MAT1002: Calculus II

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§10.1 Infinite Sequences

Overview of Chapter 10

Two key topics in the analysis and computation of functions:

- derivative of a function
- integral of a function

A third key topic: **infinite series**

- express numbers

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

- express functions

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots$$

- approximate a function using the first few terms
- represent a function as a polynomial with infinitely many terms (evaluate, differentiate, integrate), e.g., $\cos x$
- has applications in science and engineering

Topics for this chapter:

How do you add infinitely many numbers (powers of x)?

Infinite sequences

Definition

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$$\{a_1, a_2, \cdots, a_n, \cdots\}$$

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- A sequence is a function whose domain is the set of integers $\{1, 2, 3, \cdots\}$, i.e., $a_1 = f(1)$, $a_2 = f(2)$, $\cdots$.
- The integer n is called the **index** of a_n.
- The index set can start with any integer number other than 1, e.g., $\{a_0, a_1, a_2, \cdots\}$, $\{b_2, b_3, b_4, \cdots\}$, $\{c_{-2}, c_{-1}, c_0, c_1 \cdots\}$ are sequences.
- We use $\{a_n\}_{n=2}^{\infty}$ to denote $\{a_2, a_3, a_4, \cdots\}$. Sometimes, we say $\{a_n\}$ if the index set is not important or is clear from the context.

Different ways to describe a sequence

- $$\{(-1)^{n+2}n^2\}_{n=1}^{\infty}$$ $\{(-1)^{n+1}(n+1)^2\}_{n=0}^{\infty}$
- $$\{-1, 4, -9, 16, \cdots\}$$

A sequence can be defined **recursively**

Fibonacci sequence

$$F_0 = 0, \; F_1 = 1, \; F_i = F_{i-1} + F_{i-2}$$

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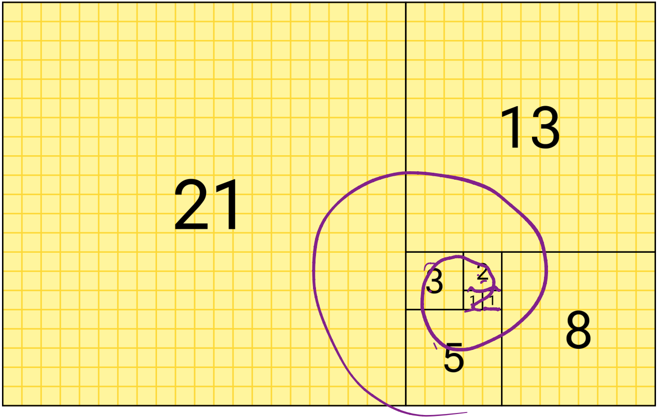


Convergence and divergence

- $$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n}, \cdots$$

  goes to 0

- $$1, 2, 3, 4, \cdots, n, \cdots$$

  goes to ∞

- $$1, -1, 1, -1, \cdots, (-1)^n, \cdots$$

  $\{(-1)^{n+1}\}_{n=1}^{\infty}$

  back & forth between ± 1

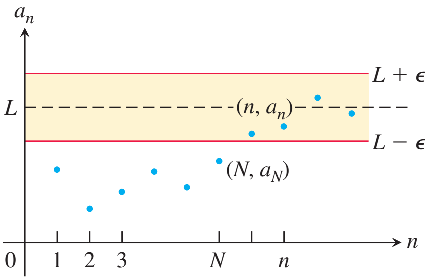
Convergence

Definition

A sequence $\{a_n\}$ **converges** to the number L if for any positive number ϵ, there exists an integer N such that for all $n > N$,

$$|a_n - L| < \epsilon.$$

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence $\{a_n\}$.



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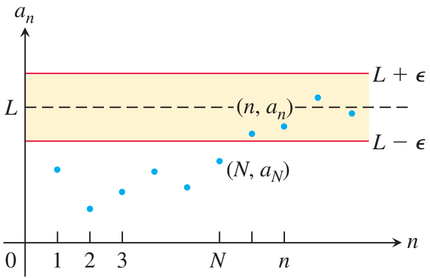


- N depends on ϵ, in general, a smaller ϵ requires a bigger N.
- $n > N$ can be replaced by $n \geq N$.

Examples:

- $$\frac{1}{n} \to 0$$

$$\left|\frac{1}{n}-0\right| \leq \varepsilon \Leftrightarrow \frac{1}{n} \leq \varepsilon \Leftrightarrow n \geq \frac{1}{\varepsilon}$$

choose $N = \lceil \frac{1}{\varepsilon} \rceil$ the smallest integer larger than $1/\varepsilon$

- $$\frac{n}{2n+1} \to \frac{1}{2}$$

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| \leq \varepsilon \Leftrightarrow \left|\frac{2n-(2n+1)}{2(2n+1)}\right| \leq \varepsilon$$

$$\Leftrightarrow \frac{1}{2(2n+1)} \leq \varepsilon$$

$$\Leftrightarrow n \geq \frac{1}{2}\left(\frac{1}{2\varepsilon} - 1\right)$$

choose $N = \lceil \frac{1}{2}(\frac{1}{2\varepsilon} - 1) \rceil$

Examples:

- $$(-1)^n \to ?$$

Assume it has a limit L

choose $\varepsilon = \frac{1}{2}$, there exists N

s.t. $\underbrace{|(-1)^n - L| \le \varepsilon}$ if $n > N$

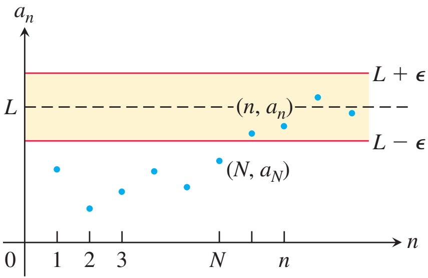
$\Rightarrow$ $|(-1) - L| \le \varepsilon$ $\Rightarrow$ $-1-\varepsilon \le L \le -1+\varepsilon$

$|1 - L| \le \varepsilon$ $\Rightarrow$ $1-\varepsilon \le L \le 1+\varepsilon$

we can't find L satisfying both.

Divergence

- Convergence: for any $\epsilon > 0$, there exists N, such that $|a_n - L| \leq \epsilon$ for any $n > N$.



- If no such number L exists, we say that $\{a_n\}$ **diverges**.

Divergence to infinity

- $\lim_{n\to\infty} a_n = \infty$ or $a_n \to \infty$

  If for every $M \in R$, there exists N such that $a_n > M$ for all $n > N$. We say that $\{a_n\}$ diverges to ∞

  Examples:

  $\{n\}$

- $\lim_{n\to\infty} a_n = -\infty$, or simply $a_n \to -\infty$

  If for every $M \in R$, there exists N such that $a_n < M$ for all $n > N$. We say that $\{a_n\}$ diverges to $-\infty$

  Examples:

  $\{-n\}$ $\{-\ln n\}$

Possible outcomes

- Convergence
- Divergence
  - Divergence to ∞ or $-\infty$.
  - Other divergence cases

$$\{1, -2, 3, -4, \cdots\}$$

$$\underbrace{\{1, 0, 2, 0, 3, 0, \cdots\}}$$

Remarks

Adding/removing/changing finitely many terms to a sequence does not change its convergence/divergence.

- $$\{7, 48, 56, -\pi, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$$
- $$\{46, 3, e^{-1}, -1, 1, -1, 1, -1, \cdots\}$$

Calculating limits of sequences

Similar to the theorems on limits of functions in Chapter 2, we have

Theorem (Theorem 1)

Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences of real numbers with $a_n \to A$ and $b_n \to B$.

- *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
- *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
- *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ *(any real number k)*
- *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
- *Quotient Rule:* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}$ *if $B \neq 0$*

Example: $\lim_{n\to\infty} \frac{n^2+1}{n^3+3} = 0$

$$\left|\frac{n^2+1}{n^3+3} - 0\right| < \varepsilon \iff n^2+1 \le \varepsilon(n^3+3)$$

$$\frac{n^2+1}{n^3+3} = \frac{\frac{n^2+1}{n^3}}{\frac{n^3+3}{n^3}} = \frac{\overset{\nearrow 0}{\frac{1}{n}} + \overset{\nearrow 0}{\frac{1}{n^3}}}{1 + \underset{\searrow 0}{\frac{3}{n^3}}}$$

$$\to \frac{0+0}{1+0} = 0$$

Remark

It can happen that $\lim_{n\to\infty}(a_n + b_n)$ exists but both $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist.

$a_n = \ln n$

$b_n = -\ln n$

$a_n + b_n = 0$

$\{a_n\}$ and $\{ca_n\}$ $(c \neq 0)$ are divergent or convergent at the same time.

$\{a_n\}$ converges $\Rightarrow$ $\{c\, a_n\}$ converges

$\Downarrow$ $c \neq 0$

$\{\frac{1}{c}(c\, a_n)\}$ converges

Sandwich theorem for sequences

Theorem (Theorem 2)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all $n \geq N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

- Example: $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

$$0 < \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{n\to\infty} 0 = 0$$

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} (-1)^n \frac{1}{n} = 0$$

$$-|a_n| \leq a_n \leq |a_n|$$

Continuous function theorem for sequences

Theorem (Theorem 3)

Let $\{a_n\}$ be a sequence with $a_n \to L$ and f is a function that is continuous at L and defined at all $\{a_n\}$, then $f(a_n) \to f(L)$.

- Example: $\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^{\lim_{n\to\infty} \frac{1}{n}} = 2^0 = 1$

Proof:

We need $|f(a_n) - f(L)| < \varepsilon$ for all $n > N$

f is continuous at L, $\Rightarrow$ $\forall \varepsilon > 0$

$\exists \delta$ s.t. $|f(x) - f(L)| < \varepsilon$

if $|x - L| < \delta$

So we need $|a_n - L| < \delta$ for all $n > N$.

which depends on δ

Using l'Hôpital's rule

Theorem (Theorem 4)

Support that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence such that $a_n = f(n)$ for all $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \Longrightarrow \lim_{n\to\infty} a_n = L.$$

$\forall \varepsilon \quad \exists M$ s.t. $|f(x) - L| < \varepsilon$

for all $x > M$

$\Rightarrow \quad |f(n) - L| < \varepsilon$

for all $n > M$

Example: $\lim_{n\to\infty} \frac{\ln n}{n}$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Example: $\lim_{n\to\infty}\left(\frac{n+1}{n-1}\right)^n$

$$\lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right)$$

$$= \lim_{x\to\infty} x \ln\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \frac{\ln\left(\frac{x+1}{x-1}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{x-1}{x+1}\left(-\frac{2}{(x-1)^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{2}{(x+1)(x-1)}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2x^2}{(x+1)(x-1)} = 2$$

Common limits (x is fixed)

- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ ✓

- $\lim_{n\to\infty} \sqrt[n]{n} = 1$

  $\sqrt[n]{n} = \lim_{n\to\infty} e^{\left(\frac{1}{n}\ln n\right)} = e^{\lim_{n\to\infty}\left(\frac{1}{n}\ln n\right)} = 1$

- $\lim_{n\to\infty} x^{\frac{1}{n}} = 1 \quad (x > 0)$

  $x^{\frac{1}{n}} = e^{\frac{1}{n}\ln x}$

- $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$

  $|x^n| < \varepsilon \iff n\ln|x| < \ln\varepsilon$

  $\iff n > \ln\varepsilon / \ln|x|$

- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any x)

  $\left(1+\frac{x}{n}\right)^n = e^{\underline{n\ln\left(1+\frac{x}{n}\right)}}$

- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any x)

  $\frac{x}{n}\underbrace{\frac{x}{n-1}\frac{x}{n-2}\frac{x}{n-3}\cdots\frac{x}{1}}_{} < \frac{x}{n}$ for large enough n

Bounded Sequences

Definition

A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

- $\{n\}$ has a lower bound, no upper bound
- $\{\frac{n}{n+1}\}$ bounded

Bounded Monotonic Sequences

Definition

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le \cdots$. The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

- $\{3\}$ monotonic
- $\{(-1)^n\}$ not monotonic
- $\{\frac{n}{n+1}\}$ $\frac{n}{n+1} = 1 - \frac{1}{n+1}$ monotonic

Theorem (Theorem 6)

A bounded and monotonic sequence converges.