MAT1002: Calculus II

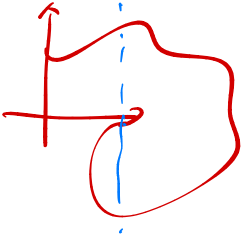
Ming Yan

§11.1 Parametrizations of Plane Curves
§11.2 Calculus with Parametric Curves
§11.3 Polar Coordinates
§11.4 Graphing Polar Coordinate Equations

Overview of Chapter 11

$f(x)$

- new ways to define curves in the plane (on the y-axis)
- *polar coordinates* using r and θ $(r\cos\theta, r\sin\theta)$
- geometric definitions and standard equations of parabolas, ellipses, and hyerperbolas



Parametric Equations: expressing both x and y as functions of t

Definition

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of $t-$values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations**.

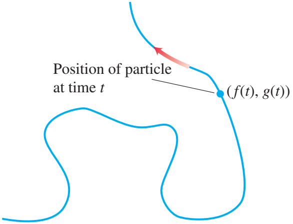


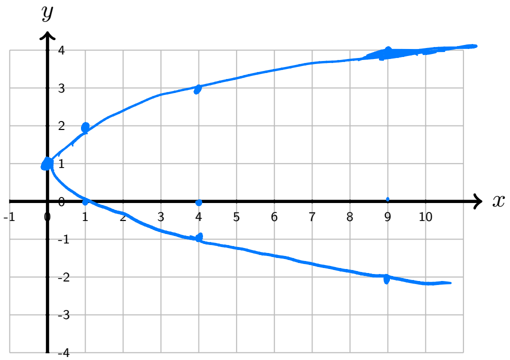
FIGURE 11.1 The curve or path traced by a particle moving in the xy-plane is not always the graph of a function or single equation.

- t: **parameter** for the curve
- I: **parameter interval**
- If $I = [a, b]$: $(f(a), g(a))$ and $(f(b), g(b))$ are the **initial point** and the **terminal point** of the curve.

Example: $x = t^2,\ y = t + 1$ for $t \in R$

$x = (t-1)^2$

$y = t$

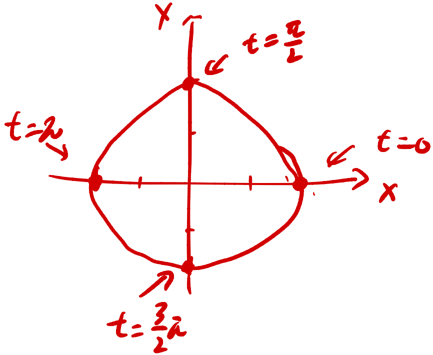


Eliminate t:

$y = t + 1 \Rightarrow t = y - 1$

$x = t^2 = (y-1)^2$

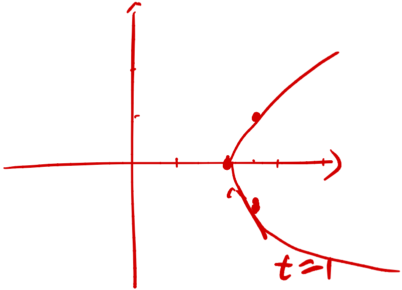
Example: $x = 2\cos t,\ y = 2\sin t$ for $t \in [0, 2\pi]$



$$x^2+y^2 = 4\cos^2 t + 4\sin^2 t = 4$$

Remark

A given curve can be represented by different parametrizations.
$\left(x = t + \frac{1}{t},\ y = t - \frac{1}{t},\ t > 0\right)$



	x	y
$t=2$	2.5	1.5
$t=\frac{1}{2}$	2.5	-1.5
$t=3$	$3+\frac{1}{3}$	$3-\frac{1}{3}$
$t=\frac{1}{3}$	$3+\frac{1}{3}$	$\frac{1}{3}-3$

$$y = t - \frac{1}{t} \Rightarrow t^2 - yt - 1 = 0$$

$$\Rightarrow t = \frac{y \pm \sqrt{y^2+4}}{2}$$

$$x = t + \frac{1}{t} = \frac{y+\sqrt{y^2+4}}{2} + \frac{2}{y+\sqrt{y^2+4}} = \frac{y^2+y^2+4+2y\sqrt{y^2+4}}{2\left(y+\sqrt{y^2+4}\right)}$$

$$= y + \frac{4}{y+\sqrt{y^2+4}}$$

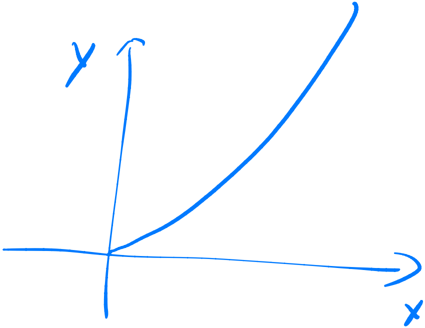
$$= y + \frac{\left(\sqrt{y^2+4}+y\right)\left(\sqrt{y^2+4}-y\right)}{\sqrt{y^2+4}+y}$$

$$= \sqrt{y^2+4} \quad \#$$

Remark

A given curve can be represented by different parametrizations.
$(x = \sqrt{t},\ y = t,\ t \geq 0)$

$x = t \quad y = t^2 \quad \Rightarrow \quad y = x^2$



Cycloids

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is call a **cycloid**.

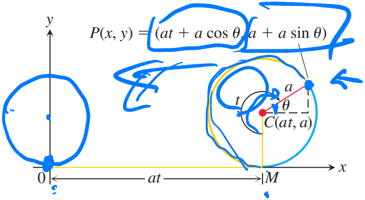


FIGURE 11.8 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

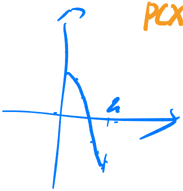
$$t + \theta = \frac{3\pi}{2}$$

a

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right)$$
$$= \cos\left(\frac{\pi}{2} - (t - \pi)\right)$$
$$= \sin(t - \pi)$$
$$= -\sin(\pi - t)$$
$$= -\sin t$$

$$\sin\theta = \sin\left(\frac{3\pi}{2} - t\right)$$
$$= \cos(t - \pi)$$
$$= \cos(\pi - t) = -\cos t$$

$$P(x, y) = (at - a\sin t,\ a - a\cos t)$$



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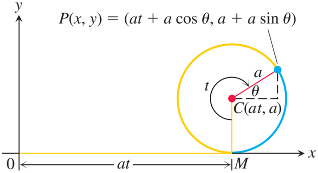

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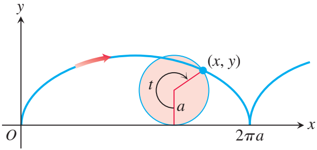

FIGURE 11.9 The cycloid curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$, for $t \geq 0$.

Cycloids

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center (its lowest position).

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This does not happen if the bob can be made to swing in a cycloid. In 1673, Christian Huygens designed a pendulum clock whose bob would swing in a cycloid.

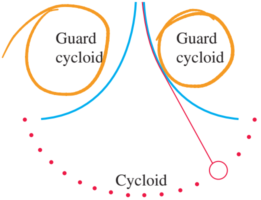

FIGURE 11.7 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

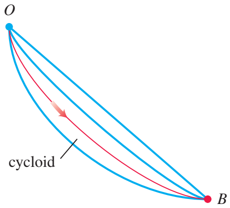

FIGURE 11.11 The cycloid is the unique curve which minimizes the time it takes for a frictionless bead to slide from point O to B.

§11.2 Preview

- slope
- length
- area

Tangents and Areas

A parameterized curve $(x, y) = (f(t), g(t))$ is **differentiable** at t if f and g are differentiable at t. If at a point on a differentiable parameterized curve, y is also a differentiable function of x, then we have the Chain Rule:

$$\boxed{\frac{dy}{dt}} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = ?$$

$$\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt}$$

$$= \frac{d\left(\frac{dy/dt}{dx/dt}\right)/dt}{dx/dt}$$

$$y^{(t)} = y(x) = y(x(t))$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$g'(t) = \frac{dy}{dx} f'(t)$$

$$\Rightarrow \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

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$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$$

Example: $(x, y) = (\sec t, \tan t)$ at the point $(\sqrt{2}, 1)$

$\tan t = \dfrac{\sin t}{\cos t}$

$t = \frac{\pi}{4}$

$\sec = 1/\cos$

- Slope

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{\cos t \cdot \cos t + \sin t \sin t}{\cos^2 t}}{-\frac{-\sin t}{\cos^2 t}} = \frac{1}{\sin t}$$

Let $t = \frac{\pi}{4}$

$$\frac{1}{\sin\frac{\pi}{4}} = \sqrt{2}$$

- Tangent line

$$(y-1) = \sqrt{2}(x - \sqrt{2})$$

$$y = 1 + \sqrt{2}(x - \sqrt{2})$$

$$= \sqrt{2}x - 1$$

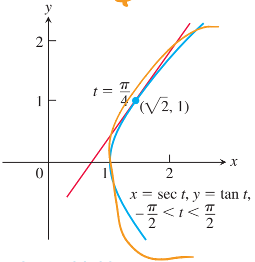


FIGURE 11.13 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

Find d^2y/dx^2 as a function of t if $x = t - t^2$ and $y = t - t^3$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$$

$$\frac{d^2y}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\left(\frac{dy}{dx}\right)/dt}{dx/dt}$$

$$\frac{d(dy/dx)}{dt} = \frac{-6t(1-2t) - (1-3t^2)\cdot(-2)}{(1-2t)^2} = \frac{-6t+12t^2+2-6t^2}{(1-2t)^2}$$

$$= \frac{6t^2-6t+2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{6t^2-6t+2}{(1-2t)^2}}{1-2t} = \frac{6t^2-6t+2}{(1-2t)^3}$$

Example: Find the area enclosed by the asteroid.

$\cos 2t = 1 - 2\sin^2 t$

$\sin^2 t = \dfrac{1-\cos 2t}{2}$

$$A = 4\int_0^1 y(x)dx$$

$x = \cos^3 t \qquad t \in [0, \frac{\pi}{2}]$

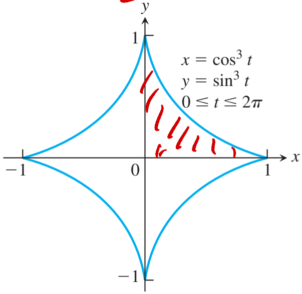


FIGURE 11.14 The astroid in Example 3.

$= 4\int_{\frac{\pi}{2}}^{0} y(x(t))\,\dfrac{dx}{dt}\,dt$

$= 4\int_{\frac{\pi}{2}}^{0} \sin^3 t \cdot 3\cos^2 t \cdot (-\sin t)\,dt$

$= 12\int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t\,dt$

$= 12\int_0^{\frac{\pi}{2}} \frac{1}{4}\sin^2 2t\,\dfrac{1-\cos 2t}{2}\,dt = \cdots$

Length of a Parametrically Defined Curve

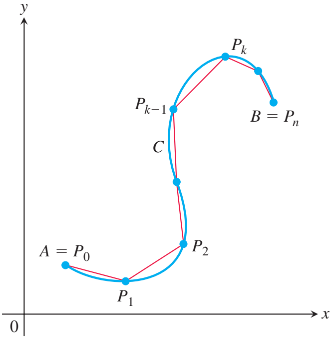


FIGURE 11.15 The length of the smooth curve C from A to B is approximated by the sum of the lengths of the polygonal path (straight-line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

$y = f(x)$

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$\int_\alpha^\beta \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

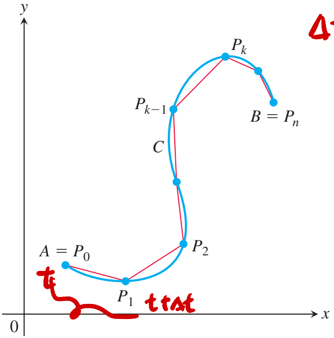
$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Length of a Parametrically Defined Curve

$P_r: (f(t+\Delta t),\ g(t+\Delta t))$

$P_0: f(t),\ g(t)$

$$\Delta t \sqrt{\left(\frac{f(t+\Delta t)-f(t)}{\Delta t}\right)^2 + \left(\frac{g(t+\Delta t)-g(t)}{\Delta t}\right)^2}$$



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Definition

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and *not simultaneously zero* on $[a, b]$, and C is traversed *exactly once* as t increases from $t = a$ to $t = b$, then the **length of** C is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt.$$

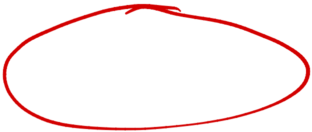
Example: Find the length of $(x, y) = (r\cos t, r\sin t)$ for $0 \le t \le 2\pi$

$2\pi r$

$$\int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\ dt$$

$$= \int_0^{2\pi} r\ dt = 2\pi r$$

Find the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Do not need the value).



$x = a\cos t \qquad y = b\sin t$

$$\int_0^{2\pi} \sqrt{a^2\sin^2 t\, t \quad b^2\cos^2 t} \quad dt$$

Length of a Curve $y = f(x)$: $\int \sqrt{1 + (f'(x))^2}\, dx$

$x = \underline{t}$

$y = \underline{f(t)}$

$$\int \sqrt{1 + (f'(t))^2}\, dt$$

The Arc Length Differential

The arc length function is

$$s(t) = \int_a^t \sqrt{[f'(z)]^2 + [g'(z)]^2}\, dz.$$

$$ds = \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

$$\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

Example: Find the centroid of the first quadrant arc

$\{0, \frac{\pi}{2}\}$

$$\left(\frac{M_x}{M}, \frac{M_y}{M}\right)$$

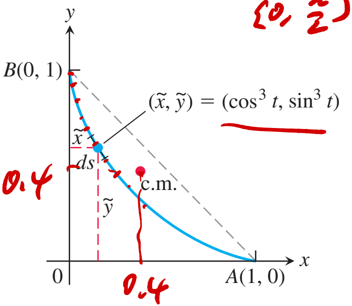


FIGURE 11.17 The centroid (c.m.) of the astroid arc in Example 7.

$$ds = 3\cos t \sin t\, dt$$

$$\boxed{M = \int_0^{\frac{\pi}{2}} 1\, ds} = \int_0^{\frac{\pi}{2}} 3\cos t \sin t\, dt = \frac{3}{2}\sin^2 t \Big|_0^{\frac{\pi}{2}} = \frac{3}{2}$$

$$M_x = \int_0^{\frac{\pi}{2}} y(t)\, ds$$

$$\int_0^{\frac{\pi}{2}} \sin^3 t \; 3\cos t \sin t \; dt$$

$$= \frac{3}{5}\sin^5 t \Big|_0^{\frac{\pi}{2}} = \frac{3}{5}$$

Areas of Surfaces of Revolution

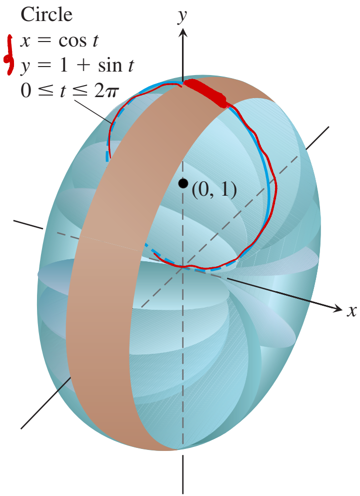


FIGURE 11.18 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

$$\int 2\pi y \cdot \cancel{\Delta s}\ ds$$

$$ds = \sqrt{(-\sin t)^2 + \cos^2 t}\ dt$$

$$= dt$$

$$\int_0^{2\pi} 2\pi (1 + \sin t)\ dt$$

$$= 4\pi^2$$

§11.3 and §11.4

- Polar coordinates and Cartesian coordinates
- Symmetry
- Slope

Definition of Polar Coordinates

We fix an **origin** O (call the **pole**) and an **initial ray** from O. Then each point P corresponds to a **polar coordinate pair** (r, θ), in which r is the directed distance from O to P, and θ is the directed angle from the initial ray to ray OP. So, we label the point P as

$$P(r, \theta)$$

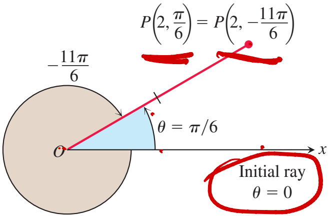


FIGURE 11.20 Polar coordinates are not unique.

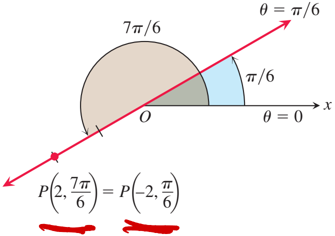
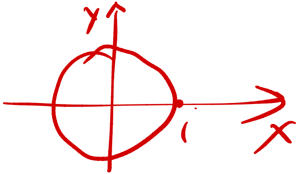


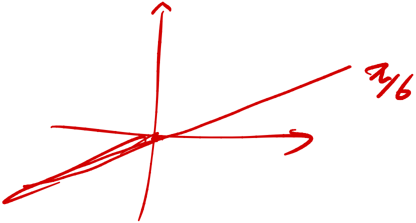
FIGURE 11.21 Polar coordinates can have negative r-values.

Polar Equations

- $r = 1$ and $r = -1$



- $\theta = \pi/6$ and $\theta = 7\pi/6$



Polar Equations

- $1 \leq r \leq 2,\ 0 \leq \theta \leq \pi/2$

- $2\pi/3 \leq \theta \leq 5\pi/6$

Relation between Polar and Cartesian Coordinates

$$x = r\cos\theta,\;\; y = r\sin\theta,$$

Relation between Polar and Cartesian Coordinates

$$x = r\cos\theta,\ \ y = r\sin\theta,$$

- $r^2 \cos\theta \sin\theta = 4$
- $r = \frac{4}{2\cos\theta - \sin\theta}$
- $r = 1 + 2r\cos\theta$