MAT1002: Calculus II

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§15.5 Triple Integrals in Rectangular Coordinates
§15.7 Triple Integrals in Cylindrical and Spherical Coordinates
§15.8 Substitutions in Multiple Integrals

Triple integrals

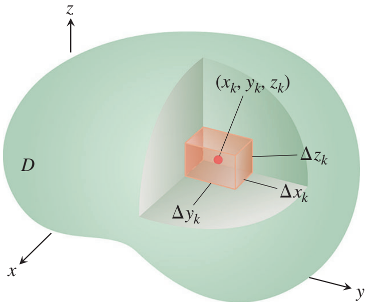


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

$$\iiint_D F(x, y, z)dxdydz$$

Definition

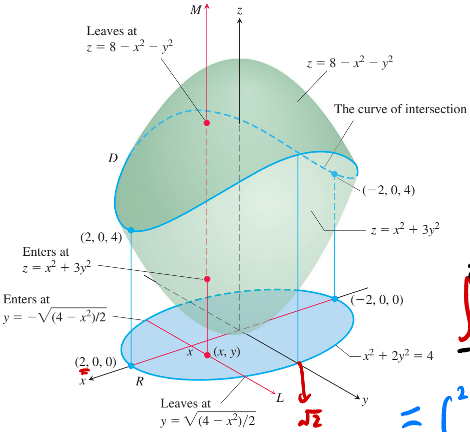
The **volume** of a closed bounded region D in space is

$$V = \iiint_D dV$$

Finding limits of integration in the order of $dz\ dy\ dx$

- Sketch the region of integration along with its "shadow" in the xy-plane.
- Find the z-limits of integration
- Find the y-limits of integration
- Find the x-limits of integration

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



$$x^2+3y^2 = 8-x^2-y^2$$

$$\Rightarrow 2x^2+4y^2 = 8$$

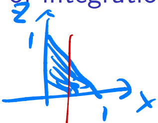
$$\Rightarrow \underbrace{x^2+2y^2 = 4}$$

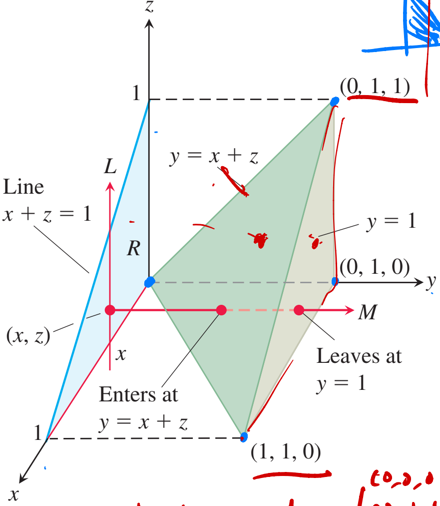
$$y = \pm\sqrt{2-x^2/2}$$

$$\int_{-2}^{2}\int_{-\sqrt{2-x^2/2}}^{\sqrt{2-x^2/2}}\int_{x^2+3y^2}^{8-x^2-y^2} 1\, dz\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{2-x^2/2}}^{\sqrt{2-x^2/2}} 8-2x^2-4y^2\, dy\, dx$$

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dydzdx$.





$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 F(x,y,z)\,dy\,dz\,dx$$

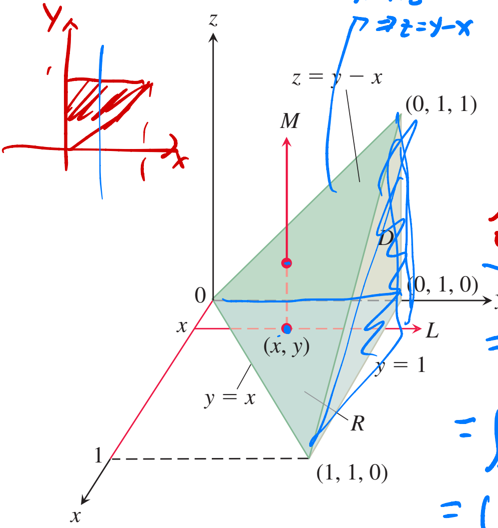
$$ax+by+cz=d \Rightarrow \begin{cases} (0,0,0) \Rightarrow d=0 \\ (0,1,1) \Rightarrow b+c=0 \\ (1,1,0) \Rightarrow a+b=0 \end{cases} \Rightarrow b=-a=-c$$

$$ax-ay+az=0$$

Integrate $F(x,y,z) = 1$ over the tetrahedron D with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, and $(0,1,1)$ in the order $dzdydx$ and $dydzdx$.

$y = x + z$

$\Rightarrow z = y - x$



$$\int_0^1 \int_x^1 \int_0^{y-x} 1 \, dz\, dy\, dx$$

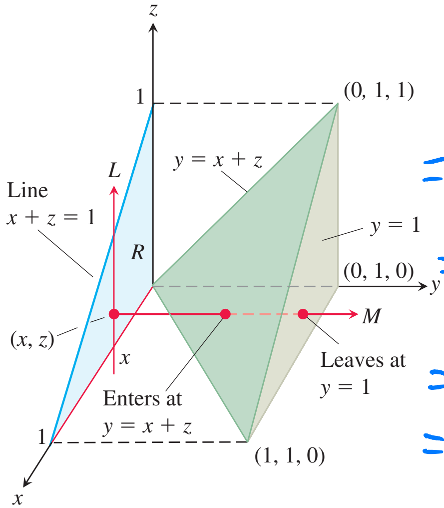
$$= \int_0^1 \int_x^1 y - x \, dy\, dx$$

$$= \int_0^1 \tfrac{1}{2}y^2 - xy \Big|_{y=x}^{y=1} dx$$

$$= \int_0^1 \tfrac{1}{2} - x - \tfrac{1}{2}x^2 + x^2 \, dx$$

$$= \tfrac{1}{2}x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{3}x^3 \Big|_{x=0}^{x=1} = \tfrac{1}{6}$$

Integrate $F(x, y, z) = 1$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$ in the order $dzdydx$ and $dydzdx$.



$$\int_0^1 \int_0^{1-x} \int_{x+z}^1 1 \, dy\, dz\, dx$$

$$= \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx$$

$$= \int_0^1 (1-x)z - \frac{1}{2}z^2 \Big|_{z=0}^{z=1-x} dx$$

$$= \int_0^1 (1-x)^2 - \frac{1}{2}(1-x)^2 \, dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2 \, dx$$

$$= -\frac{1}{6}(1-x)^3 \Big|_{x=0}^{x=1} = \frac{1}{6}$$

Average value of a function in the 3D space

$$\text{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV$$

Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

$$\int_0^2 \int_0^2 \int_0^2 1 \, dx\,dy\,dz = 8$$

$$\int_0^2 \int_0^2 \int_0^2 xyz \, dx\,dy\,dz$$

$$= \int_0^2 \int_0^2 \frac{1}{2}x^2 yz \Big|_{x=0}^{x=2} dy\,dz = \int_0^2 \int_0^2 2yz \, dy\,dz$$

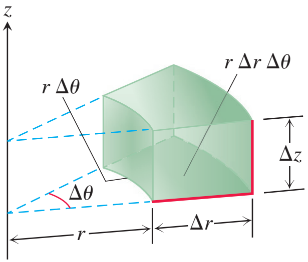
$$= \int_0^2 y^2 z \Big|_{y=0}^{y=2} dz = \int_0^2 4z\,dz = 2z^2 \Big|_{z=0}^{z=2} = 8$$

Integration in cylindrical coordinates

Definition

Cylindrical coordinates represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

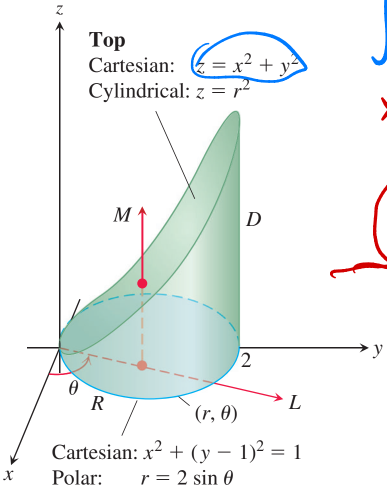
- r and θ are polar coordinates for the vertical projection of P onto the xy-plane.
- z is the rectangular vertical coordinate.

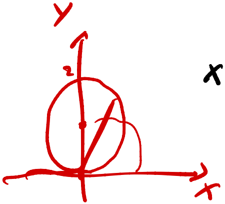


$$\iiint_D f\,dz\,r\,dr\,d\theta$$

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

$$\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r,\theta,z)\, \boxed{r}\, dz\, dr\, d\theta$$





$x^2 + (y-1)^2 = 1$

$\Rightarrow x^2 + y^2 - 2y + 1 = 1$

$\Rightarrow r^2 - 2r\sin\theta = 0$

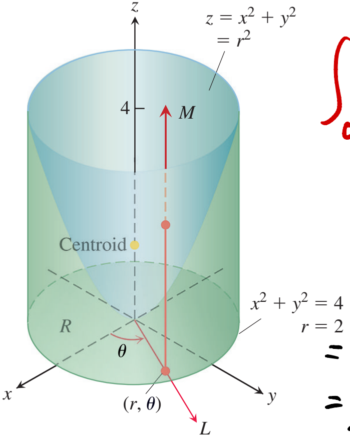
$\Rightarrow r = 2\sin\theta$

How to integrate in cylindrical coordinates

z, r, θ

- Sketch the region of integration along with its "shadow" in the xy-plane.
- Find the z-limits of integration
- Find the r-limits of integration
- Find the θ-limits of integration

Find the centroid of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.



$$\int_0^{2\pi}\int_0^{2}\int_0^{r^2} 1\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r(r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} r^4\Big|_{r=0}^{r=2}\,d\theta = 8\pi$$

$$\int_0^{2\pi}\int_0^{2}\int_0^{r^2} z\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} \frac{1}{2}(r^2)^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} r^6\Big|_{r=0}^{r=2}\,d\theta = \frac{2^6}{12}\cdot 2\pi = \frac{32}{3}\pi$$

Spherical coordinates and integration

Definition

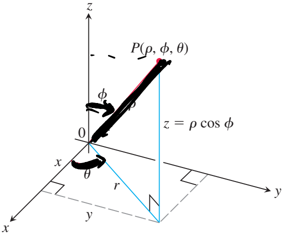
Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ),

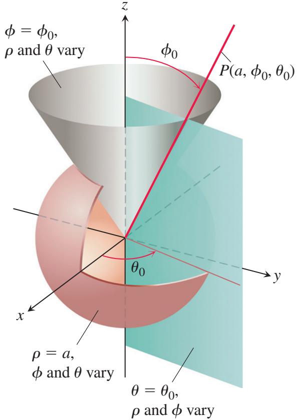
- ρ is the distance from P to the origin $(\rho \geq 0)$.
- ϕ is the angle $\vec{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
- θ is the angle from the cylindrical coordinate.

$$z = \rho \cos \phi$$

$$y = \rho \sin \phi \cos \theta$$

$$x = \rho \sin \phi \sin \theta$$





Equations relating spherical, cartesian, and cylindrical coordinates

$$\begin{aligned}
r &= \rho \sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta \\
z &= \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta \\
\rho &= \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}
\end{aligned}$$

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

$$\underbrace{x^2+y^2+z^2}_{\rho^2} - 2z + 1 = 1$$

$$\Rightarrow \quad \rho^2 - 2\rho\cos\phi = 0$$

$$\Rightarrow \quad \underbrace{\rho = 2\cos\phi}_{\rho}$$

Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$

$$\rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\cos\theta)^2}$$

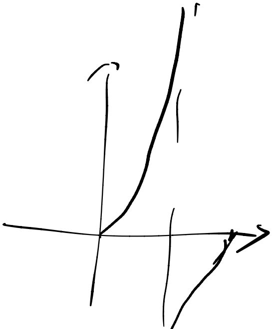
$$\Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi}$$

$$\rho \geq 0 \qquad \phi \in [0, \pi]$$

$$\Rightarrow \rho\cos\phi = \rho\sin\phi$$

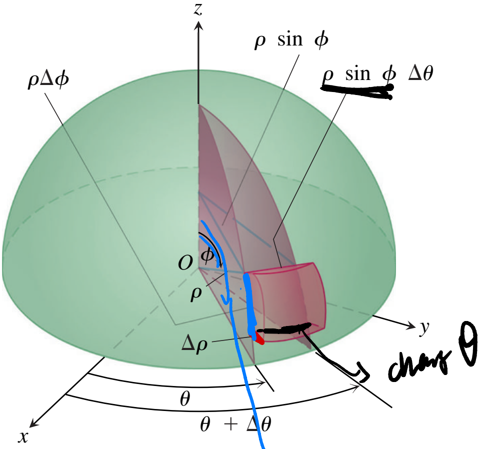
$$\Rightarrow \tan\phi = 1$$

$$\Rightarrow \phi = \frac{\pi}{4}$$



Triple integrals in spherical coordinates

$\Delta\rho \cdot \rho\Delta\phi \; \rho\sin\phi\,\Delta\theta$



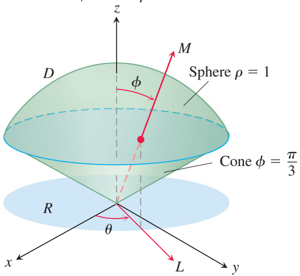
$$\iiint_D f(\rho,\phi,\theta)\boxed{\rho^2 \sin\phi}\,d\rho\, d\phi\, d\theta$$

fixed θ

How to integrate in spherical coordinates

- Sketch the region of integration along with its "shadow" in the xy-plane.
- Find the ρ-limits of integration
- Find the ϕ-limits of integration
- Find the θ-limits of integration

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.



$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

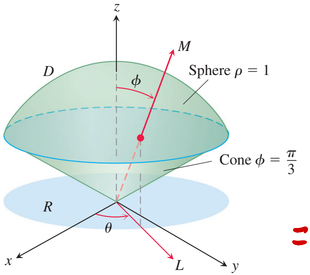
$$= \int_0^{2\pi}\int_0^{\pi/3} \underline{\frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=1}} \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos\phi \Big|_{\phi=0}^{\phi=\pi/3} \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{3} - \frac{1}{3}\cos\frac{\pi}{3}\right) d\theta = \frac{2\pi}{6} = \frac{\pi}{3}$$

A solid of constant density $\delta = 1$ occupies the region D in the previous example. Find the solid's moment of inertia about the z-axis.



$$\iiint_D \underline{(x^2+y^2)}\,dV$$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \underbrace{\rho^2\sin^2\phi}\ \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{5}\rho^5 \sin^3\phi \Big|_{\rho=0}^{\rho=1} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{5}\sin^3\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{5}(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{5}\cos\phi + \frac{1}{15}\cos^3\phi \Big|_{\phi=0}^{\phi=\pi/3} d\theta$$

Summary: coordinate conversion formulas

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

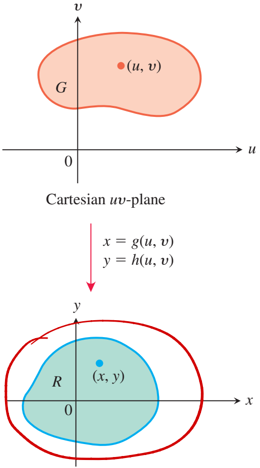
$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Substitutions in double integrals

$x = g(u)$ $dx = g'(u)\,du$

Recall the one-dimensional case

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(u))g'(u)du$$



Cartesian xy-plane

Suppose that a region G in the uv-plane is transformed into the region R in the xy-plane by

$$x = g(u, v), \quad y = h(u, v).$$

We assume the transformation is one-to-one

on the interior of G.

- **image** of G: R
- **preimage** of R: G .

Jacobian

Definition

The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u,v), y = h(u,v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Theorem (Theorem 3)

Suppose that $f(x,y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u,v), y = h(u,v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

$\vec{k}$

$u \to u + \Delta u$

$v \to v + \Delta v$

$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) \cdot \Delta u$ $\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0\right)$

$\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) \Delta v$ $\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0\right)$

Find the Jacobian for the polar coordinate transformation $\underbrace{x = r\cos\theta}$, $y = r\sin\theta$.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$dxdy \Rightarrow r\,dr\,d\theta$$

Example: evaluate

$$= \int_0^4 \frac{1}{2}x^2 - \frac{y}{2}x \Big|_{x=(\frac{y}{2})}^{x=(\frac{y}{2})+1} dy = \int_0^4 \frac{1}{2}(2\frac{y}{2}+1) - \frac{y}{2} \; dy$$

$$= \int_0^4 \frac{1}{2}y + \frac{1}{2} - \frac{y}{2} \; dy$$

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2} dx dy$$

by the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}.$$

$$2x = 2u + y = 2u + 2v$$

$$x = u + v$$

$$y = 2v$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$\int_0^2 \int_v^{v+1} u \cdot 2 \, du \, dv$$

$$= \int_0^2 u^2 \Big|_0^{v+1} dv$$

$$= \int_0^2 (2v+1) \, dv$$

$$= v^2 + v \Big|_{v=0}^{v=2} \Rightarrow 6$$

Example: evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by the transformation

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}.$$

$$\int$$

$$\int_0^2 \int_0^1 u \cdot 2 \, du \, dv$$

$$= \int_0^2 u^2 \Big|_{u=0}^{u=1} dv = 2$$

$$0 \le y \le 4 \implies 0 \le v = \frac{y}{2} \le 2$$

$$\frac{y}{2} \le x \le \frac{y}{2}+1 \implies 0 \le \underbrace{x - \frac{y}{2}}_{u} \le 1$$

Example: evaluate

$$\frac{d(y,x)}{d(v,u)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9}+\frac{2}{9} = \frac{1}{3}$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx.$$

$$\left.\begin{aligned} u &= x+y \\ v &= y-2x \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x &= \tfrac{1}{3}(u-v) \\ y &= \tfrac{1}{3}(2u+v) \end{aligned}\right.$$

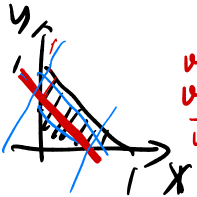
$$0 \le x \le 1 \Rightarrow 0 \le \tfrac{1}{3}(u-v) \le 1$$

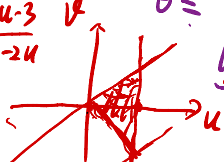
$$0 \le y \le 1-x \Rightarrow 0 \le \tfrac{1}{3}(2u+v) \le 1-\tfrac{1}{3}(u-v)$$

$$\Rightarrow \quad 0 \le u-v \le 3, \quad 0 \le 2u+v, \quad u \le 1$$

$$v \le u, \quad v \ge u-3, \quad v \ge -2u$$

$$\int_0^1 \int_{-2u}^{u} \sqrt{u}\, v^2 \tfrac{1}{3}\, dv\, du = \int_0^1 \sqrt{u}\, \tfrac{1}{9} v^3 \Big|_{-2u}^{u} du = \int_0^1 \sqrt{u}\, \tfrac{1}{9}(u^3+8u^3)\, du = \int_0^1 \sqrt{u}\, u^3\, du = \int_0^1 u^{3.5}\, du = \frac{1}{4.5}$$





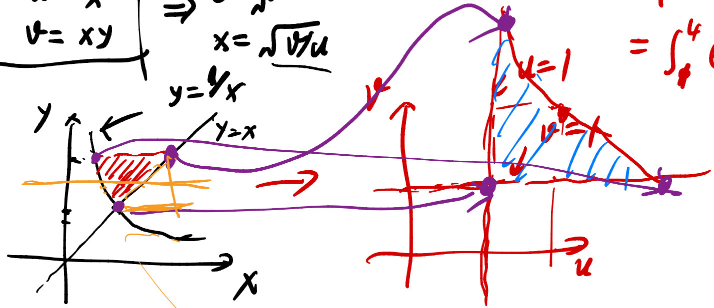
Example: evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx.$$

Nonlinear transformation: evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$u = \frac{y}{x}, \quad v = xy \quad \Rightarrow \quad y = \sqrt{uv}, \quad x = \sqrt{v/u}$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{v/u}} \cdot \frac{v(-1)}{u^2} & \frac{1}{2\sqrt{v}\sqrt{u}} \\ \frac{1}{2\sqrt{u}}\sqrt{v} & \frac{1}{2\sqrt{v}}\sqrt{u} \end{vmatrix} = -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u}$$

$y = 2 \Rightarrow \sqrt{uv} = 2$

$\Rightarrow uv = 4$

$$\int_1^4 \int_1^{4/v} \sqrt{u}\, e^{\sqrt{v}}\, du\, dv \quad \left(-\frac{1}{2u}\right)$$

$$= \int_1^4 e^{\sqrt{v}} \sqrt{u} \Big|_{u=1}^{u=4/v} dv$$

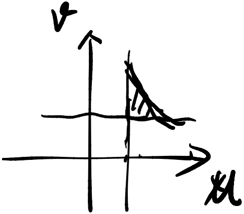
$$= \int_1^4 e^{\sqrt{v}} \left(\frac{2}{\sqrt{v}} - 1\right) dv$$

Nonlinear transformation: evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ v & u \end{vmatrix} = -\frac{v}{u} - \frac{v}{u}$$

$$u = \sqrt{\frac{y}{x}}, \quad v = \sqrt{xy} \quad \Rightarrow \quad y = uv, \quad x = v/u$$



$$\int_1^2 \int_1^{2/v} \cancel{u}\, e^{v} \left(\frac{2v}{\cancel{u}}\right) du\, dv$$

$$= \int_1^2 e^{v} \cdot 2v \left(\frac{2}{v} - 1\right) dv$$

$$= \int_1^2 4e^{v}\, dv - \int_1^2 e^{v} \cdot 2v\, dv$$

Substitutions in triple integrals

$$J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

Cylindrical coordinates

$$x=r\cos\theta,\ y=r\sin\theta,\ z=z.$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=\cos\theta\begin{vmatrix}r\cos\theta & 0\\ 0 & 1\end{vmatrix}-r\sin\theta\begin{vmatrix}0 & \sin\theta\\ 1 & 0\end{vmatrix}$$

$$=r\cos^2\theta+r\sin^2\theta$$

$$=r$$

Substitutions in triple integrals

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Spherical coordinates

$$x = \rho \sin\phi \cos\theta, \;\; y = \rho \sin\phi \sin\theta, \;\; z = \rho\cos\phi.$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \begin{matrix} \sin\phi\sin\theta \\ \cos\phi \end{matrix}$$

$$= \sin\phi\cos\theta\,(\rho^2\sin^2\phi\cos\theta) + \rho\cos\phi\cos\theta\,(\rho\sin\phi\cos\phi\cos\theta)$$

$$- \rho\sin\phi\sin\theta\,(-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta)$$

$$= \rho^2\sin\phi\cos^2\theta + \rho^2\sin\phi\sin^2\theta = \rho^2\sin\phi$$

Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \underbrace{\left(\frac{2x-y}{2}+\frac{z}{3}\right)}_{u+w} dxdydz$$

with transformation

$$u=(2x-y)/2,\ v=y/2,\ w=z/3. \Rightarrow \begin{cases} x = u+\frac{y}{2} = u+v \\ y = 2v \\ z = 3w \end{cases}$$

$0 \le z \le 3 \Rightarrow 0 \le w \le 1$

$0 \le y \le 4 \Rightarrow 0 \le v \le 2$

$\frac{y}{2} \le x \le \frac{y}{2}+1 \Rightarrow 0 \le \underbrace{x-\frac{y}{2}}_{u} \le 1 \Rightarrow 0 \le u \le 1$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

$$\int_0^1 \int_0^2 \int_0^1 (u+w)6 \, du\,dv\,dw = \cdots$$