MAT1002: Calculus II

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§14.6 Tangent Planes and Differentials
§14.7 Extreme Values and Saddle Points

Chain rule for paths in the plane

Consider the level curve $f(x, y) = c$. Assume it has a parametric form $\vec{r}(t)$, then

$$c = f(\vec{r}(t)) = f(x(t), y(t)).$$

Taking the derivative with respect to t on both sides, we have

$$0 = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$$

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$$\vec{OP} = \vec{OP_0} + t \cdot \vec{r}'(t)$$

- **tangent line** at the point P_0 on the curve $\vec{r}(t)$: the line through P_0 in the direction of $\vec{r}'$.
- **tangent line** at the point P_0 on the level curve $f(x,y)=c$: a line through P_0 orthogonal to ∇f.

$$\nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle = 0$$

Chain rule for paths in the space

curve $\vec{r}(t)$ on the level surface.

Consider a parameterized curve $\vec{r}(t)$ such that

$$\boxed{c = f(\vec{r}(t))} = f(x(t), y(t), z(t)).$$

Taking the derivative with respect to t on both sides, we have

$$\underbrace{0 = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)}$$

$$\nabla f(\vec{r}(t)) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Chain rule for paths in the space

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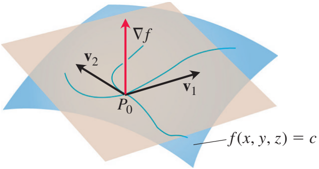
- **tangent line** at the point P_0 on the curve $\vec{r}(t)$ such that $f(\vec{r}(t)) = c$: a line through P_0 orthogonal to ∇f.

Tangent planes and normal lines

Definition

- **tangent plane** to the level surface $f(x, y, z) = c$ at the point $P_0(x_0, y_0, z_0)$: the plane through P_0 normal to $\nabla f|_{P_0}$.
- **normal line** of the surface at P_0: the line through P_0 parallel to $\nabla f|_{P_0}$.



- Tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

- Normal line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \cdot \vec{\nabla} f(x_0, y_0, z_0)$$

Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1, 2, 4)$.

$\nabla f = \langle 2x, 2y, 1 \rangle$

$\nabla f(1, 2, 4) = \langle 2, 4, 1 \rangle$

tangent plane :

$2(x-1) + 4(y-2) + 1(z-4) = 0$

$\Rightarrow 2x + 4y + z = 2 + 8 + 4 = 14$

normal line

$\langle x, y, z \rangle = \langle 1, 2, 4 \rangle + t \langle 2, 4, 1 \rangle$

Plane tangent to a surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

it is a level surface.

$g(x, y, z) := f(x, y) - z = 0$

$\nabla g = \langle f_x, f_y, -1 \rangle$

Tangent plane

$f_x (x - x_0) + f_y (y - y_0) + (-1)(z - f(x_0, y_0)) = 0$

$\Rightarrow f_x \cdot x + f_y \cdot y - z = f_x \cdot x_0 + f_y y_0 - f(x_0, y_0)$

Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

$g(x, y, z) := x \cos y - ye^x - z = 0$

$\nabla g(x, y, z) = \langle \cos y - ye^x, -x \sin y - e^x, -1 \rangle$

$\nabla g(0, 0, 0) = \langle 1, -1, -1 \rangle$

plane $(x - 0) - (y - 0) - (z - 0) = 0$

$\Rightarrow x - y - z = 0$

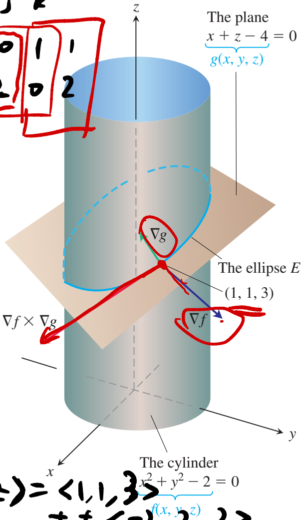
Find the parametric equations for the line tangent to an ellipse E

$$f(x,y,z) = x^2 + y^2 - 2 = 0, \quad g(x,y,z) = x + z - 4 = 0$$

(1,1,3)

meet in an ellipse E.

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{array}$$



E is a curve on the plane. g(x,y,z) = 0

so the direction is normal to ∇g (1,1,3)

= <1, 0, 1>

Similarly, the direction is normal to ∇f(1,1,3)

= <2, 2, 0>

So one direction is <1,0,1> × <2,2,0>

= <-2, 2, 2>

(x,y,z) = <1,1,3> + t <-2,2,2>

Estimating the change in f in a direction $\vec{u}$

$$df = (\nabla f|_{P_0} \cdot \vec{u})ds$$

estimates the change in the function value of f when we move a small distance ds from a point P_0 in the direction $\vec{u}$.

Estimating the change in f in a direction $\vec{u}$

$$df = (\nabla f|_{P_0} \cdot \vec{u})ds$$

estimates the change in the function value of f when we move a small distance ds from a point P_0 in the direction $\vec{u}$.

Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point moves 0.1 unit from $P_0(0, 1, 0)$ toward $P_1(2, 2, -2)$.

$$\overrightarrow{P_0P_1} = \langle 2, 2-1, -2 \rangle = \langle 2, 1, -2 \rangle$$

$$\vec{u} = \frac{1}{3}\langle 2, 1, -2 \rangle$$

$$\nabla f = \langle y\cos x, \sin x + 2z, 2y \rangle$$

$$\nabla f(0,1,0) = \langle 1, 0, 2 \rangle$$

$$\nabla f \cdot \vec{u} = \frac{1}{3}(2-4) = -\frac{2}{3}$$

so the change is around $-\frac{2}{3} \times 0.1$

Linearize a function of ONE variables

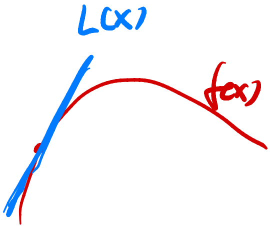
The **linearization** of $f(x)$ at x_0 where f is *differentiable* is the function

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

The approximation

$$f(x) \approx L(x)$$

is the **standard linear approximation** of f at x_0.



Linearize a function of TWO variables

The **linearization** of $f(x, y)$ at (x_0, y_0) where f is *differentiable* is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

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The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at $(3, 2)$.

$$\nabla f(x,y) = \langle 2x - y, -x + y \rangle$$

$$\nabla f(3,2) = \langle 6 - 2, -3 + 2 \rangle = \langle 4, -1 \rangle$$

$$L(x,y) = f(3,2) + 4(x-3) - (y-2)$$

$$= 9 - 6 + 2 + 3 + 4x - 12 - y + 2$$

$$= 4x - y - 2$$

The error $f(x,y) - L(x,y)$ in the standard linear approximation

The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

Definition

If we move from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

in the linearization of f is called the **total differential** of f.

$$\vec{\nabla} f \cdot \vec{u} \times \sqrt{dx^2 + dy^2} = \vec{\nabla} f \cdot \frac{\langle dx, dy \rangle}{\sqrt{dx^2 + dy^2}} \times \sqrt{dx^2 + dy^2}$$

$$= \vec{\nabla} f \cdot \langle dx, dy \rangle$$

Example

A cylindrical can is designed to have a radius of 1 cm and a height of 5 cm, but the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

$$V = \pi r^2 \cdot h$$

$$V_r = 2\pi r h$$

$$V_h = \pi r^2$$

$$dV = V_r(1,5) \cdot dr + V_h(1,5) \cdot dh$$

$$= 10\pi \cdot 0.03 + \pi \cdot (-0.1)$$

$$= 0.2\pi$$

Example

A cylindrical can is designed to have a radius of 1 cm and a height of 5 cm. How sensitive are the cans' volumes to small variations in height and radius?

$$\nabla f(1,5) = \langle 10\pi, \pi \rangle$$

Functions of more than two variables

- The linearization of $f(x, y, z)$ at $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

- Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region where the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the error

$$|E(x, y, z)| = |f(x, y, z) - L(x, y, z)| \leq \frac{M}{2}(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

- If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the total differential

$$df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$$

gives a good approximation of the resulting change in f.

Example

Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3\sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region

$$R: \ |x-2| \le 0.01, \ |y-1| \le 0.02, \ |z| \le 0.01.$$

$\nabla f = \langle 2x - y, \ -x, \ 3\cos z \rangle$

$f_{xx} = 2$, $f_{xy} = -1$, $f_{xz} = 0$

$f_{yy} = 0$, $f_{yz} = 0$, $f_{zz} = -3\sin z$

we can let $M = 3$

If $|z| \le 0.01$ $|f_{zz}| = 3|\sin z| < 2$

so we can let $M = 2$ and the upper

bound for the error is $\frac{M}{2}(|x-2| + |y-1| + |z|)^2 \le 0.04^2$

How to find extrem values?

- $f(x)$ on $x \in [a, b]$.
- $f(x, y)$ on $(x, y) \in [a, b] \times [c, d]$.

if $f(x,y) = g_1(x)$

Derivative tests for local extreme values

Definition

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

- $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an *open disk* centered at (a, b).
- $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an *open disk* centered at (a, b).

Theorem (Theorem 10)

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

If $\quad f_x(a, b) > 0$

$\quad f(a+dx, b)$

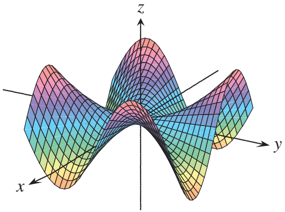
Definition (**critical point**)

critical pointof f: *interior point* of $f(x, y)$ where both $f_x = f_y = 0$ or where one or both of f_x and f_y do not exist.

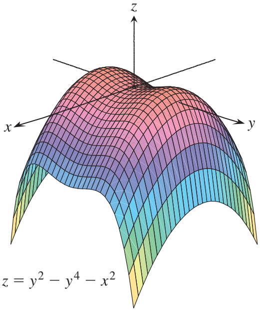
Saddle point

Definition (**saddle point**)

saddle point $(a, b, f(a, b))$ of the surface $z = f(x, y)$: in every open disk centered at a *critical point* (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$.



$z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$



$z = y^2 - y^4 - x^2$

$f(x) = x^3$ when $x = 0$

Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$

$\nabla f = \langle 2x, 2y-4 \rangle$

let $\nabla f = 0 \Rightarrow \begin{matrix} 2x = 0 \\ 2y-4 = 0 \end{matrix} \Rightarrow \begin{matrix} x = 0 \\ y = 2 \end{matrix}$

$(0, 2)$ is a critical point

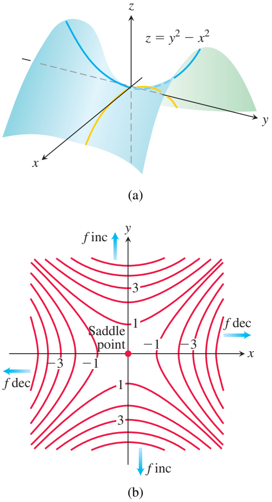
$f(x, y) = x^2 + (y-2)^2 + 5$

$\geq 5 = f(0, 2)$

so $(0, 2)$ is a local minimum.

with value 5

Find the local extreme values of $f(x,y) = y^2 - x^2$



(a)
(b)

$\nabla f = \langle -2x, 2y \rangle$

let $\nabla f = 0 \Rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$

for any disk $x^2 + y^2 \le a^2$
with $a > 0$

$f(a,0) = -a^2 < f(0,0)$

$f(0,a) = a^2 > f(0,0)$

so $(0,0)$ is a saddle point.

Second derivative test for local extreme values

critical point

Suppose that $f(x,y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$. Then

- f has a saddle point at (a,b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a,b).
- f has a local maximum at (a,b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).
- f has a local minimum at (a,b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a,b).
- the test is inconclusive at (a,b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a,b). In this case, we must find another way to determine the behavior of f at (a,b).

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$f_{xx}f_{yy} - f_{xy}^2$

< 0 saddle

$> 0 \Rightarrow f_{xx}f_{yy} > 0$

$f_{xx} > 0$ local minimum

$f_{yy} < 0$ local maximum.

$= 0$ undetermined

Find the local extreme values of

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

$\nabla f = \langle y - 2x - 2, \; x - 2y - 2 \rangle$

Let $\nabla f = 0 \Rightarrow \begin{cases} y - 2x - 2 = 0 \\ x - 2y - 2 = 0 \end{cases} \Rightarrow x = y = -2$

$f_{xx} = -2 \qquad f_{xy} = 1$

$f_{yy} = -2$

$f_{xx} f_{yy} - f_{xy}^2 = (-2)^2 - 1 > 0$

$f_{xx} < 0$

$\Rightarrow$ local maximum

Find the local extreme values of

$$f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$

$\nabla f = \langle -6x + 6y,\ 6y - 6y^2 + 6x \rangle$

Let $\nabla f = 0 \Rightarrow \begin{cases} -6x + 6y = 0 \\ 6y - 6y^2 + 6x = 0 \end{cases} \Rightarrow \begin{cases} 12y - 6y^2 = 0 \\ x = y \end{cases}$

$\Rightarrow x = 0,\ y = 0$

$x = 2,\ y = 2$

$f_{xx} = -6$

$f_{xy} = 6$

$f_{yy} = 6 - 12y$

$x = y = 0$	$x = y = 2$
$f_{xx} f_{yy} - f_{xy}^2$	$f_{xx} f_{yy} - f_{xy}^2$
$= -36 - 6^2$	$= (-6)(-18) - 6^2$
< 0	> 0
Saddle point	$f_{xx} < 0$
	Local maximum

Find critical points of $f(x,y) = 10xye^{-(x^2+y^2)}$ and classify them

$\nabla f = \langle 10ye^{-(x^2+y^2)} + 10xye^{-(x^2+y^2)} \cdot (-2x),$

$10xe^{-(x^2+y^2)} + 10xye^{-(x^2+y^2)} \cdot (-2y) \rangle$

Let $\nabla f = 0$ $\Rightarrow$ $10y + 10xy(-2x) = 0$

$10x + 10xy(-2y) = 0$

$\Rightarrow$ $y - 2x^2y = 0$

$x - 2xy^2 = 0$

$\Rightarrow \begin{cases} y = 0 \quad x = 0 \\ x = \frac{1}{\sqrt{2}} \quad y = \frac{1}{\sqrt{2}}, \ y = -\frac{1}{\sqrt{2}} \\ x = -\frac{1}{\sqrt{2}} \quad y = \frac{1}{\sqrt{2}}, \ y = -\frac{1}{\sqrt{2}} \end{cases}$

$f_{xx} = -40xye^{-(x^2+y^2)} + (10y + 10xy(-2x)) \cdot e^{-(x^2+y^2)}(-2x)$

$f_{yy} = -40xye^{-(x^2+y^2)} + (10x + 10xy(-2y)) \cdot e^{-(x^2+y^2)}(-2y)$

$f_{xy} = (10 - 20x^2)e^{-(x^2+y^2)} + (10y + 10xy(-2x))e^{-(x^2+y^2)}(-2y)$

$x = y = 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = 0 - 10^2$

saddle point.

$x = y = \frac{1}{\sqrt{2}}$ $\quad x = y = -\frac{1}{\sqrt{2}}$

$f_{xx} f_{yy} - f_{xy}^2 = (-20)(-20)e^{-1}$

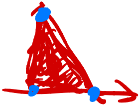
Local maximum

$x = \frac{1}{\sqrt{2}}$ $y = -\frac{1}{\sqrt{2}}$, $\quad x = -\frac{1}{\sqrt{2}}$ $y = \frac{1}{\sqrt{2}}$

Local maximum

Absolute maxima and minima on closed bounded regions

- Check critical points
- Check boundary points
- Compare values.



Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

$\nabla f = \langle 2-2x, 4-2y \rangle$ Let $\nabla f = 0 \Rightarrow x=1 \quad y=2$

$f_{xx} = -2 \quad f_{yy} = -2, \quad f_{xy} = 0$ $\quad (1,2)$ is a local maximum

$x=0 \quad 0 \le y \le 9$

$f_y = 4-2y \Rightarrow y=2$

$(0,2)$

$y=0 \quad 0 \le x \le 9$

$f_x = 2-2x \Rightarrow x=1$

$(1,0)$

$x+y=9$ $\quad x=t$, $y=9-t$

$g(t) = f(t, 9-t)$

$= 2 + 2t + 4(9-t) - t^2 - (9-t)^2$

$= 2 + 2t + 36 - 4t - t^2 - 81 - t^2 + 18t$

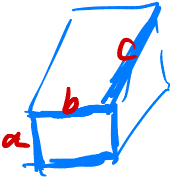
$g'(t) = 16 - 4t \Rightarrow t = 4$

$(4,5)$

$(0,0) \quad (0,9), \quad (9,0)$

Example

A delivery company accepts only rectangular boxes, the sum of whose length and girth (perimeter of a cross-section) does not exceed 270 cm. Find the dimensions of an acceptable box of the largest volume.



$$2a + 2b + c \le 270$$

maximum $a \cdot b \cdot c$

maximum $a \cdot b\,(270 - 2a - 2b)$

$$\nabla f = \langle b(270-2a-2b) + ab(-2),\; a(270-2a-2b) + ab(-2) \rangle$$

Let $\nabla f = 0 \Rightarrow \begin{cases} a = b = 0 \\ a = 0 \quad b = 135 \\ a = 135 \quad b = 0 \\ a = b = 45 \end{cases}$

$$\boxed{\begin{aligned} 270 - 4a - 2b &= 0 \\ 270 - 2a - 4b &= 0 \end{aligned}}$$

Summary of max-min tests

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of f

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**

ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive**