Lecture 3
Euclid

GFN1000 In Dialogue with Nature

Content

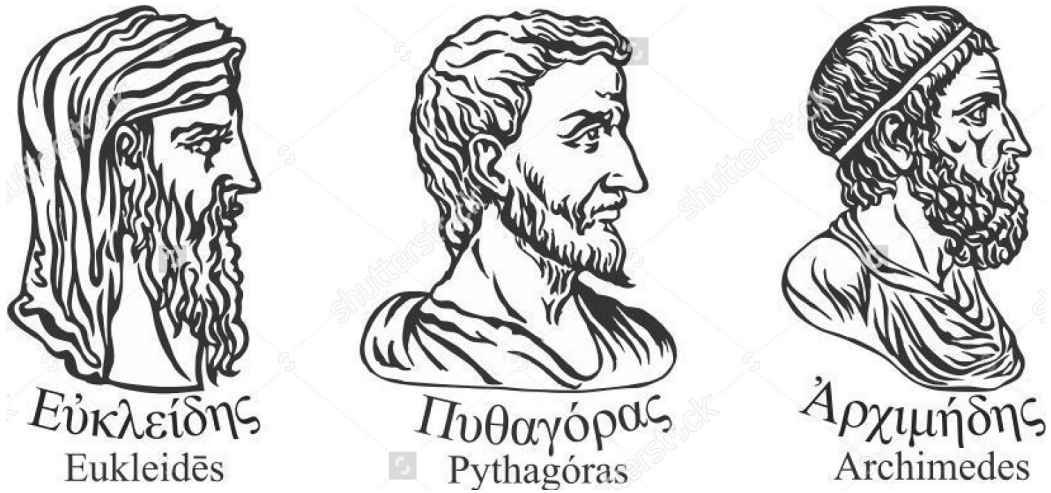
- Ancient Mathematics
  - Egyptian and Babylonian Mathematics
  - Greek Mathematics
- Euclid's Elements
  - Definitions, Axioms, Propositions
  - Greek Geometry
- Broader Implications
  - Modern Science
  - Philosophy and Social Affairs

Babylonian and Egyptian Mathematics

- The mathematical cultures of the Egyptians and the Babylonians in Mesopotamia concerned themselves with **practical** matters
  - Commercial transactions
  - Computation of the height of a pyramid
  - Geometry: Geo (earth) and metron (measure)

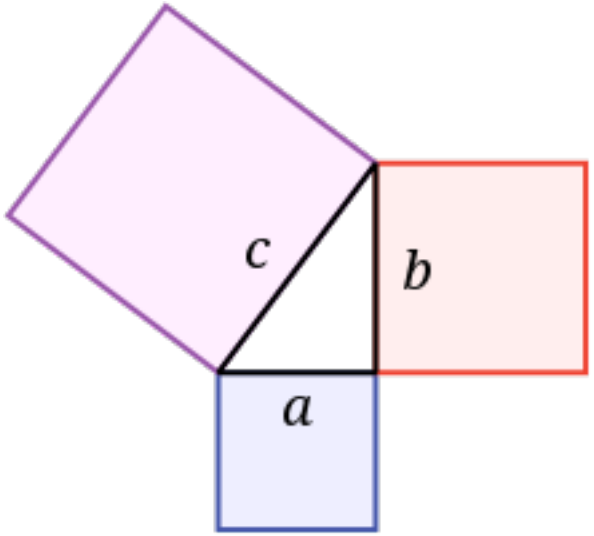
Greek Mathematics

- Mathematics → μάθημα, máthēma
- The study of mathematics for its own sake (rather than for practical purposes)
- A systematic study of mathematics
  - The use of generalized mathematical theories and proofs
  - The use of abstraction and logic
- Intrinsic aesthetics and inner beauty



Pythagoras (570-495 BCE)

- Greek philosopher, mathematician, founder of Pythagoreanism
- Discovered the foundations of musical tuning
- Pythagorean theorem
- The Pythagorean school: a philosophical and political school



Pythagoreanism

- Number as a fundamental aspect of reality
  - “Everything is number.” (positive integers)
  - “The elements of numbers to be the elements of all things.”
- Mathematization of nature and modern science
  - Copernicus, Kepler, Newton

FIGURE 1: The Seven Liberal Arts

THE TRIVIUM:
The three arts of language pertaining to the mind

Logic.....art of thinking
Grammar .....art of inventing and combining symbols
Rhetoric.....art of communication

THE QUADRIVIUM:
The four arts of quantity pertaining to matter

Discrete quantity or number

Arithmetic .....theory of number
Music.....application of the theory of number

Continuous quantity

Geometry .....theory of space
Astronomy .....application of the theory of space

Euclid of Alexandria (~300 BCE)

- Greek mathematician, “Father of Geometry.”
- Euclid’s principal concern is to set mathematics on a sure logical foundation.
- Euclid played a crucial role in the development of mathematics by incorporating the mathematical knowledge of his time into a **systematic** form and presenting the old mathematics in a thoroughly **rigid**, **organized, and logical** way.

Euclid's Elements

- A mathematical and geometric treatise composed of thirteen parts or "books." Each book is arranged around a particular topic or theme.
  - Books I – IV & VI are on plane geometry.
  - Books V & X are about magnitudes and ratios.
  - Books VII – IX are about whole numbers.
  - Books XI – XIII are about solid geometry.
- Each book begins with a series of (1) definitions, (2) permitted geometric or arithmetic procedures (or operations), and (3) common assumptions.

Definitions

- Book I begins with a list of 23 definitions of concepts that can be agreed upon.
  - i.e., points, lines, planes, etc.
- The reader would know precisely what his terms meant – leaving no room for ambiguity.

Definitions

1. A *point* is that which has no part.
2. A *line* is breadthless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.
5. A *surface* is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A *plane surface* is a surface which lies evenly with the straight lines on itself.
8. A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called *rectilineal*.
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is *right*, and the straight line standing on the other is called a *perpendicular* to that on which it stands.
11. An *obtuse angle* is an angle greater than a right angle.
12. An *acute angle* is an angle less than a right angle.

Axioms

- 5 **common notions** that everyone would agree with.
  - Self-evident statements, i.e., common senses
- 5 **postulates** relating to geometry, which Euclid took to be intuitively true.
  - Claim something is *possible*
  - Without proof or justification

Postulates

Let the following be postulated:

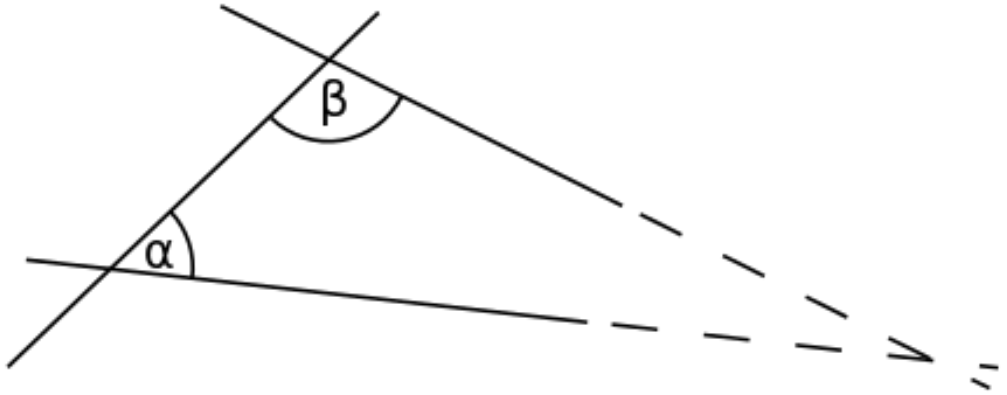
1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

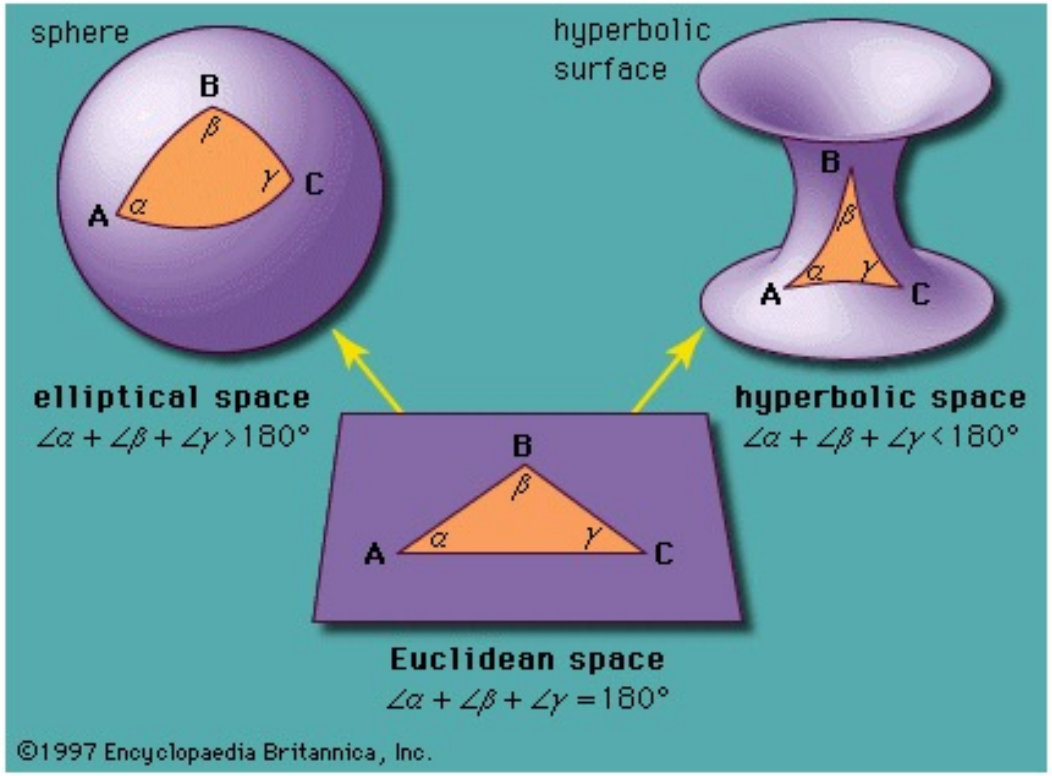
Common Notions

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

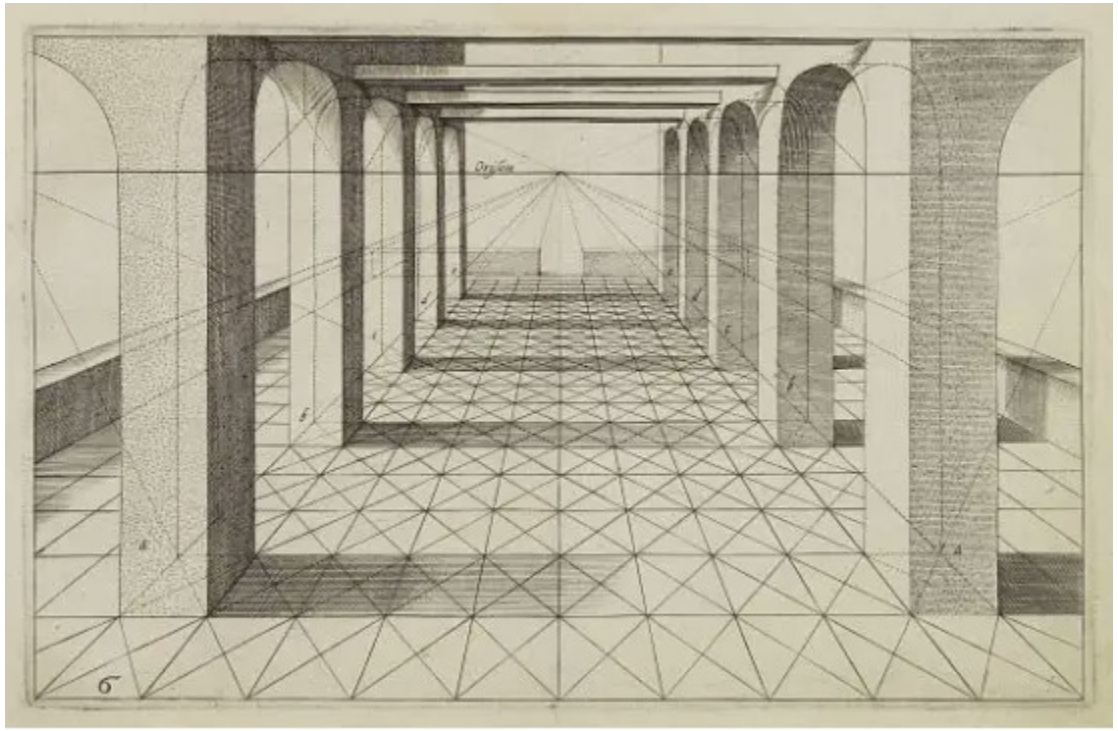
The Parallel Postulate

- If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.





Linear Perspective in Renaissance Art



Holy Trinity (Masaccio)

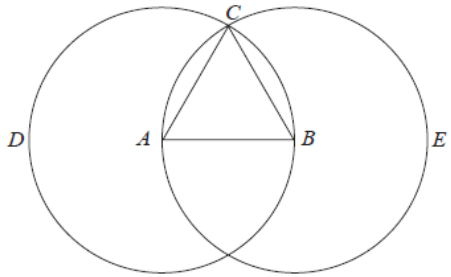
Propositions

- Propositions are deduced from definitions, postulates, common notions, and other propositions.
- Compared to the postulates or common notions, these are sometimes not straightforward or even violate our common senses.

Propositions

Proposition 1.

On a given finite straight line to construct an equilateral triangle.



Let *AB* be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line *AB*.

With centre *A* and distance *AB* let the circle *BCD* be described; [Post. 3]

again, with centre *B* and distance *BA* let the circle *ACE* be described; [Post. 3]

and from the point *C*, in which the circles cut one another, to the points *A*, *B* let the straight lines *CA*, *CB* be joined. [Post. 1]

Now, since the point *A* is the centre of the circle *CDB*,

AC is equal to *AB*. [Def. 15]

Again, since the point *B* is the centre of the circle *CAE*,

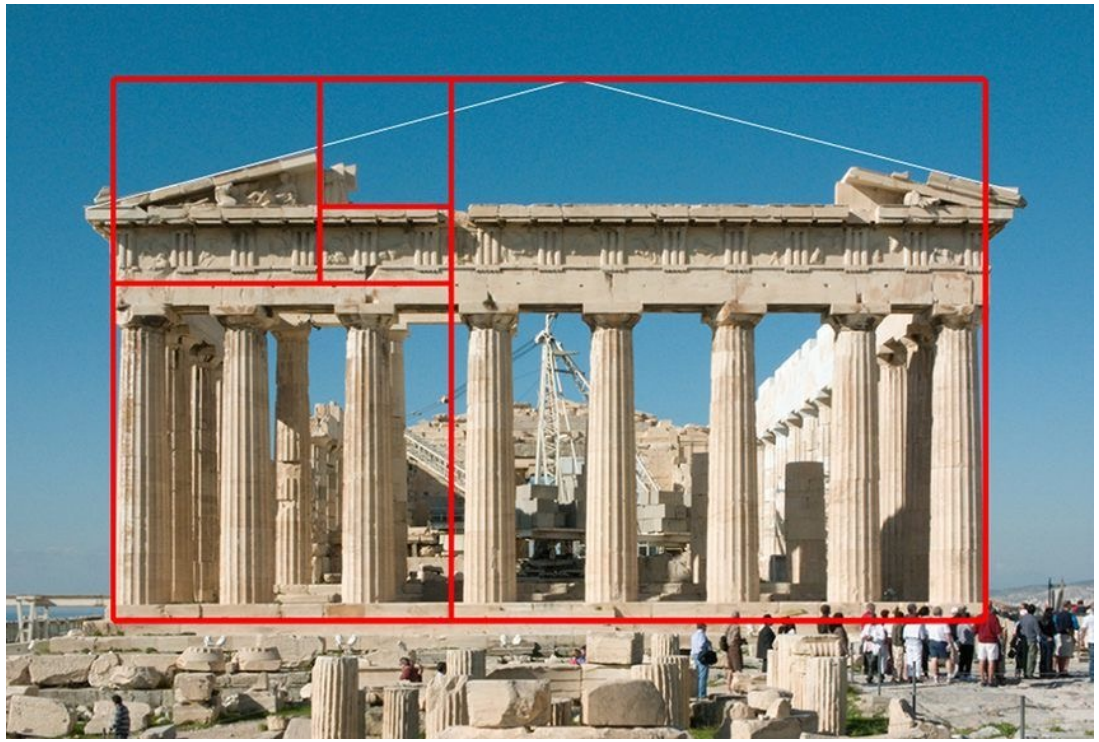
BC is equal to *BA*. [Def. 15]

But *CA* was also proved equal to *AB*;

therefore each of the straight lines *CA*, *CB* is equal to *AB*.

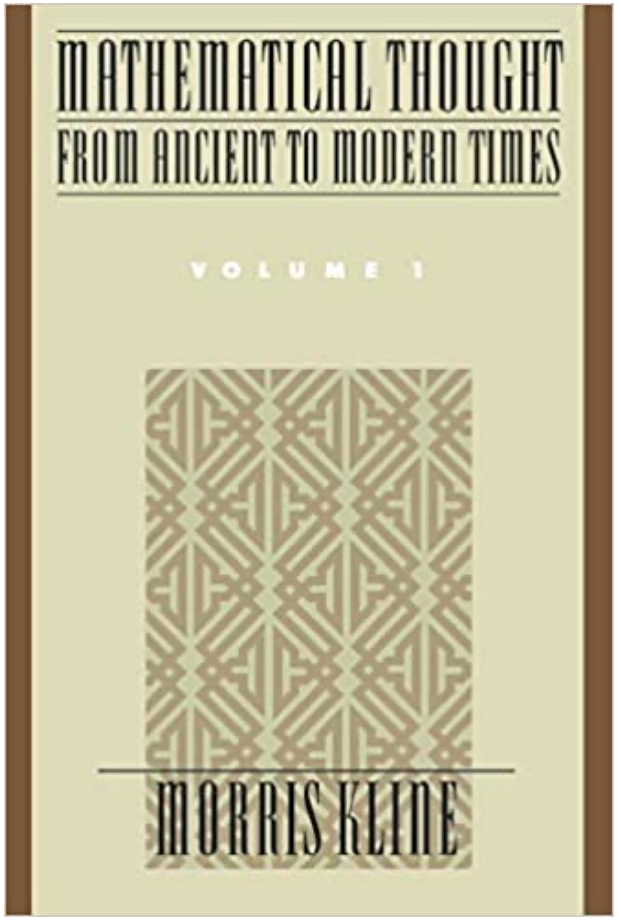
Greek Geometry

- Impressive logical rigor
- Purely geometric - as opposed to numerical - nature
- Skillful organization in presenting and developing mathematical propositions
- Pure thought, ideal, immaterial, and eternal
- Reliance on compass and straightedge



Philosophy of Mathematics

- “They are not thinking about these figures but of *those things which the figures represent*; thus it is the square **in itself** and the diameter **in itself** which are the matter of their arguments, not that which they draw; similarly, when they model or draw objects, which may themselves have images in shadows or in water, they use them in turn as images, endeavoring to see those **absolute objects** which cannot be seen otherwise than by thought.”

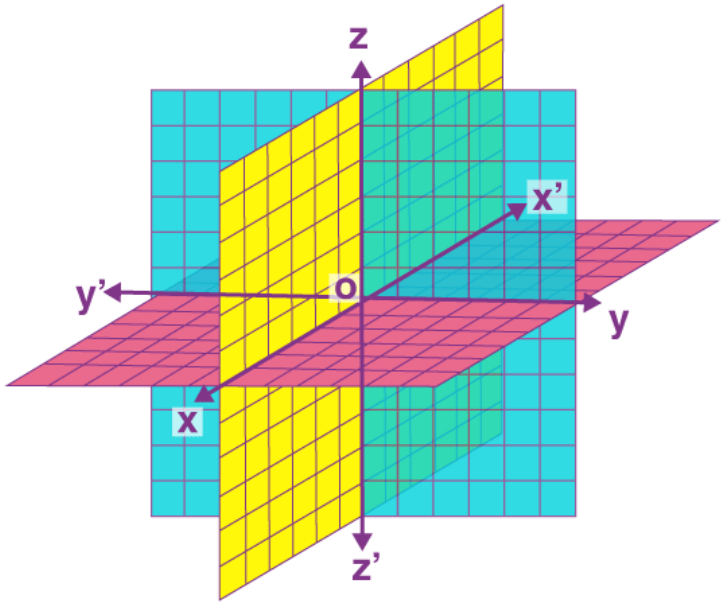


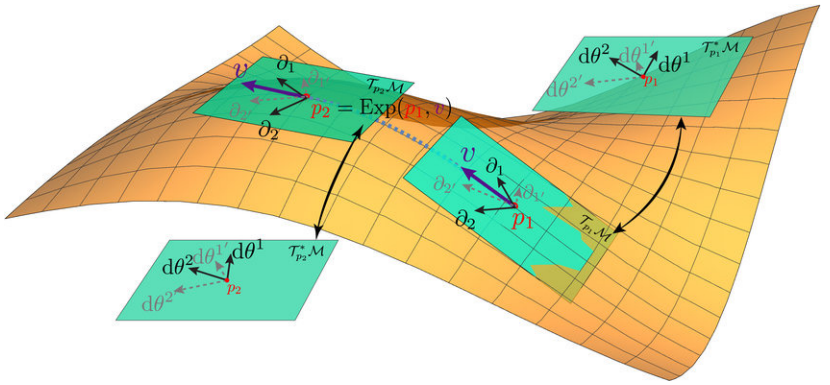
Euclid in China

- Assisted by Xu Guangqi (1562-1633), Matteo Ricci in 1607 began translating Euclid's Elements into Chinese. The first six books of Euclid's Elements were published in Beijing in 1614.
- Li Shanlan (1811-1882) collaborated with the British missionary Alexander Wylie and completed the translation of Euclid's Elements.

Geometry after Euclid

- Analytical geometry
  - Cartesian Coordinates
  - A systematic link between Euclidean geometry and algebra
- Topology
  - Bernhard Riemann
  - Properties of spaces that are invariant under any continuous deformation



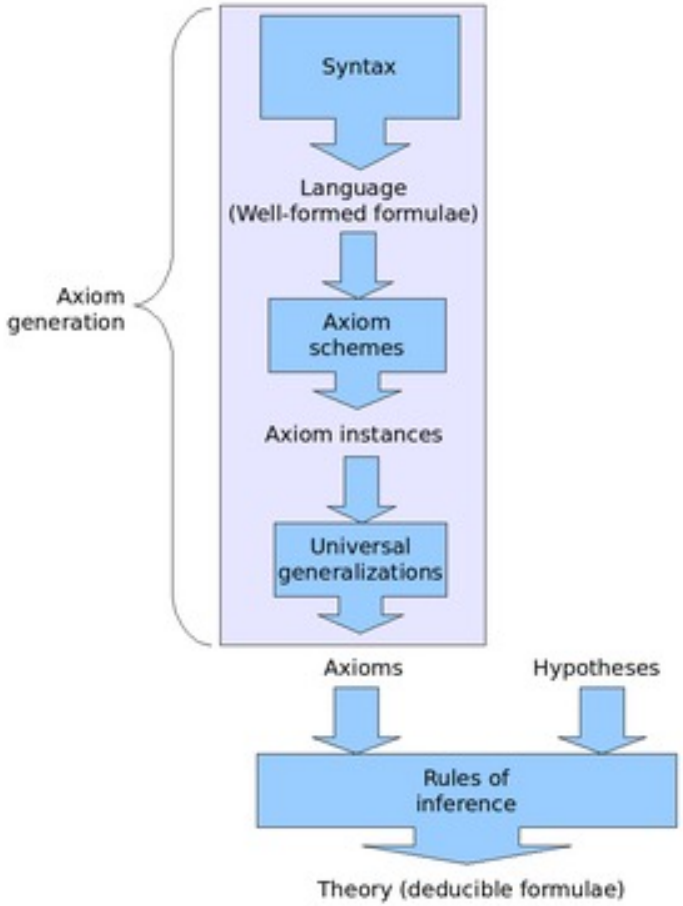


Euclid's Influence on Modern Science

- "The universe cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures." **Galileo**
- "It is the glory of geometry that so much is accomplished with so few principles that are obtained elsewhere." **Isaac Newton**, Preface of *Principia*
- "If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker." **Albert Einstein**

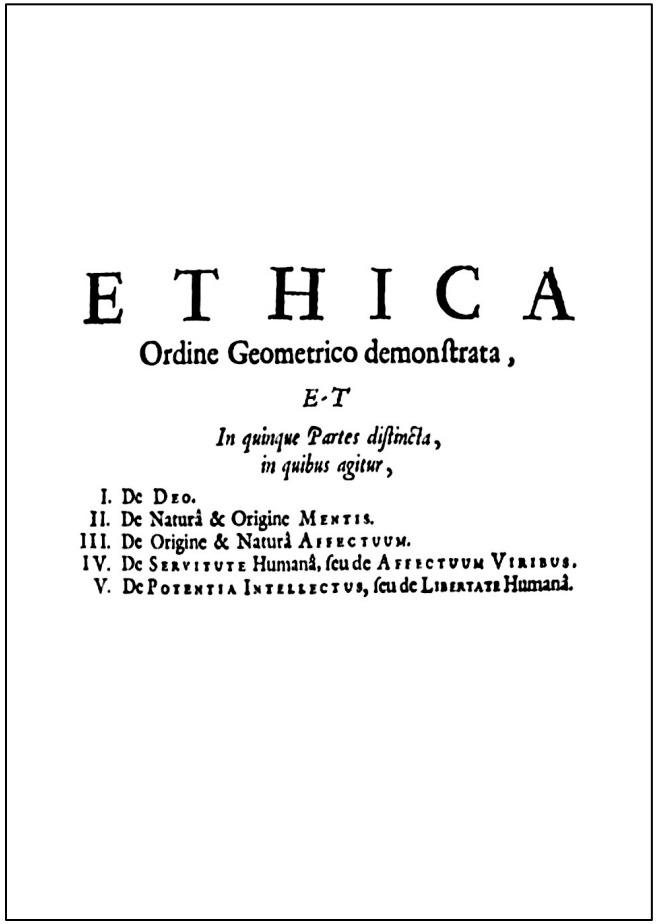
Axiomatic Systems: The Case of Principia

- Make Definitions (statements that explain the meaning of the terms)
  - Mass, inertia, centripetal force, etc.
- State Axioms (the truth that cannot be proved but can only be accepted)
  - Newton's three laws of motion
- Prove Propositions and Results
  - Projectile motion, planetary elliptic orbits, etc.



Geometric Method in Philosophy

- Descartes philosophical method
  - Moving from basic principles to complex conclusions.
- Spinoza (1632-1677) uses the geometrical method in his *Ethics: Demonstrated in Geometrical Order*
  - "I treated God and the mind, and I shall consider human actions and appetites just as if it were a question of lines, planes, and bodies."


E T H I C A
Ordine Geometrico demonſtrata,
E·T
In quinque Partes diſtincta,
in quibus agitur,
I. De Deo.
II. De Naturâ & Origine Mentis.
III. De Origine & Naturâ Affectuum.
IV. De Servitute Humanâ, ſeu de Affectuum Viribus.
V. De Potentia Intellectus, ſeu de Libertate Humanâ.

Logical Thinking in Social Affairs

- The Declaration of Independence is based on “self-evident” axioms used to prove the colonies are justified in forming the US.

1. All men are created equal and have unalienable Rights;
2. Governments are to secure those Rights;
3. Governments get their powers from the consent of the governed;
4. It is the right of the governed to change Government if it is destructive.