PHY1001: Mechanics (Week 4)

In this chapter, we are going to learn two new concepts, momentum and impulse, and a new conservation law, conservation of momentum. We will use these concepts to study problems of collisions and rocket propulsion.

1 Linear Momentum

In the last week, we re-expressed Newton's second law, $\vec{F} = m\vec{a}$ in terms of the work-energy theorem through integration. This theorem led us to the concept of the law of conservation of energy.

However, we did not fully use the information of the vector equation $\vec{F} = m\vec{a}$, since vectors have not only magnitudes but also directions.

Let us now return to $\vec{F} = m\vec{a}$ and see yet another useful way to restate this fundamental law.

- Let us write $\vec{F}_{\text{net}} = m\dfrac{d\vec{v}}{dt}$.
- Assuming that m of the considered object is a constant, $\vec{F}_{\text{net}} = \dfrac{d(m\vec{v})}{dt}$.
- Define the quantity $m\vec{v}$ as the momentum, or linear momentum, $\vec{P} \equiv m\vec{v}$. Then Newton's second law becomes
$$\vec{F}_{\text{net}} = \frac{d\vec{P}}{dt}. \tag{1}$$
- The net force acting on an object equals the time rate of momentum change.
- This is, in fact, the original form of Newton's second law. Newton called momentum the "quantity of motion", which tells us the physical meaning of momentum.
- The integral form gives us the impulse-momentum theorem
$$\underbrace{\vec{J} \equiv \int_{t_1}^{t_2} \vec{F}_{\text{net}} dt}_{\text{Definition of Impulse}} = \vec{P}_2 - \vec{P}_1, \tag{2}$$
$$\text{Define} \quad \vec{J} \equiv \int_{t_1}^{t_2} \vec{F}_{\text{net}} dt \equiv \vec{F}_{avg}(t_2 - t_1). \tag{3}$$
The change in momentum of an object during a time interval equals the impulse of the net force. This theorem, similar to the work-energy theorem (scalar equation), is also an integral principle. In contrast, Newton's second law $\vec{F}_{\text{net}} = \dfrac{d\vec{P}}{dt}$ is a differential principle.
- The impulse-momentum theorem is an integral vector equation while the work-energy theorem is an integral scalar equation.
- If $F_{\text{net}} = 0$, we can conclude that $\vec{P}$ is a constant, namely a conserved quantity. This is known as the conservation of momentum. The conservation of momentum is extremely useful when we study a system of many particles.

2 Conservation of Momentum

In order to study systems which consist of many particles, we first introduce some new terminology.

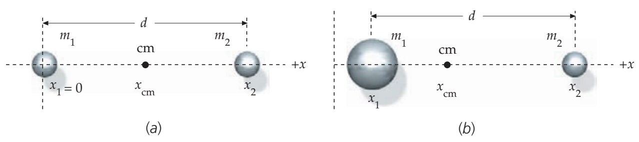
- Internal force: the forces that the particles of the system exert on each other.
- External force: the forces exerted on any part of the system by some outside object.
- When there is no external force, the system is regarded as an isolated system with no external influences.
- Because internal forces always come in pairs in terms of actions and reactions according to Newton's third law, we can write down Newton's second law for an isolated system as
$$\sum \vec{F}_{\text{internal}} = 0 = \frac{d\vec{P}}{dt}, \tag{4}$$
where $\vec{P}$ now is the total momentum of the isolated system.
- For example, for two body system $\vec{P} = \vec{P}_A + \vec{P}_B$ and
$$F_{\text{A on B}} = \frac{d\vec{P}_B}{dt}, \quad F_{\text{B on A}} = \frac{d\vec{P}_A}{dt}, \tag{5}$$
where $F_{\text{A on B}} + F_{\text{B on A}} = 0$.
- For a system of many particles $\vec{P} = \vec{P}_1 + \vec{P}_2 + \cdots = m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots = \sum_i m_i\vec{v}_i$.
- The principle of conservation of momentum: Therefore, this shows that the total momentum $\vec{P}$ of an isolated system is a conserved quantity. This is of course only valid in the inertial frame of reference.
- This principle is sometimes more general than the conservation of mechanical energy, which requires that the internal forces to be conservative. But the conservation of momentum is valid even when the internal forces are not conservative, as long as there are no external forces.
- Sometimes, when the time interval is very short, $F_{ext}\Delta t$ is so small (compared to momentum change Δp) that we can neglect $F_{ext}\Delta t$. Therefore, we can approximately say that the momentum is conserved in such process. For example, frictions can usually be neglected during collisions.

- Momentum is a vector.

$$\vec{P} = P_x\hat{\imath} + P_y\hat{\jmath} + P_z\hat{k}. \tag{6}$$

Often we need to project it onto certain directions (x, y, or z) to consider the change of momentum or momentum conservation.

3 The Center of Mass



The concept of center of mass is very useful when we discuss momentum conservation of a system of many particles. For a system of two particles, we define

$$(m_1+m_2)x_{cm} = m_1x_1+m_2x_2 \quad \Rightarrow \quad x_{cm} = \frac{m_1x_1+m_2x_2}{m_1+m_2}. \tag{7}$$

- Let $x_1 = 0$, $x_2 = d$, then $x_{cm} = \dfrac{m_2 d}{m_1+m_2}$;
- If $m_1 = m_2$, then $x_{cm} = \dfrac{d}{2}$ in the middle of two objects.
- If $m_1 \gg m_2$, $x_{cm} \to 0$; If $m_1 \ll m_2$, $x_{cm} \to d$. Center of mass tends to get close to the heavy object.

For a system which consists of many particles in 3 dimension space, we have

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{m_1+m_2+\cdots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}. \tag{8}$$

Taking time derivative of the above equation, we get the velocity of the center of mass

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots}{m_1+m_2+\cdots} = \frac{\sum_i m_i\vec{v}_i}{\sum_i m_i}. \tag{9}$$

By defining M as the total mass $\sum\limits_i m_i\vec{v}_i$, we can write

$$M\vec{v}_{cm} = \sum_i m_i\vec{v}_i = \vec{P}. \tag{10}$$

The right side is simply the total momentum $\vec{P}$ of the system. From this we can learn that the total momentum is also equal to the total mass of the system times the velocity of the center of mass.

It is important to note that the total momentum is always conserved for an isolated system which experiences no external force. No matter how complicated the internal forces or interactions are, the total momentum of an isolated system always remains the same. This is very useful when we study collisions.

4 Collisions

Collision is the event in which two or more bodies exert forces on each other in about a relatively short time. If the *external* force is negligible, momentum is then conserved in collisions.

- *Elastic Collision* is the collision which conserves kinetic energy.

$$\frac{1}{2}m_A\vec{v}_A^2 + \frac{1}{2}m_B\vec{v}_B^2 = \frac{1}{2}m_A\vec{v}_A'^2 + \frac{1}{2}m_B\vec{v}_B'^2, \tag{11}$$

- *Inelastic Collision* is the collision which does not conserve kinetic energy.
- *Complete Inelastic Collision* is the collision that the colliding objects stick together after the collision. In this type of collision, the system loses the maximum amount of kinetic energy. Also in this case $\vec{v}_{cm} = \vec{v}_A' = \vec{v}_B'$, because they stick together.

$$\vec{v}_{cm} = v_A' = v_B' = \frac{m_A\vec{v}_A + m_B\vec{v}_B}{m_A+m_B} \tag{12}$$

Elastic collision is a bit complicated in general. Let us study the special example of one dimensional elastic collision with $v_B = 0$, therefore

$$m_Av_A = m_Av_A' + m_Av_B'$$
$$\frac{1}{2}m_Av_A^2 = \frac{1}{2}m_Av_A'^2 + \frac{1}{2}m_Bv_B'^2.$$

It is then straightforward to find that

$$v_A' = \frac{m_A-m_B}{m_A+m_B}v_A$$
$$v_B' = \frac{2m_A}{m_A+m_B}v_A.$$

Several interesting cases:

1. $m_A = m_B$, then A and B exchange velocities.
2. $m_A \ll m_B$, A is bounced back with the same speed. (bouncing off from a heavy wall?)
3. $m_A \gg m_B$, B is bounced forward with twice the speed of A. (Kicking a soccer ball?)

4.1 From the Center of Mass

In the above discussion, we did not use the concept of COM. Now let us try to understand the collision from the perspective of COM systems in arbitrary dimensions.

$$m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v}_A' + m_A\vec{v}_B' = (m_A+m_B)\vec{v}_{cm} \tag{13}$$

Let us look at the kinetic energy. Initial kinetic energy:

$$\frac{1}{2}m_Av_A^2 + \frac{1}{2}m_Bv_B^2$$
$$= \frac{1}{2}(m_A+m_B)\vec{v}_{cm}^2 + \frac{1}{2}\frac{m_Am_B}{m_A+m_B}(\vec{v}_A-\vec{v}_B)^2. \tag{14}$$

Final kinetic energy:

$$\frac{1}{2}m_A v_A'^2 + \frac{1}{2}m_B v_B'^2 = \frac{1}{2}(m_A + m_B)\vec{v}_{cm}^2 + \frac{1}{2}\frac{m_A m_B}{m_A + m_B}(\vec{v}_A' - \vec{v}_B')^2. \quad (15)$$

1. When $(\vec{v}_A' - \vec{v}_B')^2 = (\vec{v}_A - \vec{v}_B)^2$, kinetic energy is conserved. Elastic scattering (collision).

2. When $(\vec{v}_A' - \vec{v}_B')^2 \neq (\vec{v}_A - \vec{v}_B)^2$, kinetic energy is not conserved. inelastic scattering (collision). Kinetic energy can either increase or decrease. If $(\vec{v}_A' - \vec{v}_B')^2 > (\vec{v}_A - \vec{v}_B)^2$, there must be some energy converted into the kinetic energy, for example, firecracker. If $(\vec{v}_A' - \vec{v}_B')^2 < (\vec{v}_A - \vec{v}_B)^2$, it means that some kinetic energy is lost during the collision. Usually, this part of the kinetic energy is converted into the internal energy of colliding bodies.

3. When $(\vec{v}_A' - \vec{v}_B')^2 = 0$, we call it completely inelastic collision since it loses the maximum amount of kinetic energy, which is $\frac{1}{2}\frac{m_A m_B}{m_A + m_B}(\vec{v}_A - \vec{v}_B)^2$. Note that due to momentum conservation, the center of mass kinetic energy can not be lost even in completely inelastic collisions.

4. For elastic collisions in general, 2-d or 3-d cases can not be solved solely from energy and momentum conservation. Extra information should be provided. For example, 2-d case has 4 unknown variables v_{Ax}', v_{Ay}' and v_{Bx}', v_{By}', but we only have three equations (one from energy, two from momentum).

Ballistic Pendulum

In Problem 3 of this week's HW, you are asked to work out the problem of the Ballistic Pendulum, which was a convenient way used to measure the speed of a fast moving bullet. It involves the use of inelastic collision and momentum conservation. You will get to do this experiment in PHY1002 next semester.

5 Rocket propulsion

Advanced Topics: Relative velocities in elastic collisions and the interesting case of Rocket propulsion.

Rotten Tomatoes

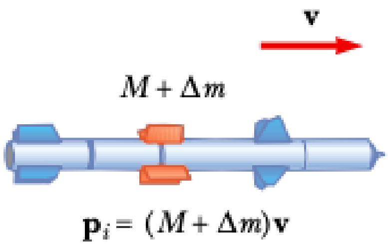
First, let us consider a ridiculous example. Suppose you have an annoying cousin named Tony. He throws a rotten tomato (each with mass $m = 0.1$ kg) with a speed of $v = 1$ m/s at you per second. So in ten seconds, Tony throws ten tomatoes which stick on your body. What is the average force exerted on your body during these ten seconds?

Use the impulse-momentum theorem

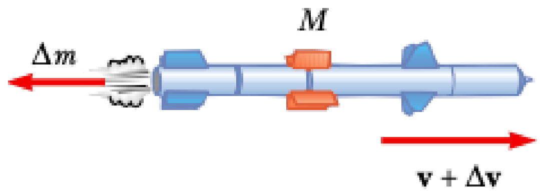
$$Nmv = J = \bar{F}\Delta t \quad (16)$$

$$\bar{F} = \frac{Nmv}{\Delta t} = \frac{10 * 0.1kg * 1m/s}{10s} = 0.1N. \quad (17)$$

In general, we find that $\bar{F} = \frac{dm}{dt}v$ with $dm/dt = 0.1kg/s$. In fact, the physics behind the rocket propulsion is similar to this ridiculous example.



(a)



(b)

In the case of Rocket propulsion, the rocket fuel (propellant) is ejected backwards from the bottom of the rocket. Therefore, the rocket is accelerated as a result of the "push," or thrust, from the exhaust. The total mass of the system is constant, but the mass of the rocket itself decreases as the propellant is ejected. We can not directly use $F = ma$ because m changes. Instead, one needs to use a more general form $F_{net} = \frac{dP}{dt}$.

Let v_{ex} (some textbooks use u to denote this relative speed) be the exhaust speed relative to the rocket. Since the burned fuel is ejected opposite to the direction of motion, then $v_{fuel} = v - v_{ex}$, where v is the velocity of the rocket.

Use $\sum F = 0 = \frac{dP}{dt}$ (neglecting the gravitation force on the surface of the earth or consider the rocket in the outer space), one finds

$$[M(v + dv) + dm(v - v_{ex})] - (M + dm)v = 0, \quad (18)$$

$$Mdv = dm v_{ex} = -dM v_{ex}, \quad \text{note } dM = -dm < 0, \quad (19)$$

$$\int_{v_i}^{v_f} dv = -v_{ex}\int_{M_i}^{M_f}\frac{dM}{M}, \quad (20)$$

where the increase in the exhaust mass dm corresponds to an equal decrease in the rocket mass dM. M_i is the mass of the rocket before ignition while M_f is the mass of the rocket when its engine shuts down after running out all the fuel.
Solution to the above equation is then

$$v_f = v_i + v_{ex}\ln M_i/M_f.$$

The increase in velocity only depends on the ratio of the mass and v_{ex}.

We usually define Thrust

$$F_{thrust} = Mdv/dt = v_{ex}dM/dt$$

for rockets since this is the reaction force that pushes the rocket upward. Side note: Just to use some numbers to put things into perspective. Falcon 9 is a partially reusable rocket designed and manufactured by SpaceX. These are the important parameters of this rocket

$$v_{ex} = u = 2.5km/s, \tag{21}$$
$$\frac{dm}{dt} = 3\times 10^3\,kg/s, \tag{22}$$
$$\text{fuel burning time}\quad \Delta t = 180\,s. \tag{23}$$

This gives the thrust about 7.5×10^6 N, which much larger than the weight of the rocket. The most powerful rocket in the world can produce 34×10^6 N of thrust force by ejecting roughly 14 tons of fuel per second. How impressive it is!

Additional question: What if the gravitational force can not be neglected? Answer:

$$v_f = v_i + v_{ex}\ln M_i/M_f - gt,$$

where velocity is v_i at $t=0$ and it is v_f at t.