PHY1001: Mechanics (Week 9)

1 Chapter 13 Gravitation

In this chapter, we will learn the classical gravity and briefly discuss the physics of black hole.

1.1 Newton's Law of Gravitation

Newton published the law of gravitation in 1687, which gave birth to the famous formula

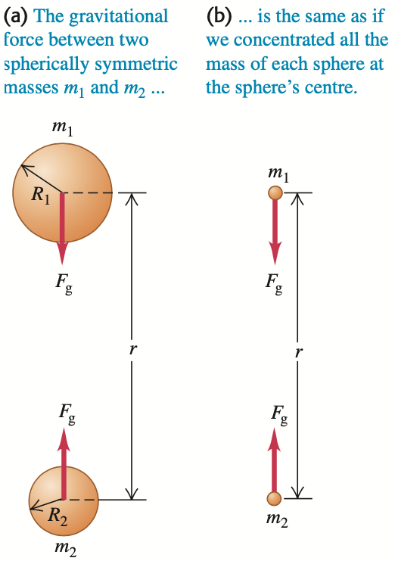
$$\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}, \tag{1}$$

where G is the gravitational constant. Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them. In 1798, Sir Henry Cavendish measured G for the first time and found

$$G = 6.67 \times 10^{-11}\mathrm{N}\cdot\mathrm{m}^2/\mathrm{kg}^2. \tag{2}$$

Newton's Law of Gravitation:

1. Two important features of the gravitational forces: a). central force; b). proportional to $1/r^2$.
2. Any two objects attract each others through gravitational forces.
3. These two forces form an action and reaction pair.
4. The gravitational effect outside any spherically symmetric mass distribution is the same as if all the mass were concentrated at its center. We will show this with the help of the concept of gravitational potential energy.

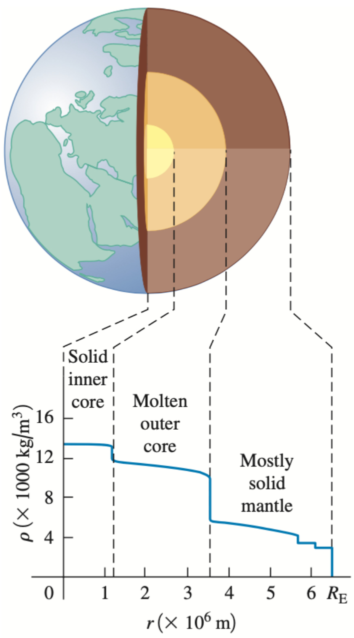


Spherically symmetric bodies are an important case because moons, planets and stars all tend to be spherical. Since all particles in a body gravitationally attract each other, the particles tend to move to minimise the distance between them. Why? (Answer: If a planet was like a cube, the gravity from the center of mass would eventually pull the matter at the corners of the cube down and put them on the edges.)

1.2 Weight

When we first talk about Weight in physics, we defined it as the attractive gravitational force exerted on it by the earth. We can now broaden the definition: The weight of a body is the total gravitational force exerted on the body by all other bodies in the universe.

1. Near the surface of the earth, we can neglect all other gravitational forces, and write
$$w = \frac{GM_Em}{R_E^2} = mg, \quad \Rightarrow \quad g = \frac{GM_E}{R_E^2}. \tag{3}$$
Above the surface of the earth $r > R_E$, thus $w = GM_Em/r^2 < mg$.
2. Near the surface of the moon, then we get
$$w = \frac{GM_mm}{R_m^2}, \tag{4}$$
which implies the gravitational acceleration on the moon $g_m = GM_m/R_m^2$.
3. "Measuring" the mass of the Earth
$$M_E = \frac{gR_E^2}{G} = \frac{9.80 \times (6.38 \times 10^6)^2}{6.67 \times 10^{-11}}kg = 5.98\times10^{24}kg.$$
4. 'Measuring" the average density of the Earth
$$\rho = \frac{M_E}{V_E} = 5.5 \times 10^3\ kg/m^3. \tag{5}$$



The surface of the earth is made up of water and rock less dense than the above density ρ. Therefore, we know that the earth cannot be uniform, and the interior of the earth must be much more dense than the surface in order that the average density be $5.5 \times 10^3\, kg/m^3$.

1.3 Gravitational Potential Energy

Similar to previous derivation of gravitational potential energy near the surface of the earth, one can use the definition of work and the work-energy theorem to redefine the gravitational potential. Use $W = \int \vec{F} \cdot d\vec{s}$ with $d\vec{s} = dr\hat{r} + rd\theta\hat{\theta}$ (recall the polar coordinate unit vectors) and Newton's gravity force, one gets

$$W_{grav} = -GM_E m \int_{r_1}^{r_2} \frac{\hat{r} \cdot \hat{r}}{r^2} dr = \frac{GM_E m}{r_2} - \frac{GM_E m}{r_2}, \tag{6}$$

which is equal to the decrease of potential energy $-\Delta U = -(U_2 - U_1)$. Therefore, we can define U as follows

$$U = -\frac{GM_E m}{r}, \tag{7}$$

where the zero point is chosen to be at infinity. (In principle, one can add any constant to the above expression to redefine the zero point without bringing any physical consequence to our calculations. Usually, it is the difference of potential energy that matters.)

1. For either straight path from or curve path r_1 to r_2, the above result remains the same.

2. Previously, we mentioned that $U = mgy$ for constant gravitational force. How can you reconcile these seemingly incompatible descriptions of gravitational potential energy? (Homework problem. Hint: Choose the surface of the earth as the zero point for potential energy and then Taylor expand near the earth's surface.)

3. Newton's gravity is conservative with $\vec{F} = -\vec{\nabla} U$ and $\vec{\nabla} \times \vec{F} = 0$. (see the criteria for conservative force.)

1.4 Escape velocity and the motion of satellites

Use gravitational potential energy, one can easily compute the escape velocity, which is the minimum velocity needed to escape the gravitational pull of the earth. Use energy conservation, one gets

$$\frac{1}{2}mv^2 - \frac{GM_E m}{R_E} = 0, \quad \Rightarrow \tag{8}$$

$$v = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2gR_E} = 1.12 \times 10^4 m/s. \tag{9}$$

Satellites can have closed orbits (circle or ellipse with $E < 0$) or open orbits (parabola with $E = 0$ or hyperbola with energy $E > 0$). Satellite never returns to its starting point but travels ever farther away from the earth if it is in an open orbit.

Simplest case: satellites in circular orbits.
Use Newton's laws, one finds the following parameters of the circular orbits

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r}, \quad \Rightarrow \quad v = \sqrt{\frac{GM_E}{r}}, \tag{10}$$

$$T = \frac{2\pi r}{v} = 2\pi \frac{r^{3/2}}{\sqrt{GM_E}}. \tag{11}$$

The total mechanical energy is

$$E = K + U = \frac{1}{2}mv^2 - \frac{GM_E m}{r} = -\frac{GM_E m}{2r} < 0, \tag{12}$$

where $U = 0$ at infinity.

1.5 Kepler's law and the motion of planets

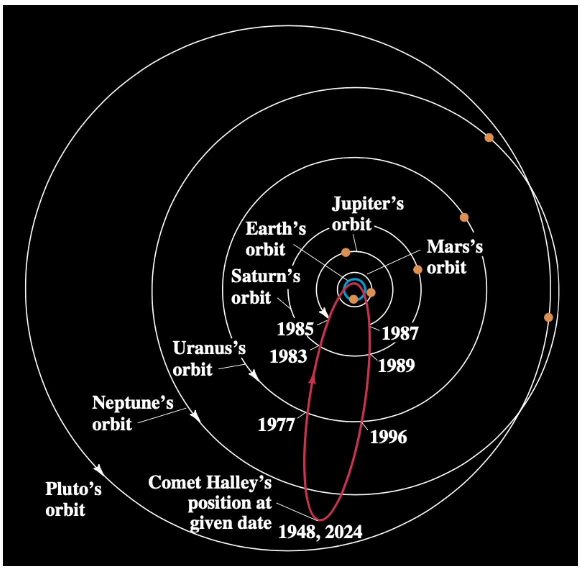
By trial and error, Kepler discovered three empirical laws that accurately described the motions of the planets in early 1600:

1. Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

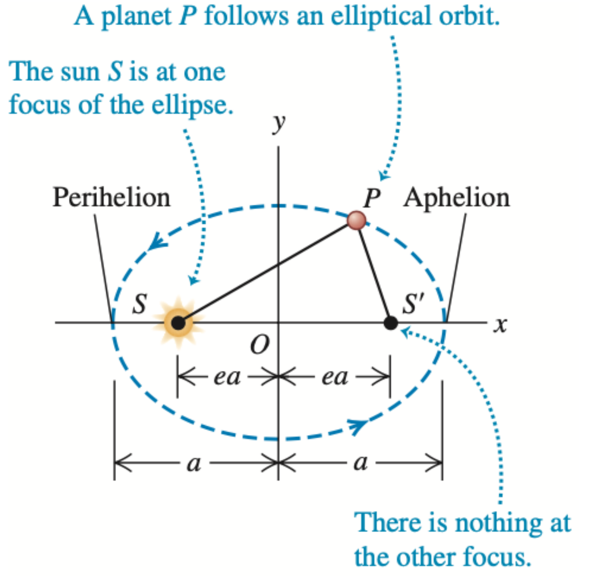
2. A line from the sun to a given planet sweeps out equal areas in equal times.

3. The periods of the planets are proportional to the 3/2 powers of the major axis lengths of their orbits.

Then 80 years later, Newton discovered that each of Kepler's laws can be derived from Newton's laws of motion and the law of gravitation.



1. Kepler's First Law: Elliptical Orbits
The name planet comes from a Greek word meaning 'wanderer', since the planets continuously change their positions in the sky relative to the background of stars. In the solar system, the planets include Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, and Neptune, with Pluto reclassified as dwarf planet.



The above figure shows the geometry of an elliptical orbit for the planet P. If $e=0$, the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e=0.017$.) The point in the planet's orbit closest to the sun is the perihelion, and the point most distant from the sun is the aphelion.

Mathematically, the above orbit can written as

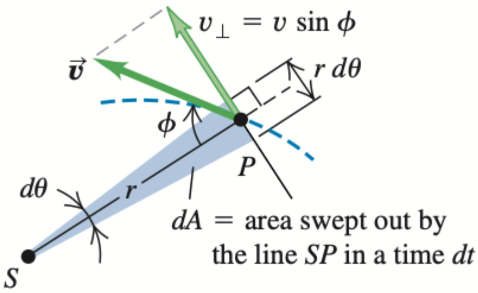
$$r=\frac{(1+e)r_0}{1+e\cos\theta}, \tag{13}$$

where $r_0=(1-e)a$ is the distance between the sun and the planet at the Perihelion.

Newton was able to show that for a body acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits must be parabolas or hyperbolas.

These results can be derived by a straightforward application of Newton's laws and the law of gravitation, together with rather lengthy derivations involving differential equations.[1]

2. Kepler's Second Law: Equal area = L conservation



$$\frac{dA}{dt}=\frac{1}{2}rv_\perp=\frac{1}{2}rv\sin\phi=\frac{1}{2m}|\vec{r}\times m\vec{v}|=\frac{L}{2m}. \tag{18}$$

Therefore, the Kepler's Second Law - that sector velocity is constant - means that angular momentum is conserved. This is because gravitational force is central force $\to d\vec{L}/dt=0$. Also, since $\vec{L}=\vec{r}\times\vec{p}$ is a conserved quantity, this means that $\vec{r}$ and $\vec{v}$ always lie in the same plane, which is the plane of the planet's orbit.

3. Kepler's Third Law:
For both circular and elliptic orbits

$$T=\frac{2\pi}{\sqrt{GM}}a^{3/2}, \quad \text{a is the semi-major-axis.} \tag{19}$$

The period T is independent of e. To derive the above result, one need to use energy and angular momentum conservation and find the velocities at the perihelion and aphelion are

$$v_p=\sqrt{\frac{1+e}{1-e}\frac{GM}{a}}, \tag{20}$$

$$v_a=\sqrt{\frac{1-e}{1+e}\frac{GM}{a}}, \tag{21}$$

respectively. Then use the geometric relation $A=\pi ab$ and $T=A/(L/2m)$.

Newtonian Synthesis: The idea that the same physics laws apply equally well in the terrestrial and celestial realms is a major intellectual development of 17th century. It has had profound effects on the way that humanity looks at the Universe - not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

[1] If you are really interested in detailed derivation, you can try to follow the hints below. Caution, the calculation is a bit tedious. Start from two equations

$$\frac{d^2r}{dt^2}-\frac{L^2}{m^2r^3}=-\frac{GM}{r^2}, \tag{14}$$

$$\text{Angular Momentum}: L=mr^2\frac{d\theta}{dt}=\text{const.} \tag{15}$$

The first equation is from 2nd law (note the $1/r^2$ dependence on RHS) and the second equation is due to the fact that gravity is central force. By eliminating the time dependence, show that the above equations can be converted to

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{GMm^2}{L^2}, \tag{16}$$

with the solution reads

$$\frac{1}{r}=\frac{GMm^2}{L^2}(1+e\cos\theta). \tag{17}$$

1.6 Black hole

Black hole: The concept of a black hole is one of the most fascinating idea in modern physics, yet the basic idea can be understood on the basis of Newtonian mechanics. Imagine when the gravitational pull of a star is so strong that even light can no longer escape from its gravitational influence, then this star should appear to be completely black since it can not emit any light of its own.

By setting the escape velocity of a black hole to $\sqrt{2GM/R_S} = c$, we can solve for the critical radius

$$R_S = \frac{2GM}{c^2}, \tag{22}$$

which is known as the Schwarzschild radius. In 1916, Karl Schwarzschild solved Einstein's equation in the general theory of relativity (GR) and found the black hole solution predicted by GR.

1. As a matter of fact, Eq. (22) does give the correct result in GR, but only because of two compensating errors used in Newtonian mechanics.

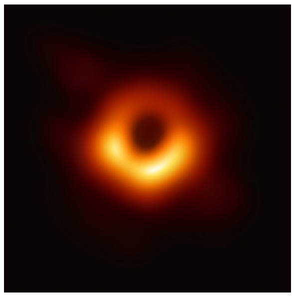
2. The surface of the sphere with radius R_S surrounding a black hole is called the event horizon: Since light can't escape from within that sphere ($R < R_S$), we can't see events occurring inside.

3. No-hair theorem: classical black holes have no hair. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other bodies), its electric charge (from the electric forces it exerts on other charged bodies), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information (for which "hair" is a metaphor) is irretrievably lost when the body collapses inside its event horizon.

4. Stephen Hawking's last paper: black holes may have "soft quantum hair".

5. Usually, a typical black hole has a mass between about 3 and 10 solar masses ($1M_\odot$ =1 solar mass = mass of the Sun). Supermassive black holes with masses millions of times the solar mass $M_\odot$ exist in the center of most galaxies, including our own Milky Way Galaxy.

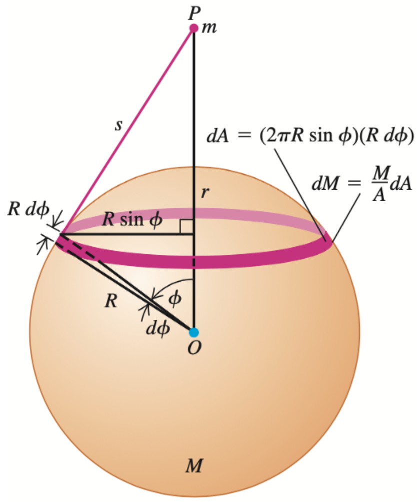
6. Evidence of black holes: a). The Event Horizon Telescope released the first direct image of a black hole in 2019. The figure below depicts the gases matter at the edge of the event horizon (displayed as orange or red) of the supermassive black hole that lies in the center of Messier 87 galaxy. b). Gravitational waves from collisions of black holes.



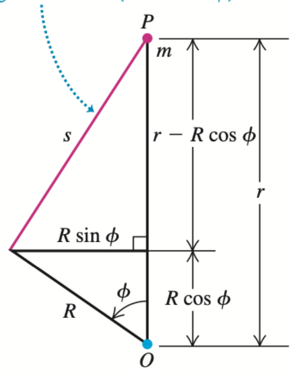
1.7 Spherical Mass Distribution

We have stated earlier that the gravity outside any spherically symmetric mass distribution is the same as if all the mass were concentrated at its center. Now let us try to prove this conclusion using calculus. (This statement is known as **the Shell Theorem**, and this section is optional. It took Newton a few years to search for a proof.)

(a) Geometry of the situation



(b) The distance s is the hypotenuse of a right triangle with sides ($r - R\cos\phi$) and $R \sin\phi$.



To tackle the general problem of two spherical masses, let us start with the simple problem of a point mass m interacting with a thin spherical shell with total mass M. We will show that when m is outside the sphere, the potential energy associated with this gravitational interaction is the same as though M were all concentrated at the centre of the sphere.

Then, we will consider the other half of the proof, which is to prove that two spherically symmetric mass distributions interact as though they were both points.

1. We start by considering a ring on the surface of the shell as shown in the above figure. In this case, all the particles that make up the ring have the same distance s from the point mass m. Thus, the potential energy between the point mass and the ring is

$$dU = -\frac{GmdM}{s}. \tag{23}$$

To proceed, we need to know the mass of the thin strip (ring) dM. Let us suppose the mass per unit area of the homogeneous spherical shell is $\sigma = \frac{M}{A}$ where

$$A = \int_0^\pi d\theta RR\sin\theta(2\pi) = 4\pi R^2$$

is the total area of the spherical shell. Therefore, we can write

$$dM = \sigma dA = \frac{M}{4\pi R^2} R\,d\theta(2\pi)R\sin\theta = \frac{M}{2}\sin\theta\,d\theta. \tag{24}$$

In addition, we need to express the distance s as

$$s = \sqrt{r^2 + R^2 - 2rR\cos\theta}. \tag{25}$$

By substituting the above result into Eq. (23), we find the differential potential energy becomes

$$dU = -\frac{GmM}{2}\frac{\sin\theta\,d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}. \tag{26}$$

To carry out the integration, we can further change the integration variable from θ to

$$u = r^2 + R^2 - 2rR\cos\theta \;\Rightarrow\; du = 2rR\sin\theta\,d\theta.$$

This allows us to rewrite Eq. (26) as

$$dU = -\frac{GmM}{2}\frac{du}{2Rr\sqrt{u}}. \tag{27}$$

Now we can integrate the differential potential energy and obtain

$$\begin{aligned} U &= -\frac{GmM}{4Rr}\int_{(r-R)^2}^{(r+R)^2}\frac{du}{\sqrt{u}} \\ &= -\frac{GmM}{4Rr}2\left[(r+R)-|r-R|\right]. \end{aligned} \tag{28}$$

It is worth noting that the lower ($(r-R)^2$) and upper ($(r+R)^2$) limits of u integration arise from $\theta = 0$ and $\theta = \pi$ (lower and upper limits of θ integral), respectively.

   (a) When $r > R$, the point mass is outside the spherical shell, thus Eq. (28) becomes

$$U = -\frac{GmM}{2Rr}\left[(r+R)-(r-R)\right] = -\frac{GmM}{r}. \tag{29}$$

   (b) When $r < R$, the point mass is inside the spherical shell, thus Eq. (28) becomes

$$U = -\frac{GmM}{2Rr}\left[(r+R)-(R-r)\right] = -\frac{GmM}{R}, \tag{30}$$

   which is a constant and independent of r.

2. We have shown that the gravitational effect of a spherical shell can be viewed from the center. In the above figures, the forces the two bodies exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts on the sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centred at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

3. In the end, let us compute the gravitational potential which is defined as $V = U/m$

$$V(r) = \begin{cases} -\dfrac{GM}{r}, & r > R \\ -\dfrac{GM}{R}, & r < R. \end{cases} \tag{31}$$

4. To compute the gravitational potential in the interior of the ball, let us consider a point inside the ball with the radius $r < R$.

   (a) All the spherical mass shell with the radius $l < r$ gives the following potential

$$V = -\frac{GM}{r}, \tag{32}$$

   where $M = \sigma 4\pi l^2$. In order to integrate over l, we need to consider the thickness of the shell and rewrite $\sigma = \rho dl$, where ρ is the mass per unit volume. Now we can obtain the contribution from

$$V_{l<r} = \int_0^r -\frac{G}{r}4\pi\rho l^2 dl = -\frac{4\pi G\rho r^2}{3}. \tag{33}$$

   (b) Similarly, the spherical shell with the radius $r < l < R$ gives

$$V_{r<l<R} = \int_r^R -\frac{G}{l}4\pi\rho l^2 dl = -2\pi G\rho(R^2 - r^2).$$

   (c) The total potential is then given by the sum of above two contributions

$$\begin{aligned} V_{total} &= -2\pi G\rho(R^2-r^2) - \frac{4\pi G\rho r^2}{3}. \\ &= -2\pi G\rho R^2 + \frac{2\pi G\rho r^2}{3} \\ &= -\frac{3}{2}\frac{GM}{R} + \frac{1}{2}\frac{GMr^2}{R^3}. \end{aligned} \tag{34}$$

   (d) In the end, the gravitational field g is

$$g(r) = -\frac{\partial V_{total}}{\partial r} = -\frac{GM}{R^3}r. \tag{35}$$

   In terms of vector analysis, since

$$\begin{aligned} \vec{\nabla} r^2 &= \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial j}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2) \\ &= 2\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = 2\vec{r}, \end{aligned} \tag{36}$$

   then the vector form of the gravitational field is

$$\vec{g}(r) = -\vec{\nabla} V_{total} = -\frac{GM}{R^3}\vec{r}. \tag{37}$$

   (e) With the gravitational field g, we can obtain the gravity force $F = mg = -\frac{GMm}{R^3}r$ for an object with mass m inside a sphere ($r < R$) with mass M and radius R.