

# Finite-Difference Time-Domain Method

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To learn and do three-dimensional electromagnetic simulation using the finite-difference time-domain (FDTD) method.

Type of material:

1. Free space
2. Complex dielectric material
3. Frequency-dependent material

Some choice that have been made:

1. The use of Normalised Units:

Maxwell's equations have been normalized by substituting

$$\tilde{E} = \sqrt{\frac{\epsilon_0}{\mu_0}} E$$

this is a system similar to Gaussian units.

The reason for using it here is the simplicity in the formulation. The  $E$  and  $H$  fields have the same order of magnitude. This has an advantage in formulating the PML.

2. Maxwell's Equations with the Flux Density:

Time-domain Maxwell's equations from which the FDTD formulation is developed. straight forward formulation:

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

formulation using the flux density:

$$\frac{\partial D}{\partial t} = \nabla \times H$$

$$D = \varepsilon_0 \varepsilon_r^* E$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

in this formulation, it is assumed that the materials being simulated are non-magnetic, that is,  $H = (1/\mu_0)B$

## Pulse propagating in free space in one-dimension

Time-dependent Maxwell's curl equations for free space:

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

simple one-dimensional case:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

## Pulse propagating in free space in one-dimension

- The formulation of Equations assume that the  $E$  and  $H$  fields are interleaved in both space and time.
- The new value of  $E_x$  is calculated from the previous value of  $E_x$  and the most recent values of  $H_y$ . This is the fundamental paradigm of the FDTD method.

governing equations,

$$\tilde{E}_x^{n+1/2}(k) = \tilde{E}_x^{n-1/2}(k) - \frac{\Delta t}{\sqrt{\epsilon_0 \mu_0} \cdot \Delta x} \left[ H_y^n \left( k + \frac{1}{2} \right) - H_y^n \left( k - \frac{1}{2} \right) \right]$$

$$H_y^{n+1} \left( k + \frac{1}{2} \right) = H_y^n \left( k + \frac{1}{2} \right) - \frac{\Delta t}{\sqrt{\epsilon_0 \mu_0} \cdot \Delta x} \left[ \tilde{E}_x^{n+1/2}(k+1) - \tilde{E}_x^{n+1/2}(k) \right]$$

## Pulse propagating in free space in one-dimension

Once the cell size  $\Delta x$  is chosen, then the time step  $\Delta t$  is determined by

$$\Delta t = \frac{\Delta x}{2 \cdot c_0},$$

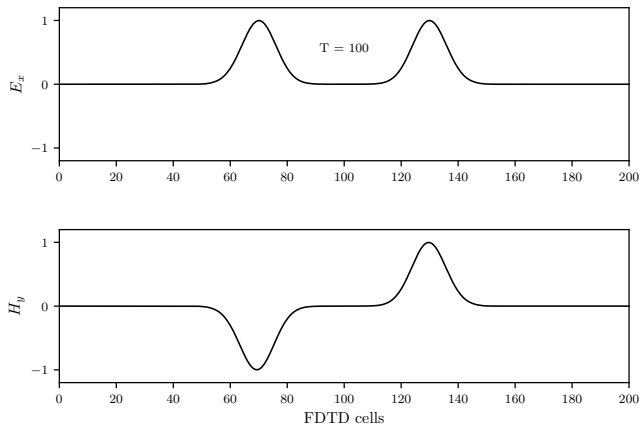
where  $c_0$  is the speed of light in free space. Therefore, remembering that  $\varepsilon_0 \mu_0 = 1/(c_0)^2$ ,

$$\frac{\Delta t}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} = \frac{\Delta x}{2 \cdot c_0} \cdot \frac{1}{\sqrt{\varepsilon_0 \mu_0} \cdot \Delta x} = \frac{1}{2}$$

$$\tilde{E}_x^{n+1/2}(k) = \tilde{E}_x^{n-1/2}(k) - \frac{1}{2} \left[ H_y^n \left( k + \frac{1}{2} \right) - H_y^n \left( k - \frac{1}{2} \right) \right]$$

$$H_y^{n+1} \left( k + \frac{1}{2} \right) = H_y^n \left( k + \frac{1}{2} \right) - \frac{1}{2} \left[ \tilde{E}_x^{n+1/2}(k+1) - \tilde{E}_x^{n+1/2}(k) \right]$$

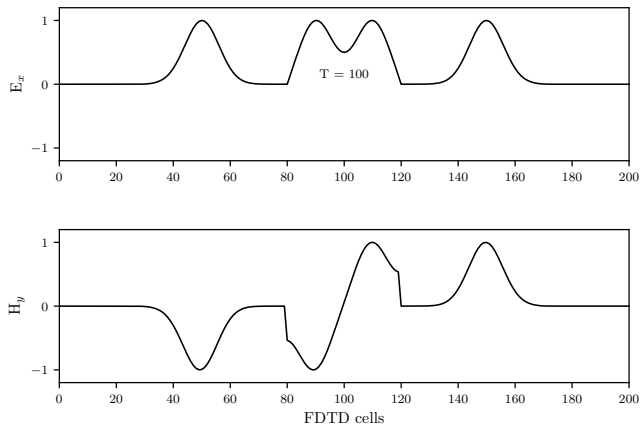
## Simulation in free space



**Figure 1:** FDTD simulation of a pulse in free space after 100 time steps. The pulse originated in the center and travels outward.

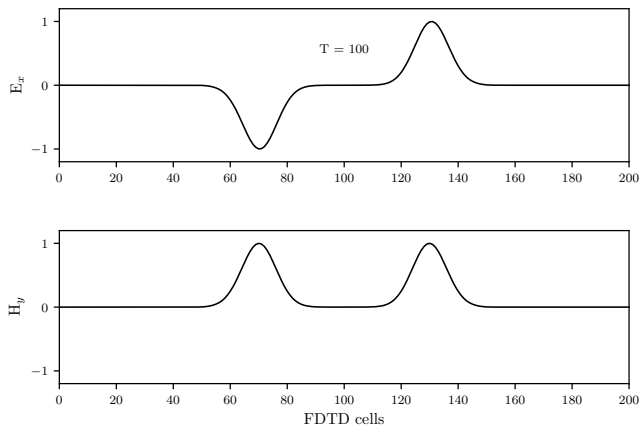


## Simulation in free space



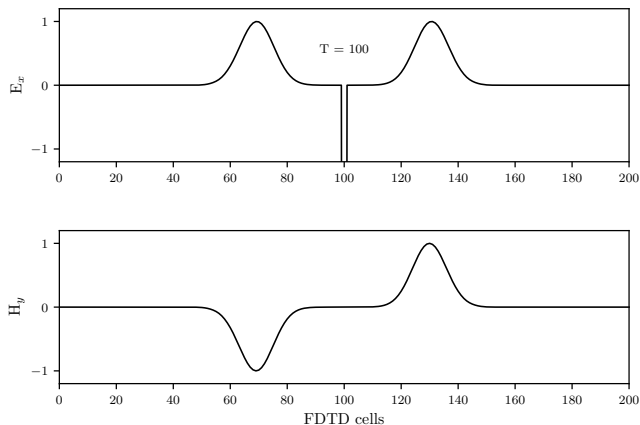
**Figure 2:** FDTD simulation of a pulse in free space after 100 time steps. It has two sources, one at  $kc - 20$  and one at  $kc + 20$

## Simulation in free space



**Figure 3:** FDTD simulation of a pulse in free space after 100 time steps. Instead of  $E_x$  as the source, use  $H_y$  at  $k = kc$  as the source

## Simulation in free space



**Figure 4:** FDTD simulation of a pulse in free space after 100 time steps. Instead of  $E_x$  as the source, use a two-point magnetic source at  $kc - 1$  and  $kc$  such that  $hy[kc - 1] = -hy[kc]$

- An EM wave propagating in free space cannot go faster than the speed of light.
- To propagate a distance of one cell requires a minimum time of  $\Delta t = \Delta x / c_0$ .
- With a two-dimensional simulation, we must allow for the propagation in the diagonal direction, which brings the requirement to  $\Delta t = \Delta x / (\sqrt{2}c_0)$ .
- With a three-dimensional simulation requires  $\Delta t = \Delta x / (\sqrt{3}c_0)$ .
- This is summarized by the Courant Condition

$$\Delta t = \frac{\Delta x}{\sqrt{n} \cdot c_0},$$

where  $n$  is the dimension of the simulation.

## Absorbing boundary condition in one dimension

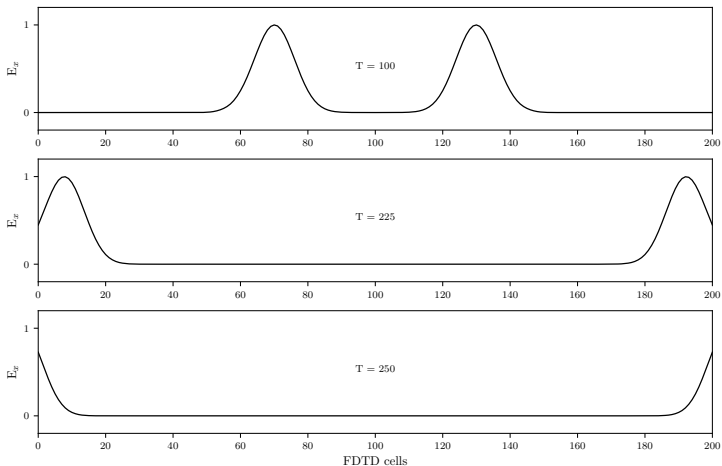
- Absorbing boundary conditions are necessary to keep outgoing  $E$  and  $H$  fields from being reflected back into the problem space.
- If a wave is going toward a boundary in free space, it is traveling at  $c_0$ , the speed of light.
- In one time step of the FDTD algorithm, it travels

$$Distance = c_0 \cdot \Delta t = c_0 \cdot \frac{\Delta x}{2 \cdot c_0} = \frac{\Delta x}{2}$$

- It takes two time steps for the field to cross one cell. Thus an acceptable boundary condition might be

$$E_x^n(0) = E_x^{n-2}(1)$$

## Absorbing boundary condition in one dimension



**Figure 5:** Simulation of an FDTD program with absorbing boundary conditions. Notice that the pulse is absorbed at the edges without reflecting anything back.

## Propagation in a Dielectric medium

To simulate a medium with a dielectric constant other than 1, we have to add the relative dielectric constant  $\epsilon_r$  to Maxwell's equations:

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 \epsilon_r} \nabla \times H$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu_0} \nabla \times E$$

simple one-dimensional case:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0 \epsilon_r} \frac{\partial H_y}{\partial z}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z}$$

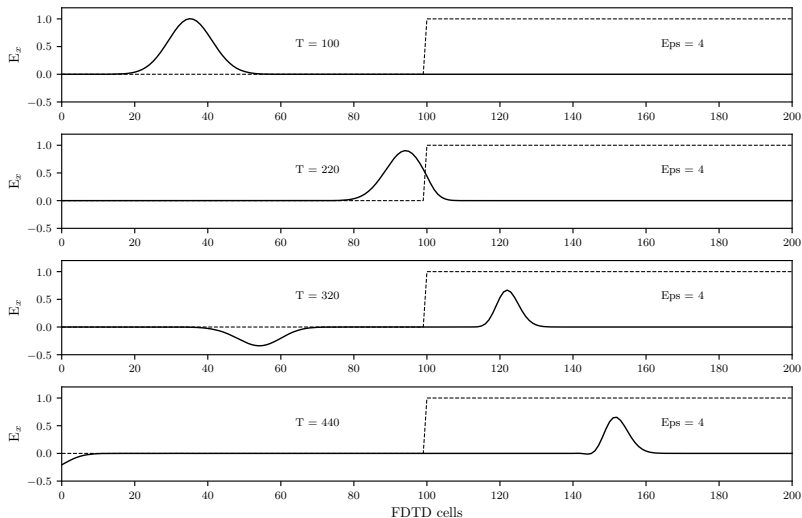
governing equations,

$$\tilde{E}_x^{n+1/2}(k) = \tilde{E}_x^{n-1/2}(k) - \frac{1}{2 \cdot \varepsilon_r} \left[ H_y^n \left( k + \frac{1}{2} \right) - H_y^n \left( k - \frac{1}{2} \right) \right]$$

$$H_y^{n+1} \left( k + \frac{1}{2} \right) = H_y^n \left( k + \frac{1}{2} \right) - \frac{1}{2} \left[ \tilde{E}_x^{n+1/2}(k+1) - \tilde{E}_x^{n+1/2}(k) \right]$$

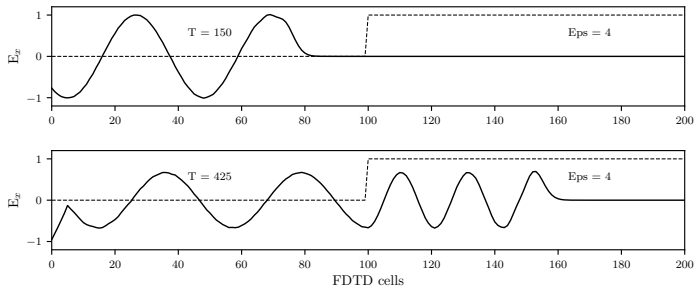


# Propagation in a Dielectric medium



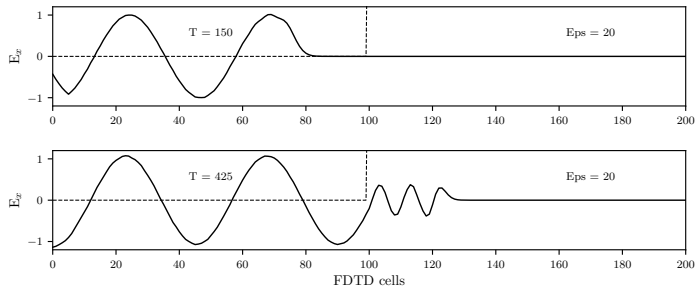
**Figure 6:** Simulation of a pulse striking dielectric material with a dielectric constant of 4. The source originates at cell number 5.

## Simulating with a sinusoidal source



**Figure 7:** Simulation of a propagating sinusoidal wave of 700 MHz striking a medium with a relative dielectric constant of  $\epsilon_r = 4$ .

## Simulating with a sinusoidal source



**Figure 8:** Simulation of a propagating sinusoidal wave of 3 GHz striking a medium with a relative dielectric constant of  $\epsilon_r = 20$ .

## Propagation in a lossy dielectric medium

general form of time-dependent Maxwell's curl equations,

$$\epsilon_r \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}$$

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_0} \nabla \times \mathbf{E}$$

$\mathbf{J}$ , the current density, can also be written as

$$\mathbf{J} = \sigma \mathbf{E}$$

where  $\sigma$  is the conductivity.

substituting,

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_r \epsilon_0} \nabla \times \mathbf{H} - \frac{\sigma}{\epsilon_r \epsilon_0} \mathbf{E}$$

simple one-dimensional equation:

$$\frac{\partial E_x(t)}{\partial t} = -\frac{1}{\epsilon_r \epsilon_0} \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\epsilon_r \epsilon_0} E_x(t)$$

## Propagation in a lossy dielectric medium

using the change of variables,

$$\frac{\partial \tilde{E}_x(t)}{\partial t} = -\frac{1}{\epsilon_r \sqrt{\mu_0 \epsilon_0}} \frac{\partial H_y(t)}{\partial z} - \frac{\sigma}{\epsilon_r \epsilon_0} \tilde{E}_x(t)$$

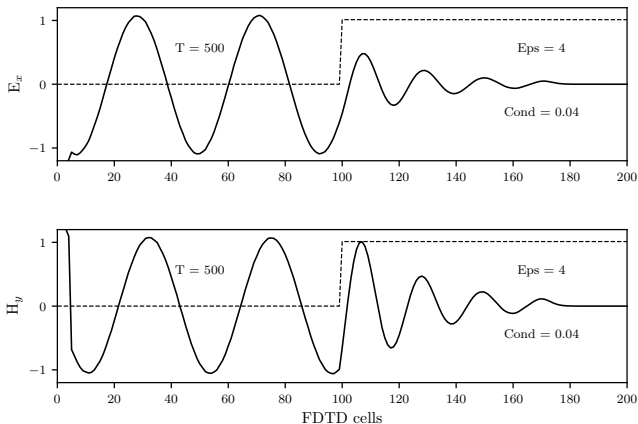
$$\frac{\partial H_y(t)}{\partial t} = -\frac{1}{\sqrt{\mu_0 \epsilon_0}} \frac{\partial \tilde{E}_x(t)}{\partial t}$$

governing equations,

$$\tilde{E}_x^{n+1/2}(k) = \frac{\left(1 - \frac{\Delta t \cdot \sigma}{2\epsilon_r \epsilon_0}\right)}{\left(1 + \frac{\Delta t \cdot \sigma}{2\epsilon_r \epsilon_0}\right)} \tilde{E}_x^{n-1/2}(k) - \frac{\left(\frac{1}{2}\right)}{\epsilon_r \left(1 + \frac{\Delta t \cdot \sigma}{2\epsilon_r \epsilon_0}\right)} \left[ H_y^n \left(k + \frac{1}{2}\right) - H_y^n \left(k - \frac{1}{2}\right) \right]$$

$$H_y^{n+1} \left(k + \frac{1}{2}\right) = H_y^n \left(k + \frac{1}{2}\right) - \frac{1}{2} \left[ \tilde{E}_x^{n+1/2}(k+1) - \tilde{E}_x^{n+1/2}(k) \right]$$

## Propagation in a lossy dielectric medium



**Figure 9:** Simulation of a propagating sinusoidal wave striking a lossy dielectric material with a dielectric constant of 4 and a conductivity of 0.04 (S/m). The source is 700 MHz and originates at cell number 5.