MAT1002: Calculus II

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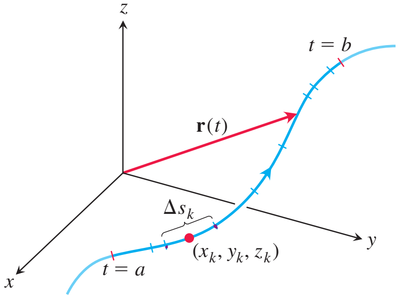
§16.1 Line Integrals
§16.2 Vector Fields and Line Integrals: Work, Circulation, and Flux

Overview of Chapter 16

- Line integrals: the work done by a force moving an object along a path; the mass of a curved wire with variable density.
- Surface integrals: the rate of flow of a fluid across a surface.
- The fundamental theorems of vector integral calculus and their mathematical consequences and physical applications.

Line integral



$f(x_k, y_k, z_k)\Delta s_k$

Definition

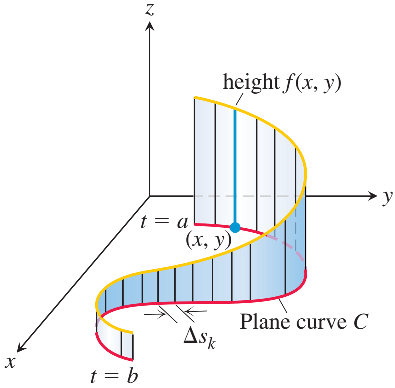
If f is defined on a curve C by $\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$, $a \le t \le b$, then the **line integral of** f **over** C is

$$\int_C f(x, y, z)ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta s_k$$

provided this limit exists.

$\int_C f(g(t), h(t), k(t))\, ds$

Line integrals in the plane $\int_C f\,ds$: area of the portion of the cylindrical surface or "wall" beneath the function $f(x,y)$



How to evaluate a line integral with curve $\vec{r}(t)$

If we have

$$s(t) = \int_a^t |\vec{v}(\tau)| d\tau, \qquad \vec{v} = \frac{d\vec{r}}{dt},$$

$ds = |\vec{v}(t)| dt$

Then we have

$$\int_C f(x,y,z) ds = \int_a^b f(g(t), h(t), k(t)) |\vec{v}(t)| dt.$$

To integrate a continuous function $f(x,y,z)$ over a curve C:

$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$

$\vec{v}(t) = g'(t)\vec{i} + h'(t)\vec{j} + k'(t)\vec{k}$

$\int_a^b f(g(t), h(t), k(t)) |\vec{v}(t)| dt$

How to evaluate a line integral with curve $\vec{r}(t)$

If we have

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Then we have

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|\vec{v}(t)|dt.$$

To integrate a continuous function $f(x,y,z)$ over a curve C:

- Find a smooth parametrization of C,

$$\vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}, \qquad a \leq t \leq b$$

- Evaluate the integral as

$$\int_C f(x,y,z)ds = \int_a^b f(g(t),h(t),k(t))|\vec{v}(t)|dt$$

Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point $(1, 1, 1)$.

$$C : \vec{r}(t) = t\,\vec{i} + t\,\vec{j} + t\,\vec{k} \qquad t \in [0, 1]$$

$$f(g(t), h(t), k(t)) = t - 3t^2 + t = 2t - 3t^2$$

$$\vec{v}(t) = 1\,\vec{i} + 1\,\vec{j} + 1\,\vec{k}$$

$$|\vec{v}(t)| = \sqrt{3}$$

$$\int_0^1 (2t - 3t^2)\sqrt{3}\, dt = 0$$

Additivity

For a piecewise smooth curve C made by joining $C_1, C_2, \cdots, C_n$,

$$\int_C ds = \int_{C_1} f ds + \int_{C_2} f ds + \cdots + \int_{C_n} f ds.$$

Integrate $f(x, y, z) = x - 3y^2 + z$ over C which goes from the origin directly to the point $(1, 1, 0)$, and from $(1, 1, 0)$ to $(1, 1, 1)$.

C_1 $\quad \vec{r}_1(t) = t\vec{i} + t\vec{j} + 0\vec{k} \quad t \in (0, 1)$

C_2 $\quad \vec{r}_2(t) = 1\vec{i} + 1\vec{j} + t\vec{k} \quad t \in (0, 1)$

$\to C_1$ $\quad f(t, t, 0) = t - 3t^2$

$|\vec{r}\,'(t)| = |1\vec{i} + 1\vec{j} + 0\vec{k}| = \sqrt{2}$

$\int_0^1 (t - 3t^2)\sqrt{2}\, dt = \left(\frac{t^2}{2} - t^3\right)\sqrt{2} \Big|_{t=0}^{t=1} = -\frac{1}{2}\sqrt{2}$

$\to C_2$ $\quad f(1, 1, t) = 1 - 3 + t = t - 2$

$|\vec{r}\,'(t)| = |0\vec{i} + 0\vec{j} + 1\vec{k}| = 1$

$\int_0^1 (t - 2)\, dt = \frac{1}{2}t^2 - 2t \Big|_0^1 = -\frac{3}{2}$

Find the line integral of $f(x, y, z) = 2xy + \sqrt{z}$ over the helix $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$, $0 \leq t \leq \pi$.

$$f(\cos t, \sin t, t) = 2\cos t \sin t + \sqrt{t}$$
$$= \sin 2t + \sqrt{t}$$

$$\vec{v}(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\vec{k}$$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\int_0^{\pi} (\sin 2t + \sqrt{t})\sqrt{2}\, dt = \cdots$$

Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

- Mass:

$$M = \int_C \delta ds, \quad \delta = \delta(x, y, z) \text{ is the density at } (x, y, z)$$

- First moments about the coordinate planes:

$$M_{yz} = \int_C x\delta ds, \quad M_{xz} = \int_C y\delta ds, \quad M_{xy} = \int_C z\delta ds$$

- Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{xz}/M, \quad \bar{z} = M_{xy}/M$$

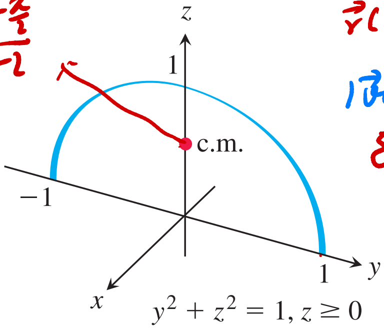
- Moments of inertia about axes and other straight lines

$$I_x = \int_C (y^2 + z^2)\delta ds, \quad I_y = \int_C (x^2 + z^2)\delta ds, \quad I_z = \int_C (x^2 + y^2)\delta ds$$

$$I_L = \int_C r^2 \delta ds, \quad r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L$$

Example: a slender metal arch, denser at the bottom than at the top, lies along the semicircle $y^2 + z^2 = 1$, $z \geq 0$, in the yz-plane. Find the center of the arch's mass if the density on the arch is $\delta(x, y, z) = 2 - z$.

$\dfrac{4-\frac{\pi}{2}}{2\pi-2}$



$\vec{r}(t) = 0\vec{i} + \cos t\,\vec{j} + \underline{\sin t}\,\vec{k}$

$|\vec{v}(t)| = 1, \quad 0 \leq t \leq \pi$

$\delta(0, \cos t, \sin t) = 2 - \sin t$

$\int_0^\pi (2 - \sin t) \cdot 1 \, dt$

$= 2\pi + \cos t \Big|_0^\pi = 2\pi - 2$

$\int_0^\pi z\,\delta\,dt = \int_0^\pi \sin t(2 - \sin t)\,dt$

$= \int_0^\pi 2\sin t\,dt - \int_0^\pi \sin^2 t\,dt$

$\sin^2 t = \dfrac{1-\cos 2t}{2}$

$= 2 \cdot 2 - \dfrac{\pi}{2} = 4 - \dfrac{\pi}{2}$

Vector fields

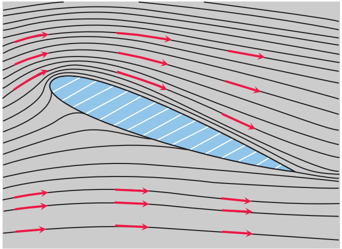


FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

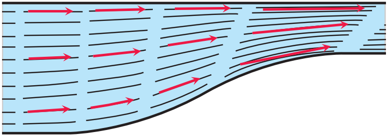


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

A **vector field** is a function that assigns a vector to each point in its domain. A vector field on a 3D domain might have a formula like

$$\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}.$$

- A field is **continuous** if its component functions M, N, and P are continuous;
- A field is **differentiable** if its component functions are differentiable.
- Example: tangent vector $\vec{T}$ and normal vector $\vec{N}$ of a curve.

Special vector fields: gradient fields

We define the **gradient field** of a differentiable function $f(x, y, z)$ to be the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}.$$

- At (x, y, z), the gradient field gives a vector pointing toward the greatest increase of f.
- Its magnitude is the directional derivative in the direction of the greatest increase of f.

Find the vector field $\vec{F}$

Suppose that the temperature T at (x, y, z) in a region of space is given by

$$T = 100 - x^2 - y^2 - z^2,$$

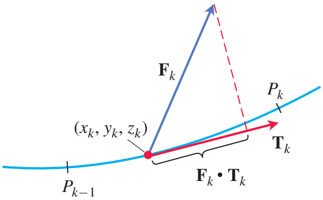
and that $\vec{F}(x, y, z)$ is the gradient of T.

$$\vec{F} = \nabla T = -2x\,\vec{i} - 2y\,\vec{j} - 2z\,\vec{k}$$

Work done by a force over a curve

Let the tangent vector $\vec{T} = d\vec{r}/ds$, a unit vector tangent to the path. The work done along the subarc from P_{k-1} to P_k shown below is approximately

$$\underbrace{\vec{F}(x_k, y_k, z_k) \cdot \vec{T}(x_k, y_k, z_k)}_{\text{the portion contributes}} \Delta s_k.$$



Definition

Let C be a smooth curve parametrized by $\vec{r}(t)$, $a \leq t \leq b$, and $\vec{F}$ be a continuous force field over a region containing C. Then, the work done in moving an object from the point $A = \vec{r}(a)$ to the point $B = \vec{r}(b)$ along C is

$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F} \cdot \vec{T} \, |\vec{r}'(t)| \, dt$$

Work done by a force over a curve

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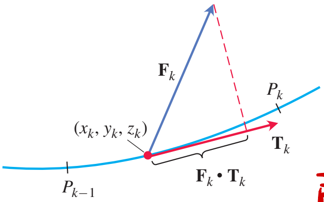

$\frac{d\vec{r}}{dt}$

$\|$

$\vec{F} \cdot \underbrace{\vec{T}\,|\vec{r}\,'(t)|}$

Definition

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$$W = \int_C \vec{F} \cdot \vec{T} ds = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt.$$

Line integrals of vector fields

Let the tangent vector $\vec{T} = d\vec{r}/ds$, a unit vector tangent to the path.

Definition

Let $\vec{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\vec{r}(t)$, $a \leq t \leq b$. The **line integral of F along C** is

$$\int_C \vec{F} \cdot \vec{T} ds = \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds} \right) ds = \int_C \vec{F} \cdot d\vec{r}.$$

Evaluate the integral of $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ along $C : \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$ in the following steps

$$\int_a^b \vec{F}(g(t), h(t), k(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

Line integrals of vector fields

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- Express the vector field $\vec{F}$ as $\vec{F}(\vec{r}(t))$
- Find the derivative vector $d\vec{r}/dt$.
- Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt.$$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$

Let $\vec{F}(x,y,z) = z\vec{i} + xy\vec{j} - y^2\vec{k}$ and $C : \vec{r}(t) = t^2\vec{i} + t\vec{j} + \sqrt{t}\vec{k}$, $0 \le t \le 1$.

$$\vec{F}(t^2, t, \sqrt{t}) = \sqrt{t}\,\vec{i} + t^3\,\vec{j} - t^2\,\vec{k}$$

$$\frac{d\vec{r}(t)}{dt} = 2t\,\vec{i} + 1\,\vec{j} + \frac{1}{2\sqrt{t}}\,\vec{k}$$

$$\vec{F} \cdot \frac{d\vec{r}(t)}{dt} = 2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}$$

$$= t^3 + \frac{3}{2}t^{\frac{3}{2}}$$

$$\int_0^1 t^3 + \frac{3}{2}t^{\frac{3}{2}}\,dt = \cdots$$

Scalar differential form: line integral with respect to dx, dy, or dz

Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ and $C : \vec{r}(t) = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k}$

$$\int_C M\vec{i} \cdot d\vec{r} = \int_C M(x, y, z)dx = \int_a^b M(g(t), h(t), k(t))g'(t)dt$$

$$\int_C N\vec{j} \cdot d\vec{r} = \int_C N(x, y, z)dy = \int_a^b N(g(t), h(t), k(t))h'(t)dt$$

$$\int_C P\vec{k} \cdot d\vec{r} = \int_C P(x, y, z)dz = \int_a^b P(g(t), h(t), k(t))k'(t)dt$$

$$\int_C Mdx + Ndy + Pdz = \int_C \vec{F} \cdot d\vec{r}$$

Different ways to write the work integral for

$\vec{F} - M\vec{i} + N\vec{j} + P\vec{k}$ over the curve $C : \vec{t} = g(t)\vec{i} + h(t)\vec{j} + k(t)\vec{k},\ a \leq t \leq b.$

$$W = \int_C \vec{F} \cdot \vec{T} ds$$

The definition

$$= \int_C \vec{F} \cdot d\vec{r}$$

Vector differential form

$$= \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

Parametric vector evaluation

$$= \int_a^b (Mg'(t) + Nh'(t) + Pk'(t))dt$$

Parametric scalar evaluation

$$= \int_a^b Mdx + Ndy + Pdz$$

Scalar differential form

Example: evaluate $\int_C -ydx + zdy + 2xdz$, where C is the helix $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$, $0 \le t \le 2\pi$

$$\vec{F} = -y\,\vec{i} + z\,\vec{j} + 2x\,\vec{k}$$

$$\vec{F}(\cos t, \sin t, t) = -\sin t\,\vec{i} + t\,\vec{j} + 2\cos t\,\vec{k}$$

$$\frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\,\vec{k}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = \sin^2 t + t\cos t + 2\cos t$$

$$\int_0^{2\pi} \sin^2 t + t\cos t + 2\cos t \, dt$$

$= \ldots$

$(t\sin t + \cos t)' = \cos t$

Find the work done by the force field $\vec{F} = (y - x^2)\vec{i} + (z - y^2)\vec{j} + (x - z^2)\vec{k}$ along the curve $\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$, $0 \le t \le 1$, from $(0,0,0)$ to $(1,1,1)$.

$$\vec{F}(\vec{r}(t)) = (t^2 - t^2)\vec{i} + (t^3 - t^4)\vec{j} + (t - t^6)\vec{k}$$

$$\frac{d\vec{r}(t)}{dt} = 1\vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6)$$

$$\int_0^1 2t(t^3 - t^4) + 3t^2(t - t^6)\, dt$$

Find the work done by the force field $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ in moving an object along the curve C: $\vec{r}(t) = \cos(\pi t)\vec{i} + t^2\vec{j} + \sin(\pi t)\vec{k},\ 0 \le t \le 1$.

$$\vec{F}(\vec{r}(t)) = \cos(\pi t)\,\vec{i} + t^2\,\vec{j} + \sin(\pi t)\,\vec{k}$$

$$\frac{d\vec{r}(t)}{dt} = -\sin(\pi t)\cdot\pi\,\vec{i} + 2t\,\vec{j} + \cos(\pi t)\cdot\pi\,\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} = \cancel{-\sin(\pi t)\cdot\pi\cos(\pi t)} + 2t^3 + \cancel{\sin(\pi t)\cos(\pi t)\pi} = 2t^3$$

$$\int_0^1 2t^3\,dt =$$

Flow integrals and circulation for velocity fields

Definition

If $\vec{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\vec{F}$, the **flow** along the curve from $A = \vec{r}(a)$ to $B = \vec{r}(b)$ is

$$\text{Flow} = \int_C \vec{F} \cdot \vec{T} ds = \int_C \vec{F} \cdot d\vec{r}.$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

A fruid's velocity field is $\vec{F} = x\vec{i} + z\vec{j} + y\vec{k}$. Find the flow along the helix $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$, $0 \le t \le \pi/2$.

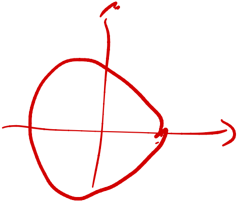
$$\vec{F}(\vec{r}(t)) = \cos t\ \vec{i} + t\ \vec{j} + \sin t\ \vec{k}$$

$$\frac{d\vec{r}(t)}{dt} = -\sin t\ \vec{i} + \cos t\ \vec{j} + 1\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} = -\sin t\cos t + t\cos t + \sin t$$

$$\int_0^{\pi/2} -\sin t\cos t + t\cos t + \sin t\ dt = \ldots$$

Find the circulation of the field $\vec{F} = (x - y)\vec{i} + x\vec{j}$ around the circle $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}$, $0 \le t \le 2\pi$.



$$\vec{F}(\vec{r}(t)) = (\cos t - \sin t)\vec{i} + \cos t\,\vec{j}$$

$$\frac{d\vec{r}(t)}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

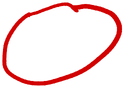
$$\vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} = -\sin t\,(\cos t - \sin t) + \cos^2 t$$

$$\int_0^{2\pi} -\sin t \cos t + 1 \, dt$$

Flux across a simple closed plane curve

Definition

A curve in the xy-plane is **simple** if it does not cross itself. When a curve starts and ends at the same point, it is a **closed curve** or **loop**.

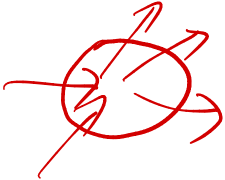


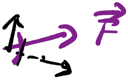
Not simple

Definition

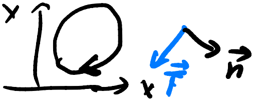
If C is a smooth, simple closed curve in the domain of a continuous vector field $\vec{F} = M(x, y)\vec{i} + N(x, y)\vec{j}$, and if $\vec{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\vec{F}$ across C is

$$\text{Flux of } \vec{F} \text{ across } C = \int_C \vec{F} \cdot \vec{n}\, ds.$$





How to evaluate the flux



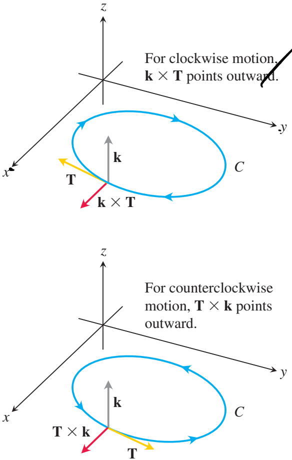


FIGURE 16.21 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Clockwise: add $\pi/2$

$\vec{T} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

$\vec{n} = \cos(\theta+\frac{\pi}{2})\,\vec{i} + \sin(\theta+\frac{\pi}{2})\,\vec{j}$

$\cos(\theta+\pi/2) = -\sin\theta, \ \sin(\theta+\pi/2) = \cos\theta$

Counterclockwise: minus $\pi/2$

$\cos(\theta-\pi/2) = \sin\theta, \ \sin(\theta-\pi/2) = -\cos\theta$

For counterclockwise motion

$$\vec{n} = \frac{dy}{ds}\vec{i} - \frac{dx}{ds}\vec{j}$$

So we have

$$\int_C \vec{F}\cdot\vec{n}ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right)ds$$
$$= \oint_C M dy - N dx.$$

$$= (-N\vec{i} + M\vec{j})\cdot d\vec{r}$$

Find the flux of $\vec{F} = (x - y)\vec{i} + x\vec{j}$ across the circle $x^2 + y^2 = 1$.

$C: \vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j} \qquad 0 \le t \le 2\pi$

$$\oint_C M\,dy - N\,dx.$$

$M(\vec{r}(t)) = \cos t - \sin t \qquad N(\vec{r}(t)) = \cos t$

$\frac{dy}{dt} = \cos t \qquad \frac{dx}{dt} = -\sin t$

$$\int_0^{2\pi} \cos t(\cos t - \sin t) + \sin t \cos t \, dt$$

$$= \int_0^{2\pi} \cos^2 t \cancel{- 2\sin t \cos t} \, dt = \pi$$

$\vec{G} = -x\vec{i} + (x - y)\vec{j} \qquad \frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$\vec{G}(\vec{r}(t)) = -\cos t\,\vec{i} + (\cos t - \sin t)\vec{j}$