MAT1002: Calculus II

Ming Yan

§16.8 The Divergence Theorem and a Unified Theory

Divergence in three dimensions

Divergence form of Green's theorem

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

We compute the net outward flux of a vector field across a simple closed curve by integrating the divergence of the field over the region enclosed by the curve.

$$\oiint_S \vec{F} \cdot \vec{n} \, d\sigma$$

$$\iiint_V \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \, dx dy dz$$

Divergence in three dimensions

Divergence form of Green's theorem

$$\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

We compute the net outward flux of a vector field across a simple closed curve by integrating the divergence of the field over the region enclosed by the curve.

The divergence theorem extends Green's theorem (flux density) to three dimensions.

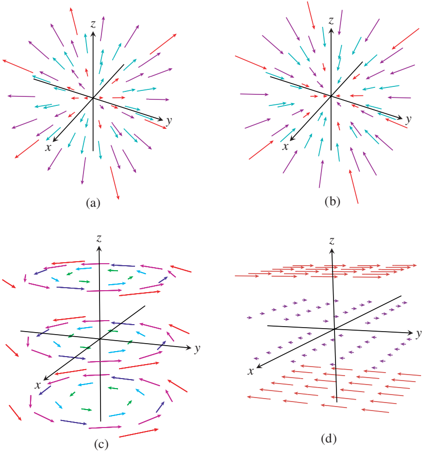
The **divergence** of a vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is the scalar function

$$\text{div}\vec{F} = \nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

Physical interpretation:

- If $\vec{F}$ is the velocity field of a flowing gas, the value of div$\vec{F}$ at (x, y, z) is the rate at which the gas is compressing or expanding at the point.
- The divergence is the flux per unit volume or **flux density** at the point. (analogous to *circulation density*)

Find their divergence and interpret their physical meaning



(a) (b) (c) (d)

- Expansion:
  $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

  $\nabla \vec{F} = 3$

- Compression:
  $\vec{F}(x,y,z) = -x\vec{i} - y\vec{j} - z\vec{k}$

  $\nabla \vec{F} = -3$

- Rotation about the z-axis:
  $\vec{F}(x,y,z) = -y\vec{i} + x\vec{j}$

  $\nabla \vec{F} = 0$

- Shearing along parallel horizontal planes:
  $\vec{F}(x,y,z) = z\vec{j}$

  $\nabla \vec{F} = 0$

Divergence theorem

Theorem (Theorem 8–Divergence theorem)

If $\vec{F}$ is a vector field whose components have continuous first partial derivatives. Let S be a piecewise smooth oriented closed surface. The flux of $\vec{F}$ across S in the direction of the surface's outward unit normal field $\vec{n}$ equals the triple integral of the divergence $\nabla \cdot \vec{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \vec{F} \cdot \vec{n} d\sigma}_{\text{outward flux}} = \underbrace{\iiint_D \nabla \cdot \vec{F} dV.}_{\text{divergence integral}}$$

Corollary

The outward flux across a piecewise smooth oriented closed surface S is zero for any vector field $\vec{F}$ having zero divergence at every point of the region enclosed by the surface.

Examples: the velocity field of a circulating incompressible liquid, constant vector fields, shearing

Verify the divergence theorem for the expanding vector field

$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$.

$$\underbrace{\oint\!\!\oint_S \vec{F}\cdot\vec{n}\, d\sigma}_{0} = \iiint_D \nabla\cdot\vec{F}\, dV$$

$$\vec{r}(\varphi, \theta) = a\sin\varphi\cos\theta\,\vec{i} + a\sin\varphi\sin\theta\,\vec{j} + a\cos\varphi\,\vec{k}$$

$$0 \le \varphi \le \pi, \quad 0 \le \theta \le 2\pi$$

$$\vec{F}\cdot\vec{n} = a\sin\varphi\cos\theta\cdot\sin\varphi\cos\theta + a\sin\varphi\sin\theta\,\sin\varphi\sin\theta + a\cos\varphi\cos\varphi$$

$$= a$$

$$\oint\!\!\oint_S a\, d\sigma = a\cdot 4\pi a^2$$

$$\nabla\cdot\vec{F} = 3$$

$$\iiint_D 3\, dV = 3\cdot\frac{4}{3}\pi a^3 = 4\pi a^3$$

Find the flux of $\vec{F} = xy\vec{i} + yz\vec{j} + xz\vec{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

$\vec{F} \cdot \vec{n}$

$x=1$, $0 \le y \le 1$, $0 \le z \le 1$	$\vec{n} = \vec{i}$	$\vec{F} \cdot \vec{n} = y$	$\int_0^1 \int_0^1 y \, dy \, dz = \frac{1}{2}$
$x=0$ $0 \le y \le 1$ $0 \le z \le 1$	$\vec{n} = -\vec{i}$	$\vec{F} \cdot \vec{n} = 0$	
$0 \le x \le 1$ $y=1$ $0 \le z \le 1$	$\vec{n} = \vec{j}$	$\vec{F} \cdot \vec{n} = z$	$\int_0^1 \int_0^1 z \, dx \, dz = \frac{1}{2}$
$0 \le x \le 1$ $y=0$ $0 \le z \le 1$	$\vec{n} = -\vec{j}$	$\vec{F} \cdot \vec{n} = 0$	
$0 \le x \le 1$ $0 \le y \le 1$ $z=1$	$\vec{n} = \vec{k}$	$\vec{F} \cdot \vec{n} = x$	$\int_0^1 \int_0^1 x \, dx \, dy = \frac{1}{2}$
$0 \le x \le 1$ $0 \le y \le 1$ $z=0$	$\vec{n} = -\vec{k}$	$\vec{F} \cdot \vec{n} = 0$	

Find the flux of $\vec{F} = xy\vec{i} + yz\vec{j} + xz\vec{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

$$\nabla \cdot \vec{F} = y + z + x$$

$$\int_0^1 \int_0^1 \underbrace{\int_0^1 (y + z + x)\,dx}\, dy\, dz$$

$$= \int_0^1 \int_0^1 (y+z) + \frac{1}{2} \quad dy\, dz$$

$$= \int_0^1 z + \frac{1}{2} + \frac{1}{2} \quad dz$$

$$= \frac{3}{2}$$

Find the flux of $\vec{F} = x^2\vec{i} + 4xyz\vec{j} + ze^x\vec{k}$ out of the box-shaped region $D : 0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1$. Verify the divergence theorem.

$x=3$	$0\le y\le 2$	$0\le z\le 1$	$\vec{F}\cdot\vec{n} = 9$	$\int_0^2\int_0^1 9\,dz\,dy = 18$
$x=0$	$0\le y\le 2$	$0\le z\le 1$	$\vec{F}\cdot\vec{n} = 0$	
$0\le x\le 3$	$y=2$	$0\le z\le 1$	$\vec{F}\cdot\vec{n} = 8xz$	$\int_0^3\int_0^1 8xz\,dz\,dx = \int_0^3 4x\,dx = 18$
$0\le x\le 3$	$y=0$	$0\le z\le 1$	$\vec{F}\cdot\vec{n} = 0$	
$0\le x\le 3$	$0\le y\le 2$	$z=1$	$\vec{F}\cdot\vec{n} = e^x$	$\int_0^3\int_0^2 e^x\,dy\,dx = 2e^x\Big\vert_{x=0}^{x=3} = 2e^3-2$
$0\le x\le 3$	$0\le y\le 2$	$z=0$	$\vec{F}\cdot\vec{n} = 0$	

Find the flux of $\vec{F} = x^2\vec{i} + 4xyz\vec{j} + ze^x\vec{k}$ out of the box-shaped region $D: 0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1$. Verify the divergence theorem.

$$\nabla \cdot \vec{F} = 2x + 4xz + e^x$$

$$\int_0^3 \int_0^2 \underbrace{\int_0^1 2x + 4xz + e^x \, dz}\, dy\, dx$$

$$= \int_0^3 \int_0^2 2x + e^x + 2x \; dy\, dx$$

$$= \int_0^3 8x + 2e^x \, dx$$

$$= 4x^2 + 2e^x \Big|_{x=0}^{x=3}$$

$$= 36 + 2e^3 - 2$$

Divergence and the curl

$\nabla \times (\nabla f) = \vec{0}$

Theorem

If $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is a vector field with continuous second partial derivatives, then

$$div(curl\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0.$$

$$\underline{\nabla \times \vec{F}} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$$\nabla\cdot(\nabla\times\vec{F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0$$

If $\underbrace{\vec{G} = \nabla \times \vec{F}}$, then $\underbrace{\int\int_S \vec{G} \cdot \vec{n} d\sigma} = 0$ for any closed surface S.

$$= \iiint_D \nabla \cdot \vec{G}\, dV = \iiint_D \nabla \cdot (\nabla \times \vec{F})\, dV = 0$$

$$\iint_S (\nabla \times \vec{F}) \cdot n\, d\sigma$$

Divergence and the curl

Theorem

If $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is a vector field with continuous second partial derivatives, then

$$div(curl\vec{F}) = \nabla \cdot (\nabla \times \vec{F}) = 0.$$

$$\nabla \times \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

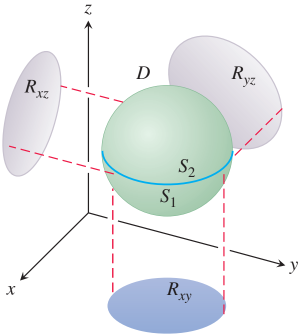
$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma + \iint_{S_2} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \oint_L \vec{F}\,d\vec{r} - \oint_L \vec{F}\cdot d\vec{r} = 0$$

If $\vec{G} = \nabla \times \vec{F}$, then $\underline{\iint_S \vec{G}\cdot\vec{n}\,d\sigma = 0}$ for any closed surface S.

Stokes' thm:

$$\oint_L \vec{F}\,d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma$$

Proof of divergence theorem for special regions (no holes/bubbles)



$$S_2 : z = f_2(x,y), \quad (x,y) \text{ in } R_{xy}$$
$$S_1 : z = f_1(x,y), \quad (x,y) \text{ in } R_{xy}$$

Let $\vec{n} = (\cos\alpha)\vec{i} + (\cos\beta)\vec{j} + (\cos\gamma)\vec{k}$, then

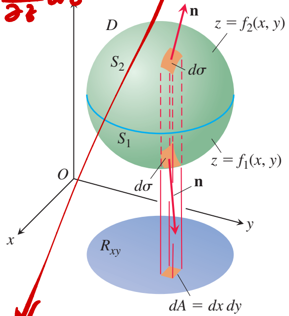
$$\vec{F}\cdot\vec{n} = M\cos\alpha + N\cos\beta + P\cos\gamma$$

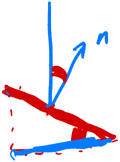
$$\iint_S (M\cos\alpha + N\cos\beta + P\cos\gamma)\,d\sigma = \iiint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dx\,dy\,dz$$

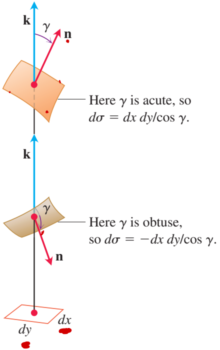
Proof of divergence theorem for special regions

We prove that $\iint_S P\cos\gamma d\sigma = \iint_{R_{xy}} P(x,y,f_2(x,y)) - P(x,y,f_1(x,y))dxdy$

$\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial P}{\partial z} dz$







$\iint_{S_1} P\cos\gamma\, d\sigma + \iint_{S_2} P\cos\gamma\, d\sigma$

$\iint_{R_{xy}} P(x,y,f_2(x,y)) \cos\gamma \frac{1}{|\cos\gamma|} dxdy$

Divergence theorem for other regions

D is the region between two concentric spheres.

The surface of the lower half D_1 consists of an outer hemisphere, an inner hemisphere, and a plane washer-shaped base.

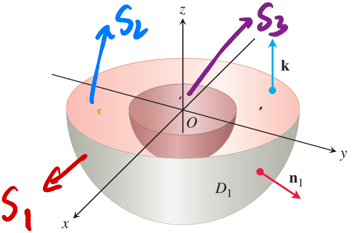


FIGURE 16.76 The lower half of the solid region between two concentric spheres.

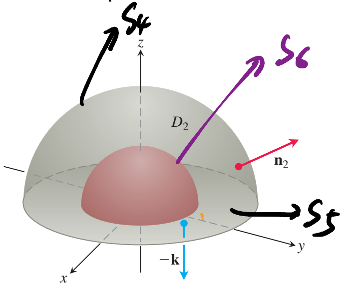


FIGURE 16.77 The upper half of the solid region between two concentric spheres.

$$\iiint_{D_1} \nabla\cdot\vec{F}\, dV = \iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_3} \vec{F}\cdot\vec{n}\, d\sigma$$

$$\iiint_{D_2} \nabla\cdot\vec{F}\, dV = \iint_{S_4} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_5} \vec{F}\cdot\vec{n}\, d\sigma + \iint_{S_6} \vec{F}\cdot\vec{n}\, d\sigma$$

Find the net outward flux of the field

$$\vec{F} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region $D : 0 < b^2 \le x^2 + y^2 + z^2 \le a^2$.

$\vec{F} \cdot \vec{n}$

S_1: $x^2+y^2+z^2 = a^2$ $\qquad \vec{F}\cdot\vec{n} = \dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{\rho^3} \cdot \dfrac{x\vec{i}+y\vec{j}+z\vec{k}}{a}$

$$= \frac{a^2}{a^4} = \frac{1}{a^2}$$

$$\iint_{S_1} \vec{F}\cdot\vec{n}\, d\sigma = 4\pi a^2/a^2 = 4\pi$$

S_2: $x^2+y^2+z^2 = b^2$ $\qquad \vec{F}\cdot\vec{n} = -\dfrac{1}{a^2}$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\, d\sigma = -4\pi$$

Find the net outward flux of the field

$$\frac{x}{\sqrt{x^2+y^2+z^2}^3}$$

$$\vec{F} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\rho^3}, \qquad \rho = \sqrt{x^2+y^2+z^2}$$

across the boundary of the region $D : 0 < b^2 \leq x^2 + y^2 + z^2 \leq a^2$.

$$\nabla \cdot \vec{F} = \frac{1}{\rho^3} + \left(-\frac{3}{2}\right) \frac{x}{(x^2+y^2+z^2)^{5/2}} \cdot 2x$$

$$+ \frac{1}{\rho^3} + \left(-\frac{3}{2}\right) \frac{y}{(x^2+y^2+z^2)^{5/2}} 2y$$

$$+ \frac{1}{\rho^3} + \left(-\frac{3}{2}\right) \frac{z}{(x^2+y^2+z^2)^{5/2}} 2z$$

$$= \frac{3}{\rho^3} - 3 \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} = \frac{3}{\rho^3} - \frac{3}{\rho^3} = 0$$

What can we learn from this example?

- $\nabla \cdot \vec{F} = 0.$
- $\iint_{S_a} \vec{F} \cdot \vec{n} d\sigma = 4\pi.$

Gauss's law: one of the four great laws of electromagnetic theory

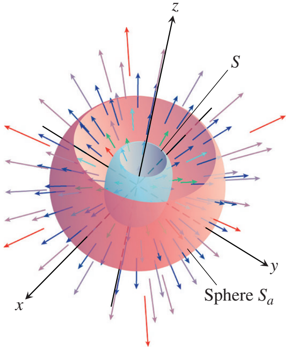
Let

$$\vec{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{|\vec{r}|^3}$$

$$\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\rho^3}$$

Then, for any closed surface that encloses the origin, we have

$$\iint_S \vec{E}\cdot\vec{n}\,d\sigma = \frac{q}{\epsilon_0}.$$



Continuity equation of hydrodynamics

Let D be a region bounded by a closed-oriented surface S. If $\vec{v}(x,y,z)$ is the velocity field of a fluid flowing smoothly through D, $\delta = \delta(t,x,y,z)$ is the fluid's density at (x,y,z) at time t, and $\vec{F} = \delta\vec{v}$, then the continuity equation of hydrodynamics states that

$$\nabla \cdot \vec{F} + \frac{\partial \delta}{\partial t} = 0.$$

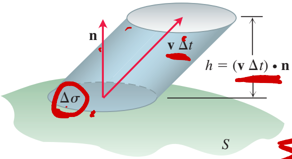


FIGURE 16.80 The fluid that flows upward through the patch $\Delta\sigma$ in a short time Δt fills a "cylinder" whose volume is approximately base $\times$ height $=$ $\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t$.

Let B be a solid sphere centered at Q:

$$\frac{\iiint_B \nabla \cdot \vec{F} dV}{\iiint_B dV} = \boxed{\frac{\iint_S \vec{F} \cdot \vec{n} d\sigma}{\iiint_B dV} = \frac{\frac{dm}{dt}}{\iiint_B dV}} = -\frac{\partial \delta}{\partial t}$$

$$\frac{\sum \Delta\sigma \cdot (\vec{v}\Delta t) \cdot \vec{n} \cdot \delta}{\Delta t} \xrightarrow{\Delta\sigma \to 0} \iint_S (\vec{v} \cdot \vec{n}\, \delta)\, d\sigma = -\frac{dm}{dt}$$

Conservation of mass

$$-\iiint_B \frac{\partial \delta}{\partial t} dV = \iiint_B \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \vec{n} d\sigma$$

Unifying

Tangential form of Green's Theorem: $\oint_C \vec{F} \cdot \vec{T} ds = \iint_R \nabla \times \vec{F} \cdot \vec{k} dA$ ✓

Stokes' Theorem: $\oint \vec{F} \cdot \vec{T} ds = \iint_S \nabla \times \vec{F} \cdot \vec{n} d\sigma$

Normal form of Green's Theorem: $\oint_C \vec{F} \cdot \vec{n} ds = \iint_R \nabla \cdot \vec{F} dA$ ✓

Divergence Theorem: $\iint_S \vec{F} \cdot \vec{n} d\sigma = \iiint_D \nabla \cdot \vec{F} dV$

Fundamental Theorem: $f(b) - f(a) = \int_a^b \frac{df}{dx} dx$

$\nabla \cdot \vec{F}$

$$\vec{F}(b) \cdot \vec{n} + \vec{F}(a) \cdot \vec{n} = \int_{[a,b]} \nabla \cdot \vec{F} dx$$

- Fundamental theorem of calculus, normal form of Green's theorem, divergence theorem
- Stokes' theorem, tangential form of Green's theorem