MAT2041 Linear Algebra and Applications

Final Exam

Time: Feb 02, 2023 Thur 8:30am - 11am   Total: 8 questions, 100 points + 5 bonus points

1. **(10 pt) Least Squares** Consider matrix A with full column rank and a vector b.

   (a) (4 points) Write down the solution of the least squares problem $\min_x \|Ax - b\|$, using a formula (just use letters A, b, not numbers). Additionally, write down a formula of $p = A\hat{x}$.

   (b) (3 points) Which fundamental subspaces with respect to A does the vector p belong to? How about the vector $r = b - p$? (Recall that the four fundamental subspaces of a given matrix are the row space, column space, null space, and the left null space of that matrix.)

   (c) (3 points) When A is a square matrix, simplify the expressions of $\hat{x}, p$ and r.

2. **(10 pt) Linear Transformation** Suppose $P_2 \triangleq \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$ is the linear space of polynomials of degree at most 2. Consider the mapping $T : P_2 \to P_2$, defined as $T(p(x)) = xp(x) + (x^2 - 1)p'(x)$ for all $p(x) \in P_2$. Here $p'(x)$ is the derivative of $p(x)$. Is T a linear transformation? Justify your answer.

3. **(16 pt) True or False** Decide whether the following statements are true or false. You do NOT need to justify (only writing T or F is enough).

   (a) If a matrix A has full row rank, then the linear system $Ax = b$ has at least one solution.

   (b) If A is an invertible square matrix, then $(A^\top)^{-1} = (A^{-1})^\top$.

   (c) The number of non-zero eigenvalues (counting multiplicity) of an $n \times n$ real matrix A is the same as the rank of A.

   (d) An $m \times n$ real matrix always has $\min\{m, n\}$ singular values (counting multiplicity).

   (e) All $n \times n$ real matrices are diagonalizable.

   (f) If A is a 2×2 matrix with characteristic polynomial $p_A(\lambda) = \lambda^2$, then $A = 0$.

   (g) If 0 is an eigenvalue of an $n \times n$ matrix A, then $\text{rank}(A) < n$.

   (h) If A can be written as the sum of k rank-1 matrices, then $\text{rank}(A) \le k$.

4. **(14 pt) Dynamics and Eigenvalues** Suppose at the n-th year's January, the number of the skilled and unskilled workers in the product line is x_n and y_n respectively. In February, the product line transfers $\frac{1}{5}$ of the total skilled workers to other apartments. In March, the product line recruits new unskilled workers to keep the total number of workers unchanged, with the number of newly recruited unskilled workers being equal to the number of transferred skilled workers. After training for nine months, in December, $\frac{3}{8}$ of the unskilled workers (including both newly recruited and old) become skilled workers.

   (a) (6 points) What is the relationship between $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}$? Write your solution in the matrix form of $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = A \begin{bmatrix} x_n \\ y_n \end{bmatrix}$.

   Remark: You need to explain briefly how you get the expression. If you directly write some equations or directly write A without explanation, you will not get full credit.

   (b) (6 points) Compute the eigenvalues and the eigenvectors of the matrix A.

   (c) (2 points) Suppose $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}$. Write down the expression of $\begin{bmatrix} x_n \\ y_n \end{bmatrix}$ for any positive integer n.

5. **(14 pt) SVD**

Let

$$\boldsymbol{A} = \left[\begin{array}{ccc} 2 & 0 & 1 \\ 0 & 2 & 1 \end{array}\right].$$

(a) (12 points) Find the singular value decomposition $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$, where $\boldsymbol{\Sigma}$ is a diagonal matrix, and $\boldsymbol{U}$ and $\boldsymbol{V}$ have orthonormal sets of column vectors.

(b) (2 points) Write A in the form $A = \sum_{j=1}^{2} \sigma_j u_j v_j^\top$, where σ_j, u_j, v_j are obtained from (a).

6. **(10 pt) Question Answering** Answer the following questions.

(a) (2 pt) During the Spring Festival, your friend in another university asked you: what topics do you learn in your linear algebra class? Answer this question, by providing at least three topics (three short expressions that describe the topic).

Remark 1: As an analogy, in calculus class, you learned "derivatives, integrals, limit"; or "theory of limit, how to take derivatives, theory of integrals"; or "limit and continuity, derivatives and mean value theorem, integrals". There is no unique answer, but you should cover major topics.

Remark 2: If your answer is "we learned transpose of matrix, symmetric matrix and basis", you will not get full credit because these three topics are too small.

(b) (2 pt) What is a geometrical interpretation of "determinant"? You can use 2-dim or 3-dim case to answer the question (surely, answering n-dim case is also acceptable).

Remark: If you don't remember a certain English word, you can use an extra sentence to describe the word.

(c) (3 pt) We have studied two definitions of the linear transformation in Euclidean space. Why do we study two? What are the benefits of each definition?

(d) (3 pt) What is a geometrical interpretation of the decomposition $A = SDS^{-1}$, where S is invertible and D is diagional? What is a geometrical interpretation of the decomposition $A = U\Sigma V^\top$, where U, V are orthogonal matrices and Σ is rectangular diagonal? You can use a 2-dimensional case to answer. Hint: Linear transformation, basis, tile.

7. **(12 pt) Eigenvalues and Eigenvectors**

Recall that a matrix $Z \in \mathbb{R}^{n\times n}$ is positive-definite iff Z is symmetric and $x^\top Zx > 0$ for any $x \in \mathbb{R}^n \backslash \{0\}$. Suppose $A, B \in \mathbb{R}^{n\times n}$ are positive-definite matrices.

(a) (5 pts) Prove: there exist positive-definite matrices X, Y such that $X^2 = A$ and $Y^2 = B$.

(b) (5 pts) For X, Y you find in (a), is it true that any eigenvalue of AB is an eigenvalue of the matrix $XYYX$? Justify your answer.

(c) (2 pts) Is it true that $x^\top ABx \geq 0$ for any $x \in \mathbb{R}^n$? Justify your answer.

8. **(14 pt + 5 bonus pt) Inequalities Related to Eigenvalues and Singular Values**

(a) (6 pt) Consider a symmetric matrix $B \in \mathbb{R}^{n\times n}$. Prove or disprove the following statement: $x^\top Bx \leq \lambda_{\max}\|x\|^2$ for any $x \in \mathbb{R}^{n\times 1}$, where $\lambda_{\max}$ is the largest eigenvalue of B.

Remark: If you want to disprove a statement, you need to provide a counter-example.

(b) (5 pt) Suppose the largest singular value of $A \in \mathbb{R}^{m\times n}$ is $\sigma_{\max}$. Prove or disprove the following statement: $\|Ax\| \leq \sigma_{\max}\|x\|$ for any $x \in \mathbb{R}^{n\times 1}$.

(c) (3 pt) Suppose the smallest singular value of $A \in \mathbb{R}^{m\times n}$ is $\sigma_{\min}$. Prove or disprove the following statement: $\|Ax\| \geq \sigma_{\min}\|x\|$ for any $x \in \mathbb{R}^{n\times 1}$.

(d) (5 bonus pt) Now we consider a math problem in deep learning. Suppose A_1, A_2, A_3 are three matrices with dimension $d_1 \times d_2, d_2 \times d_3, d_3 \times n$ respectively. Can you find proper α_1, α_2 so that $\alpha_1\|x\| \leq \|A_1A_2A_3x\| \leq \alpha_2\|x\|$ for any $x \in \mathbb{R}^{n\times 1}$? These α_1, α_2 should be positive, purely decided by singular values of A_1, A_2, A_3, and can make the inequalities tight (more specifically, for any A_1, A_2, A_3, there exist two x's so that two inequalities become equalities respectively).

If you answer "yes", provide proof; if you answer "no", provide extra restrictions (the weaker, the better) on the set of A_1, A_2, A_3 so that such α_1, α_2 can be found.