Estimating the Parameter for the Exponential Distribution

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Introduction

- <u>Estimators:</u>
  - Maximum Likelihood: Inverse of the Mean
  - Inverse of the Median
  - Exponential-Gamma Estimator with Jeffrey’s Prior
- <u>Confidence Intervals:</u>
  - Delta method, using the MLE
  - Exact distribution (Chi-squared distribution)
  - Credible interval (interval with highest posterior density (HPD))
- <u>Approach</u>:
  - R programming
  - Comparison criteria:
    - We created 1000 simulations in R to determine the MSE, variance and bias for each estimator. We assessed estimator performance over different parameter and n-size values.
    - We created 1000 simulations in R to determine the coverage probabilities for different confidence intervals. We assessed coverage performance over different parameter and n-size values.

Maximum Likelihood Estimator

- Using the likelihood function to find an estimate of theta. The likelihood function is given by:

$$L(\theta;\ x_1, \dots., x_n) = \theta^n \exp(-\theta \textstyle\sum_{j=1}^{n} x_j)$$

- Log likelihood:

$$l(\theta;\ x_1, \dots., x_n) = n ln(\theta) - \theta \sum_{j=1}^{n} x_j$$

- Giving us the MLE, which is the point where the log is at a maximum:

$$\hat{\theta} = \frac{n}{\sum_{j=1}^{n} x_j}$$

Inverse of the Median

- The median of a random variable X is a number that satisfies:

$$F_x\mu = 1/2$$

- The median of the Exponential Function is given by:

$$F_x(x) = \Pr(X \le x) = \int_0^x \theta e^{-\theta t} dt = 1 - e^{-\theta t}$$

$$For\ median, solve\ for\ m:\ 1 - e^{-\theta m} = {}^{1}/_{2}$$

$$e^{-\theta m} = {}^{1}/_{2} \rightarrow -\theta m = \ln\left(\frac{1}{2}\right) = -ln2$$

$$Therefore, m = \frac{ln2}{\theta}$$

- This method uses the sample median as an estimate of theta:

$$\theta = {}^{ln2}/_{m}$$

Jeffrey's Prior Bayesian Estimate

- Mathematical Derivation:

$$X|\theta \sim Exponential(\theta)$$

$$\theta \sim Gamma(\alpha, \beta)$$

$$p(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$L(\theta|x) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$p(\theta|x) \propto p(\theta)L(\theta|x)$$

$$\propto \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$\propto \theta^{\alpha+n-1} e^{-(\beta+\sum_{i=1}^{n} x_i)\theta}$$

$$\theta|x \sim Gamma(\alpha + n\,, \beta + \sum_{i=1}^{n} x_i)$$

Jeffrey's Prior Bayesian Estimate (continued)

- Jeffrey's Prior:

$$\theta \sim Gamma(1, 0)$$

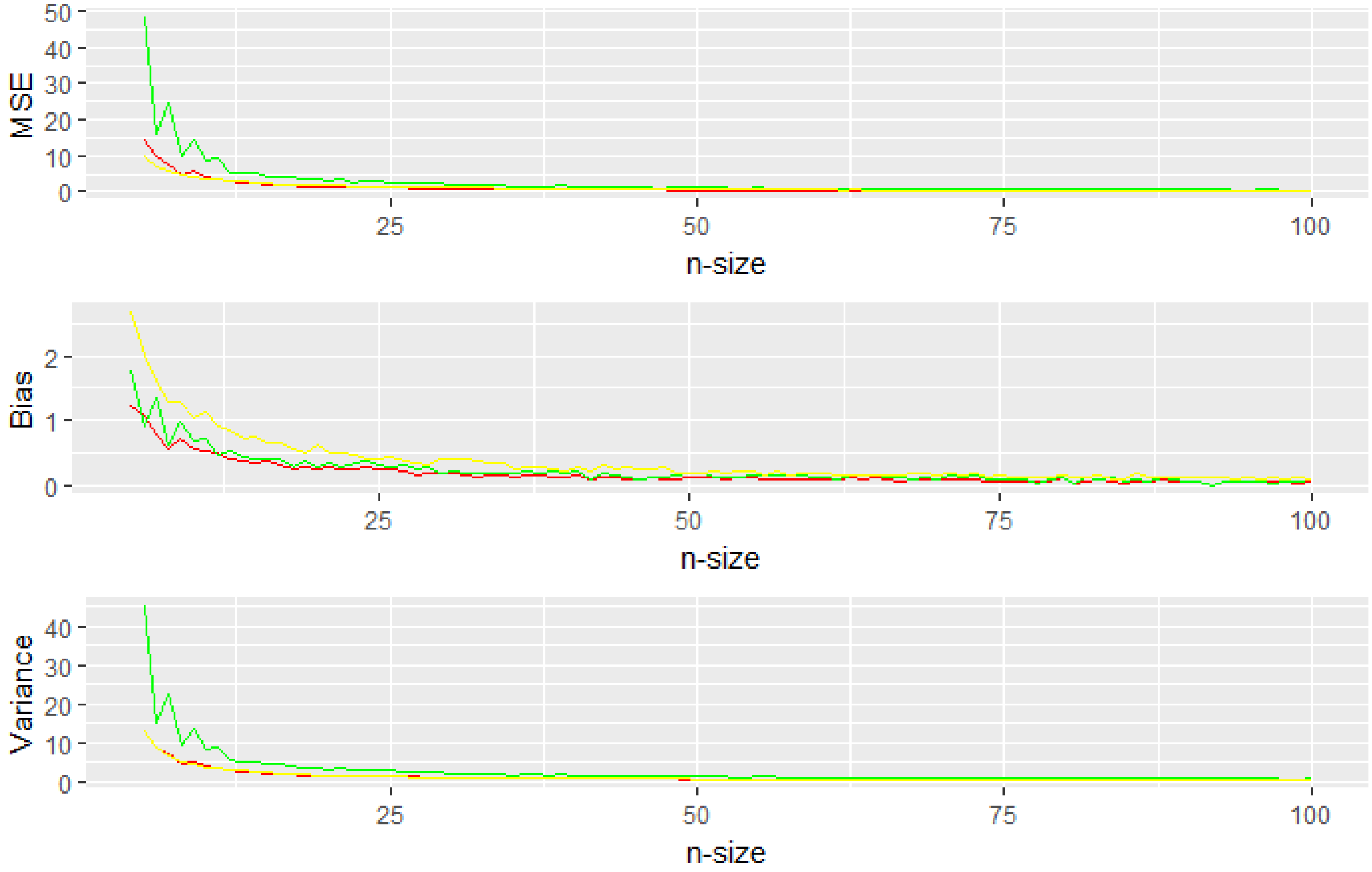
$$\theta | x \sim Gamma(1 + n\,, \sum_{i=1}^{n} x_i)$$

- Expected Value:

$$E(\theta|x) = \frac{1+n}{\sum_{i=1}^{n} x_i}$$

$$V(\theta|x) = \frac{1+n}{(\sum_{i=1}^{n} x_i)^2}$$

Estimates MSE, bias, variance as N increases



θ=5 for all graphs

Estimator	
Jefrrey's Prior Bayesian Estimate =	
MLE =	
Inverse of the median =	

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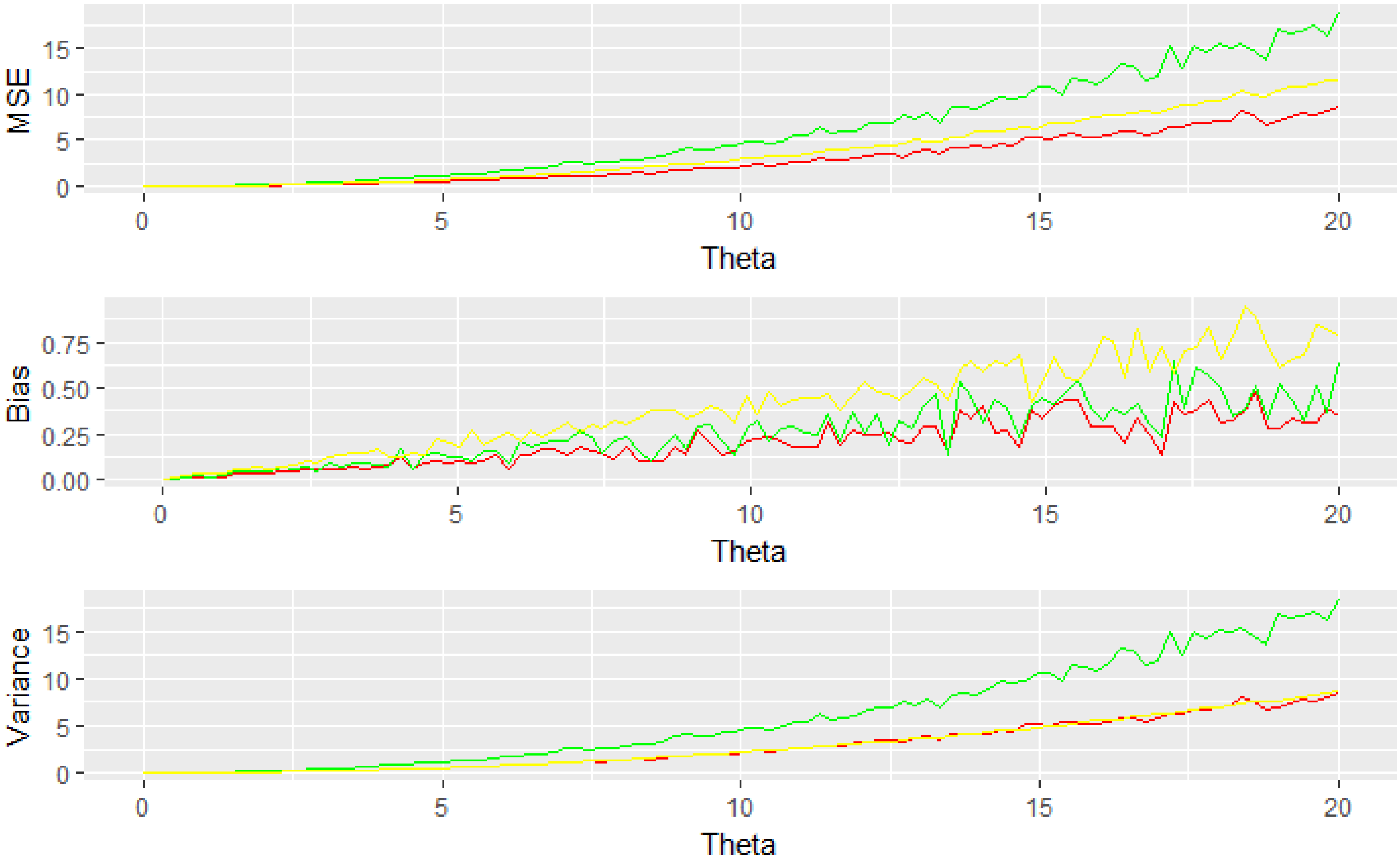
Estimates MSE, bias, variance as θ changes



N=50 for all graphs

Estimator	
Jefrey's Prior Bayesian Estimate =	
MLE =	
Inverse of the median =	

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Delta method, using the MLE

- Using normal distribution to approximate the CI (Central limit theorem!)
- From MLE theory: $\bar{x} \sim N\left(\frac{1}{\theta}, \frac{1}{\theta^2 n}\right)$

$$g(x) = \frac{1}{x} \quad g'(x) = -1/x^2$$

$$\frac{1}{\bar{x}} \sim N(\theta, \frac{1}{\theta^2 n}(-\theta^2)^2)$$

$$\frac{1}{\bar{x}} \sim N\left(\theta, \frac{\theta^2}{n}\right) \rightarrow \frac{\sqrt{n}({}^1/_{\bar{x}} - \theta)}{\theta} \sim N(0,1)$$

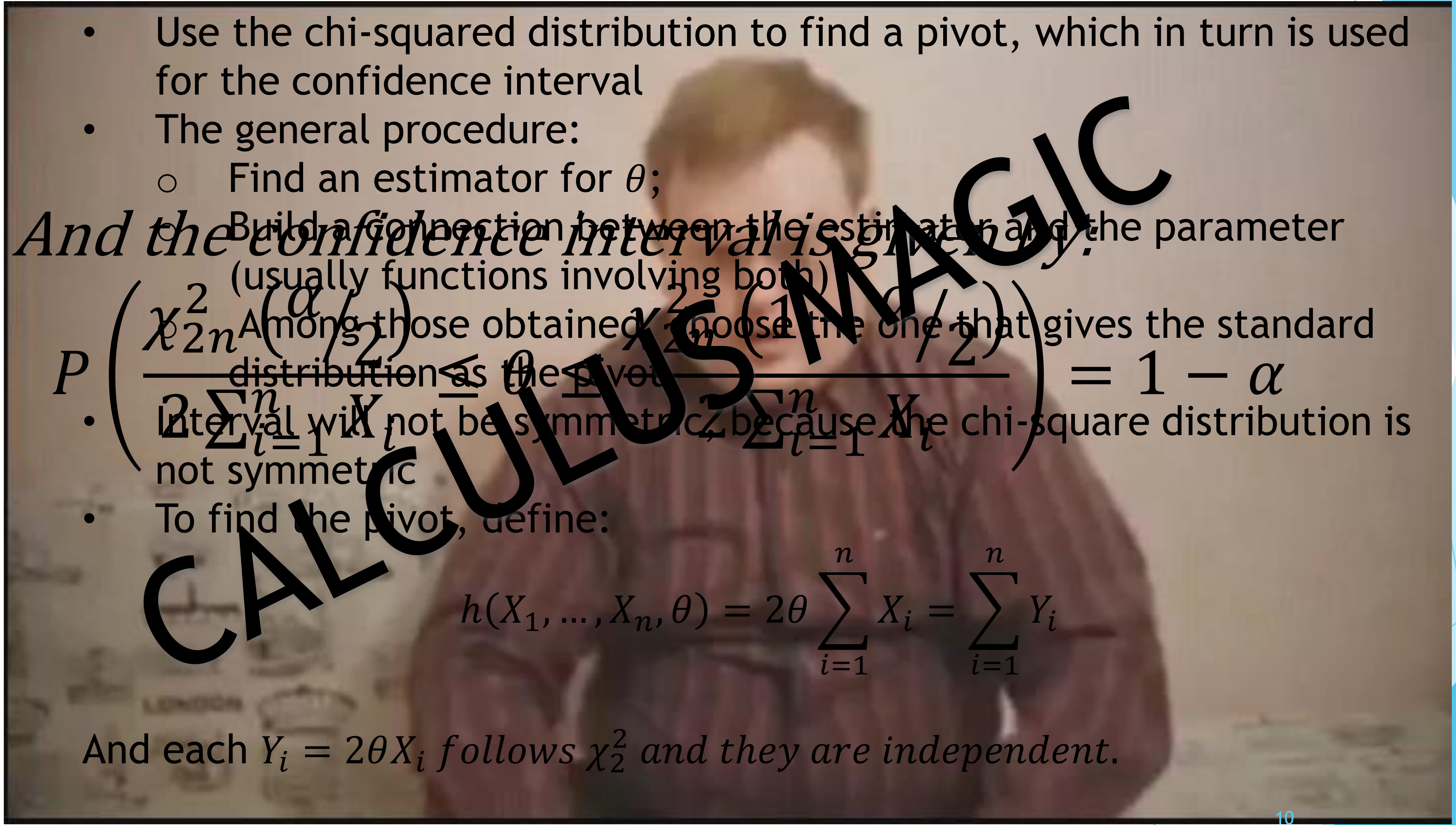
- The confidence interval is given by:

$$-1.96 \leq \frac{\sqrt{n}({}^1/_{\bar{x}} - \theta)}{\theta} \leq 1.96$$

$$\frac{-1.96\theta}{\sqrt{n}} \leq ({}^1/_{\bar{x}} - \theta) \leq \frac{1.96\theta}{\sqrt{n}}$$

$$\frac{1}{\bar{x}} - \frac{1.96}{\bar{x}\sqrt{n}} \leq \theta \leq \frac{1}{\bar{x}} + \frac{1.96}{\bar{x}\sqrt{n}}$$

Exact distribution (Chi-squared dist.)



- Use the chi-squared distribution to find a pivot, which in turn is used for the confidence interval
- The general procedure:
  - Find an estimator for θ;
  - Build a connection between the estimator and the parameter (usually functions involving both)
  - Among those obtained, choose the one that gives the standard distribution as the pivot
- Interval will not be symmetric, because the chi-square distribution is not symmetric
- To find the pivot, define:

$$h(X_1, \dots, X_n, \theta) = 2\theta \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$$

And each $Y_i = 2\theta X_i$ *follows* χ^2_2 *and they are independent.*

Then…

And the confidence interval is given by:

$$P\left(\frac{\chi^2_{2n}\left(\alpha/2\right)}{2\sum_{i=1}^{n} X_i} \le \theta \le \frac{\chi^2_{2n}\left(1-\alpha/2\right)}{2\sum_{i=1}^{n} X_i}\right) = 1 - \alpha$$

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Credible interval

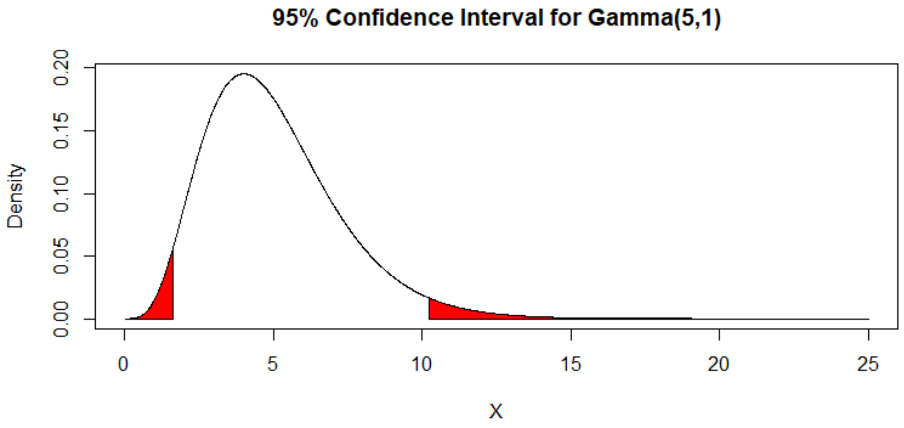
- The credible interval uses the highest posterior density (HPD), which comes from the Gamma distribution:

$$\theta|x \sim Gamma(1+n, \sum_{i=1}^{n} x_i)$$

- The credible interval follows:

$$1-\alpha = Pr\left[v_{\frac{\alpha}{2}} < \theta < v_{1-\frac{\alpha}{2}}\right]$$

Where $v_{\frac{\alpha}{2}}$ and $v_{1-\frac{\alpha}{2}}$ are quantiles of the Gamma distribution



Using the credible interval, there is a 95% chance the true value falls in the interval range, without repeated sampling.

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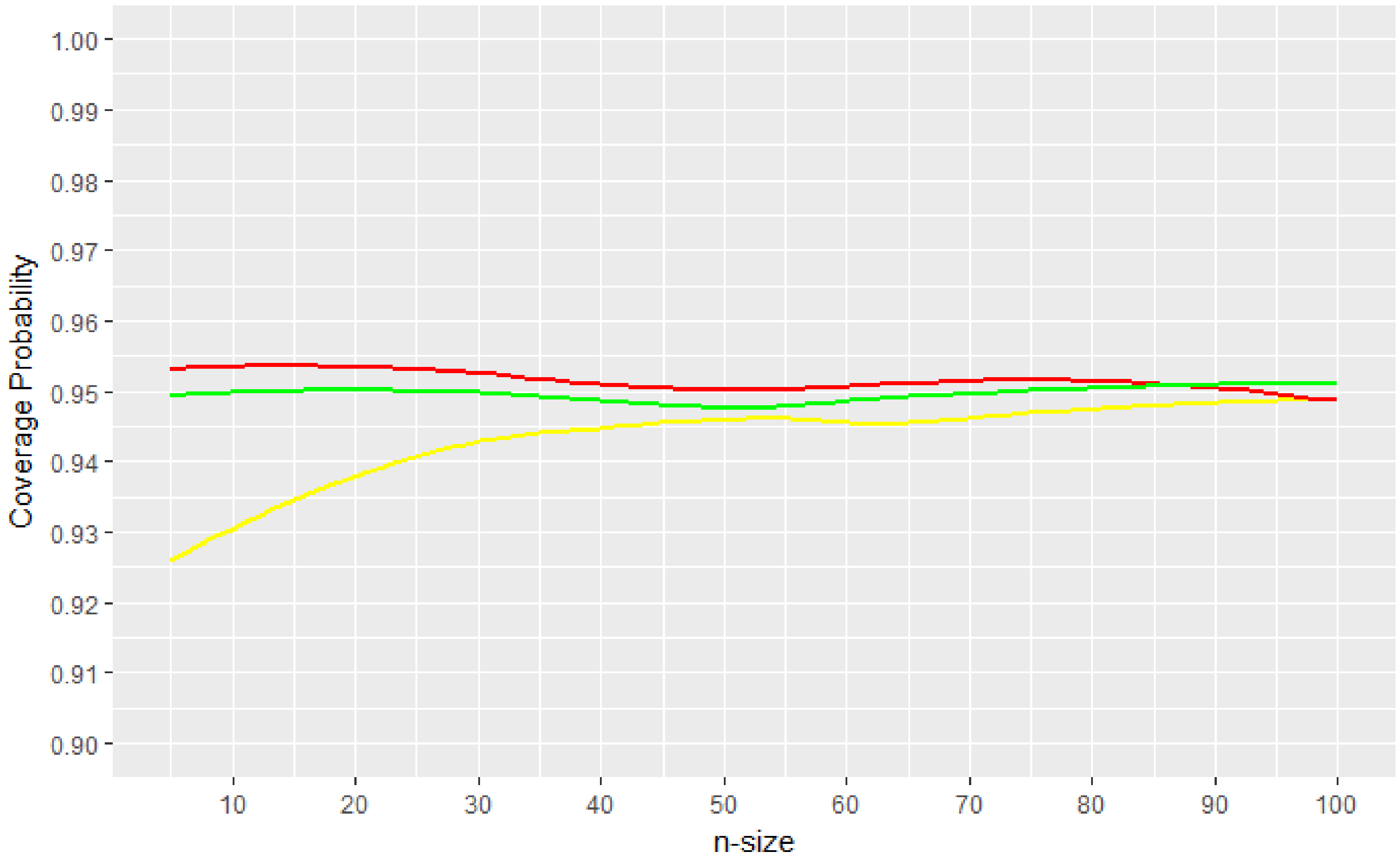
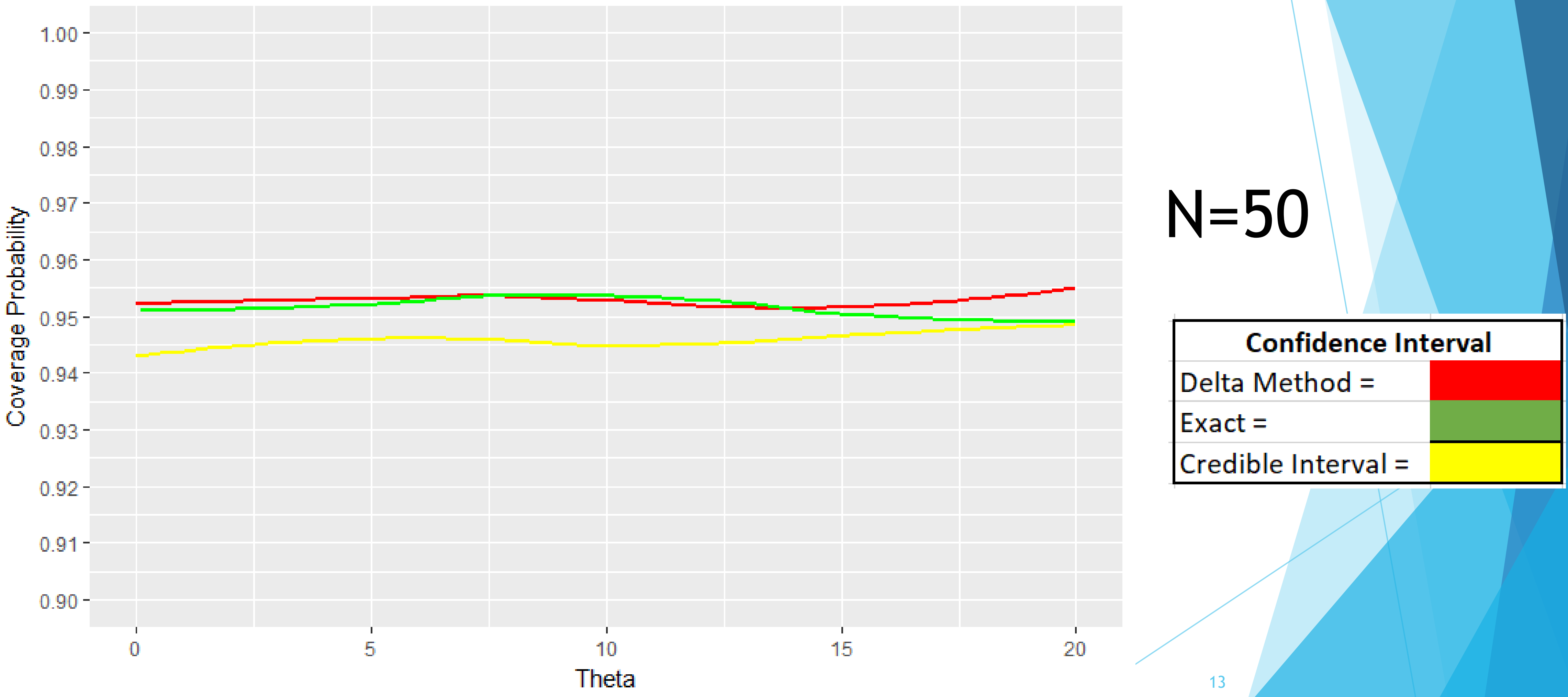
Confidence Interval Coverage as n increases



$\theta=5$

Confidence Interval	
Delta Method =	
Exact =	
Credible Interval =	

Confidence Interval Coverage with changing θ



N=50

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Conclusions

- Estimators:
  - All estimators are asymptotically unbiased
  - For small sample sizes, the inverse of the median is not a good estimator
  - Jeffrey's and the inverse of the median are biased for small sample sizes
  - Large θ results in poor estimates for all estimators
- Confidence Interval:
  - The credible interval is liberal for small sample sizes; yet gets close to .95 as n increases
  - Because the exponential distribution has a continuous outcome, the exact confidence interval is optimally conservative
  - The coverage probabilities begin to converge for n>45
  - For a fixed sample size, the delta and exact method perform about the same until θ = 15
  - The credible interval appears more liberal, yet seems to approach the same coverage probability as the exact method as θ increases