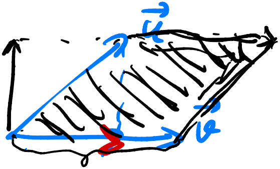

MAT1002: Calculus II

$\vec{u} \times \vec{v} = |\vec{u}| \cdot |\vec{v}| \sin\theta \; \vec{w}$ $\qquad \|\vec{w}\| = 1$

Ming Yan

§12.4 The Cross Product
§12.5 Lines and Planes in Space

$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

§12.4 The Cross Product

In a plane, we can use slope and angle of inclination to describe a line. How do you describe a plane in space?
They determine a plane if $\vec{u}$ and $\vec{v}$ are not parallel. We select a unit vector $\vec{n}$ perpendicular to the plane by the **right-hand rule**.

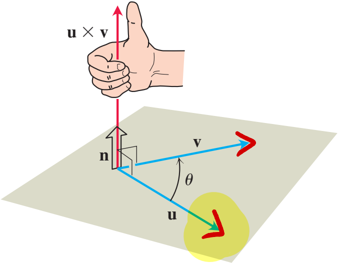

FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

Definition

The **cross/vector product** $\vec{u} \times \vec{v}$ is the vector

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}$$

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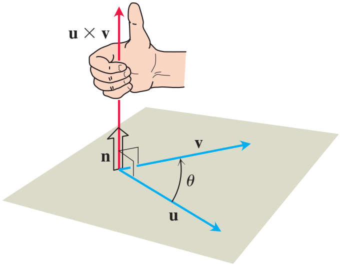

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Definition

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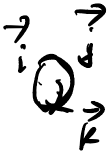
$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\vec{n}$$

- $\boxed{\vec{i} \times \vec{j} = \vec{k}} = -(\vec{j} \times \vec{i})$
- $\vec{j} \times \vec{k} = \vec{i} = -(\vec{k} \times \vec{j})$
- $\vec{k} \times \vec{i} = \vec{j} = -(\vec{i} \times \vec{k})$
- $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$



§12.4 The Cross Product

In a plane, we can use slope and angle of inclination to describe a line. How do you describe a plane in space?
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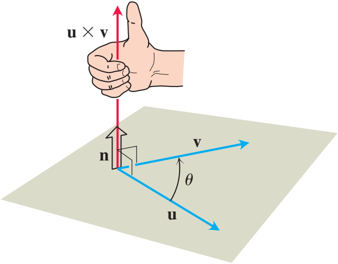

FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

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The **cross/vector product** $\vec{u} \times \vec{v}$ is the vector

$$\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin\theta)\vec{n}$$

- $\vec{i} \times \vec{j} = \vec{k} = -(\vec{j} \times \vec{i})$
- $\vec{j} \times \vec{k} = \vec{i} = -(\vec{k} \times \vec{j})$
- $\vec{k} \times \vec{i} = \vec{j} = -(\vec{i} \times \vec{k})$
- $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$

Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$

$|\vec{u} \times \vec{v}|$ is the area of a parallelogram

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}||\sin\theta||\vec{n}| = \underbrace{|\vec{u}||\vec{v}||\sin\theta|}$$

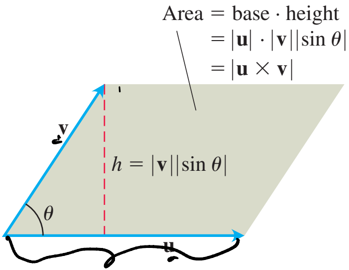
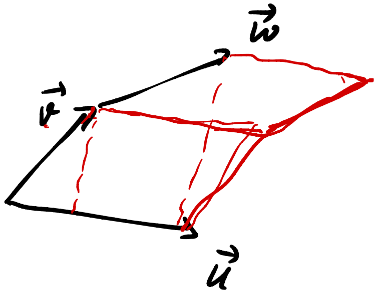


FIGURE 12.30 The parallelogram determined by **u** and **v**.

Some Properties of the Cross Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors and $r,\ s$ are scalars, then

- $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{0} \cdot \vec{u} = \vec{0}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$ B
- $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$ A



Determinant formula for $\vec{a} \times \vec{b}$

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

$$= \cancel{a_1 b_1 \vec{i}\times\vec{i}} + a_1 b_2 \vec{i}\times\vec{j} + a_1 b_3 \vec{i}\times\vec{k}$$

$$+ a_2 b_1 \vec{j}\times\vec{i} \qquad + a_2 b_3 \vec{j}\times\vec{k}$$

$$+ a_3 b_1 \vec{k}\times\vec{i} + a_3 b_2 \vec{k}\times\vec{j}$$

$$= (a_1 b_2 - a_2 b_1)\vec{k} + (a_3 b_1 - a_1 b_3)\vec{j}$$

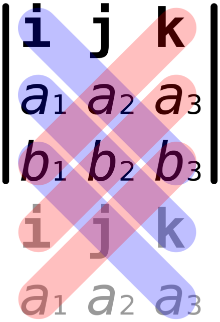
$$+ (a_2 b_3 - a_3 b_2)\vec{i}$$

Determinant formula for $\vec{a} \times \vec{b}$

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

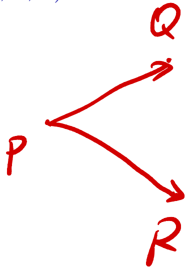
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \begin{matrix} \vec{i} \\ a_1 \\ b_1 \end{matrix}$$

$$\begin{aligned} &+\mathbf{i}a_2b_3 \\ &+a_1b_2\mathbf{k} \\ &+b_1\mathbf{j}a_3 \\ &-b_1a_2\mathbf{k} \\ &-\mathbf{i}b_2a_3 \\ &-a_1\mathbf{j}b_3 \end{aligned}$$



Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$

$\vec{PQ} \times \vec{PR}$



$$\vec{PQ} = \langle 2-1,\ 1-(-1),\ -1 \rangle = \langle 1,\ 2,\ -1 \rangle$$

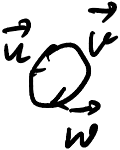
$$\vec{PR} = \langle -1-1,\ 1-(-1),\ 2 \rangle = \langle -2,\ 2,\ 2 \rangle$$

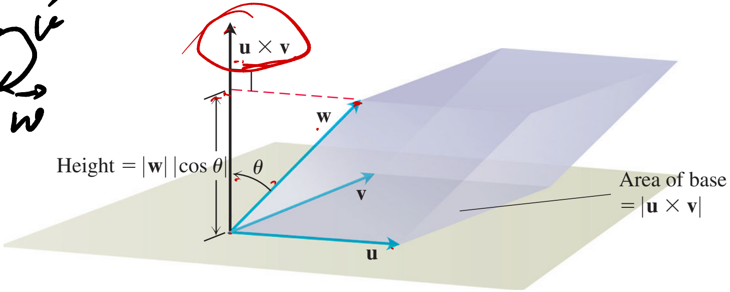
$$\left|\begin{array}{ccc|c} \vec{i} & \vec{j} & \vec{k} & \\ 1 & 2 & -1 & 1 \\ -2 & 2 & 2 & -2 \end{array}\right.$$

$$\vec{PQ} \times \vec{PR} = 6\vec{i} + 0\vec{j} + 6\vec{k}$$

Triple scalar or box product

$$|(\vec{u} \times \vec{v}) \cdot \vec{w}| = |\vec{u} \times \vec{v}||\vec{w}||\cos\theta|$$





$$\begin{aligned} \text{Volume} &= \text{area of base} \cdot \text{height} \\ &= |\mathbf{u} \times \mathbf{v}||\mathbf{w}||\cos\theta| \\ &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \end{aligned}$$

FIGURE 12.34 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

Calculating the Triple Scalar Product as a Determinant

$u_1 \leftrightarrow w_1$ $\quad$ w, v, u $\quad$ w, u, v

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

u_1, v_1, w_1

$$(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{v}\times\vec{w})\cdot\vec{u} = (\vec{w}\times\vec{u})\cdot\vec{v}$$

$$u_1\cdot(v_2w_3 - v_3w_2) + u_2\cdot(v_3w_1 - v_1w_3) + u_3\cdot(v_1w_2 - v_2w_1)$$

$$= \underbrace{(u_1\vec{i} + u_2\vec{j} + u_3\vec{k})}\cdot\big((v_2w_3 - v_3w_2)\vec{i} + (v_3w_1 - v_1w_3)\vec{j} + (v_1w_2 - v_2w_1)\vec{k}\big)$$

$$= \vec{u}\cdot(\vec{v}\times\vec{w})$$

$$= (\vec{v}\times\vec{w})\cdot\vec{u} = (\vec{u}\times\vec{v})\cdot\vec{w}$$

Properties of the Cross Product

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors and $r,\ s$ are scalars, then

- $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$
- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{0} \cdot \vec{u} = \vec{0}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$
- $\vec{u} \times \underbrace{(\vec{v} \times \vec{w})} = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

$\vec{u} \times \vec{s}$

since $\vec{s}$ perpendicular to $\vec{v}$ and $\vec{w}$

so $\vec{u} \times \vec{s} = a\vec{v} + b\vec{w}$

$\vec{u} \cdot (\vec{u} \times \vec{s}) = \underline{a}\, \vec{u} \cdot \vec{v} + \underline{b}\, \vec{u} \cdot \vec{w} = 0$

§12.5 Lines and Planes in Space

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v}$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot t$$

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

Example: Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\langle 2, 4, -2 \rangle$

$$x(t) = -2 + 2t$$

$$y(t) = 0 + 4t$$

$$z(t) = 4 - 2t$$

Example: Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$

$\overrightarrow{PQ} = \langle 1-(-3), -1-2, 4-(-3) \rangle$

$= \langle 4, -3, 7 \rangle$

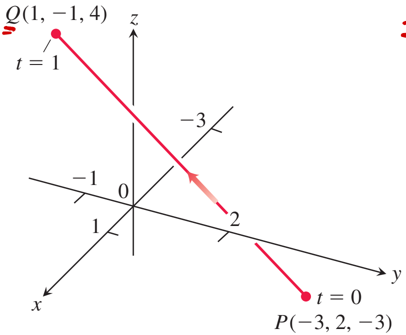


FIGURE 12.37 Example 3 derives a parametrization of line segment PQ. The arrow shows the direction of increasing t.

$x(t) = -3 + 4t$

$y(t) = 2 - 3t$

$z(t) = -3 + 7t$

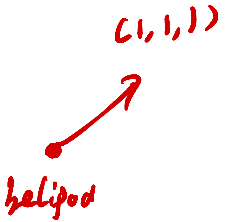
$t \in [0, 1]$

$0 \le t \le 1$

A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of $60m/s$. What is the position of the helicopter after $10s$?

$(600, 600, 600)$?

$|(600, 600, 600)| = 600 \times \sqrt{3}$



$$600 \cdot \frac{\langle 1, 1, 1 \rangle}{|\langle 1, 1, 1 \rangle|}$$

$$= \left\langle \frac{600}{\sqrt{3}}, \frac{600}{\sqrt{3}}, \frac{600}{\sqrt{3}} \right\rangle$$

Distance from a Point S to a Line Through P Parallel to $\vec{v}$

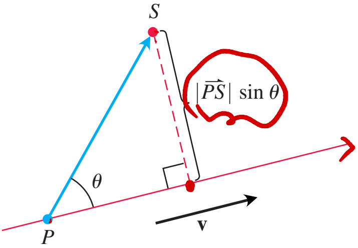


FIGURE 12.38 The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}|\sin\theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

$$|\vec{PS}\times\vec{v}|$$
$$= \underline{|\vec{PS}|}\cdot|\vec{v}|\cdot\underline{|\sin\theta|}$$

$$\frac{|\vec{PS}\times\vec{v}|}{|\vec{v}|}$$

$$\vec{PS} = \langle 1-1,\ 1-3,\ 5-0\rangle$$
$$= \langle 0,\ -2,\ 5\rangle$$

Find the distance from the point $S(1, 1, 5)$ to the line

$$L : x = 1 + t,\ \ y = 3 - t,\ \ z = 2t$$

$$\vec{PS}\times\vec{v} = \vec{i} + 5\vec{j} + 2\vec{k}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} \begin{matrix} \vec{i} \\ 0 \\ 1 \end{matrix}$$

$$P = \{1, 3, 0\}$$
$$\vec{v} = \langle 1, -1, 2\rangle$$

$$\frac{|\vec{PS}\times\vec{v}|}{|\vec{v}|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \sqrt{5}$$

An Equation for a Plane in Space

A plane is determined by a point on the plane and its "tilt" or orientation.

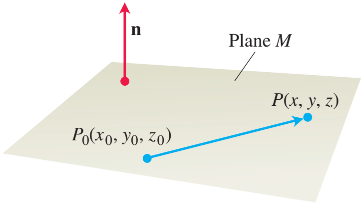


FIGURE 12.39 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

The plane through $P_0(x_0, y_0, z_0)$ normal to $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k} = \langle A, B, C \rangle$ has

- **Vector equation:** $\vec{n} \cdot \overrightarrow{P_0P} = 0$
- **Component equation:** $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
- **Component equation simplified:** $Ax + By + Cz = Ax_0 + By_0 + Cz_0$

Example: Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\vec{n} = 5\vec{i} + 2\vec{j} - \vec{k}$

$$5 \cdot (x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$

$$\Rightarrow 5x + 2y - z = -15 - 7$$

$$\Rightarrow 5x + 2y - z = -22$$

Example: Find an equation for the plane through $A(0,0,1)$, $B(2,0,0)$, and $C(0,3,0)$.

$$ax + by + cz = d \iff \frac{a}{d}x + \frac{b}{d}y + \frac{c}{d}z = 1$$

$$A(0,0,1) \Rightarrow c = d$$

$$B(2,0,0) \Rightarrow 2a = d$$

$$C(0,3,0) \Rightarrow 3b = d$$

Let $d = 1 \Rightarrow c = 1 \quad a = \frac{1}{2} \quad b = \frac{1}{3}$

$$\frac{1}{2}x + \frac{1}{3}y + z = 1$$

Lines of Intersection

Two planes that are not parallel intersect in a line.

$\vec{v}$ in both planes

so $\vec{n}_1 \perp \vec{v}$ $(\vec{n}_1 \cdot \vec{v} = 0)$

$\vec{n}_2 \perp \vec{v}$ $(\vec{n}_2 \cdot \vec{v} = 0)$

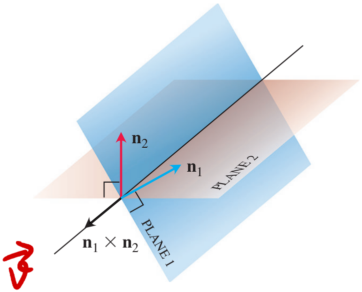


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

so $\vec{v} = \vec{n}_1 \times \vec{n}_2$

$= \dots$

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

$\vec{n}_1 = \langle 3, -6, -2 \rangle$ $\quad$ $\vec{n}_2 = \langle 2, 1, -2 \rangle$

Example: Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect

Find a point on the line

$$\begin{cases} 3x - 6y - 2z = 15 \\ 2x + y - 2z = 5 \end{cases}$$

Let $z = 0$ so we have

$$\begin{cases} 3x - 6y = 15 \\ 2x + y = 5 \end{cases} \Rightarrow \begin{cases} x - 2y = 5 \\ 2x + y = 5 \end{cases}$$

$$\Rightarrow x = 3 \quad y = -1$$

$(3, -1, 0)$

Example

Find the point where the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

intersect the plane $3x + 2y + 6z = 6$.

$$3 \cdot \left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$$

$$\Rightarrow 8 + 6t - 4t + 6 + 6t = 6$$

$$\Rightarrow t = -1$$

$$x = \frac{2}{3} \qquad y = 2 \qquad z = 0$$

Distance from a Point to a Plane

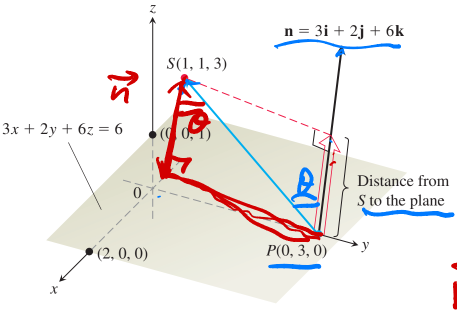


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$ (Example 11).

$$|\overrightarrow{PS}||\cos\theta|$$

$$= \frac{|\overrightarrow{PS} \cdot \vec{n}|}{|\vec{n}|}$$

$$\overrightarrow{PS} = \langle 1-0,\ 1-3,\ 3-0 \rangle$$

$$\vec{n} = \langle 3,\ 2,\ 6 \rangle$$

Angles Between Planes

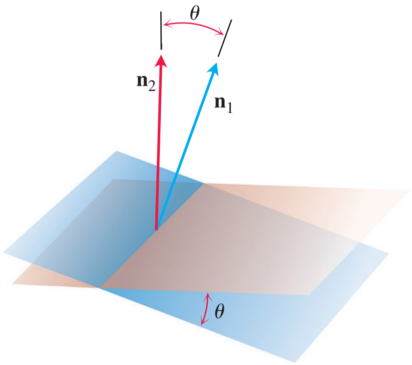


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.