MAT1002: Calculus II

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§16.5 Surfaces and Area

Curves and surfaces

Curves in the 2D plane

- Explicit form: $y = f(x)$
- Implicit form: $F(x, y) = 0$
- Parametric vector form: $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$, $a \leq t \leq b$

Surfaces in the 3D space

- Explicit form: $z = f(x, y)$
- Implicit form: $F(x, y, z) = 0$.
- Parametric vector form: ?

Parametrizations of surfaces

Suppose that

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$$

is a continuous vector function defined on a region R in the uv-plane and one-to-one on the *interior* of R.

- The range of $\vec{r}$ is the **surface** S defined or traced by $\vec{r}$.
- The equation, together with the domain R, constitutes a **parametrization** of the surface.
- u and v are **parameters**, and R is the **parameter domain**.

Example: Find a parametrization of the cone using (r,θ)

$$z = \sqrt{x^2+y^2}, \quad 0 \le z \le 1.$$

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + \sqrt{x^2+y^2}\,\vec{k}$ $\quad R = \{(x,y) : x^2+y^2 \le 1\}$

$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\,\vec{k}$ $\quad R = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$ using (ϕ, θ).

$$\vec{r}(\phi, \theta) = a \sin\phi \cos\theta \ \vec{i} + a \sin\phi \sin\theta \ \vec{j} + a \cos\phi \ \vec{k}$$

$$R = \left\{ (\phi, \theta) : \begin{array}{l} 0 \leq \phi \leq \pi \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

Find a parametrization of the cylinder

$$\underbrace{x^2 + (y-3)^2 = 9,}\qquad 0 \le z \le 5.$$

$x = 3\cos\theta$

$y = 3 + 3\sin\theta$

$$\vec{r}(\theta, z) = 3\cos\theta\,\vec{i} + (3 + 3\sin\theta)\,\vec{j} + z\,\vec{k}$$

$$R = \left\{ (\theta, z) : \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le z \le 5 \end{array} \right\}$$

Smooth surface

A curved surface S is expressed as

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k},\ a \le u \le b,\ c \le v \le d.$$

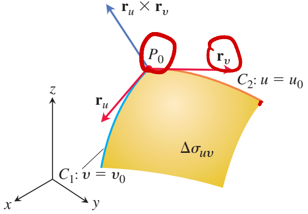
The partial derivatives are

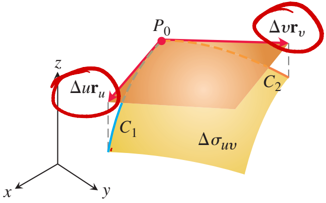
$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial f}{\partial u}\vec{i} + \frac{\partial g}{\partial u}\vec{j} + \frac{\partial h}{\partial u}\vec{k}$$
$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial f}{\partial v}\vec{i} + \frac{\partial g}{\partial v}\vec{j} + \frac{\partial h}{\partial v}\vec{k}$$

Definition

A parametrized surface $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$ is **smooth** if $\vec{r}_u$ and $\vec{r}_v$ are continuous and $\vec{r}_u \times \vec{r}_v$ is never zero on the interior of the parameter domain (make sure that a tangent plane exists).

Surface area





$|\vec{r}_u \times \vec{r}_v|\Delta u \Delta v$

Definition

The **area** of the smooth surface

$$\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k},\ a \le u \le b,\ c \le v \le d$$

is

$$A = \iint_R |\vec{r}_u \times \vec{r}_v| dA = \int_c^d \int_a^b |\vec{r}_u \times \vec{r}_v| du dv = \int_c^d \int_a^b d\sigma.$$

Find the surface area of the cone

$$z = \sqrt{x^2 + y^2},\ 0 \le z \le 1.$$

We have

$$\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + r\vec{k}, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

$$\vec{r}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + 1\vec{k}$$

$$\vec{r}_\theta = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j} + 0\vec{k}$$

$$\vec{r}_r \times \vec{r}_\theta = -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r(\cos^2\theta + \sin^2\theta)\vec{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{r^2 + r^2} = \sqrt{2}\,r$$

$$\int_0^{2\pi}\int_0^1 \sqrt{2}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\,d\theta = \sqrt{2}\pi$$

Find the surface area of a sphere of radius a.

We have

$$\vec{r}(\phi,\theta) = (a\sin\phi\cos\theta)\vec{i} + (a\sin\phi\sin\theta)\vec{j} + (a\cos\phi)\vec{k}, \quad 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi,$$

and $|\vec{r}_\phi \times \vec{r}_\theta| = a^2 \sin\phi$.

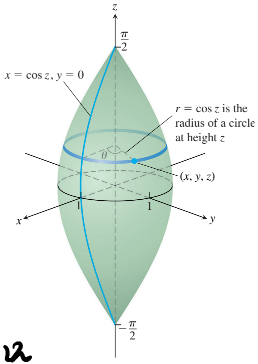
$$\int_0^{\pi}\int_0^{2\pi} a^2 \sin\phi \; d\theta \, d\phi$$

$$= \int_0^{\pi} 2a^2\pi \sin\phi \; d\phi$$

$$= 4\pi a^2$$

Let S be the "football" surface formed by rotating the curve

$x = \cos z,\ y = 0, -\pi/2 \le z \le \pi/2$ around the z-axis. Find a parametrization for S and compute its surface area. Note
$\int \sqrt{1+x^2}dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln(\sqrt{x^2+1}+x) + c$



$$\vec{r}(z,\theta) = \cos z\cos\theta\,\vec{i} + \cos z\sin\theta\,\vec{j} + z\vec{k}$$

$$\vec{r}_z = -\sin z\cos\theta\,\vec{i} - \sin z\sin\theta\,\vec{j} + 1\vec{k}$$

$$\vec{r}_\theta = -\cos z\sin\theta\,\vec{i} + \cos z\cos\theta\,\vec{j} + 0\vec{k}$$

$$\vec{r}_z \times \vec{r}_\theta = -\cos z\cos\theta\,\vec{i} - \cos z\sin\theta\,\vec{j} + (-\sin z\cos\theta\cos z\cos\theta - \sin z\sin\theta\cos z\sin\theta)\vec{k}$$

$$= -\cos z\cos\theta\,\vec{i} - \cos z\sin\theta\,\vec{j} - \sin z\cos z\,\vec{k}$$

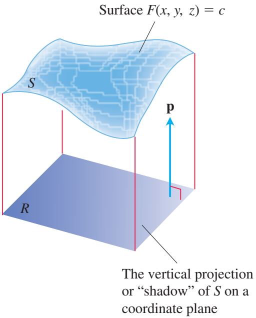
$$|\vec{r}_z \times \vec{r}_\theta| = \sqrt{\cos^2 z + \sin^2 z\cos^2 z}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\sqrt{\cos^2 z + \sin^2 z\cos^2 z}\,d\theta dz$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos z\sqrt{1+\sin^2 z}\,dz \cdot 2\pi$$

$$= \dots$$

Implicit surface



$$F(x, y, z) = c$$

Assume $z = f(x, y)$

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$

$\vec{r}_x = \vec{i} + 0\vec{j} + f_x\vec{k}$

$\vec{r}_y = 0\vec{i} + \vec{j} + f_y\vec{k}$

$|\vec{r}_x \times \vec{r}_y| = |-f_x\vec{i} - f_y\vec{j} + \vec{k}|$

$= \sqrt{(f_x)^2 + (f_y)^2 + 1}$

$= \dfrac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} = \dfrac{|\nabla F|}{|F_z|}$

$F(x, y, f(x,y)) = c$

$F_x + F_z \cdot f_x = 0 \Rightarrow f_x = -\dfrac{F_x}{F_z}$

$f_y = -\dfrac{F_y}{F_z}$

$$d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} dx dy$$

Formula for the surface area of an implicit surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} \, dA$$

where $\vec{p} = \vec{i}$, $\vec{j}$, or $\vec{k}$ is normal to R and $\nabla F \cdot \vec{p} \neq 0$.

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

$z = x^2 + y^2$

$R = \{(x, y) : x^2 + y^2 \le 4\}$

$$d\sigma = \frac{|\nabla F|}{|\nabla F \cdot \vec{k}|} dx\,dy = \frac{\sqrt{4x^2 + 4y^2 + 1}}{1}$$

$$\nabla F = 2x\vec{i} + 2y\vec{j} - 1\vec{k}$$

$$\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\,dy = \int_0^{2\pi} \underbrace{\int_0^2 \sqrt{4r^2 + 1}\, r\,dr}\, d\theta$$

$$= \cdots$$

Derive the surface area differential $d\sigma$ of the surface $z = f(x, y)$ over a region R in the xy-plane (a) parametrically, and (b) implicitly.

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

$$F = f(x, y) - z$$

$$\nabla F = f_x \vec{i} + f_y \vec{j} - \vec{k}$$

$$\frac{|\nabla F|}{|\nabla F \cdot \vec{k}|} = \frac{\sqrt{f_x^2 + f_y^2 + 1}}{1}$$

$$= \sqrt{f_x^2 + f_y^2 + 1}$$