香港中文大學（深圳）
The Chinese University of Hong Kong, Shenzhen

PHY1001 Mechanics

2021-2022 Term 2

Final Examination

May 17th, 2022;   Time Allowed: 3 Hours

NAME (print) ______________________________

CUHKSZ ID ______________________________

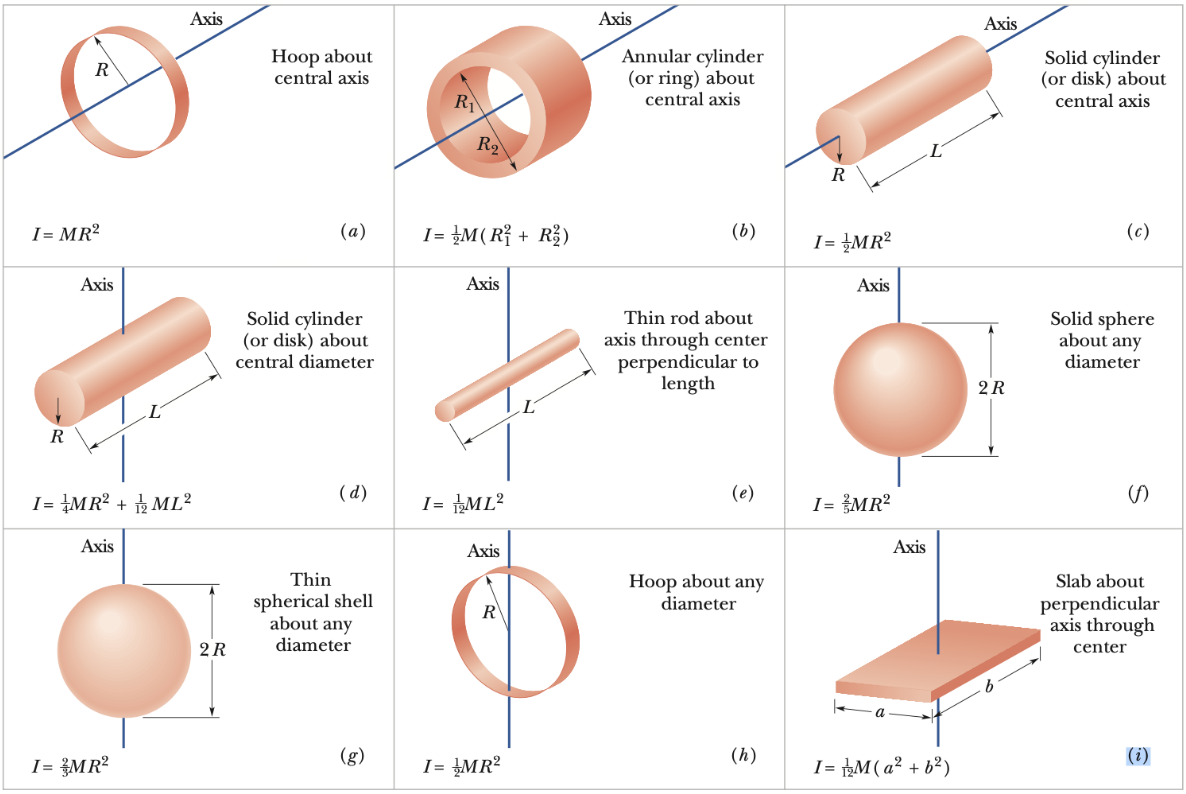
ZOOM/Seat No. ______________________________

- **Show all your work**. Correct answers with little supporting work will not be given credit.
- Closed Book Exam: One piece of double-sided A4 reference paper, a scientific calculator, and a paper-based dictionary are allowed.
- Students who are late for more than 30 minutes will NOT be admitted.
- The total points are 120 points. You need to finish ALL the questions in 3 hours (180 minutes).

#1	#2	#3	#4	#5	#6	#7	#8	#9	#10	#11	#12	TOTAL
10	10	10	10	10	10	10	10	10	10	10	10	120

Summary of Basic Calculus:

$$\frac{d}{dx}x^n = nx^{n-1}, \qquad \frac{d}{dx}e^{ax} = ae^{ax}, \qquad \frac{d}{dx}\ln ax = \frac{1}{x},$$
$$\frac{d}{dx}\sin kx = k\cos kx, \qquad \frac{d}{dx}\cos kx = -k\sin kx,$$
$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v,$$
$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C,$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{2}\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} + C,$$
$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}} + C, \quad \text{where } C \text{ is a constant.}$$



Bernoulli's equation

$$P + \rho g y + \frac{1}{2}\rho v^2 = \text{Const.}$$

SHO: $\quad m\frac{d^2x}{dt^2} = -Kx$

Spring mass system

1. Figure 1 shows a stationary horizontal nonuniform bar suspended by two massless cords. As shown in the figure, $\alpha = 15°$, and $\beta = 45°$. The length L and mass M of the bar are 1.00 m and 1.00 kg, respectively. ($\sin 15° = 0.259$, $\cos 15° = 0.966$; $\sin 45° = \cos 45° = 0.707$) **(10 pts)**

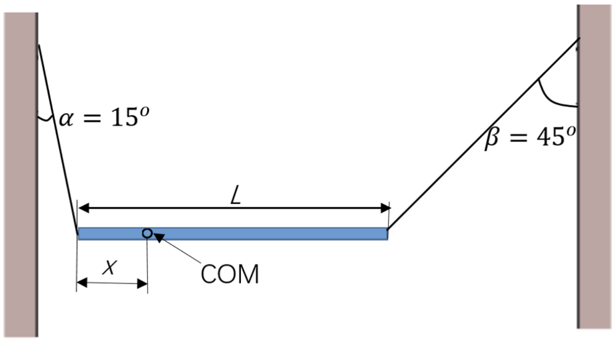


Figure 1

(a) Find the position of the COM (center of mass) of the bar, i.e., the distance x. **(6 pts)**

Solution:

a) The bar is in equilibrium, so the total forces and the total torques acting on it are zero. Let T_L and T_R be the tensions in wires 1 and 2, respectively. The equilibrium equations are

Vertical direction: $T_L cos\alpha + T_R\, cos\beta = mg$ (1) 1 pt

Horizontal direction: $T_L sin\alpha = T_R\, sin\beta$ (2) 1 pt

Torque about the left end: $mgx = T_R\, cos\beta L$ (3) 1 pt

There are three unknowns in the above three equations. We can eliminate T_1 and T_2, and then solve for x, which gives

$$x = \frac{sin\alpha \cdot cos\beta}{\sin(\alpha+\beta)} L = 0.211(\text{m}) \quad (4)$$ 3 pts

(b) Find the tension forces T_L of the left cord and T_R of the right cord. **(4 pts)**

b) Inserting (4) into (3), we have

$$T_R = \frac{mgx}{Lcos\beta} = 2.93\ (\text{N}) \quad (5)$$ 2 pts

Inserting (5) into (2), we have

$$T_L = \frac{T_R\, sin\beta}{sin\alpha} = 8.00(\text{N}) \quad (6)$$ 2 pts

2. A thin, uniform rod has a length L and a mass M. A small particle of mass m is placed a distance a from the center of the rod, as shown in Figure 2. **(10 pts)**

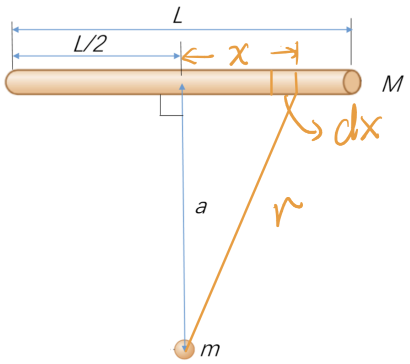


Figure 2

(a) Consider a small element of the rod, the potential energy $dU = -\frac{GmdM}{r} = -\frac{GmM}{L}\frac{dx}{r}$. Show the total potential energy between the rod and the particle is **(5 pts)**

$$U = -\frac{GMm}{L}\ln\frac{\sqrt{\frac{L^2}{4}+a^2}+\frac{L}{2}}{\sqrt{\frac{L^2}{4}+a^2}-\frac{L}{2}}. \qquad \text{(See integral identities on Page 1.)}$$

Solution: Potential energy.

$$U = \int dU = -\int \frac{GmdM}{r} = -\frac{GMm}{L}\int_{-L/2}^{L/2}\frac{dx}{r} \quad \text{—— 1pt}$$

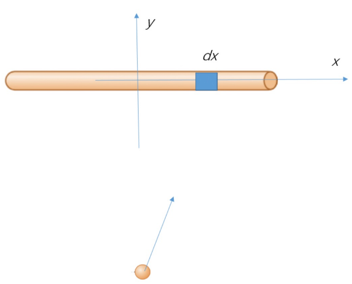
$$= -\frac{GMm}{L}\int_{-L/2}^{L/2}\frac{dx}{\sqrt{x^2+a^2}} \quad \text{—— 2pt}$$

use $\int dx \frac{1}{\sqrt{x^2+a^2}} = \frac{1}{2}\ln\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x} + C.$ —— 1pt

$$= -\frac{GMm}{L}\ln\frac{\sqrt{\frac{L^2}{4}+a^2}+\frac{L}{2}}{\sqrt{\frac{L^2}{4}+a^2}-\frac{L}{2}} \quad \text{—— 1pt}$$

(b) Show that the magnitude of the gravitational force F between the rod and the particle is given by **(5 pts)**

$$F = \frac{GMm}{a\sqrt{\frac{L^2}{4}+a^2}}. \qquad \text{(See integral identities on Page 1.)}$$



We consider an infinitesimal element dx, with a coordinate x in the rod, the y component of the gravitational force between this element and the mass m is

$$\mathrm{d}F_y = \frac{G\cdot\frac{M}{L}dx\cdot m}{x^2+a^2}\frac{a}{\sqrt{x^2+a^2}} \qquad \text{2 pts}$$

Considering the symmetry of the rod, the total gravitational force will be along the y direction, and

$$F_y = 2\int_0^{L/2}\frac{GMma}{L}\frac{1}{(x^2+a^2)^{3/2}}dx \qquad \text{2 pts}$$

$$= \frac{GMm}{a\sqrt{a^2+L^2/4}} \qquad \text{1 pt}$$

3. Figure 3 shows the top view of a horizontal pipe placed on a flat surface with the following structural parameters: $d_1 = 8.0$ cm, $d_2 = 1.0$ cm, and $d_3 = 2.0$ cm. The flowing velocities of the fluid going through Sections 1 and 2 are $v_1 = 2.0$ m/s and $v_2 = 8.0$ m/s, respectively. (No need to consider the gravitational effects.) **(10 pts)**

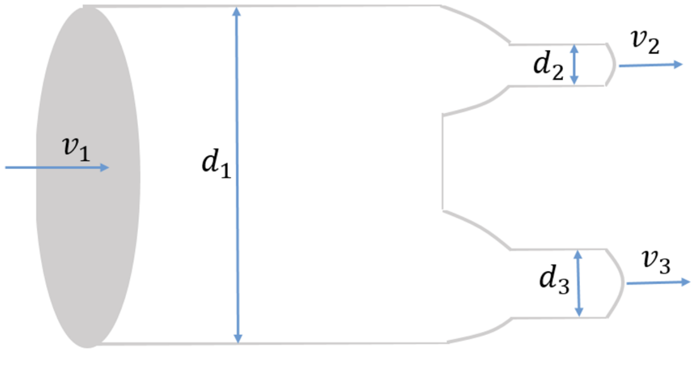


Figure 3

(a) Find the volume flow rate (volume per second) crossing this pipe. **(5 pts)**

Solution:

(1) The volume flowing rate is

$$R_v = Av \quad \text{2 pts}$$

$$= \frac{\pi}{4} d_1^2 v_1 \quad \text{2 pts}$$

$$= 1.0 \times 10^{-2}\ m^3/s \quad \text{1 pts}$$

(Correct result with one of the above steps get full credit.)

(b) Use the continuity equation, find the flow velocity v_3. **(5 pts)**

(2) The volume flowing-in rate equals the volume flowing-out rate, and we have

$$\frac{\pi}{4} d_1^2 v_1 = \frac{\pi}{4} d_2^2 v_2 + \frac{\pi}{4} d_3^2 v_3 \quad \text{2 pts}$$

So that

$$v_3 = \frac{d_1^2 v_1 - d_2^2 v_2}{d_3^2} \quad \text{2 pts}$$

$$= 30\ m/s \quad \text{1 pt}$$

4. A simple harmonic spring-mass system with four springs in parallel is shown in Figure 4. The structural parameters are as follows: $m = 1.00$ kg, $k_1 = k_2 = k_3 = k_4 = 1.00$ N/m. **(10 pts)**

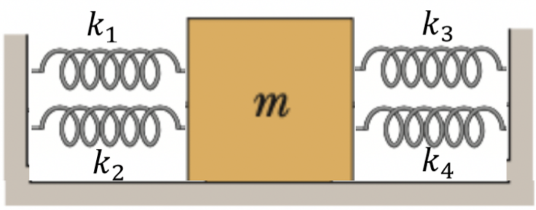


Figure 4

(a) Suppose the mass has a small displacement x, write down the equation of motion of the spring-mass system according to Newton's 2nd law. **(3 pts)**

Newton's second law

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = -(k_1+k_2+k_3+k_4)x \quad \text{—— 3pts}$$
$$= -4k_1x \equiv -k_{eff}x$$

As to other incorrect results

$$m\frac{d^2x}{dt^2} = -\#kx \text{ , give 1pt}$$

(b) Find the angular frequency ω of this oscillator? **(3 pts)**

Angular Frequency:

$$\omega = \sqrt{\frac{k_{eff}}{m}} \quad \text{—— 2pts}$$

$$= \sqrt{\frac{4k_1}{m}} = 2.00 \text{ rad/s.} \quad \text{—— 1pt}$$

(c) Find the frequency and period of this oscillator. **(4 pts)**

Frequency $f = \frac{\omega}{2\pi}$ —— 1pt

$= 0.318$ HZ —— 1pt

Period $T = \frac{2\pi}{\omega}$ —— 1pt

$= 3.14$ s. —— 1pt

5. A linear damped oscillator is shown in Figure 5. **(10 pts)**

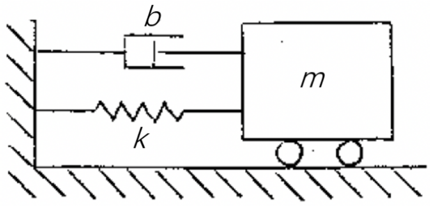


Figure 5

(a) Given the equation of motion (EOM) of the damped oscillator

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx,$$

show that $x(t) = A(t)\cos(\omega t) = A_0 e^{-\frac{b}{2m}t}\cos(\omega t)$ satisfies the above EOM if ω is properly chosen. (For simplicity, we set the phase constant δ to 0.) **(4 pts)**

Solution: Given $X(t) = A_0 e^{-\frac{b}{2m}t}\cos\omega t$

then $\frac{dx(t)}{dt} = -\frac{b}{2m}A_0 e^{-\frac{b}{2m}t}\cos\omega t - \omega A_0 e^{-\frac{b}{2m}t}\sin\omega t$ —— 1pt

$\frac{d^2x(t)}{dt^2} = \left(\frac{b}{2m}\right)^2 A(t)\cos\omega t + 2\omega\frac{b}{2m}A(t)\sin(\omega t)$ —— 1pt

$- \omega^2 A(t)\cos\omega t$

To satisfy the EOM. $\Rightarrow$ $m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$ —— 1pt

$\Rightarrow \frac{b^2}{4m} - m\omega^2 = \frac{b^2}{2m} - k \Rightarrow \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$ —— 1pt.

Suppose the parameters are given as follows for the parts below, $m = 1.00$ kg, $k = 1.00$ N/m, $b = 1.00$ kg/s, and the amplitude of oscillation $A_0 = 0.100$ m at $t = 0$.

(b) Find the value of the angular frequency ω of this damped oscillator. **(2 pts)**

From part a). $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{1 - \frac{1}{4}}$ rad/s

$= 0.866$ rad/s. —— 2pt

(c) Find the initial velocity v_0 (dx/dt at $t = 0$) of this damped oscillator. **(2 pts)**

$V_0 = \frac{dx}{dt}\Big|_{t=0} = -\frac{b}{2m}A_0 e^{-\frac{b}{2m}t}\cos\omega t\Big|_{t=0} - \omega A_0 e^{-\frac{b}{2m}t}\sin\omega t\Big|_{t=0}$ —— 1pt

$= -\frac{b}{2m}A_0 = -0.500 \times 0.100$ m/s $= -5.00\times10^{-2}$ m/s —— 1pt.

(d) Find the amplitude of the damped oscillation $A(t)$ at $t = 2.00$ s. **(2 pts)**

$A(t) = A_0 e^{-\frac{b}{2m}t}\Big|_{t=2.00s}$ —— 1pt

$= 0.100\,m\, e^{-1.00} = 3.68\times10^{-2}\,m$ —— 1pt.

6. Transverse wave. **(10 pts)**

The wave function of a transverse wave traveling along a very long string is

$$y(x,t) = (6.00\times10^{-2}\,\mathrm{m})\cos(20.0\pi\,\mathrm{m}^{-1}x - 120\pi\,\mathrm{s}^{-1}t),$$

(a) What is the amplitude (A) of this wave? **(1 pts)**

$A = 6.00\times10^{-2}\,m$ —— 1pt

(b) In what direction does this wave travel? (In the $+x$ or $-x$ direction) **(1 pts)**

In the $+x$ direction —— 1pt

(c) What is the wave's speed v? **(1 pts)**

$k = 20.0\pi\,(m^{-1})$, $\omega = 120\pi\,(s^{-1})$

$V = \frac{\omega}{k} = \frac{120\pi}{20.0\pi} = 6.00\,m/s$ —— 1pt

(d) Find the wavelength λ, frequency f, and period T of this wave. **(3 pts)**

$k = \frac{2\pi}{\lambda} = 20.0\pi\,(m^{-1})$

$\Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{20.0\pi} = 0.100(m)$

$f = \frac{\omega}{2\pi} = \frac{120\pi(s^{-1})}{2\pi} = 60.0\,Hz$

$T = \frac{1}{f} = \frac{1}{60Hz} = 0.0167\,(s).$

1pt

(e) What is the maximum oscillation speed of any point on the string? **(2 pts)**

$V_y = \frac{\partial y}{\partial t} = \omega A\sin(kx-\omega t)$ —— 1pt

maximum speed $= \omega A = 120\pi\times6.00\times10^{-2}\,m/s$

$= 22.6\,m/s.$ —— 1pt

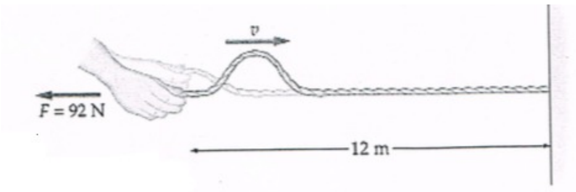
(f) Given the mass per unit length $\mu = 0.05$ kg/m, how much average power $P_{av} = \frac{1}{2}\mu v\omega^2A^2$ must be supplied to the string to generate this sinusoidal wave? **(2 pts)**

$P_{av} = \frac{1}{2}\mu v\omega^2A^2 = \frac{1}{2}\mu v\,(V_{y\,max})^2$ —— 1pt

$= \underbrace{77\,(Watt)}_{} \simeq \underbrace{8\times10^1\,(Watt)}_{}$

either is fine —— 1pt

7. A 12-meter-long rope is pulled tight with a tension of 92 N as shown below. When one end of the rope is given a "thunk" (disturbance), it takes 1.0 s for the disturbance to propagate to the other end. **(10 pts)**



(a) Is this wave on the rope transverse or longitudinal? Explain why? **(2 pts)**

Transverse — 1pt

Because the disturbance (displacement) is transverse to the direction that wave travels. — 1pt

(b) What is the speed of the wave v? **(2 pts)**

$$V = \frac{L}{t} = \frac{12\,m}{1.0s} = 12\ m/s.$$

1pt

(c) What is the linear density (mass per length, μ) of the string? **(3 pts)**

use the formula $V = \sqrt{\frac{F}{\mu}}$ — 1pt

$\Rightarrow \mu = \frac{F}{V^2}$ — 1pt

$= 92\,N / (12\,m/s)^2 = 0.64\ kg/m$ — 1pt

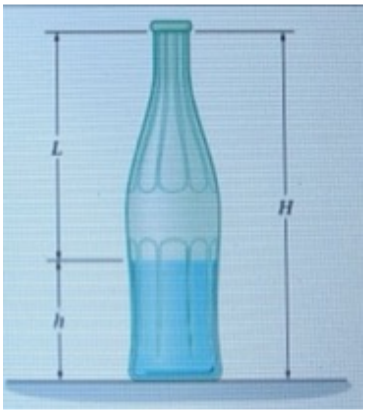
(d) What is the total mass of the rope? **(3 pts)**

Mass of rope

$M = \mu L$ — 2pts

$= 0.64 \cdot 12 = 7.7\ (kg).$ — 1pt

8. A soda bottle with some water inside can be used as a musical instrument. To tune it properly, the **fundamental frequency** must be 440.0 Hz. The sound speed (v) is 343 m/s. Treat the bottle as a pipe that is closed at one end and open at the other end. **(10 pts)**



(a) Is this sound wave transverse or longitudinal? Explain why? **(2 pts)**

Longitudinal wave —— 1pt

Sound wave propagates in the air with oscillation along the direction of propagation.

(b) Treat the above bottle as a pipe of length of L with **only one open end**, what are the wave lengths when the condition for resonance (standing wave) is satisfied? **(2 pts)**

With one open end and one closed end, one has the antinode and the node correspondingly.

2pts $L = n\frac{\lambda}{2} + \frac{\lambda}{4}$ with $n = 0, 1, 2 \cdots$

or $\lambda = \frac{4L}{(2n+1)} = \frac{4L}{m}$ with $m = 1, 3, 5 \ldots$ odd

(c) If the bottle is $H = 26.0$ cm tall, how high h should it be filled with water to produce the fundamental mode (the first harmonic) of the desired frequency? **(3 pts)**

For the first harmonic, $n=0$ or $m=1$.

1pt —— $\lambda = 4L \Rightarrow f = \frac{V}{\lambda} = \frac{v}{4L}$. —— 1pt

$\Rightarrow L = \frac{V}{4f}$ and $h = H - \frac{V}{4f} \doteq 0.065\,m \doteq 6.5\,cm$ 1pt

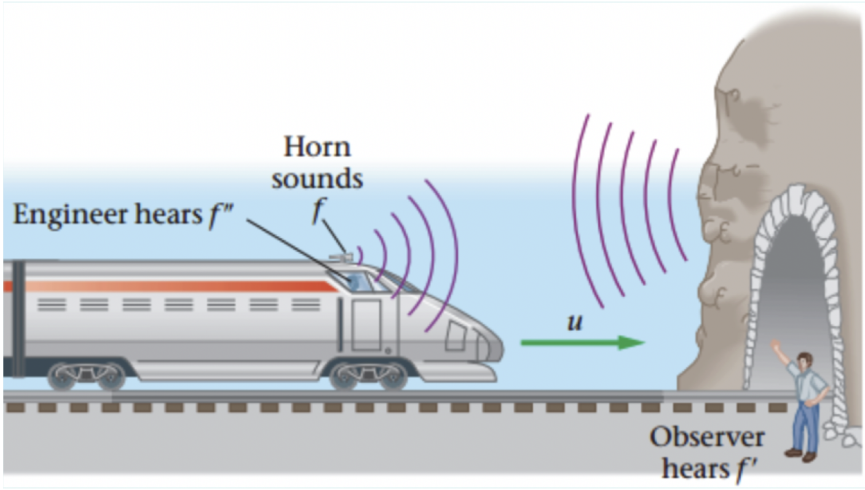
(d) What is the frequency of the next harmonic for this bottle? **(3 pts)**

For the next harmonic, $n=1$, or, $m=3$

1pt $\lambda_2 = 4L/3 = \lambda_1/3$; $f_2 = \frac{V}{\lambda_2}$ 1pt

$\Rightarrow f_2 = \frac{V}{4L} \cdot 3 = 3f_1 = 3 \times 440 = 1.32 \times 10^3\,Hz$ 1pt

9. A train sounds its horn as it approaches a tunnel in a cliff. The horn produces a tone of $f = 650.0$ Hz (when it is at rest), and the train travels with a speed of $u = 21.2$ m/s. The sound speed (v) is 343m/s. (**Suppose that the tunnel is narrow enough and only the reflection from the cliff needs to be considered.**) **(10 pts)**



Doppler effect for detected frequency: $\frac{v \pm v_D}{v \pm v_S} f$.

(a) Find the frequency f' of the sound **directly from the train horn** heard by an observer standing near the tunnel entrance. **(4 pts)**

Sound source (train horn) approaches the observer, thus

$$f' = \frac{v}{v-u} f \quad (\text{set } V_D = 0,\ V_S = u) \quad \text{— 2pt}$$

$$= \frac{343}{343-21.2} \cdot 650\,\text{Hz} = 693 (\text{Hz}) \quad \text{— 2pt}$$

(b) The sound from the horn reflects from the cliff back to the engineer on the train. What is the frequency of the reflected sound? **(2 pts)**

The cliff is at rest. The reflection simply means the cliff takes the sound wave, reverses its direction, then emits it towards the train.

Frequency is the same as $f' = 693$ (Hz) — 2pt

(c) What is the frequency f'' that the engineer on the train hears? **(4 pts)**

For the engineer on the train, he is now the new (moving) listener.

$$f'' = \frac{V+u}{V} f' \quad (\text{set } V_D = u,\ V_S = 0) \quad \text{— 2pt}$$

$$= \frac{V+u}{V-u} f = 736 (\text{Hz}) \quad \text{— 2pt}$$

10. Fig. 10 shows a stream of water flowing through a hole at depth h in a tank holding water to height H.

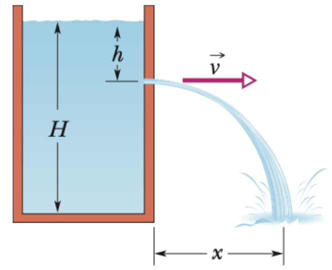


Fig. 10

(a) Find the water speed v when it leaves the hole. **(3 pts)**

Use Bernoulli's equation on the water surface and the hole

$$P_0 + \frac{1}{2}\rho v_0^2 + \rho g h = P_0 + \frac{1}{2}\rho v^2 + 0 \quad \text{—— 2pt}$$

$$v_0 = \frac{S}{S_0} v \ll v.$$

$$\Rightarrow \quad v = \sqrt{2gh} \quad \text{—— 1pt}$$

*with necessary steps, full credits can be given to correct result.

(b) Suppose $\vec{v}$ is horizontal, at what distance x does the stream strike the floor? **(4 pts)**

Solution: Water exits with horizontal velocity $\sqrt{2gh}$.

Trajectory: $\begin{cases} x = \sqrt{2gh}\, t & \text{—— 1pt} \\ y = \frac{1}{2} g t^2 & \text{—— 1pt} \end{cases}$

set $y = H - h \Rightarrow t = \sqrt{\frac{2(H-h)}{g}}$ —— 1pt

$\Rightarrow x = 2\sqrt{h(H-h)}$. —— 1pt

when the stream strikes the floor.

(c) At what depth h should a hole be made to maximize x? **(3 pts)**

Treat x as a function of h.

set $\frac{dx}{dh} = 0 \Rightarrow$ —— 1pt

$$\frac{H-h-h}{\sqrt{h(H-h)}} = 0 \quad \text{—— 1pt}$$

$$\Rightarrow \quad h = \frac{1}{2}H. \quad \text{—— 1pt}$$

* with necessary steps, full credits can be given to $h = \frac{1}{2}H$.

11. A point particle of mass m and speed v collides elastically with the end of a uniform thin rod of mass M and length L on a frictionless horizontal plane as shown below. After the collision, the point particle of mass m becomes stationary (at rest). **(10 pts)**

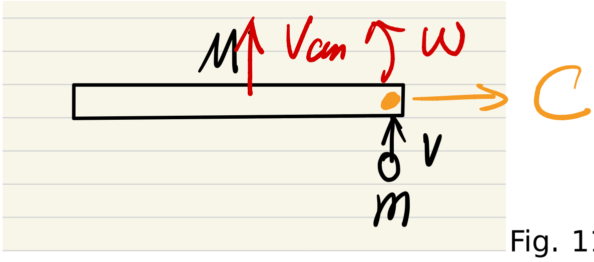


Fig. 11

(a) Find mass ratio M/m that can let this occur. **(8 pts)**

Method 1:

Choose COM of the rod as the axis.

Angular momentum conservation ———— 2pts.

$$mv\frac{L}{2} = I_{cm}\omega \qquad ①$$

Momentum conservation ———— 2pts

$$mv = MV_{cm} \qquad ②$$

Kinetic energy conservation ———— 2pts

$$\frac{1}{2}mv^2 = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \quad \text{with } I_{cm} = \frac{1}{12}ML^2 \qquad ③$$

$$① + ② + ③ \Rightarrow \quad M = 4m \quad \text{———— 2pts.}$$

Method 2. Choose C point as the axis. $0 = -MV_c\frac{L}{2} + I_{cm}\omega$

(b) Find the COM velocity v_{cm} and angular velocity ω of the rod after the collision. **(2 pts)**

putting $M/m = 4$ into eq ②.

$$\Rightarrow \quad V_{cm} = \frac{1}{4}V \quad \text{———— 1pt}$$

putting M/m into eq ①

$$\Rightarrow \quad \omega = \frac{6m}{M}\frac{v}{L} = \frac{3}{2}\frac{V}{L}. \quad \text{———— 1pt.}$$

12. Laplace-Runge-Lenz (LRL) vector(Don't Panic! This problem is long, but it is not as hard as it seems to be.)

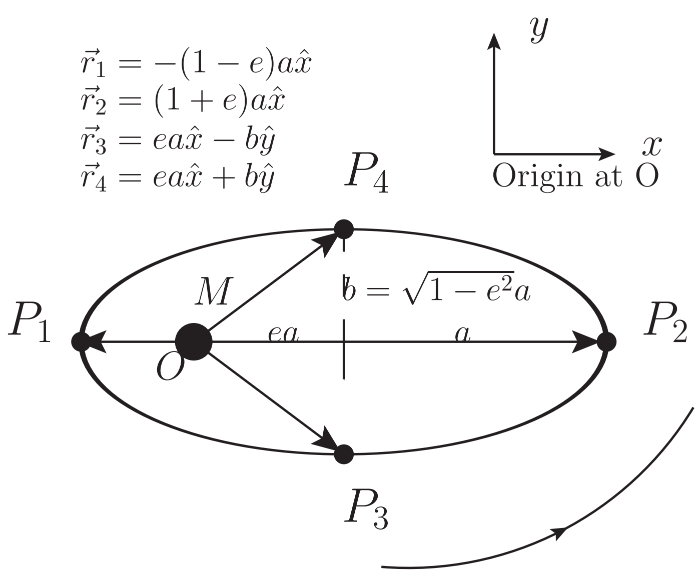
The Laplace-Runge-Lenz (LRL) vector is an additional conserved quantity in Newtonian gravity. For two celestial bodies interacting with Newton's gravitational force

$$\vec{F}_g = -\frac{GMm}{r^2}\frac{\vec{r}}{r} = -\frac{GMm}{r^2}\hat{r} \quad \text{with unit vector} \quad \hat{r} \equiv \frac{\vec{r}}{r},$$

the LRL vector is a constant vector, meaning that it is a constant no matter where it is calculated on the orbit. For the star-planet system shown below, the LRL vector is defined as

$$\vec{A} = \vec{v} \times \vec{L} - GMm\frac{\vec{r}}{r} = \vec{v} \times \vec{L} - GMm\hat{r},$$

where $\vec{v}$ stands for the velocity of rotating planet and $\vec{L}$ represents its angular momentum.



Consider the elliptic orbit of the planet as shown above, the planet with mass m is rotating counter-clockwise about the star with mass M. To simplify the calculation, assume that the star is so massive that it is approximately sitting at rest at point O. The semi-major axis of this orbit is a and the eccentricity of the orbit is e. Several geometric relations ($\hat{x}$ and $\hat{y}$ are unit vectors) that may be useful to your calculation are provided in the figure as well. **(10 pts)**

(a) Find the direction of the angular momentum $\vec{L} \equiv \vec{r} \times \vec{p}$ of the planet. **(1 pts)**
Is $\vec{L}$ conserved? (YES/NO) **(1 pts)**

Rotating counter-clock-wise $\vec{r} \times \vec{p}$ is in the $+\hat{z}$ direction: Perpendicular to the paper and pointing upward —— 1pt.

Yes. L is conserved. (centered force: $\vec{F_g}$)
1pt

(b) Use the right-hand rule or other methods, explain that the LRL vector $\vec{A}$ is within the $x-y$ plane. **(1 pts)**

$$\vec{A} = \vec{v} \times \vec{L} - GMm\,\hat{r}$$

Since $\vec{L}$ is in $+\hat{z}$ direction, then $\vec{v}\times\vec{L}$ is in the $(x-y)$ plane, and $\hat{r}$ is also in $x-y$ plane. — 1pt

(c) Find the expressions for the speed of the planet at the perihelion (P_1) and aphelion (P_2), respectively. **(2 pts)**

At the perihelion (P_1)

$$V_p = \sqrt{\frac{GM}{a}\frac{1+e}{1-e}} \text{ ——— } 1pt$$

At the aphelion (P_2).

$$V_a = \sqrt{\frac{GM}{a}\frac{1-e}{1+e}} \text{ ——— } 1pt$$

(d) Find the magnitude and the direction of the LRL vector $\vec{A}$ at either P_1 or P_2. **(2 pts)**

Consider $\vec{A}$ at P_1 (result is the same for P_2).

$$\vec{A} = (-\hat{x})\sqrt{\frac{GM}{a}\frac{1+e}{1-e}}\left(m\sqrt{\frac{GM}{a}\frac{1+e}{1-e}}\,\underbrace{r_1}_{(1-e)a}\right) + GMm\,\hat{x}$$

$$= (-\hat{x})V_p(mV_p r_1) - GMm\,\vec{r}_1/r_1$$

$$= GMm(1+e)(-\hat{x}) + GMm\,\hat{x}$$

$$= (-\hat{x})\,GMme \longrightarrow 1pt \text{ magnitude!}$$

1pt direction!

(e) Find the LRL vector $\vec{A}$ at either P_3 or P_4. **(1 pts)**

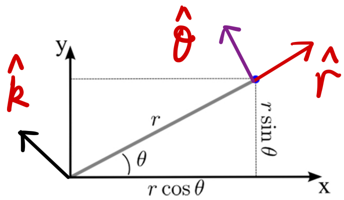
Consider $\vec{A}$ at P_3, first find

$$V_3 = \frac{m V_p r_1}{m b} = \frac{\sqrt{\frac{GM}{a}\frac{1+e}{1-e}}\cdot(1-e)a}{\sqrt{1-e^2}\,a} = \sqrt{\frac{GM}{a}}.$$

$$\vec{A} = (-\hat{y})V_3 \cdot L - GMm\,\hat{r}_3$$
$$= (-\hat{y})\,GMm\sqrt{(1-e^2)} - GMm\left[\frac{ea}{a}\hat{x} - \frac{b}{a}\hat{y}\right] \rightarrow 1pt$$
$$= -GMme\,\hat{x}$$

(f) Prove that the LRL vector $\vec{A}$ is conserved for any points on this orbit. **(2 pts)**

Hint 1: Show that $\frac{d\vec{A}}{dt} = 0$ in the polar (cylindrical) coordinate. As shown below, the three unit vectors satisfy the following relations



$$\hat{r}\times\hat{\theta} = \hat{k},\quad \hat{k}\times\hat{r} = \hat{\theta},\quad \hat{\theta}\times\hat{k} = \hat{r};$$
$$\frac{d\hat{r}}{dt} = \omega\hat{\theta},\quad \frac{d\hat{\theta}}{dt} = -\omega\hat{r},\quad \frac{d\hat{k}}{dt} = 0.$$

Here $\hat{k}$ is the unit vector in the z-direction perpendicular to the $x-y$ plane.

Hint 2: If you do not want to work in the polar coordinate, you may use the following two identities: $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ and $\frac{dr^2}{dt} = 2r\frac{dr}{dt} = \frac{d(\vec{r}\cdot\vec{r})}{dt} = 2\vec{r}\cdot\frac{d\vec{r}}{dt}$.

Use Hint 1. First method: To show $\frac{d\vec{A}}{dt} = 0$. let us first study $\frac{d(\vec{V}\times\vec{L})}{dt}$.

Note $\frac{d\vec{L}}{dt} = 0$ and $\vec{L} = mr^2\vec{\omega}$ —— 1pt

$$\frac{d(\vec{V}\times\vec{L})}{dt} = \frac{d(\vec{V}\times mr^2\vec{\omega})}{dt} = \frac{d\vec{p}}{dt}\times(r^2\vec{\omega}) = -\frac{GMm}{\cancel{r^2}}\frac{\vec{r}}{r}\times(\cancel{r^2}\vec{\omega})$$

1pt —— $= -GMm(\hat{r}\times\vec{\omega}) = GMm\omega(\hat{k}\times\hat{r}) = GMm\omega\hat{\theta}$

Finally, $\frac{d\vec{A}}{dt} = GMm\omega\hat{\theta} - GMm\frac{d\hat{r}}{dt} = GMm\omega\hat{\theta} - GMm\omega\hat{\theta} = 0$.

Use Hint 2. Second Method: use $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (a\cdot b)\vec{c}$ 1pt

$\Rightarrow \frac{d(\vec{V}\times\vec{L})}{dt} = \vec{a}\times(\vec{r}\times\vec{p}) = -\frac{GMm}{r^2}\hat{r}\times\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)$

use $\boxed{r\frac{dr}{dt} = \vec{r}\cdot\frac{d\vec{r}}{dt}}$ $= -\frac{GMm}{r^3}\left[\vec{r}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) - r^2\frac{d\vec{r}}{dt}\right] = +GMm\frac{d[\vec{r}/r]}{dt}$ 1pt

At last $\frac{d\vec{A}}{dt} = GMm\frac{d\hat{r}}{dt} - GMm\frac{d\hat{r}}{dt} = 0$.