MAT1002 Midterm Examination

Saturday, March 23, 2024

Time: 9:30 - 11:30 AM

Notes and Instructions

1. No books, no notes, no dictionaries, and no calculators.

2. The maximum possible score of this examination is ***120****.*

3. There are ***13*** *questions (with parts), which are worth 128 points in total.* ***This means that you do not have to answer all the questions to get the full score****.*

*4. The symbol [**N**] at the beginning of a question indicates that the question is worth N points.*

5. Write down your solutions on the ***answer book****.*

6. Show your intermediate steps ***except Questions 1 and 2*** *— answers without intermediate steps will receive minimal (or even no) marks.*

7. Express irrational numbers in exact forms instead of decimal forms; e.g., write $\sqrt{2}$ instead of 1.414..., and write $\ln 2$ instead of 0.693....

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MAT1002 Midterm Questions

1. [**12**] True or False? No explanation is required.

   (i) The series $\sum_{n=1}^{\infty} \frac{4n}{n^4+1}$ converges.

   (ii) If $\sum_{n=1}^{\infty} a_n$ converges and a_n is non-negative for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n^3$ also converges.

   (iii) The magnitude of the cross product of two vectors in the 3D space is equal to the product of their magnitudes.

   (iv) The cross product of two vectors in the 3D space is undefined if both vectors lie on the same line.

   (v) The cross product is commutative, meaning that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in the 3D space.

   (vi) The polar curve given by $r = 2\cos\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, is a circle of radius 1 centered at $(1, 0)$.

2. [**12**] Short questions: no intermediate step is required.

   (i) Assume that a_n converges. Find the limit of a_n as $n \to \infty$, where

   $$a_1 = 2, \quad a_{n+1} = \sqrt{a_n + 1} \text{ for } n \geq 1.$$

   (ii) Determine the area of the parallelogram formed by the vectors $\mathbf{u} = \langle 2, -3, 1 \rangle$ and $\mathbf{v} = \langle 1, -1, 3 \rangle$.

   (iii) Find the projection of the vector $\mathbf{u} = \langle 2, -3, 1 \rangle$ onto the vector $\mathbf{v} = \langle 1, -1, 3 \rangle$.

   (iv) Find the principal unit normal $\mathbf{N}$ of the circle $(x-4)^2 + (y-8)^2 = 9$ at the point $(x, y) = (4, 11)$.

3. [**6**] Consider the following statement:

$$\sum_{n=1}^{\infty} a_n b_n \text{ converges if } \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n \text{ both converge.}$$

Is it true or false? If it is true, prove it. Otherwise, give a counterexample.

4. [**20**] For each of the following series, determine whether it converges absolutely, converges conditionally, or diverges. Justify your answer.

(i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{2024}{n}\right)^n$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(20n+24)\big(\ln(1+n)\big)^2}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{1}{2}n^3\pi\right)}{n\sqrt{n}}$

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$

5. [**5**] Find the surface area of the surface generated by rotating the following parametric curve about x-axis:

$$x = 9 + 2t^2; \qquad y = 4t; \qquad t \in [0, 2].$$

6. [**5**] Find the equation of the plane passing through the points $P(1, -2, 3)$, $Q(3, 1, -1)$, and $R(-2, 4, 0)$, and express it in the form $Ax + By + Cz = D$.

7. [**4**] Consider the curve on the xy-plane given by

$$\mathbf{r}(t) = (t-1)^3\mathbf{i} + \cos(\pi t)\mathbf{j}, \quad -\infty < t < \infty.$$

Determine if the curve is smooth. If not, find the locations of all the cusps (i.e., "non-smooth points").

8. [**2+5+5**] Consider the polar curve given by

$$r = 1 - \theta^2, \quad \theta \in [-1, 1].$$

(i) Sketch the curve.

(ii) Compute the arc length of the curve.

(iii) Compute the area of the region bounded by this curve.

9. [**4+2+6**] Consider a particle traveling along the curve given by

$$\mathbf{r}(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, \quad t \geq 0.$$

(a) Find the first point in time t_0 at which it hits the plane given by

$$2x + 3y + 3z = 5.$$

(b) Determine the coordinates of the point of impact at time t_0.

(c) Determine the angle of incidence of the impact, i.e., the acute angle θ_0 between the tangent to the curve and the normal to the plane. (Since no calculator is allowed, you can give your answer in terms of $\cos\theta_0$.)

10. [**6+6+3**] The trajectory of a moving particle is a curve given by

$$\mathbf{r}(t) = \sqrt{2}\cos(t)\mathbf{i} + f(t)\mathbf{j} + t\mathbf{k}, \quad t \geq 0$$

starting from time $t = 0$, where the function $f(t)$ is given by

$$f(t) = \int_0^t \sqrt{1 + \cos(2\tau)}\, d\tau, \quad t \geq 0.$$

(a) Compute the time T it takes for it to travel a distance of s_0 along the curve, and its mean speed (i.e., average speed) v over the time interval $[0, T]$.

(b) Compute the straight-line distance $c(t)$ at any time t of the particle from the origin, and find its rate of change $\dfrac{d}{dt}c(t)$.

(c) Briefly, explain why $\dfrac{d}{dt}c(t)$ is not the same as the speed $\left|\dfrac{d}{dt}\mathbf{r}(t)\right|$. No calculation is required.

11. [**5+3**] Consider the function

$$F(x) = \int_0^x \cos(\sqrt{t})\, dt.$$

(a) Find a power series representation of $F(x)$ (centered at 0).

(b) Consider approximating $F(1)$ by taking a partial sum of the series in (a). By the theory of alternating series approximation, what is the number N of terms that you would need to take in the sum so that the error is less than 0.001? Take N as small as possible.

12. [**5+5**] Consider the function

$$f(x) = \frac{1}{(1-x)^2}.$$

(a) Find the Taylor series of $f(x)$ centered at 0.

(b) Determine ALL values of x for which the series in (a) converges.

13. [**7**] Determine whether the following series converges or diverges. If it converges, find the limit; otherwise, explain why it diverges.

$$\sum_{n=1}^{\infty} \frac{3^n + 2n}{4^n}.$$