Slide 7–LU decomposition

MAT2040 Linear Algebra

Upper triangular matrix

Definition 7.1: (**upper triangular matrix**)

$A = (a_{ij})_{n \times n}$ is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$.

A 4×4 upper triangular matrix is given as follows: $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$

$*$ is arbitrary number.

Lower triangular matrix

$A = (a_{ij})_{n\times n}$ is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$.

A 4×4 upper triangular matrix is given as follows: $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$

$*$ is arbitrary number

Note: the diagonal entries could be zero for upper triangular matrix and lower triangular matrix.

In lecture 2, we talked about for a square matrix with good property (here good property actually means that in every step of row reduction, the diagonal entry always be nonzero and without row exchange), a series of elementary row operation type III can be used to transform this square matrix to upper triangular form.

Recall: Illustration of the procedure for 4×4 matrix without row exchange and the diagonal entries are all nonzero:
Using elementary row operation type III:

$$\text{Step 1: } A \triangleq \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \to \begin{bmatrix} \# & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$$

$$\text{Step 2: } \begin{bmatrix} \# & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \to \begin{bmatrix} \# & * & * & * \\ 0 & \# & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

$$\text{Step 3: } \begin{bmatrix} \# & * & * & * \\ 0 & \# & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix} \to \begin{bmatrix} \# & * & * & * \\ 0 & \# & * & * \\ 0 & 0 & \# & * \\ 0 & 0 & 0 & \# \end{bmatrix} \triangleq U$$

$\#$ is nonzero number, $*$ is arbitrary number

For $n \times n$ matrix A with good property, one can use a series of elementary row operation type III $op_1, \cdots, op_k$ (the corresponding elementary matrices are $E_1, \cdots, E_k$) to transform it into an upper triangular form U.
Suppose $A \xrightarrow{op_1} A_1 \xrightarrow{op_2} A_2 \xrightarrow{op_3} \cdots \xrightarrow{op_k} A_k = U$.
By using the properties of elementary matrices, one has
$E_1 A = A_1, E_2 A_1 = A_2, \cdots, E_k A_{k-1} = A_k = U$,
then $E_k E_{k-1} \cdots E_1 A = U$
Thus, $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U = LU$ where $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is a lower triangular matrix.
This is because that $E_i (i = 1, \cdots, k)$ are lower triangular matrices, $E_i^{-1} (i = 1, \cdots, k)$ are also lower triangular matrices, the production $E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is also a lower triangular matrix.

$A = LU$ is called the LU-decomposition.

Recall Gaussian elimination to obtain upper triangular form

Example 7.2 Take the invertible matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to -(-3)R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

$$L_3(L_2 L_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

where U is the upper triangular matrix and $L_3(L_2 L_1 A) = U$.

$$
\begin{aligned}
A &= L_1^{-1} L_2^{-1} L_3^{-1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \\
&= LU
\end{aligned}
$$

L is the lower triangular matrix.
$A = LU$ is called the **LU decomposition.**

Check the lower triangular entries of L, together with the elementary row operations. What do you observe?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

The entries below the diagonal of the unit lower triangular matrix L are the multipliers during the Gaussian Elimination process.

Do we really need to calculate L through finding the inverse of elementary matrices? NO!

When using the elementary row operations to transform A to an upper triangular form, we can obtain L simultaneously.

For the example 7.2

$$A = \begin{bmatrix} \boxed{2} & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

Start with $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Step 1: $\boxed{2}$ is the first pivot corresponding to elimination of first variable, now set the entries in first column of L below the number 1 equal to multipliers during the elimination in the first step. Multipliers are $1/2$ and 2 for second row and third row, respectively.

Update L: $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Step 2: Perform elementary row operations for first column

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -\frac{1}{2}R_1 + R_2} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

$\boxed{3}$ is the second pivot corresponding to elimination of second variable, set the entries in second column of L below the number 1 equal to the multiplier during the elimination in the second step. Multiplier is -3 for the third row.

Update L: $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

Now performing elementary row operations for second column to obtain the upper triangular form

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & -9 & 5 \end{bmatrix} \xrightarrow{R_3 \to 3R_2 + R_3} \begin{bmatrix} 2 & 4 & 2 \\ 0 & \boxed{3} & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$$

One can check that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = LU$$

Keep tracking the multipliers during the Gaussian Elimination process, one can obtain the LU decomposition simultaneously.

Application of LU decomposition to solve linear system

Example 7.3 Find the solution of following system

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 15 \end{bmatrix}$$

$A = LU$, thus

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 15 \end{bmatrix}$$

For $A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$, one can first solve $L\mathbf{y} = \mathbf{b}$ using **forward substitution**, then solve $U\mathbf{x} = \mathbf{y}$ by **backward substitution**.

First solve the linear system $L\mathbf{y} = \mathbf{b}$ by using **forward substitution**:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 15 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}$$

Then solve the linear system $U\mathbf{x} = \mathbf{y}$ by using **backward substitution**:

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

In software, Matlab's command $x = A \setminus b$ use LU-decomposition, along with forward and backward substitution to solve the linear system when the matrix A is nonsingular.

For general square matrices, we have the following result.

Theorem 7.4 (**LU decomposition for a square matrix**)
If A is a square matrix, then there exists a permutation matrix P such that PA has the LU decomposition, i.e., $PA = LU$, where L is a unit lower triangular matrix (a lower triangular matrix whose diagonal entries are all 1's), U is an upper triangular matrix.

Proof is skipped.

Remark: For $PA = LU$, when U is an upper triangular matrix with nonzero diagonal entries, then A is nonsingular. However, if one of the diagonal entries of U is zero, then A is singular.

Example 7.5 Take the nonsingular matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix}$$

$$L_2 L_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

We find row exchange is needed at this stage! Let $P = E_{R_2 R_3}$, then

$$P L_2 L_1 A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

But this does not yields the form $PA = LU$.

How to get the the form $PA = LU$?

Idea: do all the row exchanges first.

Starting from A, if we do the row exchange for row 2 and row 3 first, then
$PA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix}$

Now start with $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\boxed{1}$ is the first pivot of $PA = \begin{bmatrix} \boxed{1} & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix}$

Set $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and perform elementary row operations for PA, one can get

$$\begin{bmatrix} \boxed{1} & 1 & 2 \\ 3 & 5 & 9 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 \to -2R_1 + R_3]{R_2 \to -3R_1 + R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & \boxed{2} & 3 \\ 0 & 0 & -1 \end{bmatrix} = U$$

Now one can check that

$$PA = L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = LU$$

Note that here A is nonsingular, the diagonal entries of U are nonzero, U can be further decomposed into

$$U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} = D\hat{U}$$

where D is a diagonal matrix with nonzero diagonal entries, $\hat{U}$ is a unit upper triangular matrix (an upper triangular matrix whose diagonal entries are all 1's).
Thus

$$PA = LU = LD\hat{U}$$

This is the LDU decomposition.

Theorem 7.6 (**LDU decomposition for a nonsingular matrix**)
If A is nonsingular, then there exists a permutation matrix P such that PA has the LDU decomposition, i.e., $PA = LDU$, where L is a unit lower triangular matrix, D is a diagonal matrix whose diagonal entries are nonzero, U is a unit upper triangular matrix.