set Theoritic Approach to probability

Random vauaible Approach

1) Trial and Event (A)

2) Sample space

collection of all events under the set theory.

3) $\text{prob}(A) = \frac{n(A)}{n(S)}$

Axioms ($p_i$)

$0 \le p_i \le 1 \ \forall i$

$\Sigma p_i = 1$

cumulative distribution functions.

$F(x) = P[X < n]$

$p_i$ = probability of $i^{th}$ function

4) Addition theorem of probability

$$P[A \cup B] = p(A) + p(B) - p(A \cap B)$$

5) Boole's inequality X

*6) Bayeu's theorem.

Bayeu's theorem.

Statement : If $A_1$ $A_2$ $A_3$ .... $A_n$ can be n events and B is any event occuring along with some of $A_i$ s then.

$$P(A_i/B) = \frac{P(A_i)\,P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\,P(B/A_i)}$$

capability - ability to fit varaiety of functions

## conditional probability

let A and B be any two events

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\, P(B/A)}{P(B)} \quad \text{when } p(B) \neq 0$$

$$\underbrace{p(A/B)}_{\text{posterior}} \propto \underbrace{p(A)}_{\text{prior}} \; \underbrace{p(B/A)}_{\text{Maximum livlihood (ML)}}$$

## Random varaible Approach.

Two types
- Discrete random varaible.
- continous random varaible.

Random variable is a varaible which can correspond to a real number in its sample space.

$s$ = set of outcomes.

$x(s) \in \mathbb{R}$

x - varaible.
s - sample space
IR - set of all reals

A discrete random variable is a random varaible where can outcome space is having finite or countable number of points.

continous random variable is a random variable. whose outcome belongs to outcome space. which has infinite or uncountable many points.

Discrete distribution
- Binomial distribution.
- poission distribution

continous distribution $\Bigg\langle$ Normal distribution (earlier) / Gaussian distribution.

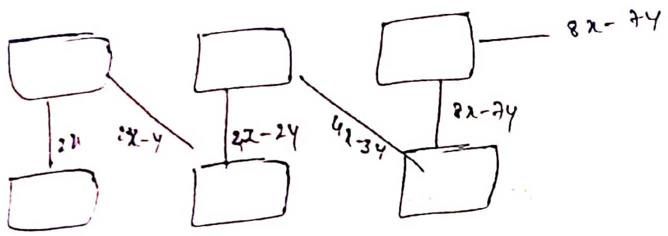
(2m) What are 3 sources of uncertainity?

Inherent stochasticity

Incomplete observatability

Incomplete modelling

stochasticity means probabilicity




$8x-74=0$

$8x=74$

$x/4 = 7/8$

Probability distribution of random variables.

when we describe the values in the range of random variaible in terms of probability of their occurence.

In other words the probability distribution of a random variable can be determined by calculating the probability of occurence of every value in the range of random variaible.

<u>Discrete case.</u>

for discrete variaibles, the term probability mass function is used to describe their distributions.

In general if a random variable x has a countable range given by :

$$R_X = \{x_1, x_2 \cdots x_n)$$

probability mass function as:

$$P_X(x) = P(X = x)$$

This also leads us to general description of the distribution in tabular format;

| $X$ | $x_1$ | $x_2$ | .... | $x_n$ |
|---|---|---|---|---|
| $P_X(x)$ | $P_1$ | $P_2$ | ..... | $P_n$. |

<u>properties of PMF</u>

1) PMF can never be more than 1 or -ve.

$$0 \leq P_X(x) \leq 1$$

2) PMF must sum to one over the entire range set of a random variable.

$$\sum_{-\infty}^{\infty} P_X(x) = \sum_{x \in R_X} P_X(x) = 1$$

3) for A, a subset of $R_X$.

$$P(X \in A) = \sum_{x \in A} P_X(x)$$

<u>Continous case.</u>

for continous variables, the term probability density function (PDF) is used to describe their distributions.

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We use the notation $f_X(x)$ to refer to the PDF of random varaible $X$. Both PMF and PDF are analogous.

## Probability mass function.

Taking the values $x_1\ x_2\ \dots\ x_n$ etc.

$p(x_i) = p_i$ is called probability mass function of a discrete random variable $x$.

i) $p_i \geq 0\ \forall i$ iii) $\sum_i p_i = 1$

ii) $p_i \leq 1\ \forall i$

* If random variable $X$ is continous then function associate with random variable $f(x)$ is called probabity density function

if $f(x) \geq 0\ \forall x$.

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

and $\forall$ real $A, B$ $\quad p[a < x < b] = \int_a^b f(x)\, dx$.

## Define cumulative distribution function (Distribution function)

if $x$ is defined as cumulative.

$$P(x) = p[X < x] = \begin{cases} \int_{-\infty} f(x)\, dx & \text{if } x \text{ is continous} \\ \sum_{i=1}^{p_i} P(x_i) & \text{if } x \text{ is discrete} \end{cases}$$

## Marginal and conditional probability mass and density function.

let $x, y$ be 2 dimensional random variable whose conbimed probabi mass function is defined as $p(x_i, y_j)$ in discrete case.

if $x, y$ is discrete the marginal probability mass function of $x$ is defined as $p(x_i) = \sum_{j=1}^{n} P(x_i, y_j)$

$$q(y_j) = \sum_{i=1}^{m} p(x_i, y_j) \qquad \sum_{j=1}^{m} \sum_{i=1}^{n} p(x_i, y_j) = 1$$

Note. if $p(x_i, y_i) = p(x_i)\, q(y_i)$ then x and y are given to be independent of

If x, y are a pair of

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

Note - If $f(x, y) = g(x)\, h(y)$ x and y are given to be independent

Denfinition of conditional pmfs and densities.

$$X\ (x_i,\ i = 1, \ldots m)$$

$$Y\ (y_j,\ j = 1, \ldots n)$$

$$p(x_i/y_i) = \frac{P(X = x_i,\ Y = y_i)}{q(y_i)} \qquad \text{for discrete.}$$

$$q(y_i/x_i) = \frac{P(X = x_i,\ Y = y_i)}{p(x_i)}$$

for continous.

$$g(x/y) = \frac{f(x, y)}{h(y)} \qquad h(y) \neq 0$$

$$h(y/x) = \frac{f(x, y)}{g(x)} \qquad g(x) \neq 0.$$

## chain rule of conditional probability

$$p[a, b, c] = p[a/b, c]\ p[b, c]$$

$$= p[a|b, c]\ p[b|c)\ p(c)$$

$$p(x_1, x_2 \cdots x_n) = p(x_1) \prod_{i=2}^{n} p(x_i \mid x_2, x_3, \ldots x_{n-1})$$

Define problems.

1) A random process given measurements of $x$ between 0 and 1.

with pdf $f(x) = \begin{cases} 12x^3 - 21x^2 + 10x. & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

i) $p[x \le 1/2)$  ii) $p[x > 1/2]$  iii) find $K \rightarrow p(x = K) = 1/2$

Sol: To find whether the given $f(x)$ is pdf.

$$\int_{-\infty}^{\infty} f(x)\, dx. = 1$$

$$\int_0^1 (12x^3 - 21x^2 + 10x)\, dx.$$

$$= 12\left(\frac{x^4}{4}\right) - 21\left(x^3/3\right) + 10\left(x^2/2\right)$$

$$= 12(1/4) - 21(1/3) + 10(1/2)$$

$$= 3 - 7 + 5 = 1$$

$$p(x < 1/2) = \int_0^{1/2} (12x^3 - 21x^2 + 10x)\, dx.$$

$$= \left(3x^4 - 7x^3 + 5x^2.\right)_0^{1/2}$$

$$= 3(1/2)^4 - 7(1/2)^3 + 5(1/2)^2$$

$$= 3(1/16) - 7(1/8) + 5(1/4) = 9/16.$$

$$P(X > 1/2) = \int_{1/2}^{1} (12x^3 - 21x^2 + 10x)\,dx$$

$$= (3x^4 - 7x^3 + 5x^2)\Big|_{1/2}^{1}$$

$$= (3 - 7 + 5) - (9/16)$$

$$= 1 - 9/16 = 16 - 9/16 = 7/16.$$

$$P(X \le K) = 1/2$$

$$\int_0^K f(x)\,dx = 1/2$$

$$3K^4 - 7K^3 + 5K^2 = 1/2$$

$$K^2(3K^2 - 7K + 5) = 1/2$$

$$K = \frac{\pm 7\sqrt{49-60}}{6}$$

$$= \frac{\pm 7\sqrt{-11}}{6}$$

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19/03/22

1) A pdf of $p(x, y) = \begin{cases} e^{-(x+y)}, & x, y > 0 \\ 0, & \text{elsewhere} \end{cases}$

$$P(1/2 < X < 2,\ 0 < Y < 4) = \int_{1/2}^{2}\int_{0}^{4} f(x, y)\,dy\,dx.$$

$$= \int_{1/2}^{2} (-e^{-y})_0^4\, e^{-x}\,dx = -\int_{1/2}^{2} (e^{-4} - 1)e^{-x} = (e^{-4} - 1)(-e^{-x})_{1/2}^{2}$$

$$= (e^{-4} - 1)(\cancel{+}e^{-2} \cancel{+} - e^{-1/2})$$

2) consider a pdf a $f(x,y) = \begin{cases} 2 & 0 < x < 1,\ 0 < y < x \\ 0 & \text{elsewhere.} \end{cases}$

find a) Marginal density b) conditional density

sol

$$g(x) = \int_0^2 f(x,y)\,dy = \int_0^x 2\,dy = 2x.$$

$$h(y) = \int_0^1 2\,dx = 2$$

$f(x,y) \neq g(x)\,h(y)$

$x$ and $y$ are not independent

$$g(x/y) = \frac{f(x,y)}{h(y)} = 2/2 = 1$$

$$h(y/x) = \frac{f(x,y)}{g(x)} = 2/2x = 1/x.$$

3) Two random variables $x$ and $y$ are joined pdfs $f(x,y) = \begin{cases} Ae^{-(2x+y)} & x,y \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$ calculate a, find marginal, conditional and cumulative densities.

Sol

$$f(x,y) = \begin{cases} Ae^{-(2x+y)} & x,y \geq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

To find A

Total prob $= 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} Ae^{-(2x+y)}\,dx\,dy = 1$$

$$A\left[\frac{e^{-2x}}{-2}\right]_0^{\infty}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = 1$$

$$\frac{A}{2}[0-1][0-1] = \frac{A}{2} = 1$$

$$\boxed{A = 2}$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^{\infty} 2(e^{-(2x+y)})\,dy$$

$$= 2e^{-2x}\left[e^{-y}/-1\right]_0^{\infty}$$

$$= 2e^{-2x}\left[-e^{-y}\right]_0^{\infty}$$

$$= 2e^{-2x}(0-1)$$

$$g(x) = 2e^{-2x}.$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} 2e^{-(2x+y)}\,dx$$

$$2e^{-y}\int_0^{\infty} e^{-2x}\,dx$$

$$= -\cancel{2}e^{-y}\left(\frac{e^{-2x}}{-\cancel{2}}\right)_0^{\infty}$$

$$= -2e^{-y}\cdot e^{-2x}(0-1)$$

$$= -e^{-y}(0-1)$$

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$g(x)\, h(y) = 2e^{-2x} \cdot e^{-y}$

$= 2e^{-(2x+y)}$ x and y are independents.

$= f(x,y)$

$$g(x/y) = \frac{f(x,y)}{h(y)} = \frac{2e^{-(2x+y)}}{e^{-y}} = 2e^{-2x}$$

$$h(y/x) = \frac{f(x,y)}{g(x)} = \frac{2e^{-(2x+y)}}{e^{-2x}} = 2e^{-y}$$

cumulative.

$p(x,y) = p[x < x, y < y)$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} 2e^{-(2x+y)}\, dx\, dy$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{-\infty}^{x}\left[\frac{e^{-y}}{-1}\right]_{-\infty}^{y}$$

$$= 2\left[\frac{e^{-2x}}{-2} - 0\right]\left[\frac{e^{-y}}{-1} - 0\right]$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)\left(\frac{e^{-y}}{-1}\right)$$

$$= 2e^{-(2x+y)}$$

$F(-1) =$ infinity. we cannot say independent or dependent in

$F(1) =$ finite. cumulative distribution.

only 2 properties
in cumulative distribution.

4) $f(x,y) = \begin{cases} 8xy & 0 \le x \le y,\ 0 \le y \le 4 \\ 0 & \text{elsewhere.} \end{cases}$ check wether x and y are independent to each other

Sol

$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= \int_{0}^{4} 8xy\,dy$

$= 8x \int_0^4 y\,dy$

$= 8x \left(y^2/2\right)_0^4$

$= 8x\left(16/2\right)$

$= 64x.$

$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$

$= \int_0^y 8xy\,dx$

$8y \int_0^y x\,dx = 8y\left(x^2/2\right)_0^y$

$= 8y \left(y^2/2\right)$

$= 8y^3/2$

$= 4y^3$

$f(x,y) = g(x)\,h(y)$

$= 64x \cdot 4y^3$

$= 256xy^3$

6) Two discrete Rv's have pdf

$p(x,y) = \dfrac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!}$, $y = 0,1,2\cdots x$, $x = 0,1,2\cdots y$, $\lambda > 0$, $0 < p < 1$

find the marginal and conditional PMFs

Sol

$g(x) = \sum_{y=0}^{x} \dfrac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!}$

$= \lambda^x e^{-\lambda} \sum_{y=0}^{x} \dfrac{p^y (1-p)^{x-y}}{y!\,(x-y)!}$

$= \dfrac{\lambda^x e^{-\lambda} x!}{x!}\left[\dfrac{(1-p)^x}{x!} + \dfrac{p(1-p)^{x-1}}{1!\,(x-1)!} + \dfrac{p^2(1-p)^{x-2}}{2!\,(x-2)!} + \cdots + \dfrac{p^x(1-p)}{x!\,0!}\right.$

$$\frac{e^{-\lambda}\lambda^{x}}{x!}\left[(1-p)^{x} + \frac{xp(1-p)^{x-1}}{1!} + \frac{x(x-1)p^{2}(1-p)^{x-2}}{2!} + \dots + \frac{\cancel{x!}\,p^{x}(1-p)^{0}}{\cancel{x!}}\right]$$

$$= \frac{e^{-\lambda}\lambda^{y}}{x!}\left(p+(1-p)\right)^{x}$$

$$g(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$h(y) = \frac{e^{-\lambda}\lambda^{y}}{y!}$$

conside a random expriment of tossing two honest dices . find the marginal prob distribution. Also find $p[2 \leq X \leq 3, \; 3 \leq Y < 6)$

| y \ x | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | $p(Y=1) = 1/6$ |
| 2 | | | | | | | $p(Y=2) = 1/6$ |
| 3 | | | $\boxed{1/36}$ | | | | |
| 4 | | | $\boxed{1/36}$ | | | | |
| 5 | | | $\boxed{1/36}$ | | | | |
| 6 | | | | | | | |
| | $p(X=1)=1/6$ | $p(X=2)=1/6$ | $p(X=3)=1/6$ | | | | |

$$P[2 < X \leq 3, \; 3 \leq Y < 6] = 3/36 = 1/12$$

$$p(x=i, y=j) = p(X=i)\,p(Y=j) \quad \text{(from table)}$$

X and Y are independent

21/03/22.

1) Find the foumula for puobability distribution of no. of head when a faiu coin is tossed 4 times.

Sol) No. of tosses of a coin $= n = 4$.

Let x be a discuete uandom vauaible denoting the no. of heads,

let p be the prob of head to appear

prob of tail to appear $= 1-p$.
$= q$

$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$.

$= {}^{4}C_{x}\, p^{x} q^{\cancel{4}-x}$.

$x = 0, 1, 2, 3, 4$

$P(X=0) = {}^{4}C_{0}\, p^{0} q^{4-0} = 1 \cdot 1 \cdot 1/2^{4} = 1/16$.

$P(X=1) = {}^{4}C_{1}\, p^{1} q^{3} = 4 \cdot \frac{1}{2} \cdot \frac{1}{8} = 1/4$

$P(X=2) = {}^{4}C_{2}\, p^{2} q^{2} =$

$3\times1$

$x \cdot 2\times1$

$\frac{n!}{(n-r)!\,r!}$ $\frac{x^{4}}{4x}$

$= 3 \cdot 6 \cdot \frac{1}{4} \cdot \frac{1}{4} = 3/8$.

$P(X=3) = {}^{4}C_{3}\, p^{3} q^{3} = 4$

$P(X=4) = {}^{4}C_{4}\, p^{4} q^{0} = 0$.

2) consider the experiment of tossing a coin ~~and~~ of being an occurence of head or tail.

Assign prob's $p$ and $q$ for head and tail respectively. define a random variable of $x$ by $x(h) = 1$ $x(t) = 0$. Define prob in $f(x)$ and $F(x)$

sol: $p(1) = 1/2$ $P[X = 0] = 1/2$

$p(0) = 1/2$ $P[X = 1] = 1$

$$F(x) = \begin{cases} 0 & \text{for } x \notin \{0,1\} \\ 1/2 \\ 1 \end{cases}$$

) Two dice are tossed $x$ denotes a RV sum of no.'s appears on dice. fin prob distribution of faclots. also find the cumulative dist of RV

Sample Space

$S = \{(1,1)\ (1,2) \ldots (1,6), (2,1) \ldots (2,6)\ (3,1) \ldots (3,6) \ldots (6,$

$n(S) = 36$.

Range of $X = \{ \overset{(1,1)}{2}, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$

$p(x = 2) = 1/36$ $p(x = 10) = 3/36$

$p(x = 3) = 2/36$ $p(x = 11) = 2/36$

$p(x = 4) = 3/36$ $p(x = 12) = 1/36$

$p(x = 5) = 4/36$

$p(x = 6) = 5/36$

$p(x = 7) = 6/36$

$p(x = 8) = 5/36$

$p(x = 9) = 4/36$

cumulative distribution.

$$F[X] = \begin{cases} 0 & x < 2 \\ 1/36 & x \leq 2 \\ 3/36 & x \leq 3 \\ 6/36 & x \leq 4 \end{cases}$$

consider the exp of tossing a fair coin tossing 4 times. Define the $X=0$ if 0 or 1, head appear. $X=1$ if two heads appear $X=2$ if 3 or 4 heads appear. find the prob fn, mean, variance.

sol 4 Heads.

$S = \{ (H,T,T,T) \ldots \}$

$n(s) = 2^4 = 16.$

$$X : \{ \underset{\substack{\downarrow \\ 0 \text{ or } 1H}}{0} \quad \underset{\substack{\downarrow \\ 2H}}{1} \quad \underset{\substack{\downarrow \\ 3 \text{ or } 4H}}{2} \}$$

$$p(X=0) = \left( {}^4C_0 (1/2)^0 (1/2)^4 \right) + \left( {}^4C_1 (1/2)^1 (1/2)^3 \right)$$

$$= 1 \cdot 1 \cdot 1/16 + \overset{3}{\cancel{24}} \cdot 1/2 \cdot 1/8$$

$$= 1/16 + 3/2 = \frac{1+24}{16} = \overset{5}{\cancel{25}}/16.$$

$$p(X=1) = {}^4C_2 (1/2)^2 (1/2)^{4-2} = {}^4C_2 (1/2)^2 (1/2)^2 = 6/16.$$

$$p(X=2) = {}^4C_3 (1/2)^3 (1/2)^{4-3} + {}^4C_4 (1/2)^4 (1/2)^{4-0} = {}^4C_3 (1/2)^3 (1/2)^1 + {}^4C_4 (1/2)^4 (1/2)^0$$

$$= 5/16.$$

Mean $= E(x) = \sum x\, p(x)$

$= 0(5/16) + 1(6/16) + 2(5/16)$

$= 16/16 = 1$

$E(x^2) = \sum x^2 p(x)$

$= 0^2 (5/16) + 1^2 (6/16) + 2^2 (5/16)$

$= 13/8.$

Variance. $= E(x^2) - [E(x)]^2$

$= 13/8 - (1)^2$

$= 13/8 - 1 = \frac{13-8}{8} = 5/8.$

$\sigma = STD(x)$

$= \sqrt{var\ x}$

$= \sqrt{5/8}$

Definition of Expectation of random varaible.

let x be a random varaible

$$E(x) = \begin{cases} \sum_i x_i\, p(x_i) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x)\, dx & \text{if } x \text{ is continous} \end{cases}$$

variance. $V(x) = E[x^2] - [E(x)]^2$ , $E[(X - E(x))^2] = var(x)$

original definitio

Definition of co-variance

List of common prob distribution and write short note on each of them.

1) Bernouli distribution.

let $x$ be binary rv $x \in \{0, 1\}$.

prob of success be '$\theta$', let $x$ be a bernouli rv

$$x \sim ber(\theta)$$

$$\text{PMF} : Ber[x/\theta] = \begin{cases} \theta & x = 1 \\ 0 & x = 0 \end{cases}$$

This is a special case of bernouli rv

$$Bin[K/n, \theta] = {}^{n}C_{K} \, \theta^{K} (1-\theta)^{n-K}.$$

Multinomial distribution.

$$Mu(x/n, \theta) = \binom{n}{x_1 \, x_2 \dots x_K} \prod_{j=1}^{K} \theta_j^{x_j}$$

Where $\theta_j \rightarrow$ side $j$ shows up

$$\binom{n}{x_1 \, x_2 \dots x_K} = \frac{n!}{x_1! \, x_2! \dots x_K!}$$

## Gaussian Distribution.

$$N[x/\mu, \sigma^2] = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2}$$

$\mu = E(x)$

$\sigma^2 = var(x)$

CDF of Gaussian rv $= \int_{-\infty}^{\infty} N(z/\gamma, \sigma^2)\, dz$.

### Interms of Error function.

$$\phi(x, \mu, \sigma) = \frac{1}{2}\left[1 + erfn\left(z/\sqrt{2}\right)\right]$$

where $erfn(z) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$

fractional calculus $\longrightarrow \frac{d^{1/2}}{dx^{1/2}}(x)$

$L[f'(t)] = SF(s) - f(0)$

$z = \frac{x - \mu}{\gamma}$.

$L\left[\int_0^t f(s)dt\right] = \frac{F(s)}{s}$

$N(z, 0) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$L[t f(t)] = -F'(s)$

$L\left[\frac{f(t)}{t}\right] = \int_s^{-s} F(s)\, ds$

iv) $p(x, \lambda) = \lambda e^{-\lambda x}, x \geq 0$.

To have a sharp point at $x = 0$.

v) laplace Distribution.

$$p(x, \mu, \nu) = \frac{1}{2\nu} e^{-\left[\frac{(x-\mu)}{\nu}\right]}$$

vi) To accomodate a sharp ~~teeth~~ of prob mass at an arbitrary point

vii) Direct distribution and empirical distribution.

To accomodate all prob mass around a single point direct delta function is defined as

$$\phi(x) = \delta(x - \mu) \; ; \; \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Written by - $\mu$ to

Empirical.

Given a dataset $D = \{x_1, x_2 \dots x_n\}$

$$P_{emp}(A) = \frac{1}{N} \sum_{1}^{\infty} \delta_{x_i}(A)$$

$$\delta_x(n) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

$$p(x) = \sum_{1}^{\infty} w_i \, \delta_{xi}(x) \qquad 0 \leq w_i \leq 1$$

$$\sum_{i=1}^{N} w_i = 1$$

Mixtures of distribution.

let a, b a stair of categorical RV

let $x$ be dummy encoding

$$x\,[(x=1)(x=2)\dots(x=K)]$$

for example if $a = 3$ it encode states as $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

it is called one hot encoding

in this representation multinoulli distribution

$$Mu(x|1,\theta) = \prod_{j=1}^{k} \theta_j^{(x_j=1)}$$

Difference between Multinomial or Multinoulli

The difference in them is absence of multinomial cofficient in multinoulli

A mixed distribution is made up of several compornments distribution. on each trail.

A sampling component identity from multinoulli distribution

$$p(x) = \sum_i P[c=i]\, P[x|c=i]$$

where $p(c)$ is multinoulli distribution.

$A, B$

$$p(A/B) = \frac{P(A \cap B)}{P(B)} \quad P(A)\, P(B/A)$$

$$P[Ai/B] = \frac{P(Ai)\, P(B/Ai)}{\sum_{i=1}^{n} P(Ai)\, P(B/Ai)}$$

4.* Explain the properties of common functions using in deeplearning process. (6m)

i) logistics sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\sigma'(x) = \sigma(x)\left[1 - \sigma(x)\right]$$

1) $\sigma(x) = \frac{1}{1+e^{-x}}$

To prove $\sigma'(x) = \sigma(x)\,[1 - \sigma(1)]$

$$\sigma'(x) = \frac{-1}{(1+e^{-x})^2}\,[-e^{-x}]$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= (1+e$$

Soft plus:

$$\nu(x) = \log[1+e^{-x}]$$

2) $1 - \sigma(x) = \sigma(-x)$

3) $\frac{d}{dx}\nu(x) = \sigma(x)$

4) $\log \sigma(x) = -\nu(-x)$

$\nu'(x) = -(x)$

$\nu^{-1}(x) = 1\log(e^{q}-1)$

$$\nu(x) = \int_{-\infty}^{x} \sigma(y)\,dy$$

$\nu(x) - \nu(-x) = x.$

A. $\sigma^{-1}(x) = \log\left(\frac{x}{1-x}\right)$

let $y = \sigma(x)$

$= \frac{1}{1+e^{-x}}$

$1+e^{-x} = \frac{1}{y}$

$e^{-x} = \frac{1}{y} - 1 = \frac{1-y}{y}$

$-x = \log\left(\frac{1-y}{y}\right)$

$x = \log\left(\frac{y}{1-y}\right)$

$\sigma^{-1}(x) = \log\left(\frac{x}{1-x}\right)$

* $\sigma(x) = \frac{1}{1+e^{-x}}$ —— ①

$\sigma(-x) = \frac{1}{1+e^{x}}$ —— ②

$1-\sigma(x) = 1 - \frac{1}{1+e^{-x}}$

$= \frac{1+e^{-x}-1}{1+e^{-x}} = \frac{e^{-x}}{1+e^{-x}}$

$= \frac{e^{-x}\cdot e^{x}}{e^{x}+e^{-x}\cdot e^{x}} = \frac{1}{1+e^{x}} = \sigma(-x)$

from ②

$\cancel{\text{softp}}$

* $\zeta(x) = \log(1+e^{-x})$

$$\zeta'(x) = \frac{1}{1+e^{-x}}\left[-e^{-x}\right]$$

$$= \frac{-e^{-x}\, e^{x}}{(1+e^{-x})\, e^{x}}$$

$$= \frac{-1}{e^{x}+1} = -\sigma(-x)$$

$$\log \sigma(x) = -\zeta(-x)$$

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

$$\log \sigma(x) = \log(1) - \log(1+e^{-x})$$

$$= 0 - \zeta(x)$$

$$-\zeta(-x) = -\log(1+e^{-x})$$

$$= \log(1+e^{-x})^{-1}$$

$$= \log\left(\frac{1}{1+e^{x}}\right)$$

$$= \log 1 - \log(1+e^{x})$$

$$= -\zeta(-x)$$

* $\vartheta(x) = \log(1+e^{-x}) = y$

$$1+e^{-x} = e^{y}$$

$$e^{-x} = e^{y}-1$$

$$-x = \log(e^{y}-1)$$

$$x = -\log(e^{y}-1)$$

As $y$ is dummy

$$\vartheta^{-1}(x) = -\log(e^{x}-1)$$

## Problem on total probability Theorem

An urn contains 10W, 3 black balls. Another urn contains 3 white and 5 black balls. 2 balls are drawn at random from the first urn and are placed in the second urn. And then one ball is taken at random from the urn 2. What is the probability that it is white ball.

Sol Mode of transfer.

Transferred balls may be 2W, 2B, 1W, 1B

let $B_1$, $B_2$, $B_3$ be the events of drawing 2W, 2B, 1W, 1B from urn 1

$$p(B_1) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{\frac{10\cdot 9}{2\cdot 1}}{\frac{13\cdot 12}{2\cdot 1}} = \frac{45}{78}$$

$$p(B_2) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{\frac{3\cdot 2}{2\cdot 1}}{\frac{13\cdot 12}{2\cdot 1}} = \frac{3}{78}$$

$$p(B_3) = \frac{{}^{10}C_1\cdot {}^{3}C_1}{{}^{13}C_2} = \frac{10\cdot 3}{\frac{13\cdot 12}{2\cdot 1}} = \frac{30}{78}$$

let A be the event of drawing one white ball from urn 2 after transfer.

Total balls in urn 2 $= 8+2 = 10$

$p(A/B_1) = 5/10$ urn 2 $= 5W, 5B$

$P(A/B_2) = 3/10$ urn 2 $= 3W, 7B$

$P(A/B_3) = 4/10$ urn 3 $= 4W, 6B$

$P(A)$ = probability of drawing 1W Ball from urn 2 after transfer

$$P(A) = P(A/B_1)\,P(B_1) + P(A/B_2)\,P(B_2) + P(A/B_3)\,P(B_3)$$

$$= \frac{5}{10} \times \frac{45}{78} + \frac{3}{10} \times \frac{3}{78} + \frac{4}{10} \times \frac{30}{78}$$

$$= 59/130$$

<u>Sample problem using Bayes's Theorem</u>

There are 3 true coins and 1 false coin with head on both sides. A coin is chosen at random and tossed 4 times. If head occurs all the 4 times what is the probability that false coin has been chosen and used.

Sol

3 TC , 1 FC

let T be the event of chosing

28/03/21

1) The joint pdf of 2D RV is $f(x,y) = xy^2 + x^2/8$ $\quad 0 \le x, y \le 1$

i) $p[X < 1)$ ii) $p( \quad < 1/2)$ iii) $p(X > \frac{3}{4}) \; Y < 1/2$ iv) $p(Y < 1/2) \; X > 1$

v) $p(X < Y)$ vi) $p(X + Y < 1)$

Sol To find $p(X<1)$

i) $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$= \int_0^1 (xy^2 + x^2/8)\, dy$

$= x \cdot \frac{y^3}{3}\Big|_0^1 + \frac{x^2}{8} \cdot y\Big|_0^1$

$= x\left(1/3 - 0\right) + \frac{x^2}{8}(1)$

$= x/3 + x^2/8$ .

$p(X<1) = \int_0^1 (x/3 + x^2/8)\, dx = \frac{x^2}{6} + \frac{x^3}{24}\Big|_0^1 = 1/6 + 1/24$

$= 5/24$

ii) $p(Y < 1/2) = \int_0^{1/2} \left( \frac{y^2}{2} + 1/24 \right) dy$

$= \frac{y^3}{6} + \frac{y}{24}\Big|_0^{1/24} = 1/6 \left( 1/8 \right) + \frac{1}{24}(1/2)$

$= \frac{1}{48} + \frac{1}{48} = 1/24$

iii) $p(X > 3/4 \mid Y < 1/2) = \frac{p(X > 3/4, Y < 1/2)}{p(Y < 1/2)}$

$= \frac{1}{p(Y<1/2)} \left[ \int_{3/4}^{1} \int_{1/2}^{0} (xy^2 + x^2/8)\, d \; dx \right.$

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$$= 1\left[\int_{3/4}^{1}\left(\frac{xy^3}{3}+\frac{x y^2}{8}\right)_0^1 dx\right.$$

$$= \frac{1}{(1/24)}\left[\int_{3/4}^{1}\left(\frac{x}{24}+\frac{y^2}{16}\right)dx\right]$$

$$= \left(\frac{y^2x^2}{2}+\frac{x^3}{24}\right)_0^{1/2}$$

v) $$p[X<Y] = \int_0^y\left(\frac{x}{24}+\frac{x^2}{16}\right)dx = \left(\frac{x^2}{48}+\frac{x^3}{48}\right)_0^y$$

$$= \frac{y^2}{48}+\frac{y^3}{48} = \frac{y^2+y^3}{48}.$$

vi) $$p(X+Y<1) = \int_0^1\int_0^{1-y}(xy^2+x^2/8)\,dx\,dy$$

$$= \int_0^1\left(\frac{y^2x^2}{2}+\frac{x^3}{24}\right)_0^{1-y}dy = \int_0^1\left(\frac{y^2(1-y)^2}{2}+\frac{(1-y)^3}{24}\right)dy.$$

$$= \int_0^1 \frac{(y^2-2y^3+y^4)}{2}+\frac{(1-y)^3}{24}\,dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}-\frac{2y^4}{4}+\frac{y^5}{5}\right)+\frac{(1-y)^4}{-4(24)}\Bigg|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}+1/5\right)+(0)$$

$$= \frac{1}{2}\left(\frac{10-15+6}{30}\right) = \frac{1}{2}(1/30) = 1/60$$

2) Find $P\left[x^2+y^2 \le a^2\right] = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}\, dy\, dx$.

$$P = \frac{2}{\pi\sigma^2}\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-\frac{1}{2\sigma^2}(x^2+y^2)}\, dy\, dx \quad \text{— } (1)$$

put $x = r\cos\theta \quad y = r\sin\theta$

$x^2+y^2 = r^2$

$dx\,dy = |J|\,dr\,d\theta$

where $J = \begin{vmatrix} \frac{dx}{dy} & \frac{\partial y}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$.

$$P = \frac{2}{\pi\sigma^2}\int_0^{\pi/2}\int_0^a e^{-\frac{r^2}{2\sigma^2}}\, r\, dr\, d\theta.$$

$$P = \frac{2}{\pi\sigma^2}\int_0^{\pi/2}\int_0^a e^{-\frac{r^2}{2\sigma^2}}\, d\left(\frac{-r^2}{2\sigma^2}\right)(-\sigma^2)$$

$$= \frac{2}{\pi+2}\int_0^{\pi/2}\left(e^{r^2/2\sigma^2}(-\sigma^2)\right)_0^a d\theta$$

$$= -\frac{2}{\pi}\int_0^{\pi/2}\left(e^{\frac{a^2}{2\sigma^2}} - 1\right) d\theta$$

$$= \frac{\cancel{2}\left(1-e^{a^2/2\sigma^2}\right)}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{\cancel{2}}\right)$$

$$P = \left(1-e^{a^2/2\sigma^2}\right)$$

3) In a binary communication system a 0 or 1 is transmitted. because of noise in system 0 can be received as 1 with probability $p$ and 1 can be received as 0 with same probability $p$. Assuming that 0 is transmited with probability $p_0$ and 1 as $q_0 = (1-p_0)$. find the prob that ~~wh~~ 1 is transmited that when 1 is received.

Sol let A be the event if 1 is transmited

$\bar{A}$ be the event if 0 is received

let B be the event if 1 is transited

$\bar{B}$ be event if 0 is received.

$$P(A/B) = \frac{P(A)\,P(B/A)}{P(A)\,P(B/A) + P(\bar{A})\,P(B/\bar{A})}$$

$$= \frac{q_0(1-p)}{q_0(1-p) + p_0(p)}$$

$$= \frac{q_0(1-p)}{q_0 - q_0 p + p_0 p}$$

30/03/22.

2) A prob that student passes exam is 0.9 given that he studied a prob that he passes exam without is 0.2. Assume that a prob that student studied is 0.75. given that a student passes the exam. what is the prob that he studied.

Sol Let A be an event denoting that a student studies

$\bar{A}$ " " " does not study

B " " " passes an exam

$\bar{B}$ " " " does not passes an exam

$p(B/A) = 0.9$

$p(B/\bar{A}) = 0.2$

$p(A) = 0.75$

$$p(A/B) = \frac{p(A)\,p(B/A)}{p(A)\,p(B/A) + p(\bar{A})\,p(B/\bar{A})}$$

$$= \frac{0.75\,(0.9)}{(0.75)(0.9) + 0.25\,(0.2)}$$

$$= \frac{0.675}{0.725} = 0.931$$

3) For a certain binary communication channel the prob that transmitted as 0 is received as 0 is 0.95. The prob that transmitted 1 is received as 1 is 0.99. The prob that 0 is transmitted is 0.4. Find the prob that i) 1 is received ii) 1 was transmitted and 1 was received

3) for a certain binary communication channel the prob that transmitted as 0 is recieved a 0 is 0.95. The prob that transmitted 1 is recieved as 1 is 0.99. The prob that 0 is transmited is 0.4. find the prob that

i) 1 is recieved ii) 1 was transmited and 1 was received

p(A) prob transmited 0 received 0 — 0.95

p(B) prob trans 1 — 0.99

$p(A) = 0.4$

$p(A/B) = 0.95$

$p(\bar{B}/\bar{A}) = 0.99$

$p(B)$

$p(B/A) = 0.4$

$$p(A/B) = \frac{p(A)\, p(B/A)}{p(A)\, p(B/A) + p(\bar{A})\, p(B/\bar{A})}$$

$0.396 + 0.6\, p(B/\bar{A}) = \frac{0.396}{0.95} = 0.4168$

$0.396 + 0.6\, p(B/\bar{A}) = 0.4168$

$0.6\, p(B/\bar{A}) = 0.0208$

$p(B/\bar{A}) = \frac{0.02}{0.6}$

$p(B/\bar{A}) = 0.0333$

$$0.95 = \frac{0.4(0.99)}{0.4(0.99) + 0.6\, p(B/\bar{A})}$$

$$0.95 = \frac{0.396}{0.396 + 0.6\, p(B/\bar{A})}$$

$P(\bar{A}) = 0.4$

$P(A$

$P(B/A) = 0.9$

$P(\bar{A}) = 0.6$

$P(B) =$