MAT1002: Calculus II

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§11.3 Polar Coordinates
§11.4 Graphing Polar Coordinate Equations
§11.5 Areas and Lengths in Polar Coordinates

§11.3 and §11.4

- Polar coordinates and Cartesian coordinates
- Symmetry
- Slope

Definition of Polar Coordinates

We fix an **origin** O (call the **pole**) and an **initial ray** from O. Then each point P corresponds to a **polar coordinate pair** (r, θ), in which r is the directed distance from O to P, and θ is the directed angle from the initial ray to ray OP. So, we label the point P as

$$P(r, \theta)$$

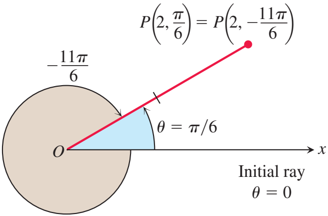


FIGURE 11.20 Polar coordinates are not unique.

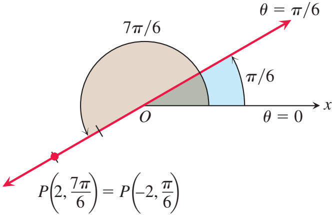


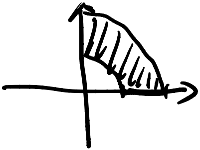
FIGURE 11.21 Polar coordinates can have negative r-values.

Polar Equations

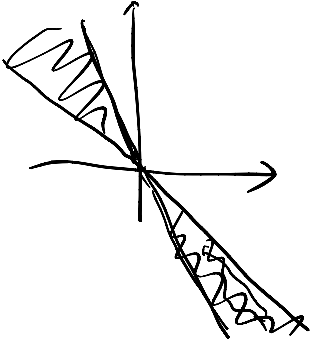
- $r = 1$ and $r = -1$
- $\theta = \pi/6$ and $\theta = 7\pi/6$

Polar Equations

- $1 \leq r \leq 2, \ 0 \leq \theta \leq \pi/2$

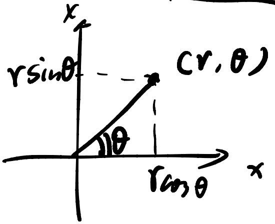


- $2\pi/3 \leq \theta \leq 5\pi/6$



Relation between Polar and Cartesian Coordinates

$$x = r\cos\theta, \;\; y = r\sin\theta,$$



Relation between Polar and Cartesian Coordinates

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

$$x = \underbrace{r\cos\theta}, \quad y = \underbrace{r\sin\theta},$$

- $r^2\cos\theta\sin\theta = 4$ $\Leftarrow$ $\underbrace{r\cos\theta}_{x}\ \underbrace{r\sin\theta}_{y} = 4$, $= 4$
- $r = \frac{4}{2\cos\theta - \sin\theta}$ $\Rightarrow$ $r(2\cos\theta - \sin\theta) = 4$

  $2r\cos\theta - r\sin\theta = 4$

  $2x - y = 4$
- $r = 1 + 2r\cos\theta$

$\theta \to \theta + \pi$

$r \to -r$

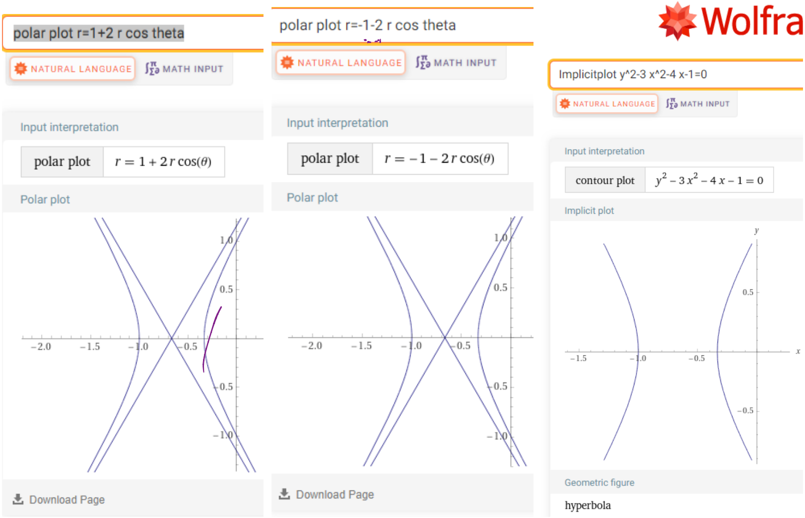
$r^2 = (1 + 2r\cos\theta)^2$

$r = -1 - 2r\cos\theta$

$\Rightarrow x^2 + y^2 = (1 + 2x)^2$

$\Rightarrow x^2 + y^2 = 1 + 4x + 4x^2 \Rightarrow 3x^2 + 4x - y^2 + 1 = 0$

$$r = 1 + 2r\cos\theta,\ r = -1 - 2r\cos\theta,\ y^2 - 3x^2 - 4x - 1 = 0$$



Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

- **Symmetry about the x-axis**: If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.27a).
- **Symmetry about the y-axis**: If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.27b).
- **Symmetry about the origin**: If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.27c).

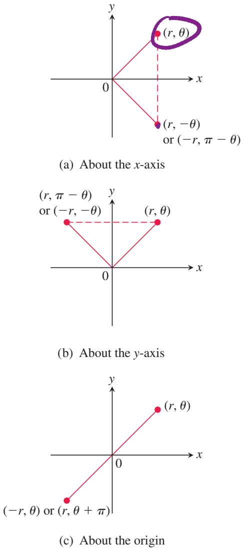


FIGURE 11.27 Three tests for symmetry in polar coordinates.

Slope of a polar curve $r = f(\theta)$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

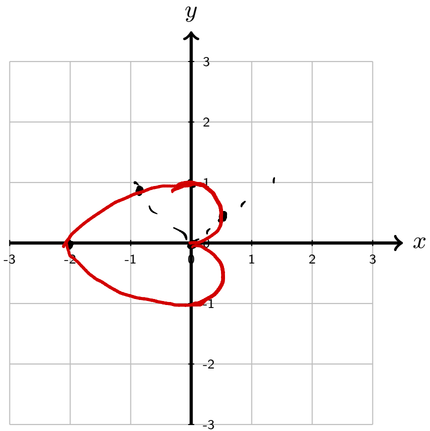
$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope at $(0, \theta_0)$ is

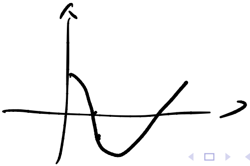
$\tan\theta_0$

Example: Graph the curve $r = 1 - \cos\theta$

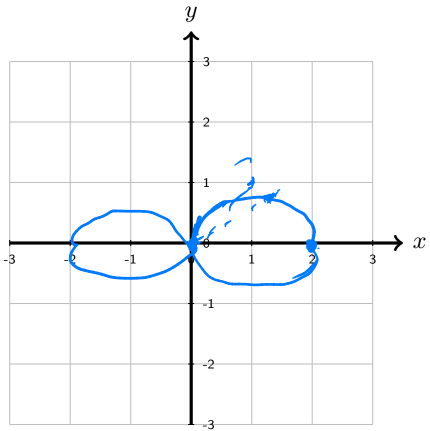


θ	r
0	0
$\frac{\pi}{4}$	$1-\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	$1+\frac{\sqrt{2}}{2}$
π	2

$\theta \to -\theta$



Example: Graph the curve $r^2 = 4\cos\theta$



$(r, -\theta)$ on the graph

$(-r, \theta)$ on the graph

$(r, \pi-\theta)$ not on -

$4\cos(\pi-\theta)$

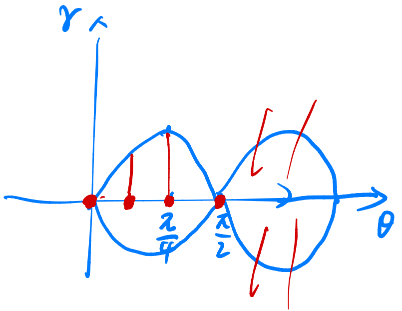
$(-r, -\theta)$ on the graph.

θ	r
0	2
$\frac{\pi}{2}$	0
$\frac{\pi}{4}$	$2\sqrt{\frac{\sqrt{2}}{2}}$
$\frac{\pi}{6}$	

Converting a Graph from the $r\theta$- to xy-Plane

- Use a table and connect some points.
- graph $r = f(\theta)$ and use it as a table.

$$r^2 = \sin 2\theta$$



Converting a Graph from the $r\theta$- to xy-Plane

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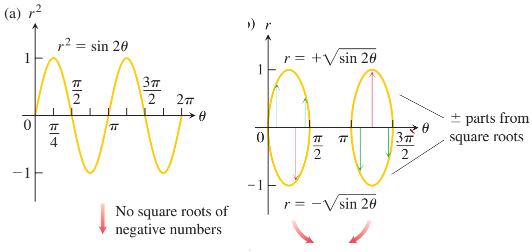

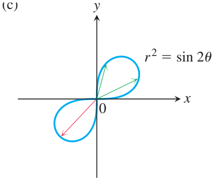

FIGURE 11.30 To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a)

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \leq \theta \leq \beta$

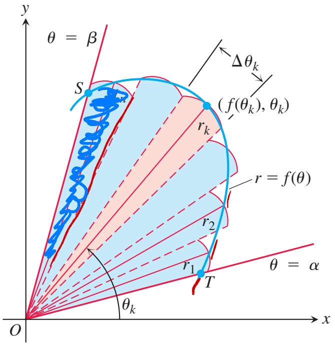

FIGURE 11.31 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$$

$\frac{1}{2} r^2 \cdot \theta$

$\sum_k \frac{1}{2} r^2(\theta_k) \, \Delta\theta$

$\int_{\alpha}^{\beta} \frac{1}{2} r^2(\theta) \, d\theta$

Example: Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

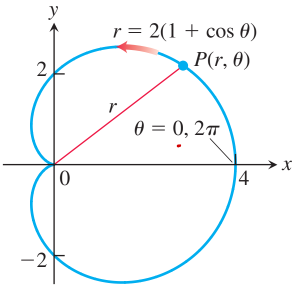

FIGURE 11.33 The cardioid in Example 1.

$$\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \cdot 2^2 (1+\cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} 2 + 4\cos\theta + 2\cos^2\theta \, d\theta$$

$$= 4\pi + 0 + 2\pi$$

$$= 6\pi$$

Area of the region $0 \le r_1(\theta) \le r_2(\theta),\ \alpha \le \theta \le \beta$

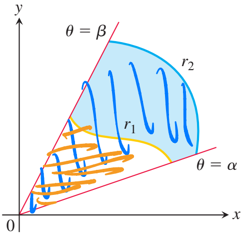

FIGURE 11.34 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

$$\int_\alpha^\beta \frac{1}{2} r_2^2(\theta)\,d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2(\theta)\,d\theta$$

$$= \int_\alpha^\beta \frac{1}{2}\left(r_2^2(\theta) - r_1^2(\theta)\right) d\theta$$

Example: Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

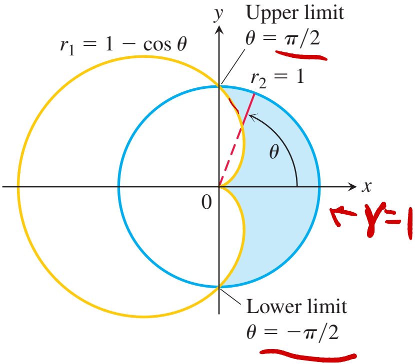

FIGURE 11.35 The region and limits of integration in Example 2.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(1 - 1 + 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta - \underline{\cos^2\theta}\, d\theta$$

$$= \frac{1}{2}\left(2\sin\theta - \frac{1}{4}\sin 2\theta - \frac{1}{2}\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2 - \frac{1}{4}\cdot\pi$$

$$2\cos^2\theta - 1 = \cos 2\theta$$

$$-\cos^2\theta = -\frac{\cos 2\theta + 1}{2}$$

Length of a Polar Curve $r = f(\theta)$

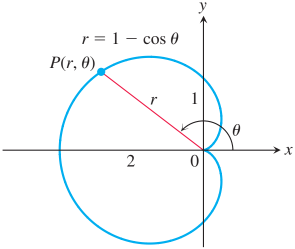

FIGURE 11.36 Calculating the length of a cardioid (Example 3).

$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$

$$\underbrace{\int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}}\, d\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (f'(\theta))^2\cos^2\theta + f^2(\theta)\sin^2\theta$$
$$- \cancel{2f'(\theta)\cos\theta f(\theta)\sin\theta} + (f'(\theta))^2\sin^2\theta$$
$$+ f^2(\theta)\cos^2\theta + \cancel{2f'(\theta)\cos\theta f(\theta)\sin\theta}$$
$$= f^2(\theta) + (f'(\theta))^2$$

Length of a Polar Curve $r = f(\theta)$

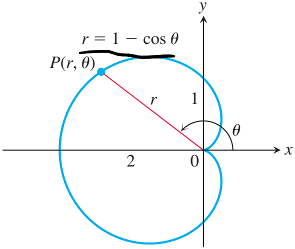

FIGURE 11.36 Calculating the length of a cardioid (Example 3).

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\frac{dr}{d\theta} = \sin\theta$$

$$L = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$= 1 - 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = -4\cos\frac{\theta}{2}\Big|_0^{2\pi} = 8$$