Slide 16-Determinants II

MAT2040 Linear Algebra

Property 16.1 (Determinant for Row or Column Interchange)
Suppose that A is a square matrix, and $A \xrightarrow{R_i \to R_j} B$, or $A \xrightarrow{C_i \to C_j} B$. Then $\det(B) = -\det(A)$.

Proof. This can also be proved by mathematical induction.

Remark for elementary row operation I:
Recall elementary row operation I:
$I_n \xrightarrow{R_i \to R_j} E_{R_iR_j}$. Thus $\det(E_{R_iR_j}) = -\det(I_n) = -1$. And $\det(E_{R_iR_j}A) = -\det(A) = \det(E_{R_iR_j})\det(A)$.

Examples 16.2

$$\det(A) = \begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} (R_1 \leftrightarrow R_4) = - \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{14} \end{vmatrix} (R_2 \leftrightarrow R_3)$$

$$= \begin{vmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{14} \end{vmatrix}$$

$$= a_{14} a_{32} a_{23} a_{41}$$

Property 16.3 Suppose that A is a square matrix. Let $A \xrightarrow{R_i \to \alpha R_i (\alpha \neq 0)} B$ or $A \xrightarrow{C_i \to \alpha C_i (\alpha \neq 0)} B$. Then $\det(B) = \alpha \det(A)$.

Proof. If the ith row of A is multiplied by $\alpha(\alpha \neq 0)$ to obtain B, then expand B along ith row, one obtains that

$$\begin{aligned}
\det(B) =& b_{i1}A_{i1} + b_{i2}A_{i2} + \cdots + b_{in}A_{in} \\
=& \alpha a_{i1}A_{i1} + \alpha a_{i1}A_{i2} + \cdots + \alpha a_{i1}A_{in} \\
=& \alpha(a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}) \\
=& \alpha \det(A).
\end{aligned}$$

The proof for column operation is similar.

Remark for elementary row operation II:

Recall elementary row operation II: $I_n \xrightarrow{R_i \to \alpha R_i (\alpha \neq 0)} E_{\alpha R_i}$.
Thus, $\det(E_{\alpha R_i}) = \alpha \det(I_n) = \alpha (\alpha \neq 0)$, and

$$\det(E_{\alpha R_i} A) = \alpha \det(A) = \det(E_{\alpha R_i}) \det(A) (\alpha \neq 0).$$

Lemma 16.4 Let $A = (a_{ij})_{n\times n}$, A_{ij} is the cofactor of a_{ij}, then

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

Proof. Define a new determinant

$$det(A^*) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \begin{array}{l} \\ \\ ith\ row \\ \\ jth\ row \\ \\ \\ \end{array} = \begin{cases} \det(A), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

On the other hand, expand along jth row for $det(A^*)$ gives

$$det(A^*) = a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}. \text{This completes the proof.}$$

Remark The cofactor A_{ij} of matrix A and matrix A^* are the same only along jth row.

Property 16.5 Suppose that A is a square matrix, and $A \xrightarrow{R_j \to \beta R_i + R_j} B$ $(i \neq j)$. Then $\det(B) = \det(A)$, i.e.,

$$\begin{array}{r} \\ \\ ith\ row \\ \\ jth\ row \\ \\ \\ \end{array} \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \cdots & \vdots \\ \beta a_{i1} + a_{j1} & \cdots & \beta a_{in} + a_{jn} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & \cdots & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \begin{array}{l} \\ \\ ith\ row \\ \\ jth\ row \\ \\ \\ \end{array}$$

Proof. Due to $A \xrightarrow{R_j \to \beta R_i + R_j} B$ $(i \neq j)$. Expanding along jth row for B, one gets that

$$\begin{aligned}
\det(B) =& b_{j1}A_{j1} + b_{j2}A_{j2} + \cdots + b_{jn}A_{jn} \\
=& (\beta a_{i1} + a_{j1})A_{j1} + (\beta a_{i2} + a_{j2})A_{j2} + \cdots + (\beta a_{in} + a_{jn})A_{jn} \\
=& \beta(a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}) \\
&+ (a_{j1}A_{j1} + a_{j2}A_{j2} + \cdots + a_{jn}A_{jn}) \\
=& \det(A)
\end{aligned}$$

where we have used the fact that $a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = 0$ since $i \neq j$.

Remark: 1. The cofactor A_{ij} of matrix A and matrix B are the same only along jth row.

2. If $A \xrightarrow{C_j \to \beta C_i + C_j} B$ $(i \neq j)$. Then $\det(B) = \det(A)$.

Remark elementary row operation III:

Recall elementary row operation III:
$I_n \xrightarrow{R_j \to \beta R_i + R_j} E_{\beta R_i + R_j}, \quad \det(E_{\beta R_i + R_j}) = \det(I_n) = 1.$ Thus,
$\det(E_{\beta R_i + R_j} A) = \det(A) = \det(E_{\beta R_i + R_j}) \det(A).$

Summary

Property 16.1, Property 16.3, Property 16.5 are the properties for determinants with respect to row operations (column operations).

(1) When using row operation I ($R_i \leftrightarrow R_j$) or column operation I ($C_i \leftrightarrow C_j$), the determinant after the operation equals (-1) times the determinant before the operation.

(2) When using row operation II ($R_i \to \alpha R_i$) ($\alpha \neq 0$)or column operation II ($C_i \to \alpha C_i$)($\alpha \neq 0$), the determinant after the operation equals the α times the determinant before the operation.

(3) When using row operation III ($R_j \to \beta R_i + R_j$) or column operation III ($C_j \to \beta C_i + C_j$), the determinant after the operation equals the determinant before the operation.

Summary for determinants of elementary matrices

Property 16.6 (**Determinants of Three Elementary Matrices**)

(1) $I_n \xrightarrow{R_i \to R_j} E_{R_iR_j}, \quad \det(E_{R_iR_j}) = -\det(I_n) = -1,$
$\det(E_{R_iR_j}A) = -\det(A) = \det(E_{R_iR_j})\det(A).$

(2) $I_n \xrightarrow{R_i \to \alpha R_i (\alpha \neq 0)} E_{\alpha R_i}, \quad \det(E_{\alpha R_i}) = \alpha\det(I_n) = \alpha(\alpha \neq 0),$
$\det(E_{\alpha R_i}A) = \alpha\det(A) = \det(E_{\alpha R_i})\det(A)(\alpha \neq 0).$

(3) $I_n \xrightarrow{R_j \to \beta R_i + R_j} E_{\beta R_i + R_j}, \quad \det(E_{\beta R_i + R_j}) = \det(I_n) = 1,$
$\det(E_{\beta R_i + R_j}A) = \det(A) = \det(E_{\beta R_i + R_j})\det(A).$

Property 16.7 (Determinant for an Elementary Matrix Multiply a Matrix) Suppose that A is a square matrix of size n and E is an elementary matrix of size n, then

$$\det(EA) = \det(E)\det(A)$$

$$\det(AE) = \det(A)\det(E)$$

Proof

For elementary matrices type I, II, III, $\det(EA) = \det(E)\det(A)$ has already been shown in previous discussion.

E is an an elementary matrix, thus E^T is also an elementary matrix. Therefore

$$\begin{aligned}\det(AE) &= \det((AE)^T) \\ &= \det(E^T A^T) \\ &= \det(E^T)\det(A^T) \\ &= \det(E)\det(A)\end{aligned}$$

Example 16.8 (1). Given $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1$, find $\begin{vmatrix} 2a_{11} & 3a_{12} & 4a_{13} \\ 2a_{21} & 3a_{22} & 4a_{23} \\ 2a_{31} & 3a_{32} & 4a_{33} \end{vmatrix}$

$$\begin{aligned}
\begin{vmatrix} 2a_{11} & 3a_{12} & 4a_{13} \\ 2a_{21} & 3a_{22} & 4a_{23} \\ 2a_{31} & 3a_{32} & 4a_{33} \end{vmatrix} &= 2\begin{vmatrix} a_{11} & 3a_{12} & 4a_{13} \\ a_{21} & 3a_{22} & 4a_{23} \\ a_{31} & 3a_{32} & 4a_{33} \end{vmatrix} \\
&= 2 * 3 \begin{vmatrix} a_{11} & a_{12} & 4a_{13} \\ a_{21} & a_{22} & 4a_{23} \\ a_{31} & a_{32} & 4a_{33} \end{vmatrix} \\
&= 2 * 3 * 4 * \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 24
\end{aligned}$$

(2). Compute

$$
\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{vmatrix}
\begin{pmatrix} R_2 \to -R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \\ R_5 \to -R_1 + R_5 \end{pmatrix}
$$

$$
= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} (\textit{expand along 3rd column})
$$

$$
=(-1)^{1+3} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{vmatrix}
$$

$$
=(1)(1)(2)(3) = 6
$$

(3). Compute

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 3 & 2 \\ 1 & 1 & 1 & 1 & 4 \end{vmatrix} \begin{pmatrix} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \\ R_5 \to -R_1 + R_5 \end{pmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} (R_2 \leftrightarrow R_3)$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = -(1)(1)(-1)(2)(3) = 6$$

(4)

$$\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} (R_1 \leftrightarrow R_2) = -\begin{vmatrix} 1 & a & 1 & 1 \\ a & 1 & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix} \begin{pmatrix} R_2 \to -aR_1 + R_2 \\ R_3 \to -R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{pmatrix}$$

$$= -\begin{vmatrix} 1 & a & 1 & 1 \\ 0 & 1-a^2 & 1-a & 1-a \\ 0 & 1-a & a-1 & 0 \\ 0 & 1-a & 0 & a-1 \end{vmatrix} (\textit{Take out } (1-a) \textit{ from } R_2\ R_3\ R_4)$$

$$= -(1-a)^3 \begin{vmatrix} 1 & a & 1 & 1 \\ 0 & 1+a & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} (R_2 \leftrightarrow R_4)$$

$$=(1-a)^3\begin{vmatrix} 1 & a & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1+a & 1 & 1 \end{vmatrix}\begin{pmatrix} R_3{\to}-R_2{+}R_3 \\ R_4{\to}-(1{+}a)R_2{+}R_4 \end{pmatrix}$$

$$=(1-a)^3\begin{vmatrix} 1 & a & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & a+2 \end{vmatrix}\left(R_4{\to}R_3{+}R_4\right)$$

$$=(1-a)^3\begin{vmatrix} 1 & a & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & a+3 \end{vmatrix}$$

$$=(1-a)^3(1)(1)(-1)(a+3) = (a-1)^3(a+3)$$

Method to compute the determinant

Using row operations (column operations) to reduce the corresponding matrix into the upper triangular form or lower triangular form.

Property 16.9

$$(I)\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ b_1 & b_2 & \cdots & b_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ c_1 & c_2 & \cdots & c_n \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

(II)

$$\begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 + c_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 + c_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n + c_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & \cdots & a_{1,i-1} & c_1 & a_{1,i+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,i-1} & c_2 & a_{2,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & c_n & a_{n,i+1} & \cdots & a_{nn} \end{vmatrix}$$

Proof. (I) Expand along the ith row. (II) Expand along the ith column.

Example Given $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1$, then

$$\det(B) = \begin{vmatrix} a_{11} + 10a_{12} + 5a_{13} & a_{12} & a_{13} \\ a_{21} + 10a_{22} + 5a_{23} & a_{22} & a_{23} \\ a_{31} + 10a_{32} + 5a_{33} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} +$$

$$\begin{vmatrix} 10a_{12} & a_{12} & a_{13} \\ 10a_{22} & a_{22} & a_{23} \\ 10a_{32} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 5a_{13} & a_{12} & a_{13} \\ 5a_{23} & a_{22} & a_{23} \\ 5a_{33} & a_{32} & a_{33} \end{vmatrix} = 1 + 0 + 0 = 1.$$

Theorem 16.10 (**Equivalent Condition for Singular Matrices**) Let $A \in \mathbb{R}^{n\times n}$, A is singular if and only if

$$\det(A) = 0$$

Proof. The matrix A can be reduced into reduced row-echelon form B with a finite number of elementary row operations. Thus,

$$B = E_k E_{k-1} \cdots E_1 A,$$

where all $E_i, i = 1, \cdots, k$ are elementary matrices.
By property 16.7, one has

$$\det(B) = \det(E_k E_{k-1} \cdots E_1 A) = \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A)$$

Since the determinants of $E_s (s = 1, \cdots, k)$ are all nonzeros, it follows that $\det(A) = 0$ if and only if $\det(B) = 0$. Thus, we only need to prove that A is singular if and only if $\det(B) = 0$.

If A is singular, then B has a row consisting entirely zeros, and hence $\det(B) = 0$.

If $\det(B) = 0$, suppose that A is nonsingular, then each row of B has a leading 1, B must be the identity matrix, and hence $\det(B) = 1$, which is a contradiction.

Corollary (Equivalent condition for a nonsingular matrix) Let

$A \in \mathbb{R}^{n \times n}$, A is nonsingular $\Leftrightarrow \det(A) \neq 0$.

Theorem 16.11 (**Determinants of Matrices Product**) If A and B are square matrices, then

$$\det(AB) = \det(A)\det(B)$$

Proof. (1) If A is singular, one has $\det(A) = 0$ by theorem 16.10, and AB is also singular by using the Theorem 8.8. Thus, $\det(AB) = 0$ by Theorem 16.10.
(2) If A is nonsingular, then A can be written as a product of elementary matrices by using the Theorem 8.7, $A = E_1 E_2 \cdots E_k$, where $E_1, \cdots, E_k$ are elementary matrices.

$$\begin{aligned}
\det(AB) &= \det(E_1 E_2 \cdots E_k B) \\
&= \det(E_1)\det(E_2 \cdots E_k B) \\
&= \det(E_1)\det(E_2)\cdots\det(E_k)\det(B) \\
&= \det(E_1 E_2 \cdots E_k)\det(B) \\
&= \det(A)\det(B)
\end{aligned}$$

where Theorem 16.7 is used.

Example 16.12 Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 & 1 \\ 5 & -5 & 4 \\ 6 & -5 & 0 \end{bmatrix}$$

$$\det(AB) = 45, \quad \det(A) = 5, \quad \det(B) = 9$$

$$\det(AB) = \det(A)\det(B)$$

Question:

Is it true that $\det(A + B) = \det(A) + \det(B)$? (No.)

Definition 16.13 (**Adjoint Matrix**) Let $A = (a_{ij})_{n\times n}$, then the adjoint matrix of A is defined as

$$\mathrm{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where $A_{ij} = (-1)^{i+j}\det(M_{ij})$ is the cofactor of a_{ij}.

Theorem 16.14 (**Adjoint Matrix**) Let $A = (a_{ij})_{n\times n}$, then $A\,\mathrm{adj}(A) = \det(A) I_n$. If $\det(A) \neq 0$, $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

Proof. By Lemma 16.4, one has

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

And (i,j)-entry of $A\,\mathrm{adj}(A)$ is $a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn}$. Thus

$$A\,\mathrm{adj}(A) = \det(A) I_n$$

In addition, if $\det(A) \neq 0$, $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$.

Methods to find the inverse of A

(1) $[A|I] \xrightarrow{\text{Row operations}} [I|A^{-1}]$

(2) $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$ (Straightforward but massive calculations)

Example 16.15 Let

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\mathrm{adj}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -7 \\ 0 & -3 & -4 \end{vmatrix}$$

$$= -\begin{vmatrix} -4 & -7 \\ -3 & -4 \end{vmatrix} = -((-4)(-4) - (-3)(-7)) = 5$$

Thus

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A) = \frac{1}{5}\begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Theorem 16.16 **(Cramer's Rule)** Let $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ ($\mathbf{a}_1, \cdots, \mathbf{a}_n$ are column vectors of A) be a nonsingular $n \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^n$ and $A_i = [\mathbf{a}_1, \cdots, \mathbf{a}_{i-1}, \mathbf{b}, \mathbf{a}_{i+1}, \cdots, \mathbf{a}_n]$, then the unique solution of $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad \text{for } i = 1, 2, \cdots, n$$

Proof.

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\mathrm{adj}(A)\mathbf{b}$$

It follows that

$$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{\det(A)} = \frac{\det(A_i)}{\det(A)}$$

where $\det(A_i) = b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}$ (expand along ith column).

Example 16.17 Solve the following linear system by using the Cramer's rule:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 5, \\ 2x_1 + 2x_2 + x_3 &= 6, \\ x_1 + 2x_2 + 3x_3 &= 9. \end{aligned}$$

We calculate:

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -4, \quad \det(A_1) = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$\det(A_2) = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4 \quad \det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

Thus

$$x_1 = \frac{\det(A_1)}{\det(A)} = 1, \quad x_2 = \frac{\det(A_2)}{\det(A)} = 1, \quad x_3 = \frac{\det(A_3)}{\det(A)} = 2$$

Appendix: proof for Theorem 16.7

Theorem 16.7 (**Determinants of Matrices Product**) If A and B are square matrices, then

$$\det(AB) = \det(A)\det(B)$$

Proof. (1) If A is singular, one has $\det(A) = 0$ by theorem 16.6, and AB is also singular by using the Theorem 8.11. Thus, $\det(AB) = 0$ by Theorem 16.6.

(2) If A is nonsingular, then A can be written as a product of elementary matrices by using the Theorem 8.7, $A = E_1 E_2 \cdots E_k$, where $E_1, \cdots, E_k$ are elementary matrices. By property 16.15, one has

$$\begin{aligned}
\det(AB) &= \det(E_1 E_2 \cdots E_k B) \\
&= \det(E_1)\det(E_2 \cdots E_k B) \\
&= \det(E_1)\det(E_2)\cdots\det(E_k)\det(B) \\
&= \det(E_1 E_2 \cdots E_k)\det(B) \\
&= \det(A)\det(B)
\end{aligned}$$

Appendix: Alternative Definition for Determinants

Definition 16.18 (Regard Permutation Matrix as reordering columns of identity matrix) A permutation matrix is a matrix formed from the identity matrix by reordering its columns or rows. Suppose that P is the permutation matrix formed by reordering the columns of I in the order $(k_1, k_2, \cdots, k_n)$, then $P = [\mathbf{e}_{k_1}, \mathbf{e}_{k_2}, \cdots, \mathbf{e}_{k_n}]$. Obviously, $\det(P) = 1$ or $\det(P) = -1$. Denote:

$$\tau_C(k_1, \cdots, k_n) = \det([\mathbf{e}_{k_1}, \mathbf{e}_{k_2}, \cdots, \mathbf{e}_{k_n}]) = \det(P)$$

Not the permutation $(k_1, k_2, \cdots, k_n)$ is odd if $\tau_C(k_1, \cdots, k_n) = -1$, and the permutation $(k_1, k_2, \cdots, k_n)$ is even if $\tau_C(k_1, \cdots, k_n) = 1$.

Appendix: Alternative Definition for Determinants

Definition 16.19 (**Regard Permutation Matrix as reordering rows of identity matrix**) Suppose that Q is the permutation matrix formed by reordering the rows of I in the order $(k_1, k_2, \cdots, k_n)^T$, then $Q = \begin{bmatrix} \vec{\mathbf{e}}_{k_1} \\ \vec{\mathbf{e}}_{k_2} \\ \vdots \\ \vec{\mathbf{e}}_{k_n} \end{bmatrix}$.

Obviously, $\det(Q) = 1$ or $\det(Q) = -1$. Denote:

$$\tau_R(k_1, \cdots, k_n) = \det\left(\begin{bmatrix} \vec{\mathbf{e}}_{k_1} \\ \vec{\mathbf{e}}_{k_2} \\ \vdots \\ \vec{\mathbf{e}}_{k_n} \end{bmatrix}\right) = \det(Q)$$

Appendix: a closed formula for Determinants

Definition 16.20 (**Alternative Definition for Determinants**) Let $A = (a_{ij})_{n\times n}$ be an $n \times n$ matrix, then the determinant is defined as

$$\det(A) = \sum_{permutations\ (k_1,k_2,\cdots,k_n)\ of\ (1,2,\cdots,n)} \tau_C(k_1,\cdots,k_n)a_{1k_1}a_{2k_2}\cdots a_{nk_n}$$

or

$$\det(A) = \sum_{permutations\ (k_1,k_2,\cdots,k_n)\ of\ (1,2,\cdots,n)} \tau_R(k_1,\cdots,k_n)a_{k_1 1}a_{k_2 2}\cdots a_{k_n n}$$

which are called the **Leibnitz formulas**. The above two equalities are equivalent. These two formulas are big formulas since the permutation of $(1, 2, \cdots, n)$ has $n!$ terms, which will result a lot of calculations. But in the case when the matrix has lots of zeros, the calculation may not that tedious.

Remark. Every term in the summation of the Leibniz formula is a multiplication of n entries of A (with size $n \times n$), where all the entries in the same term must be taken from different rows and different columns.

Example 16.21 Calculate

$$\det(A) = \begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Observation: only one nonzero term in first row, which is a_{14} in the fourth column; besides the four column, only one nonzero term a_{23} in second row (in the third column); besides the third and four columns, only one nonzero term a_{32} in third row (in the second column); besides the second, third and four columns, only one nonzero term a_{41} in fourth row (in the first column). Thus, only one nonzero term in the summation when using the Leibnitz formula for this example.

$$\det(A) = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} a_{14}a_{23}a_{32}a_{41} = a_{14}a_{23}a_{32}a_{41}$$