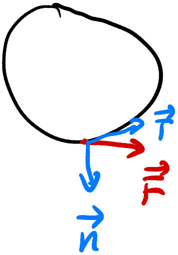
MAT1002: Calculus II



Ming Yan

§16.7 Stokes' Theorem

The curl vector

Green's theorem (in a plane)

$\frac{d\vec{r}}{ds} = \vec{T}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

It measures the rotation rate of $\vec{F}$ around an axis parallel to $\vec{k}$.
How to extend it to a general surface?

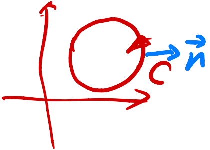
$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T}\, ds$$

C in 3D

The curl vector

Green's theorem (in a plane)

$\operatorname{curl}\vec{F}\cdot\vec{k}$



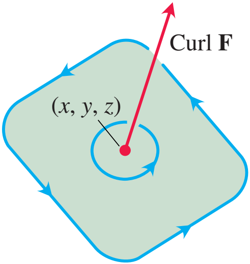
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

It measures the rotation rate of $\vec{F}$ around an axis parallel to $\vec{k}$.

How to extend it to a general surface?

$\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + \oint_C N\,dy + \oint_C P\,dz$



When viewed looking down, the vector points in the direction for which the rotation is counterclockwise.

The **curl vector** for the vector field $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ is

$$\operatorname{curl}\ \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

Curl vector

Define (pronounced 'del')

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Then the curl of $\vec{F}$ is

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \quad \begin{matrix} \\ \frac{\partial}{\partial x} \\ M \end{matrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

Find the curl of $\vec{F} = (x^2 - z)\vec{i} + xe^z\vec{j} + xy\vec{k}$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{array}{c} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z & xe^z & xy \end{vmatrix} \end{array} \begin{array}{c} \\ \frac{\partial}{\partial x} \\ x^2 - z \end{array}$$

$$= (x - xe^z)\vec{i} + (-1 - y)\vec{j}$$

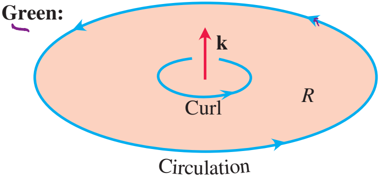
$$+ (e^z)\vec{k}$$

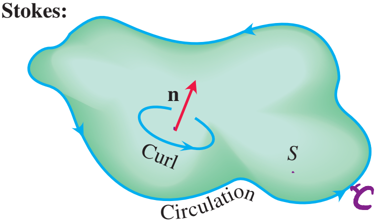
Stokes' theorem

Theorem (Theorem 6 - Stokes' Theorem)

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\vec{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\vec{n}$ equals the integral of the curl vector field $\nabla \times \vec{F}$ over S:

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\textit{counterclockwise}} = \underbrace{\iint_S \nabla \times \vec{F} \cdot \vec{n}\, d\sigma}_{\textit{curl integral}}$$





Stokes' theorem

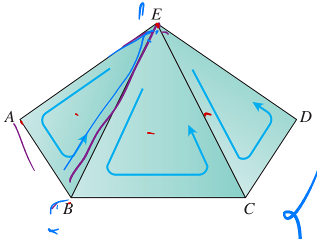
- Surface independent

$$\iint_{S_1} \nabla \times \vec{F} \cdot \vec{n}_1 d\sigma = \iint_{S_2} \nabla \times \vec{F} \cdot \vec{n}_2 d\sigma.$$

- Analogous to a path independent for $\vec{F} := \nabla f$.
- Green's theorem is a special case of Stokes' theorem. Let C be a curve in the xy-plan, oriented counterclockwise, and R is the region in the xy-plane bounded by C. Then we have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint_S \nabla \times \vec{F} \cdot \vec{k} d\sigma$$

Proof of Stokes' theorem for polyhedral surfaces



$$\underbrace{\oint_{EAB} \vec{F}\cdot d\vec{r} = \iint_{EAB} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma}$$

$$\oint_{EBC} \vec{F}\cdot d\vec{r} = \iint_{EBC} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

$$\oint_{ECD} \vec{F}\cdot d\vec{r} = \iint_{ECD} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

$$\oint_{EABCD} \vec{F}\cdot d\vec{r} = \iint_{EABCD} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

Stokes' theorem for surfaces with holes

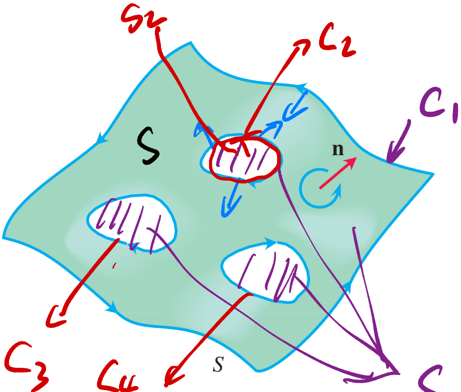


FIGURE 16.67 Stokes' Theorem also holds for oriented surfaces with holes. Consistent with the orientation of S, the outer curve is traversed counterclockwise around **n** and the inner curves surrounding the holes are traversed clockwise.

$$\iint_{S_5} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \oint_{C_1} \vec{F}\cdot d\vec{r}$$

$$\oint_{C_2} \vec{F}\cdot d\vec{r} = \iint_{S_2} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

$$\Rightarrow \iint_{S} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma = \iint_{S_5} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma - \iint_{S_2+S_3+S_4} \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$

Stokes' theorem for surfaces with holes

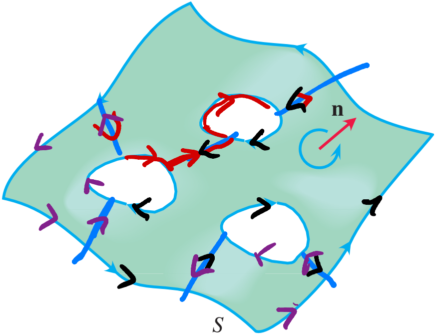
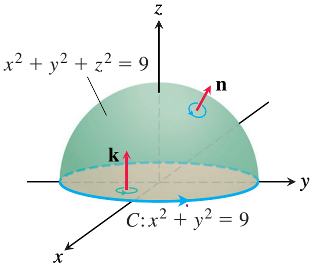


FIGURE 16.67 Stokes' Theorem also holds for oriented surfaces with holes. Consistent with the orientation of S, the outer curve is traversed counterclockwise around $\mathbf{n}$ and the inner curves surrounding the holes are traversed clockwise.

Verify Stokes' theorem: the hemisphere $S: x^2+y^2+z^2=9,\ z\geq 0$, its bounding circle $C: x^2+y^2=9,\ z=0$, and the field $\vec{F} = y\vec{i} - x\vec{j}$.



$$\vec{r}(t) = \underline{3\cos t}\,\vec{i} + \underline{3\sin t}\,\vec{j}, \quad 0 \leq t \leq 2\pi$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} 3\sin t\cdot(-3\sin t) - 3\cos t\cdot 3\cos t \, dt$$

$$= \int_0^{2\pi} -9\, dt = -18\pi$$

$$d\sigma = \frac{|\nabla F|}{|\nabla F\cdot \vec{k}|}\, dA = \frac{2\sqrt{x^2+y^2+z^2}}{2|z|}\, dA = \frac{3}{z}\, dA$$

$$F(x,y,z) = x^2+y^2+z^2$$

$$d\sigma = \frac{3}{z}\, dA$$

$$\iint_S \underline{\nabla\times\vec{F}}\cdot\vec{n}\, d\sigma$$

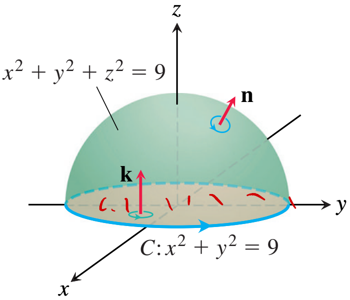
$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + (-2)\vec{k}$$

$$\vec{n} = (x\vec{i} + y\vec{j} + z\vec{k})\frac{1}{3}$$

$$\iint_S \frac{-2z}{3}\, d\sigma$$

$$= \iint_R \frac{-2}{3} z\cdot\frac{3}{z}\, dA = -18\pi$$

Calculate the circulation around the bounding circle C from the previous example using the surface $x^2 + y^2 \le 9,\ z = 0$.

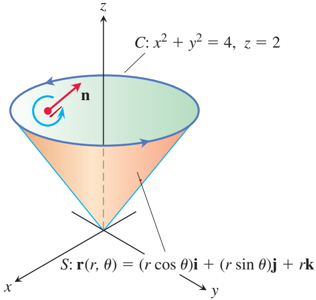


$$d\sigma = dA$$

$$\nabla \times \vec{F} \cdot \vec{k} = -2$$

$$\iint_R (-2)\, dA = -18\pi$$

Find the circulation of $\vec{F} = (x^2 - y)\vec{i} + 4z\vec{j} + x^2\vec{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise as viewed from above.



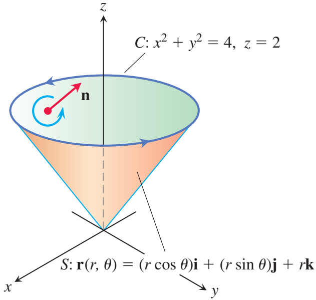
$$
\begin{aligned}
\vec{r}(r,\theta) =& r\cos\theta\vec{i} + r\sin\theta\vec{j} + r\vec{k}, \\
|\vec{r}_r \times \vec{r}_\theta| =& -r\cos\theta\vec{i} - r\sin\theta\vec{j} + r\vec{k} \\
\vec{n} =& -\frac{1}{\sqrt{2}}\cos\theta\vec{i} - \frac{1}{\sqrt{2}}\sin\theta\vec{j} + \frac{1}{\sqrt{2}}\vec{k} \\
\nabla \times \vec{F} =& -4\vec{i} - 2x\vec{j} + \vec{k} \\
=& -4\vec{i} - 2r\cos\theta\vec{j} + \vec{k}
\end{aligned}
$$

$$(\nabla \times \vec{F}) \cdot \vec{n} = 2\sqrt{2}\cos\theta + \sqrt{2}\, r\sin\theta\cos\theta + \frac{1}{\sqrt{2}}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla\times\vec{F} \cdot \vec{n}\, d\sigma = \int_0^2 \int_0^{2\pi} \left(2\sqrt{2}\cos\theta + \sqrt{2}\, r\sin\theta\cos\theta + \frac{1}{\sqrt{2}}\right)\sqrt{2}\, r\, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} (4r\cos\theta + 2r^2\sin\theta\cos\theta + r)\, d\theta\, dr = 2\pi \cdot 2 = 4\pi$$

Find the circulation of $\vec{F} = (x^2 - y)\vec{i} + 4z\vec{j} + x^2\vec{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise as viewed from above.



$$
\begin{aligned}
\vec{r}(r,\theta) =& r\cos\theta\vec{i} + r\sin\theta\vec{j} + r\vec{k},\\
\vec{r}_r \times \vec{r}_\theta =& -r\cos\theta\vec{i} - r\sin\theta\vec{j} + r\vec{k}\\
\vec{n} =& -\frac{1}{\sqrt{2}}\cos\theta\vec{i} - \frac{1}{\sqrt{2}}\sin\theta\vec{j} + \frac{1}{\sqrt{2}}\vec{k}\\
\nabla \times \vec{F} =& -4\vec{i} - 2x\vec{j} + \vec{k}\\
=& -4\vec{i} - 2r\cos\theta\vec{j} + \vec{k}
\end{aligned}
$$

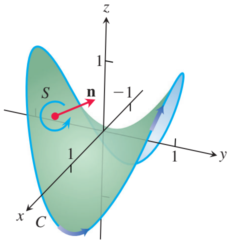
$\oint_C \vec{F}\cdot d\vec{r}$ $\quad$ $\vec{r}(\theta) = 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} + 2\vec{k}$

$$\int_0^{2\pi} (4\cos^2\theta - 2\sin\theta)\cdot(-2\sin\theta) + 8\cdot 2\cos\theta + 0 \, d\theta$$

$$= \int_0^{2\pi} -8\sin\theta\cos^2\theta + 4\sin^2\theta + 16\cos\theta \, d\theta = 4\pi$$

Verify Stokes' theorem for S using the vector field $\vec{F} = y\vec{i} - x\vec{j} + x^2\vec{k}$.

Find a parametrization for the surface S formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around the z-axis.



$$\vec{r}(r,\theta) = r\cos\theta\vec{i} + r\sin\theta\vec{j} + r^2(\sin^2\theta - \cos^2\theta)\vec{k},$$
$$0 \le \theta \le 2\pi, \ 0 \le r \le 1$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = 0\vec{i} - 2x\vec{j} - 2\vec{k}$$

$$\vec{r}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + 2r(\sin^2\theta - \cos^2\theta)\vec{k}$$

$$\vec{r}_\theta = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j} + r^2(2\sin\theta\cos\theta + 2\sin\theta\cos\theta)\vec{k}$$

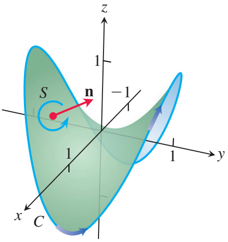
$$\iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma$$
$$= \iint_R \nabla\times\vec{F}\cdot(\vec{r}_r\times\vec{r}_\theta)\,dA$$

$$\vec{r}_r\times\vec{r}_\theta = \ ? \ \vec{i} + \big(-2r^2\sin\theta(\sin^2\theta - \cos^2\theta) - r^2\cos\theta\, 4\sin\theta\cos\theta\big)\vec{j} + (r\cos^2\theta + r\sin^2\theta)\vec{k}$$

$$= \ \cdots\ \vec{i} - 2r^2\sin\theta\,\vec{j} + r\,\vec{k}$$

Verify Stokes' theorem for S using the vector field $\vec{F} = y\vec{i} - x\vec{j} + x^2\vec{k}$.

Find a parametrization for the surface S formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around the z-axis.



$$\vec{r}(r,\theta) = r\cos\theta\vec{i} + r\sin\theta\vec{j} + r^2(\sin^2\theta - \cos^2\theta)\vec{k},$$
$$0 \le \theta \le 2\pi,\ 0 \le r \le 1$$

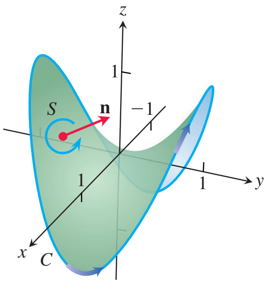
$$\nabla \times \vec{F} =$$

$$\iint_R \nabla\times\vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta)\,dA = \int_0^{2\pi}\int_0^1 -2r\cos\theta(-2r^2\sin\theta) - 2r\,dr\,d\theta$$

$$= -2\pi$$

Verify Stokes' theorem for S using the vector field $\vec{F} = y\vec{i} - x\vec{j} + x^2\vec{k}$.

Find a parametrization for the boundary curve C of the surface S.



$$\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j} + (\sin^2 t - \cos^2 t)\vec{k},$$
$$0 \le t \le 2\pi$$

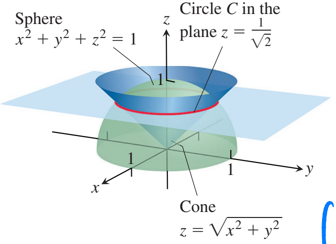
$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin t\,(-\sin t) - \cos t(\cos t) + \cos^2 t\,(4\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} -1\,dt = -2\pi$$

Calculate the circulation of the vector field

$$\vec{F} = (x^2 + z)\vec{i} + (y^2 + 2x)\vec{j} + (z^2 - y)\vec{k}$$

along the curve of intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the cone $z = \sqrt{x^2 + y^2}$ traversed in the counterclockwise direction around the z-axis when viewed from above.



$$x^2 + y^2 + x^2 + y^2 = 1 \Rightarrow x^2 + y^2 = \frac{1}{2}$$

$$z = \frac{1}{\sqrt{2}}$$

$$\vec{r}(\theta) = \frac{1}{\sqrt{2}}\cos\theta\,\vec{i} + \frac{1}{\sqrt{2}}\sin\theta\,\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$$

$$\int_0^{2\pi} \left(\frac{\cos^2\theta}{2} + \frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\sin\theta\right) + \left(\frac{\sin^2\theta}{2} + \sqrt{2}\cos\theta\right)\frac{1}{\sqrt{2}}\cos\theta \, d\theta$$

$$= \int_0^{2\pi} -\frac{\cos^2\theta}{2\sqrt{2}}\sin\theta - \frac{1}{2}\sin\theta + \frac{\sin^2\theta\cos\theta}{\sqrt{2}} + \cos^2\theta \, d\theta$$

$$= \pi$$

Find $\nabla \times \vec{F}$ and relate it to the circulation density $\frac{1}{\pi\rho^2}\oint_C \vec{F}\cdot d\vec{r}$.

A fluid of constant density rotates around the z-axis with velocity $\vec{F} = w(-y\vec{i} + x\vec{j})$, where w is the *angular velocity* of the rotation.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 2w\vec{k}$$

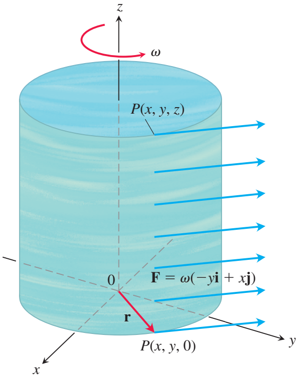


FIGURE 16.63 A steady rotational flow parallel to the xy-plane, with constant angular velocity ω in the positive (counter-clockwise) direction (Example 8).

For circle C of radius ρ

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma =$$

$$= 2w\cdot\pi\rho^2$$

$$\frac{\oint \vec{F}\cdot d\vec{r}}{\pi\rho^2} = \frac{\iint_S \nabla\times\vec{F}\cdot\vec{n}\,d\sigma}{\pi\cdot\rho^2} = 2w$$

Paddle wheel interpretation of $\nabla \times \vec{F}$

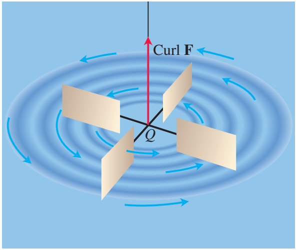


FIGURE 16.62 A small paddle wheel in a fluid spins fastest at point Q when its axle points in the direction of curl **F**.

We fix a point Q in the region R and a direction $\vec{u}$. Take C as a circle of radius ρ, with center at Q, whose plane is normal to $\vec{u}$.

If $\nabla \times \vec{F}$ is continuous at Q, the average value of the $\vec{u}$-component of $\nabla \times \vec{F}$ over the circular disk S bounded by C approaches the $\vec{u}$-component of $\nabla \times \vec{F}$ at Q as the radius $\rho \to 0$:

$$(\nabla \times \vec{F} \cdot \vec{u})_Q = \lim_{\rho\to 0} \frac{1}{\pi\rho^2} \iint_S \nabla \times \vec{F} \cdot \vec{u}\, d\sigma \qquad \vec{u} = \vec{n}$$

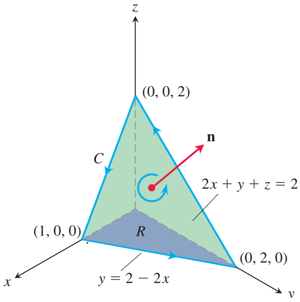
$$= \lim_{\rho\to 0} \frac{1}{\pi\rho^2} \oint_C \vec{F} \cdot d\vec{r}.$$

circulation density: the circulation around C divided by the area of the disk

$$\frac{1}{\pi\rho^2} \oint_C \vec{F} \cdot d\vec{r}$$

Use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ with $\vec{F} = xz\vec{i} + xy\vec{j} + 3xz\vec{k}$

C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant, traversed counterclockwise as viewed from above.



$$\vec{n} = (2\vec{i} + \vec{j} + \vec{k})\frac{1}{\sqrt{6}}$$

$$\nabla\times\vec{F} = \begin{array}{c} \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xy & 3xz \end{array} \right| \end{array} \begin{array}{c} \\ \frac{\partial}{\partial x} \\ xz \end{array}$$

$$= 0\vec{i} + (x - 3z)\vec{j} + y\vec{k}$$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (2 - 2x - y)\vec{k}$$

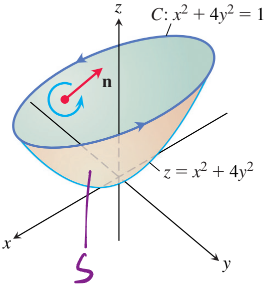
$$0 \le y \le 2 - 2x, \quad 0 \le x \le 1$$

$$\int_0^1\int_0^{2-2x} (\nabla\times\vec{F})\cdot\vec{n}\,\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dy\,dx$$

$$= \int_0^1\int_0^{2-2x} (x - 3z + y)/\sqrt{6}\cdot\sqrt{6}\;dy\,dx = \int_0^1\int_0^{2-2x} (x + y - 3(2 - 2x - y))\,dy\,dx$$

Find the flux of $\nabla \times \vec{F}$ across S in the direction $\vec{n}$ for $\vec{F} = y\vec{i} - xz\vec{j} + xz^2\vec{k}$

The surface S is the elliptical paraboloid $z = x^2 + 4y^2$ lying beneath the plane $z = 1$. We define the orientation of S by taking the inner normal vector $\vec{n}$ to the surface, which is the normal having a positive $\vec{k}$-component.



$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$$

$$= \oint_C \vec{F} \cdot d\vec{r}$$

$x^2 + 4y^2 = 1 \qquad z = 1$

$$\vec{r} = \cos\theta\, \vec{i} + \frac{1}{2}\sin\theta\, \vec{j} + \vec{k}$$

$$\int_0^{2\pi} \frac{1}{2}\sin\theta\,(-\sin\theta) - \cos\theta\left(\frac{1}{2}\cos\theta\right) d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}\sin^2\theta - \frac{1}{2}\cos^2\theta\, d\theta = -\pi$$

An important identity

$$\nabla \times \nabla f = \vec{0}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i}$$

$$+ \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j}$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$$= 0$$

Curl $\vec{F} = 0$ related to the closed-loop property

Theorem

If $\nabla \times F = \vec{0}$ at every point of a simply connected open region D in space, then on any piecewise smooth closed path C in D,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} d\sigma = 0.$$

$$\underbrace{\oint_C \nabla f \cdot d\vec{r}} = \iint_S (\nabla \times \nabla f) \cdot \vec{n} \, d\sigma = 0$$

$$\oint_C \underbrace{\left(f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \right)}_{\text{path independence.}} dt$$

Summary

Theorem 2,
Section 16.3

$\mathbf{F}$ conservative on D $\iff$ $\mathbf{F} = \nabla f$ on D

Theorem 3,
Section 16.3

$\Updownarrow$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

over any closed path in D

$\Downarrow$

Vector identity (Eq. 8)
(continuous second
partial derivatives)

$\Longleftarrow$ $\nabla \times \mathbf{F} = \mathbf{0}$ throughout D

Theorem 7
Domain's simple
connectivity and
Stokes' Theorem

depends on
2nd derivatives