MAT1002: Calculus II

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§14.4 The Chain Rule
§14.5 Directional Derivatives and Gradient Vectors

§14.4 The Chain Rule (proof)

If $w = f(x)$ and $x = g(t)$ are differentiable functions, let $w(t) = f(g(t))$ and we have

$$\frac{dw}{dt} = f'(g(t))g'(t) = \frac{dw}{dx}\frac{dx}{dt}$$

§14.4 The Chain Rule (proof)

Theorem (Theorem 5 Chain Rule For Functions of One Independent Variable and Two Intermediate Variables)

If $w = f(x, y)$ *is differentiable and if* $x = x(t), y = y(t)$ *are differentiable functions of* t*, then the composite* $w = f(x(t), y(t))$ *is a differentiable function of* t *and*

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{dw}{dt} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t+\Delta t), y(t))}{\Delta t}$$

$$+ \underbrace{\lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t)) - f(x(t), y(t))}{\Delta t}}_{f_x(x(t), y(t))\, x'(t)}$$

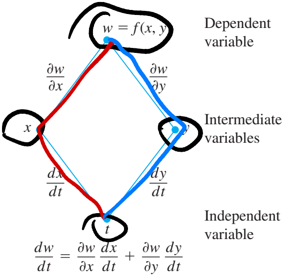
$$\lim_{\Delta t \to 0} f_y(x(t+\Delta t), y(t))\, y'(t)$$

$$= f_y(x(t), y(t)) \cdot y'(t)$$

Chain Rule

To remember the Chain Rule, picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$w = xy$ and $x = \cos t$, $y = \sin t$ at $t = \pi/2$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} x'(t) + \frac{\partial w}{\partial y} y'(t)$$

$$= y\, x'(t) + x\, y'(t)$$

$$= \sin t \cdot (-\sin t) + \cos t \cdot \cos t$$

$$= \cos^2 t - \sin^2 t$$

$$w = xy = \cos t \sin t$$

$$\frac{dw}{dt} = -\sin t \cdot \sin t + \cos t \cdot \cos t$$

Let $t = \pi/2$ $\quad \sin \pi/2 = 1$

$\cos \pi/2 = 0$

$$\frac{dw}{dt}(\pi/2) = -1$$

Functions of Three Variables

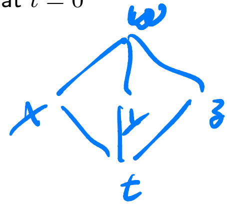
Theorem (Theorem 6 Chain Rule For Functions of One Independent Variable and Three Intermediate Variables)

If $w = f(x, y, z)$ is differentiable and if $x = x(t), y = y(t), z = z(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t), z(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Find dw/dt for

$$w = xy + z, \quad x = \cos t,\ y = \sin t,\ z = t,\ \text{at } t = 0$$



$$\frac{dw}{dt} = y\cdot(-\sin t) + x\cos t + 1\cdot 1$$
$$= -\sin^2 t + \cos^2 t + 1$$

Let $t=0$

$$\frac{dw}{dt}(0) = 1+1 = 2$$

Functions Defined on Surfaces

Theorem (Theorem 7 Chain Rule for Two Independent Variables and Three Intermediate Variables)

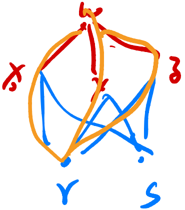
Suppose that

$$w = f(x, y, z) = f(g(r, s), h(r, s), k(r, s)).$$

If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}$$



Functions Defined on Surfaces

Theorem (Theorem 7 Chain Rule for Two Independent Variables and Three Intermediate Variables)

Suppose that

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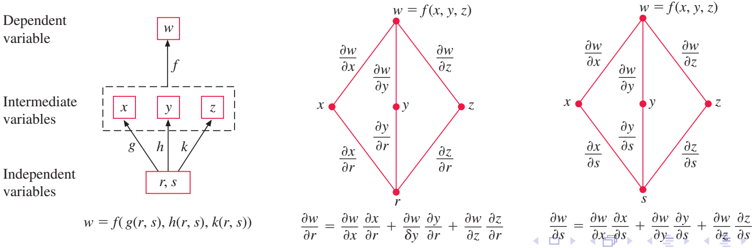
If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by

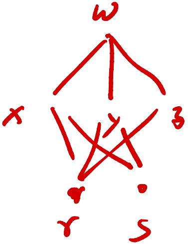
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



Example: $w = \frac{r}{s} + 2r^2 + 2\ln s + 4r^2$

$$w = x + 2y + z^2,\ x = \frac{r}{s},\ y = r^2 + \ln s,\ z = 2r$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2$$

$$= \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= 1 \cdot \left(-\frac{r}{s^2}\right) + 2 \cdot \frac{1}{s} = -\frac{r}{s^2} + \frac{2}{s}$$

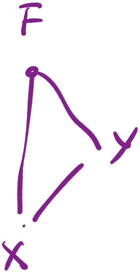
Implicit Differentiation Revisited

Suppose that

- The function $F(x, y)$ is differentiable and
- The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Find dy/dx

$$\underbrace{F(x, h(x)) = 0}$$



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Implicit Differentiation Revisited

Suppose that

- The function $F(x, y)$ is differentiable and
- The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Find dy/dx

Theorem (Theorem 8)

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example

$$\underbrace{y^2 - x^2 - \sin xy}_{} = 0$$

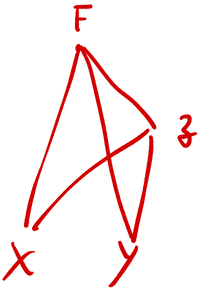
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - \cos xy \cdot y}{2y - \cos xy \cdot x} = \frac{2x + \cos xy \cdot y}{2y - \cos xy \cdot x}$$

Three Dimension

Suppose that

- The function $F(x, y, z)$ is differentiable and
- The equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, say $z = f(x, y)$.

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$



$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Three Dimension

Suppose that

- The function $F(x, y, z)$ is differentiable and
- The equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, say $z = f(x, y)$.

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

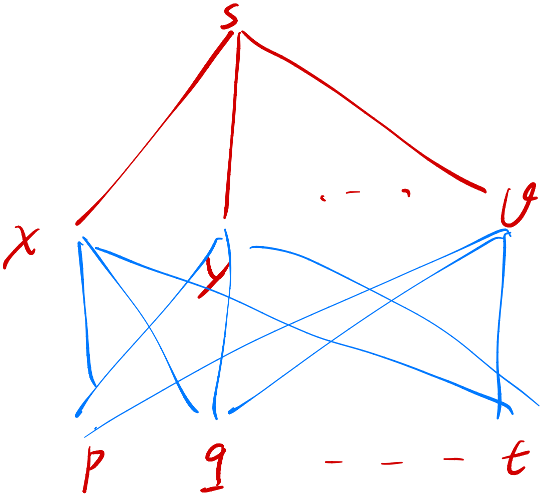
$$x^3 + z^2 + ye^{xz} + z\cos y = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + ye^{xz}z}{2z + ye^{xz}x + \cos y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + ye^{xz}x + \cos y}$$

Chain Rule for Functions of Many Variables

$s = f(x, y, \cdots, v)$ and $x, y, \cdots, v$ are functions of $p, q, \cdots, t$



§14.5 Directional Derivatives and Gradient Vectors

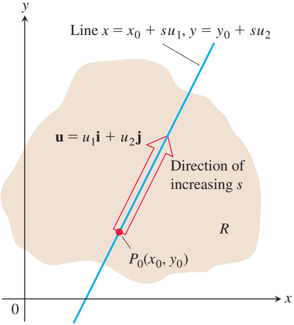


FIGURE 14.27 The rate of change of f in the direction of $\mathbf{u}$ at a point P_0 is the rate at which f changes along this line at P_0.

Definition

The **derivative of** f **at** $P_0(x_0, y_0)$ **in the direction of the unit vector** $\vec{u} = u_1\vec{i} + u_2\vec{j}$ is the number

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

The **directional derivative** is also denoted as

$$(D_{\vec{u}}f)_{P_0}$$

$$f(x_0+su_1, y_0+su_2)$$

$$(D_{\vec{u}}f)_{P_0} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$= \frac{\partial f}{\partial x}\cdot u_1 + \frac{\partial f}{\partial y}\cdot u_2 = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle u_1, u_2 \rangle$$

Interpretation of the Directional Derivative

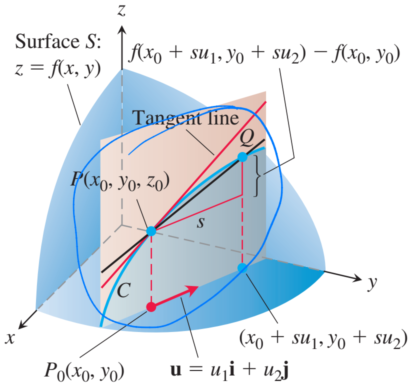


FIGURE 14.28 The slope of the trace curve C at P_0 is $\lim\limits_{Q \to P}$ slope (PQ); this is the directional derivative

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (D_{\mathbf{u}} f)_{P_0}.$$

Find the derivative of

$$f(x,y) = x^2 + xy$$

at $P_0(1,2)$ in the direction of the unit vector $\vec{u} = (1/\sqrt{2})\vec{i} + (1/\sqrt{2})\vec{j}$

$$f\left(1+\frac{1}{\sqrt{2}}s,\ 2+\frac{1}{\sqrt{2}}s\right) = \left(1+\frac{s}{\sqrt{2}}\right)^2 + \left(1+\frac{s}{\sqrt{2}}\right)\left(2+\frac{s}{\sqrt{2}}\right)$$

$$= 1 + \frac{s^2}{2} + \sqrt{2}s + 2 + \sqrt{2}s + \frac{s}{\sqrt{2}} + \frac{s^2}{2}$$

$$= \underbrace{3 + \left(2\sqrt{2}+\frac{1}{\sqrt{2}}\right)s + s^2}$$

$$\left.\frac{df}{ds}\right|_{s=0} = \left(2\sqrt{2}+\frac{1}{\sqrt{2}}\right) + 2s\Big|_{s=0} = 2\sqrt{2}+\frac{1}{\sqrt{2}}$$

$$\left.\frac{\partial f}{\partial x}\right|_{P_0} = 2x+y = 4$$

$$\left.\frac{\partial f}{\partial y}\right|_{P_0} = x = 1$$

$$\frac{\partial f}{\partial x}\cdot\frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}\frac{1}{\sqrt{2}}$$

$$= 4\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2}+\frac{1}{\sqrt{2}}$$

Calculation and Gradients

Definition

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Let $(x, y) = (x_0, y_0) + s(u_1, u_2)$,

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = \quad \nabla f \cdot \langle u_1, u_2 \rangle$$

Calculation and Gradients

Definition

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

obtained by evaluating the partial derivatives of f at P_0.

Let $(x, y) = (x_0, y_0) + s(u_1, u_2)$,

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} =$$

Theorem (Theorem 9)

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0} = (\nabla f)_{P_0} \cdot \vec{u},$$

the dot product of the gradient ∇f at P_0 and $\vec{u}$. In brief, $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.

Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\vec{v} = 3\vec{i} - 4\vec{j}$

- Normalize $\vec{v}$

$$|\vec{v}| = \sqrt{3^2 + (-4)^2} = 5$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$

- Find ∇f

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = (e^y - \sin(xy)\cdot y)\vec{i} + (xe^y - \sin(xy)x)\vec{j}$$

Let $(x,y) = (2,0)$

$$\nabla f = 1\vec{i} + 2\vec{j}$$

- $$(D_{\vec{u}} f)_{P_0} = \langle 1, 2 \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = \frac{3}{5} - \frac{8}{5} = -1$$

Properties

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is

$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.

3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Find the directions in which $f(x,y) = (x^2/2) + (y^2/2)$

- increases most rapidly at the point $(1,1)$. $\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$
- decreases most rapidly at the point $(1,1)$ $-\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$
- what are the directions of zero change in f at $(1,1)$

$\nabla f = x\vec{i} + y\vec{j}$

If $(x,y) = (1,1)$, $\nabla f = \vec{i} + \vec{j}$

$\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

$\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

Gradients and Tangents to Level Curves

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\vec{r} = g(t)\vec{i} + h(t)\vec{j}$

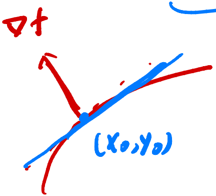
$$f(\vec{r}(t)) = c$$

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

$$\langle f_x, f_y \rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = 0 \Rightarrow \nabla f(g(t), h(t)) \cdot \frac{d\vec{r}}{dt} = 0$$

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0).

What is the $\underbrace{\textbf{tangent line to the level curve}}$ of $f(x, y)$ at $P_0(x_0, y_0)$?



$$\nabla f \cdot \langle x - x_0, y - y_0 \rangle = 0$$

$$\nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle = 0$$

Example: Find an equation for the tangent line to the ellipse

$$\underbrace{\frac{x^2}{4} + y^2 = 2}$$

at the point $(-2, 1)$.

level curve of $f(x,y) = \frac{x^2}{4} + y^2$ for $c = 2$

$\nabla f(x,y) = \langle \frac{x}{2}, 2y \rangle$

let $\langle x_0, y_0 \rangle = \langle -2, 1 \rangle$

then $\nabla f(x_0, y_0) = \langle -1, 2 \rangle$

the equation is $-1(x-(-2)) + 2(y-1) = 0$

$\Rightarrow -x - 2 + 2y - 2 = 0 \Rightarrow -x + 2y = 4$

Algebra Rules for Gradient

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

Example: $\nabla(fg)$

$$f(x,y) = x - y, \qquad g(x,y) = 3y$$

$$f(x,y)\, g(x,y) = (x-y)\cdot 3y = \underline{3xy - 3y^2}$$

$$\nabla(fg) = \langle 3y \;, 3x - 6y \rangle$$

$$\nabla(fg) = \underline{\nabla f}\cdot g + f \cdot \underline{\nabla g}$$

$$\nabla f = \langle 1, -1 \rangle$$

$$\nabla g = \langle 0, 3 \rangle$$

$$\nabla(fg) = \langle 1, -1 \rangle\, 3y + (x-y)\langle 0, 3 \rangle$$

$$= \langle 3y, \underline{-3y + 3(x-y)} \rangle$$

$$= \langle 3y, 3x - 6y \rangle$$

$\nabla f = \langle 3x^2 - y^2, -2xy, -1 \rangle$

$|\vec{v}| = \sqrt{4 + 9 + 36} = 7$

Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$. In what directions do f change most rapidly at P_0, and what is the rate of change in these directions?

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{7}\vec{i} - \frac{3}{7}\vec{j} + \frac{6}{7}\vec{k}$

$\nabla f(1,1,0) = \langle 2, -2, -1 \rangle$

$\nabla f(x) \cdot \vec{u} = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$

Chain Rule for Paths

$$w = f(\vec{r}(t)) = f(x(t), y(t), z(t))$$

Find the derivative along the path

$$\frac{d}{dt} f(\vec{r}(t)) = \quad f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$