Rollnumber: PH15M015

1. Prove the Equioscillation theorem.

**Equioscillation theorem:** Let  $f \in C[-1,1]$  and p(x) be a polynomial whose degree doesn't exceed n. p minimizes  $||f-p||_{\infty}$  iff f-p equioscillates at n+2 points.

- 2. Consider the function f(x) = |x| on the interval [-1, 1].
  - Prove that of all polynomials whose degree doesn't exceed 3,  $p(x) = x^2 + \frac{1}{8}$  is the best approximation in the  $\|\cdot\|_{\infty}$  norm.
  - Interpolate the function using 4 Legendre nodes and Chebyshev nodes. Call the polynomials obtained as  $p_L(x)$  and  $p_C(x)$ .
  - Fill in the table below. You should be able to complete the table by hand.

Approximation	$\  \cdot \ _2$	$\ \cdot\ _{\infty}$
f(x) - p(x)		
$f(x) - p_L(x)$		
$f(x) - p_C(x)$		

Comment on the errors you obtain using the different norm. Which one is optimal under the  $\|\cdot\|_2$  and  $\|\cdot\|_{\infty}$ ? For each norm, order the different approximations in increasing order of accuracy.

3. Error estimate of Gaussian quadrature: Prove that

$$\int_{a}^{b} w(x)f(x)dx - \sum_{i=1}^{n} w_{i}f(x_{i}) = \frac{f^{(2n)}(\xi)}{(2n)!} \|p_{n}(x)\|_{w}^{2}$$

for some  $\xi \in (a, b)$  and  $p_n(x)$  is the monic orthogonal polynomial corresponding to the weight function w(x).

4. Let f(x) be periodic function on [0,1] of the form

$$f(x) = a_0 + \sum_{k=1}^{n} (a_k \cos(2k\pi x) + b_k \sin(2k\pi x))$$

Take the case of n = 20 and  $a_k, bk$  are uniformly distributed on [-1, 1]. Approximate  $\int_{-1}^{1} f(x)dx$  using trapezoidal rule with k points where  $k \in \{1, 2, ..., 80\}$ . Compare the error with the exact integral and comment on the result you obtain. Prove that the trapezoidal rule give you the exact integral for k > n.

5. Evaluate  $\int_{-1}^{1} e^{-x^2} dx$  using Gaussian quadrature with n nodes, where  $n \in \{3, 4, 5, ..., 51\}$ . Plot the absolute error as a function of N on a log-log plot.