STA2001 Probability and Statistics (I)

Lecture 7

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Review

- Negative binomial distribution with parameter p and r:

  X, the number of Bernoulli trials at which the rth success is observed, and its pmf takes the form of

  $$\text{pmf: } f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \; x \in \overline{S} = \{r, r+1, \cdots\}$$

- Poisson distribution with parameter $\lambda > 0$:

  X, the number of occurrences of an event in <u>a unit interval</u> and its pmf takes the form of

  $$\text{pmf: } f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \; x \in \overline{S} = \{0, 1, \cdots\}$$

Chapter 3 Continuous Distribution

Section 3.1 Random Variable of Continuous Type

Continuous RV

Recall that a RV $X : S \to \overline{S}$ is called a discrete RV if $\overline{S}$ contains finite or countably infinite number of outcomes.

Now we consider RVs with $\overline{S}$ that is an interval or unions of intervals, which are quite common (e.g., velocity of a vehicle traveling along the high way)

Discrete RV vs. Continuous RV

RV X is a function $X : S \to \overline{S} \subseteq R$

Discrete RV:

pmf $f(x) : \overline{S} \to (0,\ 1]$

1. $f(x) > 0$
2. $\sum_{x \in \overline{S}} f(x) = 1$
3. $P(X \in A) = \sum_{x \in A} f(x)$

Continuous RV:

Continuous RV

Definition

A RV X with $\overline{S}$ that is an interval or unions of intervals is said to be continuous RV, if there exists a function $f(x)$:$\overline{S}$ $\rightarrow (0, \infty)$ such that

1. $f(x) > 0, \quad x \in \overline{S}$
2. $\int_{\overline{S}} f(x)dx = 1$
3. If $[a, b] \subseteq \overline{S}$

$$P(a \leq X \leq b) \triangleq \int_a^b f(x)dx$$

f is the so called probability density function (pdf).

Discrete RV vs. Continuous RV

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Continuous RV:

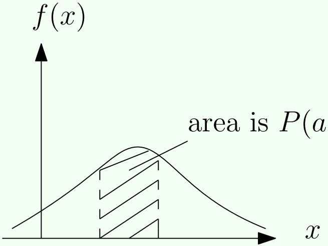
pdf $f(x) : \overline{S} \rightarrow (0, \infty)$

1. $f(x) > 0$
2. $\int_{\overline{S}} f(x) dx = 1$
3. $P(X \in A) = \int_A f(x) dx$

Interpretation of pdf

Interpretation

1. $$P(a \leq X \leq b) = \int_a^b f(x)dx$$

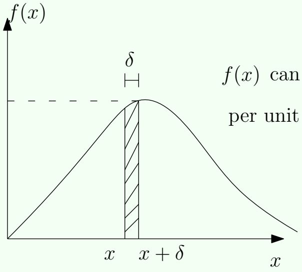


Interpretation of pdf

Interpretation

2.

$$P(x \leq X \leq x + \delta) = \int_x^{x+\delta} f(t)dt \approx f(x)\delta$$



$f(x)$ can be viewed as the probability mass

per unit length near x

Remarks

1. We often extend the domain of $f(x)$ from $\overline{S}$ to R and let $f(x) = 0$, $x \notin \overline{S}$. In this case, $f(x) : R \to [0, \infty)$ and $\overline{S}$ is called the support of X.

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$$\begin{cases} f(x) \geq 0, \quad x \in R \\ \int_{-\infty}^{\infty} f(x)dx = 1 \\ P(a \leq X \leq b) = \int_a^b f(x)dx \end{cases}$$

Remarks

2. For any single value a, $P(X = a) = \int_a^a f(x)dx = 0$.

   Therefore, including or excluding the end points of an interval has no effect on its probability:

   $$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

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3. pdf needs not to be continuous

   $$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < 1, \quad 2 < x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

4. pdf needs not to be bounded, e.g., the Gamma distribution

Cumulative distribution function

Definition

cdf $F(x) : R \to [0, 1]$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$$

1. $F(x)$ is nondecreasing
2. relation between the probability function and the cdf

$$P(a \leq X \leq b) = F(b) - F(a)$$

3. relation between the pdf and the cdf

$$f(x) = F'(x)$$

for those values of x at which $F(x)$ is differentiable

Example 1 [Uniform Distribution]

Let the RV X denote the outcome when a point is selected randomly from $[a, b]$ with $-\infty < a < b < \infty$.

Define the pdf of X

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

What is the cdf of X?

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$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

What is the cdf of X?

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$

Uniform Distribution

$$\text{For any } x \in [a, b], \qquad P(X \leq x) = \frac{x - a}{b - a}$$

implies the probability of selecting a point from the interval $[a, x]$ is proportional to the length of $[a, x]$. Such distribution is called uniform distribution and denoted by $X \sim U(a, b)$.

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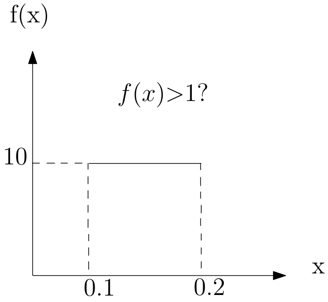
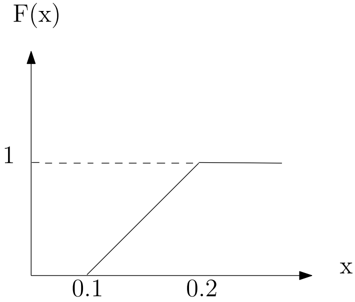
For example, let $X \sim U(0.1, 0.2)$





Example 2, page 96

Let Y be a continuous RV with pdf $g(y) = 2y$, $0 < y < 1$.

What is the cdf of Y, $P(\frac{1}{2} < Y \leq \frac{3}{4})$, $P(\frac{1}{4} < Y < 2)$?

Example 2, page 96

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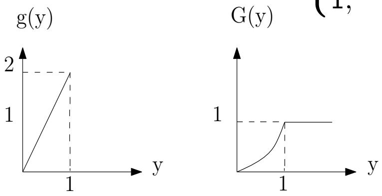
What is the cdf of Y, $P(\frac{1}{2} < Y \le \frac{3}{4})$, $P(\frac{1}{4} < Y < 2)$?

$$G(y) = P(Y \le y) = \int_{-\infty}^{y} g(t)dt = \begin{cases} 0, & y \le 0 \\ y^2, & 0 < y < 1 \\ 1, & y \ge 1 \end{cases}$$



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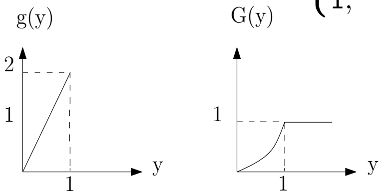


$$P(\frac{1}{2} < Y \le \frac{3}{4}) = G(\frac{3}{4}) - G(\frac{1}{2}) = \frac{5}{16}$$
$$P(\frac{1}{4} < Y < 2) = G(2) - G(\frac{1}{4}) = \frac{15}{16}$$

Mathematical Expectation

Mathematical Expectation

Let X be a continuous RV with pdf $f(x) : \overline{S} \to (0, \infty)$. If $\int_{\overline{S}} g(x)f(x)dx$ exists, it is called the mathematical expectation for $g(X)$ and denoted by

$$E[g(X)] = \int_{\overline{S}} g(x)f(x)dx$$

If the range of X is extended from $\overline{S}$ to R with $f(x) = 0$ for $x \notin \overline{S}$, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Expectation is a linear operator [Theorem 2.2-1, page 60].

$$E[c_1 g_1(X) + c_2 g_2(X)] = c_1 E[g_1(X)] + c_2 E[g_2(X)]$$

Special Mathematical Expectations

1. $[g(X) = X]$: Mean of X, $E[X] = \int_{\overline{S}} xf(x)dx$
2. $[g(X) = (X - E[X])^2]$: Variance of X,

$$Var[X] = E[(X - E[X])^2] = \int_{\overline{S}} (x - E[X])^2 f(x)dx$$

3. $[g(X) = X^r]$, Moments of X:

$$E[X^r] = \int_{\overline{S}} x^r f(x)dx$$

Special Mathematical Expectations

4. $[g(X) = e^{tX}]$: Moment generating function (mgf). If there exists $h > 0$, such that

$$M(t) = E[e^{tX}] = \int_S e^{tx} f(x)dx, \quad -h < t < h \text{ for some } h > 0$$

Mgf determines the distribution of X and all moments exist and are finite

$$M^{(r)}(0) = E[X^r]$$

which can be used to derive the mean and variance of a RV X

$$E[X] = M'(0), \quad Var[X] = M''(0) - (M'(0))^2$$

Example 3, page 98

Let X have the pdf

$$f(x) = \begin{cases} \frac{1}{100}, & 0 < x < 100 \\ 0, & \text{otherwise.} \end{cases} \Leftrightarrow X \sim U(0, 100)$$

Example 3, page 98

Let X have the pdf

$$f(x) = \begin{cases} \frac{1}{100}, & 0 < x < 100 \\ 0, & \text{otherwise.} \end{cases} \Leftrightarrow X \sim U(0, 100)$$

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^{100} x \frac{1}{100} dx = \frac{1}{100} \cdot \frac{1}{2} x^2 \Big|_0^{100} = 50 \end{aligned}$$

$$Var[X] = E[(X - E[X])^2] = \int_0^{100} (x - 50)^2 \frac{1}{100} dx = \frac{2500}{3}.$$

Mean and Variance for $U(a, b)$

Actually, for $X \sim U(a, b)$

$$E[X] = \frac{a+b}{2}, \quad Var[X] = \frac{(b-a)^2}{12},$$

They can be derived by

1. the definition
2. the mgf technique?

$$M(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

It does not work as usual and is skipped.

Example 4, page 99

Question

Let X be a continuous RV and have the pdf

$$f(x) = \begin{cases} xe^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

$E[X]$ and $Var[X]$?

Example 4, page 99

Question

Let X be a continuous RV and have the pdf

$$f(x) = \begin{cases} xe^{-x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

$E[X]$ and $Var[X]$?

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \int_{0}^{\infty} xe^{-x} e^{tx} dx$$

$$= \int_{0}^{\infty} xe^{-(1-t)x} dx = \left[-\frac{xe^{-(1-t)x}}{1-t} - \frac{e^{-(1-t)x}}{(1-t)^2} \right] \Big|_{0}^{\infty}$$

Example 4, page 99

$$M(t) = \lim_{b\to\infty}\left[-\frac{be^{-(1-t)b}}{1-t} - \frac{e^{-(1-t)b}}{(1-t)^2}\right] + \frac{1}{(1-t)^2}$$

$$\overset{\text{when} t<1, i.e., 1-t>0}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=} \frac{1}{(1-t)^2}$$

$$M'(t) = 2\cdot\frac{1}{(1-t)^3} \Rightarrow M'(0) = 2$$

$$M''(t) = 6\cdot\frac{1}{(1-t)^4} \Rightarrow M''(0) = 6$$

$$E[X] = M'(0) = 2,$$

$$Var[X] = E[X^2] - (E[X])^2 = M''(0) - (M'(0))^2 = 2$$

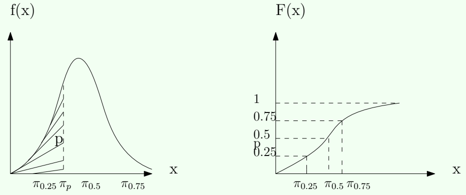
(100p)th percentile

Definition

It is a number π_p such that the area under $f(x)$ to the left of π_p is p. That is

$$p = \int_{-\infty}^{\pi_p} f(x)dx = F(\pi_p)$$

The 50th percentile is called the median. The 25th and 75th percentiles are called the first and third quantiles, respectively. The median is also called the 2nd quantile.



Example 5

Let X be a continuous RV with the pdf

$$f(x) = \frac{3x^2}{4^3} e^{-\left(\frac{x}{4}\right)^3}, \quad 0 < x < \infty$$

What is $\pi_{0.3}$?

Example 5

Let X be a continuous RV with the pdf

$$f(x) = \frac{3x^2}{4^3} e^{-(\frac{x}{4})^3}, \quad 0 < x < \infty$$

What is $\pi_{0.3}$?

$$F(x) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & -\infty < x < 0 \\ 1 - e^{(-\frac{x}{4})^3}, & 0 \le x < \infty \end{cases}$$

$$F(\pi_{0.3}) = P(X \le \pi_{0.3}) = 0.3$$

$$\Rightarrow 1 - e^{(-\frac{\pi_{0.3}}{4})^3} = 0.3, \quad \ln 0.7 = (-\frac{\pi_{0.3}}{4})^3$$

$$\Rightarrow \pi_{0.3} = -4(\ln 0.7)^{\frac{1}{3}} = 2.84$$