Slide 17-Linear Transformation I

MAT2040 Linear Algebra

Definition 17.1 (Linear transformation) Let V, W be two vector spaces, and the mapping L from V to W is said to be a linear transformation if the following condition is satisfied:

$$L(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2), \quad \forall \alpha_1,\ \alpha_2 \in \mathbb{R}, \quad \forall \mathbf{v}_1,\ \mathbf{v}_2 \in V.(*)$$

V is called the **domain** of the linear transformation, and W is called the **codomain** of the linear transformation. In particular, if $V = W$, then L is a linear operator on V.

Remark

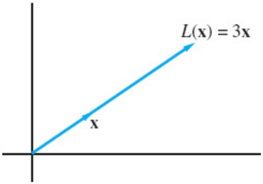
It can be easily checked that the above condition (*) is equivalent to the following two conditions:

(1) $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}), \forall\ \mathbf{u}, \mathbf{v} \in V$

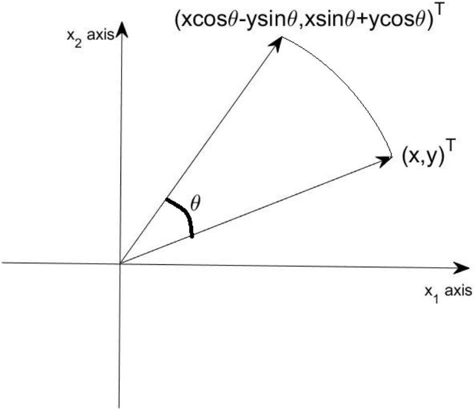
(2) $L(\alpha\mathbf{u}) = \alpha L(\mathbf{u}), \forall\ \mathbf{u} \in V, \alpha \in \mathbb{R}$

Example 17.2

(a) $L : \mathbb{R}^2 \to \mathbb{R}^2$. Stretching or shrinking: $L((x, y)^T) = (\alpha x, \alpha y)^T (\alpha > 0)$.



(b) $L : \mathbb{R}^2 \to \mathbb{R}^2$. Rotation:
$L((x,y)^T) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)^T$. (Rotate in anticlockwise by angle θ)



Example 17.3 Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as:

$$L((x, y)^T) = x - y$$

then L is a linear transformation, the following can be easily checked:

$$\begin{aligned}
&L(\alpha_1(x_1, y_1)^T + \alpha_2(x_2, y_2)^T) \\
=&L((\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2)^T) \\
=&\alpha_1 x_1 + \alpha_2 x_2 - (\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1(x_1 - y_1) + \alpha_2(x_2 - y_2) \\
=&\alpha_1 L((x_1, y_1)^T) + \alpha_2 L((x_2, y_2)^T)
\end{aligned}$$

Example 17.4

Let $L : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined as:

$$L((x,y)^T) = \sqrt{x^2 + y^2}$$

then L is not a linear transformation since

$$L(-(x,y)^T) = L((-x,-y)^T)$$
$$= \sqrt{(-x)^2 + (-y)^2} = \sqrt{x^2 + y^2} \neq -L((x,y)^T).$$

Question: How about the mapping: $L((x,y)^T) = \sqrt[3]{x^3 + y^3}$?

Recall: **Lemma 12.17:** Let V be a vector space with a basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, and let $\mathbf{x}, \mathbf{y} \in V$. For any $\alpha, \beta \in \mathbb{R}$, one has

$$[\alpha \mathbf{x} + \beta \mathbf{y}]_{\mathcal{U}} = \alpha[\mathbf{x}]_{\mathcal{U}} + \beta[\mathbf{y}]_{\mathcal{U}}$$

Therefore, $[\cdot]_{\mathcal{U}}$ is a linear transformation from V to $\mathbb{R}$, where $[\cdot]_{\mathcal{U}}$ is the operator of taking the coordinate w.r.t. basis $\mathcal{U}$.

Property 17.5 **(Property of Linear transformation)** Let L be a linear transformation from V to W, then $\forall\ \alpha_i \in \mathbb{R}, \forall\ \mathbf{u}_i \in V$
(1) $L(\mathbf{0}_V) = \mathbf{0}_W$ ($\mathbf{0}_V$ is the zero vector in V and $\mathbf{0}_W$ is the zero vector in W)
(2) $L(\alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n) = \alpha_1 L(\mathbf{u}_1) + \cdots + \alpha_n L(\mathbf{u}_n)$
(3) $L(-\mathbf{u}) = -L(\mathbf{u})$

Proof. (1) $L(\alpha\mathbf{u}) = \alpha L(\mathbf{u})$, let $\alpha = 0$, then $L(\mathbf{0}_V) = \mathbf{0}_W$.
(2) It can be proved by mathematical induction. $n = 1$ is valid, suppose it is valid for $n = k$, then

$$\begin{aligned} & L((\alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k) + \alpha_{k+1}\mathbf{u}_{k+1}) \\ =& L(\alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k) + L(\alpha_{k+1}\mathbf{u}_{k+1}) \\ =& \alpha_1 L(\mathbf{u}_1) + \cdots + \alpha_k L(\mathbf{u}_k) + \alpha_{k+1} L(\mathbf{u}_{k+1}) \end{aligned}$$

(3) Note that $\mathbf{0}_W = L(\mathbf{0}_V) = L(\mathbf{u} + (-\mathbf{u})) = L(\mathbf{u}) + L(-\mathbf{u})$, this gives $L(-\mathbf{u}) = -L(\mathbf{u})$

Example 17.6 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping defined as:

$$T([x, y, z]^T) = [3x + 2y, 0, 2x + y - 1]^T$$

T is not a linear transformation since
$T((0, 0, 0)^T) = [0, 0, -1]^T \neq (0, 0, 0)^T$

Example 17.7 Let $T : P_3 \to \mathbb{R}^{2\times 2}$ be a mapping defined as:

$$T(ax^3 + bx^2 + cx + d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It is an exercise to show that T is a linear transformation.

Definition 17.8 (Kernel of the Linear transformation) Let L be a linear transformation from V to W, then the kernel of L, denoted by $\ker(L)$ is defined as

$$\ker(L) = \{\mathbf{v} \in V | L(\mathbf{v}) = \mathbf{0}_W\}$$

Definition 17.9 (Image and Range) Let L be a linear transformation from V to W and let S be a subspace of V, the **image** of S, denoted by $L(S)$, is defined by

$$L(S) = \{\mathbf{w} \in W | \exists\ \mathbf{v} \in S,\ s.t.\ L(\mathbf{v}) = \mathbf{w}\}$$

The image of the entire vector space V, i.e., $L(V)$ is called the **range** of L.

Theorem 17.10 **(Kernel and Image are Subspaces)** Let L be a linear transformation from V to W, and let S be a subspace of V, then

(1) $\mathrm{Ker}(L)$ is a subspace of V.

(2) $L(S)$ is a subspace of W.

Proof. Skipped. see Steven's book P174-175.

Example 17.11

(1) $L : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation, $L([x, y]^T) = [x, y, 0]^T$, Ker($L$)=$\{[0, 0]^T\}$, $L(\mathbb{R}^2) = \{[x, y, 0]^T | x, y \in \mathbb{R}\}$.

(2) $L : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, $L([x, y, z]^T) = [x, y]^T$, Ker($L$)=$\{[0, 0, z]^T | z \in \mathbb{R}\}$, $L(\mathbb{R}^3) = \mathbb{R}^2$.

(3) Identity transformation I from vector space V to V, Ker(I)=$\{\mathbf{0}\}$, $I(V) = V$.

Example 17.12 Define $L : \mathbb{R}^2 \to \mathbb{R}^3$ as:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \\ x+y \end{bmatrix}$$

Notice that

$$L(\mathbf{u}) = L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A\mathbf{u}$$

$$\begin{aligned} & L(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2) \\ =& A(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2) \\ =& \alpha_1 A\mathbf{u}_1 + \alpha_2 A\mathbf{u}_2 \\ =& \alpha_1 L(\mathbf{u}_1) + \alpha_2 L(\mathbf{u}_2) \end{aligned}$$

L is the linear transformation.

Remark: any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ will be associated with a matrix.

Theorem 17.13 (Matrix Representation for linear transformation between Eulerian vector spaces w.r.t. standard bases) If L is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, there is a $m \times n$ matrix A such that

$$L(\mathbf{x}) = A\mathbf{x}$$

for each $\mathbf{x} \in \mathbb{R}^n$. In fact, the jth column vector of $A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$ is given by

$$\mathbf{a}_j = L(\mathbf{e}_j), \quad j = 1, 2, \cdots, n$$

where $\mathcal{E}_n = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.

Remark 1. Linear transformation and its matrix representation is completely characterized by its action on a basis of its domain.
Remark 2. The matrix A in this theorem is the matrix representation of the linear transformation L w.r.t. standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

Proof. For $j = 1, 2, \cdots, n$, define:

$$\mathbf{a}_j = L(\mathbf{e}_j), \quad j = 1, 2, \cdots, n$$

and let

$$A = [\mathbf{a}_1, \cdots, \mathbf{a}_n]$$

$\forall\ \mathbf{x} = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$, $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, then

$$\begin{aligned} L(\mathbf{x}) =& L(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\ =& x_1 L(\mathbf{e}_1) + \cdots + x_n L(\mathbf{e}_n) \\ =& x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n \\ =& [\mathbf{a}_1, \cdots, \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x} \end{aligned}$$

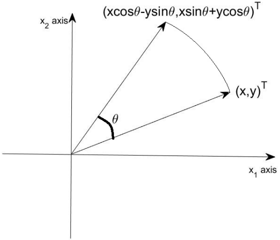
Remark: Indeed, $[L(\mathbf{x})]_{\mathcal{E}_m} = A[\mathbf{x}]_{\mathcal{E}_n}$, $\mathcal{E}_n$ is the standard basis of $\mathbb{R}^n$ and $\mathcal{E}_m$ is the standard basis of $\mathbb{R}^m$.

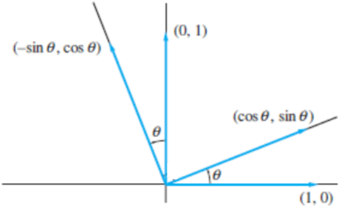
Example 17.14 Define the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix}$$

L is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$, it is completed characterized by its action on the standard basis of $\mathbb{R}^3$.

The matrix A can be constructed as follows: the first column is $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the second column is $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the third column is $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.





Example 17.15 Let L be the linear operator on $\mathbb{R}^2$ that rotates each vector (starting point is the origin) by an angle θ in anti-clockwise (counterclockwise).

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ is the standard basis of $\mathbb{R}^2$. Now look for the action on this standard basis.

Since $L\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $L\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$.

The matrix representation of L w.r.t the standard bases will be

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

And the rotation linear transform is

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$