MAT1002: Calculus II

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§10.4 Comparison Tests
§10.5 Absolute Convergence; The Ratio and Root Tests
§10.6 Alternating Series and Conditional Convergence

Review

- geometric series $a \neq 0$ $\Sigma a r^n$ converges if and only if $|r|<1$
- harmonic series $\Sigma \frac{1}{n}$ diverges
- p-series $\Sigma \frac{1}{n^p}$ converges if and only if $p>1$

We will use the existing convergence results of some series.

$\Sigma \frac{1}{n(n+1)}$ Converges

The Comparison Test

Theorem (Theorem 10)

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with **nonnegative** *terms. Suppose that for some integer N*

$$d_n \leq a_n \leq c_n, \qquad \forall n > N$$

- *If $\sum c_n$ converges, then $\sum a_n$ also converges*
- *If $\sum d_n$ diverges, then $\sum a_n$ also diverges*

Examples

- $\sum_{n=1}^{\infty} \frac{5}{5n-1}$

$$\frac{5}{5n-1} > \frac{5}{5n} = \frac{1}{n}$$

$\sum \frac{1}{n}$ diverges, so $\sum \frac{5}{5n-1}$ diverges

- $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\frac{1}{n(n-1)\cdots 1} \le \frac{1}{n(n-1)}$$

if $n > 2$

$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges

so $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

$$\sum_{n=1}^{\infty} \frac{5}{5n+1}$$

$$\frac{5}{5n+1} > \frac{5}{5(n+1)}$$

$\sum \frac{1}{n+1}$ diverges

So $\sum_{n=1}^{\infty} \frac{5}{5n+1}$ diverges

The Limit Comparison Test

Theorem (Theorem 11)

Suppose that $a_n > 0$ *and* $b_n > 0$ *for* $n \geq N$

- *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *then both* $\sum a_n$ *and* $\sum b_n$ *converge or both diverge.*
- *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
- *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

$\exists N$ s.t. $\frac{a_n}{b_n} \in \left(\frac{c}{2}, \frac{3}{2}c\right) \quad \forall n > N$

$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3}{2}c \Leftrightarrow \frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$

$\exists N$ s.t. $\frac{a_n}{b_n} \in [0, \varepsilon) \quad \forall n > N$

$0 \leq \frac{a_n}{b_n} < \varepsilon \Leftrightarrow 0 \leq a_n < \varepsilon \cdot b_n$

$\exists N$ s.t. $\frac{a_n}{b_n} > 1 \quad \forall n > N$

$\Rightarrow b_n < a_n$

Example

- $\sum \frac{2n+1}{n^2+2n+1}$ $\to$ $\sim \frac{2}{n}$ diverges,

$$\frac{\frac{2n+1}{n^2+2n+1}}{\frac{1}{n}} = \frac{2n^2+n}{n^2+2n+1} \to 2$$

diverges because $\sum \frac{1}{n}$ diverges.

- $\sum \frac{1+n\ln n}{n^2+5}$

$$\frac{\frac{1+n\ln n}{n^2+5}}{\frac{1}{n}} = \frac{n+n^2\ln n}{n^2+5} \to +\infty$$

diverges because $\sum \frac{1}{n}$ diverges

Example

- $\sum \frac{\ln n}{n^{1.1}}$

$$\frac{\frac{\ln n}{n^{1.1}}}{\frac{1}{n^{1.1}}} = \ln n \to +\infty$$

$$\frac{\frac{\ln n}{n^{1.1}}}{\frac{1}{n}} = \frac{\ln n}{n^{0.1}} \to 0$$

$$\lim_{n\to\infty} \frac{\ln n}{n^{0.1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{0.1 n^{-0.9}}$$

$$= \lim_{n\to\infty} \frac{10}{n^{0.1}}$$

$$= 0$$

$$\frac{\frac{\ln n}{n^{1.1}}}{\frac{1}{n^{1.05}}} = \frac{\ln n}{n^{0.05}} \to 0$$

since $\sum \frac{1}{n^{1.05}}$ converges,

so $\sum \frac{\ln n}{n^{1.1}}$ converges.

What happens with negative terms?

Definition

A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if $\sum |a_n|$ converges.

Theorem (Theorem 11)

An absolutely convergent series converges, i.e., if $\sum |a_n|$ converges, then $\sum a_n$ converges.

Hint: Use the combing rule.

$$a_n = \max\{0, a_n\} + \min\{0, a_n\}$$

$$= \underbrace{\max\{0, a_n\}}_{b_n} - \underbrace{\max\{0, -a_n\}}_{c_n}$$

$$0 \le b_n \le |a_n| \qquad 0 \le c_n \le |a_n|$$

Example

- $\sum \frac{\sin n}{n^2}$

$$\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$$

since, $\sum \frac{1}{n^2}$ converges.

then $\sum \left|\frac{\sin n}{n^2}\right|$ converges

comparison test.

so $\sum \frac{\sin n}{n^2}$ converges.

The Ratio Test (Important)

$a_n = \frac{1}{n}$ or $a_n = \frac{1}{n^2}$

Theorem (Theorem 13)

Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$$

- *If $\rho < 1$, then the series $\sum a_n$ converges.*
- *If $\rho > 1$ or $\rho = \infty$, the series diverges* → use nth term test for divergence
- *If $\rho = 1$, the test is inconclusive.*

$\exists N$ s.t. $\left| \frac{a_{n+1}}{a_n} \right| < \rho + \varepsilon < 1 \quad \forall n > N$

$$|a_{N+k}| < (\rho+\varepsilon)|a_{N+k-1}|$$
$$< (\rho+\varepsilon)^2 |a_{N+k-2}| < \cdots < (\rho+\varepsilon)^k |a_N|$$

$$\sum_{n=N}^{\infty} |a_n| \le \underbrace{\sum_{k=0}^{\infty} (\rho+\varepsilon)^k |a_N|}_{\text{converges}}$$

Example

- $\sum \frac{(2n)!}{n!n!}$

$$\frac{\frac{(2n+2)!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}} = \frac{\frac{(2n+2)!}{(2n)!}}{\frac{(n+1)!}{n!}\,\frac{(n+1)!}{n!}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \to 4$$

diverges

- $\sum \frac{4^n n!n!}{(2n)!}$

$$\frac{\frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!}}{\frac{4^n\,n!\,n!}{(2n)!}} = \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} \to 1$$

Converges?

$$a_n = \begin{cases} \frac{n}{2^n} & n \text{ odd} \\ \frac{1}{2^n}, & n \text{ even} \end{cases}$$

$$\frac{a_{n+1}}{a_n} = \begin{cases} \dfrac{\frac{n+1}{2^{n+1}}}{\frac{1}{2^n}} = \dfrac{n+1}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{\frac{1}{2^{n+1}}}{\frac{n}{2^n}} = \dfrac{1}{2n} & \text{if } n \text{ is odd} \end{cases}$$

The Root Test

Theorem (Theorem 14)

Suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

- *If $\rho < 1$, then the series converges*
- *If $\rho > 1$ or $\rho = \infty$, then the series diverges*
- *If $\rho = 1$, then the test is inconclusive.*

$\exists N$ s.t. $\sqrt[n]{|a_n|}$

Converges?

$$a_n = \begin{cases} \frac{n}{2^n} & n \text{ odd} \\ \frac{1}{2^n}, & n \text{ even} \end{cases}$$

Example:

- $\sum \frac{n^2}{2^n}$

- $\sum \frac{2^n}{3^n}$

Alternating Series

An **alternating series** has terms that are alternatively positive and negative.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

The Alternating Series Test

Theorem (Theorem 15)

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ *converges if the following are satisfied*

- $u_n \geq u_{n+1} > 0$ *for* $n \geq N$.
- $u_n \to 0$.

The Alternating Series Test

Theorem (Theorem 15)

The series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ *converges if the following are satisfied*

- $u_n \geq u_{n+1} > 0$ *for* $n \geq N$.
- $u_n \to 0$.

WLOG, assume that $N = 1$. Consider s_{2m}

- s_{2m} is increasing, while s_{2m+1} is decreasing.
- $0 \leq s_{2m} \leq s_{2m+1} \leq u_1$.
- Both s_{2m} and s_{2m+1} converge, and $u_{2m+1} \to 0$.

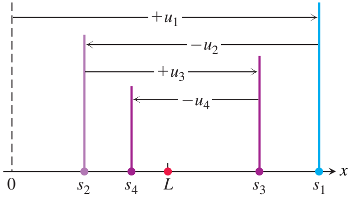

FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

Examples

- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Use derivative to check monotonicity

Define $f(x)$ such that $f(n) = u_n$.

$$u_n = \frac{10n}{n^2 + 16}$$

Alternating Series Estimation Theorem

Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions of Theorem 15, then for $n \geq N$, then we have

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the total sum L of the series with an error whose absolute values is less than u_{n+1}. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder $L - s_n$ has the same sign as the first unused term.

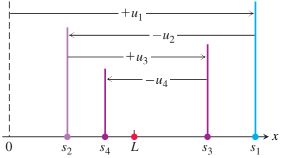

FIGURE 10.13 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

Example

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

$$s_8=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}=\frac{85}{128}=0.6640625,\ s_9=0.66796875$$