# STA2001 Probability and Statistics (I)

## Lecture 18

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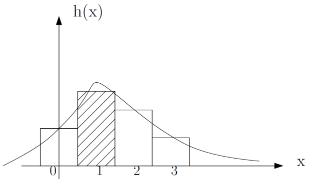
# Review of Last Lecture

**Histogram of and Approximation for discrete distribution**

Consider a discrete RV $Y$ with pmf $f(y) : \overline{S} \to (0, 1]$ with $\overline{S} = \{0, 1, \cdots, n\}$.Then the histogram for $Y$ is

$$h(y) = f(k), y \in (k - \frac{1}{2}, k + \frac{1}{2}), k = 0, 1, \cdots, n$$

The area below the histogram corresponds to probability, which make the histogram has similar property as the pdf of continuous distribution.




If it is possible to find a continuous distribution with pdf "close" to the histogram of the discrete distribution, then we can compute the probability of discrete distribution approximately by using the continuous distribution.

# Review of Last Lecture

**Half-unit correction for continuity**

Now, if $Y = \sum_{i=1}^{n} X_i$, where $X_1, \cdots, X_n$ are i.i.d. random sample drawn from discrete distributions with mean $\mu$ and variance $\sigma^2$.

| | | |
|---|---|---|
| $P(Y = k)$ | $\approx$ | $P(k - \frac{1}{2} < Y < k + \frac{1}{2})$ |
| discrete RV | | approximate by continuous RV |
| pmf $f(y)$ | | by CLT for large $n$, $Y$ can be approximated by $N(n\mu, n\sigma^2)$ in the sense that the pdf of the normal distribution is close to the histogram of $Y$ |
| hard to calculate | | easy to calculate |

# Review of Last Lecture

**[Chebyshev's inequality]**

If the RV $X$ has a finite mean $\mu$ and finite nonzero variance $\sigma^2$, then for every $k \geq 1$,

$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

If $\varepsilon = k\sigma$, then

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$$

# Convergence in Probability

**Definition**

A sequence of RVs $Z_1, Z_2, \cdots$, is said to converge in probability to a RV $Z$, often denoted by, $Z_n \xrightarrow{p} Z$, if for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|Z_n - Z| \geq \varepsilon) = 0.$$

# Example 1

Assume that $Z_n$ has an exponential distribution with $\theta = 1/n$, $n = 1, 2, \cdots$. Then show that $Z_n \xrightarrow{p} 0$, i.e., the sequence of RVs $Z_1, Z_2, \cdots$, converges in probability to $Z = 0$.

# Example 1

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For any $\epsilon > 0$,

$$\begin{aligned} \lim_{n\to\infty} P\big(|Z_n - 0| \geq \epsilon\big) &= \lim_{n\to\infty} P\big(Z_n \geq \epsilon\big) \\ &= \lim_{n\to\infty} e^{-n\epsilon} \\ &= 0, \end{aligned}$$

where the first equation is true because $Z_n \geq 0$, and the second equation is obtained by using the fact that $Z_n$ has an exponential distribution with $\theta = 1/n$.

# Theorem [Law of Large Number]

**Theorem (Law of Large Numbers)**

Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n$ drawn from a distribution with finite mean $\mu$ and finite nonzero variance, and let $\overline{X}$ be the sample mean. Then $\overline{X}$ converges in probability to $\mu$, i.e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\overline{X} - \mu\right| \geq \varepsilon\right) = 0$$

# Proof of Law of Large Number

Note that

$$E(\overline{X}) = \mu, \quad Var(\overline{X}) = \frac{1}{n}\sigma^2$$

By the Chebyshev's inequality, i.e., Corollary 5.8-1, for every $\varepsilon > 0$.

$$P(|\overline{X} - \mu| \geq \varepsilon) \leq \frac{1}{\varepsilon^2}\frac{\sigma^2}{n}$$

# Proof of Law of Large Number

Taking limits on both sides yield

$$0 \leq \lim_{n\to\infty} P(|\overline{X} - \mu| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{\varepsilon^2 n} = 0$$

$$\Rightarrow \lim_{n\to\infty} P(|\overline{X} - \mu| \geq \varepsilon) = 0$$

or equivalently $\quad \displaystyle\lim_{n\to\infty} P(|\overline{X} - \mu| < \varepsilon) = 1$

## Section 5.9 Limiting Moment Generating Functions

# Motivation

Binomial distribution $b(n, p)$ can be approximated by the Poisson distribution with $\lambda = np$ when $n$ is large and $p$ is fairly small:

- the approximation is good if $n \geq 20$ and $p \leq 0.05$
- the approximation is very good if $n \geq 100$ and $p \leq 0.1$
- the approximation becomes better with larger $n$ and smaller $p$

# Example 1, page 227

**Question**

Let $Y \sim b(50, 1/25)$. Q: $P(Y \leq 1)$?

# Example 1, page 227

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Let $Y \sim b(50, 1/25)$. Q: $P(Y \leq 1)$?

1. By definition,

$$P(Y \leq 1) = P(Y = 0) + P(Y = 1)$$
$$= \left(\frac{24}{25}\right)^{50} + 50\left(\frac{1}{25}\right)\left(\frac{24}{25}\right)^{49} = 0.4$$

2. By approximation with Poisson distribution $\lambda = np = 2$

$$P(Y \leq 1) \approx \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} = 3e^{-2} = 0.406$$

# Why? and an Interesting Observation

First, recall the mgf of $b(n, p)$ is $M(t) = (1 - p + pe^t)^n$.

Then, we will consider the limit of $M(t) = (1 - p + pe^t)^n$ as $n \to \infty$ such that $np = \lambda$ is a constant.

$$M(t) = (1 - \frac{\lambda}{n} + \frac{\lambda}{n}e^t)^n = [1 + \frac{\lambda(e^t - 1)}{n}]^n$$

Since $\lim_{n\to\infty}(1 + \frac{b}{n})^n = e^b$

$$\lim_{n\to\infty} M(t) = e^{\lambda(e^t-1)} \to \text{mgf for Poisson distribution}$$

# Theorem 5.9-1, page 226

## Limiting mgf technique

Let $\{M_n(t)\}_{n=1}^{\infty}$ be a sequence of mgfs for $t$ in an open interval around $t=0$. If $\lim_{n\to\infty} M_n(t) = M(t)$, for $t$ in the open interval around $t=0$. Then the sequence of RVs

$$Z_n \xrightarrow{d} Z,$$

where $M_n(t)$ and $M(t)$ are mgfs of $Z_n$ and $Z$, respectively.

$$\begin{array}{ccc} \lim_{n\to\infty} M_n(t) & = & M(t) \\ \updownarrow & & \updownarrow \\ Z_n & \xrightarrow{d} & Z \end{array}$$

# Convergence of $b(n, p)$

$b(n, p)$ converges in distribution to

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How to understand?

**Convergence of $b(n, p)$**

1. as $n \to \infty$ with $\lambda = np$ being a constant, and let $Z_n \sim b(n, p)$ and $Z \sim \text{Poisson}(\lambda)$, then $Z_n \xrightarrow{d} Z$.

2. as $n \to \infty$ with $p$ being a constant, and let $Z_n \sim b(n, p)$ and $Z \sim N(0, 1)$, then
$$\frac{Z_n/n - p}{\sqrt{p(1-p)/n}} \xrightarrow{d} Z$$

# Proof of CLT

**CLT**

Let $\overline{X}$ be the sample mean of the random sample of size $n$, $X_1$,$X_2$ ,$\cdots$,$X_n$ from a distribution with a finite mean $\mu$ and a finite nonzero variance $\sigma^2$, then as $n \to \infty$, the random variable $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ converge in distribution to $N(0,1)$.

The idea of the proof:

1. Let

$$W_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}},$$

   and then show the mgf of $W_n$, say $M_n(t)$, converges to the mgf of $N(0,1)$ for $t$ in an open interval around $t = 0$
2. By Theorem 5.9-1, $W_n \xrightarrow{d} N(0,1)$

# Proof of CLT, page 208

$$
\begin{aligned}
E[e^{tW_n}] &= E\left\{\exp\left[t\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - n\mu)}{\sigma/\sqrt{n}}\right]\right\} \\
&= E\left\{\exp\left[\frac{t}{\sqrt{n}}\frac{\sum_{i=1}^{n}(X_i - n\mu)}{\sigma}\right]\right\} \\
&= E\left\{\exp\left(\frac{t}{\sqrt{n}}\frac{X_1-\mu}{\sigma}\right)\cdots\exp\left(\frac{t}{\sqrt{n}}\frac{X_n-\mu}{\sigma}\right)\right\} \\
&= E\left\{\exp\left(\frac{t}{\sqrt{n}}\frac{X_1-\mu}{\sigma}\right)\right\}\cdots E\left\{\exp\left(\frac{t}{\sqrt{n}}\frac{X_n-\mu}{\sigma}\right)\right\} \\
&\text{[independence]}
\end{aligned}
$$

# Proof of CLT, page 208

Let

$$Z_i = \frac{X_i - \mu}{\sigma}, \quad i = 1, \cdots, n$$

Then $Z_1, \cdots, Z_n$ are i.i.d..

Let

$$M(t) = E\left\{\exp\left(tZ_i\right)\right\}, \quad |t| < h$$

be the common mgf for $Z_i, i = 1, \cdots, n$.

Then

$$E[e^{tW_n}] = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n, \quad \left|\frac{t}{\sqrt{n}}\right| < h.$$

# Proof of CLT, page 208

Now consider $M(t)$. Actually, $Z_1, \ldots, Z_n$ are i.i.d. with mean 0 and variance 1. Then

$$M(0) = 1, M'(0) = 0, M''(0) = 1.$$

By using Taylor's expansion, there exists $|t_1| \leq |t|$ such that

$$\begin{aligned} M(t) &= M(0) + M'(0)t + \frac{1}{2}M''(t_1)t^2 \\ &= 1 + \frac{1}{2}M''(t_1)t^2 = 1 + \frac{1}{2}t^2 + \frac{1}{2}t^2[M''(t_1) - 1]. \end{aligned}$$

# Proof of CLT, page 208

Then

$$E(e^{tW_n}) = \left[M(\frac{t}{\sqrt{n}})\right]^n = \left[1 + \frac{1}{2}\frac{t^2}{n} + \frac{1}{2}\frac{t^2}{n}[M''(t_1) - 1]\right]^n,$$
$$\left|\frac{t}{\sqrt{n}}\right| < h, \quad |t_1| \le \frac{|t|}{\sqrt{n}}.$$

Since $M''(t)$ is continuous at $t = 0$ and as $n \to \infty, t_1 \to 0$,

$$\lim_{n\to\infty} M''(t_1) - 1 = 1 - 1 = 0.$$

Note that

$$\lim_{n\to\infty}\left(1 + \frac{b}{n}\right)^n = e^b$$

# Proof of CLT, page 208

we have

$$\lim_{n\to\infty} E(e^{tW_n}) = \lim_{n\to\infty}\left[1+\frac{1}{2}\frac{t^2}{n}+\frac{1}{2}\frac{t^2}{n}[M''(t_1)-1]\right]^n$$
$$= e^{\frac{t^2}{2}} \to \text{mgf of } N(0,1)$$

By Theorem 5.9-7,

$$W_n = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$