MAT2041

Midterm Exam

Time: Nov/5/23 Sun 4:30pm - 6:30pm University Stadium (Sports Hall)

DURATION OF EXAMINATION: 2 hours

Your exam sheet shall include 6 **problems**. If not, notify the instructors.

1. **(18 pt) Solving a Linear System of Equations**
   Consider the linear system:

$$-x_1 + 4x_2 + 2x_3 + 4x_4 = 1 \tag{1}$$
$$3x_1 - 6x_3 + 4x_4 = 4 \tag{2}$$
$$2x_2 + 2x_4 = 3. \tag{3}$$

(a) (2 pt) Write out the *coefficient matrix* $\boldsymbol{A}$ of this system.

$$A = \begin{bmatrix} -1 & 4 & 2 & 4 \\ 3 & 0 & -6 & 4 \\ 0 & 2 & 0 & 2 \end{bmatrix}. \tag{4}$$

if coefficient matrix is wrong, 2pt

(b) (2+4 pt) Write out the *augmented matrix* for this system and calculate its *row-reduced echelon form.*

$$\begin{bmatrix} -1 & 4 & 2 & 4 & 1 \\ 3 & 0 & -6 & 4 & 4 \\ 0 & 2 & 0 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -2 & 0 & 5 \\ 0 & 1 & 0 & 0 & 17/4 \\ 0 & 0 & 0 & 1 & -11/4 \end{bmatrix} \tag{5}$$

(c) (4 pt) Write out the solution set.
Note: You shall write in the form of $\mathbf{x}_p + \text{span}\{\mathbf{v}_1, , \ldots, \mathbf{v}_k\}$, where $\mathbf{x}_p, \mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors that you need to find out. Not in the form get 2pt and null space correct 2pt

Solution:

$$x_1 = 2x_3 + 5, \tag{6}$$
$$x_2 = 17/4, \tag{7}$$
$$x_4 = -11/4, \tag{8}$$

Thus we have

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 17/4 \\ 0 \\ -11/4 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{9}$$

where $x_3 = c$.

(d) (4 pt) What is the *rank* of the coefficient matrix $\boldsymbol{A}$? Justify your answer.

rank(A) = 3. It has 3 linearly independent column vectors according to the RREF form.
4 pt for every claiming r(A) =3

(e) (2 pt) Find a basis of the null space of A.

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \tag{10}$$

2. **(10 pt) Question Answering**

(a) (2 pt) If a matrix $A \in \mathbb{R}^{m\times n}$ has full rank, what can be the number of solutions for the linear system $Ax = 0$? Provide a brief reason for your answer.

(b) (3 pt) Is it true that any subspace of $\mathbb{R}^n$ can be the column space of a certain matrix? Provide a brief reason for your answer.

(c) (2 pt) Consider the following row operation: taking the square of each entry. For instance, $[1, 2]$ will be transformed to $[2, 4]$. We do not call this operation an elementary row operation because it will change the solution set of the corresponding linear system. Please provide an example where such an operation changes the solution set.

(d) (3 pt) Answer whether there exists a 5×5 matrix satisfying three requirements below: i) it is a permutation matrix; ii) it is an upper triangular matrix; iii) it is not an identity matrix. If such a matrix exists, give an example; if not, provide a brief reason.

Solution:

(a) Since $\text{rank}(A) = \min\{m, n\}$, consider two conditions $m \geq n$ and $n < m$.

(1) If $m \geq n$, use the independence of column vectors to prove the only solution $x = 0$.

(2) If $m < n$, here are $n - m \geq 1$ number of free variables, the number of solution is infinity.

Note: 1 pt = 1 case, an absurd proof will make the case invalid.

(b) True. Proof: Consider $V = \mathbb{R}^n$, for $\forall b \in V$, using Proposition 12.1, we can always find the identity matrix I that have $Ib = b \in V$ holds. Hence, $V = C(I)$.

Then for arbitrary subspace $V_m \subset \mathbb{R}^n$ with $span\{v_1, v_2, ..., v_m\}, v_i \in \mathbb{R}^n$ WLOG. We can find a matrix $A = [v_1, v_2, ..., v_m] \in \mathbb{R}^{n\times m}$. According to the Definition 12.3, $V_m = C(A)$. Since $V_m = C(A) \subseteq C(I) = \mathbb{R}^n$, it is also easy to find that A is linearly expressed by I. Q.E.D.

Note: 1 pt for right judgement, 2 pt for proof.

(c) e.g.

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 0.5 \\ x_2 = 0.25 \end{cases}$$

After squared operation:

$$\begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 0.25 \\ x_2 = 0.0625 \end{cases}$$

Note: Right case = 2 pts.

(d) Not exist. Proof: Consider a permutation matrix $P_\pi \neq I_5$. According to the definition of permutation matrix, let $P_\pi = [e_{\pi(1)}, e_{\pi(1)}, ..., e_{\pi(5)}]$ where e_i is the vector that only takes 1 at the entry in row i with 0 to other entries. No matter how many cycles ≥ 2 there are, we can always find such a pair of vectors

$$\pi(i) < i, \pi(j) > j$$

which is contradicted to the upper triangular matrix form. Q.E.D.

Note: Right judgement = 1 pts, reasonable proof = 2pts. Just consider the cycle 2 condition is not enough. Here is another Proof. Combining $\text{rank}(A) = 5$ (condi. i)), and $A_{ij} = 0$ for each $i > j$ (condi. ii)). We can use matrix form to show that the upper and lower corner are 1, i.e., $a_{11} = a_{55} = 1$. Because each row and column of A has only one 1 (condi. i)), we can only consider a 3×3 matrix from a_{22} to a_{44}. Following this way, we have $a_{11} = a_{22} = a_{33} = a_{44} = a_{55} = 1$ and other $a_{ij} = 0$, which is contradicted to the condi. iii). Q.E.D.
Also, some students used such a proof form: Assume here exists such a matrix A that satisfies these three conditions. And we should have such a conclusion: For any 5×5 upper triangular

matrix U, AU is also a upper triangular matrix (condi. ii)). But for condi. iii), we know that at least two rows for U have occurred permutation in AU, which cannot keep a upper triangular form. Contradiction. But here I give a penalty of 1 pts, because the conclusion that the product of upper triangular matrices is equal to the upper triangular matrix is not basic enough, and needs further proof.

3. **(16 pt) Matrix Product, Independence and Inverse**

(a) (3 pt) Suppose $A = \begin{bmatrix} 1 & 5 & 2 \\ 1 & 0 & 1 \\ 3 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 5 & 1 \\ 13 & -2 & 0 \end{bmatrix}$. Compute AB and $B^\top A^\top$.

(b) (4 pt) Find the largest possible number of independent vectors among

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(c) (4 pt) Consider a matrix $A \in \mathbb{R}^{m\times n}$. Suppose $A = \left[\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\right] = \left[\mathbf{u}_{(1)}^T, \mathbf{u}_{(2)}^T, ..., \mathbf{u}_{(m)}^T\right]^T$. Express the condition $A^T A = I$ (I is the identity matrix) in at least two forms by the vectors $\mathbf{a}_i, i = 1, \ldots, n; \mathbf{u}_{(j)}, j = 1, \ldots, m$.
Remark: You cannot use the symbol "A" in the expressions of the condition.

(d) (5 pt) Consider a partitioned matrix $A \in \mathbb{R}^{(n+k)\times(n+k)}$ given by $A = \begin{bmatrix} B & C \\ \mathbf{0} & E \end{bmatrix}$ where $B \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{n\times k}, E \in \mathbb{R}^{k\times k}$. Suppose B and E are both invertible matrices. Does the inverse of A exist? If so, find its inverse; if not, explain why. (Note: To show a certain matrix G is the inverse of A, you need to verify both the left and right product with A are identity matrices.)

Solution: (a) **3 pts**:
Number of columns in A = number of rows in $B = 3$, so AB is defined.

$$AB = \begin{bmatrix} 28 & 21 & 8 \\ 15 & -2 & 3 \\ 58 & -18 & 7 \end{bmatrix},$$

(1 pt)

$$B^T A^T = \begin{bmatrix} 28 & 15 & 58 \\ 21 & -2 & -18 \\ 8 & 3 & 7 \end{bmatrix}.$$

(2 pts, 1 pt if write B^T and A^T correctly but the final answer is wrong OR they transpose the wrong AB.)

(b)**4 pts**:
Since $\mathbf{v}_4 = \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_6 = \mathbf{v}_3 - \mathbf{v}_2$, there are at most three independent vectors among these: furthermore, applying row reduction to the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$ gives three pivots, showing that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are independent. (4 pts for correct answer, no need to write steps; 0 if anything goes wrong)

(c)**4 pts**:

$$\mathbf{a}_i^T \mathbf{a}_j = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\sum_{k=1}^{m} (\mathbf{u}_{(k)}^T \mathbf{u}_{(k)})_{i,j} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

(2 pts for each. For each expression, 1 for correct format, 1 for specifying the value of entries)

(d)**5 pts**:
Given that the block matrix A is of the form:

$$\begin{bmatrix} B & C \\ \mathbf{0} & E \end{bmatrix}$$

where B and E are invertible matrices. We need to find the inverse of A. We can define a matrix $G \in \mathbb{R}^{(n+k)\times(n+k)}$ as: $G = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$, where $X \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{n\times k}, Z \in \mathbb{R}^{k\times n}, W \in \mathbb{R}^{k\times k}$.
Now, give the left product of A and G as:

$$AG = \begin{bmatrix} B & C \\ \mathbf{0} & E \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} BX + CZ & BY + CW \\ EZ & EW \end{bmatrix}$$

If we let $Ab = I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$Solve the following equations to find X, Y, Z, and W.

$$\begin{cases} BX + CZ & = I \\ BY + CW & = 0 \\ EZ & = 0 \\ EW & = I \end{cases}$$

Since B and E are invertible, then we have

$$\begin{cases} X & = B^{-1} \\ Y & = -B^{-1}CE^{-1} \\ Z & = 0 \\ W & = E^{-1} \end{cases}$$

After finding the values, it should be shown that the obtained $G = \begin{bmatrix} B^{-1} & -B^{-1}CE^{-1} \\ \mathbf{0} & E^{-1} \end{bmatrix}$ satisfies $AG = I$.
Then we consider the right product GA

$$GA = \begin{bmatrix} B^{-1}B & B^{-1}C - B^{-1}CE^{-1}E \\ \mathbf{0} & E^{-1}E \end{bmatrix} = I$$

This proof confirms that A is an invertible matrix if B and E are invertible and also provides the explicit expression for the inverse of A.
(1 pt for yes, 2 pts for G and 1 pt for each left and right product)

4. **(16 pt) True or False** You do NOT need to justify. (Only writing T or F is enough)

   (a) For any $A \in \mathbb{R}^{2\times 3}$, the linear system $Ax = 0$ has infinitely many solutions.

   Solution: **T**rue

   (b) Suppose $A \in \mathbb{R}^{n\times n}$. The linear system $Ax = 0$ has a unique solution iff the linear system $Ax = b$ has at least one solution for any $b \in \mathbf{R}^{n\times 1}$.

   Solution: **T**rue

   (c) If A and B are invertible matrices, then $A + B$ is also invertible.

   Solution: **F**alse

   (d) The inverse of an invertible upper triangular matrix is a lower triangular matrix.

   Solution: **F**alse

   (e) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three vectors in $\mathbf{R}^n$. Then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ iff $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

   Solution: **F**alse

(f) The solution set of a linear system cannot be $\mathbb{R}_+$. Here, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers.

Solution: **T**rue

(g) Suppose $\{u_1, u_2\}$ is a spanning set of a linear space V, then for any $v \in V$, there exist unique $a_1, a_2 \in \mathbb{R}$, s.t. $v = a_1u_1 + a_2u_2$.

Solution: **F**alse

(h) $\text{span}(\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}) = \text{span}(\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}, \mathbf{w} - \mathbf{u}\})$ for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

Solution: **F**alse

5. **(10 pt) Linear Space, Column Space and Null Space**

Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a square matrix. Denote $\mathbf{A}^2$ as the matrix product of $\mathbf{A}$ and $\mathbf{A}$.

(a) (5 pt) Prove two statements: The null space of $\mathbf{A}$ is contained in the null space of $\mathbf{A}^2$, i.e., $N(\mathbf{A}) \subseteq N(\mathbf{A}^2)$. The column space of $\mathbf{A}^2$ is contained in the column space of $\mathbf{A}$, i.e., $C(\mathbf{A}^2) \subseteq C(\mathbf{A})$.

(b) (5 pt) Let $\mathbf{A}^2 = \mathbf{0}$. Prove that the column space of $\mathbf{A}^\top$ is contained in the null space of $\mathbf{A}^\top$, i.e., $C(\mathbf{A}^\top) \subseteq N(\mathbf{A}^\top)$.

Solution:

(a) $\mathbf{x} \in N(\mathbf{A}) \Rightarrow \mathbf{Ax} = \mathbf{0} \Rightarrow \mathbf{A^2x} = \mathbf{A0} = \mathbf{0} \Rightarrow \mathbf{x} \in N(\mathbf{A^2})$;
$\mathbf{x} \in C(\mathbf{A^2}) \Rightarrow \mathbf{x} = \mathbf{A^2y}$ for some $\mathbf{y} \in \mathbb{R}^n \Rightarrow \mathbf{x} = \mathbf{A}(\mathbf{Ay}) \Rightarrow \mathbf{x} \in C(\mathbf{A})$.

(b) $\mathbf{A}^2 = \mathbf{0} \Rightarrow (\mathbf{A}^\top)^2 = (\mathbf{A^2})^\top = \mathbf{0}$;
Thus, $\mathbf{x} \in C(\mathbf{A}^\top) \Rightarrow \mathbf{x} = \mathbf{A}^\top\mathbf{y}$ for some $\mathbf{y} \in \mathbb{R}^n \Rightarrow \mathbf{A}^\top\mathbf{x} = (\mathbf{A}^\top)^\mathbf{2}\mathbf{y} = \mathbf{0} \Rightarrow \mathbf{x} \in N(\mathbf{A}^\top)$.

6. **(20 pt + 5 bonus pt) Linear Independence and Rank**

(6.1) Recall: a matrix $M \in \mathbb{R}^{p\times q}$ has full column rank iff $\text{rank}(M) = q$.

(a) (5 pt) Suppose $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$ and $AB = I_n$. Prove that the columns of B are linearly independent.

(b) (5 pt) Suppose $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$. Prove: If A and B have full column rank, AB also has full column rank.

Hint: Based on the definition of linear independence, the columns of a matrix $A \in \mathbb{R}^{m\times n}$ are linearly independent iff $Ax = 0$ has a unique solution $x = 0$.

(6.2) Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$.

(a) (5 pt) Prove that $a_i + b_i \in C([A, B])$, where $C(\cdot)$ is the column space of a matrix, $i = 1, 2, \cdots, n$. Here a_i and b_i are the i-th columns of A and B respectively.
Remark: As a reminder, $[A, B]$ is an $m \times 2n$ matrix paritioned into two blocks.

(b) (5 pt) Prove that $\text{rank}(A + B) \leq \text{rank}([A, B])$.

(6.3) (5 bonus pt) Suppose $A \in \mathbb{R}^{m\times n}$, $B = uv^\top \in \mathbb{R}^{m\times n}$, where u, v are vectors. Find a sufficient and necessary condition for $\text{rank}(A + uv^\top) = \text{rank}([A, uv^\top])$. The simpler and cleaner condition you obtain, the more points you get.

Solution:

1. (a) Suppose that there exists some $x \in \mathbb{R}^p$ such that $Bx = 0$, then we have $0 = ABx = x$, which implies that columns of B are linearly independent.

   (b) Suppose that there exists some $x \in \mathbb{R}^p$ such that $ABx = 0$, then $Bx = 0$ since B is of full column rank, implying $x = 0$ for B being full column rank. Thus, AB is of full column rank.

2. (a) **Proof:**

- The column space of a matrix consists of all possible linear combinations of its columns.
- For each i, the vector $a_i + b_i$ is the sum of the i-th columns of A and B.
- Since a_i is a column in A and b_i is a column in B, and $[A, B]$ includes all columns of both A and B, the sum $a_i + b_i$ can be expressed as a linear combination of the columns of $[A, B]$.
- Therefore, $a_i + b_i \in C([A, B])$ for each $i = 1, 2, \cdots, n$.

(b) **Proof:**

- The rank of $A + B$ is the dimension of the column space of $A + B$.
- Each column of $A + B$ can be written as $a_i + b_i$.
- From the result of part (a), each $a_i + b_i$ is in $C([A, B])$.
- Since every column of $A + B$ is in the column space of $[A, B]$, the column space of $A + B$ is a subset of the column space of $[A, B]$.
- The rank (dimension of the column space) of a matrix is less than or equal to the rank of a matrix whose column space includes it.
- Therefore, $\text{rank}(A + B) \leq \text{rank}([A, B])$.

3.