# MAT1002: Calculus II

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§12.1 Three-Dimensional Coordinate Systems
§12.2 Vectors
§12.3 The Dot Product

# Overview of Chapter 12

Chapter 12 is foundational to multivariable calculus.

- an analytic geometry to describe three-dimensional space: 3D coordinate systems and vectors
- study motion in space and vector fields: lines, planes, curves, surfaces,
- applications in science, engineering, operations research, economics, and higher mathematics

# Three-Dimensional Coordinate Systems

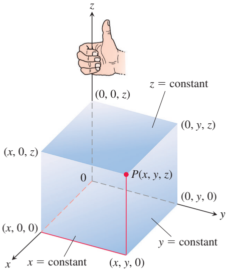



**FIGURE 12.1** The Cartesian coordinate system is right-handed.

# Three-Dimensional Coordinate Systems

- **Cartesian coordinates**: $(x, y, z)$
- $xy$-**plane**: $(x, y, 0)$
- **origin**: $(0, 0, 0)$
- **octants**

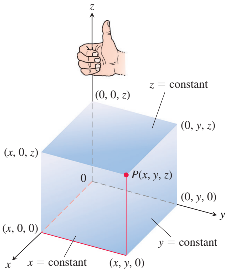



**FIGURE 12.1** The Cartesian coordinate system is right-handed.

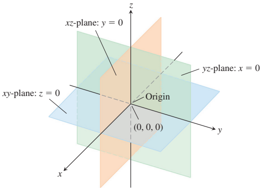



**FIGURE 12.2** The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

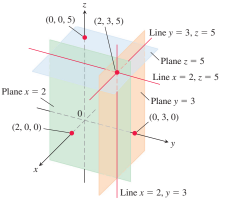



**FIGURE 12.3** The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.

- $z \geq 0$ above and includes xy-plane
- $x = -3$ a plane parallel to yz-plane
- $z = 0,\ x \leq 0,\ y \leq 0$
- $x \geq 0,\ y \geq 0,\ z \geq 0$
- $-1 \leq y \leq 1$
- $y = -2,\ z = 2$
- $x^2 + y^2 = 4,\ z = 3$

## Distance and Spheres in 3D Space

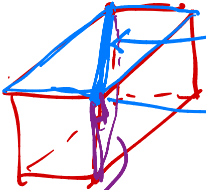
$(x_1, y_1)$ $(x_2, y_2)$

$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_1-y_2)^2+(z_1-z_2)^2}$$

$(x_1, y_1, z_1)$

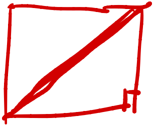


$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$(x_2, y_2, z_1)$

$(x_2, y_2, z_2)$

$(x_2, y_2)$



$(x_1, y_2)$

$$\sqrt{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}^{\,2}+(z_1-z_2)^2}$$

# Distance and Spheres in 3D Space

The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

The Standard Equation for the Sphere of Radius $a$ and Center $(x_0, y_0, z_0)$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

# Distance and Spheres in 3D Space

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The Standard Equation for the Sphere of Radius $a$ and Center $(x_0, y_0, z_0)$

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\underbrace{(x+\tfrac{3}{2})^2}\ \underbrace{+y^2}\ + \underbrace{(z-2)^2} = \underbrace{(\tfrac{3}{2})^2 + 4 \ - 1}$$

$$\frac{9}{4} + 3 = \frac{21}{4}$$

$$(x_0, y_0, z_0) = (-\tfrac{3}{2}, 0, 2) \qquad a = \sqrt{\frac{21}{4}}$$

## Geometric interpretations of inequalities and equations

- $x^2 + y^2 + z^2 < 4$ → solid ball without sphere
- $x^2 + y^2 + z^2 \le 4$ → solid ball with sphere
- $x^2 + y^2 + z^2 \ge 4$
- $x^2 + y^2 + z^2 = 4,\ z \le 0$

↑ half of the sphere not above the xy-plane

# Vector

Vectors represent things with **magnitude** and **direction** in the plane or space.

- **vector/directed line segment**: represent quantity such as force, displacement, velocity

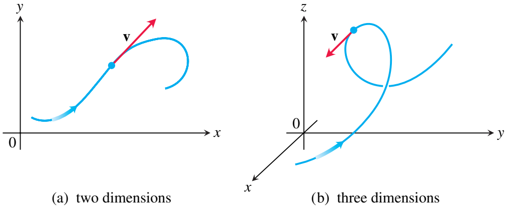


**FIGURE 12.8** The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

## Definition

The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** $A$ and **terminal point** $B$, and its length is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

## Component Form

### Definition

If $\vec{v}$ is a **two-dimensional** vector in the plane equal to the vector with the initial point at the origin and terminal point $(v_1, v_2)$, then the **component form** of $\vec{v}$ is

$$\vec{v} = \langle v_1, v_2 \rangle.$$

If $\vec{v}$ is a **three-dimensional** vector in the space equal to the vector with the initial point at the origin and terminal point $(v_1, v_2, v_3)$, then the **component form** of $\vec{v}$ is

$$\vec{v} = \langle v_1, v_2, v_3 \rangle.$$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, what is the component form of $\overrightarrow{PQ}$?

$$\overrightarrow{PQ} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

## Component Form

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Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, what is the component form of $\overrightarrow{PQ}$?

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The **magnitude** or **length** of the vector $\vec{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

# Vector Algebra Operations

## Definition

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in space and $k$ is a scalar in $R$.

- Addition: $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
- Scalar multiplication: $k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

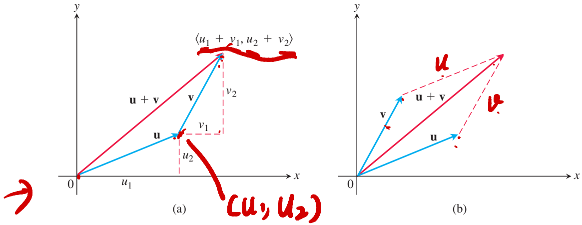



**FIGURE 12.12** (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition in which both vectors are in standard position.

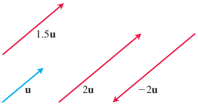
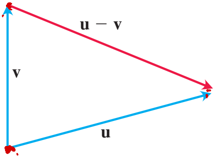



**FIGURE 12.13** Scalar multiples of $\mathbf{u}$.

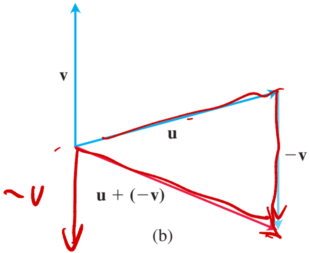
# Difference of Two Vectors $\vec{u} - \vec{v}$

$$(\vec{u} - \vec{v}) + \vec{v} = \vec{u}$$




(a)

$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$$




(b)

**FIGURE 12.14** (a) The vector **u** − **v**, when added to **v**, gives **u**. (b) **u** − **v** = **u** + (−**v**).

# Properties of Vector Operations

Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be vectors and $a$, $b$ are scalars.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- $\vec{u} + \vec{0} = \vec{0} + \vec{u}$ $= \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $0\vec{u} = \vec{0}$
- $1\vec{u} = \vec{u}$
- $a(b\vec{u}) = (ab)\vec{u}$
- $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- $(a + b)\vec{u} = a\vec{u} + b\vec{u}$

# Unit Vectors/Direction

A vector $\vec{v}$ of length 1 is called a **unit vector**. The standard unit vectors are

$$\vec{i} = \langle 1, 0, 0\rangle,\ \vec{j} = \langle 0, 1, 0\rangle,\ \vec{k} = \langle 0, 0, 1\rangle$$

We have

$$\vec{v} = v_1\,\vec{i} + v_2\,\vec{j} + v_3\,\vec{k}$$

$$= \langle v_1, v_2, v_3\rangle$$

$$= \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle = \langle v_1, v_2, v_3\rangle$$

Find a unit vector $\vec{u}$ in the direction of the vector from $P(1, 0, 1)$ to $Q(3, 2, 0)$.

$$\overrightarrow{PQ} = \langle 3-1, 2-0, 0-1\rangle = \langle 2, 2, -1\rangle$$

$$|\overrightarrow{PQ}| = \sqrt{2^2+2^2+(-1)^2} = 3$$

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle$$

$$\left|\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\right| = \sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2} = 1$$

## Length and Direction

If $\vec{v} \neq \vec{0}$, then

- $\frac{\vec{v}}{|\vec{v}|}$ is the direction of $\vec{v}$
- the equation $\vec{v} = |\vec{v}| \frac{\vec{v}}{|\vec{v}|}$ expresses $\vec{v}$ as its length times its direction.

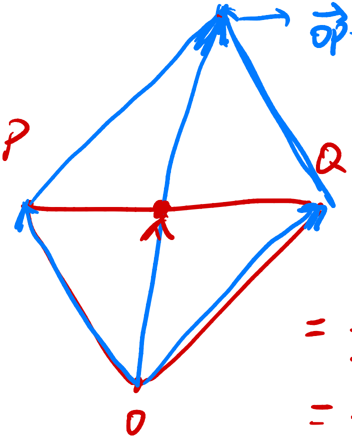
$$\vec{v} = \underbrace{|\vec{v}|}_{\text{Length}} \cdot \underbrace{\frac{\vec{v}}{|\vec{v}|}}_{\text{direction}}$$

unit vector

# Midpoint of a Line Segment

The **midpoint** $M$ of the line segment joining points $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$




$\vec{OP} + \vec{OQ}$

$$\frac{1}{2}(\vec{OP} + \vec{OQ})$$

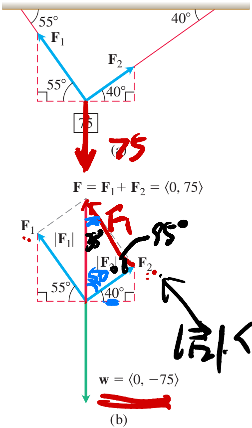
$$= \frac{1}{2}(\langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle)$$

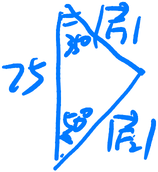
$$= \left\langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right\rangle$$

## Applications

A 75-N weight is suspended by two wires, as shown in Figure 12.18a. Find the forces $\vec{F}_1$ and $\vec{F}_2$ acting in both wires.

First find $|\vec{F}_1|$ $|\vec{F}_2|$




Find directions by the angel

$$\underbrace{F_1 + F_2 = \langle 0, 75 \rangle}$$

$|\vec{F}_2| \langle \cos 40°, \sin 40° \rangle$

**FIGURE 12.18** The suspended weight in Example 9.

# §12.3 The Dot Product

If a force $\vec{F}$ is applied to a particle moving along a path, we often need to know the magnitude of the force in the direction of motion.
If $\vec{v}$ is parallel to the tangent line to the path at the point where $\vec{F}$ is applied, then we want the magnitude of $\vec{F}$ in the direction of $\vec{v}$.

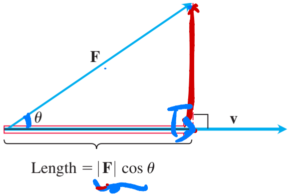
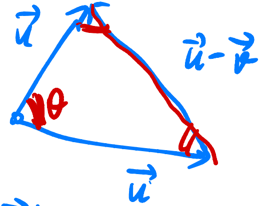



**FIGURE 12.19** The magnitude of the force **F** in the direction of vector **v** is the length $|\mathbf{F}|\cos\theta$ of the projection of **F** onto **v**.

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$




$$-\frac{|\vec{u}-\vec{v}|^2-|\vec{u}|^2-|\vec{v}|}{2|\vec{v}|} = |\vec{u}|\cos\theta$$

# §12.3 The Dot Product

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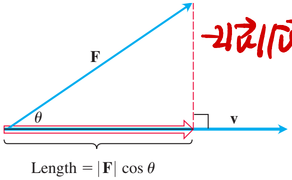



**FIGURE 12.19** The magnitude of the force **F** in the direction of vector **v** is the length $|\mathbf{F}|\cos\theta$ of the projection of **F** onto **v**.

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

$-2|\vec{u}||\vec{v}|\cos\theta$

$= |\vec{u}-\vec{v}|^2 - |\vec{u}|^2 - |\vec{v}|^2$

$= (u_1-v_1)^2 + (u_2-v_2)^2 - (u_1^2+u_2^2) - (v_1^2+v_2^2)$

$= \cancel{u_1^2} - 2u_1v_1 + \cancel{v_1^2} + \cancel{u_2^2} - 2u_2v_2 + \cancel{v_2^2} - \cancel{u_1^2} - \cancel{u_2^2} - \cancel{v_1^2} - \cancel{v_2^2}$

$= -2u_1v_1 - 2u_2v_2$

$-2u_1v_1$

# Angle Between Two Vectors

## Definition

The **dot product (inner product)** $\vec{u} \cdot \vec{v}$ of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

## Theorem (Theorem 1)

*The angle $\theta$ between two* **nonzero** *vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by*

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

$$2|\vec{u}| \cdot |\vec{v}| \cdot \cos\theta = 2\,\vec{u} \cdot \vec{v}$$

Example: Find the angle between $\vec{u} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\vec{v} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

$$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$$

$$\vec{u} \cdot \vec{v} = 1 \times 6 + (-2) \times 3 + (-2) \times 2 = -4$$

$$|\vec{u}| = \sqrt{1 + (-2)^2 + (-2)^2} = 3$$

$$|\vec{v}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$$

$36 + 9 + 4$

$$\cos\theta = \frac{-4}{3 \times 7}$$

$$\theta = \cos^{-1}\left(-\frac{4}{21}\right)$$

Example: Find the angle $\theta$ in the triangle $ABC$ determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$.

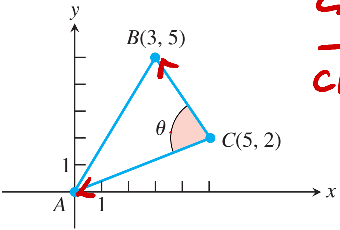



**FIGURE 12.22** The triangle in Example 3.

$$\vec{CA} = \langle 0-5, 0-2\rangle = \langle -5, -2\rangle$$

$$\vec{CB} = \langle 3-5, 5-2\rangle = \langle -2, 3\rangle$$

$$\vec{CA}\cdot\vec{CB} = (-5)\times(-2) + (-2)\times 3$$

$$= 4$$

$$|\vec{CA}| = \sqrt{25+4} = \sqrt{29}$$

$$|\vec{CB}| = \sqrt{4+9} = \sqrt{13}$$

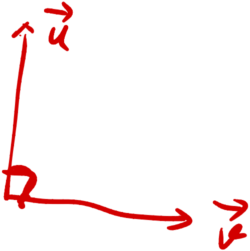
$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{29}\sqrt{13}}\right)$$

# Orthogonal Vectors

## Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$.

$\cos\theta = 0$



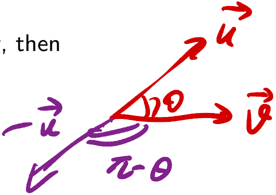
Orthogonal Vectors: $\langle 3, -2 \rangle$ and $\langle 4, 6 \rangle$

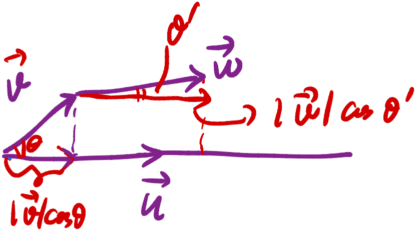
$$3 \times 4 + (-2) \times 6 = 0$$

# Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors and $c$ is a scalar, then

- $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
- $(c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$
- $\boxed{\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}}$
- $\vec{u}\cdot\vec{u} = |\vec{u}|^2$
- $\vec{0}\cdot\vec{u} = 0$



# Projecting one Vector onto another

$\vec{u} \cdot \vec{v}$

The vector projection of $\vec{u}$ onto $\vec{v}$ is the vector

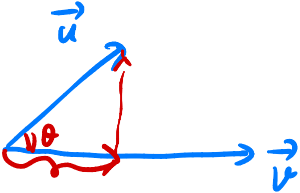
$$\text{proj}_{\vec{v}}\vec{u} = (|\vec{u}|\cos\theta)\frac{\vec{v}}{|\vec{v}|} = \frac{|\vec{u}||\vec{v}|\cos\theta}{|\vec{v}|^2}\,\vec{v}$$

The **scalar component** of $\vec{u}$ in the direction of $\vec{v}$ is the scalar

$$|\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \vec{u}\cdot\frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|}$$

$$= \left(\vec{u}\cdot\frac{\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$




$$|\vec{u}|\cos\theta\,\frac{\vec{v}}{|\vec{v}|}$$

Find the vector projection of $\vec{u} = 6\vec{i} + 3\vec{j} + 2\vec{k}$ onto $\vec{v} = \vec{i} - 2\vec{j} - 2\vec{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

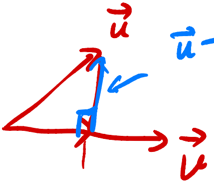
$$\vec{u} \cdot \vec{v} = 6 \times 1 + 3 \times (-2) + 2 \times (-2)$$
$$= -4$$
$$|\vec{v}| = \sqrt{1 + (-2)^2 + (-2)^2} = 3$$
$$\mathrm{Proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{-4}{9} \cdot \langle 1, -2, -2 \rangle$$
$$= \langle -\tfrac{4}{9}, \tfrac{8}{9}, \tfrac{8}{9} \rangle$$

scalar component $\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{-4}{3}$

$$\frac{\vec{v}}{|\vec{v}|} = \langle \tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{-2}{3} \rangle$$

$$\mathrm{Proj}_{\vec{v}} \vec{u} = -\frac{4}{3} \cdot \langle \tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{-2}{3} \rangle = \langle -\tfrac{4}{9}, \tfrac{8}{9}, \tfrac{8}{9} \rangle$$

Verify that $\vec{u} - \text{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$.




$\vec{u} - \text{Proj}_{\vec{v}}\vec{u}$

$$(\vec{u} - \text{Proj}_{\vec{v}}\vec{u}) \cdot \vec{v} = \vec{u}\cdot\vec{v} - \text{Proj}_{\vec{v}}\vec{u} \cdot \vec{v}$$

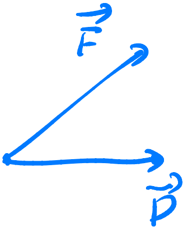
$$= \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\,\frac{\vec{v}}{|\vec{v}|} \cdot \vec{v}$$

$$= \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} = 0$$

# Work (More in Chapter 16)

The **work** done by a constant force $\vec{F}$ acting through a displacement $\vec{D}$ is

$$W = \vec{F} \cdot \vec{D}$$




$$\mathrm{Proj}_{\vec{D}} \vec{F} \cdot \vec{D} = \left( \frac{\vec{F} \cdot \vec{D}}{|\vec{D}|^2} \vec{D} \right) \cdot \vec{D}$$

$$= \frac{\vec{F} \cdot \vec{D}}{|\vec{D}|^2} \left( \vec{D} \cdot \vec{D} \right) = \vec{F} \cdot \vec{D}$$