MAT1002: Calculus II

Ming Yan

§10.2 Infinite Series
§10.3 The Integral Test

Infinite Series

Definition

An **infinite series**, or **series**, is the sum of an sequence

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Example:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \quad = 2$$

1

$2-1 \quad 1$

$2-\frac{1}{2}$

$2-\frac{1}{4}$

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Consider its n**th partial sum**

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

which is a sequence.

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n$$

Definitions

Given a series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The number a_n is the n**th term** of the series. If the sequence of partial sums converges to a limit L, we say that the series **converges** and its **sum** is L. In this case, we also write

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of partial sums of the series s_n does not converge, we say that the series **diverges**.

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{k=1}^{\infty} a_k, \quad \sum a_n$$

Geometric Series ($a \neq 0$ and r are fixed)

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

- $r = 1$

  $S_N = a \cdot n$ diverges

- $r \neq 1$

  $S_N = \sum_{n=1}^{N} ar^{n-1} = a + ar + \cdots + ar^{N-1}$

  $r \cdot S_N = \sum_{n=1}^{N} ar^n = ar + \cdots + ar^N$

  $= ar^N + S_N - a$

  $\Rightarrow (r-1) S_N = ar^N - a \Rightarrow S_N = \dfrac{ar^N - a}{r-1} = \dfrac{a(1-r^N)}{1-r}$

  it converges to $\dfrac{a}{1-r}$ if $|r| < 1$

Example:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5\left(\frac{-1}{4}\right)^n$$

$\left|\frac{-1}{4}\right| < 1$ so it converges to

first term

$$\frac{5\cdot\left(\frac{-1}{4}\right)^0}{1-\left(\frac{-1}{4}\right)} = \frac{5}{\frac{5}{4}} = 4$$

Telescoping Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1} - \cancel{\frac{1}{2}} \cancel{+ \frac{1}{2}} - \cancel{\frac{1}{3}} \cancel{+ \frac{1}{3}} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$$

Telescoping Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

The nth-Term Test for a Divergent Series

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$ diverges because $\frac{n+1}{n} \to 1 \neq 0$

Theorem (Theorem 7)

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. necessary condition for convergence.

$a_n = S_n - S_{n-1}$

$\sum_{n=1}^{\infty} a_n$ converges to $L \Rightarrow S_n \to L$

$S_{n-1} \to L$

The nth-Term Test for a Divergent Series

$$\sum_{n=1}^{\infty} \frac{n+1}{n}$$

Theorem (Theorem 7)

If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Corollary

$\sum_{n=1}^{\infty} a_n$ *diverges if* $\lim_{n\to\infty} a_n$ *fails to exists or is different from zero.*

sufficient condition for divergence.

Examples: Divergence or Convergence

- $\sum_{n=1}^{\infty} n^2$

  diverges because $n^2 \to +\infty$

- $\sum_{n=1}^{\infty} (-1)^{n+1}$

  diverges because $(-1)^{n+1}$ does not converge

- $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$

  diverges because $\frac{-n}{2n+5} \to -\frac{1}{2}$

$\lim\limits_{n\to\infty} a_n = 0$ does not mean $\sum\limits_{n=1}^{\infty} a_n$ converges

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty \quad \text{diverges}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \frac{1}{16}$$

Combing Series

Theorem (Theorem 8)

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

- *Sum rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
- *Difference rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
- *Constant Multiple rule:* $\sum(ka_n) = k\sum a_n = kA$

Corollary

- If $\sum a_n$ diverges, then $k\sum a_n$ for $k \neq 0$?

$a_n = (a_n + b_n) - b_n$

- If $\sum a_n$ diverges and $\sum b_n$ converges, then $\sum(a_n + b_n)$? diverge

$\sum_{i=1}^{n} a_i \to$ diverges

$\sum_{i=1}^{n} b_i \to L$

$\sum_{i=1}^{n} (a_i + b_i) \to$ ~~converges~~

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Corollary

- *If $\sum a_n$ diverges, then $k\sum a_n$ for $k \neq 0$?*

$\Rightarrow$
- *If $\sum a_n$ diverges and $\sum b_n$ converges, then $\sum(a_n + b_n)$?*

If $\sum(a_n+b_n)$ converges, then by the difference rule $\sum a_n = \sum(a_n+b_n) - \sum b_n$ converges which contradicts the assumption $\sum a_n$ diverges.

Example: $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$ $= \sum_{n=1}^{\infty} \frac{3^{n-1}}{6^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$

$$= \frac{\frac{3^0}{6^0}}{1-\frac{3}{6}} - \frac{\frac{1}{6^0}}{1-\frac{1}{6}}$$

$$= \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{6}}$$

$$= 2 - \frac{6}{5} = 0.8$$

Adding or Deleting Terms

Adding or deleting a finite number of terms does not change the series's convergence or divergence. but it changes the sum.

Reindexing

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}, \qquad \sum_{n=0}^{\infty} \frac{1}{2^{n}}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}$$

$$= \frac{\frac{1}{2^{5-5}}}{1-\frac{1}{2}}$$

$$= \frac{1}{1-\frac{1}{2}}$$

$$= 2$$

Whether a series converges or not?

- §10.3 Integral Test
- §10.4 Comparison Tests
- §10.5 The ratio and Root Tests

When a series converges, we may not know its sum. In this case, we can only estimate it.

Nondecreasing Partial Sums

Consider $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ for all n. We have

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots$$

Nondecreasing Partial Sums

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Corollary (of Theorem 6)

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums $\{s_n\}$ are bounded from above.

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Corollary (of Theorem 6)

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums $\{s_n\}$ are bounded from above.

- Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or not?

diverges

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or not? converges

$$\frac{1}{n^2} \geq \frac{1}{n(n+1)}$$

$$\frac{1}{n^2} \leq \frac{1}{(n-1)n} \quad \text{for } n \geq 2$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=2}^{\infty} \frac{1}{(n-1)n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right)$$

$$= 1$$

The Integral Test

Theorem (Theorem 9–The integral test)

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$. Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge.

series converges $\Leftarrow$ $\int_N^{\infty} f(x)\,dx$ converges

series diverge $\Leftarrow$ $\int_N^{\infty} f(x)\,dx$ diverges

The Integral Test

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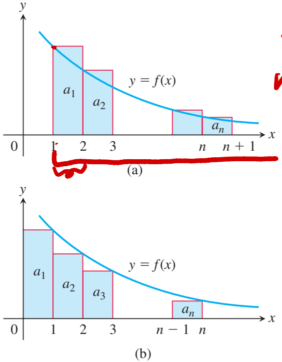


FIGURE 10.11 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

$$\sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} f(x)\,dx \leq \sum_{n=N}^{\infty} a_n$$

if $\sum_{n=1}^{\infty} a_n$ converges

$\Rightarrow \int_1^{\infty} f(x)\,dx$ converges

if $\sum_{n=1}^{\infty} a_n$ diverges

$\Rightarrow \int_1^{\infty} f(x)\,dx = +\infty$

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p} \Leftrightarrow \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } 0 \le p \le 1 \end{cases}$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx$$

$$= \begin{cases} \left. \frac{1}{-p+1} x^{-p+1} \right|_1^{\infty} & \text{if } p \neq 1 \\ \left. \ln x \right|_1^{\infty} & \text{if } p = 1 \end{cases}$$

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\downarrow$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan x \Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Example: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$dt = \frac{1}{x} dx$

$$\int_2^{\infty} \frac{1}{x \ln x} \, dx \overset{\text{Let } t = \ln x}{=\!=\!=} \int_{\ln 2}^{\infty} \frac{1}{t} \, dt$$

$$= \ln t \Big|_{\ln 2}^{\infty} \quad \text{diverges.}$$

$$= \ln(\ln x)\Big|_2^{\infty} \quad =$$

Integral Test: Approximation

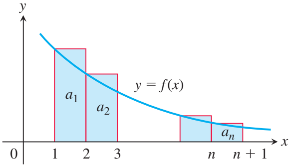
Though we can show that the series converges, we can not easily find the total sum S. But, we can estimate it with the partial sum s_n with the **remainder**

$$R_n = S - \boxed{s_n} = \underbrace{a_{n+1} + a_{n+2} + a_{n+3} + \cdots}$$

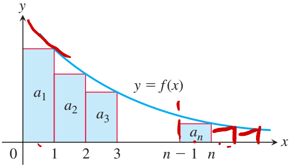
$$\geq \int_{n+1}^{+\infty} f(x)\,dx \qquad \leq \int_n^{+\infty} f(x)\,dx$$

$$a_{n+1} = \int_n^{n+1} f(n+1)\,dx \leq \int_n^{n+1} f(x)\,dx$$

$$a_{n+1} = \int_{n+1}^{n+2} f(n+1)\,dx \geq \int_{n+1}^{n+2} f(x)\,dx$$



(a)



(b)

FIGURE 10.11 Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with s_{10}

$$s_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.54977$$

$$\frac{1}{11} = \int_{11}^{+\infty} \frac{1}{x^2}\,dx \le \sum_{n=11}^{\infty} \frac{1}{n^2} \le \int_{10}^{+\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{10}^{+\infty} = \frac{1}{10}$$

1.54977

0.08

$\frac{1}{11}$ 1.62977

$\sim$ 1.64977

1.63977

Estimate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with s_{10}

$$s_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{100} \approx 1.54977$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.64493$$

Review

- Infinite series, geometric series, harmonic series, p-series
- n-th term test for a divergent series
- Combing series
- Integral test for convergence
- Integral test to estimate the total sum