MAT1002: Calculus II

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§13.2 Integrals of Vector Functions; Projectile Motion
§13.3 Arc Length in Space
§13.4 Curvature and Normal Vectors of a Curve

Overview of Chapter 13

In Chapter 12, we learned

- vectors and geometry of space

Chapter 13 introduces the calculus of vector-valued functions.

- the derivative is a vector
- the integral is also a vector
- describe paths and motions of objects
- introduces new scalars

Integrals of Vector Functions

Definition

The **indefinite integral** of $\vec{r}$ with respect to t is the set of all antiderivatives of $\vec{r}$, denoted as $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then

$$\underbrace{\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}.}$$

Example:

$$\int ((\cos t)\vec{i} + \vec{j} - 2t\vec{k})dt$$

$$= \sin t\,\vec{i} + t\,\vec{j} - t^2\vec{k} + \vec{C}$$

Integrals of Vector Functions

Definition

The **indefinite integral** of $\vec{r}$ with respect to t is the set of all antiderivatives of $\vec{r}$, denoted as $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}.$$

Definition

If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a, b]$, then so is $\vec{r}$, and the **definite integral** of $\vec{r}$ from a to b is

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}$$

Fundamental Theorem of Calculus:

$$\int_a^b \vec{r}(t)dt = \vec{R}(t)\Big]_a^b = \vec{R}(b) - \vec{R}(a).$$

Example:

Suppose we do not know the path of a hang glider, but only its acceleration vector $\vec{a}(t) = -(3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$. We also know that the glider departed from the point $(4, 0, 0)$ with velocity $\vec{v}(0) = 3\vec{j}$ at time $t = 0$. Find the glider's position as a function $\vec{r}(t)$ of t.

since $\frac{d\vec{v}(t)}{dt} = \vec{a}(t)$

$\vec{v}(t) = -3\sin t\ \vec{i} + 3\cos t\ \vec{j} + 2t\ \vec{k} + \vec{C}_1$

let $t=0$ $\vec{v}(0) = 3\vec{j} + \vec{C}_1$ so $\vec{C}_1 = \vec{0}$

$\vec{r}(t) = 3\cos t\ \vec{i} + 3\sin t\ \vec{j} + t^2\vec{k} + \vec{C}_2$

let $t=0$ $\vec{r}(0) = 3\vec{i} + \vec{C}_2$ so $\vec{C}_2 = \langle 1, 0, 0\rangle$

$\vec{r}(t) = (3\cos t + 1)\vec{i} + 3\sin t\ \vec{j} + t^2\vec{k}$

Ideal Projectile Motion

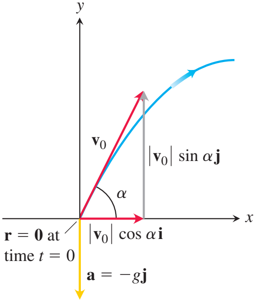
$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -g\vec{j}$$

$$\vec{v}(t) = -gt\,\vec{j} + \vec{C}_1$$

let $t=0$

$$\vec{v}(0) = \vec{C}_1 = \vec{v}_0 = |\vec{v}_0|\cos\alpha\,\vec{i} + |\vec{v}_0|\sin\alpha\,\vec{j}$$

$$\vec{r}(t) = -\frac{gt^2}{2}\vec{j} + |\vec{v}_0|\cos\alpha\cdot t\,\vec{i} + |\vec{v}_0|\sin\alpha\cdot t\,\vec{j} + \vec{C}_2$$



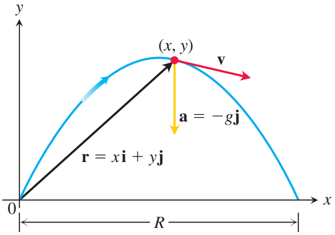
(a)

let $t=0$

$$\vec{r}(0) = \vec{C}_2 = \vec{0}$$

Ideal Projectile Motion

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j}$$



Horizontal range
(b)

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

- Maximum height

$$\vec{r}(t) = v_0 \cos\alpha \, t \, \vec{i} + \underbrace{\left(v_0 \sin\alpha \, t - \frac{g t^2}{2}\right)}\vec{j}$$

$$v_0 \sin\alpha - g t = 0 \Rightarrow t = \frac{v_0 \sin\alpha}{g}$$

Max. height is $v_0 \sin\alpha \cdot \frac{v_0 \sin\alpha}{g} - \frac{g}{2} \cdot \frac{v_0^2 \sin^2\alpha}{g^2} = \frac{v_0^2 \sin^2\alpha}{2g}$

- Flight time

$$v_0 \sin\alpha \, t - \frac{g t^2}{2} = 0 \Rightarrow t = 0 \text{ or } \underbrace{t = \frac{2 v_0 \sin\alpha}{g}}$$

Flight time is $\frac{2 v_0 \sin\alpha}{g}$

- Range

Range is $v_0 \cos\alpha \cdot \frac{2 v_0 \sin\alpha}{g} = \frac{2 v_0^2 \sin\alpha \cos\alpha}{g}$

Projectile Motion with Wind Gusts

A baseball is hit when it is 1m above the ground. It leaves the bat with an initial speed of 50 m/s, making an angle of $20°$ with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-2.5\vec{i}$ (m/s) to the ball's initial velocity (2.5 m/s $= 9$ km/h).

take derivative of the y-component

- Find a vector equation (position vector) for the baseball path.
- How high does the baseball go, and when does it reach maximum height?
- : If the ball is not caught, find its range and flight time.

$\vec{a} = -g\vec{j}$ $\quad\quad \vec{v}(t) = -gt\vec{j} + \vec{v}_0$ $\quad\quad \vec{v}_0 = (50\cos 20° - 2.5)\vec{i} + 50\sin 20°\,\vec{j}$

$$\vec{r}(t) = (50\cos 20° - 2.5)t\,\vec{i} + \left(50\sin 20°\, t - g\frac{t^2}{2}\right)\vec{j} + \vec{r}(0)$$

$\vec{r}(0) = 1\vec{j}$

$$\vec{r}(t) = (50\cos 20° - 2.5)t\,\vec{i} + \underbrace{\left(50\sin 20°\, t - g\frac{t^2}{2} + 1\right)}\vec{j}$$

§13.3 Arc Length in Space

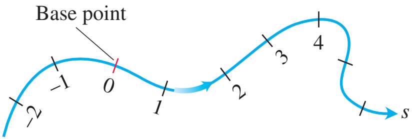


FIGURE 13.12 Smooth curves can be scaled like number lines, the coordinate of each point being its directed distance along the curve from a preselected base point.

Definition

The **length** of a smooth curve $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_a^b |\vec{v}| dt$$

Example: A glider is soaring upward along the helix $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

$$\vec{v}(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

$$|\vec{v}(t)| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$\int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

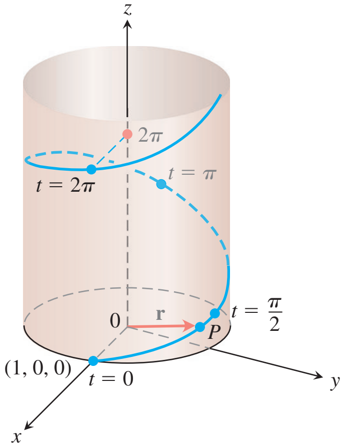


FIGURE 13.13 The helix in Example 1, $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$.

Arc Length Parameter with Base Point $P(t_0)$

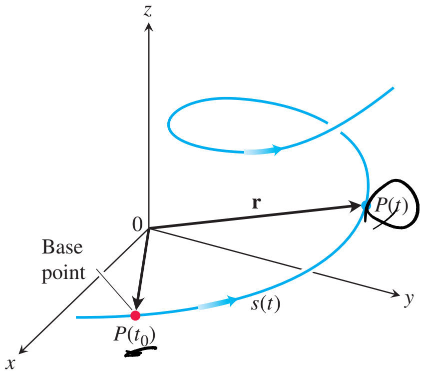


FIGURE 13.14 The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is

$$\underbrace{s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau.}$$

$s(t)$

$\uparrow$

$s'(t) = |v(t)|$

Speed on a Smooth Curve

$$\frac{ds}{dt} = |\vec{v}|$$

Unit Tangent Vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

Unit Tangent Vector

$s(t)$ $\quad \vec{r}(t) \quad \vec{r}(s)$

$\underbrace{\vec{r}(t) \quad \vec{r}(s)} = \vec{r}(s(t))$

$\vec{v}(t) \quad \vec{v}(s) = \frac{d\vec{r}}{ds} = \frac{dr/dt}{ds/dt}$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

Example:

$$\vec{r}(t) = (1 + 3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k}$$

$$\vec{v}(t) = -3\sin t\,\vec{i} + 3\cos t\,\vec{j} + 2t\,\vec{k}$$

$$|\vec{v}(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2} = \sqrt{9+4t^2}$$

$$\vec{T} = \frac{-3\sin t}{\sqrt{9+4t^2}}\vec{i} + \frac{3\cos t}{\sqrt{9+4t^2}}\vec{j} + \frac{2t}{\sqrt{9+4t^2}}\vec{k}$$

$\frac{d\vec{r}}{ds} = \vec{T}$

$$\vec{T} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{d\vec{r}}{ds}$$

$$d\vec{r}/dt = d\vec{r}/ds \cdot \underbrace{ds/dt}$$

$$|\vec{v}(t)| = \left|\frac{d\vec{r}(t)}{dt}\right|$$

§13.4 Curvature and Normal Vectors of a Curve

Definition

If $\vec{T}$ is the unit tangent vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right|$$

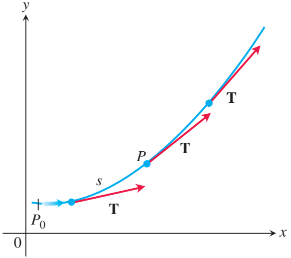


FIGURE 13.17 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P.

Formula for Calculating Curvature

If $\vec{r}(t)$ is a smooth curve, then the curvature (given $\vec{T} = \vec{v}/|\vec{v}|$) is the scalar function

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}/dt}{ds/dt} \right|$$

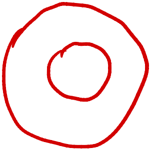
$$= \frac{\left| \frac{d\vec{T}}{dt} \right|}{|\vec{v}|}$$

Constant curvature: straight lines and circles

straight lines: $\vec{T}$ is a constant

$$\frac{d\vec{T}}{ds} = \vec{0} \quad \text{so} \quad k = 0$$

Circles



$$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + 0\vec{k}$$

$$\vec{v}(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j}$$

$$|\vec{v}(t)| = 2$$

$$\vec{T}(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$k = \frac{\sqrt{\cos^2 t + \sin^2 t}}{2} = \frac{1}{2}$$

Principal Unit Normal

$|\vec{T}| = 1$ $\vec{T} \cdot \frac{d\vec{T}}{dt} = 0$

We know that $d\vec{T}/ds$ is orthogonal to $\vec{T}$ because the unit tangent vector $|\vec{T}| = 1$. We can define a new unit vector.

Definition

At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\vec{N} = \frac{1}{\kappa} \boxed{\frac{d\vec{T}}{ds}}$$

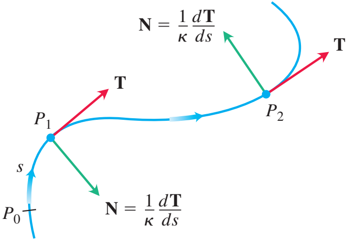


FIGURE 13.19 The vector $d\mathbf{T}/ds$, normal to the curve, always points in the direction in which $\mathbf{T}$ is turning. The unit normal vector $\mathbf{N}$ is the direction of $d\mathbf{T}/ds$.

Formula for Calculating $\vec{N}$

If $\vec{r}(t)$ is a smooth curve, then the principal unit normal (given $\vec{T} = \vec{v}/|\vec{v}|$) is

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$$

$$\vec{N} = \frac{\frac{d\vec{T}}{ds}}{\left|\frac{d\vec{T}}{ds}\right|} = \frac{\frac{d\vec{T}/dt}{\cancel{ds/dt}}}{\left|\frac{d\vec{T}/dt}{\cancel{ds/dt}}\right|} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

Example: Find $\vec{T}$ and $\vec{N}$ for the circular motion

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{r}(t) = (\cos 2t)\vec{i} + (\sin 2t)\vec{j}$$

$$\vec{v}(t) = -2\sin 2t\,\vec{i} + 2\cos 2t\,\vec{j}$$

$$|\vec{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = -\sin 2t\,\vec{i} + \cos 2t\,\vec{j}$$

$$\frac{d\vec{T}}{dt} = -2\cos 2t\,\vec{i} - 2\sin 2t\,\vec{j}$$

$$\vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|} = -\cos 2t\,\vec{i} - \sin 2t\,\vec{j}$$

Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

- is tangent to the curve at P (has the same tangent line the curve has)
- has the same curvature the curve has at P
- has a center that lies toward the concave or inner side of the curve (as in Figure 13.20).

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which is $\boxed{\rho = \frac{1}{\kappa}}$. The **center of curvature** of the curve at P is the center of the circle of curvature.

$\vec{OP} + \frac{1}{\kappa}\vec{N}$

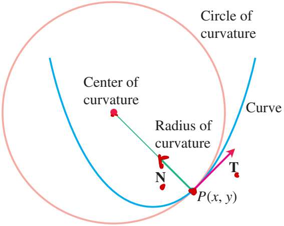


FIGURE 13.20 The center of the osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Example: Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

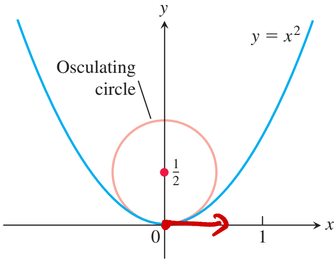


FIGURE 13.21 The osculating circle for the parabola $y = x^2$ at the origin (Example 4).

$x = t \quad y = t^2$

$\vec{r}(t) = t\vec{i} + t^2\vec{j}$

$\vec{v}(t) = \vec{i} + 2t\vec{j}$

$\vec{T}(t) = \frac{1}{\sqrt{1+4t^2}}\vec{i} + \frac{2t}{\sqrt{1+4t^2}}\vec{j}$

$$\frac{d\vec{T}}{dt} = -\frac{1}{2}\frac{8t}{\sqrt{1+4t^2}^3}\vec{i} + \left(\frac{2}{\sqrt{1+4t^2}^3} - \frac{1}{2}\frac{2t\cdot 8t}{\sqrt{1+4t^2}^3}\right)\vec{j}$$

Let $t=0$

$\vec{T} = \vec{i}$

$\frac{d\vec{T}}{dt} = 2\vec{j}$

$\vec{N} = \vec{j}$

$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}/dt|}{|\vec{v}(t)|} = 2$

so the circle is

$x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2$

Curvature and Normal Vectors for Space Curves

Given a position vector $\vec{r}(t)$ and s is the arc length parameter of the curve.

- Unit tangent vector:

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$$

- Curvature:

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = \frac{1}{|\vec{v}|} \left| \frac{d\vec{T}}{dt} \right|$$

- Principal unit normal:

$$\vec{N} = \frac{1}{\kappa} \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

Example: Find the curvature and the principal unit normal for the helix

$$\vec{r}(t) = (a\cos t)\vec{i} + (a\sin t)\vec{j} + bt\vec{k}, \qquad a, b \geq 0, \qquad \underline{a^2 + b^2} \neq 0.$$

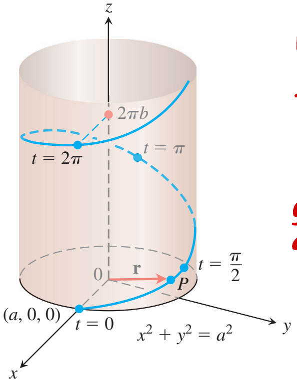


FIGURE 13.22 The helix

$$\vec{v}(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\vec{k}$$

$$|\vec{v}(t)| = \sqrt{a^2+b^2}$$

$$\vec{T}(t) = \frac{-a\sin t}{\sqrt{a^2+b^2}}\vec{i} + \frac{a\cos t}{\sqrt{a^2+b^2}}\vec{j} + \frac{b}{\sqrt{a^2+b^2}}\vec{k}$$

$$\frac{d\vec{T}}{dt} = -\frac{a\cos t}{\sqrt{a^2+b^2}}\vec{i} + \frac{-a\sin t}{\sqrt{a^2+b^2}}\vec{j} + 0\vec{k}$$

$$\vec{N} = -\cos t\,\vec{i} - \sin t\,\vec{j}$$

$$\kappa = \frac{|d\vec{T}/dt|}{|\vec{v}(t)|} = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}}$$

$$= \frac{a}{a^2+b^2}$$