MAT1002: Calculus II

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§14.1 Functions of Several Variables
§14.2 Limits and Continuity in Higher Dimensions

Overview of Chapter 14

Many functions depend on more than one independent variable.

- The volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and height. It is a function of two variables r and h.
- The monthly payment on a home mortgage is a function of the principal borrowed P, the interest rate i, and the term t of the loan.

Chapter 14 extends the basic ideas of single-variable calculus to functions of several variables.

§14.1 Functions of Several Variables

Definition

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \cdots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \cdots, x_n)$$

to each element in D.

- **domain**: the set D.
- **range**: the set of w-values taken on by f.
- **dependent/output variable**: w.
- **independent/input variables**: $\{x_1, \cdots, x_n\}$.

Domains and Ranges

- $z = \sqrt{y - x^2}$

  Domain $\{(x,y): y - x^2 \geq 0\}$

  Ranges $\{z: z \geq 0\}$

- $z = \frac{1}{xy}$

  Domain $\{(x,y): x \neq 0, y \neq 0\}$

  Range $\{z: z \neq 0\}$

- $z = \sin xy$

  Domain $(x,y) \in R^2$

  Range $[-1, 1]$

- $w = \sqrt{x^2 + y^2 + z^2}$

  Domain $(x,y,z) \in R^3$

  Range $w \geq 0$

- $w = \frac{1}{x^2+y^2+z^2}$

  Domain $\{(x,y,z) \in R^3 : x^2+y^2+z^2 \neq 0\}$

  Range $w > 0$

- $w = xy \ln z$

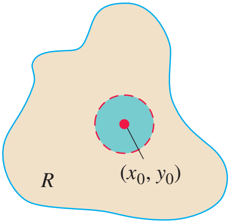
  Domain $\{(x,y,z) \in R^3 : z > 0\}$

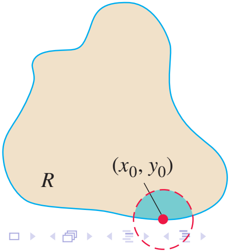
  Range R

Interior Points and Boundary Points

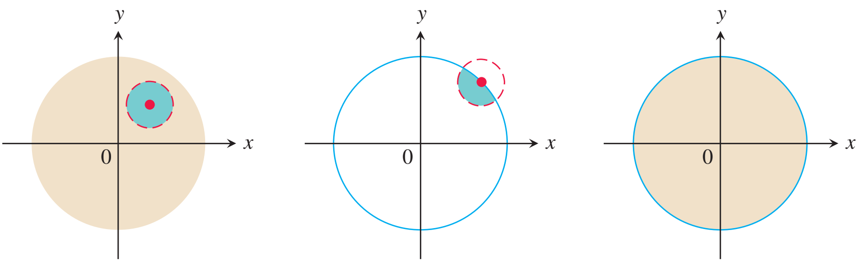
Definition

- A point (x_0, y_0) is an **interior point** of R if it is the center of the disk of positive radius that lies entirely in R.
- A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point does not need to belong to R.)
- The interior points of a region, as a set, make up the **interior** of the region.
- The region's boundary points make up its **boundary**.
- A region is **open** if it consists entirely of interior points.
- A region is **closed** if it contains all its boundary points.





Examples



$\{(x, y) \mid x^2 + y^2 < 1\}$
Open unit disk.
Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$
Boundary of unit disk. (The unit circle.)

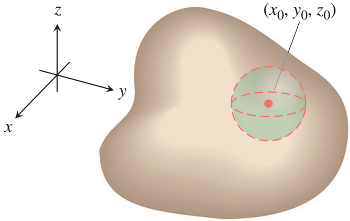
$\{(x, y) \mid x^2 + y^2 \leq 1\}$
Closed unit disk.
Contains all boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

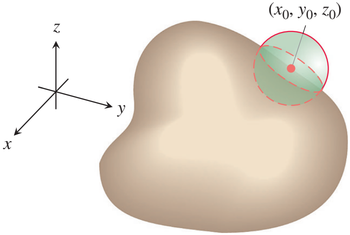
Interior and Boundary

Definition

- A point (x_0, y_0, z_0) is an **interior point** of R if it is the center of a solid ball that lies entirely in R.
- A point (x_0, y_0, z_0) is a **boundary point** of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside R. (The boundary point does not need to belong to R.)
- The interior points of R make up the **interior** of the region.
- The region's boundary points make up its **boundary**.
- A region is **open** if it consists entirely of interior points.
- A region is **closed** if it contains all its boundary points.



(a) Interior point



(b) Boundary point

Bounded and Unbounded

Definition

- A region in the plane is **bounded** if it lies inside a disk of finite radius.
- A region is **unbounded** if it is not bounded.

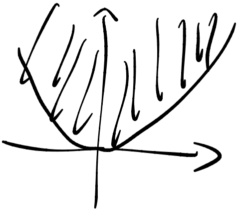
Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$. Interior, boundary, open/closed, bounded/unbounded

Domain $\{(x, y) \in R^2 : y - x^2 \geq 0\}$ closed unbounded

Interior : $\{(x, y) \in R^2 : y - x^2 > 0\}$ open unbounded

boundary $\{(x, y) \in R^2 : y - x^2 = 0\}$

closed, unbounded



Computer Graphing

$$w = \cos(1.7 \times 10^{-2} t - 0.6x) e^{-0.6x}$$

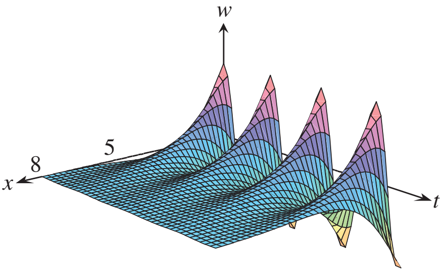


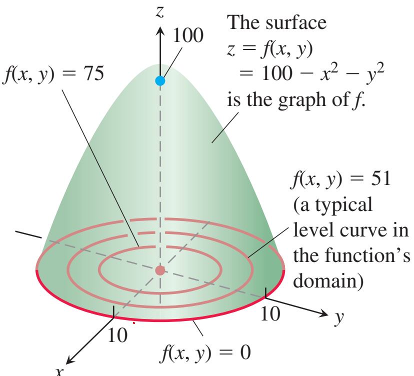
FIGURE 14.10 This graph shows the seasonal variation of the temperature below ground as a fraction of surface temperature (Example 5).

Graphs and Level Curves of Functions

Definition

- **level curve** of f: the set of points where the function $f(x,y)$ has a constant value $f(x,y)=c$.
- **graph** of f (**surface** $z=f(x,y)$): The set of all points $(x,y,f(x,y))$ in space, for all (x,y) in the domain of f.

Graph $f(x,y)=100-x^2-y^2$ and plot the level curves $f(x,y)=0$, $f(x,y)=51$, and $f(x,y)=75$ in the domain of f in the plane.



Contour Curves and Level Curves

- **level curve** of f: the set of points where the function $f(x,y)$ has a constant value $f(x,y) = c$.
- **contour curve** of f: the set of all points (x,y,c) in space, for all (x,y) such that $f(x,y) = c$.

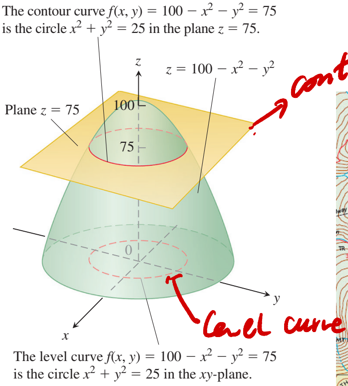


FIGURE 14.6 A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

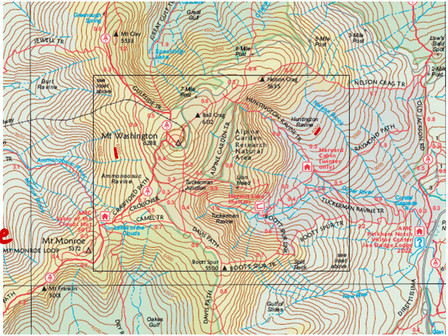
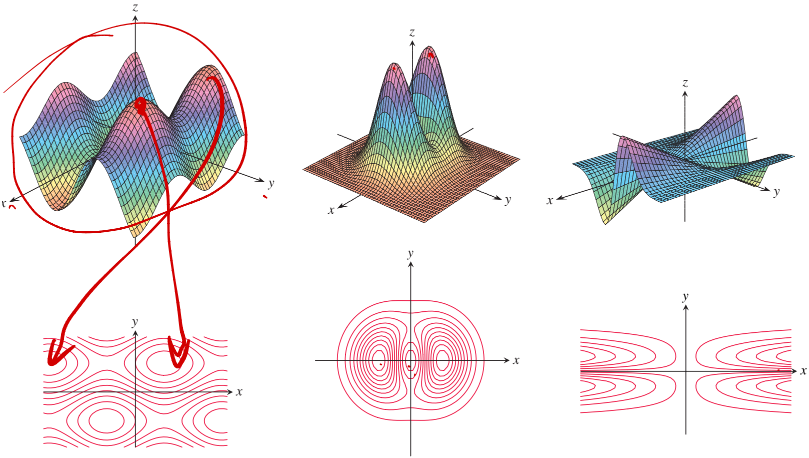


FIGURE 14.7 Contours on Mt. Washington in New Hampshire. (Reprinted by permission of the Appalachian Mountain Club.)

Computer Graphing



(a) $z = \sin x + 2\sin y$ (b) $z = (4x^2 + y^2)e^{-x^2-y^2}$ (c) $z = xye^{-y^2}$

FIGURE 14.11 Computer-generated graphs and level curves of typical functions of two variables.

Level Surface

Definition

- **level surface** of f: the set of points (x, y, z) in space where the function of three independent variables has a constant value $f(x, y, z) = c$

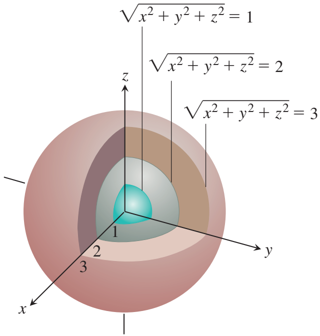


FIGURE 14.8 The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres (Example 4).

§14.2 Limits and Continuity in Higher Dimensions

Definition

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| \leq \epsilon, \text{ whenever } 0 \leq \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Properties of Limits of Functions

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M},\quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

Examples

- $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - 0 + 3}{0 + 0 - 1} = -3$$

- $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = 5$$

Examples

- $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = 0$$ $x \geq 0$, $y \geq 0$

$x^2 - xy = x(x - y)$

$= x(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

$$\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = x(\sqrt{x} + \sqrt{y})$$

Examples

- $$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$$

$$\left|\frac{4xy^2}{x^2+y^2}\right| \leq |4x| < \varepsilon$$

$$\Rightarrow |x| < \frac{\varepsilon}{4}$$

Sol: Let $\delta = \frac{\varepsilon}{4}$, then for any (x,y)

s.t. $|(x,y)| = \sqrt{x^2+y^2} < \delta$

we have $|x| \leq \sqrt{x^2+y^2} < \frac{\varepsilon}{4}$

so we have $\left|\frac{4xy^2}{x^2+y^2} - 0\right| \leq |4x| < \varepsilon$ $\square$

Examples

- $$\lim_{(x,y)\to(0,0)} \frac{y}{x}$$ does not exist

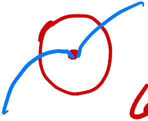
for any $\delta > 0$, we can find (x,y)

s.t. $|(x,y) - (0,0)| < \delta$

and $\frac{y}{x} = k$ for any $k \in R$

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.



$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

Let $y = kx$

$$f(x, kx) = \frac{2x^2 \cdot kx}{x^4 + k^2x^2} = \frac{2k\,x^3}{x^4 + k^2x^2} = \frac{2kx}{x^2 + k^2}$$

Let $y = kx^2$

$$f(x, kx^2) = \frac{2x^2 \cdot kx^2}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}$$

Limit does not exist!

Continuity

Definition

A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

- f is defined at (x_0, y_0)
- $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists
- $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$

A function is **continuous** if it is continuous at every point of its domain.

Example

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Let $y = kx$ then

$$f(x, kx) = \frac{2x \cdot kx}{x^2 + k^2x^2} = \frac{2k}{1+k^2}$$

so $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist.

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

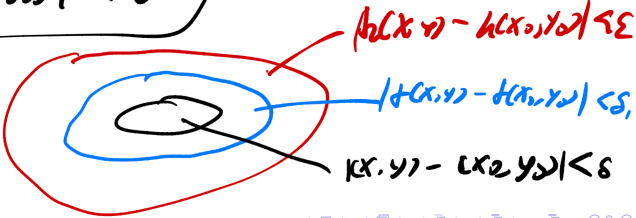
$$|h(x,y) - h(x_0,y_0)| < \varepsilon$$

$$\boxed{\begin{array}{l} |g(f(x,y)) - g(f(x_0,y_0))| < \varepsilon \\ \text{if } |f(x,y) - f(x_0,y_0)| < \delta_1 \end{array}}$$

continuity of g

$$\boxed{\begin{array}{l} |f(x,y) - f(x_0,y_0)| < \delta_1 \\ \text{if } |(x,y) - (x_0,y_0)| < \delta \end{array}}$$

$\Leftarrow$ continuity of f



Extreme Values of Continuous Functions on Closed, Bounded Sets

A continuous function $w = f(x, y, z)$ must take on absolute maximum and minimum values on any closed, bounded set (solid ball or cube, spherical shell, rectangular solid) on which it is defined.