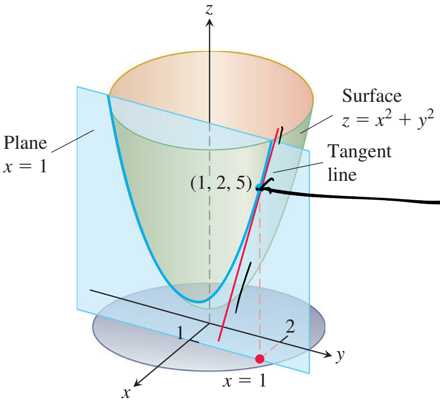
MAT1002: Calculus II

Ming Yan

§14.3 Partial Derivatives and Differentiability

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.



$z = 1 + y^2$

$\frac{dz}{dy} = 2y$

FIGURE 14.19 The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point $(1, 2, 5)$ (Example 5).

Partial Derivatives

$z = f(x, y_0)$

it becomes a function of x only.

$\frac{dz}{dx}$ is computed assuming $y = y_0$

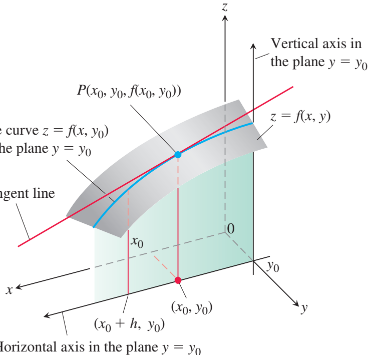


FIGURE 14.16 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

Partial Derivatives

Definition

The **partial derivative of** $f(x, y)$ **with respect to** x **at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

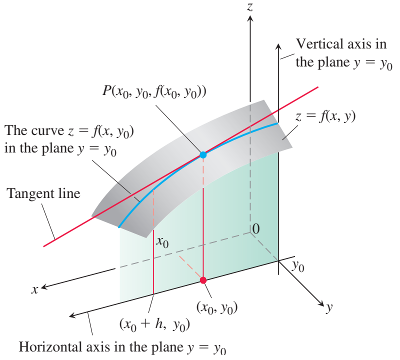


FIGURE 14.16 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

Partial Derivatives

Definition

The **partial derivative of** $f(x, y)$ **with respect to** x **at the point** (x_0, y_0) is

$$\frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Other notations for partial derivatives include

$$\frac{d}{dx} f(x, y_0)\Big|_{x=x_0}, \quad f_x(x_0, y_0)$$

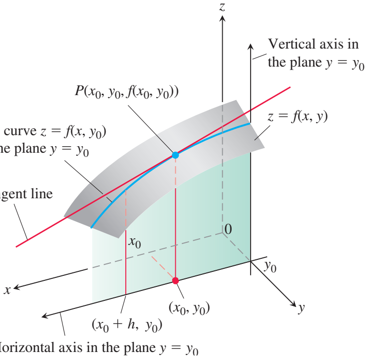


FIGURE 14.16 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

Partial Derivatives

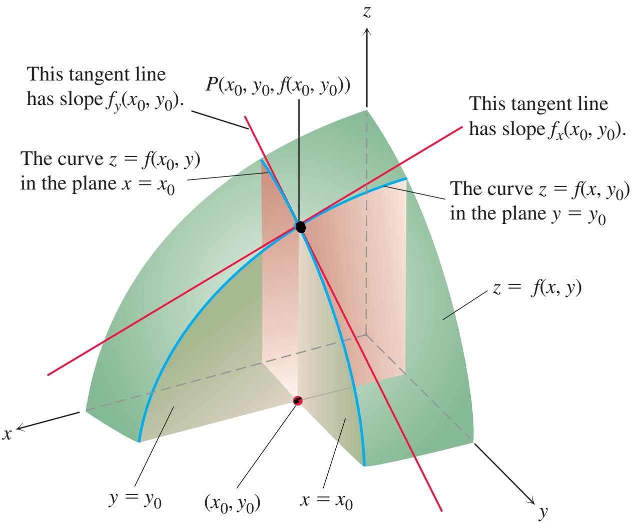


FIGURE 14.18 Figures 14.16 and 14.17 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Example: Partial derivatives of $f(x,y) = x^2 + 3xy + y - 1$ at $(4,-5)$

$$f_x(x,y) = 2x + 3y$$

$$f_y(x,y) = 3x + 1$$

so $f_x(4,-5) = 2\times 4 + 3\times(-5) = -7$

$f_y(4,-5) = 3\times 4 + 1 \quad = 13$

Example: Partial derivatives of $f(x, y) = y \sin xy$

$$f_x(x,y) = y\cos xy \cdot y = y^2\cos xy$$

$$f_y(x,y) = \sin xy + y\cos xy \cdot x$$

$$= \sin xy + xy\cos xy$$

Example: Partial derivatives of

$$f(x,y) = \frac{2y}{y+\cos x} = \underbrace{2y\,(y+\cos x)^{-1}}$$

$$f_x(x,y) = 2y\cdot(-1)\,(y+\cos x)^{-2}(-\sin x)$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y(x,y) = 2\,(y+\cos x)^{-1} + 2y\,(-1)(y+\cos x)^{-2}$$

$$= \frac{2(y+\cos x) - 2y}{(y+\cos x)^2} = \frac{2\cos x}{(y+\cos x)^2}$$

Implicit differentiation: Find $\partial z/\partial x$ of $\underbrace{yz - \ln z = x + y}$

$$y\, z(x,y) - \ln z(x,y) = x + y$$

$$f(x,y) := y\, \underline{z(x,y)} - \ln \underline{z(x,y)} - x - y = 0$$

$$\frac{\partial f}{\partial x} = \underbrace{y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x}} - 1 = 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz - 1}$$

$$\frac{\partial f}{\partial y} = z + y\frac{\partial z}{\partial y} - \frac{1}{z}\frac{\partial z}{\partial y} - 1 = 0$$

$$\frac{\partial z}{\partial y} = \frac{1 - z}{y - \frac{1}{z}} = \frac{z^2 - z}{yz - 1}$$

Example: Find $\partial f/\partial z$ for $f(x,y,z) = x\sin(y+3z)$

$$f_z(x,y,z) = x\cos(y+3z)\,3$$
$$= 3x\cos(y+3z)$$

$$f_x(x,y,z) = \sin(y+3z)$$

$$f_y(x,y,z) = x\cos(y+3z)$$

Second-Order Partial Derivatives

$\dfrac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}$ $(f_y)_x$

$$\frac{\partial^2 f}{\partial x^2},\ f_{xx},\qquad \frac{\partial^2 f}{\partial y^2},\ f_{yy},\qquad \frac{\partial^2 f}{\partial x \partial y},\ f_{yx},\qquad \frac{\partial^2 f}{\partial y \partial x},\ f_{xy}$$

Find the second-order derivatives for $f(x, y) = x\cos y + ye^x$

$f_x = \cos y + ye^x$

$f_y = -x\sin y + e^x$

$f_{xx} = ye^x$

$f_{xy} = -\sin y + e^x$

$f_{yx} = -\sin y + e^x$

$f_{yy} = -x\cos y$

The Mixed Derivative Theorem

Theorem (Theorem 2[1])

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b).}$$

Example: Find w_{yx} for

$$w = xy + \frac{e^y}{y^2+1}$$

$w_{xy} = 1$

$w_x = y$

$w_y = x + \frac{e^y(y^2+1) - e^y(\cdot 2y)}{(y^2+1)^2} = x + \frac{e^y(y-1)^2}{(y^2+1)^2}$

$w_{yx} = 1$

[1] French mathematician Alexis Clairaut

Partial Derivatives of Higher Order

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}, \quad \frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$$

Find f_{yxyz} for $f(x, y, z) = 1 - 2xy^2z + x^2y$

$f_y(x, y, z) = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

Partial Derivatives and Continuity

$f(x,0) = 1$

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

$y = kx$ $\quad f(x,kx) = 0$ if $x \neq 0$, $k \neq 0$

- What is the limit of f as (x,y) approaches $(0,0)$ along the line $y = x$. lim is 0
- Is f continuous at the origin? No. f does not have a limit at $(0,0)$.
- Do partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist?

$$\frac{\partial f}{\partial x}(0,0) = 0$$

$$\frac{\partial f}{\partial y}(0,0) = 0$$

Differentiability

Definition

- (1D) A function $y = f(x)$ is **differentiable** at $x = x_0$, then we have

$$\Delta y \equiv f(x_0 + \Delta x) - f(x_0) = f'(x_0)\Delta x + \underbrace{\epsilon \Delta x}_{o(\Delta x)}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x} = f'(x_0) + \lim_{\Delta x \to 0} \varepsilon$$

Let $\varepsilon_1 = \frac{y_0}{x_0^2} - \frac{y_0}{x_0(x_0+\Delta x)}$

$\varepsilon_2 = -\frac{\Delta x}{x_0(x_0+\Delta x)}$

Definition

- A function $z = f(x,y)$ is **differentiable** at (x_0, y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and Δz satisfies an equation of the form

$$\begin{aligned}\Delta z \equiv& f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)\\ =& f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y\end{aligned}$$

  in which both $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.
- f is **differentiable** if it is differentiable at every point in its domain, and we say that its graph is a **smooth surface**.

$f(x,y) = \frac{y}{x}$

$\Delta z = \frac{y_0+\Delta y}{x_0+\Delta x} - \frac{y_0}{x_0}$

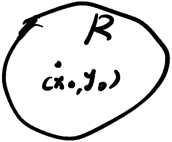
$f_x = -\frac{y}{x^2} \qquad f_y = \frac{1}{x}$

$\Delta z - f_x \Delta x - f_y \Delta y = \underbrace{\frac{y_0+\Delta y}{x_0+\Delta x} - \frac{y_0}{x_0}} + \frac{y_0}{x_0^2}\Delta x - \frac{1}{x_0}\Delta y$

$= \frac{\cancel{x_0 \Delta y} - y_0 \Delta x}{x_0(x_0+\Delta x)} + \frac{y_0}{x_0^2}\Delta x - \frac{\cancel{x_0\Delta y} + \Delta x \Delta y}{x_0(x_0+\Delta x)}$

Theorem (Theorem 3 The Increment Theorem for Functions of Two Variables)

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then $f(x, y)$ is differentiable at (x_0, y_0).



$$f_x = \begin{cases} 0 & \text{if } xy \neq 0 \\ 0 & y = 0 \\ \text{undefined} & y \neq 0 \ \ x = 0 \end{cases}$$

$$f_y = \begin{cases} 0 & \text{if } xy \neq 0 \\ 0 & \text{if } x = 0 \\ \text{undefined} & \text{if } x \neq 0 \ \ y = 0 \end{cases}$$

$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

$f(x,y)$ is differentiable for $xy \neq 0$ ✓

$(x = 0 \ \ y = 0)$ ✗

Differentiability

Theorem (Theorem 3 The Increment Theorem for Functions of Two Variables)

Suppose that the first partial derivatives of $f(x, y)$ *are defined throughout an open region* R *containing the point* (x_0, y_0) *and that* f_x *and* f_y *are continuous at* (x_0, y_0)*. Then* $f(x, y)$ *is differentiable at* (x_0, y_0)*.*

Corollary

If the partial derivatives f_x *and* f_y *of a function* $f(x, y)$ *are continuous throughout an open region* R*, then* f *is differentiable at every point of* R*.*

Differentiability

Theorem (Theorem 3 The Increment Theorem for Functions of Two Variables)

Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then $f(x, y)$ is differentiable at (x_0, y_0).

Corollary

If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Theorem (Theorem 4 Differentiability Implies Continuity)

If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).