MAT1002: Calculus II

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§12.6 Cylinders and Quadric Surfaces
§13.1 Curves in Space and Their Tangents

12.6 Cylinders and Quadric Surfaces

A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder.

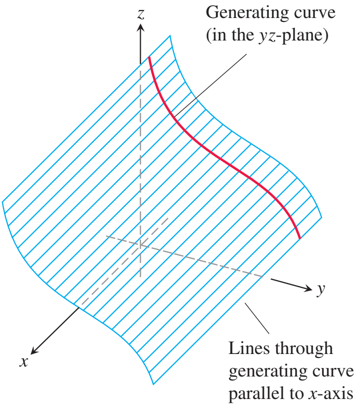

FIGURE 12.43 A cylinder and generating curve.

Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2, \ z = 0$

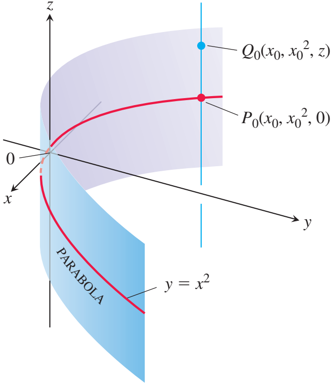

FIGURE 12.44 Every point of the cylinder in Example 1 has coordinates of the form (x_0, x_0^2, z). We call it "the cylinder $y = x^2$."

$y = x^2$

any z values

Quadric Surfaces

A **quadric surface** is the graph in space of a second-degree equaion in x, y, and z. We consider a special case where the equation is in the form

$$Ax^2 + By^2 + Cz^2 + Dz = E$$

It covers **ellipsoids, paraboloid, elliptical cones,** and **hyperboloids**.

$$x^2 + 2xy + 2y^2 = 1$$

$$(x+y)^2 + y^2 = 1$$

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- $$Ax^2 + By^2 + Cz^2 = E$$
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Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

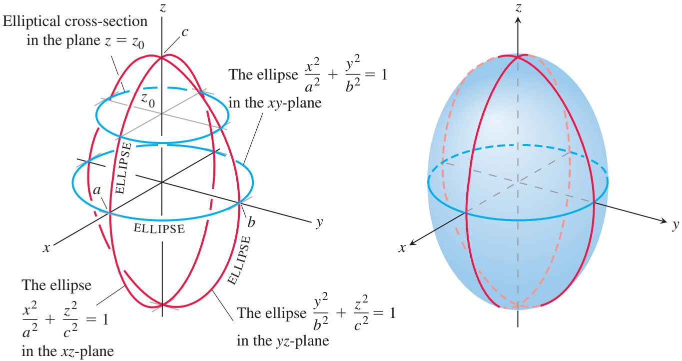


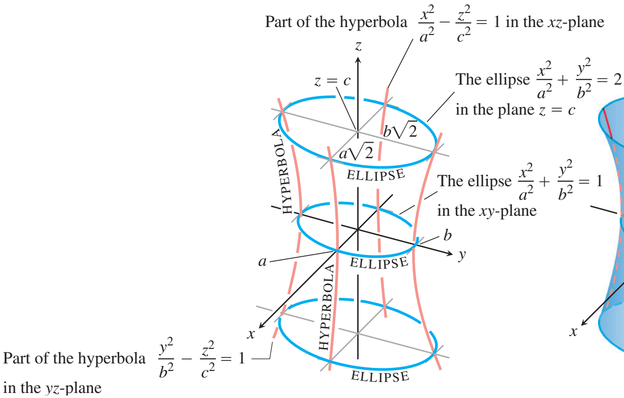
FIGURE 12.45 The ellipsoid

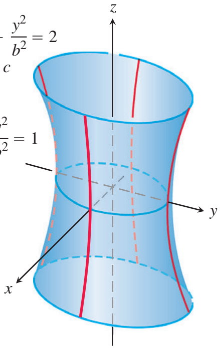
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

Hyperboloid of one sheet

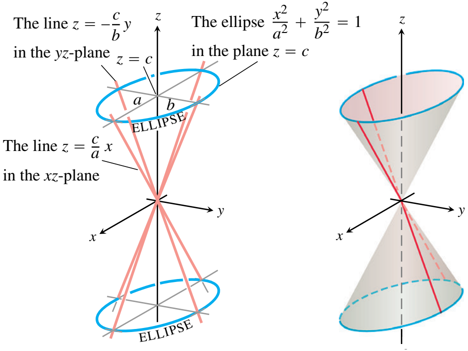
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$





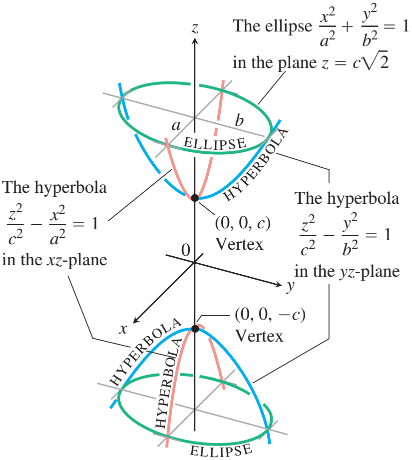
Elliptical cone

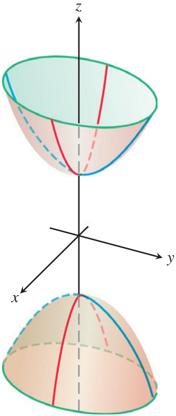
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



Hyperboloid of two sheets

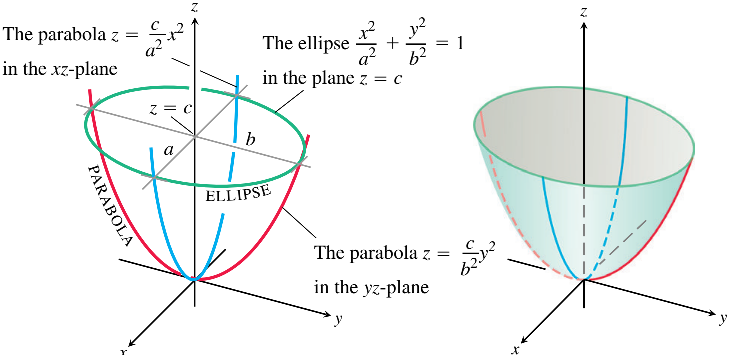
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$





Elliptical paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$



Hyperbolic paraboloid

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}$$

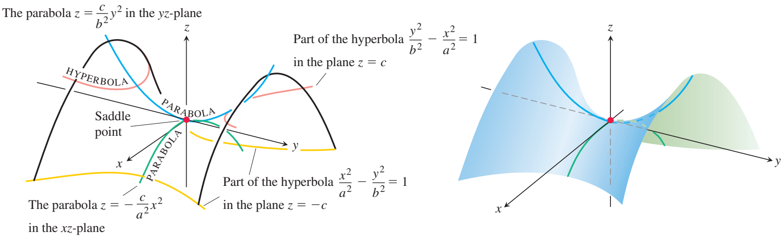


FIGURE 12.46 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c, c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

§13.1 Curves in Space and Their Tangents

Consider

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I.$$

The points $(x, y, z) = (f(t), g(t), z(t)),\ t \in I$ make up the **curve** in space that is called the particle's **path**. It has an equivalent form

$$\vec{r}(t) = \overrightarrow{OP} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

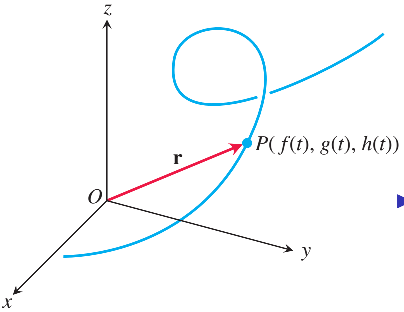


FIGURE 13.1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

- **Component functions**: f, g, and h

Graph the vector function

$$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$$

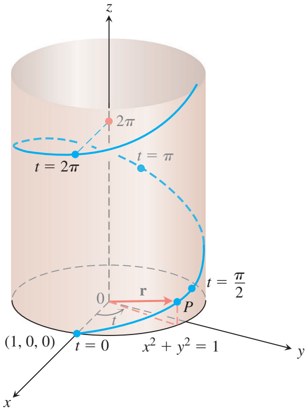


FIGURE 13.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).

Limits and Continuity

Definition

Let $\vec{r}(t) = \overrightarrow{OP} = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ be a vector function with domain D, and $\vec{L}$ a vector. We say that $\vec{r}$ has **limit** $\vec{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\vec{r}(t) - \vec{L}| < \epsilon, \text{ whenever } 0 < |t - t_0| < \delta.$$

Definition

A vector function $\vec{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$. The function is **continuous** if continuous over its interval domain.

Derivatives

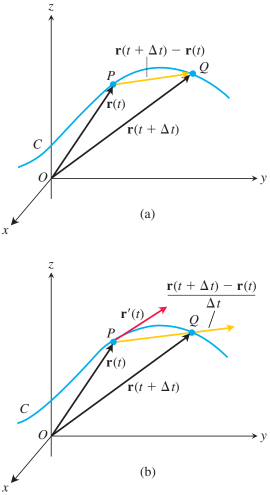


FIGURE 13.5 As $\Delta t \to 0$, the point Q approaches the point P along the curve C. In the limit, the vector $\overrightarrow{PQ}\,/\Delta t$ becomes the tangent vector $\mathbf{r}'(t)$.

Definition

The vector function

$$\vec{r}(t) = f(t)\vec{t} + g(t)\vec{j} + h(t)\vec{k}$$

has a **derivative (is differentiable) at** t if f, g, and h have derivatives at t. The derivative is the vector function

$$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k}$$

Motion

Definition

If $\vec{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\vec{v}(t) = \frac{d\vec{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\vec{v}$ is the **direction of motion**, the magnitude of $\vec{v}$ is the particle's **speed**, and the derivative $\vec{a} = \frac{d\vec{v}}{dt}$ when it exists, is the particle's **acceleration vector.** In summary,

- Velocity is the derivative of position: $\vec{v}(t) = \frac{d\vec{r}}{dt}$
- Speed is the magnitude of velocity $|\vec{v}|$
- The unit vector $\frac{\vec{v}}{|\vec{v}|}$ is the direction of motion at time t
- Acceleration is the derivative of velocity: $\vec{a}(t) = \frac{d\vec{v}}{dt}$

Example: Find the velocity, speed, and acceleration

$$\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + 5\cos^2 t\vec{k}$$

Differentiation Rules

Let $\vec{u}$ and $\vec{v}$ be differentiable vector functions of t, $\vec{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

- Constant Function Rule: $\frac{d}{dt}\vec{C} = \vec{0}$
- Scalar Multiple Rules: $\frac{d}{dt}[c\vec{u}(t)]$
- $\frac{d}{dt}[f(t)\vec{u}(t)]$
- Sum Rule: $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)]$
- Difference Rule: $\frac{d}{dt}[\vec{u}(t) - \vec{v}(t)]$
- Dot Product Rule: $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)]$
- Cross Product Rule: $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)]$
- Chain Rule: $\frac{d}{dt}[\vec{u}(f(t))]$

Vector functions of constant length

If $\vec{r}(t)$ is a differentiable vector function of t of constant length ($|\vec{r}(t)| = c$), then