MAT1002: Calculus II

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§10.10 The Binomial Series and Applications of Taylor Series

The Benefit of Taylor Series

- Evaluate integrals by giving them as Taylor series. $\int \sin x^2 dx$
- Evaluate limits that lead to indeterminate forms
- Extend the exponential function from real to complex numbers

The Binomial Series for Powers and Roots

For a given constant m, computer the Taylor series generated by $\underbrace{f(x) = (1+x)^m}$ at $x = 0$.

The Binomial Series for Powers and Roots

$f^{(k+1)}(x) = m(m-1)(m-2)\cdots(m-k)(1+x)^{m-k-1}$

For a given constant m, computer the Taylor series generated by $f(x) = (1+x)^m$ at $x = 0$.

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots$$

This is called **binomial series**, which converges absolutely for $|x| < 1$.

- If m is a positive integer or 0,
- If m is not a positive integer or 0,

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^k \cdots$$

$$\left| \frac{\frac{m(m-1)(m-2)\cdots(m-k+1)(m-k)}{(k+1)!}x^{k+1}}{\frac{m(m-1)\cdots(m-k+1)}{k!}x^k} \right| = \left| \frac{m-k}{k} x \right| \to |x| \text{ as } k \to +\infty$$

The Binomial Series

For $|x| < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\boxed{\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}.}$$

- $\sqrt{1+x}$ $\quad m = \frac{1}{2}$

$$= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} x^k$$

The Binomial Series

For $|x| < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}.$$

- $\sqrt{1+x}$
- $\sqrt{1-x^2}$
- $\sqrt{1-\frac{1}{x}}$

let $t = -x^2$ $\quad \sqrt{1+t} = (1+t)^{\frac{1}{2}}$

$$= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} t^k$$

$$= 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (-x^2)^k$$

$$\| $$

$$1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \left(-\frac{1}{x}\right)^k$$

Evaluating Nonelementary Integrals $\int \sin x^2 dx$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} = \frac{x^2}{1} - \frac{x^6}{3!} + \frac{x^{10}}{5!}$$

$$\int \sin x^2 dx = \int \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} dx$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + C$$

Estimate $\int_0^1 \sin x^2 dx$ with an error of less than 0.001

$$\sin x^2 = x^2 - \frac{x^6}{3!} + R_9$$

$$R_9 \leq \cancel{\frac{1 \cdot x^{10}}{10!}} \quad \frac{x^{10}}{5!}$$

$$\int_0^1 R_9 \, dx \leq \int_0^1 \frac{x^{10}}{5!} dx = \frac{1}{11 \times 5!}$$

$$= \frac{1}{11 \times 5 \times 4 \times 3 \times 2} < \frac{1}{10^3}$$

So we can have

$$\int_0^1 x^2 - \frac{x^6}{3!} dx = \frac{1}{3} - \frac{1}{7 \times 3!}$$

Arctangents

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\tan^{-1}x = \int \sum_{k=0}^{\infty} (-1)^k x^{2k}\, dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + 0$$

Evaluating Indeterminate Forms. $\lim_{x\to 1} \frac{\ln x}{x-1}$ $= \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$

$$\ln x = \ln 1 + 1(x-1) + \underbrace{o(x-1)}$$

$$\frac{\ln x}{x-1} = \frac{(x-1)+o(x-1)}{x-1}$$

$$= 1 + \frac{o(x-1)}{x-1} \to 1 \quad \text{as} \quad x \to 1$$

$$\ln x = \ln 1 + 1(x-1) + O((x-1)^2)$$

$$\frac{\ln x}{x-1} = \frac{(x-1)+O((x-1)^2)}{x-1} \to 1 \quad \text{as} \quad x \to 1$$

Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0} \frac{x-\sin x}{\sin x \cdot x} = \lim_{x\to 0} \frac{1-\cos x}{\cos x\cdot x+\sin x}$

$$= \lim_{x\to 0} \frac{\sin x}{-\sin x\cdot x+\cos x+\cos x} = 0$$

$$\sin x = x - \frac{x^3}{3!} + o(x^3)$$

$$\lim_{x\to 0} \frac{x - x + \frac{x^3}{3!} + o(x^3)}{\left(x-\frac{x^3}{3!}+o(x^3)\right)x} = \lim_{x\to 0} \frac{\frac{x^3}{3!}}{x^2}$$

$$= 0$$

Euler's Identity $\underbrace{e^{i\theta} = \cos\theta + \underline{i}\sin\theta}$

$i = \sqrt{-1}$ $\quad i^2 = -1$

$$e^{i\theta} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} (i\theta)^k$$

$$= 1 + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (i\theta)^{2k+1} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} (i\theta)^{2k}$$

$$= 1 + \sum_{k=0}^{\infty} \frac{i(-1)^k}{(2k+1)!} \theta^{2k+1} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \theta^{2k}$$

$$= \cos\theta + i\sin\theta$$

Euler's Identity $e^{i\theta} = \cos\theta + i\sin\theta$

$$\boxed{e^{i\pi} + 1 = 0}$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$= -1 + i0$$

Frequently Used Taylor Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \le 1$$

$$\int \frac{1}{1+x^2}\,dx$$