Slide 5-Matrices Algebra II

MAT2040 Linear Algebra

Example 5.1

$$(1) \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then

$$AB = \left[A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \left[B \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B \begin{bmatrix} 1 \\ 0 \end{bmatrix} B \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remark

(1) Even both AB and BA exist, AB and BA are generally in different sizes.

(2) $AB = O$ does not implies that $A = O$ or $B = O$.

Definition 5.2 (**Diagonal Matrix**) The square matrix $A = (a_{ij})_{n\times n}$ is called a diagonal matrix if $a_{ij} = 0$ whenever $i \neq j$. A is denoted by $A = \mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})$.
In particular, $I = I_n = \mathrm{diag}(1, 1, \cdots, 1)$ is the **identity matrix** of size n.

Example 5.3

$$\mathrm{diag}(1, 2, -5, 3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Property 5.4 **(Multiplication of Identity Matrix)**

$$AI_n = I_nA = A, \quad \forall A \in \mathbb{R}^{n\times n}$$

Remark The following conclusions can be easily checked

1. if $A \in \mathbb{R}^{m\times n}$, then $I_mA = A$ and $AI_n = A$.
2. let $A = (a_{ij})_{m\times n}$, then $AO_{n\times l} = O_{m\times l}$ and $O_{k\times m}A = O_{k\times n}$.

Matrix Partition

Sometimes, it will often be convenient to think about matrices defined in terms of other matrices.

For example, we already saw augmented matrices, defined in terms of a coefficient matrix and a vector of righthand sides.

$$[A \,|\, \mathbf{b}].$$

We also saw matrix A defined in terms of its column vectors or its row vectors:

$$A = [\mathbf{a}_1, \mathbf{a}_2, \ \ldots, \ \mathbf{a}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}.$$

Matrix Partition

Example 5.5

$$P_{11} = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}, \quad P_{22} = \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix}$$

Now define

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

This means that

$$P = \left[\begin{array}{ccc|cc} -1 & -1 & -1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \\ \hline 1 & 1 & 1 & -2 & -2 \\ -2 & -2 & -2 & -3 & -3 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

We can consider that P is a matrix which is partitioned with the blocks $P_{11}, P_{12}, P_{21}, P_{22}$.

When doing matrix partition, we just have to make sure the blocks are the right sizes, i.e., blocks in the same (block) row need to have the same number rows, blocks in the same (block) column need to have the same number of columns.

Definition 5.6

The matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{st} \end{bmatrix}$$

is a partition of matrix with $s \times t$ blocks if the matrices A_{ij} satisfies

(1) For each fixed i, the number of rows of all A_{ij} are equal.

(2) For each fixed j, the number of columns of all A_{ij} are equal.

The matrix A_{ij} is called the (i, j)-block of A.

Example 5.7

If

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 1 & 0 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 1 & 5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 7 & -2 & 3 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 2 \end{bmatrix}$$

then

$$A = \left[\begin{array}{cc|ccc|c} 1 & 2 & 5 & -1 & 3 & 4 \\ 3 & 4 & -2 & 1 & 0 & 6 \\ \hline 1 & 5 & 7 & -2 & 3 & 2 \end{array}\right]$$

has the $(1, 2)$-block A_{12} and $(2, 3)$-block A_{23}. Moreover, the number of rows of all A_{1j} is 2, and the number of columns of all A_{i3} is 1.

Matrices Multiplication by Partition I

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix. If B is partitioned into columns $B = [\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_r]$, then

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \cdots, A\mathbf{b}_r].$$

And if A is partitioned into rows

$$\begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

then

$$AB = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix} B = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{bmatrix}.$$

Matrices Multiplication by Partition I

Suppose $B = \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix}$, and $\vec{\mathbf{a}}_i = (a_{i1}, a_{i2}, \cdots, a_{in}), i = 1, \cdots, m$, and now we can compute $\vec{\mathbf{a}}_i B$.

Indeed, this can also be done directly by using the matrix-partition,

$$\vec{\mathbf{a}}_i B = [a_{i1}|a_{i2}|\cdots|a_{in}] \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix} = a_{i1}\vec{\mathbf{b}}_1 + a_{i2}\vec{\mathbf{b}}_2 + \cdots + a_{in}\vec{\mathbf{b}}_n, i = 1, \cdots, m$$

which is a linear combination of row vectors $\vec{\mathbf{b}}_1, \cdots, \vec{\mathbf{b}}_n$.

From $AB = \begin{bmatrix} \vec{\mathbf{a}}_1 B \\ \vec{\mathbf{a}}_2 B \\ \vdots \\ \vec{\mathbf{a}}_m B \end{bmatrix}$, one can see that each row vector of AB is the a linear combination of row vectors of B.

Matrices Multiplication by Partition II

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix.

Case 1. If B is partitioned into two blocks $B = [B_1, B_2]$, where B_1 is a $n \times t$ matrix and B_2 is a $n \times (r - t)$ matrix, then

$$AB = [AB_1, AB_2]$$

Case 2. If A is partitioned into two blocks $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where A_1 is a $k \times n$ matrix and A_2 is a $(m - k) \times n$ matrix, then

$$AB = \begin{bmatrix} A_1 B \\ A_2 B \end{bmatrix}$$

Matrices Multiplication by Partition III

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix.

Case 3. A, B are both partitioned matrices with two blocks $A = \begin{bmatrix} A_1, A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ where A_1 is a $m \times s$ matrix, A_2 is a $m \times (n-s)$ matrix, B_1 is a $s \times r$ matrix and B_2 is a $(n-s) \times r$ matrix,

$$AB = A_1 B_1 + A_2 B_2$$

Matrices Multiplication by Partition IV

Suppose that A is a $m \times n$ matrix and B is a $n \times r$ matrix.

Case 4. A, B are partitioned as follows:

$$A = \begin{array}{c} \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{array}{l} \}\, k \\ \}\, m-k \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{s} & \underbrace{\qquad}_{n-s} \end{array} \end{array} \qquad B = \begin{array}{c} \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] \begin{array}{l} \}\, s \\ \}\, n-s \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{t} & \underbrace{\qquad}_{r-t} \end{array} \end{array}$$

Let

$$A_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} & B_{22} \end{bmatrix}$$

Then

$$
\begin{aligned}
AB &= [A_1, A_2] \left[\begin{array}{c} B_1 \\ B_2 \end{array} \right] \\
&= A_1 B_1 + A_2 B_2 \\
&= \left[\begin{array}{c} A_{11} \\ A_{21} \end{array} \right] B_1 + \left[\begin{array}{c} A_{12} \\ A_{22} \end{array} \right] B_2 \\
&= \left[\begin{array}{c} A_{11} B_1 \\ A_{21} B_1 \end{array} \right] + \left[\begin{array}{c} A_{12} B_2 \\ A_{22} B_2 \end{array} \right] \\
&= \left[\begin{array}{cc} A_{11} B_{11} & A_{11} B_{12} \\ A_{21} B_{11} & A_{21} B_{12} \end{array} \right] + \left[\begin{array}{cc} A_{12} B_{21} & A_{12} B_{22} \\ A_{22} B_{21} & A_{22} B_{22} \end{array} \right] \\
&= \left[\begin{array}{cc} A_{11} B_{11} + A_{12} B_{21} & A_{11} B_{12} + A_{12} B_{22} \\ A_{21} B_{11} + A_{22} B_{21} & A_{21} B_{12} + A_{22} B_{22} \end{array} \right]
\end{aligned}
$$

Remark The matrix multiplication in blocks is in a similar way as the matrix multiplication with operations for entries, but need to be careful with the multiplication order.

Example 5.8

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{array}\right]$$

$$B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \left[\begin{array}{cc} 9 & -3 \\ \hline -1 & 0 \\ 2 & 1 \end{array}\right]$$

$$C = AB$$

Then

$$C = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix},$$

$$
\begin{aligned}
C_{11} = A_{11}B_{11} + A_{12}B_{21} &= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 9 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 9 & -3 \\ 36 & -12 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 43 & -6 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C_{21} = A_{21}B_{11} + A_{22}B_{21} &= \begin{bmatrix} 7 \end{bmatrix} \begin{bmatrix} 9 & -3 \end{bmatrix} + \begin{bmatrix} 8 & 9 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 63 & -21 \end{bmatrix} + \begin{bmatrix} 10 & 9 \end{bmatrix} = \begin{bmatrix} 73 & -12 \end{bmatrix}
\end{aligned}
$$

Matrix with blocks which are zero or identity matrix
Example

$$A = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right] = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & I_2 \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 3 & 2 & 2 \\ \hline 6 & 3 & 1 & 1 \\ 12 & 6 & 2 & 2 \end{array}\right]$$

When a matrix has some blocks which are identity matrices or zero matrices, using block matrix-multiplication will great simplify the calculation.

Suppose that $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $m \times n$ matrix, $B = \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix}$ is a $n \times p$ matrix, then

$$AB = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix} = \mathbf{a}_1\vec{\mathbf{b}}_1 + \mathbf{a}_2\vec{\mathbf{b}}_2 + \cdots + \mathbf{a}_n\vec{\mathbf{b}}_n$$

where each $\mathbf{a}_i\vec{\mathbf{b}}_i$ is a $m \times p$ matrix.

Example 5.10

Given

$$X = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute XY

$$\begin{aligned} XY = \left[\begin{array}{c|c} 3 & 1 \\ 2 & 4 \\ 1 & 2 \end{array}\right] \left[\begin{array}{ccc} 1 & 2 & 3 \\ \hline 2 & 4 & 1 \end{array}\right] &= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 1 \\ 8 & 16 & 4 \\ 4 & 8 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 10 \\ 5 & 10 & 5 \end{bmatrix} \end{aligned}$$

Definition (Outer Product of Two Vectors) Let $\mathbf{x}$ and $\mathbf{y}$ are two column vectors (the length of $\mathbf{x}$ is m, the length of $\mathbf{y}$ is n), the product $\mathbf{x}\mathbf{y}^T$ (called **outer product**) will result in a matrix.

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & & & \\ x_m y_1 & x_n y_2 & \cdots & x_m y_n \end{bmatrix}$$

Recall: $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ be an $m \times n$ matrix, $B = \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix}$ is a $n \times p$ matrix,

then

$$AB = [\mathbf{a}_1, \cdots, \mathbf{a}_n] \begin{bmatrix} \vec{\mathbf{b}}_1 \\ \vec{\mathbf{b}}_2 \\ \vdots \\ \vec{\mathbf{b}}_n \end{bmatrix} = \mathbf{a}_1\vec{\mathbf{b}}_1 + \mathbf{a}_2\vec{\mathbf{b}}_2 + \cdots + \mathbf{a}_n\vec{\mathbf{b}}_n$$

where each $\mathbf{a}_i\vec{\mathbf{b}}_i$ is an out product of two vectors.

Definition 5.11 **(Transpose of Matrix)** Let $A = (a_{ij})_{m \times n}$, then the transpose of A is the matrix $B = (b_{ij})_{n \times m}$, where $b_{ij} = a_{ji}(i = 1, \cdots, m, j = 1, \cdots, n)$. Notation: $B = A^T$.
Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\mathbf{a}_1, \cdots, \mathbf{a}_n] = \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{a}}_2 \\ \vdots \\ \vec{\mathbf{a}}_m \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} = \left[\vec{\mathbf{a}}_1^T,\ \vec{\mathbf{a}}_2^T,\ \ldots\ ,\ \vec{\mathbf{a}}_m^T\right]$$

Example 5.12

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 4 & -1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$$

Theorem 5.13 (**Properties of Matrix transpose**) Let $A, B \in \mathbb{R}^{m\times n}$, $\alpha \in \mathbb{R}$, then
(1) $(A+B)^T = A^T + B^T$.
(2) $(\alpha A)^T = \alpha A^T$.
(3) $(A^T)^T = A$.

Proof. Only show (1), other are exercises.

(1) The (i,j)-entry of $(A+B)^T$ is $a_{ji} + b_{ji}$, and the (i,j)-entry of $A^T + B^T$ is $a_{ji} + b_{ji}$.

Theorem 5.14 **(The transpose for the product of Matrices)** Let $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}$, then $(AB)^T = B^T A^T$.

Proof. The (i,j)-entry of AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$, and the (i,j)-entry of $(AB)^T$ is $\sum_{k=1}^{n} a_{jk}b_{ki}$.

(i,k)-entry of B^T is b_{ki}, and (k,j)-entry of A^T is a_{jk}, and (i,j)-entry of B^TA^T is $\sum_{k=1}^{n} b_{ki}a_{jk} = \sum_{k=1}^{n} a_{jk}b_{ki}$.

Thus, (i,j)-entry of $(AB)^T$ and B^TA^T are the same.

Example 5.15

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

thus

$$(AB)^T = \begin{bmatrix} 3 & -7 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ -1 & 4 \end{bmatrix} = [3, -7]$$

Thus

$$(AB)^T = B^T A^T$$

Definition 5.16 (**Symmetric Matrix**) If a matrix A satisfies $A = A^T$, we call A is symmetric.

Example 5.17

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 1 \end{bmatrix}$$

Remark 1. Symmetric matrix must be a square matrix. (Suppose $A \in \mathbb{R}^{m \times n}$, then A^T is a $n \times m$ matrix, $A = A^T$ implies that $m = n$)

Remark 2. Note that the entries of a symmetric matrix A are symmetric across the diagonal line.

Exercise

For any $m \times n$ matrix A, show that $A^T A$ and AA^T are two symmetric matrices.

Definition 5.18 (**Skew-symmetric Matrix**) If a matrix A satisfies $A^T = -A$, we call A is skew-symmetric or anti-symmetric.

Example 5.20

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & 6 \\ -5 & -6 & 0 \end{bmatrix}$$

Property 5.19 Any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$