1. TFFTF

2. (i) B
   (ii) A
   (iii) C

3. (i) -2

   (ii) 3

   (iii) $f(x) = x+1$

   (iv) -5

   (v) AC

4. (i) $$\lim_{x\to 0}\frac{x\cot 5x}{\sin^2 x\,\cot^2 3x} = \lim_{x\to 0}\frac{x\,\cos(5x)\,\sin^2(3x)}{\sin^2 x\,\cos^2(3x)\,\sin(5x)}$$

$$= \underbrace{\lim_{x\to 0}\frac{\cos(5x)}{\cos^2(3x)}}_{1}\;\underbrace{\lim_{x\to 0}\frac{x}{\sin x}}_{1}\;\underbrace{\lim_{x\to 0}\frac{x}{\sin x}}_{1}\;\lim_{x\to 0}\frac{\sin 3x}{x}\;\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$$

$$= 3\lim_{x\to 0}\frac{\sin 3x}{3x}\;\lim_{x\to 0}\frac{\sin 3x}{3x}\;\lim_{x\to 0}\frac{5x}{\sin 5x}\left(\frac{3}{5}\right) = \frac{9}{5}$$

(ii) $$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1+x^2}}{\sqrt{1+x}-1} = \lim_{x\to 0}\frac{\left(\sqrt{1+x}-\sqrt{1+x^2}\right)\left(\sqrt{1+x}+1\right)}{x}$$

$$= \lim_{x\to 0}\frac{(1+x-1-x^2)\left(\sqrt{1+x}+1\right)}{x\left(\sqrt{1+x}+\sqrt{1+x^2}\right)} = \lim_{x\to 0}\frac{(1-x)\left(\sqrt{1+x}+1\right)}{\sqrt{1+x}+\sqrt{1+x^2}}$$

$$= \frac{1\cdot 2}{1+1} = 1.$$

(iii) Let $f(t) := \sin\sqrt{t}$. By MVT, $\exists\, c\in(x-1,x+1)$ such that

$$f(x+1)-f(x-1) = f'(c)\,(x+1-(x-1)) = 2f'(c) = \frac{\cos\sqrt{c}}{\sqrt{c}}.$$

As $x\to\infty$, $c\to\infty$, so

$$L = \lim_{x\to\infty}\left[f(x+1)-f(x-1)\right] = \lim_{x\to\infty}\frac{1}{\sqrt{c}}\cos\sqrt{c} = \lim_{c\to\infty}\underbrace{\frac{1}{\sqrt{c}}}_{\to 0}\;\underbrace{\cos\sqrt{c}}_{\text{bounded}}$$

$$= 0$$

Alternatively, can use a trigonometric identity.

Use the trigonometric identity, we have

$$\sin\sqrt{x+1}-\sin\sqrt{x-1} = 2\sin\frac{\sqrt{x+1}-\sqrt{x-1}}{2}\cos\frac{\sqrt{x+1}+\sqrt{x-1}}{2}$$

Note that by continuity of sin,

$$\begin{aligned}\lim_{x\to+\infty}\sin\frac{\sqrt{x+1}-\sqrt{x-1}}{2} &= \lim_{x\to+\infty}\sin\frac{1}{\sqrt{x+1}+\sqrt{x-1}}\\ &= 0.\end{aligned}$$

And the cosine function is bounded, therefore

$$\lim_{x\to+\infty}(\sin\sqrt{x+1}-\sin\sqrt{x-1}) = 0.$$

5. Since $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3+5x^2-7}{(x+1)(x-1)} = -\infty$

and $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^3+5x^2-7}{(x+1)(x-1)} = -\infty$

the vertical asymptotes are $x=-1$ and $x=1$.

As $x\to\infty$, $A = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1+5/x-7/x^3}{1-1/x^2} = 1$

$B = \lim_{x\to\infty}(f(x)-Ax) = \lim_{x\to\infty} \frac{x^3+5x^2-7-x^3+x}{x^2-1} = 5$

$\therefore$ $y=x+5$ is the oblique asymptote as $x\to\infty$.

As $x\to-\infty$, we get the same limits

$\therefore$ $y=x+5$ is the oblique asymptote as $x\to-\infty$.

Alternatively, since $f(x) = x+5+\frac{x-2}{x^2-1}$

$$\begin{array}{r|l} & x+5 \\ \hline x^2-1 & x^3+5x^2-7 \\ & x^3-x \\ \hline & 5x^2+x-7 \\ & 5x^2-5 \\ \hline & x-2 \end{array}$$

and $\lim_{x\to\pm\infty} \frac{x-2}{x^2-1} = \lim_{x\to\pm\infty} \frac{1-2/x}{x-1/x} = 0$,

$y=x+5$ is the oblique asymptote as $x\to\infty$

and " " " " $x\to-\infty$.

6. First,

$$f'(t) = \begin{cases} t-2, & t \in [0,2) \\ -t+2, & t \in (2,\infty) \end{cases}.$$

Since $f'_-(2) = \left(\frac{1}{2}(t-2)^2+4\right)'\Big|_{t=2} = 2-2=0$

& $f'_+(2) = \left(-\frac{1}{2}(t-2)^2+4\right)'\Big|_{t=2} = -2+2=0$,

we have $f'(2)=0$. Hence

$$f'(t) = \begin{cases} t-2, & t \in [0,2] \\ -t+2, & t \in [2,\infty) \end{cases}.$$

*: In the notation here, f' means one-sided derivative at $t=0$.

Now $f''(t) = \begin{cases} 1, & t \in [0,2) \\ -1, & t \in (2,\infty) \end{cases}$.

Since $f''_-(2) = (t-2)'\Big|_{t=2} = 1$ and

$f''_+(2) = (-t+2)'\Big|_{t=2} = -1$,

$f''_-(2) \neq f''_+(2)$, so f'' is not defined (or does not exist) at $t=2$.

7. • For $x \neq 0$, $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$.

• Since $\lim_{x\to 0} \underbrace{2x}_{\to 0}\underbrace{\sin\frac{1}{x}}_{\text{bounded}} = 0$ but $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist,

$\lim_{x\to 0} f'(x)$ does not exists.

• Since $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0$,

$f'(0)$ exists.

• Since $\lim_{x\to 0} f'(x) \neq f'(0)$, f' is not continuous at $x=0$.

• For g, if $x \neq 0$, then $g'(x) = 3x^2\sin\frac{1}{x} - x\cos\frac{1}{x}$.

• Since $\lim_{x\to 0} 3x^2\sin\frac{1}{x} = 0 = \lim_{x\to 0} x\cos\frac{1}{x}$ by

"$0\cdot$ bounded $\to 0$", $\lim_{x\to 0} g'(x) = 0$.

• Since $g'(0) = \lim_{h\to 0}\frac{g(h)-g(0)}{h} = \lim_{h\to 0} h^2\sin\frac{1}{h} = 0$,

$\lim_{x\to 0} g'(x) = g'(0)$, so g' is continuous at $x=0$.

8. (i) Apply $\frac{d}{dx}$ to both sides :

$$x^2(-1)y' + 2x(2-y) = 3y^2 \cdot y'. \qquad (*)$$

For $(x,y) = (1,1)$, we have $-y' + 2 = 3y' \Rightarrow y' = \frac{1}{2}$.

Tangent line is $\quad y = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$.

(ii) Apply $\frac{d}{dx}$ to both sides of $(*)$:

$$-(x^2 \cdot y'' + 2xy') + 2x(-1)y' + 2(2-y) = 3y^2y'' + 6y \cdot y' \cdot y'. \quad (**)$$

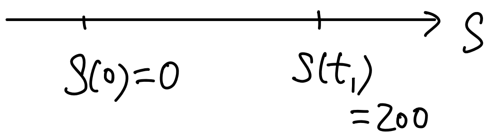
At $(x,y) = (1,1)$, $\quad y' = \frac{1}{2}$, so

$$(**) \Rightarrow -(y''+1) - 1 + 2 = 3y'' + \frac{3}{2} \cdot 1$$

$$\Rightarrow \qquad -\frac{3}{2} = 4y''$$

$$\Rightarrow \qquad y'' = -\frac{3}{8}.$$

9.



$S'(0) = v_0 = ?$

initial velocity

$S''(t) = -16$

$\Rightarrow S'(t) = -16t + v_0$, ①

$\Rightarrow S(t) = -8t^2 + v_0 t + S(0) = -8t^2 + v_0 t$. ②

Suppose car stops at t_1. Then

① $\Rightarrow$ $0 = S'(t_1) = -16t_1 + v_0$

$\Rightarrow$ $t_1 = v_0/16$

② $\Rightarrow$ $200 = S(t_1) = S\left(\frac{v_0}{16}\right) = -8 \cdot \frac{v_0^2}{256} + \frac{v_0^2}{16}$

$= \frac{v_0^2}{32}$

$\Rightarrow v_0 = 80$ (ft/s).

10. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

		0		2	
f''	$+$		$-$		$+$
Concavity of f	up		down		up

(a) f is concave up on $(-\infty, 0)$ and $(2, \infty)$

" down " $(0, 2)$

(b) By (a), points of inflection are $(0,0)$ and $(2,-16)$

(or $x=0$ and $x=2$).

(c) $f'(x)=0 \iff \underbrace{x=0 \text{ or } x=3}_{\text{all critical points}}$.

$f''(3) > 0 \Rightarrow x=3$ gives a local minimum

($(3,-27)$ is OK.)

Since

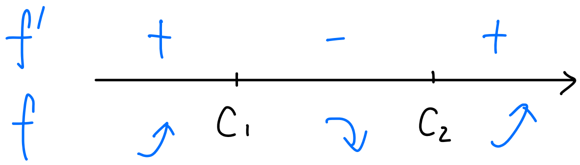
		0		3
f'	$-$		$-$	
f	$\searrow$		$\searrow$	

, $x=0$ gives no local extremum:

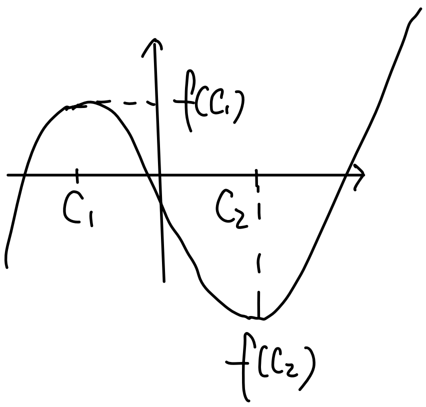
No local max.

11. Let $f(x) := x^3 - 2x + C$. Then $f'(x) = 3x^2 - 2$

$$\therefore \; f'(x) = 0 \iff x = C_2 := \sqrt{\tfrac{2}{3}} \;\text{ or }\; x = C_1 := -\sqrt{\tfrac{2}{3}}$$



$$\Rightarrow \begin{cases} f(C_1) \text{ is a local max} \\ f(C_2) \;\;''\;\;\;''\;\;\;''\;\; \text{min} \end{cases}$$



(i) $\iff f(C_1) < 0 \;\text{ or }\; f(C_2) > 0$

$\iff -\left(\tfrac{2}{3}\right)^{3/2} + 2\left(\tfrac{2}{3}\right)^{1/2} + C < 0 \;\text{ or }\; \left(\tfrac{2}{3}\right)^{3/2} - 2\left(\tfrac{2}{3}\right)^{1/2} + C > 0.$

$\iff C < \underbrace{\left(\tfrac{2}{3}\right)^{3/2} - 2\left(\tfrac{2}{3}\right)^{1/2}}_{\left(= -\frac{4\sqrt{2}}{3\sqrt{3}} \approx -1.089\right)} \;\text{ or }\; C > \underbrace{-\left(\tfrac{2}{3}\right)^{3/2} + 2\left(\tfrac{2}{3}\right)^{1/2}}_{\left(\approx 1.089\right)}$

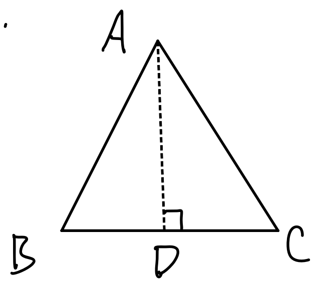
(ii) $\iff f(C_1) = 0 \;\text{ or }\; f(C_2) = 0$

$\iff C = \pm \frac{4\sqrt{2}}{3\sqrt{3}} \quad (\approx \pm 1.089)$

(iii) $\iff$ not (i) & not (ii)

$\iff C \in \left(-\frac{4\sqrt{2}}{3\sqrt{3}}, \frac{4\sqrt{2}}{3\sqrt{3}}\right)$

12.



$$|AD|^2 = L_1^2 - \left[\frac{1}{2}(L-2L_1)\right]^2$$
$$= L_1^2 - \frac{1}{4}(L^2 - 4LL_1 + 4L_1^2)$$
$$= L_1^2 - \frac{1}{4}L^2 + LL_1 - L_1^2$$
$$= L_1L - \frac{1}{4}L^2 \qquad \frac{1}{2}(L)$$

$$\therefore \text{Area} = A(L_1) = \frac{1}{2}|BC||AD| = \frac{1}{2}(L-2L_1)\sqrt{\left(L_1L - \frac{1}{4}L^2\right)}$$

$$\Rightarrow A'(L_1) = \frac{1}{2}\left[(-2)\sqrt{\left(L_1L - \frac{1}{4}L^2\right)} + (L-2L_1)\frac{L}{\sqrt{\left(L_1L - \frac{1}{4}L^2\right)}}\right]$$

$$= \frac{-2\left(L_1L - \frac{1}{4}L^2\right) + L^2 - 2LL_1}{2\sqrt{\left(L_1L - \frac{1}{4}L^2\right)}}$$

So $A'(L_1) = 0 \iff L_1 = L/3$

Since A is continuous on $\left[\frac{L}{4}, \frac{L}{2}\right]$ and $A\left(\frac{L}{4}\right) = A\left(\frac{L}{2}\right) = 0$,

$|AD| = 0$

$L_1 = L/3$ indeed gives a global maximum of A.

13. (i) $f(1) \approx f(0) + f'(0)$

(ii) By the MVT, for some $b \in (0,1)$,

(*) $f(1) = f(0) + f'(b)(1-0) = f(0) + f'(b)$.

① If $b = \frac{1}{2}$, done: take $A = 0$ and c to be any number in $(0,1)$.

② If $b > \frac{1}{2}$, then by MVT (applied on f'),

(**) $f'(b) = f'(\frac{1}{2}) + f''(c)(b - \frac{1}{2})$ for some $c \in (\frac{1}{2}, b)$

Taking $A := b - \frac{1}{2}$, (*) and (**) imply the desired equation.

③ Similar to ②.

$\square$