MAT1002: Calculus II

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§10.8 Taylor and Maclaurin Series
§10.9 Convergence of Taylor Series

Recall

Theorem (Theorem 21)

If $\sum c_n(x-a)^n$ *has radius of convergence* $R > 0$*, it defines a function*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \text{ on the interval } a - R < x < a + R$$

The function f *has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series terms by term:*

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2}$$

It tells us that within its interval of convergence, the sum of a power series is a continuous function with derivatives of all orders. How about the other way around?

Series Representation

Assume that f is the sum of a power series about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots$$

$$a_1 + 2a_2(x-a) + \cdots$$

$$f(a) = a_0$$

$$f'(a) = a_1 = 1\,a_1$$

$$f''(a) = 2a_2 = 2\cdot 1\, a_2$$

$$f'''(a) = 6a_3 = 3\cdot 2\, a_3 = 3\cdot 2\cdot 1\, a_3 \Rightarrow a_3 = \frac{f^{(3)}(a)}{3!}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

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- $f(a) = a_0$
- $f'(a) = 1! \cdot a_1$
- $f''(a) = 2! \cdot a_2$
- $f'''(a) = 3! \cdot a_3$

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$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor and Maclaurin Series

Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then **its Taylor series at** $x = a$ is

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}}{n!}(x-a)^n+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The **Maclaurin series of** f is the Taylor series at $x = 0$, or

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k.$$

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Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $x = 2$. When does it converge to $1/x$?

$f(2) = \frac{1}{2}$ $\quad f'(x) = -\frac{1}{x^2}$ $\quad f''(x) = \frac{2}{x^3}$

$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$

$f^{(n)}(2) = (-1)^n \frac{n!}{2^{n+1}}$ $\quad$ Taylor series $\sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^{k+1} k!}(x-2)^k$

$f(x) = \frac{1}{x}$ at $x = 2$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{k+1} \cancel{k!}} (x-2)^k$$

Ratio
$$\frac{\left| \frac{\cancel{(-1)^{k+1}}}{2^{k+2} \cancel{(k+1)!}} (x-2)^{k+1} \right|}{\left| \frac{\cancel{(-1)^{k}}}{2^{k+1} \cancel{k!}} (x-2)^{k} \right|} = \frac{|x-2|}{2 \cancel{(k+1)}} \cancel{\to 0}$$

Let $\frac{|x-2|}{2} < 1 \Rightarrow |x-2| < 2 \Rightarrow 0 < x < 4$

when $x = 0$ $\sum_{k=0}^{\infty} \frac{(-1)^k \cdot (-2)^k}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{2} = +\infty$

$x = 4$ $\sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2}$ does not converge.

Taylor Polynomials

Definition

Let f be a function with derivatives of order up to N in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order** n generated by f at $x = a$ is

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Example: Taylor series and Taylor polynomials generated by $f(x) = e^x$ at $x = 0$

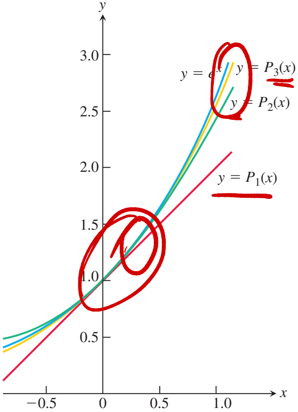

FIGURE 10.17 The graph of $f(x) = e^x$ and its Taylor polynomials

$P_1(x) = 1 + x$

$P_2(x) = 1 + x + (x^2/2!)$

$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!)$.

Notice the very close agreement near the center $x = 0$ (Example 2).

Example: Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$

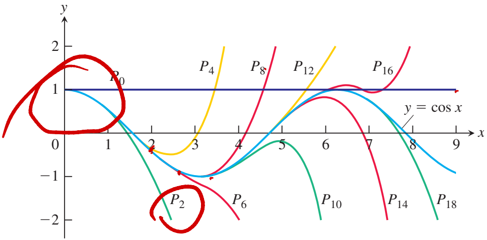

FIGURE 10.18 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{k=0}^{n} \frac{(-x^2)^k}{(2k)!}$$

converge to $\cos x$ as $n \to \infty$. We can deduce the behavior of $\cos x$ arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

Root test:

$$\cancel{\sqrt[n]{\frac{|(-x^2)^k|}{(2k)!}}}$$

Ratio test

$$\frac{\dfrac{|(-x^2)^{k+1}|}{(2k+2)!}}{\dfrac{|(-x^2)^k|}{(2k)!}} = \frac{x^2}{(2k+2)(2k+1)} \to 0$$

Example:

$f(0) = 0$

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

$$\lim_{x \to 0} \frac{e^{-1/x^2} - 0}{x} = \cancel{\lim_{x \to 0} e^{-1/x^2} \cdot 2 \frac{1}{x^3}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{y \to \infty} \frac{y}{e^{y^2}} = \lim_{y \to \infty} \frac{1}{e^{y^2} \cdot 2y} = 0$$

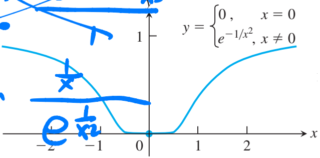

FIGURE 10.19 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4). Therefore its Taylor series, which is zero everywhere, is not the function itself.

$$f'(x) = \begin{cases} 0 & x = 0 \\ e^{-1/x^2} \cdot 2 \frac{1}{x^3} & x \neq 0 \end{cases}$$

$$f''(x) = \begin{cases} 0 \\ \boxed{} \end{cases}$$

$$f^{(n)}(0) = 0$$

Taylor series $\Sigma 0$

$$R_n(x) = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}$$

Questions

- When does the Taylor series converge to its generating function?
- How accurately does a Taylor polynomial approximate the function on a given interval?

Taylor's Theorem

If f and its first n derivatives $f', f'', \cdots, f^{(n)}$ are continuous on an open interval I containing a, and $f^{(n)}$ is differentiable on the same interval, then for given $x \in I$, there exist a number c between a and x such that

$$\begin{aligned} f(x) =& f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &+ \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \\ =& f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &+ R_n(x). \end{aligned}$$

The function $R_n(x)$ is the **remainder of order** n or the **error term** for using $P_n(x)$ approximate f over the interval I.

$$f(x) = f(a) + f'(c)(x-a)$$

$$\frac{f(x) - f(a)}{x-a} = f'(c)$$

$$\frac{f'(x) - f'(a)}{x-a} = f''(c)$$

$$\Rightarrow \quad f'(x) = f'(a) + f''(c)(x-a)$$

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

$$\int_a^x f'(a) + f''(c)(t-a)\,dt$$

$$= f'(a)(x-a) + \frac{1}{2} f''(c)(t-a)^2 \Big|_{t=a}^{t=x}$$

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The function $R_n(x)$ is the **remainder of order** n or the **error term** for using $P_n(x)$ approximate f over the interval I.
If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series P_n generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example: $f(x) = e^x$ at $x = 0$.

$f^{(n)}(x) = e^x$ $f^{(n)}(0) = 1$

Taylor's series $\sum_{k=0}^{\infty} \frac{1}{k!} x^k$

$1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$

$|R_n(x)| = \frac{f^{(n+1)}(c)}{(n+1)!} |x|^{n+1} = \frac{e^c}{(n+1)!} |x|^{n+1}$

$\leq \frac{e^x}{(n+1)!} |x|^{n+1} \to 0$

as $n \to +\infty$

The Remainder Estimation Theorem

Theorem (Theorem 24)

If there is a positive constant M such that $|f^{(n+1)}(t)| \le M$ (n) for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M \frac{|x-a|^{n+1}}{(n+1)!}$$

If this inequality (the same M) holds for every n and the order conditions of Taylor's theorem are satisfied by f, then the series converges to $f(x)$.

Example:

$\sin x$ at $x = 0$.

$f(0) = 0$ $\quad f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$ $\quad f^{(3)}(x) = -\cos x$ $\quad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(0) = 0$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^k = (-1)^{(n-1)/2} & \text{if } n = 2k+1 \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}(x-0)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

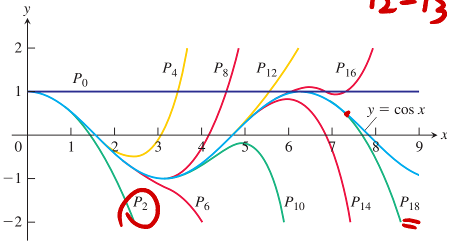
① Ratio test

$$\frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} = \frac{x^2}{(2n+2)(2n+3)} \to 0$$

as $n \to \infty$

For what x can we replace $\cos x$ by $\underbrace{1-(x^2/2!)}_{P_2 = P_3}$ with an error of magnitude no greater than 3×10^{-4}

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$



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$$R_3 = \frac{\cos(c)}{4!}(x-0)^4$$

$$\le \frac{1}{4!} x^4$$

we can let

$$\frac{1}{4!} x^4 \le 3 \times 10^{-4}$$

$$x^4 \le \underbrace{4 \times 3 \times 2 \times 3} \times 10^{-4}$$

$$|x| \le \sqrt[4]{72} \times 10^{-1}$$