MAT1002: Calculus II

Ming Yan

§15.4 Double Integrals in Polar Form
§15.5 Triple Integrals in Rectangular Coordinates

Integrals in polar coordinates

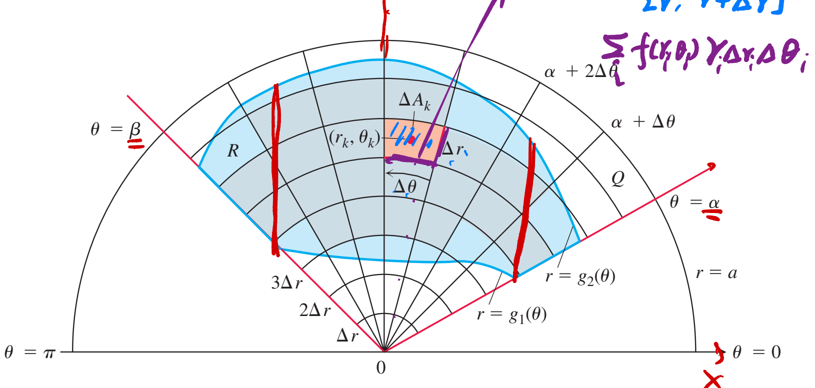


FIGURE 15.22 The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. The partition of Q by circular arcs and rays induces a partition of R.

$\alpha \le \theta \le \beta$

$g_1(\theta) \le r \le g_2(\theta)$

$$\iint_R f(r,\theta)\, r \, dr\, d\theta$$

$0 \le \theta \le 2\pi$
$0 \le r \le 1$

$$\int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}\, d\theta = \pi$$

Integrals in polar coordinates

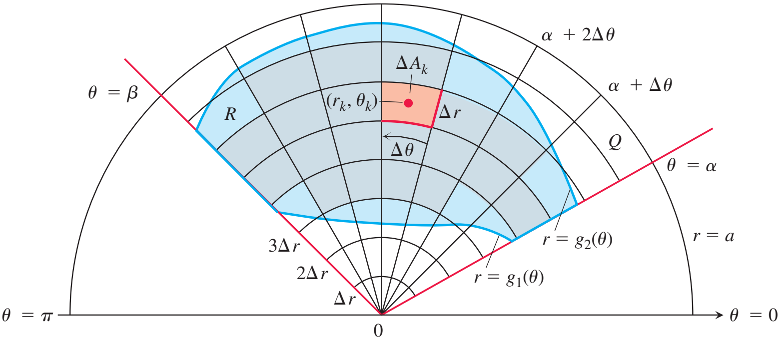
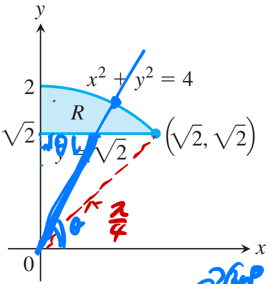


FIGURE 15.22 The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$. The partition of Q by circular arcs and rays induces a partition of R.

$$\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)r\,dr\,d\theta.$$

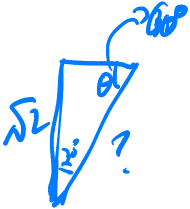
Finding limits of integration

- Sketch the region of integration
- Find the r-limits of integration
- Find the θ-limits of integration



$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\sqrt{2}}{\sin\theta}}^{2} r\,dr\,d\theta$$

$$\frac{\sqrt{2}}{?} = \sin\theta$$



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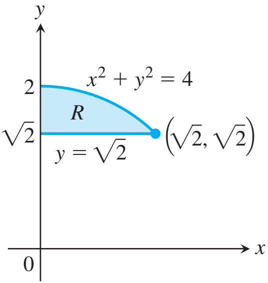


$$\iint_R f(r,\theta)dA = \int_{\pi/4}^{\pi/2} \int_{\sqrt{2}\csc\theta}^{2} f(r,\theta) r dr d\theta$$

Example

Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta$$

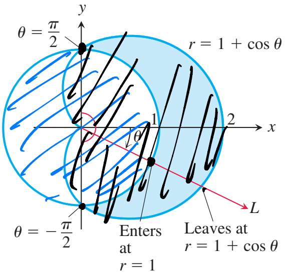


FIGURE 15.25 Finding the limits of integration in polar coordinates for the region in Example 1.

Area in polar coordinates ($f = 1$)

$$\iint_R r\,dr\,d\theta$$

Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

$r = 2\sqrt{\cos(2\theta)}$

$r = -2\sqrt{\cos(2\theta)}$

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

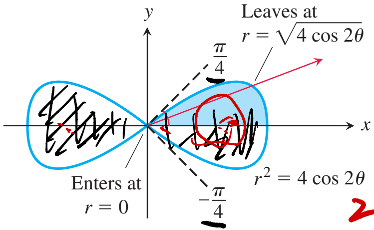


FIGURE 15.26 To integrate over the shaded region, we run r from 0 to $\sqrt{4\cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

$$2 \times \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{2\sqrt{\cos(2\theta)}} r\,dr\,d\theta$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \Big|_{r=0}^{r=2\sqrt{\cos(2\theta)}} d\theta$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos(2\theta)\,d\theta$$

$$= 2\sin(2\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2 + 2 = 4$$

Changing cartesian integral to polar integral

$$\int_0^{+\infty} e^{-x^2}dx \int_0^{+\infty} e^{-y^2}dy = \int_0^{+\infty}\int_0^{+\infty} e^{-x^2-y^2}dx\,dy$$

$$\iint_R f(x,y)\,dy\,dx = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{+\infty} e^{-r^2}\,r\,dr\,d\theta$$

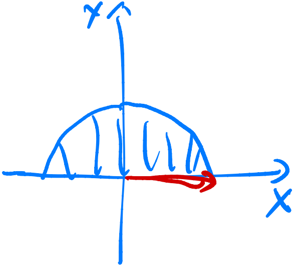
$$\int_0^1 e^{x^2}dx \,?$$

$$\int_0^{+\infty} e^{-x^2}dx$$

Evaluate

$$\iint_R e^{x^2+y^2}\,dy\,dx$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.



$$\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx \;?$$

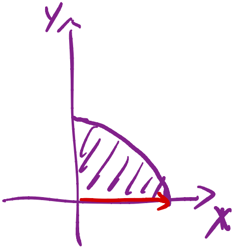
$$\int_0^{\pi}\int_0^1 e^{r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}e^{r^2}\Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e-1)\,d\theta$$

$$= \frac{\pi}{2}(e-1)$$

Evaluate the integral



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx.$$

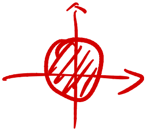
$$= \int_0^1 x^2 y + \frac{1}{3} y^3 \Big|_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 x^2\sqrt{1-x^2} + \frac{1}{3}\sqrt{1-x^2}^{\,3}\, dx$$

$$\int_0^{\pi/2} \int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\, d\theta = \frac{\pi}{8}$$

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane



$$\int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{9}{2} r^2 - \frac{1}{4} r^4 \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} - \frac{1}{4}\right) d\theta$$

$$= 9\pi - \frac{\pi}{2}$$

$$= 8.5\pi$$

Find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 = 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$

$$\frac{\pi \cdot r^2}{2\pi}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$\int_1^{\sqrt{3}} \int_{y/\sqrt{3}}^{\sqrt{4-y^2}} 1 \, dx \, dy$$

$$= \int_1^{\sqrt{3}} \left(\sqrt{4-y^2} - \frac{y}{\sqrt{3}}\right) dy$$

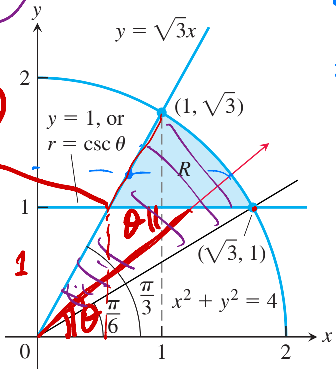


FIGURE 15.29 The region R in Example 6.

$$\int_{\pi/6}^{\pi/3} \int_{\frac{1}{\sin\theta}}^{2} r \, dr \, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{2} r^2 \Big|_{r=\frac{1}{\sin\theta}}^{r=2} d\theta$$

$$= \int_{\pi/6}^{\pi/3} 2 - \frac{1}{2\sin^2\theta} \, d\theta$$

$$= 2\theta + \frac{1}{2}\cot\theta \Big|_{\theta=\pi/6}^{\theta=\pi/3}$$

$$\frac{\cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{1/2}{\sqrt{3}/2} \qquad 1 = r \cdot \sin\theta$$

$$\frac{1}{2}\cot\frac{\pi}{3} - \frac{1}{2}\cot\frac{\pi}{6} = \frac{1}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

Triple integrals

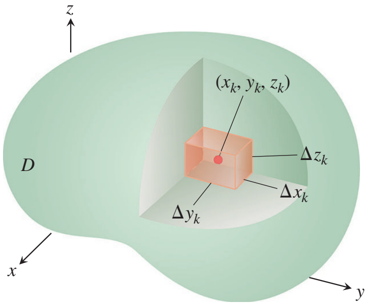


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

$$\iiint_D F(x,y,z)dxdydz$$

Definition

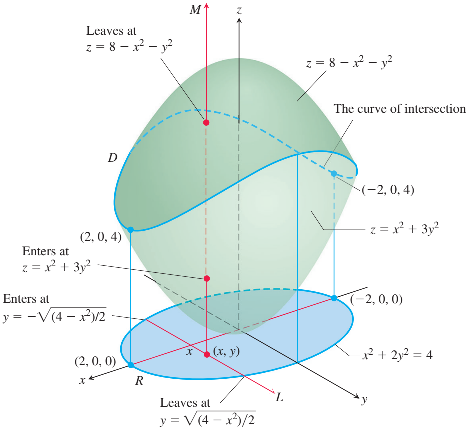
The **volume** of a closed bounded region D in space is

$$V = \iiint_D dV$$

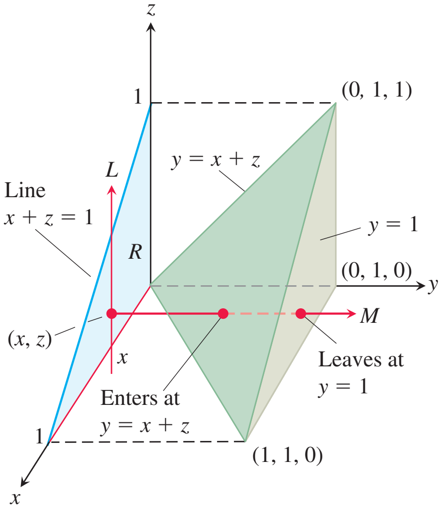
Finding limits of integration in the order of $dz\ dy\ dx$

- Sketch the region of integration along with its "shadow" in the xy-plane.
- Find the z-limits of integration
- Find the y-limits of integration
- Find the x-limits of integration

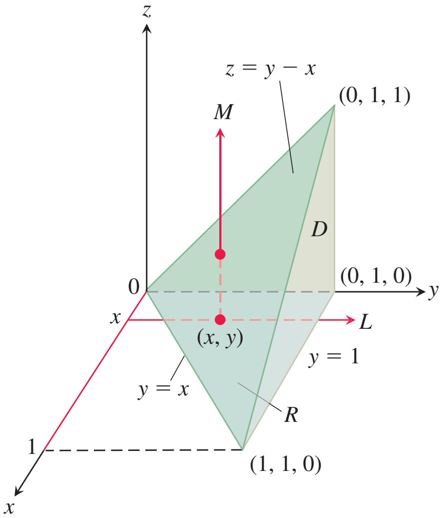
Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dydzdx$.



Integrate $F(x, y, z) = 1$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$ in the order $dzdydx$ and $dydzdx$.



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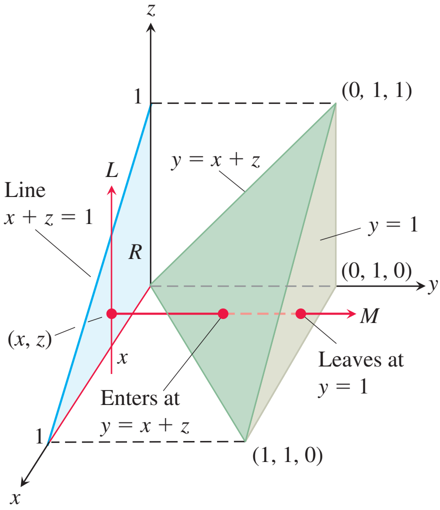

Average value of a function in the 3D space

$$\text{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV$$

Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.