STA2001 Probability and Statistics (I)

Lecture 10

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Chapter 4. Bivariate Distribution

Section 4.1 Bivariate Distribution of Discrete Type

Motivation

Very often, we are interested to study two random experiments jointly, each of whose outcome is a scalar, or a random experiment whose outcome is a pair of two scalars.

1. observe college students to obtain information such as height x and weight y.

2. observe high school students to obtain information such as rank x and score of college entrance examination y.

Motivation

Very often, we are interested to study two random experiments jointly, each of whose outcome is a scalar, or a random experiment whose outcome is a pair of two scalars.

1. observe college students to obtain information such as height x and weight y.

2. observe high school students to obtain information such as rank x and score of college entrance examination y.

- a random experiment whose outcome is a scalar,
  $\rightarrow$ univariate RV
- two random experiments jointly each of whose outcome is a scalar, or a random experiment whose outcome is a pair of two scalars, $\rightarrow$ bivariate RV

Bivariate RV

Definition

Let (X, Y) be a pair of RVs with their range denoted by $\overline{S} \subseteq R^2$. Then (X, Y) or X and Y is said to be a bivariate RV. If $\overline{S}$ is finite or countably infinite, then (X, Y) is said to be a discrete bivariate RV.

Moreover, let $\overline{S_X} \subseteq R$ and $\overline{S_Y} \subseteq R$ denote the range of X and Y, respectively.

$$\begin{aligned} \overline{S} &= \{\text{all possible values of } (X, Y)\} \\ \overline{S_X} &= \{\text{all possible values of } X\} = \left\{x | (x, y) \in \overline{S}\right\} \\ \overline{S_Y} &= \{\text{all possible values of } Y\} = \left\{y | (x, y) \in \overline{S}\right\} \end{aligned}$$

Then, it holds that

$$\overline{S} \subseteq \overline{S_X} \times \overline{S_Y} = \left\{(x, y) | x \in \overline{S_X}, y \in \overline{S_Y}\right\}$$

Example 1, Page 134

Roll a pair of 4-sided fair dice. Then the original sample space

$$S = \left\{ \begin{array}{llll} (1,1), & (1,2), & (1,3), & (1,4), \\ (2,1), & (2,2), & (2,3), & (2,4), \\ (3,1), & (3,2), & (3,3), & (3,4), \\ (4,1), & (4,2), & (4,3), & (4,4) \end{array} \right\},$$

where the two numbers in each pair represent the outcome of the first die and the second die, respectively.

Example 1, Page 134

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Now let X denote the smaller and Y the larger outcome of the pair of dice, e.g., if the outcome is (3,2) or (2,3), then $X = 2$, $Y = 3$.

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Now let X denote the smaller and Y the larger outcome of the pair of dice, e.g., if the outcome is (3,2) or (2,3), then $X = 2$, $Y = 3$. Sample space $\overline{S} \subseteq \overline{S_X} \times \overline{S_Y}$:

$$\overline{S}_X = \overline{S}_Y = \{1,2,3,4\}, \overline{S} = \left\{ \begin{array}{llll} (1,1), & (1,2), & (1,3), & (1,4), \\ & (2,2), & (2,3), & (2,4), \\ & & (3,3), & (3,4), \\ & & & (4,4) \end{array} \right\},$$

where the two numbers in each pair represent the possible values of X and Y, respectively.

Joint pmf

Definition

The function $f(x, y) : \overline{S} \to (0, 1]$ is called the joint probability mass function (joint pmf) of X and Y or (X, Y), if

1. $f(x, y) > 0$ for $(x, y) \in \overline{S}$,
2. $\sum_{(x,y)\in\overline{S}} f(x, y) = 1$,
3. For $A \subseteq \overline{S}$,

$$P[(X, Y) \in A] \triangleq P(\{(X, Y) \in A\}) = \sum_{(x,y)\in A} f(x, y)$$

which defines the probability function for a set A. In particular, taking $A = \{(x, y)\}$ yields the probability of $X = x$ and $Y = y$, i.e.,

$$P(X = x, Y = y) = f(x, y)$$

Example 1 [Continued]

Question:

$$P(X = 2, Y = 3) = ?,\ \ P(X = 2, Y = 2) = ?$$

Example 1 [Continued]

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$$P(X = 2, Y = 3) = ?,\ \ P(X = 2, Y = 2) = ?$$

$$P(X = 2, Y = 3) = \frac{1}{16} + \frac{1}{16} = \frac{2}{16}$$

$$P(X = 2, Y = 2) = \frac{1}{16}$$

Question: What is the joint pmf $f(x, y)$?

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Question: What is the joint pmf $f(x,y)$?

$$\overline{S} = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ & (2,2) & (2,3) & (2,4) \\ & & (3,3) & (3,4) \\ & & & (4,4) \end{matrix} \right\}$$

$$f(x,y) = \begin{cases} \frac{2}{16}, & 1 \le x < y \le 4 \\ \frac{1}{16}, & 1 \le x = y \le 4. \end{cases}$$

A Remark on Computation of The Probability

$$\text{For } A \subseteq \overline{S},\ P[(X,Y) \in A] \triangleq P(\{(X,Y) \in A\}) = \sum_{(x,y)\in A} f(x,y),$$

where the double summation can be split into 2 single summation.
Let

$$A_X = \{x|(x,y) \in A\}, A_Y(x) = \{y|(x,y) \in A\}, \text{for } x \in A_X$$

Then

$$P((X,Y) \in A) = \sum_{x\in A_X} \sum_{y\in A_Y(x)} f(x,y)$$

Let

$$A_Y = \{y|(x,y) \in A\}, A_X(y) = \{x|(x,y) \in A\}, \text{for } y \in A_Y$$

Then

$$P((X,Y) \in A) = \sum_{y\in A_Y} \sum_{x\in A_X(y)} f(x,y)$$

Marginal pmf

Definition

Let (X, Y) be a bivariate RV or X and Y be two RVs and have the joint pmf $f(x, y) : \overline{S} \to (0, 1]$. Sometimes, we are interested in the pmf of X or Y alone, which is called the marginal pmf of X or Y and described by

For $x \in \overline{S_X}$,

$$
\begin{aligned}
f_X(x) = P_X(X = x) &\stackrel{\Delta}{=} P\left(\{X = x, Y \in \overline{S_Y}(x)\}\right) \\
&= \sum_{y \in \overline{S_Y}(x)} f(x, y)
\end{aligned}
$$

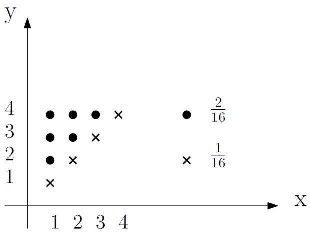
where

$$\overline{S_Y}(x) = \left\{y | (x, y) \in \overline{S}\right\} \text{ for the given } x \in \overline{S_X}.$$

Example 1 [Continued]

$$f(x,y) = \begin{cases} \frac{2}{16}, & 1 \le x < y \le 4 \\ \frac{1}{16}, & 1 \le x = y \le 4. \end{cases}$$

What is $f_X(x), f_Y(y)$?



Example 1 [Continued]

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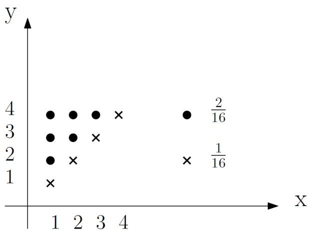

What is $f_X(x), f_Y(y)$?

First, $\overline{S_X} = \overline{S_Y} = \{1,2,3,4\}$.

$$f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x,y), x \in \overline{S_X} = \{1,2,3,4\}$$

$$\Longrightarrow f_X(1) = \frac{7}{16}, \quad f_X(2) = \frac{5}{16}, \quad f_X(3) = \frac{3}{16}, \quad f_X(4) = \frac{1}{16}$$

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For $y \in \overline{S_Y}$,

$$\begin{aligned} f_Y(y) = P_Y(Y = y) &\stackrel{\Delta}{=} P\left(\{X \in \overline{S_X}(y), Y = y\}\right) \\ &= \sum_{x \in \overline{S_X}(y)} f(x, y) \end{aligned}$$

where

$$\overline{S_X}(y) = \{x | (x, y) \in \overline{S}\} \text{ for the given } y \in \overline{S_Y}.$$

Example 1 [Continued]

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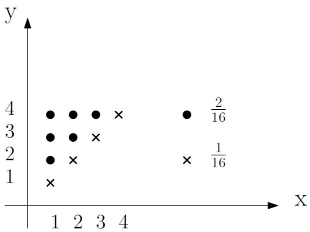

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$$f_Y(y) = \sum_{x \in \overline{S_X}(y)} f(x,y), y \in \overline{S_Y} = \{1,2,3,4\}$$

$$\Longrightarrow f_Y(1) = \frac{1}{16}, \quad f_Y(2) = \frac{3}{16}, \quad f_Y(3) = \frac{5}{16}, \quad f_Y(4) = \frac{7}{16}$$

Remarks on Marginal pmf

It is crucial to understand the following definitions

$\overline{S}, \overline{S_X}, \overline{S_Y}, \overline{S_X}(y), \overline{S_Y}(x)$

$$
\begin{aligned}
\overline{S} &= \{\text{all possible values of } (X, Y)\} \\
\overline{S_X} &= \{\text{all possible values of } X\} = \left\{x | (x, y) \in \overline{S}\right\} \\
\overline{S_Y} &= \{\text{all possible values of } Y\} = \left\{y | (x, y) \in \overline{S}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{S_X}(y) &= \left\{x | (x, y) \in \overline{S}\right\} \text{ for a given y} \in \overline{S_Y} \\
\overline{S_Y}(x) &= \left\{y | (x, y) \in \overline{S}\right\} \text{ for a given x} \in \overline{S_X}
\end{aligned}
$$

Trinomial Distribution

Description: The random experiment has three mutually exclusive and exhaustive outcomes:

- "perfect",
- "second"
- "defective"

We repeat the experiment n independent times, and moreover, the probabilities

- p_X: the probability of "perfect",
- p_Y: the probability of "second"
- p_Z: the probability of "defective"

remain the same for each repetition. Such n repetitions can be called a trinomial experiment.

For the trinomial experiment, we are interested in the number of perfects, the number of seconds and the number of defectives.

Trinomial Distribution

For the n trinomial trials, we let

- X be number of perfects,
- Y be number of seconds,
- $Z = n - X - Y$ be the number of defectives

We are interested in the joint pmf of (X, Y), $f(x, y) : \overline{S} \rightarrow \mathbb{R}^2$

- $\overline{S} = \{(x, y) | x + y \leq n, x = 0, 1, \cdots, n, y = 0, 1, \cdots, n\}$
- $f(x, y) = P(X = x, Y = y)$ which is the probability of having x perfects, y seconds, and $n - x - y$ defectives

Trinomial Distribution

Joint pmf: to calculate $f(x, y) = P(X = x, Y = y)$,

- the probability for each way of having x perfects, y seconds, and $n - x - y$ defectives is

$$p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

- the total number of ways of having x perfects, y seconds, and $n - x - y$ defectives is

$$\binom{n}{x, y, n-x-y} = \frac{n!}{x!y!(n-x-y)!}$$

Therefore, the joint pmf for trinomial distribution is

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}, (x, y) \in \overline{S}$$

It's called trinomial distribution because of the trinomial expansion.

Trinomial Distribution

$$
\begin{aligned}
(a+b+c)^n &= \sum_{x=0}^{n} \binom{n}{x} a^x (b+c)^{n-x} \\
&= \sum_{x=0}^{n} \binom{n}{x} a^x \sum_{y=0}^{n-x} \binom{n-x}{y} b^y c^{n-x-y} \\
&= \sum_{x=0}^{n} \sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} a^x b^y c^{n-x-y}
\end{aligned}
$$

Marginal pmf: to calculate $f_X(x)$ or $f_Y(y)$

Trinomial Distribution

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\end{aligned}
$$

Marginal pmf: to calculate $f_X(x)$ or $f_Y(y)$

$$
\begin{aligned}
f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x,y) &= \sum_{y=0}^{n-x} \binom{n}{x} \binom{n-x}{y} p_X^x p_Y^y (1-p_X-p_Y)^{n-x-y} \\
&= \binom{n}{x} p_X^x (1-p_X)^{n-x}
\end{aligned}
$$

Without summing, we know $X \sim b(n, p_X)$ and $Y \sim b(n, p_Y)$

Independent Random Variables

Definition

The random variables X and Y are said to be independent if for every $x \in \overline{S_X}$ and $y \in \overline{S_Y}$

$$f(x, y) = f_X(x) f_Y(y)$$

or equivalently,

$$P(X = x, Y = y) = P_X(X = x) P_Y(Y = y).$$

X and Y are said to be dependent if otherwise.

When X and Y are independent,

$$\overline{S} = \overline{S_X} \times \overline{S_Y}, \quad \overline{S} \text{ is said to be rectangular}$$

which is a necessary condition for independence of X and Y.

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or equivalently,

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The definition of independent RVs has root in the definition of independent events.

$$A = \{X = x, Y \in \overline{S_Y}(x)\}, B = \{X \in \overline{S_X}(y), Y = y\}$$

X and Y are independent if and only if A and B are independent.

Example 2, Page 135

Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{21}, \quad x = 1, 2, 3, \quad y = 1, 2.$$

$$\overline{S} = \{(x, y) | x = 1, 2, 3, \quad y = 1, 2.\}$$

$$f : \overline{S} \longrightarrow (0, 1] \text{ with } \overline{S_X} = \{1, 2, 3\}, \quad \overline{S_Y} = \{1, 2\}.$$

Question

Are X and Y independent or dependent?

Example 2, Page 135

$$f_X(x) = \sum_{y \in \overline{S_Y}(x)} f(x,y) = \sum_{y=1}^{2} \frac{x+y}{21} = \frac{2x+3}{21}, \quad x = 1,2,3.$$

$$f_Y(y) = \sum_{x \in \overline{S_X}(y)} f(x,y) = \sum_{x=1}^{3} \frac{x+y}{21} = \frac{3y+6}{21}, \quad y = 1,2$$

$$f(x,y) = \frac{x+y}{21} \neq \frac{2x+3}{21} \cdot \frac{3y+6}{21} = f_X(x) f_Y(y)$$

$$\Rightarrow X \text{ and } Y \text{ are dependent}$$

What is the implication of independent RVs?

Implication of Independent RVs

Implication of independent RVs

For any $A \subset \overline{S_X}$ and $B \subset \overline{S_Y}$, the two events $X \in A$ and $Y \in B$ are independent.

We only need to show $P(A \cap B) = P(A)P(B)$:

$$
\begin{aligned}
P(A \cap B) = P(X \in A, Y \in B) &= \sum_{x \in A, y \in B} f(x, y) \\
&= \sum_{x \in A} \sum_{y \in B} f_X(x) f_Y(y) \\
&= \sum_{x \in A} f_X(x) \sum_{y \in B} f_Y(y) \\
&= P(X \in A) P(Y \in B) = P(A)P(B)
\end{aligned}
$$

Mathematical Expectation

Let X and Y be discrete RVs with their joint pmf

$$f(x, y) : \overline{S} \to (0, 1]$$

Consider a function $g(X, Y)$ of X and Y.

Then the expectation of $g(X, Y)$ is

$$E[g(X, Y)] = \sum_{(x,y)\in\overline{S}} g(x, y)f(x, y)$$

Mathematical Expectation

When $g(X, Y) = X, \quad E[X]$ is the mean of X

When $g(X, Y) = (X - E[X])^2$,

$E[(X - E[X])^2]$ is the variance of X

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When $g(X, Y) = (X - E[X])^2$,

$E[(X - E[X])^2]$ is the variance of X

There are seemingly two ways to calculate $E[X]$:

$$\begin{cases} E[X] = \overbrace{\sum_{x \in \overline{S_X}} x f_x(x)}^{\text{Marginal pmf}} \\ E[X] = \underbrace{\sum_{(x,y) \in \overline{S}} x f(x, y)}_{\text{Joint pmf}} \end{cases}$$

Mathematical Expectation

$$\text{When } g(X,Y) = X, \quad E[X] \text{ is the mean of } X$$

$$\text{When } g(X,Y) = (X - E[X])^2,$$

$$E[(X - E[X])^2] \text{ is the variance of } X$$

There seems two ways to calculate $E[X]$:

$$\text{Equivalent} \begin{cases} E(X) = \overbrace{\sum_{x \in \overline{S_X}} x f_x(x)}^{\text{Marginal pmf}} \\ E(X) = \underbrace{\sum_{(x,y) \in \overline{S}} x f(x,y)}_{\text{Joint pmf}} = \sum_{x \in \overline{S_X}} x \underbrace{\sum_{y \in \overline{S_Y}(x)} f(x,y)}_{= f_x(x)} \end{cases}$$

Example 1, [Page 134] —– Revisited

Question

Recall that X and Y are discrete RVs with joint pmf

$f(x, y) : \overline{S} \to (0, 1]$ with $\overline{S_X} = \overline{S_Y} = \{1, 2, 3, 4\}$

$$f(x, y) = \begin{cases} \frac{2}{16} & 1 \leq x < y \leq 4 \\ \frac{1}{16} & 1 \leq x = y \leq 4 \end{cases}$$

What is $E[X + Y]$?

Example 1, [Page 134] —– Revisited

$$
\begin{aligned}
E(X+Y) &= \sum_{(x,y)\in\overline{S}} (x+y) f(x,y) \\
&= \sum_{1\le x=y\le 4} (x+y)\frac{1}{16} + \sum_{1\le x<y\le 4} (x+y)\frac{2}{16} \\
&= \sum_{x=1}^{4} (2x)\frac{1}{16} + \sum_{x=1}^{4} \sum_{y\in\overline{S_Y}(x),x<y} (x+y)\frac{2}{16}
\end{aligned}
$$

Note: the expectation is w.r.t all random variable, i.e. X and Y.