STA2001 Probability and Statistics (I)

Lecture 2

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Review

- Random experiment, Sample space, Event and An event has occurred
- Set Theory
- $P(A) = \lim_{n\to\infty} \frac{\mathcal{N}(A)}{n}$
- Probability function is a function that assigns $P(A)$ to an event A, $A \subseteq S$

  1. $P(A) \geq 0$
  2. $P(S) = 1$
  3. $A_1, A_2, \cdots$ are countable and mutually exclusive events

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

Review

- For random experiments that satisfy

  Assumption 1: S contains m possible outcomes

  $$e_k, \quad k = 1, 2, \cdots, m, \quad i.e., \quad S = \{e_1, e_2, \cdots, e_m\}.$$

  Assumption 2: The m outcomes are "equally likely"

  $$P(\{e_k\}) = \frac{1}{m}, \quad k = 1, \cdots, m.$$

  $$P(A) = \frac{N(A)}{N(S)},$$

  where $N(X)$ is the number of outcomes in $X \subseteq S$.

Ordered Sample and Sampling

Definition[Ordered sample of size r]

If r objects are selected from a set of n objects and if the order of selection is noted, then the selected set of r objects is called **ordered sample of size r.**

Definition[Sampling with replacement]

Occurs when an object is selected and then replaced before the next object is selected (n^r).

Definition[Sampling without replacement]

Occurs when an object is not replaced after it has been selected (${}_nP_r$).

Example 2 (Revisited)

The number of 4-letter words with different letters

$${}_{26}P_4 \longrightarrow \text{sampling without replacement}$$

The number of 4-letter words which can have the same letters

$$26^4 \longrightarrow \text{sampling with replacement}$$

Combination of n objects taken r at a time

Motivation

Sometimes, the order of selection is not important and we are only interested in the number of subsets of size r, i.e., **unordered sample of size** r, taken from a set of n different objects.

Instead to solve the problem in a direct way, we solve the problem in an indirect way and **we consider permutation of n objects taken r at a time by multiplication principle**.

Combination of n objects taken r at a time

1. $\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.r}} \rightarrow {}_nP_r$

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2. $\rightarrow \underset{X}{\boxed{\text{unordered subset of size r}}} \rightarrow \underset{r!}{\boxed{\text{permutation of r objects}}}$

$$\Rightarrow X \times r! =_n P_r \Rightarrow X = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} \triangleq {}_nC_r$$
$$= \binom{n}{r} = \binom{n}{n-r} = {}_nC_{n-r}$$

Definition: Each of the ${}_nC_r$ unordered subsets is called **a combination of n objects taken r at a time**.

Example 3

$$ {}_5P_2 = 5 \times 4. $$

Alternatively,

$$ \binom{5}{2} \times 2! = \frac{5!}{3!2!} \times 2! = 5 \times 4 $$

Example 4

The number of possible 5-card hands drawn from a deck of 52 playing cards is

$$ {}_{52}C_5 = \binom{52}{5} $$

- The number $\binom{n}{r}$ is often called binomial coefficients, because in binomial expansion

$$ (a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = (a+b)(a+b)\cdots(a+b) $$

Distinguishable Permutation of objects of two types

Motivation

Consider permutation of n objects of two types: r of one type and $(n-r)$ of the other type.

Instead to solve the problem in a direct way, we solve the problem in an indirect way and **we consider permutation of n different objects by multiplication principle**.

Distinguishable Permutation

1. $\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.n}} \rightarrow \quad n!$

---

2. $\rightarrow \boxed{\text{permute } n \text{ objects of two types}} \rightarrow$

X

$\boxed{\text{permute } r \text{ objects of one type}} \rightarrow$

$r!$

$\boxed{\text{permute } (n-r) \text{ objects of the other type}}$

$(n-r)!$

$$n! = X \cdot r! \cdot (n-r)! \Rightarrow X = {}_nC_r = \binom{n}{r}$$

Definition: Each of the ${}_nC_r$ permutations of n objects of two types

with r of one type and $(n-r)$ of the other type.

Example

Question

Flip a coin 10 times and the sequence of heads and tails is observed. What is the number of possible 10 tuples with 4 heads and 6 tails?

Example

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Flip a coin 10 times and the sequence of heads and tails is observed. What is the number of possible 10 tuples with 4 heads and 6 tails?

The number of possible 10 tuples with 4 heads and 6 tails is $\binom{10}{4}$

because it is a distinguishable permutation of 10 objects of two

types: 4 of one type and 6 of the other type.

Distinguishable permutation of objects of m types

Consider a set of n objects of m types:

n_1 of one type, n_2 of one type, $\cdots$, n_m of one type, where

$n_1 + n_2 + \cdots + n_m = n$

What's the number of distinguishable permutation of these n objects?

Distinguishable permutation of objects of m types

1. permutation of n different objects $n!$

$$\rightarrow \boxed{\text{pos.1}} \rightarrow \boxed{\text{pos.2}} \rightarrow \cdots \rightarrow \boxed{\text{pos.n}} \rightarrow \qquad \text{n!}$$

2.

$$\rightarrow \quad \underset{X}{\boxed{\text{permute } n \text{ objects of } m \text{ types}}} \quad \rightarrow$$

$$\underset{n_1!}{\boxed{\text{permute } n_1 \text{ objects of type } 1}} \quad \rightarrow$$

$$\vdots$$

$$\underset{n_m!}{\boxed{\text{permute } n_m \text{ objects of type } m}} \quad \rightarrow$$

$$n! = X \cdot n_1! \cdots n_m! \Rightarrow X = \frac{n!}{n_1! \cdots n_m!}$$

Section 1.3 Conditional Probability

Motivation Example

Consider a number of tulip bulbs

	Early(E)	Late(L)	Totals
Red(R)	5	8	13
Yellow(Y)	3	4	7
Totals	8	12	20

Experiment 1: Select one bulb randomly.

- Sample space $S = \{\text{all bulbs}\}$.
- Assumption: all bulbs are "equally likely".

Consider the event $R = \{\text{the selected bulb is red}\}$, what is $P(R)$?

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Consider the event $R = \{\text{the selected bulb is red}\}$, what is $P(R)$?

$$P(R) = \frac{N(R)}{N(S)} = \frac{13}{20}$$

Motivation Example

Consider a number of tulip bulbs

	Early(E)	Late(L)	Totals
Red(R)	5	8	13
Yellow(Y)	3	4	7
Totals	8	12	20

Experiment 2: Select one bulb from the ones that bloom early.

- Sample space reduces to $E = \{\text{all bulbs that bloom early}\}$.
- Assumption: all bulbs are "equally likely".

Consider the event $R = \{\text{the selected bulb is red}\}$, what is the probability of the event R, denoted by $P(R|E)$?

Motivation Example

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Experiment 2: Select one bulb from the ones that bloom early.

- Sample space reduces to $E = \{\text{all bulbs that bloom early}\}$.
- Assumption: all bulbs are "equally likely".

Consider the event $R = \{\text{the selected bulb is red}\}$, what is the probability of the event R, denoted by $P(R|E)$?

$$P(R|E) = \frac{N(R \cap E)}{N(E)} = \frac{5}{8}$$

We have defined a new probability function associated with the reduced sample space E.

Motivation Example

We study the problem of how to define a new probability function associated with a reduced sample space $E \subseteq S$, where S is the original sample space.

1. We have defined the probability function associated with the reduced sample space E directly.
2. We can also define it by linking to the probability function associated with the original sample space S.

Motivation Example

Under the assumptions that

1. S is finite
2. All outcomes are "equally likely"

the above example give us the idea

$$P(R|E) = \frac{N(R \cap E)}{N(E)} = \frac{N(R \cap E)/N(S)}{N(E)/N(S)} = \frac{P(R \cap E)}{P(E)}$$

leading to the next definition

Conditional Probability

Definition

The conditional probability of an event A, given that the event B has occurred, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

- B is the sample space for $P(A|B)$
- Independent of Assumptions 1 & 2 on the previous slide.

Conditional Probability

Conditional probability satisfies the probability axioms

1. $P(A|B) \geq 0$.

2. $P(B|B) = 1$.

3. If $A_1, A_2, A_3, \cdots$ are countable and mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots | B) = P(A_1|B) + P(A_2|B) + \cdots$$

Example 2

$$P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cap B) = 0.3,$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = 0.6$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.4} = 0.75$$

Can $P(A|B) > 1$ or $P(A|B) < 0$?

Example 2

$$P(A) = 0.4, \quad P(B) = 0.5, \quad P(A \cap B) = 0.3,$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = 0.6$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.4} = 0.75$$

Can $P(A|B) > 1$ or $P(A|B) < 0$?

No, $P(A|B)$ is a probability function.

Example 3 (Shooting Game)

Question

25 balloons of which, 10 are yellow, 8 red, 7 green.

$$A = \{\text{the first balloon shot is yellow}\}$$

$$B = \{\text{the second balloon shot is yellow}\}$$

What is the probability that the first two balloons shot are all yellow?

Example 3 (Shooting Game)

Question

25 balloons of which, 10 are yellow, 8 red, 7 green.

$$A = \{\text{the first balloon shot is yellow}\}$$

$$B = \{\text{the second balloon shot is yellow}\}$$

What is the probability that the first two balloons shot are all yellow?

$$P(A) = \frac{10}{25}, \quad P(B|A) = \frac{9}{24}$$

$$\Rightarrow P(A \cap B) = P(A)P(B|A) = \frac{10}{25} \cdot \frac{9}{24}$$

Multiplication Rule

Definition

The probability that two events, A and B both occur is given by the multiplication rule

$$P(A \cap B) = P(A)P(B|A), \quad \text{provided} \quad P(A) > 0$$

or by

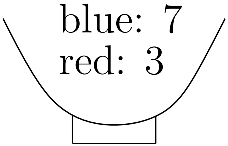
$$P(A \cap B) = P(B)P(A|B), \quad \text{provided} \quad P(B) > 0$$

Example 4

Question

A bowl contains 10 chips in total, 7 blue and 3 red. Drawn 2 chips successively at random and without replacement. What is the probability that the 1st draw is red and the 2nd draw is blue?

Example 4



$$A = \{\text{1st draw is red}\}$$

$$B = \{\text{2nd draw is blue}\}$$

$$P(A) = \frac{3}{10}, \quad P(B|A) = \frac{7}{9}$$

$$P(A \cap B) = P(B|A) \cdot P(A) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$$

Multiplication Rule for Three Events

Definition

The probability that three events, A, B and C all occur is given by the multiplication rule

$$P(A \cap B \cap C) = P((A \cap B) \cap C) = P(A \cap B)P(C|A \cap B)$$

$$\text{where} \quad P(A \cap B) = P(A)P(B|A)$$

$$\Rightarrow P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Induction principle can be used to derive the cases for more than three events.

Example 5

Question

Roll a pair of 4-sided dice and observe the sum of the dice

$$A = \{\text{a sum of 3 is rolled}\}$$

$$B = \{\text{a sum of 3 or a sum of 5 is rolled}\}$$

$$C = \{\text{a sum of 3 is rolled before a sum of 5 is rolled}\}$$

What are $P(A), P(B), P(C)$?

Example 5

Consider $P(A)$ and $P(B)$:

the sample space $S = \{(1,1),(1,2),\cdots,(4,4)\}$

$$P(A) = \frac{N(A)}{N(S)} = \frac{2}{16}, \quad P(B) = \frac{N(B)}{N(S)} = \frac{6}{16}$$

Consider P(C):

- Method 1 [by definition]:
  A. Figure out the simplified random experiment

  B. Figure out the corresponding sample space and the event

Example 5

For A, repeat the experiment of rolling a pair of 4-sided dice and record the sum of dice. For each repetition, we keep rolling the dice till we see either a sum of 3 or a sum of 5. Then we stop because we have an answer to the problem whether a sum of 3 is rolled before a sum of 5 is rolled.

Example 5

For instance

$$\text{Repetition } 1: 2, 4, 6, 3.$$
$$\text{Repetition } 2: 8, 6, 7, 4, 5$$
$$\text{Repetition } 3: 6, 5.$$

The sums other than 3 and 5 do not matter and we can remove them.

Repetition 1: a sum of 3 first

Repetition 2: a sum of 5 first

Repetition 3: a sum of 5 first

The problem reduces to roll the pair of dice (that gives the sum either 3 or 5) once and compute the probability that the sum is a 3.

Example 5

For B, the reduced sample space

$$S_r = \left\{ \begin{matrix} (1,2),(2,1) \\ (2,3),(3,2) \\ (1,4),(4,1) \end{matrix} \right\} \text{give a sum of 3 or 5}$$

$$P(C) = P(\{\text{roll the pair of dice once and the sum is a 3}\})$$

$$= \frac{N(\{\text{roll the pair of dice once and the sum is 3}\})}{N(S_r)}$$

$$= \frac{2}{6}$$

Example 5

- Method 2 [by conditional probability]:

$$P(C) = P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/16}{6/16} = \frac{2}{6}$$

Note that the event "$A|B$" is the same as event "C".

This is because

A. Event C is concerned with the cases where the sum is either a 3 or a 5. 'B happened" means that the sum is either a 3 or a 5.

B. If B happened then $A|B$ is nothing but the event "roll the pair of dice (that gives the sum either 3 or 5) once, and the sum is 3".