Finite-Difference Time-Domain Method

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Introduction

To learn and do three-dimensional electromagnetic simulation using the finite-difference time-domain (FDTD) method.

Type of material:

1. Free space
2. Complex dielectric material
3. Frequency-dependent material

Formulation i

Some choice that have been made:

1. The use of Normalised Units:
   Maxwell's equations have been normalized by substituting

$$\widetilde{E} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E$$

   this is a system similar to Gaussian units.
   The reason for using it here is the simplicity in the formulation. The E and H fields have the same order of magnitude. This has an advantage in formulating the PML.
2. Maxwell's Equations with the Flux Density:
   Time-domain Maxwell's equations from which the FDTD formulation is developed. straight forward formulation:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

Formulation ii

formulation using the flux density:

$$\frac{\partial D}{\partial t} = \boldsymbol{\nabla} \times H$$

$$D = \varepsilon_0 \varepsilon_r^* E$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \boldsymbol{\nabla} \times E$$

in this formulation, it is assumed that the materials being simulated are non-magnetic, that is, $H = (1/\mu_0)B$

Pulse propagating in free space in one-dimension

Time-dependent Maxwell's curl equations for free space:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

simple one-dimensional case:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0} \frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

Pulse propagating in free space in one-dimension

- The formulation of Equations assume that the E and H fields are interleaved in both space and time.
- The new value of E_x is calculated from the previous value of E_x and the most resent values of H_y. This is the fundamental paradigm of the FDTD method.

governing equations,

$$\widetilde{E}_x^{n+1/2}(k) = \widetilde{E}_x^{n-1/2}(k) - \frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x}\left[H_y^n\left(k+\frac{1}{2}\right) - H_y^n\left(k-\frac{1}{2}\right)\right]$$

$$H_y^{n+1}\left(k+\frac{1}{2}\right) = H_y^n\left(k+\frac{1}{2}\right) - \frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k)\right]$$

Pulse propagating in free space in one-dimension

Once the cell size Δx is choosen, then the time step Δt is determined by

$$\Delta t = \frac{\Delta x}{2 \cdot c_0},$$

where c_0 is the speed of light in free space. Therefore, remembering that $\varepsilon_0 \mu_0 = 1/(c_0)^2$,

$$\frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} = \frac{\Delta x}{2 \cdot c_0} \cdot \frac{1}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} = \frac{1}{2}$$

$$\widetilde{E}_x^{n+1/2}(k) = \widetilde{E}_x^{n-1/2}(k) - \frac{1}{2}\left[H_y^n\left(k + \frac{1}{2}\right) - H_y^n\left(k - \frac{1}{2}\right)\right]$$

$$H_y^{n+1}\left(k + \frac{1}{2}\right) = H_y^n\left(k + \frac{1}{2}\right) - \frac{1}{2}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k)\right]$$

Simulation in free space

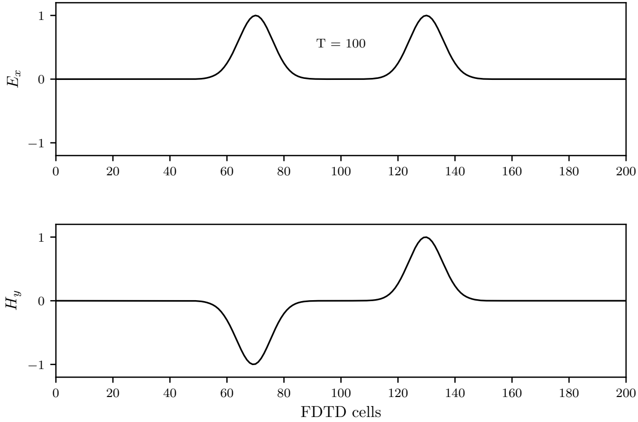

Figure 1: FDTD simulation of a pulse in free space after 100 time steps. The pulse originated in the center and travels outward.

Simulation in free space

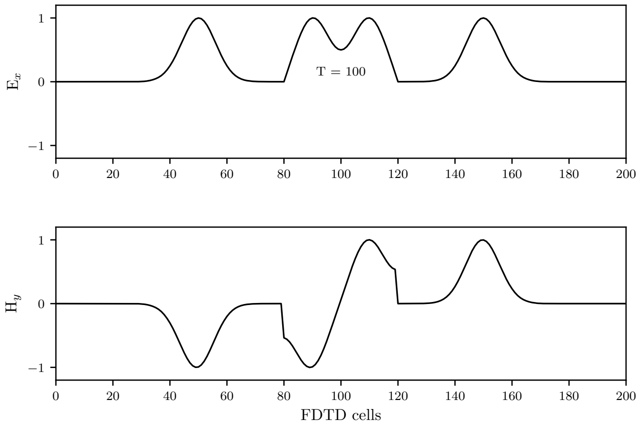

Figure 2: FDTD simulation of a pulse in free space after 100 time steps. It has two sources, one at $kc - 20$ and one at $kc + 20$

Simulation in free space

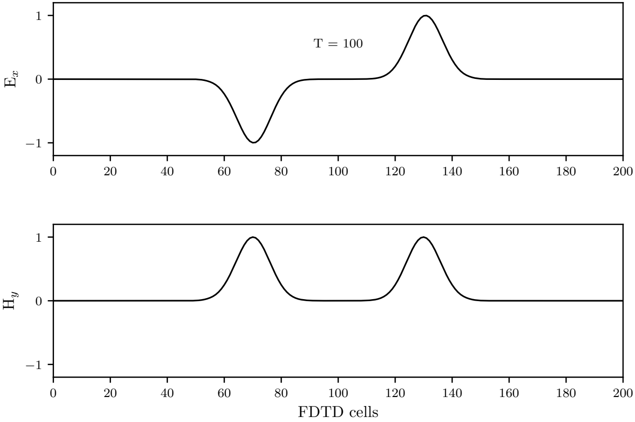

Figure 3: FDTD simulation of a pulse in free space after 100 time steps. Instead of E_x as the source, use H_y at $k = kc$ as the source

Simulation in free space

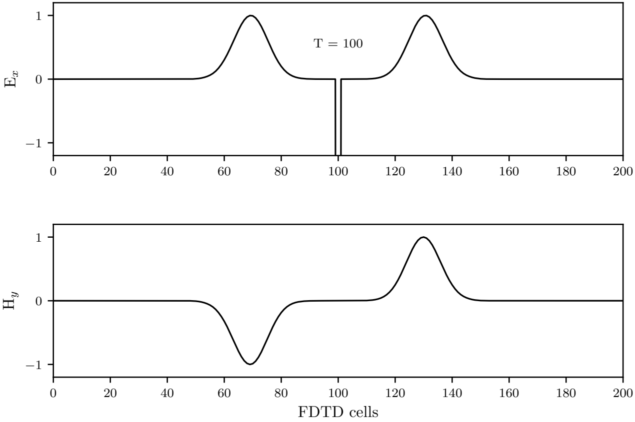

Figure 4: FDTD simulation of a pulse in free space after 100 time steps. Instead of E_x as the source, use a two-point magnetic source at $kc - 1$ and kc such that $hy[kc - 1] = -hy[kc]$

Stability and the FDTD method

- An EM wave propagating in free space cannot go faster than the speed of light.
- To propagate a distance of one cell requires a minimum time of $\Delta t = \Delta x / c_0$.
- With a two-dimensional simulation, we must allow for the propagation in the diagonal direction, which brings the requirement to $\Delta t = \Delta x / (\sqrt{2} c_0)$.
- With a three-dimensional simulation requires $\Delta t = \Delta x / (\sqrt{3} c_0)$.
- This is summarized by the Courant Condition

$$\Delta t = \frac{\Delta x}{\sqrt{n} \cdot c_0},$$

where n is the dimension of the simulation.

Absorbing boundary condition in one dimension

- Absorbing boundary conditions are necessary to keep outgoing E and H fields from being reflected back into the problem space.
- If a wave is going toward a boundary in free space, it is traveling at c_0, the speed of light.
- In one time step of the FDTD algorithem, it travels

$$Distance = c_0 \cdot \Delta t = c_0 \cdot \frac{\Delta x}{2 \cdot c_0} = \frac{\Delta x}{2}$$

- It takes two time steps for the field to cross one cell. Thus an acceptable boundary condition might be

$$E_x^n(0) = E_x^{n-2}(1)$$

Absorbing boundary condition in one dimension

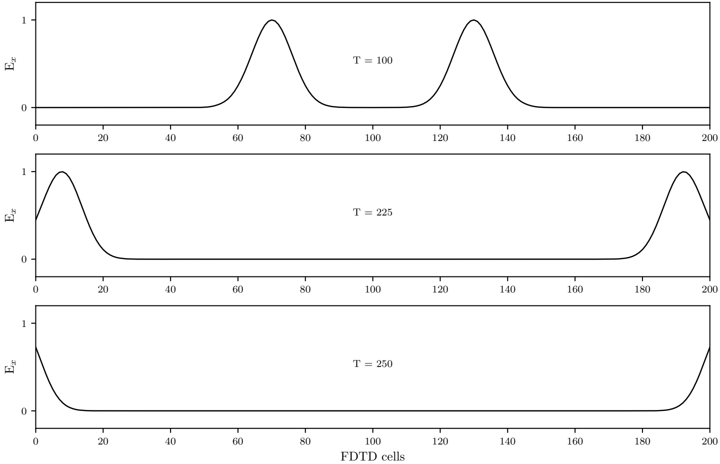

Figure 5: Simulation of an FDTD program with absorbing boundary conditions. Notice that the pulse is absorbed at the edges without reflecting anything back.

Propagation in a Dielectric medium

To simulate a medium with a dielectric constant other than 1, we have to add the relative dielectric constant ε_r to Maxwell's equations:

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

simple one-dimensional case:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0 \varepsilon_r} \frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

Propagation in a Dielectric medium

governing equations,

$$\widetilde{E}_x^{n+1/2}(k) = \widetilde{E}_x^{n-1/2}(k) - \frac{1}{2 \cdot \varepsilon_r}\left[H_y^n\left(k+\frac{1}{2}\right) - H_y^n\left(k-\frac{1}{2}\right)\right]$$

$$H_y^{n+1}\left(k+\frac{1}{2}\right) = H_y^n\left(k+\frac{1}{2}\right) - \frac{1}{2}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k)\right]$$

Propagation in a Dielectric medium

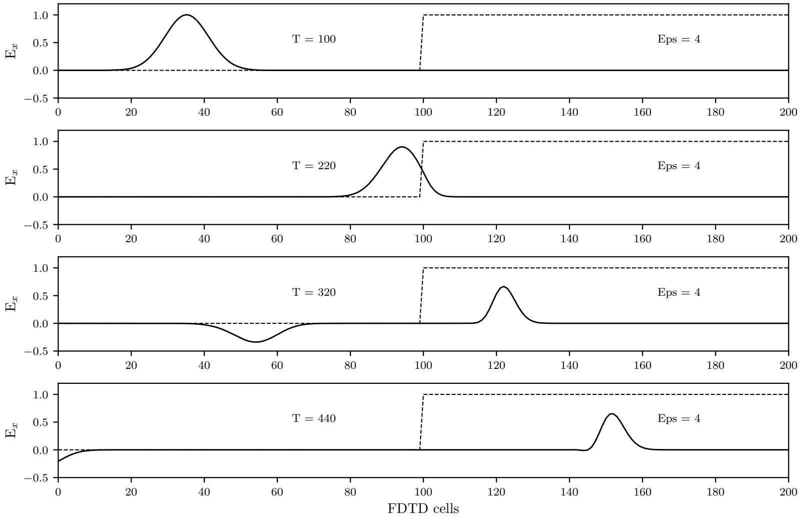

Figure 6: Simulation of a pulse striking dielectric material with a dielectric constant of 4. The source originates at cell number 5.

Simulating with a sinusoidal source

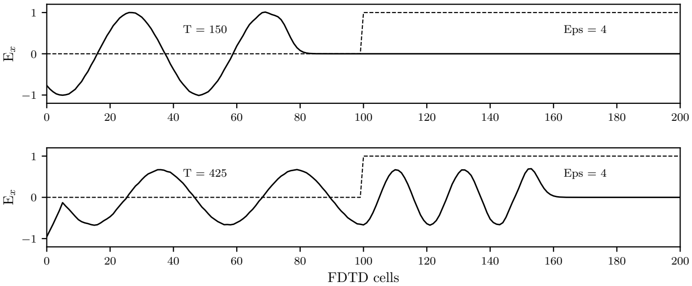

Figure 7: Simulation of a propagating sinusoidal wave of 700 MHz striking a medium with a relative dielectric constant of $\varepsilon_r = 4$.

Simulating with a sinusoidal source

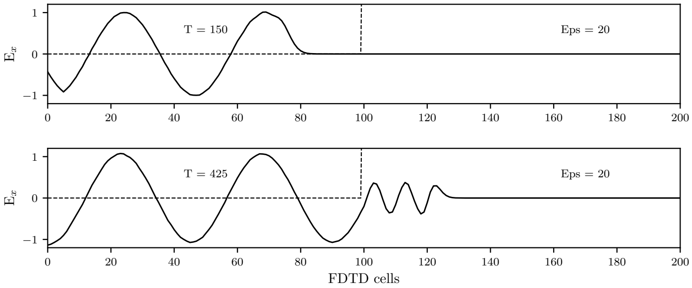

Figure 8: Simulation of a propagating sinusoidal wave of 3 GHz striking a medium with a relative dielectric constant of $\varepsilon_r = 20$.

Propagation in a lossy dielectric medium

general form of time-dependent Maxwell's curl equations,

$$\varepsilon_r \varepsilon_0 \frac{\partial E}{\partial t} = \nabla \times H - J$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

J, the current density, can also be written as

$$J = \sigma E$$

where σ is the conductivity.

substituting,

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0} \nabla \times H - \frac{\sigma}{\varepsilon_r \varepsilon_0} E$$

simple one-dimensional equation:

$$\frac{\partial E_x(t)}{\partial t} = -\frac{1}{\varepsilon_r \varepsilon_0} \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\varepsilon_r \varepsilon_0} E_x(t)$$

Propagation in a lossy dielectric medium

using the change of variables,

$$\frac{\partial \widetilde{E}_x(t)}{\partial t} = -\frac{1}{\varepsilon_r\sqrt{\mu_0\varepsilon_0}}\frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\varepsilon_r\varepsilon_0}\widetilde{E}_x(t)$$

$$\frac{\partial H_y(t)}{\partial t} = -\frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{\partial \widetilde{E}_x(t)}{\partial t}$$

governing equations,

$$\widetilde{E}_x^{n+1/2}(k) = \frac{\left(1 - \dfrac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right)}{\left(1 + \dfrac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right)}\widetilde{E}_x^{n-1/2}(k) - \frac{\left(\dfrac{1}{2}\right)}{\varepsilon_r\left(1 + \dfrac{\Delta t\cdot\sigma}{2\varepsilon_r\varepsilon_0}\right)}\left[H_y^n\left(k+\frac{1}{2}\right) - H_y^n\left(k-\frac{1}{2}\right)\right]$$

$$H_y^{n+1}\left(k+\frac{1}{2}\right) = H_y^n\left(k+\frac{1}{2}\right) - \frac{1}{2}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k)\right]$$

Propagation in a lossy dielectric medium

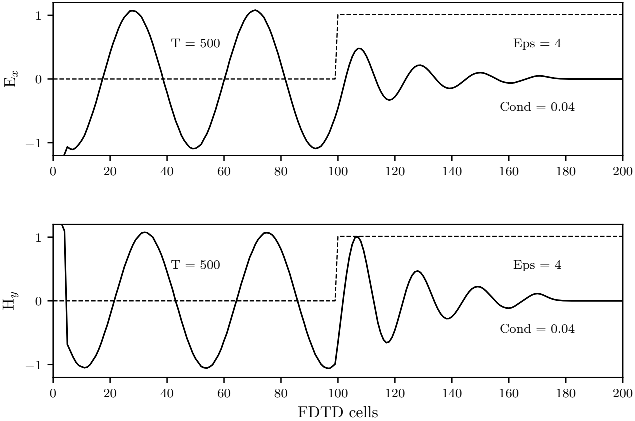

Figure 9: Simulation of a propagating sinusoidal wave striking a lossy dielectric material with a dielectric constant of 4 and a conductivity of 0.04 (S/m). The source is 700 MHz and originates at cell number 5.

Reformulation using the flux density

general form of Maxwell's equations

$$\frac{\partial D}{\partial t} = \nabla \times H$$

$$D(\omega) = \varepsilon_0 \cdot \varepsilon_r^*(\omega) \cdot E(\omega)$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

where D is the electric flux density.

normalizing these equations, using

$$\widetilde{E} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot E$$

$$\widetilde{D} = \sqrt{\frac{1}{\varepsilon_0 \mu_0}} \cdot D$$

Reformulation using the flux density

which leads to

$$\frac{\partial \widetilde{D}}{\partial t} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times H$$

$$\widetilde{D}(\omega) = \varepsilon_r^*(\omega) \cdot \widetilde{E}(\omega)$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\sqrt{\varepsilon_0 \mu_0}} \nabla \times \widetilde{E}$$

We have to get $\widetilde{D}(\omega) = \varepsilon_r^*(\omega) \cdot \widetilde{E}(\omega)$ into a time-domain difference equation for implementation into FDTD. We will assume that we are dealing with a lossy dielectric medium of the form

$$\varepsilon_r^*(\omega) = \varepsilon_r + \frac{\sigma}{j\omega \cdot \varepsilon_0}$$

substituting the above equation,

$$\widetilde{D}(\omega) = \varepsilon_r \widetilde{E}(\omega) + \frac{\sigma}{j\omega \cdot \varepsilon_0} \widetilde{E}(\omega)$$

Reformulation using the flux density

Fourier theory tells us that $1/(j\omega)$ in the frequency domain is integration in the time domain, so the above equation becomes

$$\widetilde{D}(t) = \varepsilon_r \widetilde{E}(t) + \frac{\sigma}{\varepsilon_0} \int_0^t \widetilde{E}(t')dt'$$

the integral will be approximated as a summation over the time steps Δt:

$$\widetilde{D}^n = \varepsilon_r \widetilde{E}^n + \frac{\sigma \cdot \Delta t}{\varepsilon_0} \sum_{i=0}^{n} \widetilde{E}^i$$

separating the E^n term from the rest of the summation:

$$\widetilde{D}^n = \varepsilon_r \widetilde{E}^n + \frac{\sigma \cdot \Delta t}{\varepsilon_0} \widetilde{E}^n + \frac{\sigma \cdot \Delta t}{\varepsilon_0} \sum_{i=0}^{n-1} \widetilde{E}^i$$

Reformulation using the flux density

We can calculate E^n, the current value of E, from the current value of D and previous values of E.

$$\widetilde{E}^n = \frac{\widetilde{D}^n - \dfrac{\sigma \cdot \Delta t}{\varepsilon_0} \displaystyle\sum_{i=0}^{n-1} \widetilde{E}^i}{\varepsilon_r + \dfrac{\sigma \cdot \Delta t}{\varepsilon_0}}$$

governing equations,

$$\widetilde{D}^{n+1/2}(k) = \widetilde{D}^{n-1/2}(k) - \frac{1}{2}\left[H_y^n\left(k + \frac{1}{2}\right) - H_y^n\left(k - \frac{1}{2}\right)\right]$$

$$H_y^{n+1}\left(k + \frac{1}{2}\right) = H_y^n\left(k + \frac{1}{2}\right) - \frac{1}{2}\left[\widetilde{E}_x^{n+1/2}(k+1) - \widetilde{E}_x^{n+1/2}(k)\right]$$