1. Consider the Vandermonde matrix V, i.e.,

$$V = \begin{bmatrix} 1 & x_0 & x_0^2 & . & . & . & x_0^n \\ 1 & x_1 & x_1^2 & . & . & . & x_1^n \\ 1 & x_2 & x_2^2 & . & . & . & x_2^n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & x_n & x_n^2 & . & . & . & x_n^n \end{bmatrix}$$

   - Show that $det(V)$ is a polynomial in the variables $x_0, x_1, ..., x_n$ with degree $\frac{n(n+1)}{2}$
   - Show that if $x_i = x_j$ for $i \neq j$, then $det(V) = 0$
   - Conclude that $(x_i - x_j)$ is a factor of $det(V)$
   - Conclude that $det(V) = C\left(\prod_{0 \leq j < i \leq n} (x_i - x_j)\right)$, where C is a constant
   - Compare the coefficient of $x_1\ x_2^2\ x_3^3\ ...\ x_n^n$ to obtain the value of C

2. Monic Legendre polynomials on $[-1, 1]$ are defined as follows:

$$q_0(x) = 1$$

$$q_1(x) = x$$

   and $q_n(x)$ is a monic polynomial of degree n such that $\int_{-1}^{1} q_n(x)q_m(x)dx = 0$ for all $m \neq n$.

   - Show that these orthogonal polynomials satisfy

$$q_{n+1}(x) = xq_n(x) - \left(\frac{n^2}{4n^2 - 1}\right) q_{n-1}(x)$$

- Prove that if $p(x)$ is a monic polynomial of degree n minimizing $||p(x)||_2$, then $p(x) = q_n(x)$
- Conclude that the Legendre nodes (i.e., the roots of the Legendre polynomial) minimize $\int_{-1}^{1} \left(\prod_{k=0}^{n} (x - xk)\right)^2 dx$

3. The Chebyshev polynomials of the first kind are defined as

$$T_n(x) = cos(narccos(x))$$

   - Show that the Chebyshev polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1 - x^2}} dx = 0$$

   - Show that the Chebyshev polynomials of the first kind satisfy the recurrence:

$$T_{n+1} = 2xT_n - T_{n-1}$$

     with $T_n(x) = 1$ and $T_1(x) = 1$.
   - Show that $T_n(x)$ is a polynomial of degree n with leading coefficient as 2^{n-1} for $n \geq 1$
   - All zeros of $T_{n+1}(x)$ are in the interval $[-1, 1]$ and given by $x_k = cos\left(\frac{2k+1}{2n+2}\pi\right)$, where $k \epsilon 0, 1, 2, ..., n$
   - Conclude that $T_n(x)$ alternates between $+1$ and -1 exactly $n + 1$ times

- Show that

$$\left|\prod_{k=0}^{n}(x-x_k)\right| \leq \frac{1}{2^n}$$

for all $x\epsilon[-1,1]$

- For any choice of nodes $y_{k}{}_{k=0}^{n}$, consider the polynomial $P_{n+1}(x)=\prod_{k=0}^{n}(x-y_k)$ and look at $F(x)=P_{n+1}(x)-\frac{T_{n+1}(x)}{2^n}$

If $|P_{n+1}(x) \leq \dfrac{1}{2^n}|$, show that $F(x)$ alternates in sign $n+2$ times on the interval $[-1,1]$. Hence, conclude that $F(x)$ has to be identically zero and therfore conclude that Chebyshev nodes of the first kind minimizes $maxx\epsilon[-1,1]\left|\prod\limits_{k=0}^{n}(x-x_k)\right|$