape of a normal pdf

Geometrical interpretation [P.166]

3. Consider $z_0 = f(x, y)$ with $0 < z_0 < \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$

- the intersection of the surface $z = f(x, y)$ with the plane $z = z_0$, which is parallel to the xy-plane
- an ellipse

$$\exp[-\frac{1}{2}q(x, y)] = z_0 \cdot 2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}$$

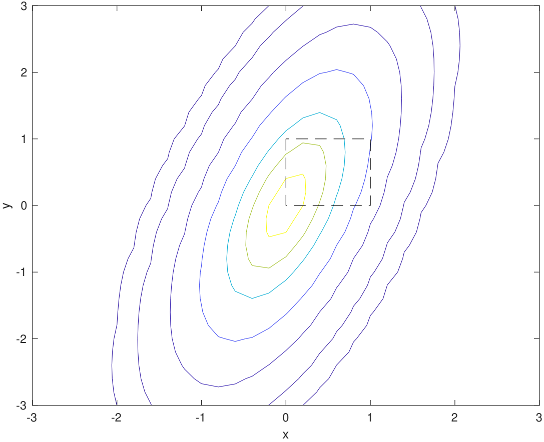
Taking logarithm yields

$$\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 = -2(1-\rho^2)\ln(z_0 2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2})$$

which corresponds to an ellipse.

Geometrical interpretation [P.166]

We can draw these ellipses for different z_0 on the xy-plane, which are called level curves or contours.



Chapter 5. Distribution of Functions of Random Variables

Section 5.1 Function of one random variable

Function of One Random Variable

Question

Let X be a RV of either discrete or continuous type. Consider a function of X, say $Y = u(X)$. Then Y is also a RV and has its pmf or pdf.

How to compute the pmf or pdf of Y?

In what follows, we consider the case where $Y = u(X)$ is a one-to-one mapping.

Discrete case

Let X be a discrete RV with pmf $f(x) : \overline{S_X} \to (0, 1]$, and $Y = u(X)$ be a one-to-one mapping with inverse $X = v(Y)$.

Then the pmf of Y, denoted by $g(y) : \overline{S_Y} \to (0, 1]$ is

- $\overline{S_Y} = \{y | y = u(x), x \in \overline{S_X}\}$
- for any $y \in \overline{S_Y}$,

$$g(y) = P(Y = y) = P(u(X) = y) = P(X = v(y)),$$

Since

$$P(X = x) = f(x), \quad g(y) = f[v(y)] \quad \text{for} \quad y \in \overline{S_Y}$$

Example 1, page 177

Question

Let X have a Poisson distribution with $\lambda = 4$ and its pmf takes the form of

$$f(x) = \frac{4^x e^{-4}}{x!}, \quad x = 0, 1, 2, \cdots$$

If $Y = \sqrt{X}$, what is the pmf $g(y)$ of Y?

Example 1, page 177

Question

Let X have a Poisson distribution with $\lambda = 4$ and its pmf takes the form of

$$f(x) = \frac{4^x e^{-4}}{x!}, \quad x = 0, 1, 2, \cdots$$

If $Y = \sqrt{X}$, what is the pmf $g(y)$ of Y?

First, $\overline{S_Y} = \{0, 1, \sqrt{2}, \sqrt{3}, \cdots\}$, and then

$$Y = u(X) = \sqrt{X} \Longrightarrow X = v(Y) = Y^2$$

$$\begin{aligned} g(y) &= P(Y = y) = P(\sqrt{X} = y) = P(X = y^2) \\ &= f(y^2) = \frac{4^{y^2} e^{-4}}{(y^2)!}, \quad y = 0, 1, \sqrt{2}, \sqrt{3}, \cdots \end{aligned}$$

Continuous Case: the idea

Question

Let X be a continuous RV with pdf $f(x) : [c_1, c_2] \to [0, \infty)$ and $Y = u(X)$ is a function of X.

Our goal is to calculate the pdf of Y, say $g(y)$.

For a continuous RV X, recall the relation between a pdf $f(x)$ and its corresponding cdf $F(x)$:

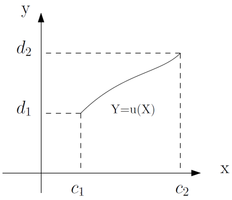
$$F(x) = \int_0^x f(t)dt \Rightarrow F'(x) = \frac{dF(x)}{dx} = f(x),$$

where $F(x)$ is differentiable.

Continuous Case: Case 1

- Case 1: $Y = u(X)$ is continuous, strictly increasing, has inverse function $X = v(Y)$, whose derivative $\frac{dv(y)}{dy}$ exists.

1. Determine the cdf of Y, $G(y)$, $y \in \overline{S_Y}$.

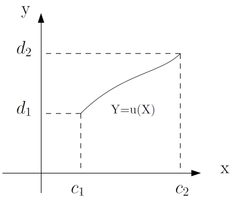


We first find the sample space of Y, $\overline{S_Y}$. Since $Y = u(X)$ is continuous and increasing, $\overline{S_Y} = [d_1,\ d_2]$ with $d_1 = u(c_1)$ and $d_2 = u(c_2)$.

$$G(y) = P(Y \leq y) = P(u(X) \leq y) = P(X \leq v(y)) = \int_{c_1}^{v(y)} f(x)dx$$

Continuous Case: Case 1

2. Determine the pdf of Y, $g(y)$, $y \in \overline{S_Y}$.



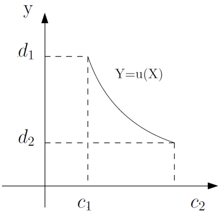
$$\begin{aligned} g(y) = G'(y) = \frac{dG(y)}{dy} &= f(v(y))v'(y) \\ &= f(v(y))\frac{dv(y)}{dy} \\ &= f(v(y))\left|\frac{dv(y)}{dy}\right|, \quad y \in \overline{S_Y} \end{aligned}$$

Continuous Case: Case 2

- Case 2: $Y = u(X)$ is continuous, strictly decreasing and has inverse function $X = v(Y)$, whose derivative $\frac{dv(y)}{dy}$ exists.

1. Determine the cdf of Y, $G(y)$, $y \in \overline{S_Y}$.

   We first find the sample space of Y, $\overline{S_Y}$. Since $Y = u(X)$ is continuous and strictly decreasing $\overline{S_Y} = [d_2,\ d_1]$ with $d_1 = u(c_1)$ and $d_2 = u(c_2)$.



$$G(y) = P(Y \leq y) = P(u(X) \leq y) = P(X \geq v(y))$$
$$= 1 - P(X \leq v(y)) = 1 - \int_{c_1}^{v(y)} f(x)dx$$

Continuous Case: Case 2

2. Determine the pdf of Y, $g(y)$, $y \in \overline{S_Y}$.

$$
\begin{aligned}
g(y) = G'(y) = \frac{dG(y)}{dy} &= -f(v(y))v'(y) \\
&= -f(v(y))\frac{dv(y)}{dy} \\
&= f(v(y))\left|\frac{dv(y)}{dy}\right| \quad y \in \overline{S_Y}
\end{aligned}
$$

Summary: for both strictly increasing and decreasing cases,

$$
g(y) = f(v(y))\left|\frac{dv(y)}{dy}\right|
$$

Example 2, page 174

Let X have the pdf

$$f(x) = 3(1-x)^2, \quad 0 < x < 1.$$

Consider $Y = (1-X)^3$ and calculate the pdf of Y, $g(y)$.

Example 2, page 174

$$Y = u(X) = (1-X)^3 \quad \longrightarrow \quad \text{continuous, strictly decreasing}$$

$$\text{Inverse function} \quad \longrightarrow \quad X = v(Y) = 1 - Y^{\frac{1}{3}}$$

1. The sample space of Y is $\overline{S_Y} = (0,1)$, since $0 < x < 1$.
2. $$g(y) = f(v(y))\left|\frac{dv(y)}{dy}\right| \quad \text{where} \quad \frac{dv(y)}{dy} = -\frac{1}{3}y^{-\frac{2}{3}}$$

$$= 3(1-(1-y^{\frac{1}{3}}))^2\left|-\frac{1}{3}y^{-\frac{2}{3}}\right| = 1,$$

$$0 < y < 1 \quad Y \sim U(0,1)$$

Theorem 5.1-1, page 175

Given a random distribution, it is possible to construct a RV such that this RV has the given random distribution.

Theorem[Random Number Generator]

Let $Y \sim U(0,1)$ and $F(x)$ have the properties of a cdf of a continuous RV with $F(\mathrm{a}) = 0, F(b) = 1$. Moreover, $F(x)$ is strictly increasing such that $F(x) : (a,b) \to [0,1]$, where a could be $-\infty$, b could be ∞. Then $X = F^{-1}(Y)$ is continuous RV with cdf $F(x)$.

Theorem 5.1-1, page 175

Proof: Idea — we need to show $P(X \leq x) = F(x)$

$$P(X \leq x) = P(F^{-1}(Y) \leq x) = P(Y \leq F(x))$$

$$\text{since} \quad \{y | F^{-1}(y) \leq x\} = \{y | y \leq F(x)\}.$$

Note that

$$Y \sim U(0,1) \Longrightarrow P(Y \leq y) \xlongequal{0<y<1} \int_0^y 1 dz = y$$

Therefore,

$$P(X \leq x) = P(Y \leq F(x)) = F(x)$$

Remarks

Theorem 5.1-1 can be used to construct a random number generator for distributions with strictly increasing cdf based on the random generator for a uniform distribution.

Random number generator

1. generator a random number y from $U(0,1)$
2. Take $x = F^{-1}(y)$

Then x is a random number generated from the distribution or RV with cdf $F(x)$.

Example 3

Question

Assume that we know how to generate a random number from $Y \sim U(0,1)$.

- Can we generate a random number from the exponential distribution with parameter θ, whose pdf is given by

$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x \geq 0$$

Example 3

Question

Assume that we know how to generate a random number from $Y \sim U(0,1)$.

- Can we generate a random number from the exponential distribution with parameter θ, whose pdf is given by
$$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x \geq 0$$
- Can we run computer simulation to collect the waiting time until the first customer that arrives at a cinema or a hospital?

Example 3

The cdf of X is

$$F(x) = P(X \leq x) = \int_0^x \frac{1}{\theta} e^{-\frac{t}{\theta}} dt = 1 - e^{-\frac{x}{\theta}}, \quad x \geq 0,$$

which is strictly increasing.

1. If we know how to generate a random number from $Y \sim U(0,1)$, say the random number generated is y.
2. $x = F^{-1}(y)$ is the random number generated from the exponential distribution with θ, where

$$x = F^{-1}(y) = -\theta \ln(1-y), \quad y \in (0,1)$$