

2.4

One-Sided Limits

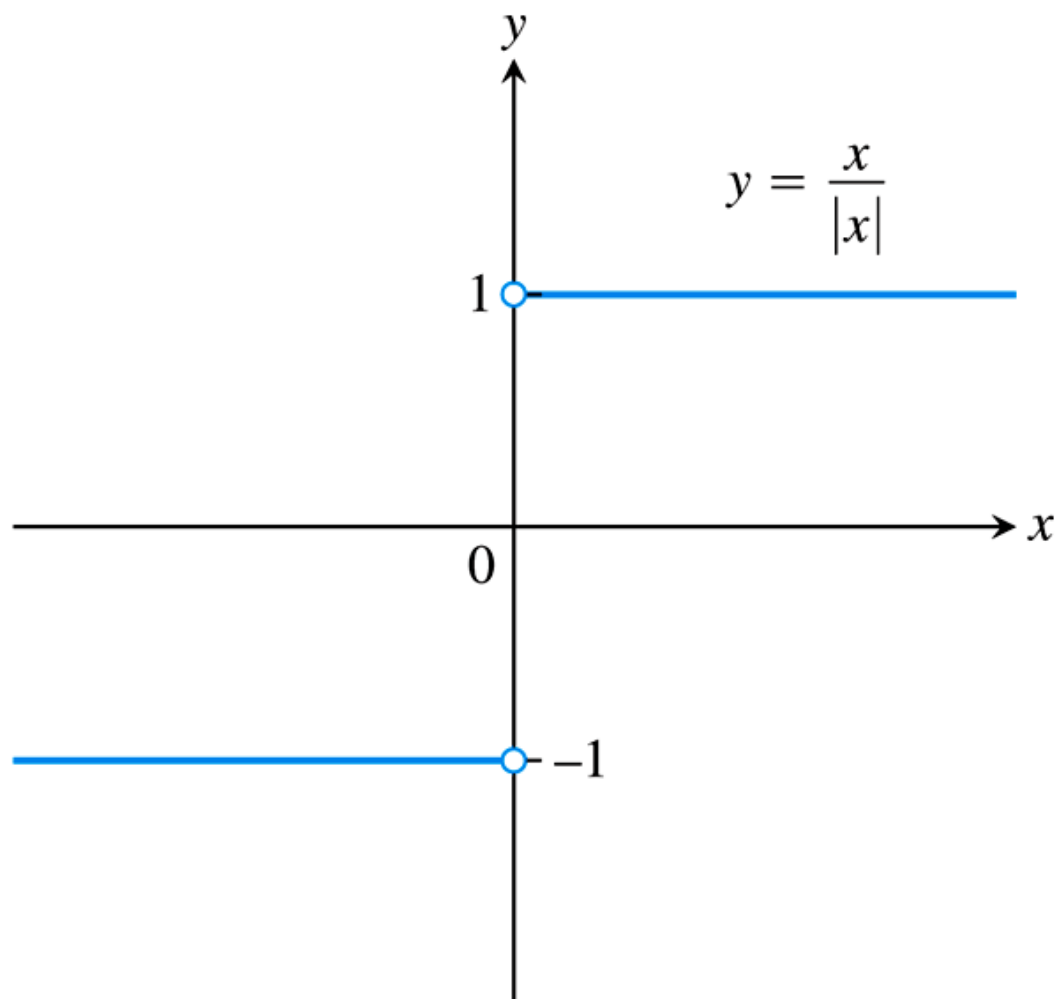


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Intuitively, if $f(x)$ is defined on an interval (c, b) , where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c . We write

$$\lim_{x \rightarrow c^+} f(x) = L.$$

The symbol " $x \rightarrow c^+$ " means that we consider only values of x greater than c .

Similarly, if $f(x)$ is defined on an interval (a, c) , where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c . We write

$$\lim_{x \rightarrow c^-} f(x) = M.$$

The symbol " $x \rightarrow c^-$ " means that we consider only x -values less than c .

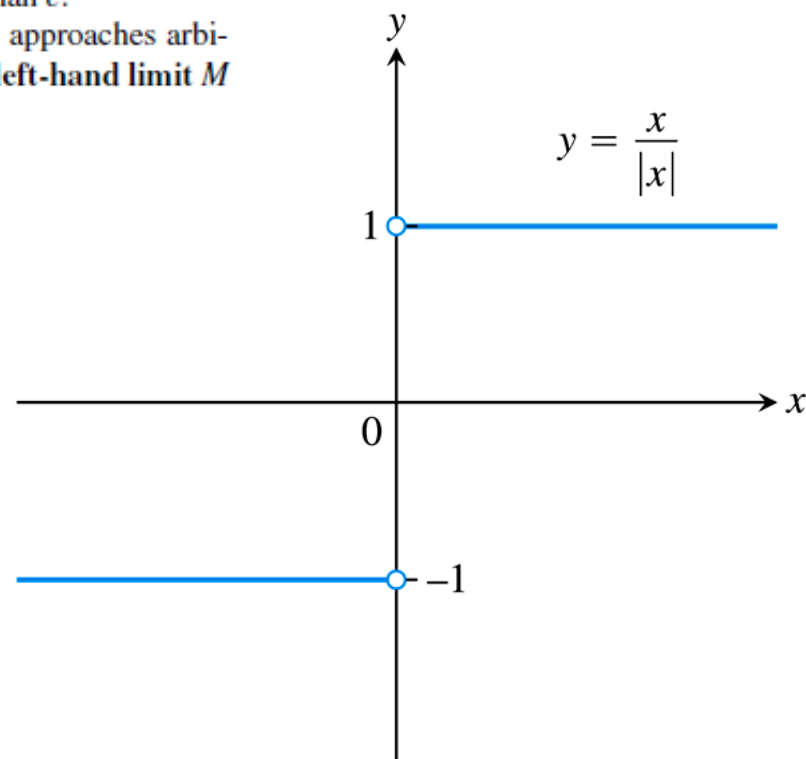
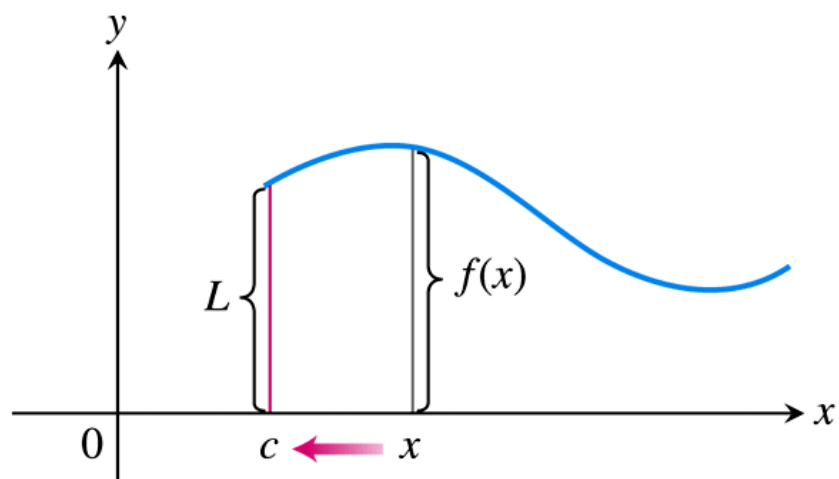
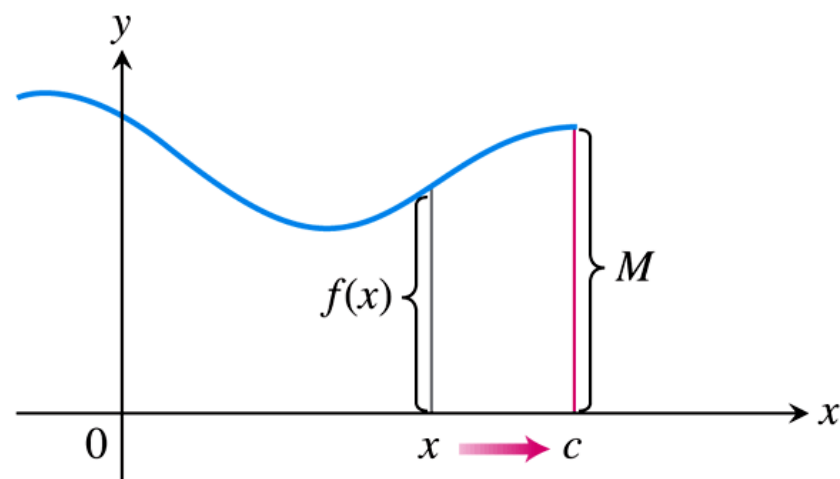


FIGURE 2.24 Different right-hand and left-hand limits at the origin.



(a) $\lim_{x \rightarrow c^+} f(x) = L$



(b) $\lim_{x \rightarrow c^-} f(x) = M$

FIGURE 2.25 (a) Right-hand limit as x approaches c . (b) Left-hand limit as x approaches c .

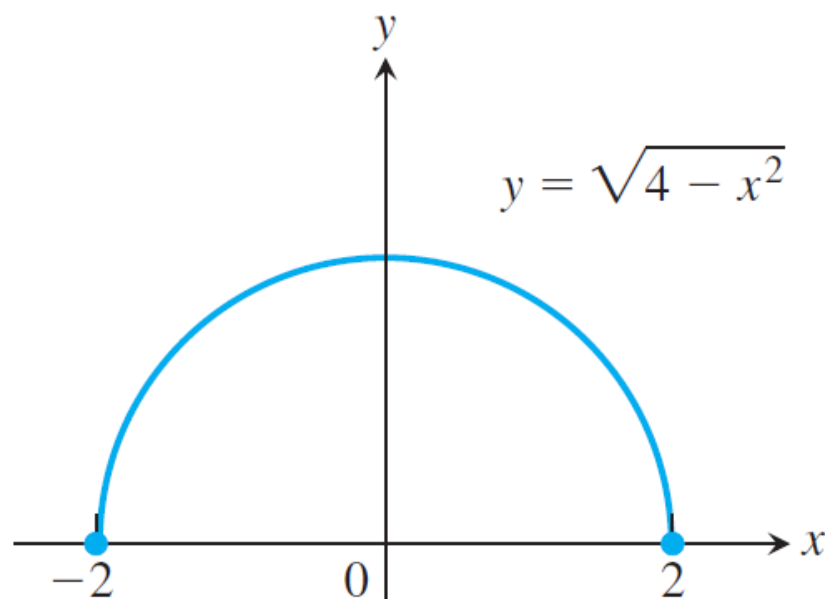


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has right-hand limit 0 at $x = -2$ and left-hand limit 0 at $x = 2$ (Example 1).

$$\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0.$$

DEFINITIONS We say that $f(x)$ has **right-hand limit** L at c , and write

$$\lim_{x \rightarrow c^+} f(x) = L \quad (\text{see Figure 2.28})$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit** L at c , and write

$$\lim_{x \rightarrow c^-} f(x) = L \quad (\text{see Figure 2.29})$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

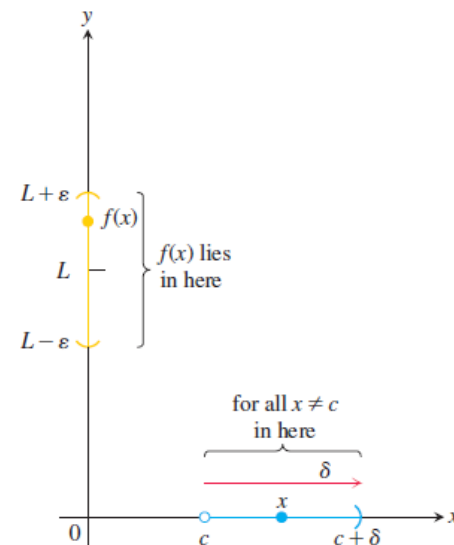


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

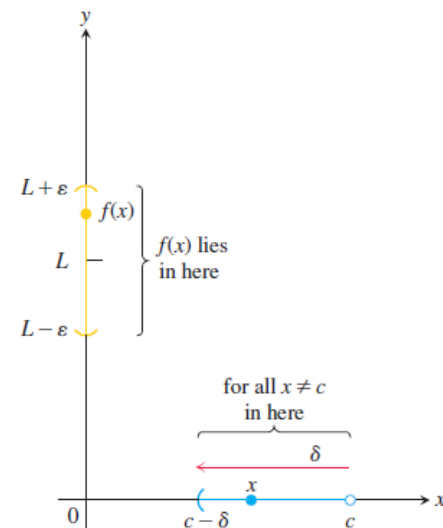


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

EXAMPLE 2 For the function graphed in Figure 2.27,

- At $x = 0$: $\lim_{x \rightarrow 0^+} f(x) = 1$,
 $\lim_{x \rightarrow 0^-} f(x)$ and $\lim_{x \rightarrow 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.
- At $x = 1$: $\lim_{x \rightarrow 1^-} f(x) = 0$ even though $f(1) = 1$,
 $\lim_{x \rightarrow 1^+} f(x) = 1$,
 $\lim_{x \rightarrow 1} f(x)$ does not exist. The right- and left-hand limits are not equal.
- At $x = 2$: $\lim_{x \rightarrow 2^-} f(x) = 1$,
 $\lim_{x \rightarrow 2^+} f(x) = 1$,
 $\lim_{x \rightarrow 2} f(x) = 1$ even though $f(2) = 2$.
- At $x = 3$: $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} f(x) = f(3) = 2$.
- At $x = 4$: $\lim_{x \rightarrow 4^-} f(x) = 1$ even though $f(4) \neq 1$,
 $\lim_{x \rightarrow 4^+} f(x)$ and $\lim_{x \rightarrow 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.
- At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$.

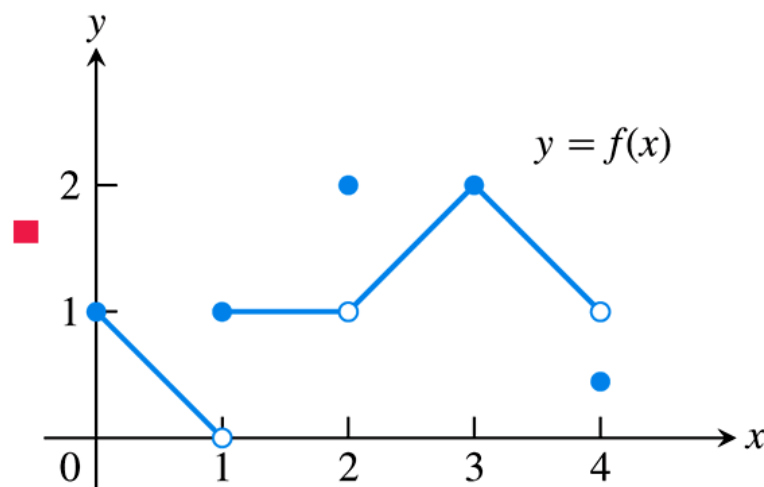


FIGURE 2.27 Graph of the function in Example 2.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

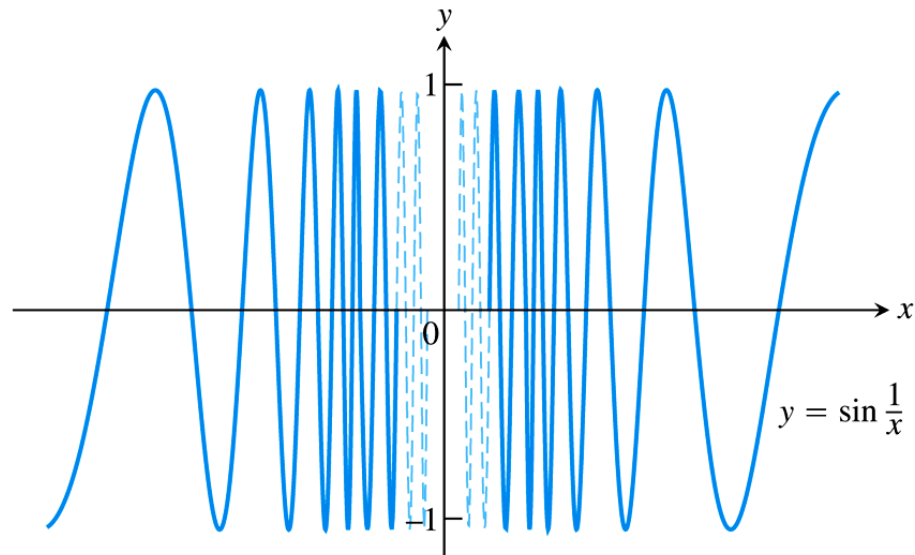
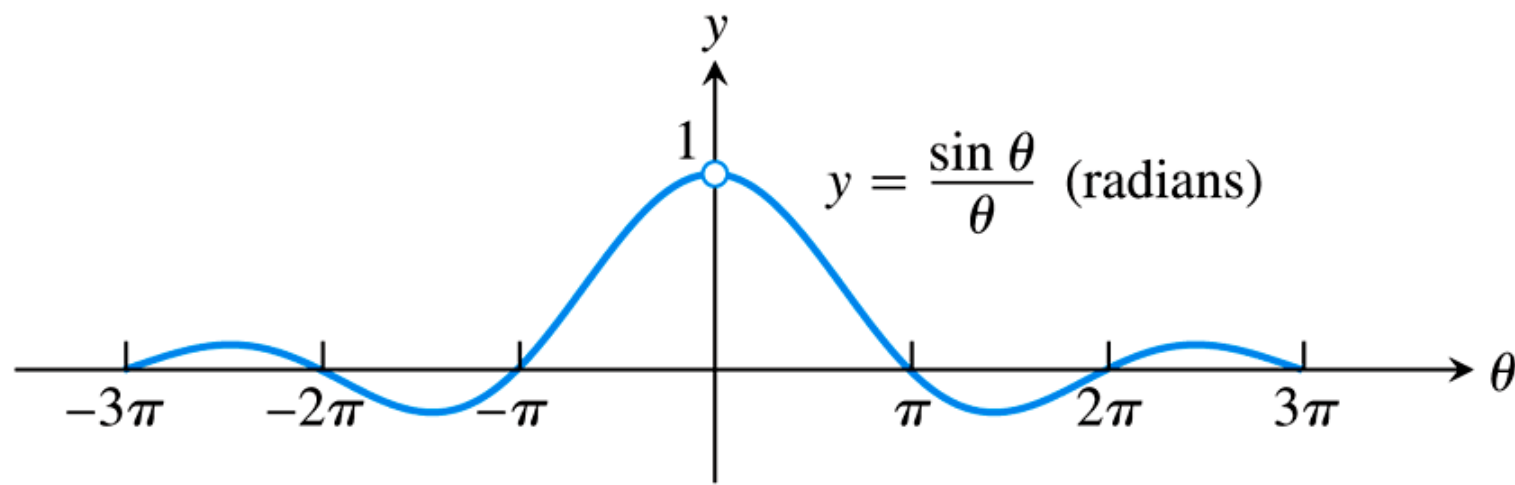


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y -axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1 . There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■



NOT TO SCALE

FIGURE 2.32 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$

Proof of Theorem 7

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \triangle OAP < \text{area sector } OAP < \text{area } \triangle OAT.$$

We can express these areas in terms of θ as follows:

$$\text{Area } \triangle OAP = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\sin \theta) = \frac{1}{2} \sin \theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \quad (2)$$

$$\text{Area } \triangle OAT = \frac{1}{2} \text{base} \times \text{height} = \frac{1}{2}(1)(\tan \theta) = \frac{1}{2} \tan \theta.$$

Thus,

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2) \sin \theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

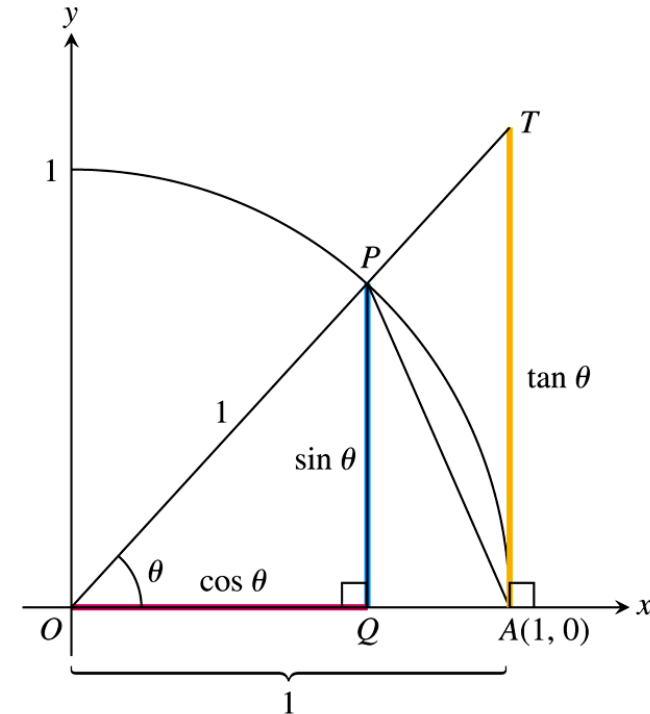


FIGURE 2.33 The figure for the proof of Theorem 7. By definition, $TA/OA = \tan \theta$, but $OA = 1$, so $TA = \tan \theta$.

Proof of Theorem 7 (Cont'd)

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin \theta}{\theta} > \cos \theta.$$

Since $\lim_{\theta \rightarrow 0^+} \cos \theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin \theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin \theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta} = 1 = \lim_{\theta \rightarrow 0^+} \frac{\sin \theta}{\theta},$$

so $\lim_{\theta \rightarrow 0} (\sin \theta)/\theta = 1$ by Theorem 6. ■

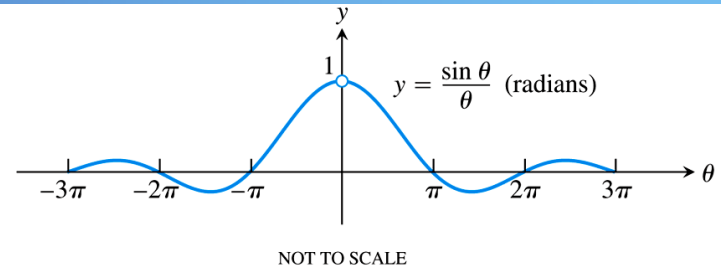


FIGURE 2.32 The graph of $f(\theta) = (\sin \theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

2.5

Continuity

Intuitively, a function is continuous when you can draw its graph without lifting your pencil

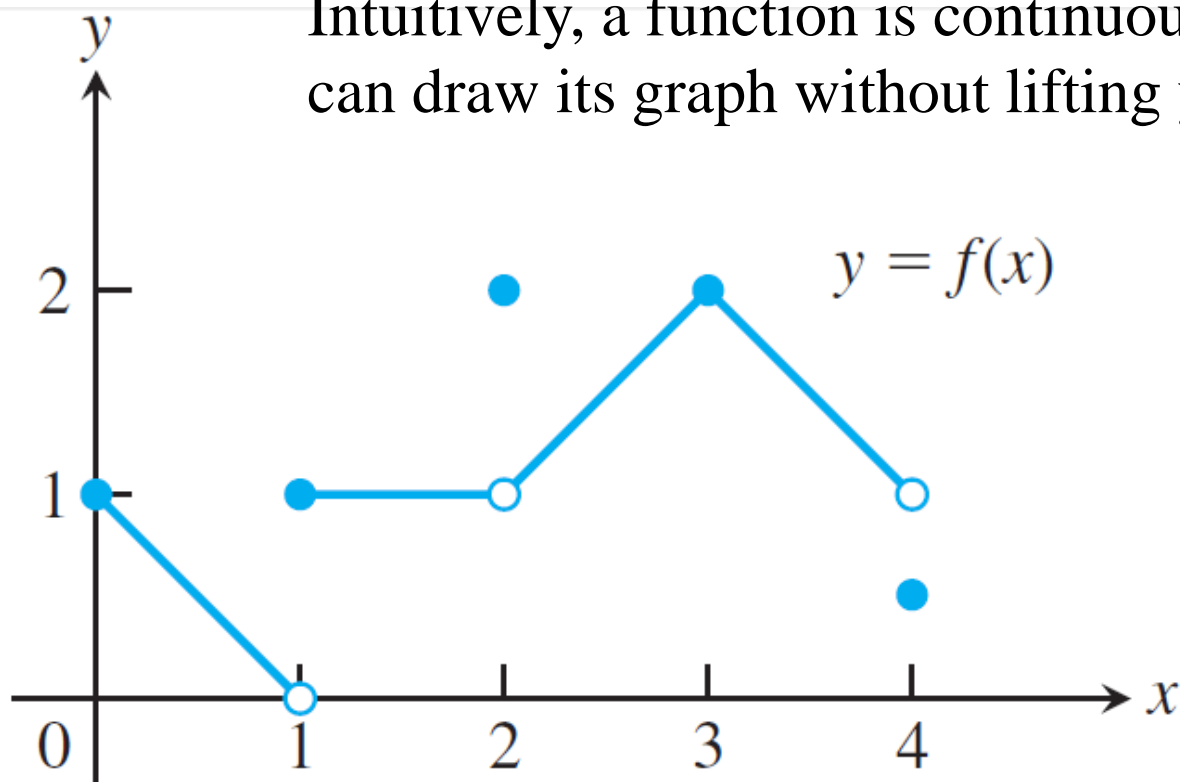


FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

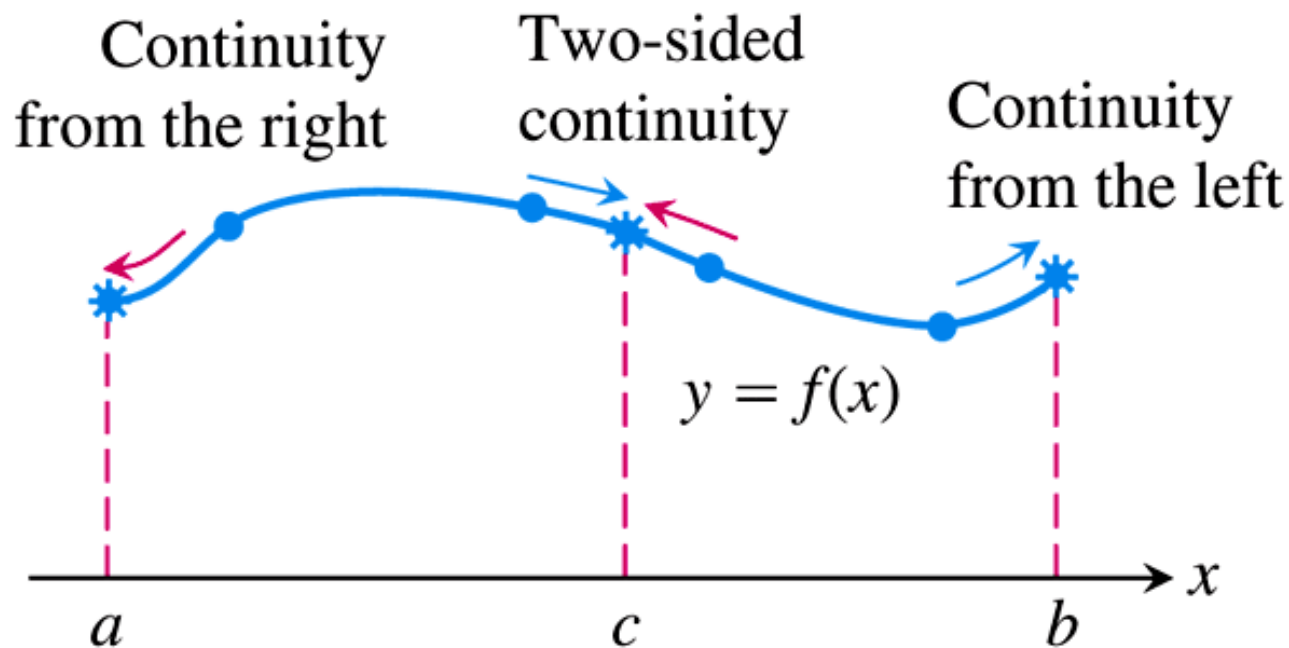


FIGURE 2.36 Continuity at points a , b , and c .

DEFINITIONS Let c be a real number on the x -axis.

The function f is **continuous at c** if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \rightarrow c^-} f(x) = f(c).$$

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.37). It is right-continuous at $x = -2$, and left-continuous at $x = 2$. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.38, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

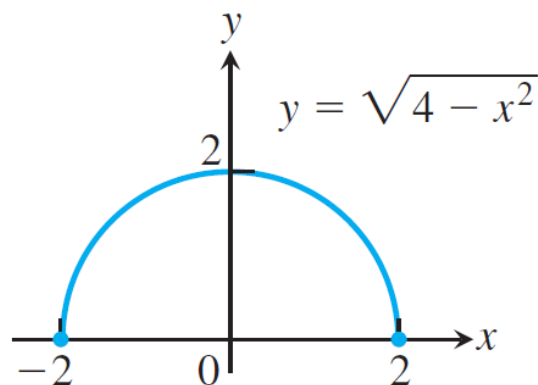


FIGURE 2.37 A function that is continuous over its domain (Example 2).

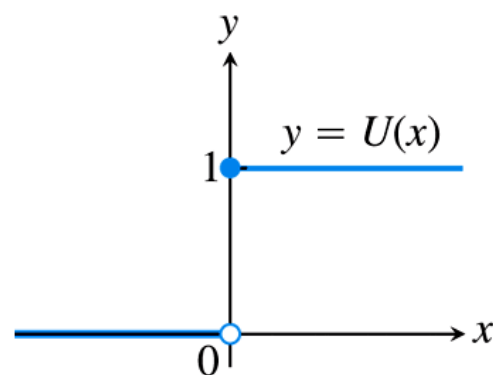


FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \rightarrow c} f(x)$ exists (f has a limit as $x \rightarrow c$).
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit equals the function value).

The floor function $\lfloor x \rfloor$, or the greatest integer function is the greatest integer less than or equal to x

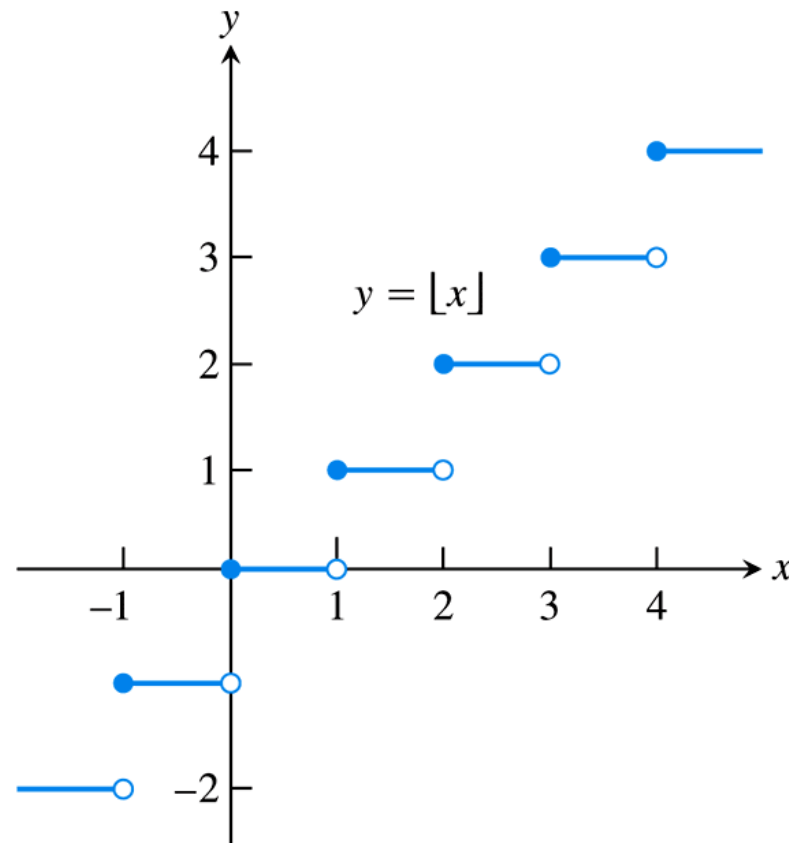


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

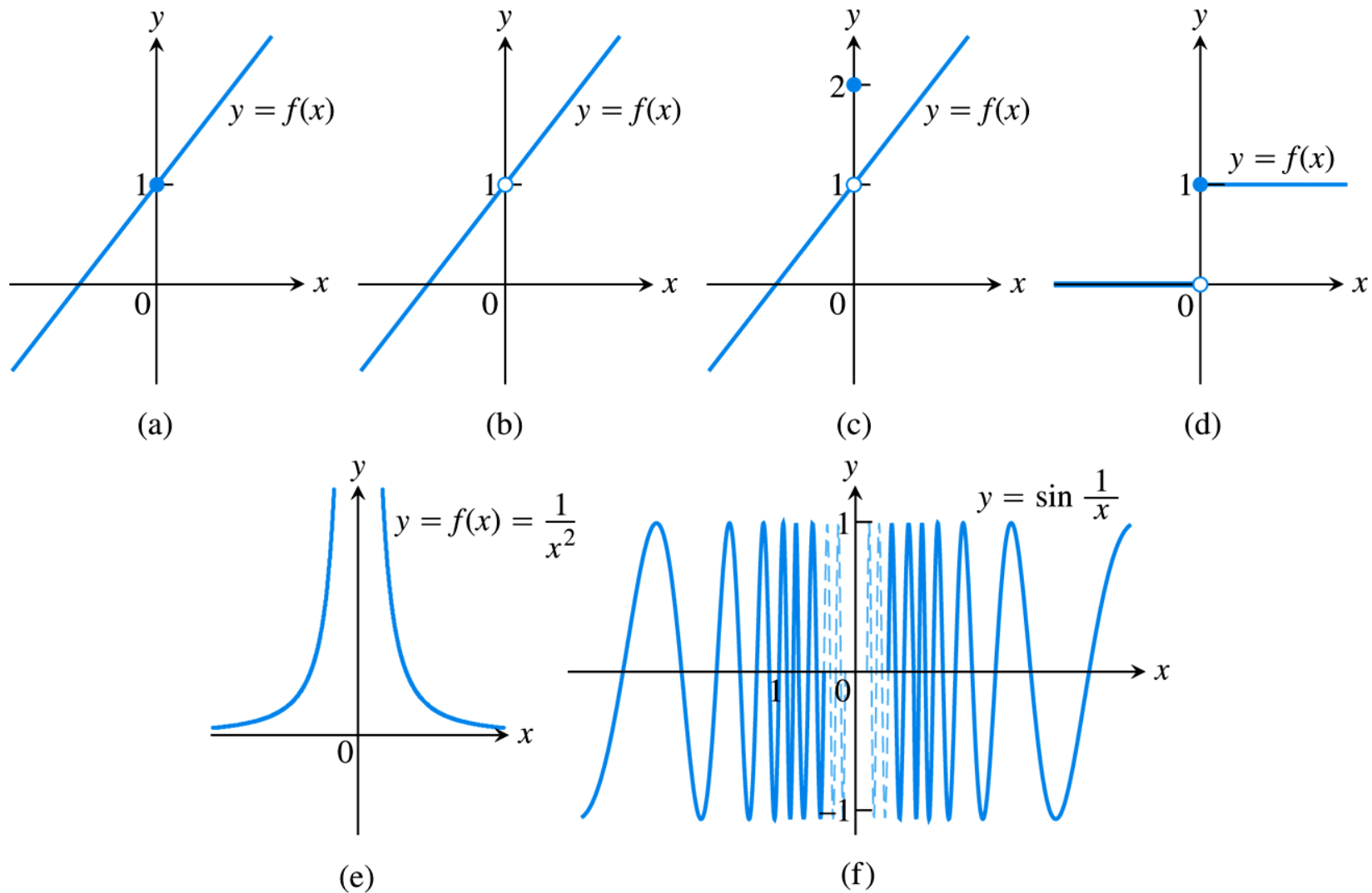


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g , provided $g(c) \neq 0$
6. *Powers:* f^n , n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an open interval containing c , where n is a positive integer

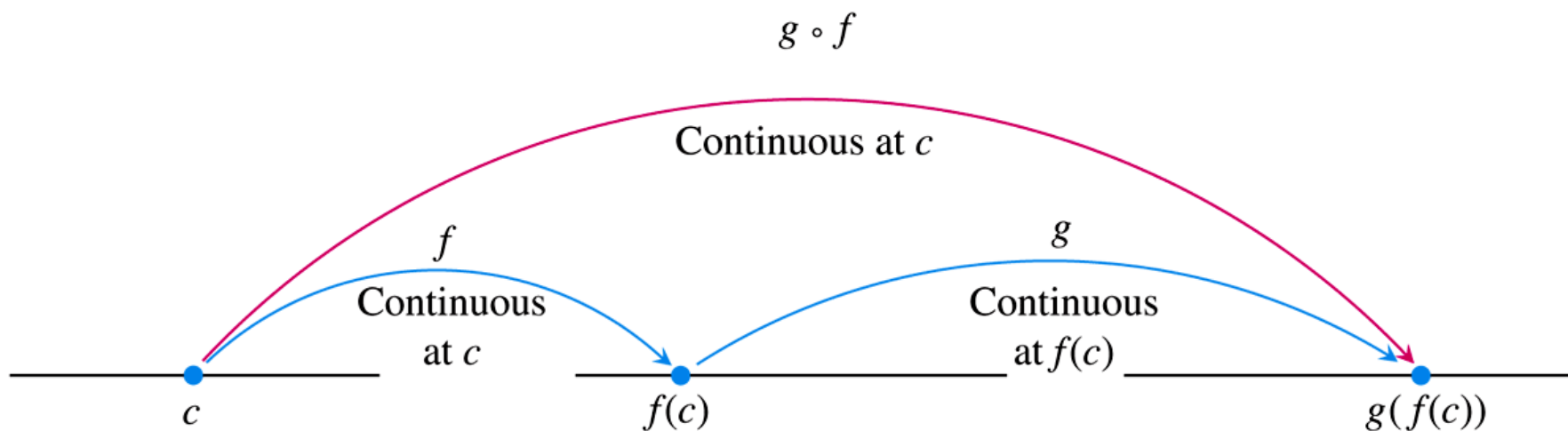


FIGURE 2.42 Composites of continuous functions are continuous.

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c .

THEOREM 10—Limits of Continuous Functions

If g is continuous at the point b and $\lim_{x \rightarrow c} f(x) = b$, then

$$\lim_{x \rightarrow c} g(f(x)) = g(b) = g(\lim_{x \rightarrow c} f(x)).$$

Proof Let $\epsilon > 0$ be given. Since g is continuous at b , there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \epsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1.$$

Since $\lim_{x \rightarrow c} f(x) = b$, there exists a $\delta > 0$ such that

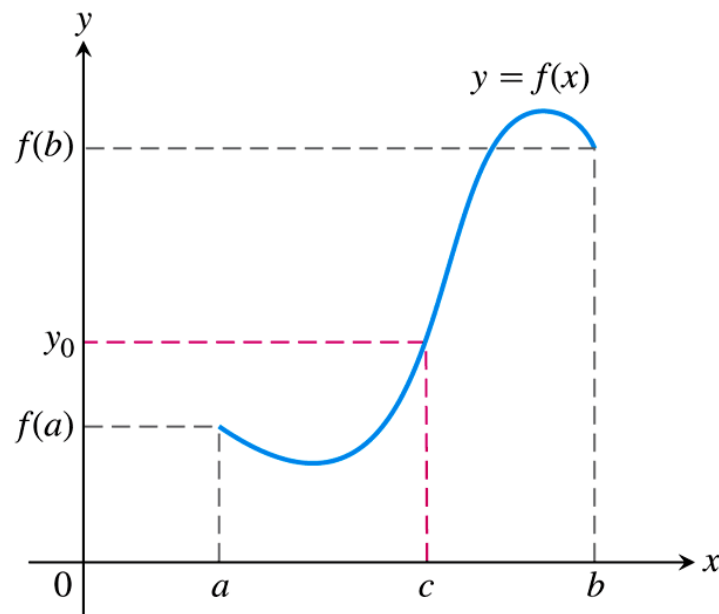
$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \epsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, this proves that $\lim_{x \rightarrow c} g(f(x)) = g(b)$. ■

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps like the graph of the greatest integer function (Figure 2.39), or separate branches like the graph of $1/x$ (Figure 2.41).

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or **zero** of the function f . The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is continuous, the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$. ■

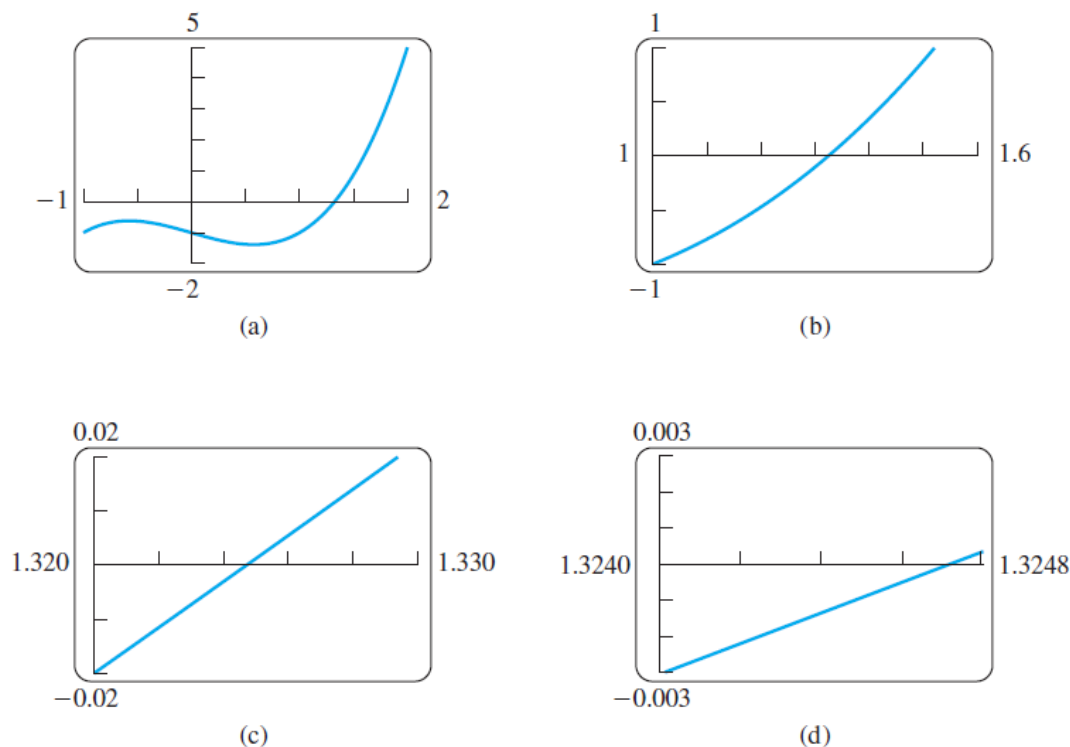


FIGURE 2.45 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

Continuous Extension to a Point

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f , to include $x = c$, creating a new function that is continuous at $x = c$.

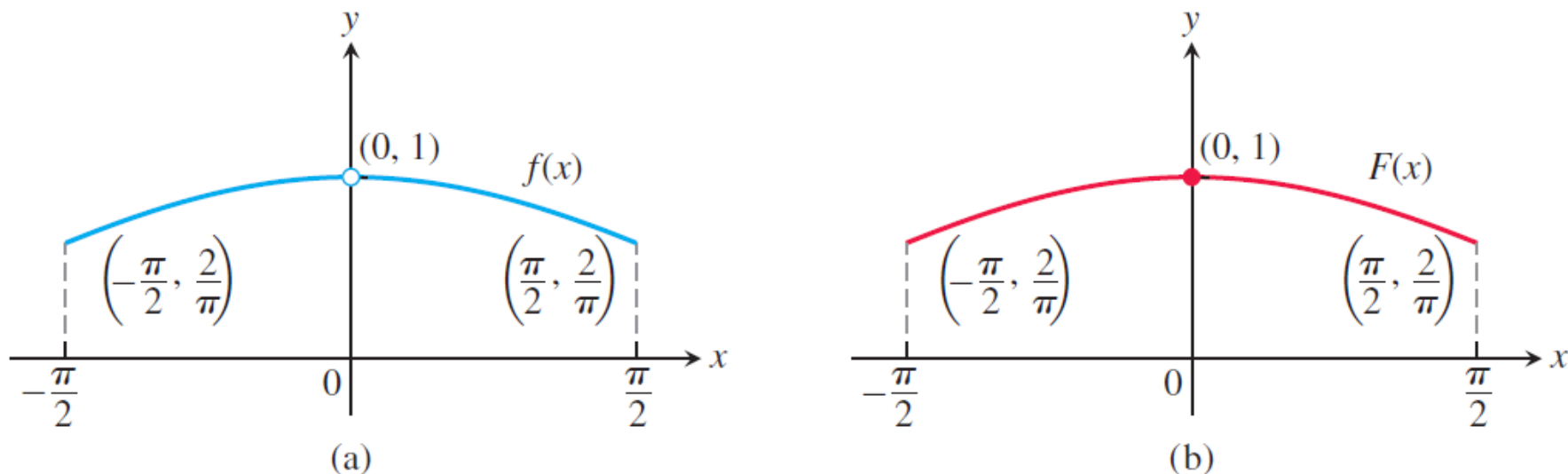


FIGURE 2.47 The graph (a) of $f(x) = (\sin x)/x$ for $-\pi/2 \leq x \leq \pi/2$ does not include the point $(0, 1)$ because the function is not defined at $x = 0$. (b) We can remove the discontinuity from the graph by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \rightarrow 0} f(x)$.

EXAMPLE 12 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \frac{x + 3}{x + 2}.$$

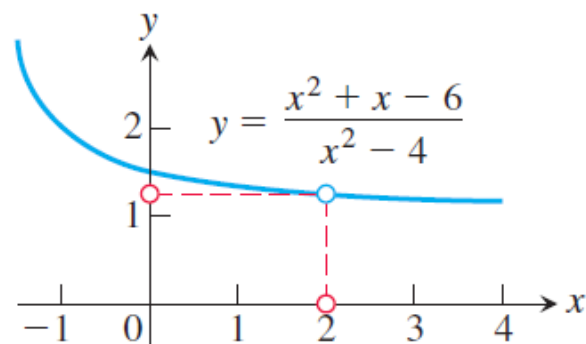
The new function

$$F(x) = \frac{x + 3}{x + 2}$$

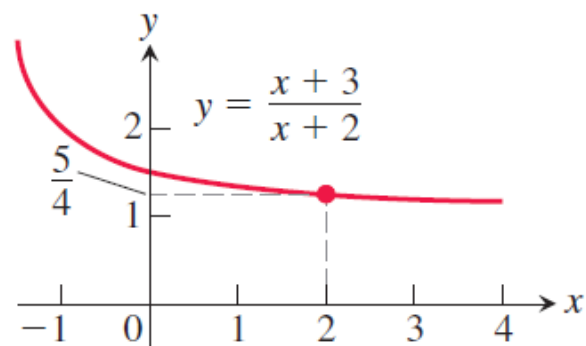
is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \rightarrow 2} f(x) = \frac{5}{4}.$$

The graph of f is shown in Figure 2.48. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f with its point of discontinuity at $x = 2$ removed. ■



(a)



(b)

FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

2.6

Limits Involving Infinity; Asymptotes of Graphs

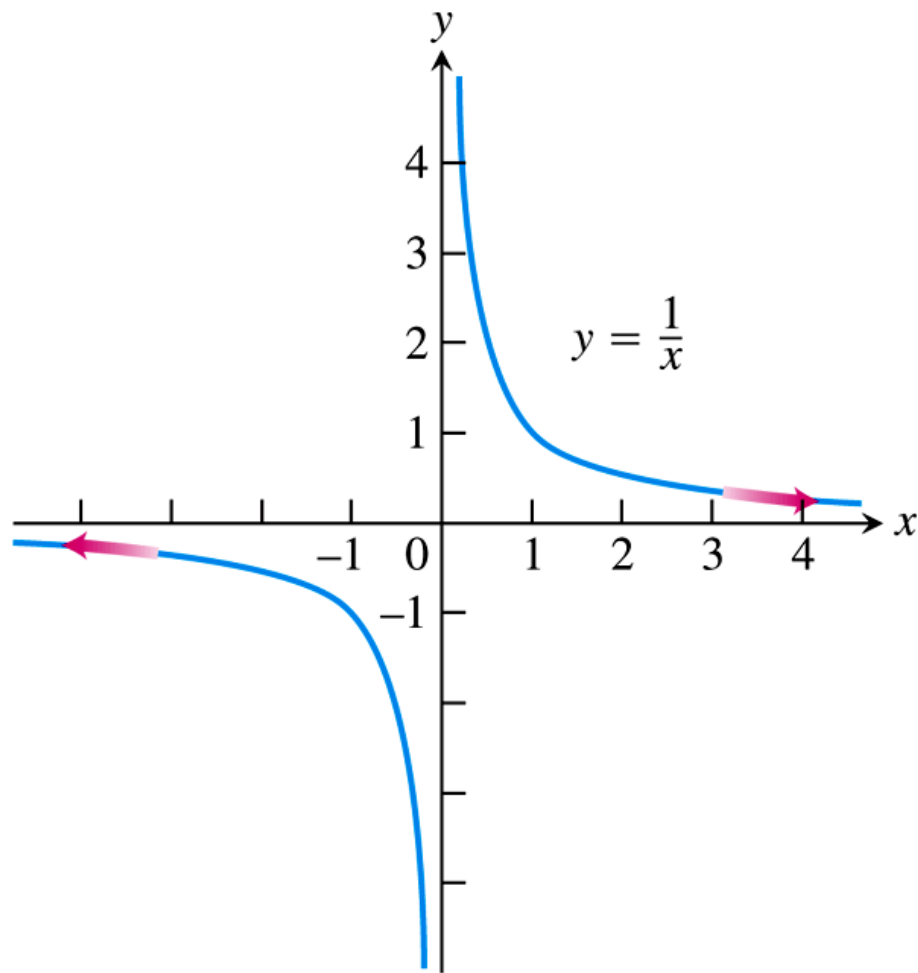


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

EXAMPLE 1 Show that

(a) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

(b) $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Solution

(a) Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \quad \Rightarrow \quad \left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \epsilon.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number (Figure 2.50). This proves $\lim_{x \rightarrow \infty} (1/x) = 0$.

(b) Let $\epsilon > 0$ be given. We must find a number N such that for all x

$$x < N \quad \Rightarrow \quad \left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \epsilon.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$ (Figure 2.50). This proves $\lim_{x \rightarrow -\infty} (1/x) = 0$.

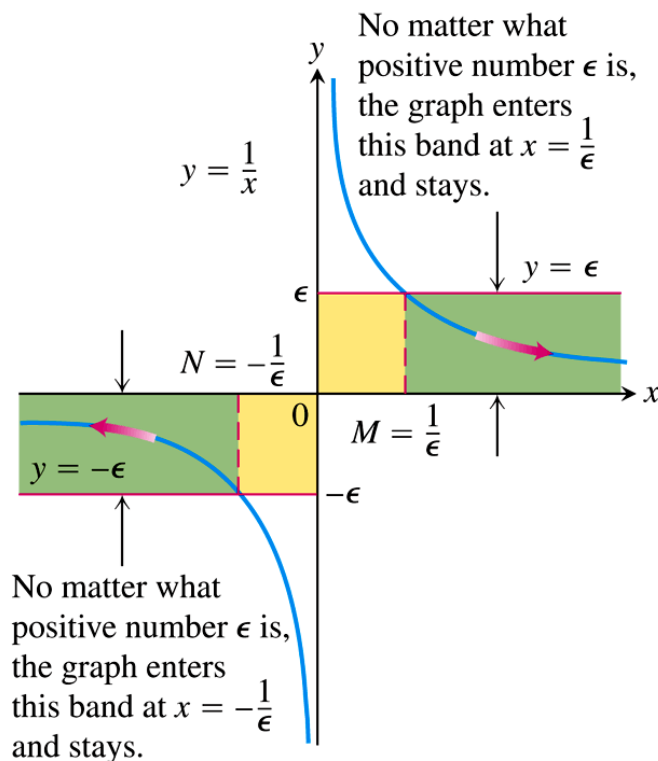


FIGURE 2.50 The geometry behind the argument in Example 1.

EXAMPLE 1 Show that

(a) $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

(b) $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$.

Solution

(a) Let $\varepsilon > 0$ be given. We must find a number M such that

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \quad \text{whenever} \quad x > M.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number (Figure 2.50).

This proves $\lim_{x \rightarrow \infty} (1/x) = 0$.

(b) Let $\varepsilon > 0$ be given. We must find a number N such that

$$\left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \quad \text{whenever} \quad x < N.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$ (Figure 2.50).

This proves $\lim_{x \rightarrow -\infty} (1/x) = 0$. ■

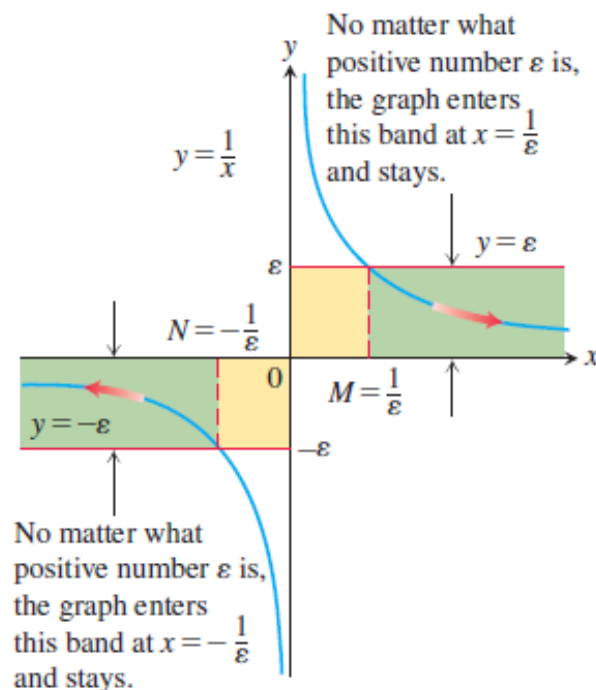



FIGURE 2.50 The geometry behind the argument in Example 1.

THEOREM 12 All the limit laws in Theorem 1 are true when we replace $\lim_{x \rightarrow c}$ by $\lim_{x \rightarrow \infty}$ or $\lim_{x \rightarrow -\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c .

$$\begin{aligned}\text{(a)} \quad \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} && \text{Sum Rule} \\ &= 5 + 0 = 5 && \text{Known limits}\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} \\ &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} && \text{Product Rule} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 && \text{Known limits}\end{aligned}$$


Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \rightarrow \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)} && \text{Divide numerator and denominator by } x^2. \\ &= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3} && \text{See Fig. 2.51.} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)} && \text{Divide numerator and denominator by } x^3. \\ &= \frac{0 + 0}{2 - 0} = 0 && \text{See Fig. 2.52.} \end{aligned}$$

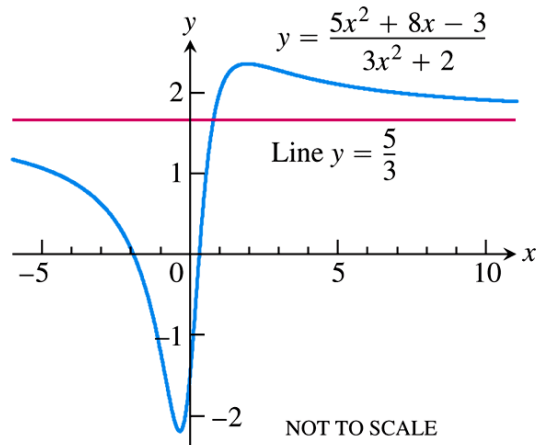


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

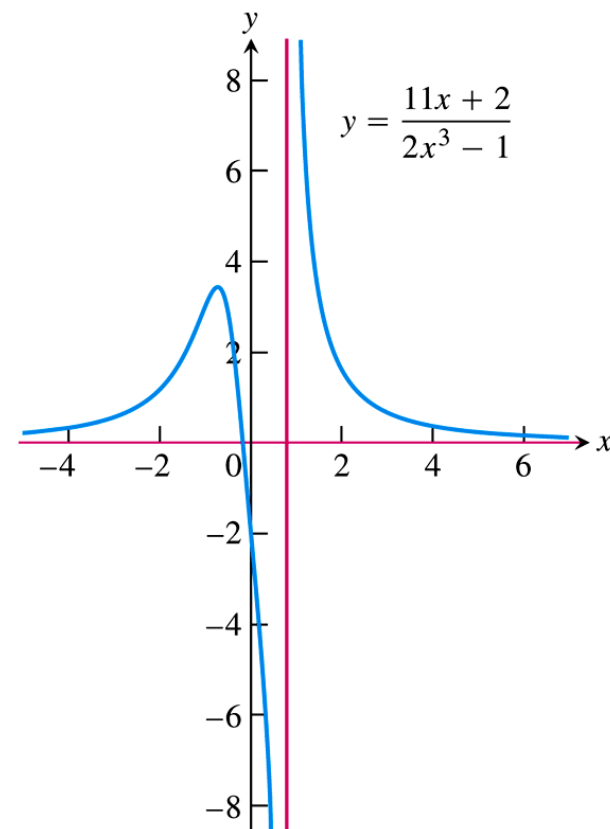


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x -axis as $|x|$ increases.

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \rightarrow \pm\infty$.

$$\text{For } x \geq 0: \lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

$$\text{For } x < 0: \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x . ■

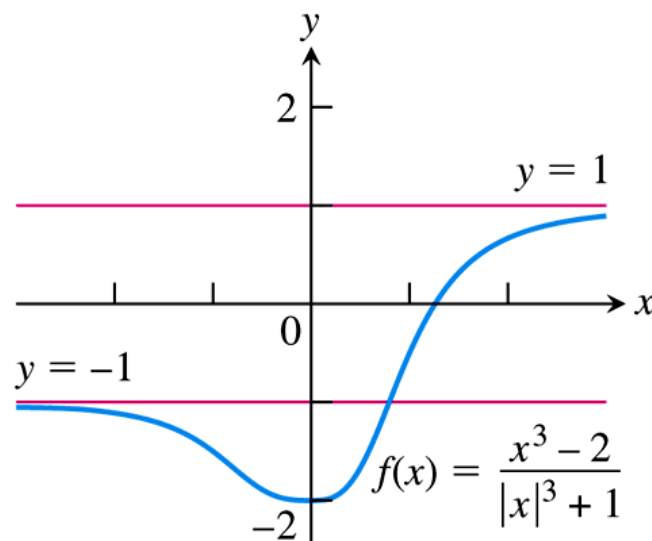


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 5 Find (a) $\lim_{x \rightarrow \infty} \sin(1/x)$ and (b) $\lim_{x \rightarrow \pm\infty} x \sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \rightarrow 0^+$ as $x \rightarrow \infty$ (see Figure 2.49). Therefore,

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$:

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.54, and we see that the line $y = 1$ is a horizontal asymptote.

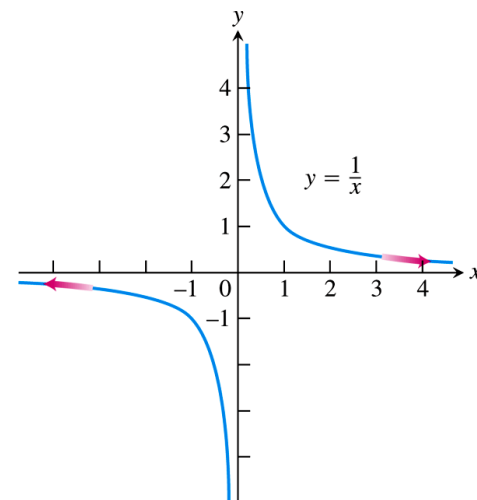


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

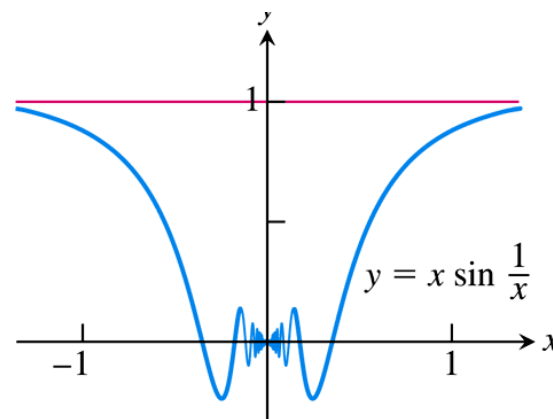


FIGURE 2.54 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 5b).

THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$

EXAMPLE 6 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. Since

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

and $\lim_{x \rightarrow \pm\infty} |1/x| = 0$, we have $\lim_{x \rightarrow \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.55).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

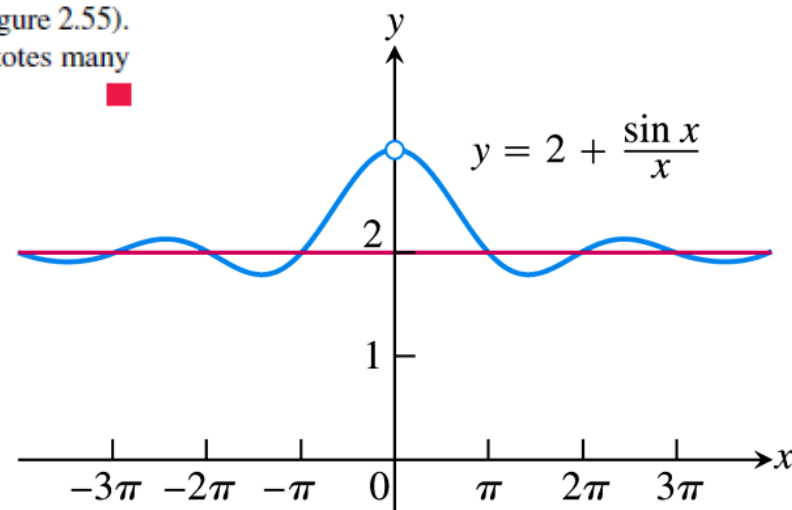


FIGURE 2.55 A curve may cross one of its asymptotes infinitely often (Example 6).

EXAMPLE 7 Find $\lim_{x \rightarrow 0^+} x \left\lfloor \frac{1}{x} \right\rfloor$.

Solution We let $t = 1/x$ so that

$$\lim_{x \rightarrow 0^+} x \left\lfloor \frac{1}{x} \right\rfloor = \lim_{t \rightarrow \infty} \frac{1}{t} \lfloor t \rfloor$$

From the graph in Figure 2.56, we see that $t - 1 \leq \lfloor t \rfloor \leq t$, which gives

$$1 - \frac{1}{t} \leq \frac{1}{t} \lfloor t \rfloor \leq 1 \quad \text{Multiply inequalities by } \frac{1}{t} > 0.$$

It follows from the Sandwich Theorem that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \lfloor t \rfloor = 1,$$

so 1 is the value of the limit we seek.

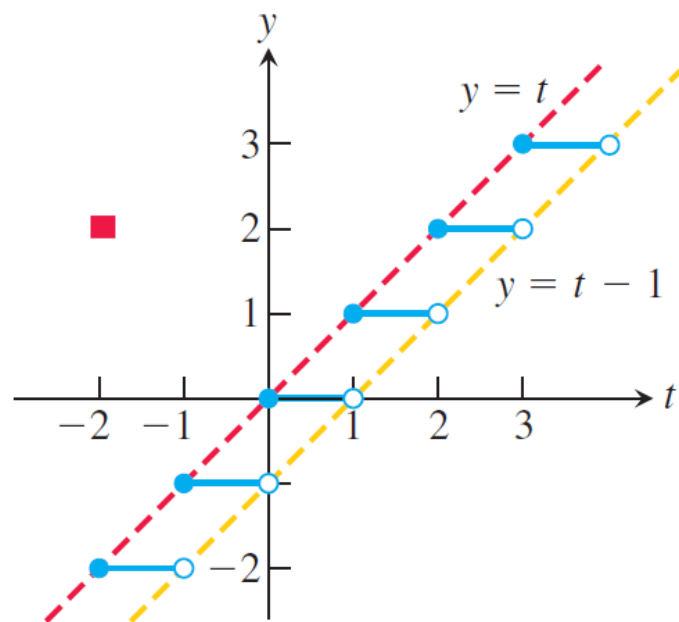


FIGURE 2.56 The graph of the greatest integer function $y = \lfloor t \rfloor$ is sandwiched between $y = t - 1$ and $y = t$.

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \rightarrow \pm\infty$.

EXAMPLE 9 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.57.

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{) x^2 - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \rightarrow \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g , goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.57). The line $y = g(x)$ is an asymptote both to the right and to the left. The next subsection will confirm that the function $f(x)$ grows arbitrarily large in absolute value as $x \rightarrow 2$ (where the denominator is zero), as shown in the graph. ■

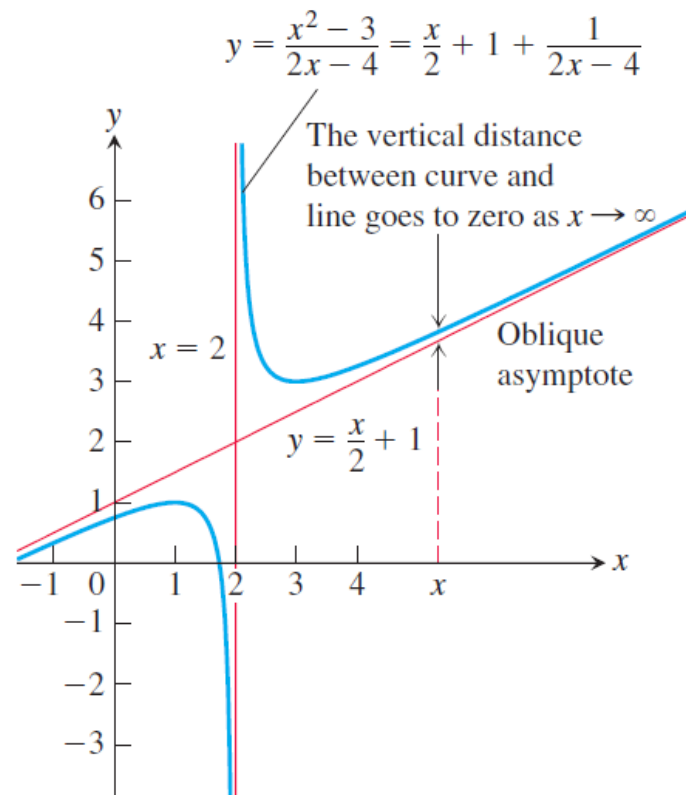


FIGURE 2.57 The graph of the function in Example 9 has an oblique asymptote.

Infinite Limits

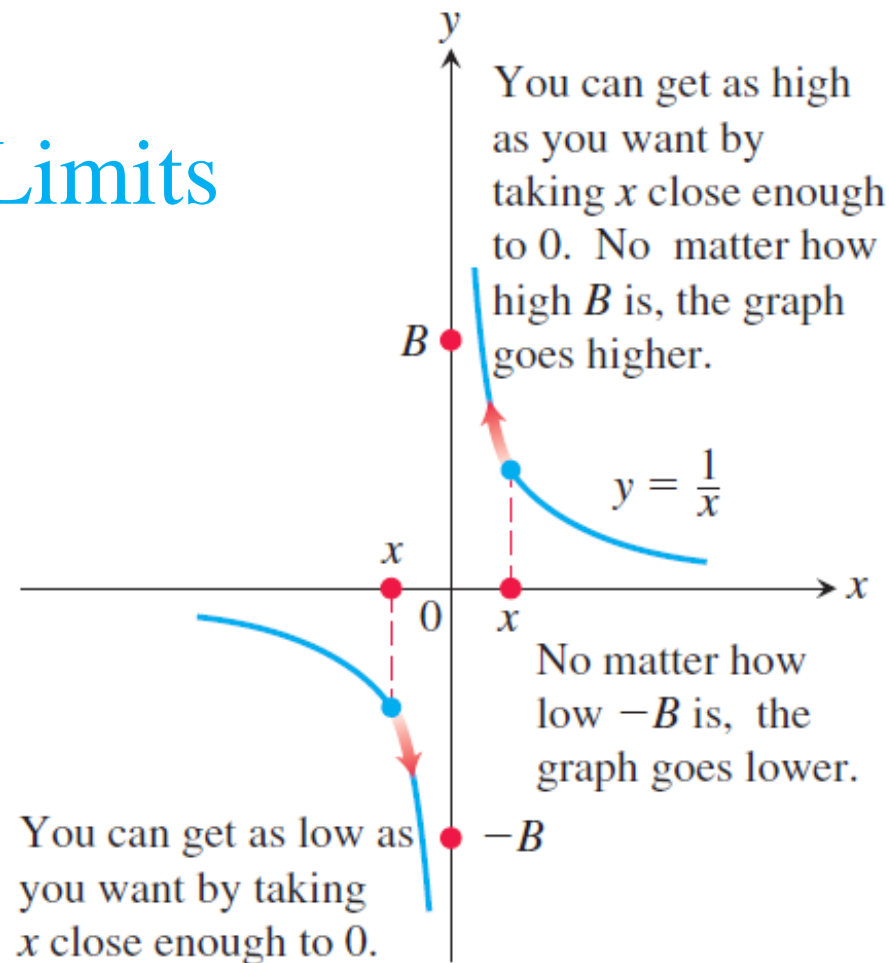


FIGURE 2.58 One-sided infinite limits:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$

EXAMPLE 10 Find $\lim_{x \rightarrow 1^+} \frac{1}{x-1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x-1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.59). Therefore, $y = 1/(x-1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x \rightarrow 1^+} \frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x-1} = -\infty.$$

Analytic Solution Think about the number $x-1$ and its reciprocal. As $x \rightarrow 1^+$, we have $(x-1) \rightarrow 0^+$ and $1/(x-1) \rightarrow \infty$. As $x \rightarrow 1^-$, we have $(x-1) \rightarrow 0^-$ and $1/(x-1) \rightarrow -\infty$. ■

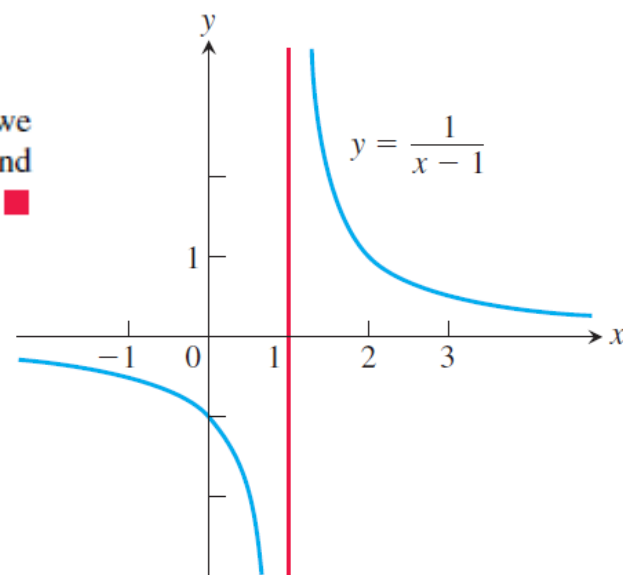


FIGURE 2.59 Near $x = 1$, the function $y = 1/(x-1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 10).

EXAMPLE 11 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \quad \text{as} \quad x \rightarrow 0.$$

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.60). This means that

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \rightarrow 0$. We have $1/x \rightarrow \infty$ if $x \rightarrow 0^+$, but $1/x \rightarrow -\infty$ if $x \rightarrow 0^-$. All we can say about $\lim_{x \rightarrow 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x \rightarrow 0} (1/x^2) = \infty$. ■

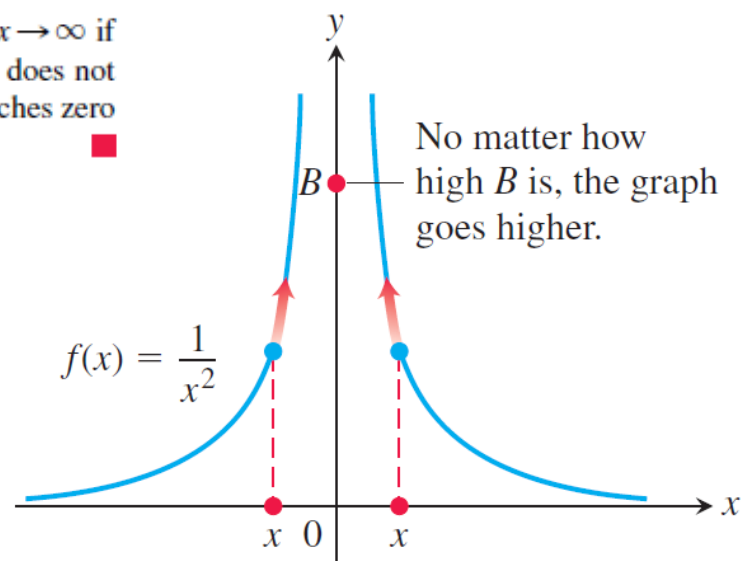


FIGURE 2.60 The graph of $f(x)$ in Example 11 approaches infinity as $x \rightarrow 0$.

EXAMPLE 13 Find $\lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

Solution We are asked to find the limit of a rational function as $x \rightarrow -\infty$, so we divide the numerator and denominator by x^2 , the highest power of x in the denominator:

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \rightarrow -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= \lim_{x \rightarrow -\infty} \frac{2x^2(x - 3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= -\infty, \quad x^{-n} \rightarrow 0, \quad x - 3 \rightarrow -\infty\end{aligned}$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \rightarrow -\infty$. ■

DEFINITIONS

1. We say that $f(x)$ approaches infinity as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) > B.$$

2. We say that $f(x)$ approaches minus infinity as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \quad \Rightarrow \quad f(x) < -B.$$

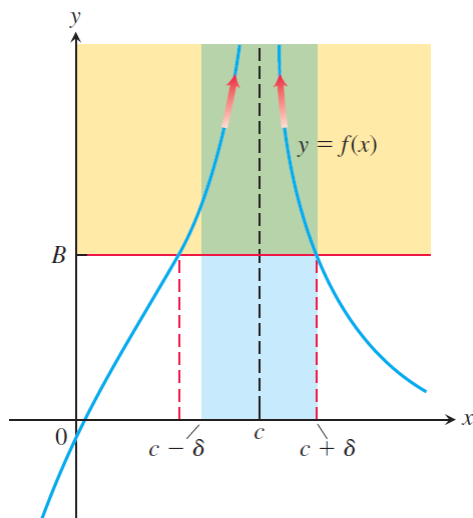


FIGURE 2.61 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

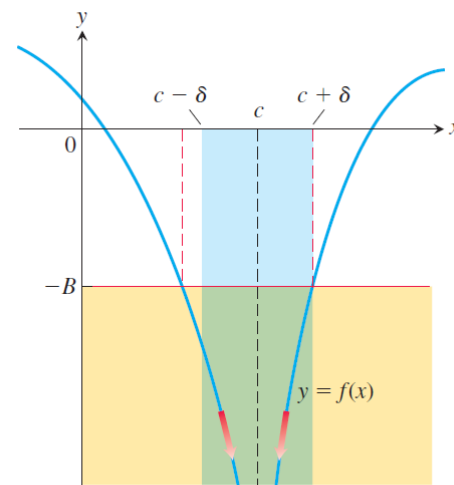


FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

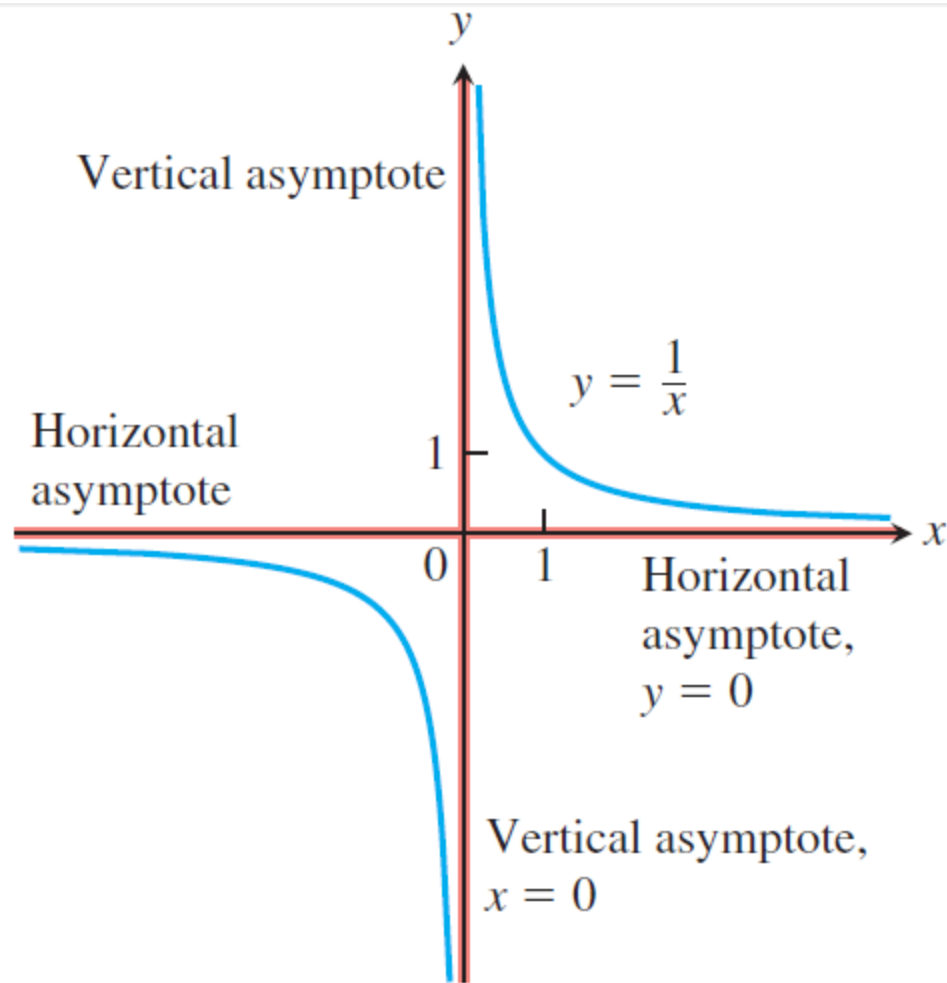


FIGURE 2.63 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

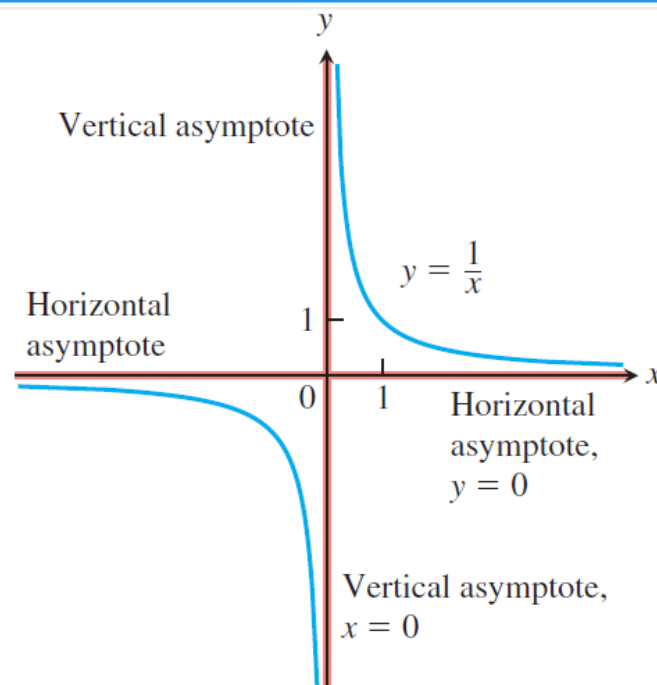


FIGURE 2.63 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

EXAMPLE 16 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and as $x \rightarrow \pm 2$, where the denominator is zero. Notice that f is an even function of x , so its graph is symmetric with respect to the y -axis.

(a) *The behavior as $x \rightarrow \pm\infty$.* Since $\lim_{x \rightarrow \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.65). Notice that the curve approaches the x -axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as $x \rightarrow \pm 2$.* Since

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

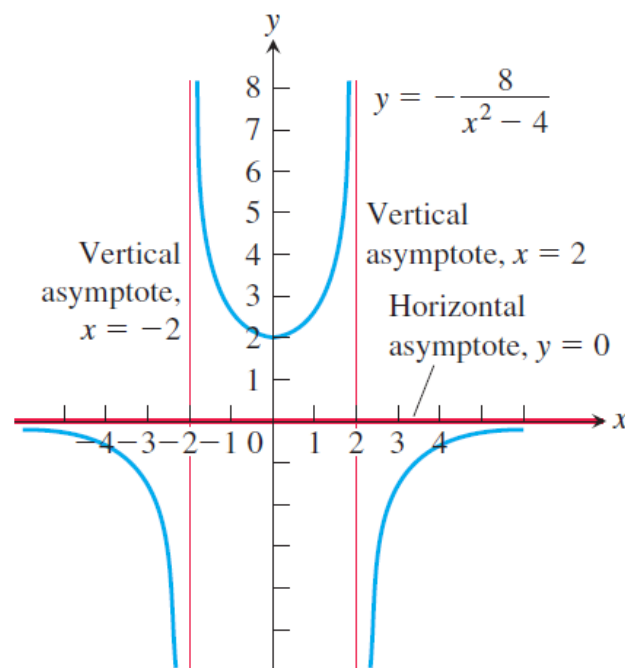


FIGURE 2.65 Graph of the function in Example 16. Notice that the curve approaches the x -axis from only one side. Asymptotes do not have to be two-sided.

3.1

Tangents and the Derivative at a Point

DEFINITIONS

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{provided the limit exists}).$$

The **tangent line** to the curve at P is the line through P with this slope.

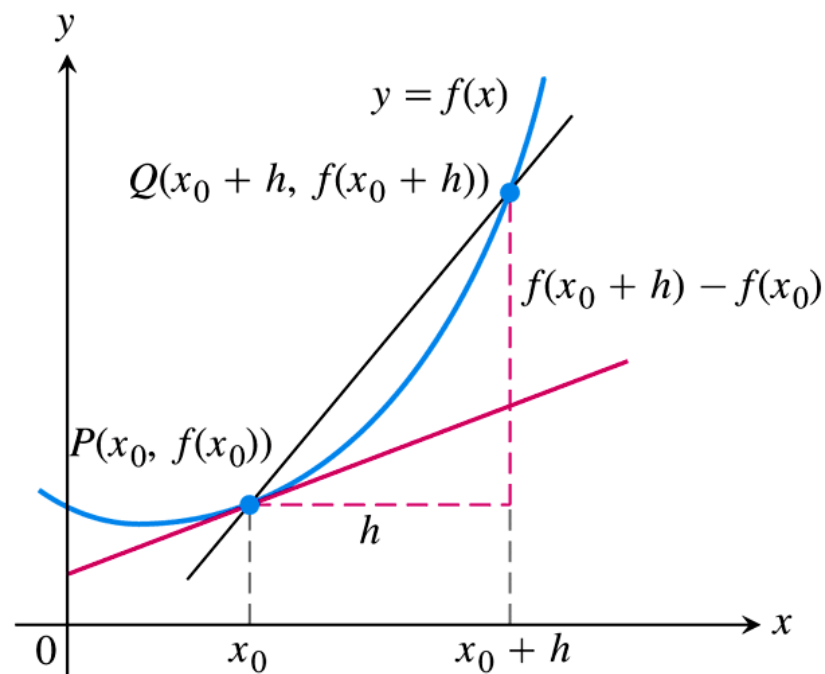


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

EXAMPLE 1

- (a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?
- (b) Where does the slope equal $-1/4$?
- (c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

- (a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)} \\ &= \lim_{h \rightarrow 0} \frac{-h}{ha(a+h)} = \lim_{h \rightarrow 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.\end{aligned}$$

Notice how we had to keep writing “ $\lim_{h \rightarrow 0}$ ” before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

- (b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be $-1/4$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 3.3).

- (c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \rightarrow 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \rightarrow 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off becoming more and more horizontal.

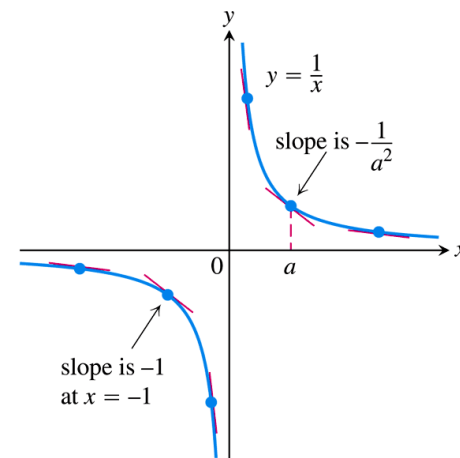


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

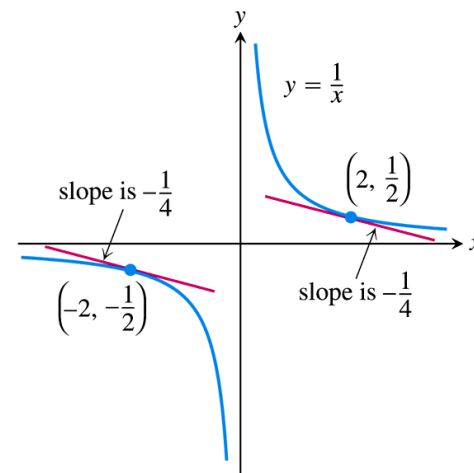


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

Difference Quotient and Derivative

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h** . If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0** , denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

Week 2

Assignment 2

2.4: #2,5,18,19,32,33,40,44

2.5: #23,30,38,45,48,55,64,65,67

2.6: #11,12,18,35,47,84,85,102

3.1: #24,35,36,38,42

The questions above need to be submitted.

Deadline: 10 PM, Friday, Sept 22 --- solutions should be submitted online on Blackboard.

Required Reading (Textbook)

- Sections 2.4, 2.5, 2.6
- Section 3.1

Quiz 1 next week (Week 3, Sept 18 – 22, in tutorials)

Scope = 2.1, 2.2, 2.4, 2.5, 2.6; three problems; 30 minutes.