$f(x, y)$ fix x $\rightarrow$ dy

or

fix y $\rightarrow$ dx

MAT1002: Calculus II

Ming Yan

§15.1 Double and Iterated Integrals over Rectangles
§15.2 Double Integrals over General Regions
§15.3 Area by Double Integration

$f(x, y)$ fix x $\int f(x,y)\,dy$

or

fix y $\int f(x,y)\,dx$

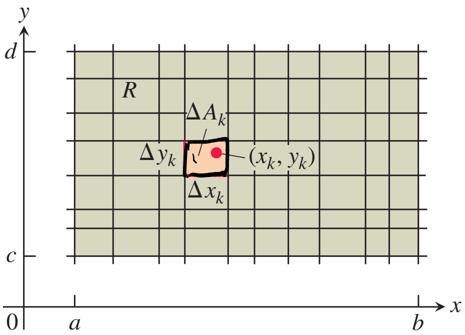
Chapter 15

- the integral of $f(x, y)$
- we need to integrate with respect to x and y in turn
- can find areas of general regions in the xy-plane
- polar coordination
- the integral of $f(x, y, z)$
- cylindrical coordinates, spherical coordinates
- can find average values, moments, and centers

Double integrals

The function $f(x,y)$ is defined on a rectangular region R

$$R: \ a \le x \le b, \ c \le y \le d.$$



$$\sum_k f(x_k)\,\Delta x_k$$

$$\Downarrow$$

$$\int_a^b f(x)\,dx$$

$$\sum_k f(x_k, y_k)\,\Delta x_k \Delta y_k$$

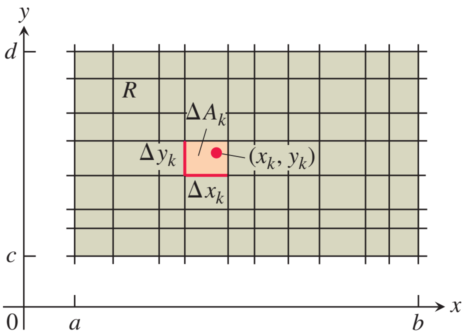
$$\Downarrow$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

Double integrals

The function $f(x, y)$ is defined on a rectangular region R

$$R: \ a \leq x \leq b, \ c \leq y \leq d.$$



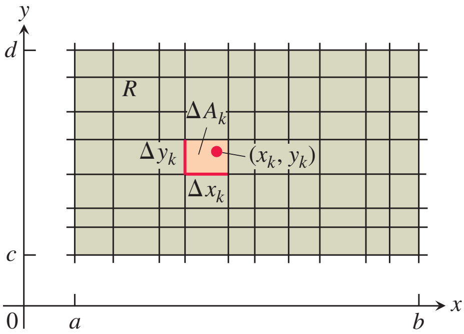
$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$
$$= \sum_{k=1}^{n} f(x_k, y_k) \Delta x_k \Delta y_k$$

$$\lim_{n \to \infty} S_n$$

Double integrals

The function $f(x,y)$ is defined on a rectangular region R

$$R: \ a \le x \le b, \ c \le y \le d.$$



$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

$$= \sum_{k=1}^{n} f(x_k, y_k)\Delta x_k \Delta y_k$$

$$\lim_{n\to\infty} S_n$$

If $\lim_{n\to\infty} S_n$ does not change no matter what choices are made, we say f is **integral**, and the limit is called the **double integral** of f over R, written as

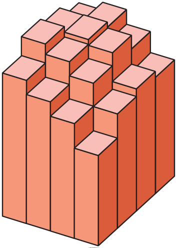
$$\iint_R f(x,y)\,dA, \qquad \text{or} \qquad \iint_R f(x,y)\,dx\,dy.$$

Which function is integrable

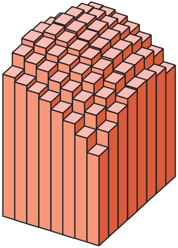
- A continuous function is integrable;
- A function that is discontinuous only on a finite number of points or smooth curves is also integrable.

Double integrals as volumes $(f \geq 0)$

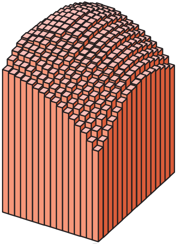
$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x,y)\,dA$$



(a) $n = 16$



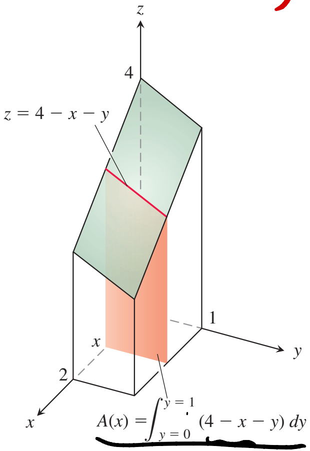
(b) $n = 64$

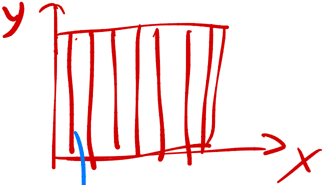


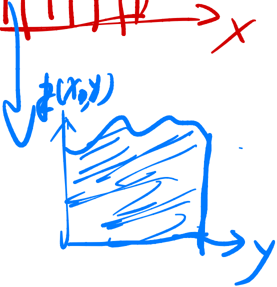
(c) $n = 256$

Fubini's theorem for calculating double integrals

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$ in the xy-plane.







$$\sum_k A(x_k)\,\Delta x_k$$

$$\downarrow$$

$$\int_a^b A(x)\,dx$$

$$A(x) = \int_c^d f(x,y)\,dy$$

$$= 4y - xy - \tfrac{1}{2}y^2 \Big|_{y=0}^{y=1} = 4 - x - \tfrac{1}{2}$$

$$\int_0^2 4 - x - \tfrac{1}{2}\,dx = 3.5x - \tfrac{1}{2}x^2 \Big|_{x=0}^{x=2}$$

$$= 7 - 2 = 5$$

FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

Fubini's theorem for calculating double integrals

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$ in the xy-plane.

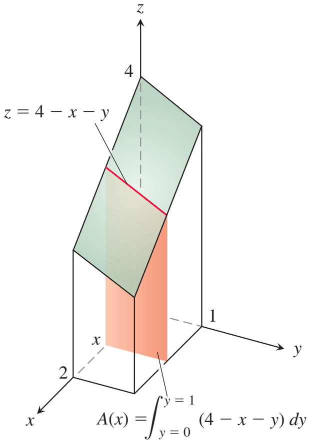


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

$$\int_{x=0}^{x=2} A(x)dx = \int_0^2 \int_0^1 (4-x-y)dydx$$

Fubini's theorem for calculating double integrals

Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$ in the xy-plane.

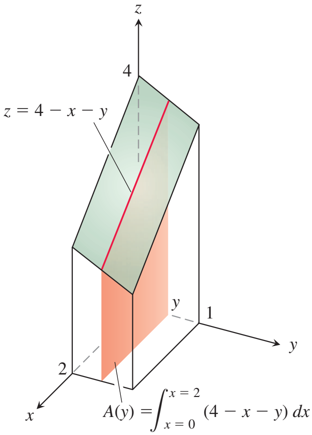


FIGURE 15.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x.

$$\int_{y=0}^{y=1} A(y)dy = \int_0^1 \int_0^2 (4 - x - y)dxdy$$

$$A(y) = \int_0^2 4 - x - y \ dx$$

$$= 4x - \tfrac{1}{2}x^2 - xy \Big|_{x=0}^{x=2}$$

$$= 8 - 2 - 2y$$

$$\int_0^1 6 - 2y \ dy = 6y - y^2 \Big|_{y=0}^{y=1}$$

$$= 6 - 1 = 5$$

Fubini's theorem (first form)

Theorem (Theorem 1)

If $f(x, y)$ is continuous throughout the region $R : a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

Example:

$$\int_0^2 \int_{-1}^1 (100 - 6x^2y)dydx.$$

$$= \int_0^2 (100y - 3x^2y^2)\Big|_{y=-1}^{y=1} dx$$

$$= \int_0^2 200\, dx = 400$$

Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R : 0 \le x \le 1, 0 \le y \le 2$

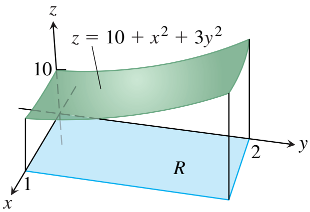


FIGURE 15.7 The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 2).

$$\underset{x}{\int_0^1} \underbrace{\underset{y}{\int_0^2} (10 + x^2 + 3y^2)\, dy}\, dx$$

$$= \int_0^1 \left(10y + x^2 y + y^3\right)\Big|_{y=0}^{y=2} dx$$

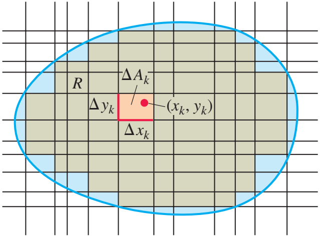
$$= \int_0^1 20 + 2x^2 + 8 \quad dx$$

$$= 28x + \frac{2}{3}x^3 \Big|_{x=0}^{x=1}$$

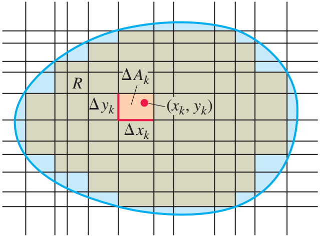
$$= 28 + \frac{2}{3}$$

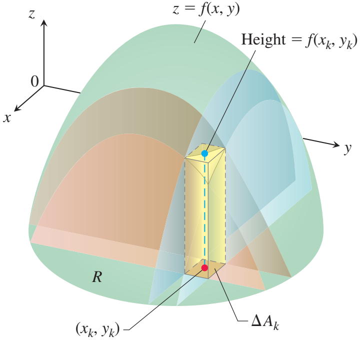
Double integrals over bounded, nonrectangular regions

$$\sum_k f(x_k, y_k)\,\Delta A_k$$



Double integrals over bounded, nonrectangular regions





$$V = \iint_R f(x, y)dA$$

Volume

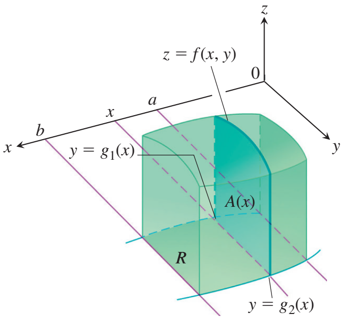


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$V = \int_a^b A(x)dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Fubini's theorem (stronger Form)

Theorem

Let $f(x,y)$ be continuous over a region R.

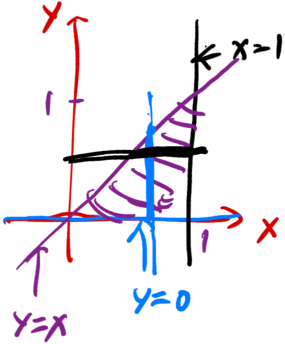
- *If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then*

$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

- *If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$ with h_1 and h_2 continuous on $[c,d]$, then*

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$$

Find the volume of the prism whose base is in the xy-plane bounded by $y = 0$, $y = x$, and $x = 1$ and whose top lies in the plane $z = 3 - x - y$.



$$\int_0^1 \int_0^x 3 - x - y \, dy \, dx$$

$$= \int_0^1 (3-x)y - \tfrac{1}{2}y^2 \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 (3-x)\cdot x - \tfrac{1}{2}x^2 \, dx$$

$$= \int_0^1 3x - \tfrac{3}{2}x^2 \, dx$$

$$= \tfrac{3}{2}x^2 - \tfrac{1}{2}x^3 \Big|_{x=0}^{x=1} = \tfrac{3}{2} - \tfrac{1}{2} = 1$$

$$\int_0^1 \int_y^1 3 - x - y \, dx \, dy$$

$$= \int_0^1 (3-y)x - \tfrac{1}{2}x^2 \Big|_{x=y}^{x=1} dy$$

$$= \int_0^1 (3-y)(1-y) - \tfrac{1}{2} + \tfrac{1}{2}y^2 \, dy = \int_0^1 3 - 4y + y^2 - \tfrac{1}{2} + \tfrac{1}{2}y^2 \, dy$$

$$= \frac{y^3}{2} - 2y^2 + \tfrac{5}{2}y \Big|_{y=0}^{y=1} = \tfrac{1}{2} - 2 + \tfrac{5}{2} = 1$$

Calculate

$$\iint_R \frac{\sin x}{x} dA$$

where R is bounded by $y = 0$, $y = x$, and $x = 1$.

$$\underbrace{\int_0^1 \int_0^x \frac{\sin x}{x}\, dy}\, dx = \int_0^1 \frac{\sin x}{x} y \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \sin x\, dx = -\cos x \Big|_{x=0}^{x=1} = 1 - \cos 1$$

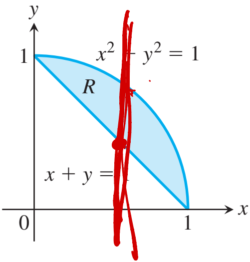
$\int_0^1 \int_y^1 \frac{\sin x}{x} dx\, dy$ not easy to compute.

Find limits of integration

Using vertical cross-sections

- Sketch the region of integration
- Find the y-limits of integration
- Find the x-limits of integration

$$\int_0^1 \int_{1-x}^{\sqrt{1-x^2}} dy\,dx$$



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- Sketch the region of integration
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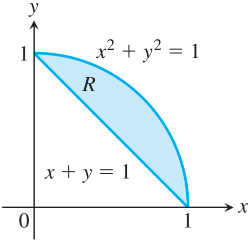


$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx, \qquad \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

and write an equivalent integral with the order of integration reversed.

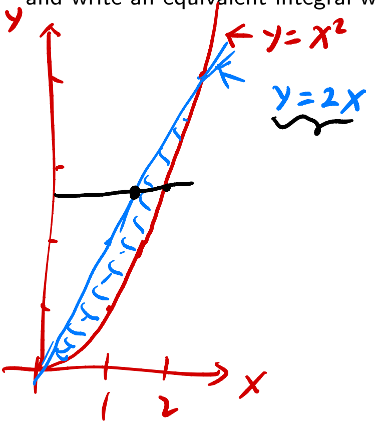
$$= \int_0^2 (4x+2)y \Big|_{y=x^2}^{y=2x} dx$$

$$= \int_0^2 (4x+2)(2x-x^2)\,dx$$

$$= \int_0^2 6x^2+4x-4x^3\,dx$$

$$= 2x^3+2x^2-x^4 \Big|_{x=0}^{x=2}$$

$$= 16+8-16 = 8$$



$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy$$

$$= \int_0^4 (2x^2+2x)\Big|_{x=y/2}^{x=\sqrt{y}} dy$$

$$= \int_0^4 2y + 2\sqrt{y} - \frac{y^2}{2} - y \; dy$$

$$= \frac{1}{2}y^2 + \frac{2}{\frac{3}{2}} y^{\frac{3}{2}} - \frac{1}{6}y^3 \Big|_{y=0}^{y=4}$$

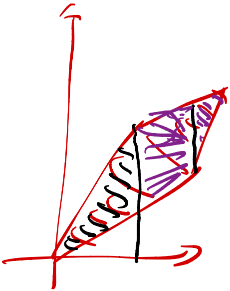
$$= 8 + \frac{4}{3}\cdot 8 - \frac{64}{6} = 8$$

Properties of double integrals

$\int\int f(x,y)\,dx\,dy$ → can have y

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\,dA = c\iint_R f(x, y)\,dA$ (any number c)



2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\,dA = \iint_R f(x, y)\,dA \pm \iint_R g(x, y)\,dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\,dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$

(b) $\displaystyle\iint_R f(x, y)\,dA \geq \iint_R g(x, y)\,dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$

4. *Additivity:* $\displaystyle\iint_R f(x, y)\,dA = \iint_{R_1} f(x, y)\,dA + \iint_{R_2} f(x, y)\,dA$

if R is the union of two nonoverlapping regions R_1 and R_2

Find the volume of the wedgelike solid that lies beneath $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

$$\int_0^2 \int_{\frac{y^2}{4}}^{\frac{y+2}{4}} 16 - x^2 - y^2 \, dx\, dy$$

$$= \int_0^2 (16-y^2)x - \frac{1}{3}x^3 \Big|_{\frac{y^2}{4}}^{\frac{y+2}{4}} dy$$

$$= \int_0^2 (16-y^2)\left(\frac{y+2}{4} - \frac{y^2}{4}\right) - \frac{1}{3}\left(\frac{y+2}{4}\right)^3 + \frac{1}{3}\left(\frac{y^2}{4}\right)^3 dy$$

$= \cdots$

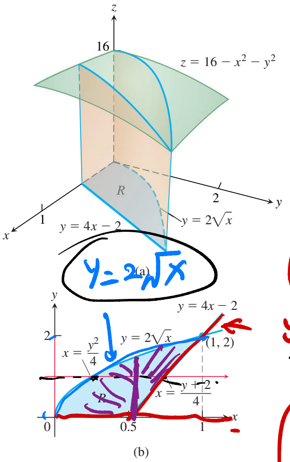


FIGURE 15.18 (a) The solid "wedge-like" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\,dy$.

$y = 4x - 2$

$$\int_0^{0.5} \int_0^{2\sqrt{x}} 16 - x^2 - y^2 \, dy\, dx$$

$$+ \int_{0.5}^{1} \int_{4x-2}^{2\sqrt{x}} 16 - x^2 - y^2 \, dy\, dx$$

$= \cdots$

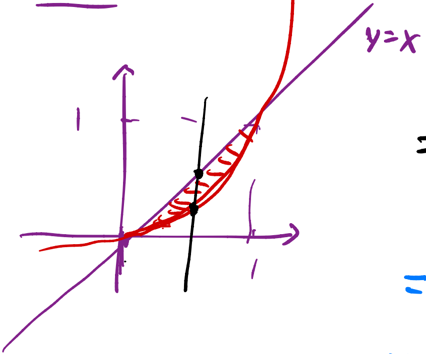
Area by double integration ($f(x, y) = 1$)

Definition

The **area** of a closed bounded plane region R is

$$\boxed{A = \iint_R dA.}$$

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.



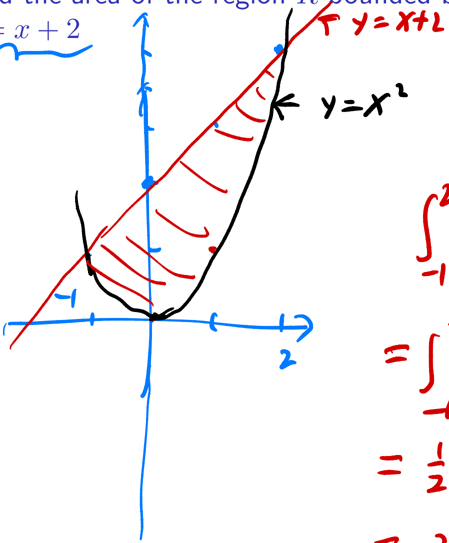
$$\int_0^1 \int_{x^2}^{x} 1 \; dy \; dx$$

$$= \boxed{\int_0^1 (x - x^2) \, dx}$$

$$= \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{1}{6}$$

Find the area of the region R bounded by the parabola $y = x^2$ and the line $y = x + 2$



$x^2 = x + 2$

$\Rightarrow x = 2 \quad x = -1$

$$\int_{-1}^{2}\int_{x^2}^{x+2} 1 \, dy \, dx$$

$$= \int_{-1}^{2} (x+2-x^2)\, dx$$

$$= \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \Big|_{-1}^{2}$$

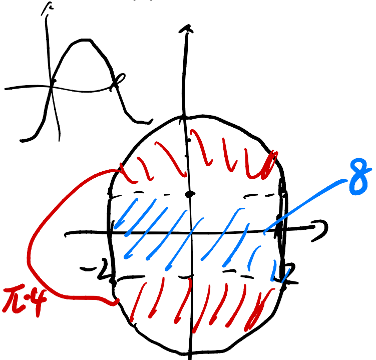
$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= 4.5$$

Find the area of the playing field described by

$$R: \quad -2 \le x \le 2, \quad -1-\sqrt{4-x^2} \le y \le 1+\sqrt{4-x^2},$$

using (a) Fubini's theorem and (b) simple geometry.



$y = 1+\sqrt{4-x^2}$

$\Rightarrow y-1 = \sqrt{4-x^2}$

$\Rightarrow \underbrace{(y-1)^2 + x^2}\, = 4$

$$\int_{-2}^{2}\int_{-1-\sqrt{4-x^2}}^{1+\sqrt{4-x^2}} 1 \; dy\, dx$$

$$= \int_{-2}^{2} 2 + 2\sqrt{4-x^2}\; dx = 8 + \int_{-2}^{2} 2\sqrt{4-x^2}\, dx$$

$$= 8 + 4\pi$$

let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x=2 \Rightarrow \theta = \pi/2$

$x=-2 \Rightarrow \theta = -\pi/2$

$$\int_{-2}^{2} 2\sqrt{4-x^2}\, dx = \int_{-\pi/2}^{\pi/2} 2\cdot 2\cos\theta \cdot 2\cos\theta\, d\theta = \int_{-\pi/2}^{\pi/2} 8\cos^2\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 4(\cos 2\theta + 1)\, d\theta$$

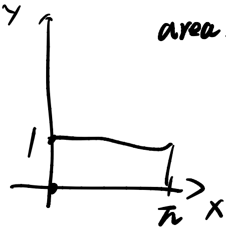
$$= 4\pi$$

Average value

$$\sum_i \frac{\Delta A_i}{\text{area of } R} f(x_i, y_i)$$

Average value of f over $R = \dfrac{1}{\text{area of } R} \displaystyle\iint_R f\,dA$

Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R : 0 \le x \le \pi,\ 0 \le y \le 1.$



area: $\displaystyle\int_0^1 \int_0^\pi 1 \, dx\, dy = \pi$

$$\int_0^1 \int_0^\pi x \cos xy \, dx\, dy$$

$$= \int_0^\pi \underbrace{\int_0^1 x \cos xy \, dy} \, dx$$

$$= \int_0^\pi \sin(xy) \Big|_{y=0}^{y=1} dx$$

$$= \int_0^\pi \sin x \, dx = -\cos x \Big|_0^\pi$$

$$= -\cos \pi + \cos 0 = 2$$

average value

$$= \frac{2}{\pi}$$