Finite Difference Time Domain

D. Saravanan

December 22, 2021

Department of Physics
Indian Institute of Technology Madras

Introduction

To learn and do three-dimensional electromagnetic simulation using the finite-difference time-domain (FDTD) method.

Type of material:

1. Free space
2. Complex dielectric material
3. Frequency-dependent material

Some choice that have been made:

1. The use of Normalised Units Maxwell's equations have been normalized by substituting

$$\widetilde{E} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E$$

   this is a system similar to Gaussian units.
   The reason for using it here is the simplicity in the formulation. The E and H fields have the same order of magnitude. This has an advantage in formulating the PML.
2. Maxwell's Equations with the Flux Density Time-domain Maxwell's equations from which the FDTD formulation is developed

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

   straight forward formulation

$$\frac{\partial D}{\partial t} = \nabla \times H$$

$$D = \varepsilon_0 \varepsilon_r^* E$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

   formulation using the flux density in this formulation, it is assumed that the materials being simulated are non-magnetic, that is, $H = (1/\mu_0)B$

Pulse propagating in free space in one dimension

Time-dependent Maxwell's curl equations for free space:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

simple one-dimensional case:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0} \frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

- The formulation of Equations assume that the E and H fields are interleaved in both space and time.
- The new value of E_x is calculated from the previous value of E_x and the most resent values of H_y. This is the fundamental paradigm of the FDTD method.

governing equations,

$$\widetilde{E}_x^{n+1/2}(k) = \widetilde{E}_x^{n-1/2}(k) - \frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} \left[H_y^n \left(k + \frac{1}{2} \right) - H_y^n \left(k - \frac{1}{2} \right) \right]$$

$$H_y^{n+1} \left(k + \frac{1}{2} \right) = H_y^n \left(k + \frac{1}{2} \right) - \frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} \left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k) \right]$$

Once the cell size Δx is choosen, then the time step Δt is determined by

$$\Delta t = \frac{\Delta x}{2 \cdot c_0},$$

where c_0 is the speed of light in free space. Therefore, remembering that $\varepsilon_0 \mu_0 = 1/(c_0)^2$,

$$\frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} = \frac{\Delta x}{2 \cdot c_0} \cdot \frac{1}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} = \frac{1}{2}$$

$$\widetilde{E}_x^{n+1/2}(k) = \widetilde{E}_x^{n-1/2}(k) - \frac{1}{2}\left[H_y^n \left(k + \frac{1}{2} \right) - H_y^n \left(k - \frac{1}{2} \right) \right]$$

$$H_y^{n+1} \left(k + \frac{1}{2} \right) = H_y^n \left(k + \frac{1}{2} \right) - \frac{1}{2}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k) \right]$$

Simulation in free space

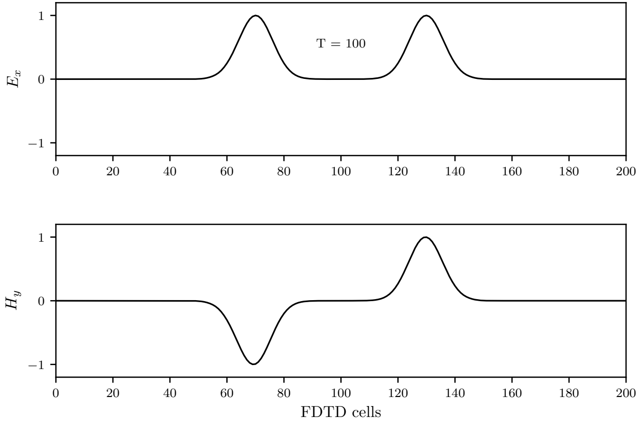

Figure 1: FDTD simulation of a pulse in free space after 100 time steps. The pulse originated in the center and travels outward.

Simulation in free space

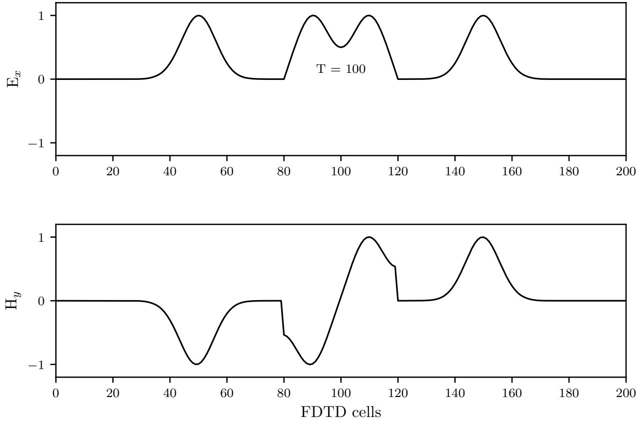

Figure 2: FDTD simulation of a pulse in free space after 100 time steps. It has two sources, one at $kc - 20$ and one at $kc + 20$

Simulation in free space

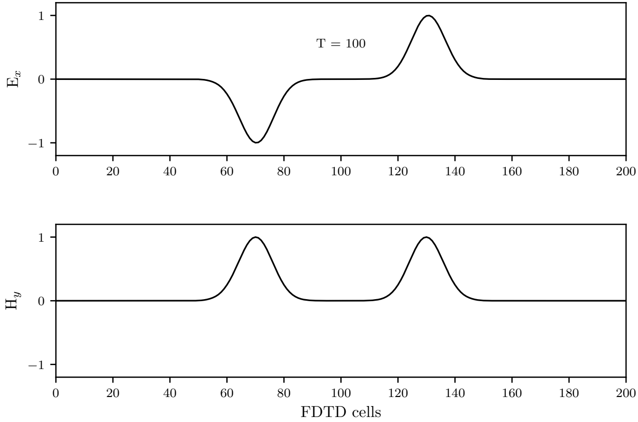

Figure 3: FDTD simulation of a pulse in free space after 100 time steps. Instead of E_x as the source, use H_y at $k = kc$ as the source

Simulation in free space

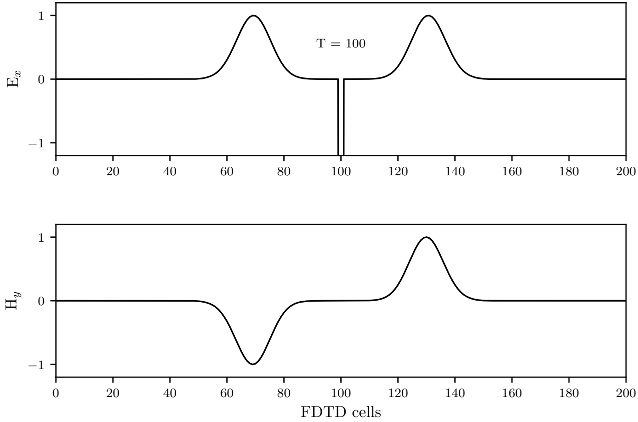

Figure 4: FDTD simulation of a pulse in free space after 100 time steps. Instead of E_x as the source, use a two-point magnetic source at $kc - 1$ and kc such that $hy[kc - 1] = -hy[kc]$

- An EM wave propagating in free space cannot go faster than the speed of light.
- To propagate a distance of one cell requires a minimum time of $\Delta t = \Delta x / c_0$.
- With a two-dimensional simulation, we must allow for the propagation in the diagonal direction, which brings the requirement to $\Delta t = \Delta x / (\sqrt{2} c_0)$.
- With a three-dimensional simulation requires $\Delta t = \Delta x / (\sqrt{3} c_0)$.
- This is summarized by the Courant Condition

$$\Delta t = \frac{\Delta x}{\sqrt{n} \cdot c_0},$$

where n is the dimension of the simulation.

Absorbing boundary condition in one dimension

- Absorbing boundary conditions are necessary to keep outgoing *E* and *H* fields from being reflected back into the problem space.
- If a wave is going toward a boundary in free space, it is traveling at c_0, the speed of light.
- In one time step of the FDTD algorithem, it travels

$$Distance = c_0 \cdot \Delta t = c_0 \cdot \frac{\Delta x}{2 \cdot c_0} = \frac{\Delta x}{2}$$

- It takes two time steps for the field to cross one cell. Thus an acceptable boundary condition might be

$$E_x^n(0) = E_x^{n-2}(1)$$

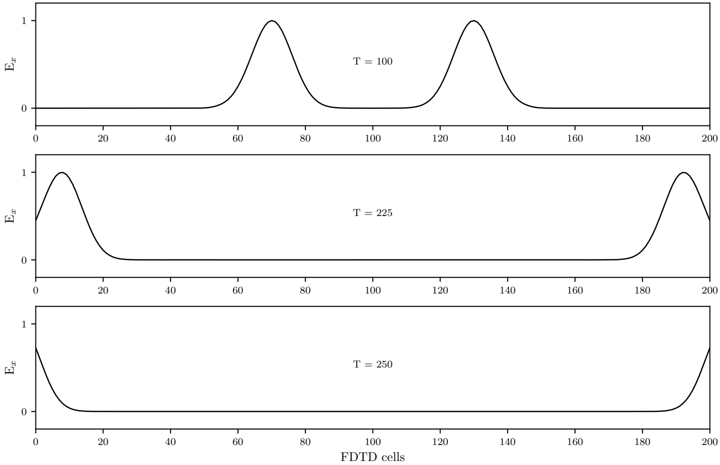

Figure 5: Simulation of an FDTD program with absorbing boundary conditions. Notice that the pulse is absorbed at the edges without reflecting anything back.

Propagation in a Dielectric medium

To simulate a medium with a dielectric constant other than 1, we have to add the relative dielectric constant ε_r to Maxwell's equations:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

simple one-dimensional case:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0 \varepsilon_r} \frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

Propagation in a Dielectric medium

governing equations,

$$\widetilde{E}_x^{n+1/2}(k) = \widetilde{E}_x^{n-1/2}(k) - \frac{1}{2 \cdot \varepsilon_r}\left[H_y^n\left(k+\frac{1}{2}\right) - H_y^n\left(k-\frac{1}{2}\right)\right]$$

$$H_y^{n+1}\left(k+\frac{1}{2}\right) = H_y^n\left(k+\frac{1}{2}\right) - \frac{1}{2}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k)\right]$$

Propagation in a Dielectric medium

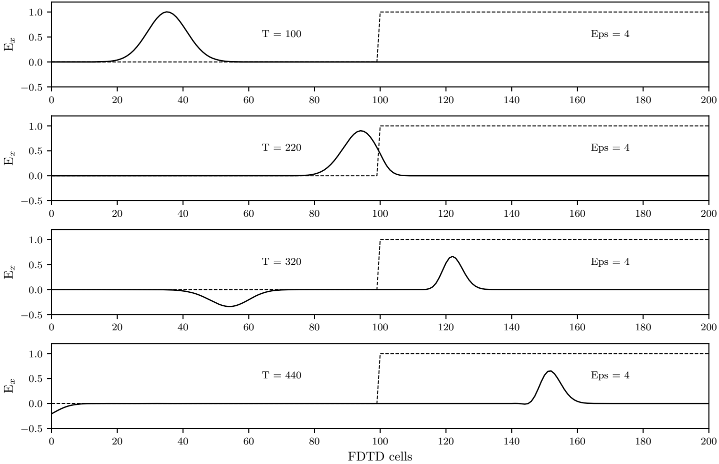

Figure 6: Simulation of a pulse striking dielectric material with a dielectric constant of 4. The source originates at cell number 5.

Simulating with a sinusoidal source

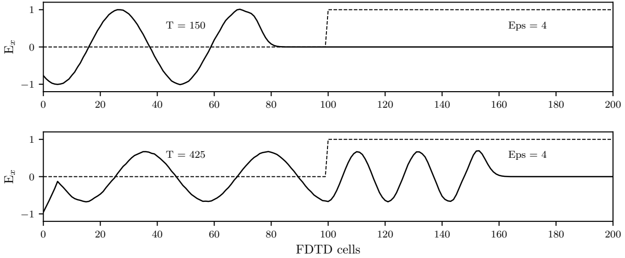

Figure 7: Simulation of a propagating sinusoidal wave of 700 MHz striking a medium with a relative dielectric constant of $\varepsilon_r = 4$.

Simulating with a sinusoidal source

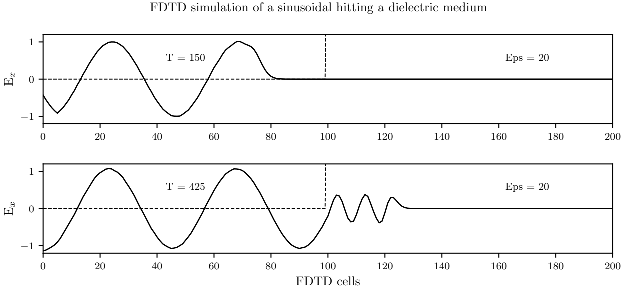

Figure 8: Simulation of a propagating sinusoidal wave of 3 GHz striking a medium with a relative dielectric constant of $\varepsilon_r = 20$.

Propagation in a lossy dielectric medium

general form of time-dependent Maxwell's curl equations,

$$\varepsilon_r \varepsilon_0 \frac{\partial E}{\partial t} = \nabla \times H - J$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

J, the current density, can also be written as

$$J = \sigma E$$

where σ is the conductivity.

substituting,

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0} \nabla \times H - \frac{\sigma}{\varepsilon_r \varepsilon_0} E$$

Propagation in a lossy dielectric medium

simple one-dimensional equation:

$$\frac{\partial E_x(t)}{\partial t} = -\frac{1}{\varepsilon_r \varepsilon_0} \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\varepsilon_r \varepsilon_0} E_x(t)$$

using the change of variables,

$$\frac{\partial \widetilde{E}_x(t)}{\partial t} = -\frac{1}{\varepsilon_r \sqrt{\mu_0 \varepsilon_0}} \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\varepsilon_r \varepsilon_0} \widetilde{E}_x(t)$$

$$\frac{\partial H_y(t)}{\partial t} = -\frac{1}{\sqrt{\mu_0 \varepsilon_0}} \frac{\partial \widetilde{E}_x(t)}{\partial t}$$

governing equations,

$$\widetilde{E}_x^{n+1/2}(k) = \frac{\left(1 - \dfrac{\Delta t \cdot \sigma}{2\varepsilon_r \varepsilon_0}\right)}{\left(1 + \dfrac{\Delta t \cdot \sigma}{2\varepsilon_r \varepsilon_0}\right)} \widetilde{E}_x^{n-1/2}(k) - \frac{\left(\dfrac{1}{2}\right)}{\varepsilon_r \left(1 + \dfrac{\Delta t \cdot \sigma}{w \varepsilon_r \varepsilon_0}\right)} \left[H_y^n\left(k + \frac{1}{2}\right) - H_y^n\left(k - \frac{1}{2}\right) \right]$$