Slide 9–Vectors I

MAT2040 Linear Algebra

Definition 9.1 (**Euclidean Vector Space**) The set defined as

$$\mathbb{R}^m = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \middle| u_i \in \mathbb{R} \right\}.$$

is called the Euclidean Vector Space.

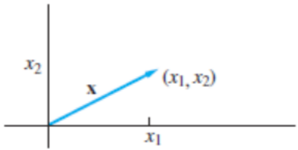
Definition 9.2 (Vector Equality, Addition/Subtraction, Scalar Multiplication)
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, then
(1) $\mathbf{u} = \mathbf{v} \Leftrightarrow u_i = v_i, \forall i = 1, \cdots, m$.
(2) $\mathbf{u} \pm \mathbf{v} = (u_i \pm v_i)_{m\times 1}$.
(3) $\alpha\mathbf{u} = (\alpha u_i)_{m\times 1}, \alpha \in \mathcal{R}$
Zero vector $\mathbf{0} = (0)_{m\times 1}$.

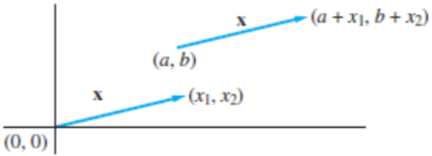
Connection with Geometry

First consider for $\mathbb{R}^2$, we can associate each geometric point (x_1, x_2) in the plane with a column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Geometrically, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be represented by a directed line segment from $(0,0)$ to (x_1, x_2), the Euclidean length of the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the length of the directed line segment from $(0,0)$ to (x_1, x_2). (This is actually called a **Cartesian coordinate system**.)

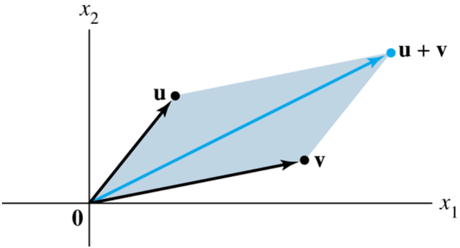
Connection with Geometry





In addition, the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can also be represented by any directed line segment from (a, b) to $(a + x_1, b + x_2)$ since the directed line segment from (a, b) to $(a + x_1, b + x_2)$ has the same Euclidean length and direction as the directed line segment from $(0, 0)$ to (x_1, x_2), that is $\sqrt{x_1^2 + x_2^2}$.

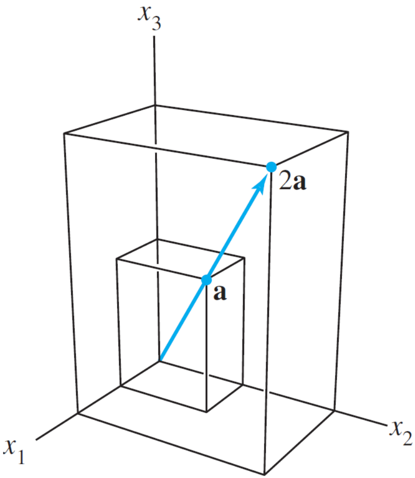
Geometric Interpretation of Vector Addition



The diagonal directed line segment in the parallelogram corresponding to the addition of two vectors **u** and **v**.

We can also do this with 3-dimensional vectors, 4-dimensional, etc. (Will be hard to draw in higher dimensions!)

Geometric Interpretation of Scaling a Vector



(picture from David Lay, Linear Algebra.)

Theorem 9.3 (**Properties of Vector Operations**)

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^m$, $\alpha, \beta \in \mathbb{R}$, then

(1)$\mathbf{u} + \mathbf{v} \in \mathbb{R}^m$.

(2) $\alpha\mathbf{u} \in \mathbb{R}^m$.

(3) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(4) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + \mathbf{v} + \mathbf{w}$.

(5) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$. (Zero vector exists)

(6) If $-\mathbf{u} = (-1)\mathbf{u}$, then $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. (Additive Inverses)

(7) $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

(8) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

(9) $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

(10) $1 \cdot \mathbf{u} = \mathbf{u}$.

Proof. Similar as Matrices operations.

Linear System Revisited: Solution in Parametric Vector Form

Example 9.4

$$
\begin{aligned}
2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 4
\end{aligned}
$$

Its augmented matrix can be reduced into reduced row-echelon form as

$$
\left[\begin{array}{cccc|c}
\boxed{1} & 0 & 3 & -2 & 4 \\
0 & \boxed{1} & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

where $D = \{1, 2\}$–indices for the pivot columns and $F = \{3, 4\}$–indices for the non pivot columns(excluding the last column in the augmented matrix).

Linear System Revisited: Solution in Parametric Vector Form

The equivalent system is

$$\begin{aligned} x_1 + 3x_3 - 2x_4 &= 4 \\ x_2 + x_3 - 3x_4 &= 0 \end{aligned}$$

Write the dependent variables in term of independent variables, the solution can be written as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 & -3x_3 & +2x_4 \\ & -x_3 & +3x_4 \\ & x_3 & \\ & & x_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

where these three vectors need to be filled.

Copying the coefficients in the column directly, we can get

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

The solution is a fixed vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ plus any linear combination of two vectors: $\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$.

We call this type of description of the solution of $A\mathbf{x} = \mathbf{b}$, a description in **parametric vector form**.

One can check that $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies the linear system, we can treat it as a **particular solution**.

While

$$x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

is the solution for the homogeneous linear system

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0 \end{aligned}$$

Definition 9.5 (Particular Solution and Homogeneous Solution)
If $\mathbf{w_0}$ is one solution of the linear system $A\mathbf{x} = \mathbf{b}$, then we can treat $\mathbf{w_0}$ as a **particular solution** for $A\mathbf{x} = \mathbf{b}$. The solution(s) of $A\mathbf{x} = \mathbf{0}$ are called the **homogeneous solutions** for the corresponding linear system $A\mathbf{x} = \mathbf{b}$.

Theorem 9.6 (**Solution Structure of the Linear System**)
Suppose $\mathbf{w}_0$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and S_h is the solution set of $A\mathbf{x} = \mathbf{0}$. Then $S = \{\mathbf{s} | \mathbf{s} = \mathbf{w}_0 + \mathbf{s}_h, \text{ for } \mathbf{s}_h \in S_h\}$ is the solution set of $A\mathbf{x} = \mathbf{b}$.

Proof. By assumption, $A\mathbf{w_0} = \mathbf{b}$. Hence, take any solution $\mathbf{y}$ of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{y} = \mathbf{b}$. Thus, $A(\mathbf{y} - \mathbf{w_0}) = A\mathbf{y} - A\mathbf{w_0} = \mathbf{b} - \mathbf{b} = \mathbf{0}$, therefore, $\mathbf{y} - \mathbf{w_0}$ is a homogeneous solution and $\mathbf{y} - \mathbf{w_0} \in S_h$.

Remark
Any solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{p} = \mathbf{w}_0 + \mathbf{y}$, where $\mathbf{w}_0$ is a **particular solution** of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}$ is the **homogeneous solution**($\mathbf{y}$ is the solution of $A\mathbf{x} = \mathbf{0}$).

Example 9.7

Suppose a linear system's augmented matrix can be reduced into reduced row-echelon form as

$$\left[\begin{array}{cccccc|c} \boxed{1} & 0 & 1 & 0 & -4 & 0 & 7 \\ 0 & \boxed{1} & -2 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where columns 1,2,4,6 are pivot columns. The original system is equivalent to the following system:

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 7 & -x_3 & +4x_5 \\ -5 & 2x_3 & +0x_5 \\ 0 & x_3 & 0x_5 \\ 2 & 0x_3 & -3x_5 \\ 0 & 0x_3 & x_5 \\ 9 & 0x_3 & 0x_5 \end{bmatrix}
$$

$$
= \begin{bmatrix} 7 \\ -5 \\ 0 \\ 2 \\ 0 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}
$$

Where $\begin{bmatrix} 7 \\ -5 \\ 0 \\ 2 \\ 0 \\ 9 \end{bmatrix}$ is the particular solution.

$x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ is the homogeneous solution.

Remark

One can show that $\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$ are **linearly independent** (the concept will be introduced in next slide).