Limits of Products and Quotients

Supposing $\lim_{x\to a} \frac{f(x)}{g(x)} = L \in \mathbb{R}$, then

$$\lim_{x\to a} f(x)h(x) = \lim_{x\to a}[g(x)h(x)]\frac{f(x)}{g(x)} = L \lim_{x\to a} g(x)h(x), \text{ and}$$

$$\lim_{x\to a} \frac{f(x)}{h(x)} = \lim_{x\to a} \frac{g(x)}{h(x)} \times \frac{f(x)}{g(x)} = L \lim_{x\to a} \frac{g(x)}{h(x)},$$

provided the limits on the right-hand side exists.

- Replacing $f(x)$ by $Lg(x)$ in a product or quotient will keep the limit unchanged, if it exists
- It does not work with sums and differences
- $\lim_{x\to a}$ can be replaced by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$
- If $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$, we write $f(x) \sim g(x)$ as $x \to a$. In this case, you can just replace $f(x)$ by $g(x)$

For $f(x) = \sin x$, $\tan x$, $e^x - 1$ and $\ln(1 + x)$, we have $f(x) \sim x$ as $x \to 0$:

- $\sin x \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
- $\tan x \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sec^2(x)}{1} = 1$
- $e^x - 1 \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$
- $\ln(1 + x) \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \ln'(1) = 1$

Example Since $\lim_{x\to 0}\frac{1-\cos(x)}{x^2} = \lim_{x\to 0}\frac{\sin(x)}{2x} = ½ \lim_{x\to 0}\frac{\sin(x)}{x} = ½$,

we have $1 - \cos x \sim x^2/2$ as $x \to 0$.

Example $\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{x^3} = \lim_{x\to 0}\frac{\sin(x)-\sin(x)\cos(x)}{x^3\cos(x)}$ (since $\tan x = \sin x/\cos x$)

$= \lim_{x\to 0}\frac{\sin(x)[1-\cos(x)]}{x^3\cos(x)} = \lim_{x\to 0}\frac{x[1-\cos(x)]}{x^3\cos(x)}$ (since $\sin x \sim x$)

$= \lim_{x\to 0}\frac{x(x^2/2)}{x^3\cos(x)} = \frac{½}{1} = ½.$ (since $1 - \cos x \sim x^2/2$)

8.2

Integration by Parts

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx,$$

leading to the **integration by parts** formula

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

EXAMPLE 1 Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x \, dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C. \qquad \blacksquare$$

EXAMPLE 2 Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}\,dx,$ $\qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x}\,dx = x \ln x - \int dx = x \ln x - x + C. \qquad \blacksquare$$

EXAMPLE 3 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x\,dx$, $du = 2x\,dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2\int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2 e^x \, dx &= x^2 e^x - 2\int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C,\end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right.

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.\end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \qquad \blacksquare$$

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x \, dx,$$

so that

$$du = (n-1)\cos^{n-2} x \, (-\sin x \, dx) \qquad \text{and} \qquad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned}
\int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx.
\end{aligned}$$

If we add

$$(n-1) \int \cos^n x \, dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n - 1}{n} \int \cos^{n-2} x \, dx. \quad \blacksquare$$

The formula found in Example 5 is called a **reduction formula** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate.

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\ &= \left[-4e^{-4} - (-0e^{-0})\right] + \int_0^4 e^{-x}\,dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - (e^{-4} - e^{-0}) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

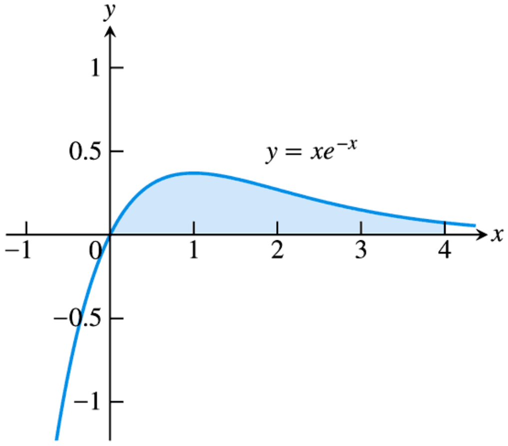

FIGURE 8.1 The region in Example 6.

8.3

Trigonometric Integrals

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) && \sin x \, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C
\end{aligned}$$

EXAMPLE 2 Evaluate

$$\int \cos^5 x \, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad \cos x \, dx = d(\sin x)$$

$$= \int (1 - u^2)^2 \, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) \, du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \qquad \blacksquare$$

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

m and *n* both even

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x) \cos 2x\, dx$$

$u = \sin 2x,$
$du = 2 \cos 2x\, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \text{cos } 2x \geq 0 \text{ on } [0, \pi/4] \\
&= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

■

Integrals of Powers of tan x and sec x

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\,dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\,dx, \qquad v = \tan x, \qquad du = \sec x \tan x\,dx.$$

Then

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\,dx)\\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \qquad \tan^2 x = \sec^2 x - 1\\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C.$$

EXAMPLE 7 Evaluate

$$\int \tan^4 x \sec^4 x \, dx.$$

Solution

$$\begin{aligned}
\int (\tan^4 x)(\sec^4 x)\, dx &= \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx && \sec^2 x = 1 + \tan^2 x \\
&= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx \\
&= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx \\
&= \int u^4\, du + \int u^6\, du = \frac{u^5}{5} + \frac{u^7}{7} + C && u = \tan x, \\
& && du = \sec^2 x\, dx \\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C
\end{aligned}$$

■

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

may be evaluated by using the following identities,

$$\sin mx \sin nx = \frac{1}{2}\left[\cos (m - n)x - \cos (m + n)x\right], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin (m - n)x + \sin (m + n)x\right], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos (m - n)x + \cos (m + n)x\right]. \tag{5}$$

EXAMPLE 8 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int \left[\sin(-2x) + \sin 8x\right] dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad \theta \; \varepsilon \; [-\pi/2, \pi/2]$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad \theta \; \varepsilon \; (-\pi/2, \pi/2)$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad \theta \; \varepsilon \; [0, \pi/2) \cup (\pi/2, \pi]$	$\sec^2\theta - 1 = \tan^2\theta$

The restrictions on θ are to ensure that the functions are invertible. If you don't restrict, then when you convert the θ back to x, for a given x, there are multiple θs; e.g., if $x = a \tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place.

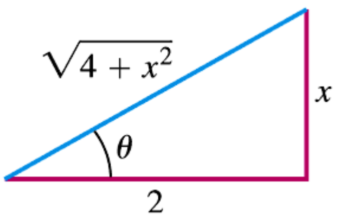

FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4+x^2}}.$$

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} && \sqrt{\sec^2\theta} = |\sec\theta| \\
&= \int \sec\theta\, d\theta && \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C. && \text{From Fig. 8.4}
\end{aligned}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

EXAMPLE 2 Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

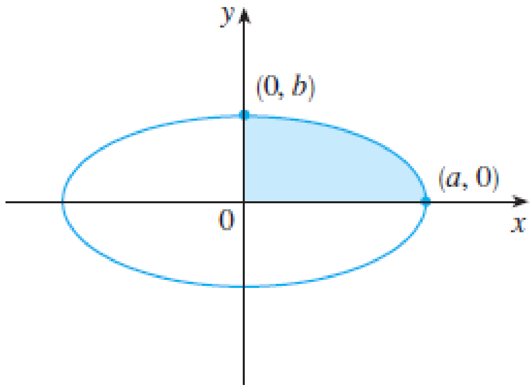

FIGURE 2

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

SOLUTION Solving the equation of the ellipse for y, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \qquad \text{or} \qquad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (see Figure 2). The part of the ellipse in the first quadrant is given by the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leqslant x \leqslant a$$

and so

$$\tfrac{1}{4}A = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

To evaluate this integral we substitute $x = a \sin \theta$. Then $dx = a \cos \theta \, d\theta$. To change the limits of integration we note that when $x = 0$, $\sin \theta = 0$, so $\theta = 0$; when $x = a$, $\sin \theta = 1$, so $\theta = \pi/2$. Also

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2\theta} = \sqrt{a^2 \cos^2\theta} = a\left|\cos \theta\right| = a \cos \theta$$

since $0 \leqslant \theta \leqslant \pi/2$. Therefore

$$\begin{aligned} A &= 4\frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = 4\frac{b}{a}\int_0^{\pi/2} a \cos \theta \cdot a \cos \theta \, d\theta \\ &= 4ab \int_0^{\pi/2} \cos^2\theta \, d\theta = 4ab \int_0^{\pi/2} \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta \\ &= 2ab\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\pi/2} = 2ab\left(\frac{\pi}{2} + 0 - 0\right) = \pi ab \end{aligned}$$

We have shown that the area of an ellipse with semiaxes a and b is πab. In particular, taking $a = b = r$, we have proved the famous formula that the area of a circle with radius r is πr^2. ■

EXAMPLE 3 Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

SOLUTION Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

So we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

To evaluate this trigonometric integral we put everything in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{\sec\theta}{\tan^2\theta} = \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

Therefore, making the substitution $u = \sin\theta$, we have

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{du}{u^2}$$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + C = -\frac{1}{4\sin\theta} + C$$

$$= -\frac{\csc\theta}{4} + C$$

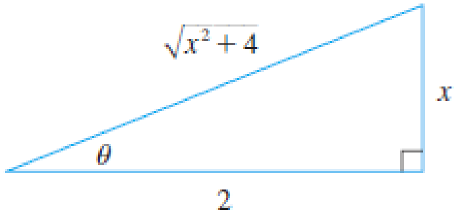

FIGURE 3

$\tan\theta = \frac{x}{2}$

We use Figure 3 to determine that $\csc\theta = \sqrt{x^2+4}/x$ and so

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + C$$

EXAMPLE 4 Find $\int \frac{x}{\sqrt{x^2 + 4}} dx$.

SOLUTION It would be possible to use the trigonometric substitution $x = 2 \tan \theta$ here (as in Example 3). But the direct substitution $u = x^2 + 4$ is simpler, because then $du = 2x\,dx$ and

$$\int \frac{x}{\sqrt{x^2 + 4}} dx = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2 + 4} + C$$

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EXAMPLE 6 Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$.

SOLUTION First we note that $(4x^2+9)^{3/2} = (\sqrt{4x^2+9})^3$ so trigonometric substitution is appropriate. Although $\sqrt{4x^2+9}$ is not quite one of the expressions in the table of trigonometric substitutions, it becomes one of them if we make the preliminary substitution $u = 2x$. When we combine this with the tangent substitution, we have $x = \frac{3}{2}\tan\theta$, which gives $dx = \frac{3}{2}\sec^2\theta\,d\theta$ and

$$\sqrt{4x^2+9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta$$

When $x = 0$, $\tan\theta = 0$, so $\theta = 0$; when $x = 3\sqrt{3}/2$, $\tan\theta = \sqrt{3}$, so $\theta = \pi/3$.

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx &= \int_0^{\pi/3} \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\,\frac{3}{2}\sec^2\theta\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{\tan^3\theta}{\sec\theta}\,d\theta = \frac{3}{16}\int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta}\,d\theta \\
&= \frac{3}{16}\int_0^{\pi/3} \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\,d\theta
\end{aligned}$$

Now we substitute $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \pi/3$, $u = \frac{1}{2}$. Therefore

$$\begin{aligned}
\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx &= -\frac{3}{16}\int_1^{1/2} \frac{1-u^2}{u^2}\,du \\
&= \frac{3}{16}\int_1^{1/2} (1-u^{-2})\,du = \frac{3}{16}\left[u + \frac{1}{u}\right]_1^{1/2} \\
&= \frac{3}{16}\left[\left(\tfrac{1}{2}+2\right) - (1+1)\right] = \frac{3}{32}
\end{aligned}$$

EXAMPLE 7 Evaluate $\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx$.

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$\begin{aligned} 3 - 2x - x^2 &= 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1) \\ &= 4 - (x + 1)^2 \end{aligned}$$

This suggests that we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{u - 1}{\sqrt{4 - u^2}}\,du$$

We now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4 - u^2} = 2\cos\theta$, so

$$\begin{aligned} \int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx &= \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta \\ &= \int (2\sin\theta - 1)\,d\theta \\ &= -2\cos\theta - \theta + C \\ &= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{2}\right) + C \end{aligned}$$

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Week 11

Assignment 11

7.8: #2,8,10,17,21(a),23(a)
8.2: #6,10,12,24,29,31,45,46,61,66,69,74
8.3: #3,8,20,29,34,35,38,44,51,64
8.4: #2,3,6,9,11,57,58

The above need to be submitted on Blackboard.
Deadline: 10 PM, Friday, Dec 1.

Required Reading (Textbook)

- Sections 7.8, 8.2, 8.3, 8.4