1. We build a computer, where the real numbers are represented using 5 digits as explained below:

S	A	B	C	E

where

- S is the sign bit; 0 is positive and 1 is negative
- A, B, C: First three significant digits in decimal expansion with decimal point occuring between A and B
- E is the exponent in base 10 with a bias of 5
- All digits after the third significant digit are chopped off
- $+0$ is represented by setting $S = 0$ and $A = 0$; B, C, E can be anything
- -0 is represented by setting $S = 1$ and $A = 0$; B, C, E can be anything
- $+\infty$ is represented by setting $S = 0$ and $A = B = C = E = 9$
- $-\infty$ is represented by setting $S = 1$ and $A = B = C = E = 9$
- Not A Number is represented by setting S to be other than 0 and 1.

For example, the number $\pi = 3.14159...$ is represented as follows. Chopping off after the third significant digit, we have $\pi = +3.14 \times 10^0$. Hence, the representation of π on our machine is:

0	3	1	4	5

The number $-0.001259...$ is represented as follows. Chopping off after the third significant digit, we have -1.25×10^{-3}. Hence, the representation of on our machine is:

1	1	2	5	2

Now answer the following questions:

(a) How many non-zero Floating Point Numbers (from now on abbreviated as FPN) can be represented by our machine?

(b) How many FPN are in the following intervals?

- $(9, 10)$
- $(10, 11)$
- $(0, 1)$

(c) Identify the smallest positive and largest positive FPN on the machine

Smallest positive floating point number:

$$S = 0 \qquad E = 1 \qquad M = 000$$

$$Trueexponent = E - bias = 1 - 5 = -4$$

$$+1.0 \times 10^{-4} = 0.0001$$

0	1	0	0	1

Largest positive floating point number:

$$S = 0 \qquad E = 8 \qquad M = 999$$

$$Trueexponent = E - bias = 8 - 5 = 3$$

$$+9.99 \times 10^3$$

0	9	9	9	8

(d) Identify the machine precision

(e) What is the smallest positive integer not representable exactly on this machine?

(f) Consider solving the following recurrence on our machine:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

with $a_1 = a_2 = 2.932$. Compute a_n for $n \epsilon 3, 4, 5, 6, 7$ on our machine (work out what the machine would do by hand). Note a_1, a_2 would be chopped to three significant digits to begin with. Next note that at each step in the recurrence $5a_n$ and $4a_{n-1}$ would be chopped down to the first three significant digits before the subtraction is performed.

Solution:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

0	2	9	3	5

Compute a_3: $a_1 = a_2 = 2.93$

$$a_3 = 5a_2 - 4a_1$$
$$= 5(2.93) - 4(2.93) = 14.65 - 11.72$$
$$= 14.6 - 11.7 = 2.90$$

Compute a_4: $a_2 = 2.93, a_3 = 2.90$

$$a_4 = 5a_3 - 4a_2$$
$$= 5(2.90) - 4(2.93) = 14.50 - 11.72$$
$$= 14.5 - 11.7 = 2.80$$

Compute a_5: $a_3 = 2.90, a_4 = 2.80$

$$a_5 = 5a_4 - 4a_3$$
$$= 5(2.80) - 4(2.90) = 14.0 - 11.6$$
$$= 14.0 - 11.6 = 2.40$$

Compute a_6: $a_4 = 2.80, a_5 = 2.40$

$$a_6 = 5a_5 - 4a_4$$
$$= 5(2.40) - 4(2.80) = 12.0 - 11.2$$
$$= 12.0 - 11.2 = 0.80$$

Compute a_7: $a_5 = 2.40, a_6 = 0.80$

$$a_7 = 5a_6 - 4a_5$$
$$= 5(0.8) - 4(2.40) = 4.0 - 9.6 = -5.6$$

2. Consider the following integral:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

(a) Obtain a recurrence for I_n in terms of I_{n-1}. (HINT: Integration by parts)
Solution:

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

$$I_{n-1} = \int_0^1 x^{2(n-1)} \sin(\pi x)\, dx$$

Integration by parts:

$$\int u\, dv = uv - \int v\, du$$

$$u = x^{2n} \qquad dv = \sin(\pi x)\, dx$$

$$du = 2nx^{2n-1}dx \qquad v = -\frac{1}{\pi}\cos(\pi x)$$

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx = uv\Big|_0^1 - \int_0^1 v\, du$$

$$= -\frac{1}{\pi}x^{2n}cos(\pi x)\Big|_0^1 + \frac{2n}{\pi}\int_0^1 x^{2n-1}cos(\pi x)\, dx$$

$$= -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi}\int_0^1 x^{2n-1}cos(\pi x)\, dx$$

$$u = x^{2n-1} \qquad dv = \cos(\pi x)\, dx$$

$$du = (2n-1)x^{2n-2}dx \qquad v = \frac{1}{\pi}\sin(\pi x)$$

$$= -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi}\left[\frac{1}{\pi}x^{2n-1}sin(\pi x)\Big|_0^1 - \frac{2n-1}{\pi}\int_0^1 x^{2n-2}sin(\pi x)\, dx\right]$$

$$= -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi}\left[\frac{1}{\pi}1^{2n-1}sin(\pi) - \frac{2n-1}{\pi}\int_0^1 x^{2n-2}sin(\pi x)\, dx\right]$$

$$I_n = -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi^2}1^{2n-1}\sin(\pi) - \frac{2n(2n-1)}{\pi^2}I_{n-1}$$

(b) Evaluate I_0 by hand

$$I_n = -\frac{1}{\pi}1^{2n}cos(\pi) + \frac{2n}{\pi^2}1^{2n-1}\sin(\pi) - \frac{2n(2n-1)}{\pi^2}I_{n-1}$$

$$I_0 = -\frac{1}{\pi}1^{0}cos(\pi) + \frac{2(0)}{\pi^2}1^{2(0)-1}\sin(\pi) - \frac{2(0)(2(0)-1)}{\pi^2}I_{0-1}$$

$$I_0 = -\frac{1}{\pi}cos(\pi) = \frac{1}{\pi} = 0.31831$$

(c) Use the recurrence to obtain I_n for $n \epsilon 1, 2, ..., 15$ in Octave

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#!/usr/bin/env octave
% Octave script to obtain I_(n) by recurrence relation

N = 15                  % Number of terms in the recurrence
I = zeros(N,1)          % initialize vector of N elements
I(1) = 0.31831          % initial condition I0 = 0.31831

for n = 2:N
    I(n) = ( (-1/pi) * 1**(2*n) * cos(pi) ) \
           + ( (2*n)/(pi**2) * 1**(2*n - 1) * sin(pi) ) \
           - ( ((2*n)*(2*n - 1))/(pi**2) * I(n-1) )
end

I =

   3.1831e-01
  -6.8709e-02
   5.2716e-01
  -2.6728e+00
   2.4691e+01
  -3.2991e+02
   6.0840e+03
  -1.4795e+05
   4.5869e+06
  -1.7661e+08
   8.2670e+09
  -4.6237e+11
   3.0451e+13
  -2.3325e+15
   2.0561e+17
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(d) Use wolframalpha to obtain I_n by directly performing the integeral for $n \epsilon 1, 2, ..., 15$

$$I_n = \int_0^1 x^{2n} \sin(\pi x)\, dx$$

$$n = 0 \qquad I_0 = \int_0^1 x^{2(0)} \sin(\pi x) dx \approx 0.63662$$

$$n = 1 \qquad I_1 = \int_0^1 x^{2(1)} \sin(\pi x) dx \approx 0.18930$$

$$n = 2 \qquad I_2 = \int_0^1 x^{2(2)} \sin(\pi x) dx \approx 0.088144$$

$$n = 3 \qquad I_3 = \int_0^1 x^{2(3)} \sin(\pi x) dx \approx 0.050384$$

$$n = 4 \qquad I_4 = \int_0^1 x^{2(4)} \sin(\pi x) dx \approx 0.032433$$

$$n = 5 \qquad I_5 = \int_0^1 x^{2(5)} \sin(\pi x) dx \approx 0.022561$$

$$n = 6 \qquad I_6 = \int_0^1 x^{2(6)} \sin(\pi x) dx \approx 0.016574$$

$$n = 7 \qquad I_7 = \int_0^1 x^{2(7)} \sin(\pi x) dx \approx 0.012679$$

$$n = 8 \qquad I_8 = \int_0^1 x^{2(8)} \sin(\pi x) dx \approx 0.010006$$

$$n = 9 \qquad I_9 = \int_0^1 x^{2(9)} \sin(\pi x) dx \approx 0.0080938$$

$$n = 10 \qquad I_{10} = \int_0^1 x^{2(10)} \sin(\pi x) dx \approx 0.0066802$$

$$n = 11 \qquad I_{11} = \int_0^1 x^{2(11)} \sin(\pi x) dx \approx 0.0056060$$

$$n = 12 \qquad I_{12} = \int_0^1 x^{2(12)} \sin(\pi x) dx \approx 0.0047708$$

$$n = 13 \qquad I_{13} = \int_0^1 x^{2(13)} \sin(\pi x) dx \approx 0.0041089$$

$$n = 14 \qquad I_{14} = \int_0^1 x^{2(14)} \sin(\pi x) dx \approx 0.0035754$$

$$n = 15 \qquad I_{15} = \int_0^1 x^{2(15)} \sin(\pi x) dx \approx 0.0031393$$

(e) Explain your observation

The values obtained from the octave script recurrence relation are different from the integral calculated.