MAT1002: Calculus II

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§14.8 Lagrange Multipliers
§14.9 Taylor's Formula for Two Variables

How do we find extreme values of a function whose domain is constrained to lie within some subset of the plane (disk, along a curve, etc)?

Find $P(x, y, z)$ on the plane $2x + y - z - 5 = 0$ closest to the origin.

a point

$(x, y, 2x+y-5)$ contains all points on the plane

$f(x,y) = x^2 + y^2 + (2x+y-5)^2$

$\nabla f = 0 \Rightarrow$ $2x + 4(2x+y-5) = 0$

$2y + 2(2x+y-5) = 0$

Constrained maxima and minima

Find point $P(x, y, z)$ on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ closest to the origin.

minimize $x^2 + y^2 + z^2$

s.t. $x^2 - z^2 - 1 = 0$

$f(x,y,z) = x^2 + y^2 + z^2$

on the cylinder $f(x,y,z) = \underbrace{y^2 + 2z^2 + 1}$

Let $y=0$ $z=0$ $x = \pm 1$

Constrained maxima and minima

Find point $P(x, y, z)$ on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ closest to the origin.

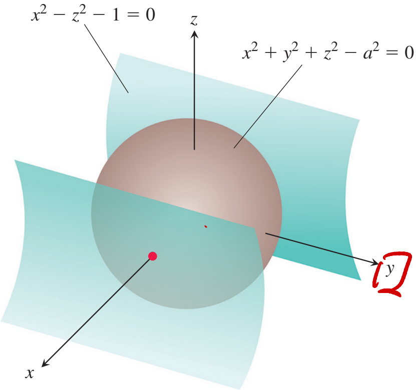
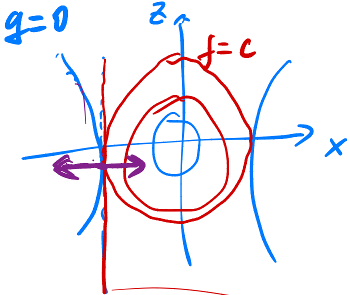


FIGURE 14.54 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).



$$\boxed{\nabla f = \lambda \nabla g}$$

$f = x^2 + y^2 + z^2$

$g = x^2 - z^2 - 1$

$$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 0 \\ -2z \end{pmatrix} \Rightarrow \cancel{x=0} \quad y = 0 \quad x \neq 0 \;\; \lambda = 1 \quad z = 0$$

$2x = 2\lambda x \to x = 0$, $\lambda = 1$

The orthogonal gradient theorem (proof)

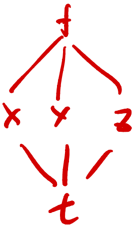
Theorem

Suppose that $f(x,y,z)$ *is differentiable in a region whose interior contains a smooth curve*

$$C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

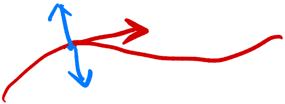
If P_0 *is a point on* C *where* f *has a local maximum or minimum relative to its values on* C *then* ∇f *is orthogonal to* C *at* P_0.

$$f(\vec{r}(t)) = f(x(t), y(t), z(t))$$



$$f_x(x,y,z)\cdot x'(t) + f_y \cdot y'(t) + f_z \cdot z'(t) = 0$$

$$\Rightarrow \langle f_x, f_y, f_z \rangle \cdot \frac{d\vec{r}(t)}{dt} = 0$$



The method of Lagrange multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq 0$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \text{ and } g(x, y, z) = 0.$$

Example: Find the greatest and smallest values of

$$f(x,y) = xy$$

takes on the ellipse

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Let $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$

$\nabla f = \begin{pmatrix} y \\ x \end{pmatrix}$ $\nabla g = \begin{pmatrix} x/4 \\ y \end{pmatrix}$

$\nabla f = \lambda \nabla g \Rightarrow \begin{array}{l} y = \lambda \cdot x/4 \\ x = \lambda y \end{array} \Rightarrow x = \lambda y = \lambda^2 x/4$

$\lambda^2 = 4 \begin{cases} \lambda = 2 \Rightarrow \frac{4y^2}{8} + \frac{y^2}{2} = 1 \Rightarrow \begin{cases} y=1 \\ x=2 \end{cases} \text{ or } \begin{cases} y=-1 \\ x=-2 \end{cases} \\ \lambda = -2 \Rightarrow \begin{cases} y=1 \\ x=-2 \end{cases} \begin{cases} y=-1 \\ x=2 \end{cases} \end{cases}$

maximum

or

$x = 0 \Rightarrow y = \lambda \cdot x/4 = 0$

minimum

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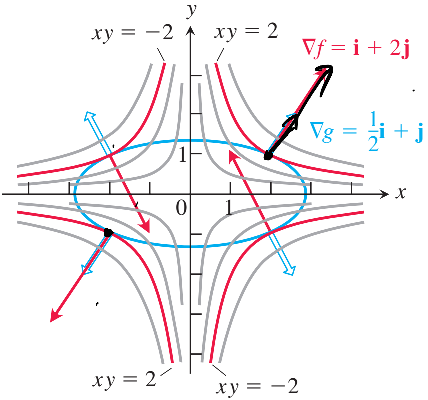
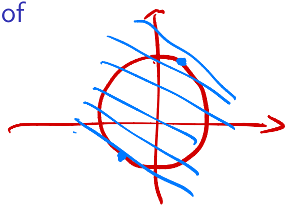


Example: Find the greatest and smallest values of

$$f(x, y) = \underline{3x + 4y}$$

takes on the circle

$$x^2 + y^2 = 1.$$



$$g(x, y) = x^2 + y^2 - 1$$

$$\nabla f = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{matrix} 3 = 2\lambda x \\ 4 = 2\lambda y \end{matrix} \Rightarrow \begin{matrix} x = \frac{3}{2\lambda} \\ y = \frac{4}{2\lambda} \end{matrix}$$

$$x^2 + y^2 = 1 \Rightarrow \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 1 \Rightarrow \lambda^2 = \frac{9}{4} + \frac{16}{4} = \frac{25}{4}$$

$$\lambda = \frac{5}{2} \text{ or } \lambda = -\frac{5}{2}$$

Example: Find the greatest and smallest values of

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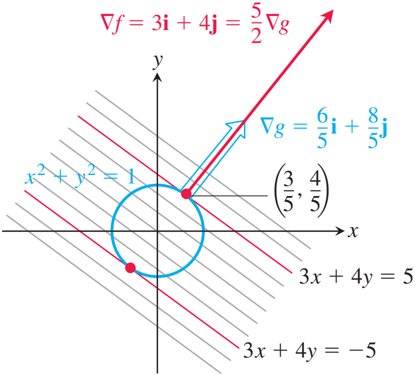
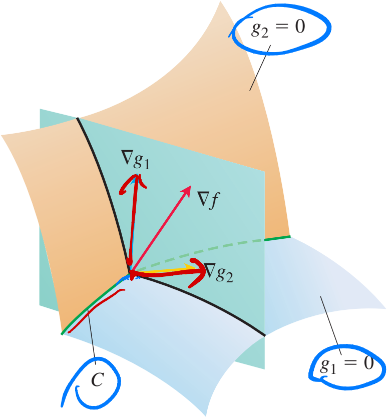
$$x^2 + y^2 = 1.$$

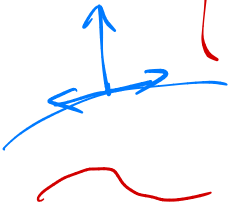


Lagrange multipliers with two constraints

$$g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0$$

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 = 0 \\ g_2 = 0 \end{cases}$$



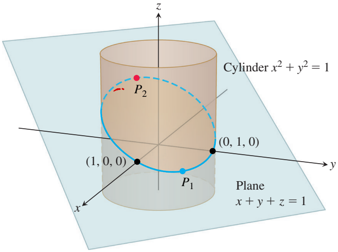


find a vector that is tangent to to C : $\vec{\nabla} g_1 \times \vec{\nabla} g_2$

∇f is orthogonal to $\vec{\nabla} g_1 \times \vec{\nabla} g_2$

Lagrange multipliers with two constraints

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in to an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



$f(x,y,z) = x^2 + y^2 + z^2$

$x + y + z = 1$

$x^2 + y^2 = 1$

$x = \cos\theta \quad y = \sin\theta \quad z = 1 - \cos\theta - \sin\theta$

$f(x,y,z) = \cos^2\theta + \sin^2\theta + (1 - \cos\theta - \sin\theta)^2$

$= 1 + 1 - 2\cos\theta - 2\sin\theta + 2\sin\theta\cos\theta + 1$

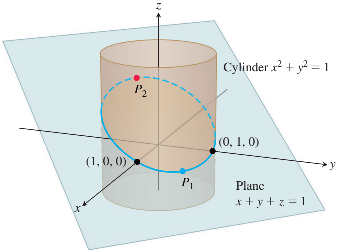
$= 3 - 2\cos\theta - 2\sin\theta + \sin 2\theta$

$\frac{df}{d\theta} = 2\sin\theta - 2\cos\theta + 2\cos 2\theta = 2\sin\theta - 2\cos\theta + 2(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

$= 2(\sin\theta - \cos\theta)(1 - \cos\theta - \sin\theta)$

Lagrange multipliers with two constraints

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



$\nabla f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$

$\nabla g_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\nabla g_2 = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$ $\quad g_1 = 0 \quad g_2 = 0$

$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

$\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$

$x + y + z = 1$

$x^2 + y^2 = 1$

$\lambda_1 = 0 \Rightarrow z = 0 \Rightarrow (x,y) = (1,0)$, $(x,y) = (0,1)$, $\cancel{(x,y) = (-1,0)}$, $\cancel{(x,y) = (0,-1)}$

$\lambda_1 \neq 0 \Rightarrow z = \frac{\lambda_1}{2}$

$\left.\begin{array}{l} 2x = \lambda_1 + \lambda_2(2x) \\ 2y = \lambda_1 + \lambda_2(2y) \end{array}\right\} \Rightarrow x = y = \underbrace{\frac{\lambda_1}{2 - 2\lambda_2}}$

$\frac{\lambda_1}{2} + \frac{\lambda_1}{1 - \lambda_2} = 1 \qquad x = y = \frac{1}{\sqrt{2}}$

$\frac{\lambda_1^2}{2(1-\lambda_2)^2} = 1 \Longrightarrow x = y = -\frac{1}{\sqrt{2}}$

Taylor's formula

We start from the single variable Taylor's formula to

- derive the Second Derivative Test for local extreme values.
- estimate error in linearizations of functions of two independent variables.

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- derive the Second Derivative Test for local extreme values.
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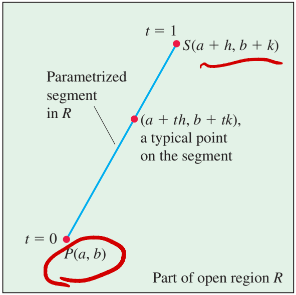
If $f_x(a,b) = f_y(a,b) = 0$. Let

$$F(t) = f(a+th, b+tk)$$

$$F'(t) = f_x(a+th, b+tk)\,h + f_y(a+th, b+tk)\,k$$



$$F'(t) = f_{xx}(a+th, b+tk)\,h^2 + f_{xy}(a+th, b+tk)\,hk + f_{yx}(a+th, b+tk)\,kh + f_{yy}(a+th, b+tk)\,k^2$$

If $f_{xx} f_{yy} - f_{xy}^2 > 0$

$f_{xx} > 0$

Derivation of the second derivative test

$f(a,b)$

$f(a+h, b+k)$

$$F(1) = F(0) + \cancel{F'(0)} + \frac{1}{2}F''(c).$$

$= 0$

If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ at (a,b)

then $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ at $(a+ch, b+ck)$

$$f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \overset{?}{>} 0$$

$$= f_{xx}\left(h - \frac{f_{xy}}{f_{xx}}k\right)^2 - \frac{f_{xy}^2}{f_{xx}}k^2 + f_{yy}k^2$$

$$= \underbrace{f_{xx}\left(h - \frac{f_{xy}}{f_{xx}}k\right)^2}_{\geq 0} + \underbrace{\frac{1}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}^2\right)k^2}_{\geq 0} > 0$$

if $(h,k) \neq (0,0)$

The error formula for linear approximations

$$F(1) = \underbrace{F(0) + F'(0)}_{} + \frac{1}{2} F''(c)(1-0)^2$$

$$F(t) = f(x_0 + th, y_0 + tk)$$

$$\frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2$$

$$F(0) = f(x_0, y_0)$$

$$F'(0) = f_x(x_0, y_0)\cdot h + f_y(x_0, y_0)\cdot k$$

$$F''(c) = f_{xx}(x_0 + ch, y_0 + ck)h^2 + 2f_{xy}(\cdots,\cdots)hk + f_{yy}(\cdots,\cdots)k^2$$

If $|f_{xx}| < M$ $\quad |f_{xy}| < M$, $\quad |f_{yy}| < M$

$$F''(c) \le Mh^2 + 2Mhk + Mk^2 = M(h+k)^2$$

Taylor's formula for functions of two variables

$$\frac{d}{dt} = \underbrace{\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)}, \quad \frac{d^2}{dt^2} = \underbrace{\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2}$$

$$h^2\frac{\partial^2}{\partial x\partial x} + hk\frac{\partial^2}{\partial x\partial y} + hk\frac{\partial^2}{\partial y\partial x}$$

$$+ k^2\frac{\partial^2}{\partial y\partial y}$$

Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin

How accurate is the approximation for $|x| \leq 0.1$ and $|y| \leq 0.1$?

$f(0,0) = 0$

$f_x = \cos x \sin y \qquad f_y = \sin x \cos y$

$f_x(0,0) = 0 \qquad f_y(0,0) = 0$

$f_{xx} = -\sin x \sin y \qquad f_{xy} = \cos x \cos y \qquad f_{yy} = -\sin x \sin y$

$f_{xx}(0,0) = 0 \qquad f_{xy}(0,0) = 1 \qquad f_{yy}(0,0) = 0$

$$f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$+ \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)(x-0)(y-0) + f_{yy}(0,0)(y-0)^2\right)$$

$$= 0 + 0 + xy$$

$f_{xxx} = -\cos x \sin y$

$|f_{xxx}| \leq 1 \qquad |f_{xxy}| \leq 1$

$|f_{xxy}| \leq 1 \qquad |f_{yyx}| \leq 1$

$$\text{error} \leq \frac{1}{3!} M \left(|x-0| + |y-0|\right)^3 \leq \frac{1}{6} \cdot 0.2^3$$