MAT1002: Calculus II

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§16.6 Surface Integrals

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Applications

- the mass of a surface
- total electrical charge on a surface
- flow of a liquid across a curved membrane

Two forms

- a scalar function over a surface
- vector fields over a surface (flux)

Green's theorem (2D)

$$\oint \vec{F} \cdot \vec{T}\, ds$$

$$\oint \vec{F} \cdot \vec{n}\, ds = \iint_R \nabla \vec{F}\, dA$$

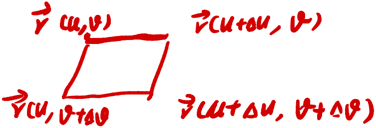
$\vec{r}(t)$: $\quad a \le t \le b$

$$\int_a^b \underbrace{f(\vec{r}(t))} \left|\frac{d\vec{r}}{dt}\right| dt$$

$$\int_a^b \vec{F} \cdot \vec{T} \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt = \int_a^b \vec{F} \cdot d\vec{r}$$

(2D) $\underbrace{\int_a^b \vec{F} \cdot \vec{n} \left|\frac{d\vec{r}}{dt}\right| dt}$

Surface integrals



Surface integral of G over the surface S

$$\iint_S G(x,y,z)\,d\sigma$$

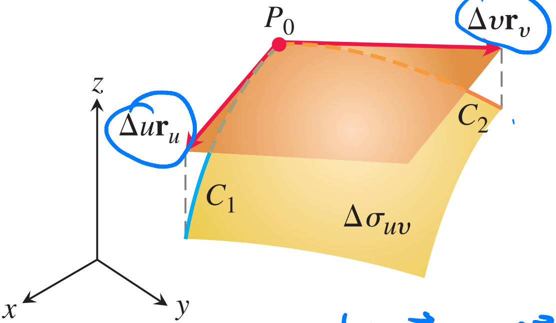
$\vec{r}(t+\Delta t)$

$\vec{r}(t)$

$$\left|\vec{r}(t+\Delta t)-\vec{r}(t)\right|$$

$$\approx\left|\frac{d\vec{r}}{dt}\cdot\Delta t\right|$$

$$=\left|\frac{d\vec{r}}{dt}\right|\Delta t$$



$$\left|\Delta u\,\vec{r}_u \times \Delta v\,\vec{r}_v\right|$$

$$=\left|\vec{r}_u \times \vec{r}_v\right|\Delta u\,\Delta v$$

$$d\sigma = \left|\vec{r}_u \times \vec{r}_v\right|du\,dv$$

Surface integrals

Surface integral of G over the surface S

$$\iint_S G(x,y,z)d\sigma$$

- If $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$, $(u,v) \in R$, we have

$$\iint_S G(x,y,z)d\sigma = \iint_R G(f(u,v), g(u,v), h(u,v))|\vec{r}_u \times \vec{r}_v|dudv.$$

Surface integrals

Surface integral of G over the surface S

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- If $\vec{r}(u,v) = f(u,v)\vec{i} + g(u,v)\vec{j} + h(u,v)\vec{k}$, $(u,v) \in R$, we have

$$\iint_S G(x,y,z)d\sigma = \iint_R G(f(u,v), g(u,v), h(u,v))|\vec{r}_u \times \vec{r}_v|dudv.$$

- $z = f(x,y)$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}dxdy.$$

- Implicitly $F(x,y,z) = c$

$$\iint_S G(x,y,z)d\sigma = \iint_R G(x,y,z)\frac{|\nabla F|}{|\nabla F \cdot \vec{p}|}dA,$$

where $\vec{p}$ is a unit vector normal to R and $\nabla F \cdot \vec{p} \neq 0$.

Integrate $G(x,y,z) = x^2$ over the cone $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$

$R = \{(x,y) : x^2+y^2 \le 1\}$

$$\vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + r\vec{k}, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{2}r$$

$G(\vec{r}(r,\theta)) = r^2\cos^2\theta$

$$\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\,\sqrt{2}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{4}\cos^2\theta\,d\theta$$

$$= \frac{\sqrt{2}}{4}\pi$$

$$\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2\sqrt{2}\,dy\,dx = \int_0^1 2\sqrt{2}\,x^2\sqrt{1-x^2}\,dx$$

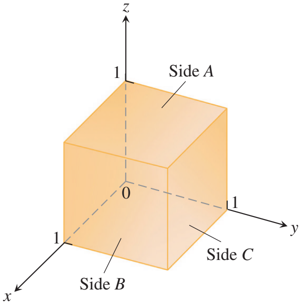
$x = \sin\theta$

$$= \int_{-\pi/2}^{\pi/2} 2\sqrt{2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= 2\sqrt{2}\cdot\frac{1}{4}\cdot\frac{1}{2}\pi = \frac{\sqrt{2}}{4}\pi$$

$$d_x = \frac{x}{\sqrt{x^2+y^2}} \qquad d_y = \frac{y}{\sqrt{x^2+y^2}}$$

Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$



Side A : $z=1 \quad 0 \le x \le 1 \quad 0 \le y \le 1$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + 1\vec{k}$$

$$\vec{r}_x = 1\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} + 0\vec{k}$$

$$|\vec{r}_x \times \vec{r}_y| = |1\vec{k}| = 1$$

$$z = f(x,y) = 1$$

$$\int_0^1 \int_0^1 xy \, dx\, dy = \frac{1}{4}$$

Side B $\quad x = 1$

$$x = f(y,z) = 1$$

$$\int_0^1 \int_0^1 yz \, dy\, dz = \frac{1}{4}$$

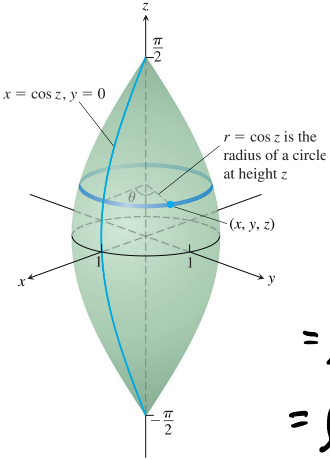
Side C $\quad y = 1$

$$y = f(x,z) = 1$$

$$\int_0^1 \int_0^1 xz \, dx\, dz = \frac{1}{4}$$

$$\boxed{\frac{3}{4}}$$

Integrate $G(x,y,z)=\sqrt{1-x^2-y^2}$ over the "football" suface S formed by rotating the curve $x=\cos z$, $y=0$, $-\pi/2\le z\le\pi/2$, around the z-axis



- $$\vec r(z,\theta)=\cos z\cos\theta\,\vec i+\cos z\sin\theta\,\vec j+z\vec k,$$
$$-\frac{\pi}{2}\le z\le\frac{\pi}{2},\quad 0\le\theta\le 2\pi.$$

$$|\vec r_z\times\vec r_\theta|=\cos z\sqrt{1+\sin^2 z}.$$

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\sqrt{1-\cos^2 z}}\ \cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin z|\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta$$

$$=\int_0^{2\pi}\underline{\int_0^{\frac{\pi}{2}} 2\sin z\cos z\sqrt{1+\sin^2 z}\,dz}\,d\theta$$

$$=\int_0^{2\pi}\frac{2}{3}(1+\sin^2 z)^{\frac{3}{2}}\Big|_{z=0}^{z=\frac{\pi}{2}}\,d\theta$$

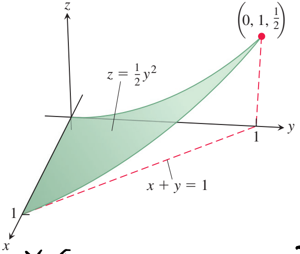
$$=\frac{2}{3}\left(2^{\frac{3}{2}}-1\right)\cdot 2\pi$$

Evaluate $\iint_S \sqrt{x(1+2z)}\,d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region $R: x \geq 0,\ y \geq 0,\ x+y \leq 1$ in the xy-plane

$$d\sigma = \sqrt{y^2+1}\,dx\,dy$$

$$z = f(x,y) = y^2/2$$

$$\sqrt{f_x^2+f_y^2+1} = \sqrt{0^2+y^2+1} = \sqrt{y^2+1}$$



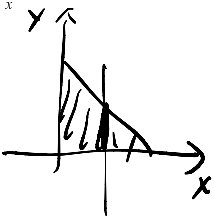
$$\int_0^1 \int_0^{1-x} \sqrt{x(1+y^2)}\,\sqrt{y^2+1}\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1+y^2)\,dy\,dx$$

$$= \int_0^1 \sqrt{x}\left(y+\frac{1}{3}y^3\right)\Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \sqrt{x}\left(1-x+\frac{1}{3}(1-x)^3\right) dx$$

$$= \ldots$$



Mass and moment formulas for very thin shells

- Mass:

$$M = \iint_S \delta d\sigma, \quad \delta = \delta(x,y,z) \text{ is the density at } (x,y,z)$$

- First moments about the coordinate planes:

$$M_{yz} = \iint_S x\delta d\sigma, \quad M_{xz} = \iint_S y\delta d\sigma, \quad M_{xy} = \iint_S z\delta d\sigma$$

- Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \quad \bar{y} = M_{xz}/M, \quad \bar{z} = M_{xy}/M$$

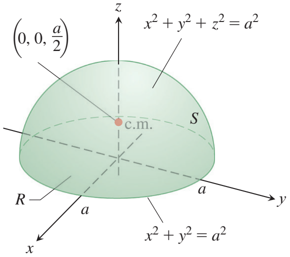
- Moments of inertia about axes and other straight lines

$$I_x = \iint_S (y^2+z^2)\delta d\sigma, \quad I_y = \iint_S (x^2+z^2)\delta d\sigma, \quad I_z = \iint_S (x^2+y^2)\delta d\sigma$$

$$I_L = \iint_S r^2 \delta d\sigma, \quad r(x,y,z) = \text{distance from the point } (x,y,z) \text{ to line } L$$

Find the center of mass of a thin hemispherical shell of radius a and constant density δ.

$$M = 2\pi a^2 \delta$$



$F(x,y,z) = x^2+y^2+z^2$

$F(x,y,z) = a^2$

$$\frac{|\nabla F|}{|\nabla F \cdot \vec{p}|} = \frac{\sqrt{(2x)^2+(2y)^2+(2z)^2}}{|2z|}$$

$\vec{p} = \vec{k}$ $\quad = \frac{a}{z}$

$d\sigma = \frac{a}{z}\, dA$

$$\iint_R x \cdot \frac{a}{z}\, dA$$

$$= \iint_R x \cdot \frac{a}{\sqrt{a^2-x^2-y^2}}\, dx\,dy = 0$$

$$\iint_R y\, \frac{a}{z}\, dA = 0$$

$$\iint_R z \cdot \frac{a}{z}\, dA = a \cdot \pi a^2$$

$$\left(0, 0, \frac{\pi a^3}{2\pi a^2}\right)$$

$$\iint_R \frac{a}{z}\, dA = \iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\, dx\,dy$$

$$= \int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2-r^2}}\, r\, dr\, d\theta$$

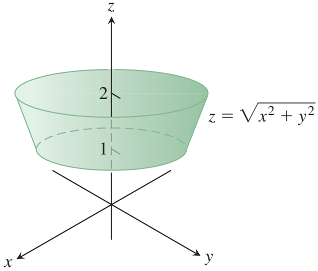
$$= \int_0^{2\pi} (-a)\sqrt{a^2-r^2}\,\Big|_{r=0}^{r=a} d\theta = 2\pi a^2$$

Find the center of mass of a thin shell of density $\delta = 1/z^2$ cut from the cone $z = \sqrt{x^2+y^2}$ by the planes $z = 1$ and $z = 2$

$1 \le r \le 2 \quad 0 \le \theta \le 2\pi$

$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\vec{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{2}r$$



$$M = \int_0^{2\pi}\int_1^2 \frac{1}{z^2}\sqrt{2}r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 \frac{\sqrt{2}r}{r^2}\,dr\,d\theta = \int_0^{2\pi} \sqrt{2}\ln r\Big|_{r=1}^{r=2} d\theta$$

$$= \sqrt{2}\ln 2 \cdot 2\pi$$

$$\int_0^{2\pi}\int_1^2 z\frac{1}{z^2}\sqrt{2}r\,dr\,d\theta = \int_0^{2\pi}\int_1^2 \sqrt{2}\,dr\,d\theta$$

$$= \sqrt{2}\cdot 2\pi$$

$$\left(0,\ 0,\ \frac{\sqrt{2}\cdot 2\pi}{\sqrt{2}\ln 2\cdot 2\pi}\right) = \left(0,\ 0,\ \frac{1}{\ln 2}\right)$$

Vector integral on a surface

The surface is in 3D. How do we generalize the line integral to 3D?

- $\int_C \vec{F} \cdot \vec{T} ds$. we talk about this for 3D

$\oint_C \vec{F} \cdot \vec{T}\, ds$ in 3D ?

- $\int_C \vec{F} \cdot \vec{n} ds$ in 2D

$\iint_S \vec{F} \cdot \vec{n}\, d\sigma$ in 3D

Orientation of a surface

A smooth surface S is **orientable** (or **two-sided**) if it is possible to define a field of unit normal vectors $\vec{n}$ on S, which varies continuously with position. By convention, we usually choose $\vec{n}$ on a closed surface to point outward.

Once $\vec{n}$ has been chosen, we say that we have **oriented** the surface, and we call it with its normal field an **oriented surface**. At any point, the vector $\vec{n}$ is called the **positive direction**.

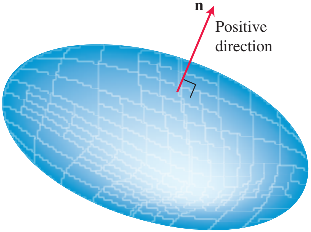


FIGURE 16.49 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

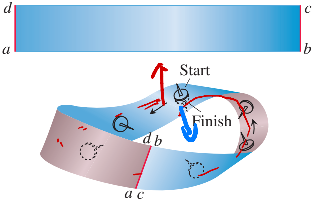


FIGURE 16.50 To make a Möbius band, take a rectangular strip of paper *abcd*, give the end *bc* a single twist, and paste the ends of the strip together to match *a* with *c* and *b* with *d*. The Möbius band is a nonorientable or one-sided surface.

Surface integrals of vector fields

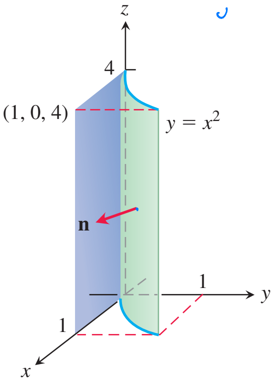
Definition

Let $\vec{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\vec{n}$ orienting S. Then the **surface integral of $\vec{F}$ over S** is

$$\iint_S \vec{F} \cdot \vec{n} d\sigma$$

The surface integral of $\vec{F}$ is also called the **flux** of the vector field across the oriented surface S.

Find the flux of $\vec{F} = yz\vec{i} + x\vec{j} - z^2\vec{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the direction $\vec{n}$ indicated in the figure.



$$\vec{r}(x,z) = x\vec{i} + x^2\vec{j} + z\vec{k},\ 0 \le x \le 1, 0 \le z \le 4$$

$$\vec{r}_x \times \vec{r}_z = 2x\vec{i} - \vec{j}$$

$$\vec{n} = \frac{2x\vec{i} - \vec{j}}{\sqrt{4x^2+1}}$$

$$\vec{F}\cdot\vec{n} = x^2 z \cdot \frac{2x}{\sqrt{4x^2+1}} - \frac{x}{\sqrt{4x^2+1}}$$

$$= \frac{2x^3 z}{\sqrt{4x^2+1}} - \frac{x}{\sqrt{4x^2+1}}$$

$$\int_0^1 \int_0^4 \left(\frac{2x^3 z}{\sqrt{4x^2+1}} - \frac{x}{\sqrt{4x^2+1}} \right) \cdot \sqrt{4x^2+1}\, dz\, dx$$

$$= \int_0^1 \int_0^4 (2x^3 z - x)\, dz\, dx = \cdots$$

$$z = f(x, y)$$

$$(x_0, y_0, f(x_0, y_0))$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

$$\vec{n} = \frac{f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} - 1\vec{k}}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\vec{r}(x, y) = x\vec{i} + y\vec{j} + f(x, y)\vec{k}$$

$$\vec{r}_x = 1\vec{i} + 0\vec{j} + f_x\vec{k}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} + f_y\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = -f_x\vec{i} - f_y\vec{j} + 1\vec{k}$$

Simplification for the parametrized case

For the parametrized case

$$d\sigma = |\vec{r}_u \times \vec{r}_v| dudv$$

and

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \iint_S \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cdot \cancel{|\vec{r}_u \times \vec{r}_v|} \, dudv$$

$$= \iint_S \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, dudv$$

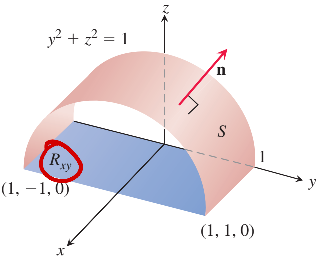
For the level surface $g(x, y, z) = c$, we have

$$\vec{n} = \pm \frac{\nabla g}{|\nabla g|}$$

$$\iint_S \vec{F} \cdot \vec{n} d\sigma = \pm \iint_S \vec{F} \cdot \frac{\nabla g}{\cancel{|\nabla g|}} \cdot \frac{\cancel{|\nabla g|}}{|\nabla g \cdot \vec{p}|} \, dA$$

$$= \pm \iint_S \vec{F} \cdot \frac{\nabla g}{|\nabla g \cdot \vec{p}|} \, dA$$

Find the flux of $\vec{F} = yz\vec{j} + z^2\vec{k}$ outward through the surface S cut frm the cylinder $y^2 + z^2 = 1$, $z \geq 0$, by the planes $x = 0$ and $x = 1$.



$$\iint_{R_{xy}} \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \int_0^1 \int_{-1}^1 \vec{F} \cdot \frac{\nabla g}{\cancel{|\nabla g|}} \frac{\cancel{|\nabla g|}}{|\nabla g \cdot \vec{k}|} \, dy\,dx$$

$g(x, y, z) = y^2 + z^2$

$\nabla g = 0\vec{i} + 2y\vec{j} + 2z\vec{k}$

$$= \int_0^1 \int_{-1}^1 (2y^2z + 2z^3)\frac{1}{2z} \, dy\,dx$$

$$= \int_0^1 \int_{-1}^1 y^2 + z^2 \, dy\,dx$$

$y^2 + z^2 = 1$

$$= \int_0^1 \int_{-1}^1 1 \, dy\,dx = 2$$