MAT1002: Calculus II

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§16.3 Path Independence, Conservative Fields, and Potential Functions
§16.4 Green's Theorem in the Plane

Line integrals in conservative fields

If A and B are two points in an open region D, the line integral of a field $\vec{F}$, defined on D, along C from A to B, usually depends on the path C. For some fields, however, the integral's value is the same for all paths from A to B.

Definition

If $\vec{F}$ is a vector field defined on D and $\vec{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for** $\vec{F}$.

Let the path C from A to B be $\vec{r}(t)$ for $a \leq t \leq b$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_a^b \underbrace{\nabla f(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}}_{\frac{d\, f(\vec{r}(t))}{dt}}\, dt$$

$$= f(\vec{r}(t)) \Big|_{t=a}^{t=b} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Line integrals in conservative fields

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Let the path C from A to B be $\vec{r}(t)$ for $a \leq t \leq b$.

$$\int_C \vec{F} \cdot d\vec{r} =$$

Theorem (Theorem 1)

Let C be a smooth curve joining the point A to the point B in the plane or space and parametrized by $\vec{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\vec{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

Suppose the force field $\vec{F} = \nabla f$ is the gradient of the function

$$f(x,y,z) = -\frac{1}{x^2+y^2+z^2}.$$

Find the work done by $\vec{F}$ in moving an object along a smooth curve C joining $(1,0,0)$ to $(0,0,2)$ that does not pass through the origin.

$$f(0,0,2) - f(1,0,0)$$

$$= -\frac{1}{4} - \left(-\frac{1}{1}\right) = \frac{3}{4}$$

Path independence

Definition

Let $\vec{F}$ be a vector field defined on an open region D, and suppose that for any two points A and B in D, the line integral $\int_C \vec{F} \cdot d\vec{r}$ along a path C from A to B is the same over all paths from A to B. Then the integral $\int_C \vec{F} \cdot d\vec{r}$ is **path independent in** D and the field $\vec{F}$ is **conservative on** D.

To make sure that the following computations and results are valid, we assume

- the curves are **piecewise smooth**.
- the vector fields $\vec{F}$ have continuous first partial derivatives.
- the domains D are **connected** (a smooth curve connects any two points). Some results require D to be **simply connected** (every loop in D can be contracted to a point in D).

connected but not simply connected (donuts);
simply connected but not connected (two disjoint domains)

Conservative fields are gradient fields

Theorem (Theorem 2)

Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field with continuous components throughout an open connected region D in space. Then $\vec{F}$ is conservative if and only if $\vec{F}$ is a gradient field ∇f for a differentiable function f.

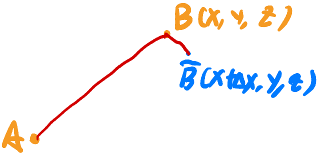
Construct the function f.

fix a point A in D

$f(B) = \int_C \vec{F}\cdot d\vec{r}(t)$ where C is a curve from A to B

$$f_x(x,y,z) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_C \vec{F}\cdot d\vec{r}(t)}{\Delta x}$$

C is from B to $\tilde{B}$



$$= \lim_{\Delta x\to 0} \frac{\int_x^{x+\Delta x} M\, dx}{\Delta x} = M(x,y,z)$$

Example: find the work done by the conservative field

$$\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k} = \nabla f, \text{ where } f(x, y, z) = xyz.$$

along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

$$f(1, 6, -4) - f(-1, 3, 9)$$

$$= -24 - (-27) = 3$$

Loop property of conservative fields

We let $\oint$ denote the integration around a closed path.

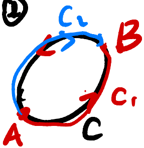
Theorem

The following statements are equivalent.

① ▶ $\oint_C \vec{F} \cdot d\vec{r} = 0$ *around every loop (that is, closed curve C) in D.*

② ▶ *The field $\vec{F}$ is conservative on D.*

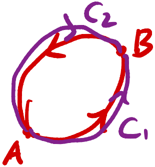
① ⇒ ②



$$\oint_C \vec{F} \cdot d\vec{r} = 0 \Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

② ⇒ ①



$$\oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = 0$$

Summary

$$\vec{F} = \nabla f \text{ on } D \Leftrightarrow \vec{F} \text{ conservative on } D \Leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 \text{ over any loop in } D$$

How do we find the function f?

Finding potentials for conservative fields

Let $\vec{F} = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\vec{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

If $\vec{F}$ is conservative then $\vec{F} = \nabla f$

$M = f_x \qquad N = f_y \qquad P = f_z$

$\frac{\partial P}{\partial y} = f_{zy} \qquad \frac{\partial N}{\partial z} = f_{yz}$

since $f_{zy} = f_{yz}$, we have $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$

The proof for conservativity is left for §16.7, which requires that "the domain is simply connected."

Show that $\vec{F} = (e^x \cos y + yz)\vec{i} + (xz - e^x \sin y)\vec{j} + (xy + z)\vec{k}$ is conservative over its natural domain and find a potential function for it.

- $M_y = -e^x \sin y + z$ $\qquad M_z = y$
  $N_x = z - e^x \sin y$ $\qquad N_z = x$
  $P_x = y$ $\qquad P_y = x$

simply connected $\Rightarrow$ $\vec{F}$ is conservative.

- since $f_x = e^x \cos y + yz$

$\Rightarrow f = e^x \cos y + xyz + g(y,z)$

$f_y = -e^x \sin y + xz + g_y = xz - e^x \sin y$

$\Rightarrow g_y = 0 \Rightarrow g(y,z) = h(z)$

$f_z = xy + h'(z) = xy + z \Rightarrow h'(z) = z$

$\Rightarrow h = \frac{1}{2}z^2 + C$

Show that $\vec{F} = (2x-3)\vec{i} - z\vec{j} + (\cos z)\vec{k}$ is not conservative.

$M_y = 0$ $\quad$ $M_z = 0$

$N_x = 0$ $\quad$ $N_z = -1$

$P_x = 0$ $\quad$ $P_y = 0$

Not conservative

Show that the vector field

$$\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + 0\vec{k}$$

satisfies the equations in the component test but is not conservative over its natural domain. Explain why this is possible.

$$M_y = \frac{-1}{x^2+y^2} + \frac{y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad M_z = 0$$

$$N_x = \frac{1}{x^2+y^2} - \frac{x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad N_z = 0$$

$$P_x = 0 \qquad P_y = 0$$

Domain is not simply connected.

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 0\vec{k} \qquad 0 \le t \le 2\pi$$

$$\vec{F}(\vec{r}(t)) = -\sin t\,\vec{i} + \cos t\,\vec{j} + 0\vec{k}$$

$$\frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j} + 0\vec{k}$$

$$\vec{F}\cdot\frac{d\vec{r}}{dt} = 1$$

$$\int_0^{2\pi} 1\,dt = 2\pi$$

Exact differential forms

Definition

Any expression $M(x, y, z)dx + N(x, y, z)dy + P(x, y, z)dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$Mdx + Ndy + Pdz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

The differential form $Mdx + Ndy + Pdz$ is exact if and only if $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is conservative.

Show that $ydx + xdy + 4dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} ydx + xdy + 4dz$$

over any path from $(1,1,1)$ to $(2,3,-1)$.

$f_x = y \Rightarrow f = xy + g(y,z)$

$f_y = x \Rightarrow x + g_y = x \Rightarrow g_y = 0 \Rightarrow g = h(z)$

$f_z = 4 \Rightarrow h'(z) = 4 \Rightarrow h(z) = 4z + C$

$\Rightarrow f = xy + 4z + C$

$$\int_{(1,1,1)}^{(2,3,-1)} ydx + xdy + 4dz = f(2,3,-1) - f(1,1,1)$$

$$= 6 - 4 - 1 - 4 = -3$$

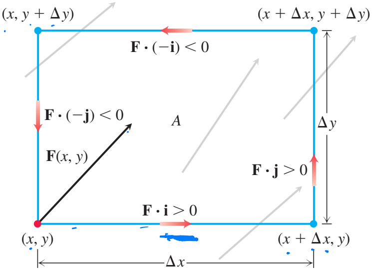
§16.4 Green's theorem in the plane

Green's Theorem allows us to convert the line integral (**closed curve** C and **nonconservative** field $\vec{F}$) into a double integral over the region enclosed by C.

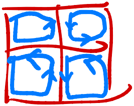
We introduce two new ideas for Green's theorem: **circulation density** around an axis perpendicular to the plane and **divergence** (or **flux density**).

Spin around an axis: the $\vec{k}$-component of curl

$\vec{F}\cdot\vec{T}$



Top and Bottom:



$$(M(x,y) - M(x,y+\Delta y))\Delta x$$

$\approx$ Bottom Top

$\approx -M_y \Delta y \Delta x$

Left and Right:

$$(N(x+\Delta x,y) - N(x,y))\Delta y$$

$\approx$ Right Left

$\approx N_x \Delta x \Delta y$

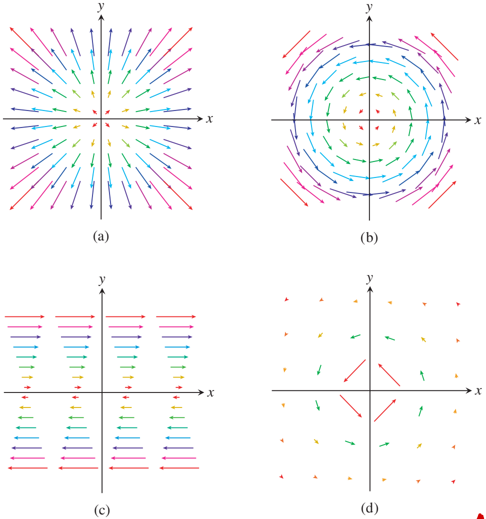
Definition

The **circulation density** of a vector field $\vec{F} = M\vec{i} + N\vec{j}$ at the point (x,y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

It is also called **the $\vec{k}$-component of the curl** (we will introduce curl later), denoted by (curl $\vec{F}$) $\cdot\, \vec{k}$.

The vector fields represent the velocity of a gas flowing in the xy-plane. Find their circulation densities and interpret their physical meanings.



- (a) Uniform expansion or compression: $\vec{F}(x,y) = cx\vec{i} + cy\vec{j}$

  $N_x - M_y = 0$

- (b) Uniform rotation: $\vec{F}(x,y) = -cy\vec{i} + cx\vec{j}$

  $N_x - M_y = c + c = 2c$

- (c) Shearing flow: $\vec{F}(x,y) = y\vec{i}$

  $N_x - M_y = 0 - 1 = -1$

- (d) Whirlpool effect: $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$

$$N_x - M_y = \frac{1}{x^2+y^2} - \frac{x \cdot 2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{2}{x^2+y^2} - \frac{2(x^2+y^2)}{(x^2+y^2)^2} = 0$$

Green's theorem (circulation-curl)

$$\oint_C \vec{F}(x,y)\cdot d\vec{r}$$

for the line integral when the simple closed curve C is traversed counterclockwise, with its positive orientation. (the region is always to the left)

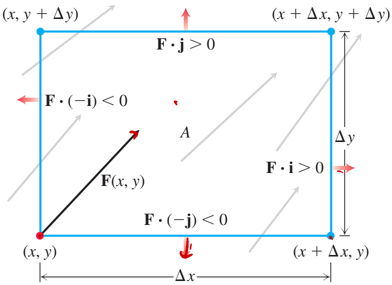
Theorem (Thm 4-Green's Theorem (Circulation-Curl /Tangential Form))

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\vec{i} + N\vec{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\vec{F}$ around C equals the double integral of (curl $\vec{F}) \cdot \vec{k}$ over R.

$$\oint_C \vec{F}\cdot\vec{T}ds = \oint_C \vec{F}\cdot d\vec{r} = \oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Divergence

$\int \vec{F}\cdot\vec{n}\,ds$



$\vec{n} = 0\vec{i} + 1\vec{j}$

Top and Bottom:

$$\big(\underbrace{N(x, y + \Delta y)}_{\text{Top}} - \underbrace{N(x, y)}_{\text{Bottom}}\big)\Delta x$$

$$\approx \quad N_y \cdot \Delta y\,\Delta x$$

Left and Right:

$\vec{n} = 1\vec{i} + 0\vec{j}$

$$\big(\underbrace{M(x + \Delta x, y)} - M(x, y)\big)\Delta y$$

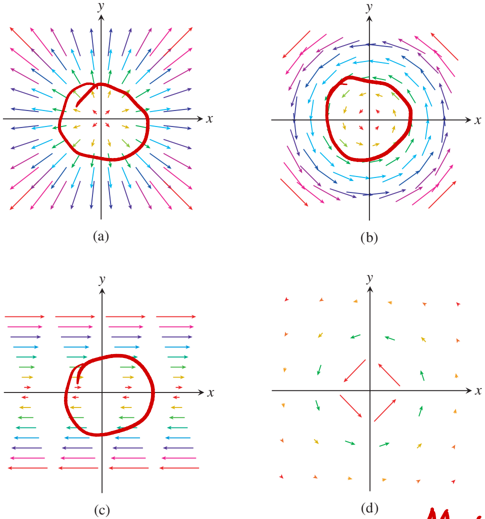
$$\approx \quad M_x\,\Delta x\,\Delta y$$

Definition

The **divergence (flux density)** of a vector field $\vec{F} = M\vec{i} + N\vec{j}$ at the point (x, y) is the scalar expression

$$\mathrm{div}\vec{F} = \underbrace{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}.$$

Find the divergence and interpret what it means for each vector field in Example 1, representing the velocity of a gas flowing in the xy plane.



- (a) Uniform expansion or compression:
  $\vec{F}(x,y) = cx\vec{i} + cy\vec{j}$

  $M_x + N_y = c + c = 2c$

- (b) Uniform rotation:
  $\vec{F}(x,y) = -cy\vec{i} + cx\vec{j}$

  $M_x + N_y = 0$

- (c) Shearing flow:
  $\vec{F}(x,y) = y\vec{i}$

  $M_x + N_y = 0$

- (d) Whirlpool effect:
  $\vec{F}(x,y) = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$

$M_x + N_y = \frac{y}{x^2+y^2}\cdot(2x) - \frac{x}{(x^2+y^2)}(\cdot 2y)$

$= 0$

Green's theorem (flux-divergence)

Theorem (Green's Theorem (Flux-Divergence or Normal Form))

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\vec{F} = M\vec{i} + N\vec{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then, the outward flux of $\vec{F}$ around C equals the double integral of div $\vec{F}$ over R.

$$\oint_C \vec{F} \cdot \vec{n} ds = \oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy$$

Green's theorem: summary

circulation-curl

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

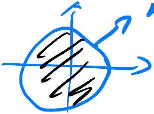
flux-divergence

$$\oint_C \vec{F} \cdot \vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\,dy$$

$$\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$$

$$\oint_C N\,dy = \iint_R N_x\,dx\,dy$$

Verify both forms of Green's theorem for the vector field



$\vec{n} = \cos t\,\vec{i} + \sin t\,\vec{j}$

$$\vec{F}(x,y) = (x-y)\vec{i} + x\vec{j}$$

and the region R bounded by the unit circle

$= \oint \vec{F} \cdot \frac{d\vec{r}}{dt} \cdot dt$

$$C: \ \vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j}, \ 0 \le t \le 2\pi.$$

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\cos t - \sin t) \times (-\sin t) + \cos t \cdot \cos t \, dt$$

$$= \int_0^{2\pi} 1 - \sin t \cos t \, dt = 2\pi$$

$$\oint \vec{F} \cdot \vec{n} \, ds = \int_0^{2\pi} (\cos t - \sin t) \cos t + \cos t \sin t \, dt$$

$$= \int_0^{2\pi} \cos^2 t \, dt = \pi$$

$$\iint_R (1 - (-1)) \, dA = 2 \iint_R dA = 2\pi$$

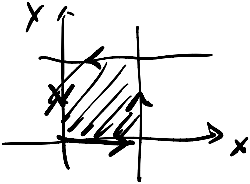
$$\iint_R 1 \, dA = \pi$$

Using Green's theorem to evaluate line integrals

Evaluate the line integral

$$\oint_C xydy - y^2 dx$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.



① circulation

$M = -y^2 \qquad N = xy$

$N_x - M_y = y + 2y = 3y$

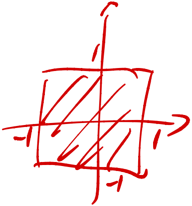
$\int_0^1 \int_0^1 3y \, dy dx = \frac{3}{2}$

② flux

$M = xy \qquad N = y^2$

$M_x + N_y = y + 2y = 3y$

Calculate the outward flux of the vector field $\vec{F}(x,y) = 2e^{xy}\vec{i} + y^3\vec{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.



$$M_x + N_y = 2e^{xy} \cdot y + 3y^2$$

$$\int_{-1}^{1} \int_{-1}^{1} 2e^{xy} \cdot y + 3y^2 \, dy \, dx$$

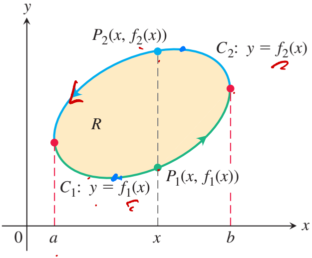
$$= \int_{-1}^{1} \underbrace{\int_{-1}^{1} 2e^{xy} \cdot y + 3y^2 \, dx}_{} \, dy$$

$$= \int_{-1}^{1} 2e^{xy} + 3xy^2 \Big|_{x=-1}^{x=1} \, dy$$

$$= \int_{-1}^{1} 2e^{y} + 3y^2 - (2e^{-y} - 3y^2) \, dy$$

$= - - -$

Proof of Green's theorem on special regions



$$\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\, dy\, dx$$

$$= \int_a^b M(x, f_2(x)) - M(x, f_1(x))\, dx$$

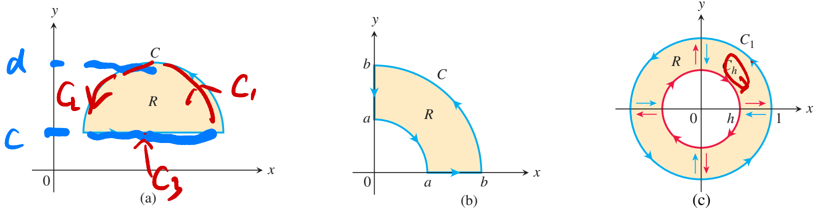
$$= -\int_b^a M(x, f_2(x))\, dx - \int_a^b M(x, f_1(x))\, dx$$

$$= -\int_{C_2} M\, dx - \int_{C_1} M\, dx$$

$$\underbrace{\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial N}{\partial x}}\, dy\, dx$$

$$\Rightarrow \underbrace{\int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\, dx}\, dy$$

More complicate regions



$$\int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x}\, dx\, dy$$

$$= \int_c^d N(g_2(y), y) - N(g_1(y), y)\, dy$$

$$\int_{C_3} N\, dy = 0$$