STA 2001 Mid-term Solution

1 Multiple choices (100 points)

- **4 points for each correct answer; -1.5 points for each incorrect answer; 0 points for no answer**
- **For each question, only choose (at most) one out of four given choices (A,B,C and D). If you choose more than one choice in one question, your answer will be incorrect and 1.5 points will be deduced.**

1. Which of the following random phenomena is the most suitable to be described by a Binomial distribution?

   (A) Counting the number of failures before the first success in a sequence of Bernoulli trials.

   (B) Counting the total number of successes in a sequence of Bernoulli trials (the order does not matter).

   (C) Counting the number of Bernoulli trials on which the rth success is observed for a given natural number r.

   (D) Counting the number of occurrences that a particular event happens in a given time interval.

   Solution: (B)

2. Which of the following random phenomena is the most suitable to be described by a Poisson distribution?

   (A) Counting the number of failures before the first success in a sequence of Bernoulli trials.

(B) Counting the total number of successes in a sequence of Bernoulli trials (the order does not matter).

(C) Counting the number of Bernoulli trials on which the rth success is observed for a given natural number r.

(D) Counting the number of occurrences that a particular event happens in a given time interval.

Solution: (D)

3. Which of the following random phenomena is the most suitable to be described by a Negative Binomial distribution?

(A) Counting the number of failures before the first success in a sequence of Bernoulli trials.

(B) Counting the total number of successes in a sequence of Bernoulli trials (the order does not matter).

(C) Counting the number of Bernoulli trials on which the rth success is observed for a given natural number r.

(D) Counting the number of occurrences that a particular event happens in a given time interval.

Solution: (C)

4. Which of the following random phenomena is the most suitable to be described by a Geometric distribution?

(A) Counting the number of failures before the first success in a sequence of Bernoulli trials.

(B) Counting the total number of successes in a sequence of Bernoulli trials (the order does not matter).

(C) Counting the number of Bernoulli trials on which the rth success is observed for a given natural number r.

(D) Counting the number of occurrences that a particular event happens in a given time interval.

Solution: (A)

5. Which of the following random phenomena is the most suitable to be described by an uniform distribution?

   (A) When a large number of outcomes are observed, the outcomes have a "bell-shaped" relative frequency distribution.

   (B) The waiting time until the first occurrence of a particular event for an approximated Poisson process.

   (C) When the outcomes have the same probability to appear.

   (D) The sum of the squares of the independent normal random variables.

   Solution: (C)

6. Which of the following random phenomena is the most suitable to be described by a normal distribution?

   (A) When a large number of outcomes are observed, the outcomes have a "bell-shaped" relative frequency distribution.

   (B) The waiting time until the first occurrence of a particular event for an approximated Poisson process.

   (C) When the outcomes have the same probability to appear.

   (D) The sum of the squares of the independent normal random variables.

   Solution: (A)

7. Which of the following random phenomena is the most suitable to be described by an exponential distribution?

   (A) When a large number of outcomes are observed, the outcomes have a "bell-shaped" relative frequency distribution.

   (B) The waiting time until the first occurrence of a particular event for an approximated Poisson process.

   (C) When the outcomes have the same probability to appear.

   (D) The sum of the squares of the independent normal random variables.

   Solution: (B)

8. For events A and B, choose the correct description below, where A' is the complement of A:

   (A) If A and B are mutually exclusive, then A' and B' are mutually exclusive.

   (B) If A and B could happen simultaneously, then A' and B' must also be able to happen simultaneously.

   (C) If A and B are mutually exclusive, $P(A) > 0$ and $P(B) > 0$, then A and B are independent.

   (D) If A and B are independent, then A' and B' are independent.

   Solution: (D)

   For **(A)**, $P(A' \cap B') = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B))$

   $\because P(A \cap B) = 0$, $\therefore P(A' \cap B') = 1 - P(A) - P(B)$ and it could be greater than 0

   For **(B)**, $P(A' \cap B') = 1 - P(A \cup B)$, when $P(A \cup B) = 1$, $P(A' \cap B') = 0$

   For **(C)**, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

   $\because P(A \cap B) = 0$, $\therefore P(A \cup B) = P(A) + P(B)$, not $P(A \cap B) = P(A)P(B)$

   For **(D)**, if $P(A \cap B) = P(A)P(B)$, then $P(A' \cap B') = 1 - P(A \cup B) = (1 - p(A)) - P(B)(1 - P(A)) = (1 - P(A))(1 - P(B))$

9. Roll a fair six-sided die three times. Let $A_1 = \{1 \text{ or } 2 \text{ on the first roll}\}$, $A_2 = \{3 \text{ or } 4 \text{ on the second roll}\}$, and $A_3 = \{5 \text{ or } 6 \text{ on the third roll}\}$. Then what is $P(A_1 \cup A_2 \cup A_3)$?

   (A) $\frac{17}{27}$

   (B) $\frac{18}{27}$

   (C) $\frac{19}{27}$

   (D) $\frac{20}{27}$

Solution: (C)

$$
\begin{aligned}
&P(A_1 \cup A_2 \cup A_3) \\
=&P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) \\
&+ P(A_1 \cap A_2 \cap A_3) \\
=&\frac{1}{3} + \frac{1}{3} + \frac{1}{3} - (\frac{1}{3})^2 - (\frac{1}{3})^2 - (\frac{1}{3})^2 + (\frac{1}{3})^3 \\
=&\frac{19}{27}
\end{aligned}
$$

10. If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, then what is $P(A \cup B)$?

(A) 0.4

(B) 0.6

(C) 0.7

(D) 0.9

Solution: (B)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.3 = 0.6.$$

11. If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, then what is $P(A \cap B')$?

(A) 0.1

(B) 0.3

(C) 0.5

(D) 0.7

Solution: (A)

$$P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1.$$

12. If $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$, then what is $P(A' \cup B')$?

(A) 0.1

(B) 0.3

(C) 0.5

(D) 0.7

Solution: (D)

$$P(A' \cup B') = P((A \cap B)') = 1 - 0.3 = 0.7.$$

13. Suppose $f(x)$ is a probability mass function of a discrete random variable X. Let S be the range space of X. Which of the following statement(s) must be correct?

(i) $f(x) = 0$ for any real number $x \in S$.

(ii) $f(x) \leq 1$ for any real number $x \in S$.

(iii) $\sum_{x \in S} f(x) = 1$

(A) Only (iii) is correct.

(B) Only (i) and (iii) are correct.

(C) Only (ii) and (iii) are correct.

(D) All are correct.

Solution: (C) By definition $f(x) = P(X = x)$.

14. Suppose $f(x)$ is a probability density function of a continuous random variable X. Let S be the range space of X. Which of the following statement(s) must be correct?

(i) $f(x) = 0$ for any real number $x \in S$.

(ii) $f(x) \leq 1$ for any real number $x \in S$.

(iii) $\int_{x \in S} f(x)dx = 1$

(A) Only (iii) is correct.

(B) Only (i) and (iii) are correct.

(C) Only (ii) and (iii) are correct.

(D) All are correct.

Solution: (A) (i) is not correct as we have $P(X = x) = 0$ but $f(x)$ can be greater than 0; (ii) is not correct, consider uniform distribution $U(0, 1/2)$ with $f(x) = 2$.

15. Let A and B be two events, which of the following statement(s) must be correct?

    (i) Even if the events A and B are mutually exclusive, A and B can be dependent.

    (ii) If $A \subset B$, A and B are always dependent.

    (iii) If $A \subset B$ and $P(A) > 0$, A and B are always dependent.

    (A) Only (i) is correct.

    (B) Only (i) and (iii) are correct.

    (C) Only (ii) and (iii) are correct.

    (D) All are correct.

    Solution: (A) (i) is correct. Since mutually exclusive means $A \cap B = \emptyset$, which implies $P(A \cap B) = 0$, while A and B are independent means $P(A \cap B) = P(A)P(B)$, which is not 0 in general. (ii) and (iii) are not correct. When $P(A) = 0$ or $P(B) = 1$, we have $P(A \cap B) = P(A)P(B)$.

16. Flip an unbiased coin eight independent times. What is the probability of getting 4 heads occurring in the 8 trials?

    (A) $\frac{1}{32}$

    (B) $\frac{35}{64}$

    (C) $\frac{35}{128}$

    (D) $\frac{35}{256}$

    Solution: (C) There are $\binom{8}{4}$ possible ways to get the required outcomes, so the probability is

    $$\binom{8}{4} \times \left(\frac{1}{2}\right)^8 = \frac{35}{128}.$$

17. Suppose an urn contains seven black balls and five white balls. We draw two balls from the urn without replacement. Assuming that each ball in the urn is equally likely to be drawn, what is the probability that both drawn balls are black?

 (A) $\frac{2}{7}$
 (B) $\frac{1}{6}$
 (C) $\frac{7}{22}$
 (D) $\frac{7}{24}$

 Solution: (C) Let F and E denote,respectively,the events that the first and second ball draw are black. our desired probability is

$$P(EF) = P(F) * P(E|F) = \frac{7}{12} * \frac{6}{11} = \frac{7}{22}$$

18. You know that a certain letter is equally likely to be in any one of the three different folders. Suppose folder i,$i = 1, 2, 3$ is quick examed. If the letter is in fact in folder i, with the probability α_i (we may have $\alpha_i < 1$) you will find it; If the letter is not in there, you will definitely not find it. Suppose you look in folder 1 and do not find the letter. What is the probability that the letter is in folder 1

 (A) $\frac{1-\alpha_1}{3-\alpha_1}$
 (B) $1 - \alpha_1$
 (C) $\frac{1-\alpha_1}{3}$
 (D) $\frac{\alpha_1}{3}$

 Solution: (A) Let $F_i, i = 1, 2, 3$ be the event that the letter is in folder i, and let E be the event that a search of folder 1 does not come up with the letter. We desire $P(F_1|E)$

$$P(F_1|E) = \frac{P(E|F_1)P(F_1)}{\sum_{i=1}^{3} P(E|F_i)P(F_i)} = \frac{(1-\alpha_1)\frac{1}{3}}{(1-\alpha_1)\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \frac{1-\alpha_1}{3-\alpha_1}$$

19. Four fair coins are flipped. If the outcomes are assumed to be independent, what is the probability that two heads and two tails are obtained?

(A) $\frac{1}{16}$
(B) $\frac{3}{8}$
(C) $\frac{3}{4}$
(D) $\frac{1}{4}$

Solution: (B) $P = \binom{4}{2}(\frac{1}{2})^2(\frac{1}{2})^2 = \frac{3}{8}$

20. Let X be uniformly distributed over (0,1), what is $E[X^3]$

(A) $\frac{1}{2}$
(B) $\frac{1}{3}$
(C) $\frac{1}{4}$
(D) $\frac{1}{8}$

Solution: (C) $E(x^3) = \int_0^1 x^3 dx = \frac{1}{4}$ since $f(x) = 1, , 0 < x < 1$

21. If immigrants to area A arrive at a Poisson rate of ten per week, and if each immigrant is of English descent with probability $\frac{1}{12}$, then what is the probability that no people of English descent will emigrate to area A during the month of February (28 days)?

(A) $e^{-\frac{10}{3}}$
(B) $e^{-\frac{6}{5}}$
(C) $0.5e^{-\frac{6}{5}}$
(D) $0.5e^{-\frac{10}{3}}$

Solution: (A) $\lambda = \mu = 10 \times 4 \times \frac{1}{4} = \frac{10}{3}$

$$P(X = 0) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{e^{-\frac{10}{3}}}{0!} = e^{\frac{-10}{3}}$$

22. Draw a ball from a bag of four balls, labelled by 1, 2, 3 and 4, uniformly at random. Define events $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 4\}$. Which claim is WRONG?

(A) A and B are independent.

(B) B and C are not independent.

(C) A, B and C are pairwise independent.

(D) A, B and C are not mutually independent.

Solution: (B) Note that $P(A) = P(B) = P(C) = \frac{1}{2}$, and $P(A \cap B) = P(A \cap C) = P(B \cap C) = P(\{1\}) = \frac{1}{4}$. Therefore (C) is correct, which implies (A) is correct and (B) is wrong.

23. Flip a coin 3 times and observe the sequence of heads and tails. For example, $\{TTH\}$ means 2 tails (T) in the first two flips and 1 head (H) in the last flip. Which of the following statement describes the event $\{THH, HTH, HHT\}$?

(A) Exactly one head

(B) At most two tails

(C) At least one head

(D) None of the above describes the event.

Solution: (D)

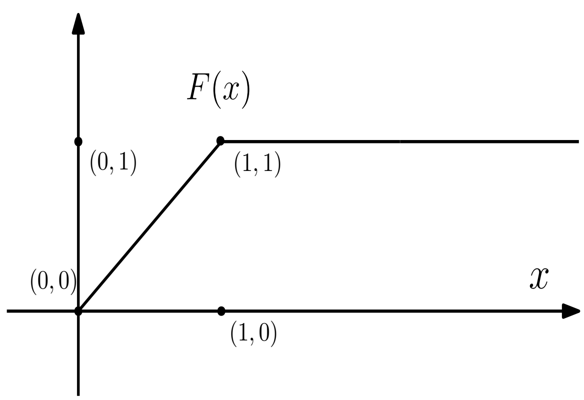

Figure 1: The cumulative distribution function $F(x)$ of X.

24. Let X be a continuous random variable and its cumulative distribution function $F(x) = P(X \leq x)$ be shown in Figure 1. Let Y be a function of X and in particular let $Y = 2X$. Then consider the probability density function $g(y)$ of Y. Which one of the statements is true:

(A)

$$g(y) = \begin{cases} 0, & y \le 0 \\ 1, & y \in (0,1] \\ 0, & y > 1 \end{cases}$$

(B)

$$g(y) = \begin{cases} 0, & y \le 0 \\ 0.5, & y \in (0,2] \\ 0, & y > 2 \end{cases}$$

(C)

$$g(y) = \begin{cases} 0, & y \le 0 \\ 0.5, & y \in (0,1] \\ 0, & y > 1 \end{cases}$$

(D)

$$g(y) = \begin{cases} 0, & y \le 1 \\ 1, & y \in (1,2] \\ 0, & y > 2 \end{cases}$$

Solution: (B) From Figure 1, it can be seen that

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x, & x \in (0,1] \\ 1, & x > 1 \end{cases}$$

We first find the distribution function for Y.

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X \le y/2) \\ &= F_X(y/2) \\ &= \begin{cases} 0, & y \le 0 \\ y/2, & y \in (0,2] \\ 1, & y > 2 \end{cases} \end{aligned}$$

Therefore, the probability density function of Y is the derivative of F_Y, which is

$$g(y) = \begin{cases} 0, & y \le 0 \\ 1/2, & y \in (0,2] \\ 0, & y > 2 \end{cases}$$

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

Figure 2: **The standard normal table**. The entries in this table provide the numerical values of $\Phi(x) = P(X \leq x)$, where X is a standard normal random variable, for x between 0 and 1.49. For example, to find $\Phi(0.72)$, we look at the row corresponding to 0.7 and the column corresponding to 0.02 so that $\Phi(0.72) = 0.7642$. When x is negative, the value of $\Phi(x)$ can be found using the formula $\Phi(x) = 1 - \Phi(-x)$.

25. The annual snowfall at a particular geometrical location is modelled as a normal random variable with a mean of $\mu = 60$ inches and a standard deviation of $\sigma = 20$. Based on the standard normal table as shown in Figure 2, calculate the probability of that this year's snow fall will be at least 80 inches. Which one of the following statements is true?

(A) 0.1587

(B) 0.3349

(C) 0.5614

(D) 0.7836

Solution: (A) Denote the annual snowfall by X. Then by assumption, $X \sim \mathcal{N}(60, 20)$. To answer the question, we have to compute

$$P(X \geq 80) = P\left(\frac{X-60}{20} \geq \frac{80-60}{20}\right) = P\left(\frac{X-60}{20} \geq 1\right)$$

Now let

$$Y = \frac{X - 60}{20}$$

Then $Y \sim \mathcal{N}(0, 1)$, i.e., Y is a standard normal distribution. So

$$P\left(Y \geq 1\right) = 1 - P\left(Y \leq 1\right) = 1 - \Phi(1)$$

Looking up the table yields that $\Phi(1) = 0.8413$ and thus $P(X \geq 80) = 1 - 0.8413 = 0.1587$.