Probability and Statistics I

Final Examination
SSE, CUHK(SZ)

May 15, 2017

Name: ____________________ Student ID: ____________________

Answer all the questions in the Answer Book. This page has no questions.

1. (8 points) Let Y_1, Y_2, Y_3 be independent random variables which have the Bernoulli distribution with the probability of success p.

   (a) (1 point) Define a new random variable $Z = Y_1 + Y_2 + Y_3$. Find the distribution of Z. Provide the details of the derivation.

   (b) (7 points) For $k = 1, 2$, let

$$X_k = \begin{cases} 1, & Y_1 + Y_2 + Y_3 = k, \\ -1 & Y_1 + Y_2 + Y_3 \neq k. \end{cases}$$

      i. (2 points) Find the joint pmf of X_1, X_2.

      ii. (2 points) Find the marginal pmfs of X_1 and X_2, respectively.

      iii. (2 points) Find the value of the success probability p that minimizes $\mathrm{E}(X_1 X_2)$.

      iv. (1 point) Compute $\mathrm{Cov}(X_1 - X_2, X_2)$.

Solution:

(a)

$$\mathrm{E}(e^{tZ}) = \mathrm{E}(e^{t(Y_1+Y_2+Y_3)}) = \mathrm{E}(e^{tY_1})\mathrm{E}(e^{tY_2})\mathrm{E}(e^{tY_3}) = (1 - p + pe^t)^3$$

Hence Z has a binomial distribution $Z = Y_1 + Y_2 + Y_3 \sim b(3, p)$

(b) i. Let $f(X_1, X_2)$ be the joint pmf of X_1 and X_2,then we have

$$\begin{aligned} f(-1,-1) &= P\{X_1 = -1, X_2 = -1\} \\ &= P\{Y_1 + Y_2 + Y_3 \neq 1, Y_1 + Y_2 + Y_3 \neq 2\} \\ &= P\{Y_1 + Y_2 + Y_3 = 0\} + P\{Y_1 + Y_2 + Y_3 = 3\} \\ &= (1-p)^3 + p^3 \end{aligned}$$

$$\begin{aligned} f(-1,1) &= P\{X_1 = -1, X_2 = 1\} \\ &= P\{Y_1 + Y_2 + Y_3 \neq 1, Y_1 + Y_2 + Y_3 = 2\} \\ &= \binom{3}{2} p^2(1-p) \\ &= 3p^2(1-p) \end{aligned}$$

$$\begin{aligned} f(1,-1) &= P\{X_1 = 1, X_2 = -1\} \\ &= P\{Y_1 + Y_2 + Y_3 = 1, Y_1 + Y_2 + Y_3 \neq 2\} \\ &= P\{Y_1 + Y_2 + Y_3 = 1\} \\ &= \binom{3}{1} p(1-p)^2 = 3p(1-p)^2 \end{aligned}$$

$$\begin{aligned} f(1,1) &= P\{X_1 = 1, X_2 = 1\} \\ &= P\{Y_1 + Y_2 + Y_3 = 1, Y_1 + Y_2 + Y_3 = 2\} = 0 \end{aligned}$$

ii. Then we could get the marginal pmf of X_k as follows:

$$f_{X1}(x) = \begin{cases} 1 - 3p + 6p^2 - 3p^3, & X_1 = -1, \\ 3p(1-p)^2, & X_1 = 1. \end{cases}$$

$$f_{X2}(x) = \begin{cases} 1 - 3p^2 + 3p^3, & X_2 = -1, \\ 3p^2(1-p), & X_2 = 1. \end{cases}$$

iii.

$$\begin{aligned} &\mathrm{E}(X_1X_2) = 1 \times P\{X_1 = -1, X_2 = -1\} + (-1) \times P\{X_1 = -1, X_2 = 1\} \\ &+ (-1) \times P\{X_1 = 1, X_2 = -1\} + 1 \times P\{X_1 = 1, X_2 = 1\} = 1 - 6p + 6p^2 \end{aligned}$$

When $p = \frac{1}{2}$, $\mathrm{E}(X_1X_2)$ gets the minimum value $-\frac{1}{2}$.

iv.

$$\begin{aligned} \mathrm{Cov}(X_1 - X_2, X_2) &= \mathrm{Cov}(X_1, X_2) - \mathrm{Cov}(X_2, X_2) \\ &= \mathrm{E}(X_1X_2) - \mathrm{E}(X_1)\mathrm{E}(X_2) - \mathrm{Var}(X_2) \\ &= -12p^2 - 24p^3 + 144p^4 - 180p^5 + 72p^6 \end{aligned}$$

2. (8 points) Let T have a student's t distribution with r degrees of freedom.

$$T = \frac{Z}{\sqrt{\frac{U}{r}}}$$

where Z has a standard normal distribution, that is $Z \sim N(0,1)$, and U has a chi-square distribution with degrees of freedom r, that is $U \sim \chi^2(r)$, and Z and U are independent.

(a) (2 points) Find $\mathrm{E}(Z)$ and $\mathrm{E}(Z^2)$.

(b) (4 points) Find $\mathrm{E}(\frac{1}{\sqrt{U}})$ and $\mathrm{E}(\frac{1}{U})$.

(c) (1 point) Show that $\mathrm{E}(T) = 0$ provided that $r \geq 2$.

(d) (1 point) Show that $\mathrm{Var}(T) = \frac{r}{r-2}$ provided that $r \geq 3$.

Solution: Please refer to Exercise 5.5-14.

(a) Because Z is $N(0,1)$, $\mathrm{E}(Z) = 0$ and $\mathrm{E}(Z^2) = 1$.

(b) Since $U \sim \chi^2(r)$ so it follows that

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{\sqrt{U}}\right] &= \int_0^\infty \frac{1}{\sqrt{u}} \frac{1}{\gamma(r/2)2^{r/2}} u^{r/2-1} e^{-u/2} du \\
&= \frac{\gamma[(r-1)/2]}{\sqrt{2}\gamma(r/2)} \int_0^\infty \frac{1}{\gamma[(r-1)/2]2^{(r-1)/2}} u^{(r-1)/2-1} e^{-u/2} du \\
&= \frac{\gamma[(r-1)/2]}{\sqrt{2}\gamma(r/2)}.
\end{aligned}$$

Note that the last integral is equal to one because the integrand is the pdf of a $\chi^2(r-1)$ random variable.

To find $E[\frac{1}{U}]$ we have

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{U}\right] &= \int_0^\infty \frac{1}{u} \frac{1}{\gamma(r/2)2^{r/2}} u^{r/2-1} e^{-u/2} du \\
&= \frac{\gamma[(r-2/2)]}{2\gamma(r/2)} \int_0^\infty \frac{1}{\gamma[(r-2)/2]2^{(r-2)/2}} u^{(r-2)/2-1} e^{-u/2} du \\
&= \frac{\gamma(r/2-1)}{2\gamma(r/2-1)(r/2-1)} = \frac{1}{r-2}.
\end{aligned}$$

Note that the last integral is equal to one because the integrand is the pdf of a $\chi^2(r-2)$ random variable.

(c)

$$\mathrm{E}[T] = \mathrm{E}\left[\frac{Z}{\sqrt{U/r}}\right] = \mathrm{E}(Z)\mathrm{E}\left[\frac{1}{\sqrt{U/r}}\right] = 0[\frac{\sqrt{r}\gamma[(r-1)/2]}{\sqrt{2}\gamma(r/2)}] = 0,$$

provided $r \geq 2$;

(d)

$$\mathrm{Var}(T) = \mathrm{E}(T^2) - 0^2 = \mathrm{E}[Z^2]\mathrm{E}[r/U] = \frac{r}{r-2},$$

provided $r \geq 3$.

3. (9 points) Let X and Y be two random variables with the joint pdf

$$f(x, y) = \frac{1}{8}, \quad 0 \leq y \leq 4,\ y \leq x \leq y+2.$$

(a) (2 points) Find $f_X(x)$, the marginal pdf of X.

(b) (1 point) Find $f_Y(y)$, the marginal pdf of Y.

(c) (2 points) Determine $h(y|x)$, the conditional pdf of Y, given that $X = x$.

(d) (1 point) Determine $g(x|y)$, the conditional pdf of X, given that $Y = y$.

(e) (2 points) Compute $\mathrm{E}(Y|x)$, the conditional mean of Y, given that $X = x$.

(f) (1 point) Compute $\mathrm{E}(X|y)$, the conditional mean of X, given that $Y = y$.

Solution: Please refer to Exercise 4.4-18.

(a)

$$f_X(x) = \begin{cases} \int_0^x \frac{1}{8} dy = \frac{x}{8}, & 0 \le x \le 2, \\ \int_{x-2}^x \frac{1}{8} dy = \frac{1}{4}, & 2 < x < 4, \\ \int_{x-2}^4 \frac{1}{8} dy = \frac{6-x}{8}, & 4 \le x \le 6. \end{cases}$$

(b)

$$f_Y(y) = \int_y^{y+2} \frac{1}{8} dx = \frac{1}{4}, \quad 0 \le y \le 4.$$

(c)

$$h(y|x) = \begin{cases} \frac{1}{x}, & 0 \le y \le x, \quad 0 \le x \le 2, \\ \frac{1}{2}, & x-2 < y < x, \quad 2 < x < 4, \\ \frac{1}{(6-x)}, & x-2 \le y \le 4, \quad 4 \le x \le 6. \end{cases}$$

(d)

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2}, \quad y \le x \le y+2, \quad 0 \le y \le 4.$$

(e)

$$E(Y|x) = \begin{cases} \int_0^x y \frac{1}{x} dy = \frac{x}{2}, & 0 \le x \le 2, \\ \int_{x-2}^x y \frac{1}{2} dy = x - 1, & 2 < x < 4, \\ \int_{x-2}^4 y \frac{1}{6-x} dy = \frac{x+2}{2}, & 4 \le x \le 6. \end{cases}$$

(f)

$$E(X|y) = \int_y^{y+2} x \frac{1}{2} dx = y + 1, \quad 0 \le y \le 4.$$

4. (10 points) Let X and Y have a bivariate normal distribution with $\mu_X = -3, \mu_Y = 10, \sigma_X^2 = 25, \sigma_Y^2 = 9$, and $\rho = 3/5$. Answer the following questions (if necessary, make use of the standard normal table shown in Figure 1).

(a) (1 point) Compute $P(-4 < X < 4)$.

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319

Figure 1: **The standard normal table**. The entries in this table provide the numerical values of $\Phi(x) = P(X \le x)$, where X is a standard normal random variable, for x between 0 and 1.49. For example, to find $\Phi(0.72)$, we look at the row corresponding to 0.7 and the column corresponding to 0.02 so that $\Phi(0.72) = 0.7642$. When x is negative, the value of $\Phi(x)$ can be found using the formula $\Phi(x) = 1 - \Phi(-x)$.

(b) (3 points) Compute $P(-4 < X < 4|Y = 13)$.

(c) (3 points) Determine $\mathrm{E}(X^2|Y = 13)$ and $\mathrm{E}[X(X-1)|Y = 13]$.

(d) (3 points) Determine $\mathrm{E}(X + Y)$, $\mathrm{Var}(X + Y)$ and $\mathrm{E}(XY)$.

Solution: Please refer to Exercise 4.5-1.

(a)

$$\begin{aligned}P(-4 < X < 4) &= \Phi(\frac{4-(-3)}{\sqrt{25}}) - \Phi(\frac{-4-(-3)}{\sqrt{25}}) \\ &= \Phi(1.4) - \Phi(-0.2) = \Phi(1.4) - (1 - \Phi(0.2)) = 0.4985\end{aligned}$$

(b)

$$\mu_{X|Y=13} = \mu_X + \rho\frac{\sigma_X}{\sigma_Y}(13 - \mu_Y) = -3 + (3/5)(5/3)(13 - 10) = 0.$$

$$\sigma^2_{X|Y=13} = \sigma^2_X(1 - \rho^2) = 25(1 - (3/5)^2) = 16.$$

$$\begin{aligned}P(-4 < X < 4|Y = 13) &= \Phi(\frac{4-0}{\sqrt{16}}) - \Phi(\frac{-4-0}{\sqrt{16}}) = \Phi(1) - \Phi(-1) \\ &= \Phi(1) - (1 - \Phi(1)) = 0.6826.\end{aligned}$$

(c)

$$E(X^2|Y = 13) = \sigma^2_{X|Y=13} + \mu^2_{x|Y=13} = 16.$$

$$E(X(X-1)|Y = 13) = E(X^2|Y = 13) - E(X|Y = 13) = 16.$$

(d)

$$E(X + Y) = E(X) + E(Y) = -3 + 10 = 7.$$

$$\mathrm{Var}(X + Y) = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y = 25 + 9 + 2 \times (3/5) \times 5 \times 3 = 52.$$

$$E(XY) = \rho\sigma_X\sigma_Y + \mu_X\mu_Y = (3/5) \times 5 \times 3 + (-3) \times 10 = -21.$$

5. (10 points) Let X be a random variable of distribution $N(0, 1)$, and define $Y = e^X$.
   (a) (3 points) Find the pdf of Y.
   (b) (2 points) Compute $P(1 < Y < 2)$, using $\ln 2 = 0.69$ and the standard normal table as shown in Figure 1.
   (c) (5 points) Find the mean and variance of Y.

Solution: Please refer to Exercises 3.3-14 and 5.1-13.

(a)

$$\begin{aligned} G(y) &= P(Y \le y) \\ &= P(e^X \le y) \\ &= P(X \le \ln y) \\ &= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx. \end{aligned}$$

$$g(y) = G(y)' = \frac{1}{y\sqrt{2\pi}} e^{-(\ln y)^2/2}.$$

(b)

$$\begin{aligned} P(1 < Y < 2) &= P(\ln 1 < X < \ln 2) \\ &= \Phi(0.69) - \Phi(0) \\ &= 0.7549 - 0.5 = 0.2549 \end{aligned}$$

(c)

$$M_X(t) = \mathrm{E}(e^{tX}) = e^{t^2/2}.$$

$$\mathrm{E}(Y) = \mathrm{E}(e^X) = M_X(1) = e^{0.5}.$$

$$\mathrm{E}(Y^2) = \mathrm{E}(e^{2X}) = M_X(2) = e^2.$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - [\mathrm{E}(Y)]^2 = e^2 - e^1.$$

6. (10 points) Let X_1 and X_2 be two independent continuous random variables and let $Y = X_1 + X_2$.
   (a) (5 points) Denote the pdf of X_1 as $f_{X_1}(x_1), x_1 \in (-\infty, \infty)$, and the pdf of X_2 as $f_{X_2}(x_2), x_2 \in (-\infty, \infty)$. Express the pdf of Y in terms of the pdfs of X_1 and X_2.
   (b) (5 points) Assume that X_1 and X_2 have the same pdf: $f(x) = e^{-x}, 0 < x < \infty$.
      i. (2 points) Use the moment-generating function technique to find the pdf of Y.
      ii. (3 points) Use your conclusion in part (a) to find the pdf of Y.

Solution:

(a)

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X_1 + X_2 \le y) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{y-x_1} f_{X_1}(x_1) f_{X_2}(x_2) dx_2 dx_1 \\ &= \int_{-\infty}^{\infty} f_{X_1}(x_1) [\int_{-\infty}^{y-x_1} f_{X_2}(x_2) dx_2] dx_1 \\ &= \int_{-\infty}^{\infty} f_{X_1}(x_1) F_{X_2}(y - x_1) dx_1 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} \\ &= \frac{d}{dy} \int_{-\infty}^{\infty} f_{X_1}(x_1) F_{X_2}(y - x_1) dx_1 \\ &= \int_{-\infty}^{\infty} f_{X_1}(x_1) \frac{dF_{X_2}}{y}(y - x_1) dx_1 \end{aligned}$$

$$= \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1, \quad y \in (-\infty, \infty)$$

(b) i. X_1 and X_2 have a gamma distribution with $\alpha = 1, \theta = 1$.

$$M_{X_1}(t) = M_{X_2}(t) = \frac{1}{1-t}, \quad t < 1.$$

$$M_Y(t) = M_{X_1}(t) M_{X_2}(t) = \frac{1}{(1-t)^2}, \quad t < 1.$$

so that Y has a gamma distribution with $\alpha = 2, \theta = 1$.

$$f_Y(y) = ye^{-y}, \quad 0 < y < \infty.$$

ii. $0 < x_1 < \infty, 0 < x_2 = y - x_1 < \infty$, so that $0 < x_1 < y$.

$$\begin{aligned} f_Y(y) &= \int_0^y f_{X_1}(x_1) f_{X_2}(y - x_1) dx_1 \\ &= \int_0^y e^{-x_1} e^{-(y-x_1)} dx_1 \\ &= \int_0^y e^{-y} dx_1 \\ &= ye^{-y}, \quad 0 < y < \infty. \end{aligned}$$

7. (10 points) Two components operate in parallel in a device, so the device fails when and only when both components fail. The lifetimes, X_1 and X_2, of the respective components are independent and identically distributed with an exponential distribution with $\theta = 2$ (i.e., the mean value is 2). The cost of operating the device is $Z = 2Y_1 + Y_2$, where $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.
   (a) (2 points) Show that $P(X_1 > a + b | X_1 > a) = P(X_1 > b)$ for $a \geq 0, b \geq 0$.
   (b) (2 points) Show that $W = X_1 + X_2$ has a gamma distribution with parameters $\alpha = 2$ and $\theta = 2$.
   (c) (4 points) Compute the pdf of Y_1 and the pdf of Y_2.
   (d) (2 points) Compute E(Z).

Solution:

(a) Please refer to Exercise 3.2-3.
Since X_1 has an exponential distribution with parameter $\theta = 2$, $X_1 \geq 0$.
If $a \geq 0$,$b \geq 0$, $P(X_1 > a + b | X_1 > a) = \frac{P(X_1 > a+b)}{P(X_1 > a)} = \frac{e^{-2(a+b)}}{e^{-2a}} = e^{-2b} = P(X_1 > b)$.

(b) Please refer to Exercise 5.4-8.

The mgf of W is

$$\mathrm{E}(e^{tW}) = \mathrm{E}(e^{t(X_1+X_2)}) = \mathrm{E}(e^{tX_1})E(e^{tX_2}) = \frac{1}{1-2t}\cdot\frac{1}{1-2t} = \frac{1}{(1-2t)^2}, t < \frac{1}{2}$$

Hence, W follows a gamma distribution with parameters $\alpha = 2$ and $\theta = 2$.

(c) Please refer to Exercise 5.3-19.

$$\begin{aligned}
F_{Y_1}(y_1) = P(Y_1 \le y_1) &= 1 - P(\min(X_1, X_2) > y_1) \\
&= 1 - P(X_1 > y_1)P(X_2 > y_1) \\
&= 1 - (1 - F_{X_1}(y_1))(1 - F_{X_2}(y_1)) \\
&= 1 - (e^{-\frac{1}{2}y_1})^2 = 1 - e^{-y_1}, \quad y_1 \ge 0 \\
\text{Hence,} \quad f_{Y_1}(y_1) &= e^{-y_1}, \quad y_1 \ge 0 \\
F_{Y_2}(y_2) = P(Y_2 \le y_2) &= P(\max(X_1, X_2) \le y_2) \\
&= P(X_1 \le y_2)P(X_2 \le y_2) = F_{X_1}(y_2)F_{X_2}(y_2) \\
&= (1 - e^{-\frac{1}{2}y_2})^2 = 1 + e^{-y_2} - 2e^{-\frac{1}{2}y_2}, \quad y_2 \ge 0 \\
\text{Hence,} \quad f_{Y_2}(y_2) &= -e^{-y_2} + e^{-\frac{1}{2}y_2}, \quad y_2 \ge 0
\end{aligned}$$

(d) Please also refer to Exercise 5.3-19.

$$\begin{aligned}
\mathrm{E}(Y_1) &= 1 \\
\mathrm{E}(Y_2) &= \int_0^{+\infty} -ye^{-y} + ye^{\frac{1}{2}y}dy \\
&= \lim_{a\to+\infty} [ye^{-y} + e^{-y} - 2ye^{-\frac{1}{2}y} - 4e^{-\frac{1}{2}y}]\Big|_0^a \\
&= 3 \\
\text{Hence,} \quad \mathrm{E}(Z) &= E(2Y_1 + Y_2) = 2 \times 1 + 3 = 5
\end{aligned}$$

Or, obviously, $\mathrm{E}(Z) = \mathrm{E}(Y_1 + Y_2) + \mathrm{E}(Y_1) = \mathrm{E}(X_1 + X_2) + \mathrm{E}(Y_1)$, and by using the conclusion of (b),we get $\mathrm{E}(Z) = 2 \times 2 + 1 = 5$.

8. (10 points) Suppose that the number of customers visiting a fast food restaurant in a given day is K, which follows a Poisson distribution with parameter λ. Assume that each customer purchases a drink with probability p, independently from other customers, and moreover, the value of p is unchanged for different value of K. Let X be the number of customers who purchase drinks, and let Y be the number of customers who do not purchase drinks. So $X + Y = K$.

   (a) (2 points) Given that $K = k$, what is the expectation of X, namely $\mathrm{E}(X|K = k)$ and the variance of X, namely $\mathrm{Var}(X|K = k)$?

   (b) (5 points) What is the pmf of X and the pmf of Y? Show that X and Y are independent. Hint: make use of the formula shown below

$$e^x = \sum_{t=0}^{\infty} \frac{x^t}{t!} \tag{1}$$

(c) (3 points) Use the normal distribution, Central Limit Theorem (CLT) and half-unit correction for continuity to approximate $P(a \le \overline{X} \le b)$, where a and b are integers and $0 < a < b$, and $\overline{X}$ is the mean of a random sample $X_1, X_2, X_3, \ldots, X_{100}$ of size 100 from the distribution of X, i.e. $\overline{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$. Express the answer in terms of the cdf Φ of the standard normal distribution $N(0,1)$.

Solution:

(a) Given that $K = k$, X follows a binomial distribution with parameters k and p, so $\mathrm{E}(X|K=k) = kp$ and $\mathrm{Var}(X|K=k) = kp(1-p)$.

(b) Let $f_K(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ be the pmf of K, which has a Poisson distribution with parameter λ.
Let $q = 1-p$, and the pmf of X, $f_X(x)$ is given by

$$\begin{aligned}
f_X(x) &= \sum_{k=0}^{\infty} P(X=x, K=k) \\
&= \sum_{k=0}^{\infty} P(X=x|K=k)P(K=k) = \sum_{k=0}^{\infty} P(X=x|K=k) f_K(k) \\
&= \sum_{k=x}^{\infty} \binom{k}{x} p^x q^{k-x} \exp(-\lambda)\frac{\lambda^k}{k!} \\
&= \frac{\exp(-\lambda)(\lambda p)^x}{x!} \sum_{k=x}^{\infty} \frac{(\lambda q)^{k-x}}{(k-x)!} \\
&= \frac{\exp(-\lambda)(\lambda p)^x}{x!} \exp(\lambda q) \qquad \text{given by the formula (1)} \\
&= \frac{\exp(-\lambda p)(\lambda p)^x}{x!} \quad x = 0, 1, 2\ldots
\end{aligned}$$

Thus, we conclude that X has a poisson distribution with parameter λp, and similarly, Y has a poisson distribution with parameter $\lambda q = \lambda(1-p)$, i.e.

$$f_Y(y) = \frac{\exp(-\lambda q)(\lambda q)^y}{y!} \quad y = 0, 1, 2\ldots$$

To find the joint pmf of X and Y, $f(x,y)$, we can use the law of total probability:

$$f(x,y) = \sum_{k=0}^{\infty} P(X=x, Y=y|K=k) f_K(k) \qquad x = 0,1,2\ldots \quad y = 0,1,2\ldots$$

But note that $P(X=x, Y=y|K=k) = 0$ if $k \neq x+y$

$$\begin{aligned} f(x,y) &= P(X=x, Y=y|K=x+y) f_K(x+y) \\ &= P(X=x|K=x+y) f_K(x+y) \\ &= \binom{x+y}{x} p^x q^y \exp(-\lambda) \frac{\lambda^{x+y}}{(x+y)!} \\ &= \frac{\exp(-\lambda p)(\lambda p)^x}{x!} \cdot \frac{\exp(-\lambda q)(\lambda q)^y}{y!} \\ &= f_X(x) f_Y(y) \end{aligned}$$

X and Y are independent, since as we saw above

$$f(x,y) = f_X(x) f_Y(y).$$

(c) Please refer to a similar question in Exercise 5.7-13.

Since X follows a Poisson distribution with parameter λp, we have $\mathrm{E}(X_i) = \lambda p$ and $\mathrm{Var}(X_i) = \lambda p$. By applying the Central Limit Theorem, we can use $N(100\lambda p, 100\lambda p)$ to approximate the distribution of $\sum_{i=1}^{100} X_i$.

$$\begin{aligned} P(a \leq \overline{X} \leq b) &= P(100a \leq \sum_{i=1}^{100} X_i \leq 100b) \\ &\approx P\left(\frac{100a - 100\lambda p - \frac{1}{2}}{\sqrt{100\lambda p}} \leq Z \leq \frac{100b - 100\lambda p + \frac{1}{2}}{\sqrt{100\lambda p}}\right) \\ &= \Phi\left(\frac{100b - 100\lambda p + \frac{1}{2}}{10\sqrt{\lambda p}}\right) - \Phi\left(\frac{100a - 100\lambda p - \frac{1}{2}}{10\sqrt{\lambda p}}\right) \end{aligned}$$

9. (15 points) This question considers a communication system, which comprises three components: an encoder, a noisy channel and a decoder. In your solution of this question, you may need the pdf of the standard normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),$$

and the cdf of the standard normal distribution

$$\Phi(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t.$$

(a) (4 points) The noisy channel has an input and an output. The input of the channel is a discrete random variable X with the uniform distribution over $\{1, -1\}$, which is generated by the encoder. The output of the channel is a continuous random variable Y defined as

$$Y = X + Z,$$

where Z has the normal distribution $N(0, \sigma^2)$, $\sigma > 0$, and X and Z are independent.

i. What is the distribution of Y given that $X = 1$? Determine the conditional pdf of Y given $X = 1$.

ii. What is the distribution of Y given that $X = -1$? Determine the conditional pdf of Y given $X = -1$.

iii. Give the pdf of Y.

(b) (6 points) After getting the output Y of the channel, the decoder makes a *decision* about the input X of the channel with respect to a *threshold* y: If $Y > y$, $\hat{X} = 1$; otherwise, $\hat{X} = -1$. The probability $P(\hat{X} \neq X)$ is called the decision error probability and is actually a function of y.

i. Find the decision error probability when $X = 1$, i.e., $P(\hat{X} \neq X | X = 1)$.

ii. Find the decision error probability when $X = -1$, i.e., $P(\hat{X} \neq X | X = -1)$.

iii. Find the decision error probability $P(\hat{X} \neq X)$.

iv. Determine the optimal value of y that minimizes the decision error probability. What is the minimum decision error probability?

(c) (5 points) Fix an integer $n > 0$. Suppose that the random variable X is transmitted n times through the channel. For the ith transmission, $i = 1, \ldots, n$, the channel takes X as the input and generates the output $Y_i = X + Z_i$, where Z_i has the normal distribution $N(0, \sigma^2)$. Moreover, $X, Z_1, Z_2, \ldots, Z_n$ are mutually independent. In this case, the decoder can use $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ to make the decision about the input X.

i. What are the mean and variance of $\overline{Y}$?

ii. Using Chebyshev's Inequality, show that $\overline{Y}$ converges to X in probability, i.e., for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\overline{Y} - X| \geq \epsilon) = 0.$$

Solution:

(a) i. Given $X = 1$, Y is $N(1, \sigma^2)$, which has pdf $f((y-1)/\sigma)/\sigma$.

ii. Given $X = -1$, Y is $N(-1, \sigma^2)$, which has pdf $f((y+1)/\sigma)/\sigma$.

iii. $[f((y-1)/\sigma) + f((y+1)/\sigma)]/(2\sigma)$.

(b) i.

$$\begin{aligned} P(\hat{X} \neq X | X = 1) &= P(\hat{X} = -1 | X = 1) \\ &= P(Y \leq y | X = 1) \\ &= \Phi((y-1)/\sigma). \end{aligned}$$

ii.

$$\begin{aligned} P(\hat{X} \neq X | X = -1) &= P(\hat{X} = 1 | X = -1) \\ &= P(Y > y | X = -1) \\ &= 1 - \Phi((y+1)/\sigma). \end{aligned}$$

iii.

$$\begin{aligned} p_{\mathrm{e}}(y) &\triangleq P(\hat{X} \neq X) \\ &= P(\hat{X} \neq X | X = 1)P(X = 1) + P(\hat{X} \neq X | X = -1)P(X = -1) \\ &= [\Phi((y-1)/\sigma) + 1 - \Phi((y+1)/\sigma)]/2. \end{aligned}$$

iv. Taking the derivative with respect to y, we have

$$p_{\mathrm{e}}'(y) = \frac{1}{2\sigma}\left(f((y-1)/\sigma) - f((y+1)/\sigma)\right).$$

Using the symmetry, bell-shape of the standard normal pdf, we have that

- when $y < 0$, $p_{\mathrm{e}}'(y) < 0$;
- when $y = 0$, $p_{\mathrm{e}}'(y) = 0$;
- when $y > 0$, $p_{\mathrm{e}}'(y) > 0$.

Hence, $p_{\mathrm{e}}(y)$ is minimized at $y = 0$ and the optimal value is $[\Phi(-1/\sigma) + 1 - \Phi(1/\sigma)]/2 = \Phi(-1/\sigma)$.

(c) Write $\overline{Y} = X + \overline{Z}$.

i. $E(\overline{Y}) = E(X) + E(\overline{Z}) = 0$. $Var(\overline{Y}) = Var(X) + Var(\overline{Z}) = 1 + \sigma^2/n$.

ii. By Chebyshev's Inequality, $P(|\overline{Y} - X| \geq \epsilon) = P(|\overline{Z} - 0| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$. Taking the limit on both sides,

$$\lim_{n\to\infty} P(|\overline{Y} - X| \geq \epsilon) \leq 0.$$

Together with $P(|\overline{Y} - X| \geq \epsilon) \geq 0$, we have $\lim_{n\to\infty} P(|\overline{Y} - X| \geq \epsilon) = 0$.

10. (10 points) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a normal distribution $N(\mu, \sigma^2)$. Suppose that it has been known that the random sample mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the random sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

are independent.

(a) (8 points) Prove that

$$\frac{n-1}{\sigma^2}S^2$$

has a chi-square distribution with degrees of freedom $n-1$, i.e.,

$$\frac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1).$$

(b) (2 points) Prove that

$$\mathrm{E}(S^2) = \sigma^2.$$

Solution:

(a) See the proof of Theorem 5.5-2 in the textbook on page 203.

(b) Note that $\mathrm{E}(X) = r$ for $X \sim \chi^2(r)$. Then it follows from $\frac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1)$ that,

$$\mathrm{E}\left(\frac{n-1}{\sigma^2} S^2\right) = n-1 \Longrightarrow \mathrm{E}(S^2) = \sigma^2.$$