MAT1002: Calculus II

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§10.6 Alternating Series and Conditional Convergence
§10.7 Power Series

Use derivative to check monotonicity

Define $f(x)$ such that $f(n) = u_n$.

$$u_n = \frac{10n}{n^2 + 16}$$

Alternating Series Estimation Theorem

Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions of Theorem 15, then for $n \geq N$, then we have

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the total sum L of the series with an error whose absolute values is less than u_{n+1}. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder $L - s_n$ has the same sign as the first unused term.

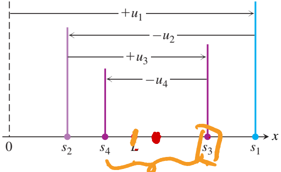


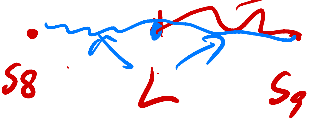
FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

Example

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$$

$$s_8 = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} = \boxed{\frac{85}{128}} = 0.6640625, \quad s_9 = 0.66796875$$

$$\frac{s_8 + s_9}{2}$$



$$|\text{error}| < \frac{1}{256} \times \frac{1}{2}$$

Conditional Convergence

$\sum \frac{1}{n^p}$ converges iff $p > 1$

Definition

A convergent series that is not absolutely convergent is **conditionally convergent.**

Consider

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

- $p > 1$ absolutely convergent.
- $0 < p < 1$ conditionally convergent.
- $p = 1$ conditionally convergent.
- $p = -1$

- $p = 0$ diverges.
- $p < 0$ $\frac{(-1)^{n-1}}{n^p}$ diverges.

Rearrangement Theorem for Absolutely Convergent Series (proof skip)

Theorem (Theorem 17)

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \cdots, b_n, \cdots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\boxed{\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n}$$

Rearrange conditional convergent series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

$1 \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{7} \quad \cdots$

$-\frac{1}{2} \quad -\frac{1}{4} \quad -\frac{1}{6} \quad -\frac{1}{8} \quad \cdots$

$2 \qquad 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$

Summary of Tests

- The nth-Term Test: If $a_n \not\to 0$, then the series diverges
- Geometric series $\sum ar^n$: If $|r| < 1$, then it converges, otherwise, it diverges.

p-series $\Sigma \frac{1}{n^p}$ converges iff $p > 1$

- Nonnegative terms:
  - Integral Test
  - Comparison Test
  - Limit Comparison test
  - Ratio Test
  - Root Test
- Series with some negative terms
  - Absolute convergent
- Alternating series
  - Conditional convergent

Power Series: infinite polynomials

Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

where the **center** a and the **coefficients** $c_0, c_1, c_2, \cdots, c_n, \cdots$ are constants.

Example

$\Sigma a r^n$

Let $c_n = 1$, we have

$\boxed{0^0 = 1}$

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \infty & \text{if } x = 1 \\ 1 & \cancel{\text{if } x = 0} \\ \frac{x^0}{1-x} = \frac{1}{1-x} & \underline{\text{other.}} \end{cases}$$

$-1 < x < 1$

it converges for $|x| < 1$

diverges for $|x| \geq 1$

Example

Let $c_n = 1$, we have

$$\sum_{n=0}^{\infty} x^n$$

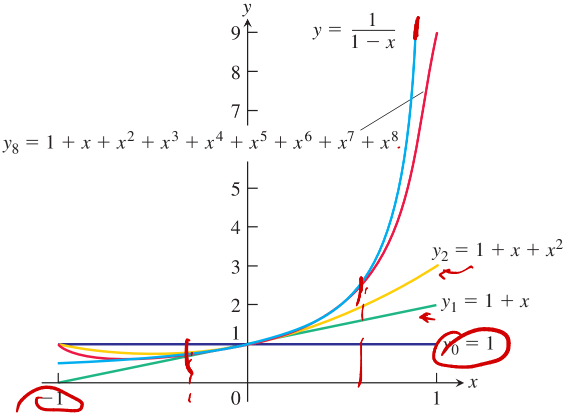


FIGURE 10.14 The graphs of $f(x) = 1/(1 - x)$ in Example 1 and four of its polynomial approximations.

Example

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-2)^n = \sum_{n=0}^{\infty} \left(\frac{-(x-2)}{2}\right)^n$$

converges iff $\left|\frac{-(x-2)}{2}\right| < 1$

$\Rightarrow |x-2| < 2 \Rightarrow 0 < x < 4$

converges to $\dfrac{1}{1-\frac{-(x-2)}{2}} = \dfrac{1}{1+\frac{x}{2}-1} = \dfrac{2}{x}$

Convergence of Power Series

- $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

Root test. $\sqrt[n]{\frac{|x|^n}{n}} = \frac{|x|}{\sqrt[n]{n}} \to |x|$

it converges if $|x| < 1$

it diverges if $|x| > 1$

it converges if $x = 1$

it diverges if $x = -1$

$(-1, 1]$

Convergence of power series

- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$

Root test

$$\sqrt[n]{\frac{|x|^{2n-1}}{2n-1}} = \frac{|x|^{(2n-1)/n}}{\sqrt[n]{2n-1}} \to x^2$$

$$= (2n-1)^{\frac{1}{n}}$$

$$\left((2n-1)^{\frac{1}{2n-1}}\right)^{\frac{2n-1}{n}}$$

Ratio test

$$\frac{\frac{x^{2(n+1)-1}}{2(n+1)-1}}{\frac{x^{2n-1}}{2n-1}} = \frac{x^2}{2n+1}\cdot(2n-1) \to x^2$$

converges if $x^2 < 1$

diverges if $x^2 > 1$

converges if $x = 1$

converges if $x = -1$

$$\sqrt[n]{2n-1} = \sqrt[n]{n}\cdot\sqrt[n]{\frac{2n-1}{n}}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n-1}$$

Convergence of power series

- $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Ratio test

$$\frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1} \to 0$$

converges for any x

Convergence Theorem for Power Series

$|a_n c^n| < 1$ for $n > N$

$\Rightarrow |a_n| < \frac{1}{c^n} \Rightarrow \sqrt[n]{|a_n|} < \frac{1}{c}$

Theorem (Theorem 18)

If the power series $\sum_{n=0}^{\infty} a_n x^n$ *converges at* $x = c > 0$, *then it converges absolutely for all* $x \in (-c, c)$, *If the series diverges at* $x = d$, *then it diverges for all* x *with* $|x| > |d|$.

$\underbrace{\sum_{n=0}^{\infty} a_n c^n}$ converges $\rightarrow$ $\underbrace{a_n c^n} \rightarrow 0$

$|a_n| \leq \left(\frac{1}{c}\right)^n$

if $|x| < c$ $\quad a_n x^n \rightarrow 0$

Root test $\sqrt[n]{|a_n||x|^n} = \sqrt[n]{|a_n|}\,|x| < 1$ if $\underbrace{\sqrt[n]{|a_n|} \leq \frac{1}{c}}$ for $n > N$

$\sum_{n=0}^{\infty} a_n d^n$ diverges $\Rightarrow \cancel{\sqrt[n]{|a_n d^n|} \geq 1}$

if $|x| > |d|$

Root test $\sqrt[n]{|a_n||x|^n} = \sqrt[n]{|a_n|}\,|x| > \sqrt[n]{|a_n|}\,|d| \geq 1$

$n \geq N$

Convergence Theorem for Power Series

Theorem (Theorem 18)

If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = c > 0$, then it converges absolutely for all $x \in (-c, c)$, If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

If it converges for x s.t. $|x| > |d|$

then it converges for $x = d$, which contradicts

the condition.

The Radius of Convergence of a Power Series

Corollary

The convergence of the series $\sum c_n(x-a)^n$ *is described by one of the following three cases:*

- *There is a positive number* R *such that the series diverges for* x *with* $|x-a| > R$ *but converges absolutely for* x *with* $|x-a| < R$. *The series may or may not converge at either of the endpoints* $x = a - R$ *and* $x = a + R$.
- *The series converges absolutely for every* x $(R = \infty)$.
- *The series converges at* $x = a$ *and diverges elsewhere* $(R = 0)$.

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x = a$ is called the **the interval of convergence**. The interval may be open, closed, or half-open.

Steps to test the convergence of a power series

- Use **Ratio Test** or **Root Test** to find the radius R.
- If R is finite and positive. Test both endpoints using **Comparison Test, Integral Test, or Alternating Series Test**

Theorem (Therem 19)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$, *and*

$$c_n = \underbrace{a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0}_{n+1 \text{ terms}} = \sum_{k=0}^{n} a_k b_{n-k}$$

then $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$:

$$\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} c_n x^n$$

Substitute a function in a convergent power series

Theorem (Theorem 20)

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

$$\underbrace{\sum_{n=0}^{\infty} (4x^2)^n}$$

$$|4x^2| < 1 \Rightarrow |x| < \frac{1}{2}$$

$$\sum_{n=0}^{\infty} 4^n x^{2n} \Rightarrow a_n = \begin{cases} 4^{n/2} x^n & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

$$\sqrt[n]{|a_n|} = \begin{cases} (4^{n/2})^{\frac{1}{n}} |x| = 2|x| & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$$

we can not directly get the convergence,

The Term-by-Term Differentiation Theorem

Theorem (Theorem 21)

If $\sum c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n, \text{ on the interval } a-R<x<a+R$$

The function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series terms by term:

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$$

Example

Find the derivative for $f'(x)$ and $f''(x)$ given

$$f(x) = 1 + x + x^2 + \cdots + x^n + = \sum_{n=0}^{\infty} x^n$$

The Term-by-Term Integration Theorem

Theorem (Theorem 22)

If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $|x-a| < R$. *Then*

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $|x-a| < R$ *and*

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $|x-a| < R$.

Example

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$