PHY1001: Mechanics (Week 11) Oscillation

In this lecture, we will discuss the general feature of periodic motion briefly and the simple harmonic motion in detail, especially various features of simple harmonic motion. We also introduce several oscillating systems and advanced topics such as the damped oscillations.

1 Describing Harmonic Oscillation

One-dimensional oscillation is one of the most interesting periodic motions. Two essential elements of Harmonic oscillation: displacement from the Equilibrium Position (EP, where $F = 0$) and restoring force (the force which always pushes or pulls the moving object back to EP).

The Restoring Force and Harmonic Oscillation When an object is displaced from the EP, there is a restoring force tends to restore the object back to the EP. Therefore, the object simply oscillates about the EP. Based on Newton's second law, simple harmonic motion (SHM) ($x(t)$ displacement at time t) can be described as follows

$$F = m\frac{d^2x}{dt^2} = -kx, \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{1}$$

$$x(t) = A\cos(\omega t + \delta), \tag{2}$$

where A — the amplitude, which is the maximum magnitude of the displacement from EP.

$\omega \equiv \frac{2\pi}{T} = 2\pi f = \sqrt{\frac{k}{m}}$, — the angular frequency (rad/s)

$T = 2\pi\sqrt{\frac{m}{k}}$ — the period (s), is the time for one cycle.

$f = \frac{1}{T}$ — frequency, number of cycles per second (Hz)

δ — Phase constant (rad) in the phase $\omega t + \delta$

- The minus sign in the above equation of motion for SHM is very important, since it means that the acceleration and displacement always have opposite signs, as the signature of the restoring force. A body that undergoes SHM (with linear restoring force $F = -kx$) is called harmonic oscillator. Not all oscillations are SHM, but SHM is certainly the most simple and useful one. The frequency and the period of SHM is independent of the amplitude.

- We can show $x(t) = A\cos(\omega t + \delta)$ is the solution to $\frac{d^2x}{dt^2} = -\omega^2 x$ by differentiating x twice with respect to t. The first and second derivative of x gives the velocity v_x and a_x, respectively

$$v_x = \frac{dx}{dt} = -\omega A\sin(\omega t + \delta), \tag{3}$$

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \delta) = -\omega^2 x \tag{4}$$

- The amplitude A and the phase constant δ can be determined from the initial position x_0 and the initial velocity v_{0x} of the system. Setting $t = 0$ in $x(t)$ and $v_x(t)$ gives (See HW 10)

$$x_0 = x(t=0) = A\cos\delta, \tag{5}$$

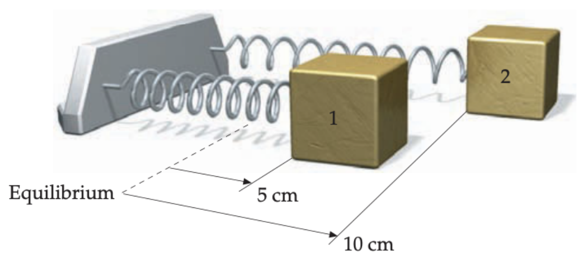
$$v_{0x} = v_x(t=0) = -\omega A\sin\delta. \tag{6}$$

Therefore, it is straightforward to find

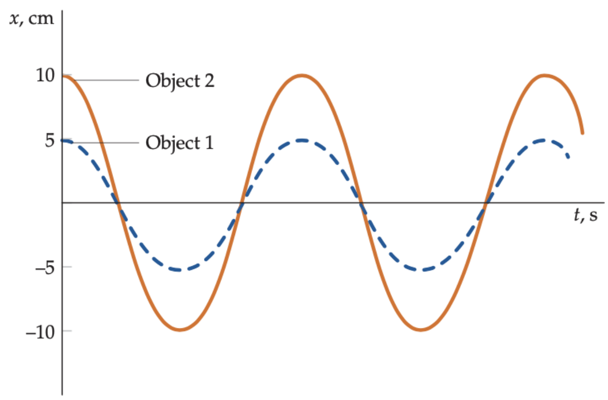
$$A = \sqrt{x_0^2 + \frac{v_{0x}^2}{\omega^2}}, \tag{7}$$

$$\tan\delta = -\frac{v_{0x}}{\omega x_0}. \tag{8}$$

- Clearly, The solution $x(t) = A\cos(\omega t + \delta) = x(t + T)$ describes the periodic motion with the period T, since $\cos(\omega t + \omega T + \delta) = \cos(\omega t + 2\pi + \delta) = \cos(\omega t + \delta)$.

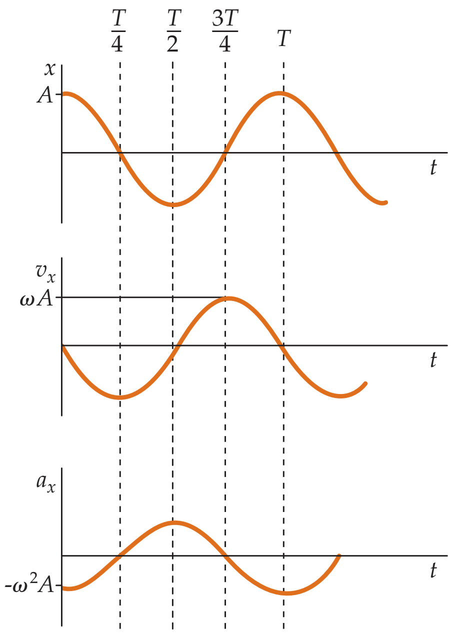


Plots of $x(t)$ for two identical oscillators with different amplitudes. For SHM, ω is independent of the amplitude.

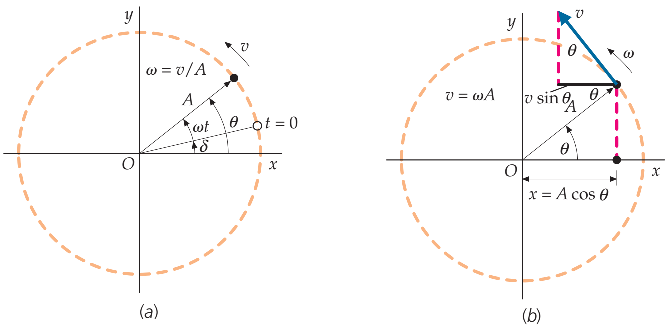


The frequency and thus the period of SHM is independent of the amplitude. This means that the pitch (which corresponds to the frequency) of music instruments does not depend on how loudly the note is played (which corresponds to the amplitude).

If we set the phase $\delta = 0$, we can plot the $x = A\cos\omega t$, $v_x = -A\omega\sin\omega t$, $a_x = -A\omega^2\cos\omega t$ as functions of time as follows.



SHM and Circular Motion



As illustrated above, SHM is the projection of uniform circular motion onto a diameter. Imagine a particle moving with constant speed v in a circle of radius A. Its angular displacement relative to the $+x$ direction is given by

$$\theta = \omega t + \delta \tag{9}$$

where δ is the initial angle and $\omega = v/A$ is the angular velocity. Thus,

$$x = A\cos\theta = A\cos(\omega t + \delta), \tag{10}$$
$$v_x = -\omega A\sin\theta = -\omega A\sin(\omega t + \delta), \tag{11}$$
$$a_x = -\omega^2 A\cos\theta = -\omega^2 A\cos(\omega t + \delta). \tag{12}$$

Similarly, $y(t) = A\sin\theta$ is also SHM with a different phase,

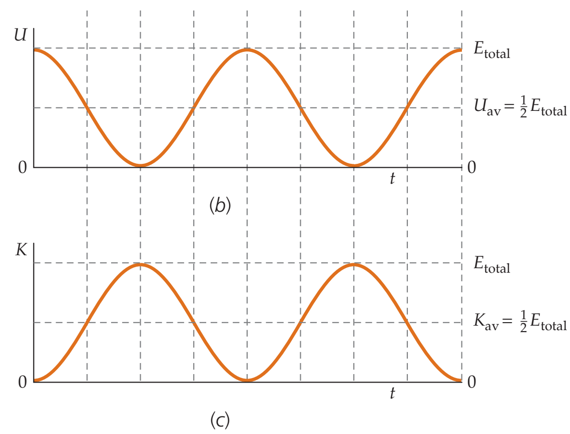
$$y = A\sin\theta = A\sin(\omega t + \delta), \tag{13}$$
$$v_y = \omega A\cos\theta = \omega A\cos(\omega t + \delta), \tag{14}$$
$$a_y = -\omega^2 A\sin\theta = -\omega^2 A\sin(\omega t + \delta). \tag{15}$$

In addition, we can describe the uniform circular motion in terms of unit vectors as follows

$$\vec{r} = x\hat{i} + y\hat{j} = A\cos(\omega t + \delta)\hat{i} + A\sin(\omega t + \delta)\hat{j}, \tag{16}$$
$$\vec{v} = v_x\hat{i} + v_y\hat{j} = -\omega A\sin(\omega t + \delta)\hat{i} + \omega A\cos(\omega t + \delta)\hat{j}, \tag{17}$$
$$\vec{a} = a_x\hat{i} + a_y\hat{j} = -\omega^2\vec{r}. \tag{18}$$

Using the above results, we can first show that $\vec{r}\cdot\vec{v} = 0$, which indicates $\vec{r} \perp \vec{v}$. Also we can find $\vec{a}$ and $\vec{r}$ are antiparallel to each other.

2 Energy in SHM



When an object on a spring undergoes simple harmonic motion, the potential energy and kinetic energy of the system vary with time. Their sum, the total mechanical energy $E = K + U$ is conserved. Let us compute the potential energy and kinetic energy as follows

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \delta), \tag{19}$$
$$K = \frac{1}{2}mv_x^2 = \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \delta). \tag{20}$$

By noting that $\omega^2 = \dfrac{k}{m}$, we can find

$$\begin{aligned} E &= K + U = \frac{1}{2}kA^2\cos^2(\omega t + \delta) + \frac{1}{2}kA^2\sin^2(\omega t + \delta) \\ &= \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2, \quad \text{with} \quad v_{\text{max}} = \omega A, \end{aligned} \tag{21}$$

which is independent of t. When $v = 0$, the total mechanical energy $E = \frac{1}{2}kA^2$ is completely given by the potential energy with the maximum displacement A. When $x = 0$, the total mechanical energy $E = \frac{1}{2}mv_{\text{max}}^2$ is completely given by the kinetic energy with the maximum velocity $v_{\text{max}} = \omega A$. At any given time, the sum of the kinetic and potential energy is a constant which is proportional to the square of the amplitude. It is also useful to note that the average of kinetic energy and potential energy in a period (cycle) are

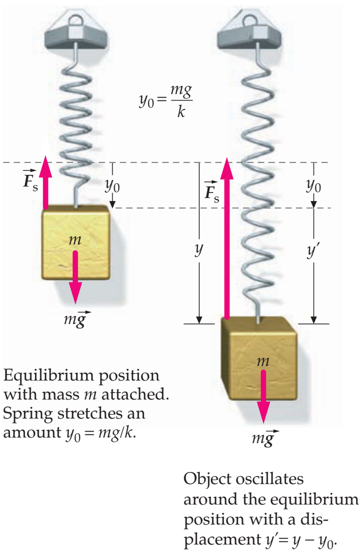
$$\langle U\rangle = \frac{1}{T}\int_0^T \frac{1}{2}kA^2\cos^2(\omega t + \delta)\,dt = \frac{1}{4}kA^2 = \frac{E}{2}, \tag{22}$$
$$\langle K\rangle = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 A^2\sin^2(\omega t + \delta)\,dt = \frac{1}{4}kA^2 = \frac{E}{2}. \tag{23}$$

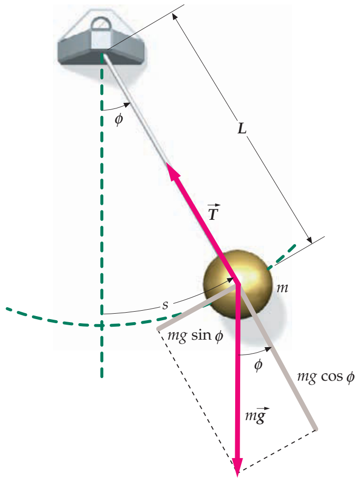
They are both equal to half of the total mechanical energy.

3 Some Oscillating Systems

To study oscillations for any physical system, it is very useful to follow the following four steps. 1. Find the EP; 2. Identify the physics laws relevant to the oscillating system and write down the corresponding equation; 3. Obtain the EOM for the oscillation and write it into the form that you are familiar with. 4. Solve the equation and fix the amplitude and phase according to the initial condition.

Object on a Vertical Spring





When an object hangs from a vertical spring, there is a downward force mg in addition to the force of the spring.

1. Find the EP $\quad \sum F_y = -ky_0 + mg = 0, \quad \Rightarrow y_0 = \dfrac{mg}{k}$
2. Write $y = y_0 + y'$ where y' stands for the oscillation about y_0. (Do not confuse this with the derivative of y.)
$$\sum F_y = -k(y_0 + y') + mg = m\frac{d^2y}{dt^2}$$
3. Obtain the Equation of Motion (EoM) for the small oscillation $\dfrac{d^2y'}{dt^2} = -\dfrac{k}{m}y'$, by noting $\dfrac{d^2y}{dt^2} = \dfrac{d^2y'}{dt^2}$ since y_0 is a constant.
4. Now one can immediately obtain the familiar solution for y'
$y' = A\cos(\omega t + \delta)$ where again $\omega = \sqrt{\dfrac{k}{m}}$. At last you may need to match your solution to initial conditions to fix A and δ if you are asked to do so.

The Simple Pendulum

A simple pendulum consists of a massless string of length L and a massive object with mass m hanging at the lower end of the string.

1. Find the EP: $\phi = 0$.
2. Consider the small oscillation near the EP along the tangential direction in terms of the angle ϕ and use the geometric relation that the arc $s = L\phi$.
$$\sum F_t = -mg\sin\phi = mL\frac{d^2\phi}{dt^2}$$
3. For small ϕ oscillations, $\sin\phi \simeq \phi$, therefore the EoM is
$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\phi.$$
4. Now one can immediately obtain the familiar solution for $\phi = \phi_0\cos(\omega t + \delta)$ where is the maximum angular displacement and $\omega = \sqrt{\dfrac{g}{L}}$. The period of the motion is thus $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{L}{g}}$. Independent of the amplitude ϕ_0.

It is useful to note that the period of a one-meter long ($L = 1$ m) simple pendulum is approximately 2 seconds. It is interesting to note that T does not depend on the mass of the pendulum as long as the string's mass is negligible.

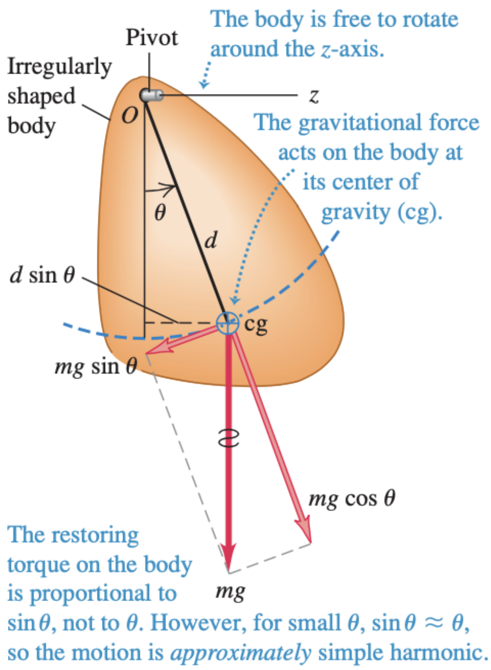
When the amplitude of a pendulum's oscillation becomes large, its motion continues to be periodic, but it is no longer a simple harmonic. In general, the angular frequency and the period depend on the amplitude of the oscillation. For an angular amplitude of ϕ_0, the period can be shown to be given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\phi_0}{2} + \frac{1}{4}\left(\frac{3}{4}\right)^2\sin^4\frac{\phi_0}{2}\cdots\right]. \quad (24)$$

If $\phi_0 = \dfrac{\pi}{4}$, the period increases by about 5%. Thus, for smaller angle, the motion can be treated as SHM.

Physical Pendulum

A physical pendulum is any real pendulum that uses an extended (close to rigid) body, as contrasted to the idealized model of the simple pendulum with all the mass concentrated at a single point. For small oscillations, analyzing the motion of a real, physical pendulum is almost as easy as for a simple pendulum.



For small oscillation, the EOM becomes

$$\tau_z = -mgd\sin\theta \simeq -mgd\theta \quad (25)$$

$$\tau_z = I\alpha_z = I\frac{d^2\theta}{dt^2} \quad (26)$$

$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta. \quad (27)$$

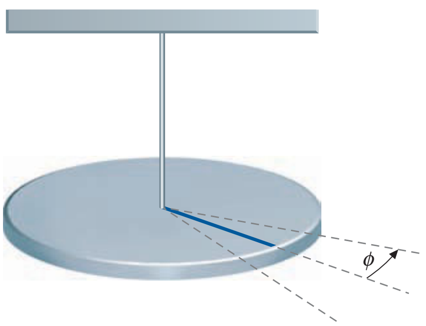
Comparing this with the standard EOM of SHM, one finds

$$\omega = \sqrt{\frac{mgd}{I}}, \quad \text{and} \quad T = 2\pi\sqrt{\frac{I}{mgd}}. \tag{28}$$

Question: A uniform rod with length L, pivoted at one end, what is the period of its motion as a pendulum?

Answer: $T = 2\pi\sqrt{2L/(3g)}$.

The Torsional Oscillator



A system that undergoes rotational oscillations in a variation of simple-harmonic motion is called a torsional oscillator.

1. Find the EP: $\phi = 0$.
2. Consider a small angular displacement ϕ of the disk from the equilibrium position, then the wire exerts a linear restoring torque τ on the disk given by $\tau = -\kappa\phi$.
3. According to Newton's second law for rotational motion, the EoM is $I\dfrac{d^2\phi}{dt^2} = -\kappa\phi$, where I is the moment of inertia of the rotating disk. This is very similar to the spring SHM case except with I in place of m, κ in place of k and ϕ in place of x.
4. Again one can immediately obtain the familiar solution for ϕ
$\phi = \phi_0 \cos(\omega t + \delta)$ where is the maximum angular displacement and $\omega = \sqrt{\dfrac{\kappa}{I}}$. The period of the motion is thus $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{\kappa}}$.

4 Advanced topic:

General Motion Near Equilibrium: In fact, for any (concave-up) potential $U(x)$, for example $U(x) = A + \frac{1}{2}B(x-x_0)^2$, there exists

$$\left.\frac{dU(x)}{dx}\right|_{x=x_0} = 0, \tag{29}$$

$$k = \left.\frac{d^2U(x)}{dx^2}\right|_{x=x_0} > 0, \tag{30}$$

where $x = x_0$ is the equilibrium position.For small oscillation about the EP at x_0, it is straightforward to find the restoring force goes like

$$\begin{aligned} F(x) &\equiv -\frac{dU(x)}{dx} \qquad (31)\\ &= -\underbrace{\left.\frac{dU(x)}{dx}\right|_{x=x_0}}_{=0,\text{minimum}} - \left.\frac{d^2U(x)}{dx^2}\right|_{x=x_0}(x-x_0)+\cdots \\ &= -k(x-x_0)+\cdots, \qquad (32) \end{aligned}$$

where $F(x)$ is expanded in terms of series of $(x-x_0)^n$, since the oscillation amplitude near equilibrium is assumed to be small. According to our analysis of the SHM, the system for a small particle of mass m oscillating back and forth near x_0 is described by SHM $x = x_0 + A\cos(\omega t + \delta)$ with A and δ determined by the initial condition.

Question: For example, consider the van der Waals interaction

$$U = U_0\left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6}\right], \tag{33}$$

find the corresponding spring constant k.

Answer: Near equilibrium, $F \simeq -kx$ with $k = 72U_0/R_0^2$ and R_0 the equilibrium position.

5 Damped Oscillation

If there is some dissipative forces on the oscillators, for example, air resistance or other frictions, then the oscillator can lose energy as it oscillates. The amplitude decreases as function of time t, and this is called damping, and this type of oscillation is called damped oscillation. Damping plays a beneficial role in the oscillations of an automobile's suspension system. More importantly, the force of wind against tall buildings can cause the top of skyscrapers to move more than a meter and cause motion sickness in people. We need to add damper (damping system) to reduce the motion.

Here is a simple example of damping with the dissipative force $f = -bv_x$

$$\sum F_x = -kx - bv_x = m\frac{d^2x}{dt^2}, \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x, \tag{34}$$

which gives a slightly different EoM as compared to SHM. How to find the solution to the above EoM?

Let us try the assumption $x(t) = A(t)\cos(\omega t + \delta)$ by simply assuming that the amplitude depends on time based on our observation of damped oscillator.

1. Assume $x(t) = A(t)\cos(\omega t + \delta)$ and try to determine $A(t)$ and ω later.

2. Let us compute $\dfrac{d^2x}{dt^2}$ and $\dfrac{dx}{dt}$ with the above assumed form of $x(t)$

$$\frac{dx(t)}{dt} = \frac{dA(t)}{dt}\cos(\omega t+\delta) - \omega A(t)\sin(\omega t+\delta), \tag{35}$$

$$\frac{d^2x(t)}{dt^2} = \frac{d^2A(t)}{dt^2}\cos(\omega t+\delta) - 2\frac{dA(t)}{dt}\omega\sin(\omega t+\delta) - \omega^2 A(t)\cos(\omega t+\delta). \tag{36}$$

3. Now let us substitute the expression of $\dfrac{d^2x}{dt^2}$ and $\dfrac{dx}{dt}$ into Eq. (34) and compare the coefficients of the cosine and sine functions. (The cosine and sine oscillations represent two orthogonal oscillations.) We can eventually obtain two equations as follows

$$\text{Coefficients of } \cos(\omega t+\delta): \quad \frac{d^2A(t)}{dt^2} - \omega^2 A(t) + \frac{b}{m}\frac{dA(t)}{dt} + \frac{k}{m}A(t) = 0, \tag{37}$$

$$\text{Coefficients of } \sin(\omega t+\delta): \quad -2\omega\frac{dA(t)}{dt} = \frac{b}{m}\omega A(t). \tag{38}$$

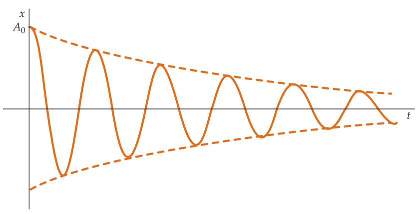
4. Let us solve the simpler equation in Eq. (38) as follows

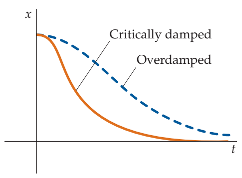
$$\frac{dA(t)}{dt} = -\frac{b}{2m}A(t) \quad\Rightarrow\quad \int_{A_0}^{A(t)}\frac{dA}{A} = -\frac{b}{2m}\int_0^t dt \quad\Rightarrow\quad \underline{A(t) = A_0 e^{-\frac{b}{2m}t}}. \tag{39}$$

5. Let us substitute the solution for $A(t)$ into Eq. (37), and reduce it into a simple algebraic equation

$$\left(\frac{b}{2m}\right)^2 A(t) - \omega^2 A(t) - \frac{b^2}{2m^2}A(t) + \frac{k}{m}A(t) = 0, \quad\Rightarrow\quad \underline{\omega^2 = \frac{k}{m} - \frac{b^2}{4m^2} = \omega_0^2\left[1 - \left(\frac{b}{2m\omega_0}\right)^2\right]}, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{40}$$

6. At the end of the day, we can reach the solution for damped oscillator $\underline{x(t) = A_0 e^{-\frac{b}{2m}t}\cos(\omega t+\delta)}$.





Some remarks:

- If the damping constant $b < 2m\omega_0$ which implies $\omega^2 > 0$, one finds that the system oscillates with an amplitude decreasing exponentially with increasing time. This motion is called underdamped as shown above in the left figure. The quantity $\tau = \dfrac{m}{b}$ has the unit of time, and it tells us how fast the amplitude decreases. Large $\tau = \dfrac{m}{b}$ implies that the amplitude decay rather slowly. For example, if $\dfrac{m}{b} \gg \dfrac{2\pi}{\omega_0}$, this means that the system can oscillate many many cycles before the amplitude decreases significantly.

- Application of damping: damping plays a beneficial role in the oscillations of an automobile's suspension system. The shock absorbers provide a velocity-dependent damping force so that when the car goes over a bump, it does not continue to bouncing forever. For optimal passenger comfort, the suspension system should be critically damped or slightly underdamped. Too much damping may be counterproductive if the car hits a second bump.

- If the damping constant $b = 2m\omega_0$ which corresponds to $\omega^2 = 0$, the motion is said to be critically damped. The system returns to equilibrium (without oscillation) very rapidly. We often use critical damping when we want a system to avoid oscillations and yet return to equilibrium quickly. If the damping constant $b < 2m\omega_0$ which corresponds to $\omega^2 < 0$, this is referred to as overdamped. The system does not oscillate at all. These two cases are shown above in the right figure. For the overdamped case the solution to Eq. (34) have the form $x(t) = C_1 e^{-\alpha_1 t} + C_2 e^{-\alpha_2 t}$. You are encouraged to work out the detail solution by yourself.

Energy in Damped Oscillations

In damped oscillation, the damping force is dissipative. Therefore, the total mechanical energy is no longer a constant but decreases continuously with time. Let us recall that the total mechanical energy of an oscillator is $E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$, together with the EoM in Eq. (34), we can differentiate the energy with respect to t and find

$$\begin{aligned}\frac{dE}{dt} &= mv_x\frac{dv_x}{dt} + kx\frac{dx}{dt} = mv_x\frac{d^2x}{dt^2} + kxv_x \\ &= v_x\left(m\frac{d^2x}{dt^2} + kx\right) = \underline{-bv_x^2} \le 0\end{aligned} \tag{41}$$

The power dissipated by the damping force equals the instantaneous rate of change of the total mechanical energy, which implies $P = \frac{dE}{dt} = Fv = -bv^2$. This is consistent with what we computed directly. The formula $\frac{dE}{dt} = -bv_x^2$ gives the rate of energy loss due to dissipative force, it indicates that the energy is always decreasing continuously.

Q factor and decay time

Let us now consider the behavior of a weakly damped ($b \ll 2m\omega_0$) oscillator. As we mentioned above that the average kinetic energy per cycle equals half the total energy

$$\langle K \rangle = \frac{1}{2} m \langle v_x^2 \rangle = \frac{1}{2} E, \quad \text{or} \quad \langle v_x^2 \rangle = \frac{E}{m}. \tag{42}$$

For a weakly damped oscillator with linear damping, the total mechanical energy decreases slowly with time. Therefore, we can approximately replace v_x^2 by $\langle v_x^2 \rangle$ and write

$$\frac{dE}{dt} = -bv_x^2 \simeq -\frac{b}{m} E, \quad \Rightarrow \quad \frac{dE}{E} = -\frac{b}{m} dt. \tag{43}$$

Upon integration, it gives

$$E = E_0 e^{-\frac{bt}{m}} = E_0 e^{-\frac{t}{\tau}} \quad \text{with} \quad \tau \equiv \frac{m}{b}. \tag{44}$$

This agree with the simple estimate from $E = \frac{1}{2} m\omega_0^2 A^2(t)$ where $A(t) = A_0 e^{-\frac{b}{2m}t}$. τ is the energy decay time constant. The physical interpretation of τ is that it is the time for the energy to decrease by a factor of e^{-1}. In weakly damped systems, the damped oscillator loses energy very slowly. From Eq. (43), we can replace dE by ΔE and dt by the period T, and find the energy loss per cycle is $\left|\frac{\Delta E}{E}\right| = \frac{b}{m} T \simeq \frac{b}{m} \frac{2\pi}{\omega_0}$. Usually we define the so-called Q-factor to describe the

$$Q \equiv \frac{2\pi}{\left|\frac{\Delta E}{E}\right|} = \omega_0 \tau. \tag{45}$$

Thus, the Q factor is inversely proportional to the fractional energy loss per cycle. The larger the Q factor is, the smaller the energy loss is per cycle. Interestingly, the same Q factor is also a measure of the sharpness of resonance.

6 Driven (Forced) Oscillation and Resonance

Now let us consider the case with damped oscillation with a periodic driving force $F(t) = F_d \cos \omega_d t$. Thus, based on what we have learnt in damped oscillations and Newton's second law, the corresponding EoM can be cast into

$$m \frac{d^2 x}{dt^2} = -b \frac{dx}{dt} - kx + F_d \cos(\omega_d t). \tag{46}$$

The solution consists of two parts, the transient solution ($x_{\text{transient}}$) and the steady-state solution (x_{steady}), which read

$$x(t) = \underbrace{A_0 e^{-\frac{b}{2m}t} \cos(\omega t + \delta)}_{x_{\text{transient}}} + \underbrace{A_d \cos(\omega_d t - \delta_d)}_{x_{\text{steady}}}, \tag{47}$$

where the amplitude A_d and the phase constant δ_d are given by

$$A_d = \frac{F_d}{\sqrt{m^2(\omega_0^2 - \omega_d^2)^2 + b^2 \omega_d^2}}, \tag{48}$$

$$\tan \delta_d = \frac{b\omega_d}{k - m\omega_d^2} = \frac{b\omega_d}{m(\omega_0^2 - \omega_d^2)}. \tag{49}$$

In the homework 11, you will be able to work out the details yourself. First, find the special steady solution by plugging $x_{\text{steady}} = A_d \cos(\omega_d t - \delta_d)$ into Eq.(46). Then you can also convince yourself that any solution of Eq.(34) can be added to x_{steady} and the sum still a solution to Eq.(46).

- The transient solution dies out when time $t \gg \tau = \frac{m}{b}$, since the amplitude decreases exponentially with time. Therefore, we can just focus on the steady-state solution.

- If we set $\omega_d = 0$, we find $\delta_d = 0$, and $A = \frac{F_d}{k}$. This is just the special case of constant force. In the case of $\omega_d \gg \omega_0$, we get $\delta_d \to \pi$.

- When $\omega_d = \omega_0$, then $\delta_d = \frac{\pi}{2}$ and $A = \frac{F_d}{b\omega_0}$. We reach the peak of the amplitude. This phenomenon is called resonance. When the driving frequency equals the natural frequency of the oscillator, the energy per cycle transferred to the oscillator is maximum. In this case, the object in resonance is always moving in the same direction of the driving force for maximum power input. (At resonance, $v_x = \omega_d A \cos(\omega_d t)$ which means $F(t) v_x(t)$ is always positive.) The natural frequency of the system ω_0 is thus called the resonance frequency. Resonance can be disastrous, see the example of Tacoma Narrows bridge.

- By plotting the amplitude square A^2, one can find that the resonance width $\Delta\omega = \frac{b}{m} = \frac{1}{\tau}$ satisfies

$$Q = \omega_0 \tau = \frac{\omega_0}{\Delta\omega}. \tag{50}$$

  Therefore, large Q factor gives very sharp resonance.