

E9 231: Digital Array Signal Processing

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1 Topics

- Adaptive Beamforming
- reintroduced Spatial Spectral Matrix Estimation
- Sample Matrix Inversion (SMI)

1.1 Spatial Spectral Matrix

$$C_X = \frac{1}{K} \sum_{k=1}^K X_k X_k^H \quad (1)$$

is an estimate of S_X . In the case of Gaussian noise, this is the MLE estimate of S_X .

$$p_X(x_k) = \frac{1}{\pi^N |S_X|} e^{-x_k^H S_X^{-1} x_k} \quad (2)$$

$$L \triangleq - \sum_{k=1}^K X_k^H S_X^{-1} X_k - K \ln |S_X| \quad (3)$$

The MLE satisfies

$$\frac{\partial L}{\partial S_X} \Big|_{S_X = \hat{S}_X} = 0 \quad (4)$$

$$L \triangleq - \sum_{k=1}^K \text{tr} \left(S_x^{-1} X_k X_k^H \right) - K \ln |S_X| = -K \text{tr} \left(S_x^{-1} C_X \right) - K \ln |S_X| \quad (5)$$

- Some results used:

$$\frac{\partial}{\partial X} \text{tr}(X^{-1} A) = \left(-X^{-1} A X^{-1} \right)^T \quad (6)$$

$$\frac{\partial}{\partial X} \ln |X| = \left(X^{-1} \right)^T \quad (7)$$

$$\frac{\partial L}{\partial S_X} = K \left(\hat{S}_X^{-1} C_X \hat{S}_X^{-1} \right)^T - K \left(\hat{S}_X^{-1} \right)^T \quad (8)$$

$$\Rightarrow \hat{S}_X^{-1} C_X \hat{S}_X^{-1} = \hat{S}_X^{-1} \quad (9)$$

$$\Rightarrow \hat{S}_X = C_X \quad (10)$$

So C_X is the MLE of S_X . It is hermitian symmetric and positive definite with w.p.1 if $K \geq N$

$$C_K \triangleq K C_X = \sum_{k=1}^K X_k X_k^H \quad (11)$$

C_K has a complex Wishart distribution.

$$p_{C_k}(C_k) = \frac{|C_k|^{K-N}}{\Gamma_N(k)|S_X|^K} e^{\text{tr}(-S_X^{-1}C_k)} \quad (12)$$

$$\Gamma_N(k) = \pi^{N \frac{(N-1)}{2}} \prod_{j=0}^{N-1} \Gamma(k-j) \quad (13)$$

Complex Wishart density, $W_N(k, S_X)$ is a generalisation of χ^2 density. When $N = 1$

$$Y \triangleq \sum_{k=1}^K |X_k|^2 / \sigma_X^2 \quad (14)$$

$$p_Y(y) = \frac{y^{K-1} e^{-y}}{\Gamma(k)} \quad (15)$$

- Some useful property of Wishart distribution:

1. If a random variable $a \in C^N$ s.t $Pr(a = 0) = 0$

$$y_1 \triangleq \frac{a^H C_k a}{a^H S_X a} \quad (16)$$

is having χ_K^2 distribution and is independent of a

$$y_2 \triangleq \frac{a^H S_X^{-1} a}{a^H C_k^{-1} a} \quad (17)$$

is having χ_{N-K+1}^2 distribution and is independent of a

2. If $B \in C^{N \times M}$ is of rank M

- (a) $B^H C_k B$ is $W_M(K, B^H S_X B)$
- (b) $[B^H C_k B]^{-1}$ is $W_M(K - N + M, [B^H C_k B]^{-1})$

- Performance improved by Forward Backward averaging and diagonal loading.

1.2 Sample Matrix Inversion

$$\hat{S}_X = \frac{1}{K} \sum_{k=1}^K X_k X_k^H \quad (18)$$

$$\hat{S}_N = \frac{1}{K} \sum_{k=1}^K N_k N_k^H \quad (19)$$

$$\hat{w}_{mvdr,SMI} = \Lambda_{SMI} \hat{S}_N^{-1} V_s \quad (20)$$

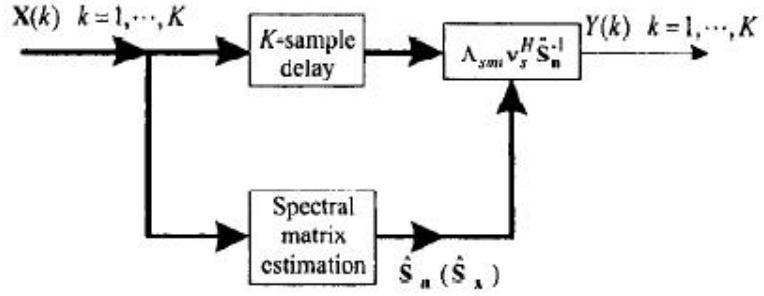


Figure 1: SMI Beamformer

$$\Lambda_{SMI} \triangleq \frac{1}{V_s^H \hat{S}_N^{-1} V_s} \quad (21)$$

Here we assumed that V_s is known.

$$\hat{w}_{mpdr,SMI} = \frac{\hat{S}_X^{-1} V_s}{V_s^H \hat{S}_X^{-1} V_s} \quad (22)$$

$$SINR_{SMI} = \frac{\sigma_s^2 |\hat{w}_{mvdr,SMI}^H V_s|^2}{\hat{w}_{mvdr,SMI}^H \hat{S}_n \hat{w}_{mvdr,SMI}} \quad (23)$$

$$SINR_{mvdr} = \frac{\sigma_s^2 |\hat{w}_{mvdr}^H V_s|^2}{\hat{w}_{mvdr}^H \hat{S}_n \hat{w}_{mvdr}} \quad (24)$$

$$\rho(k) \triangleq \frac{SINR_{SMI}}{SINR_{mvdr}} \quad (25)$$

where $0 \leq \rho(k) \leq 1$. It can be shown that if noise interference is gaussian, $f_\rho(\rho)$ is β distributed.

$$f_\rho(\rho) = \frac{K!(1-p)^{N-2} \rho^{K+1-N}}{(N-2)!(K+1-N)!} \quad (26)$$

where $N \geq 2$, $K \geq N$ and $0 \leq \rho \leq 1$

$$E(\rho) = \frac{K+2-N}{K+1} \quad (27)$$

$$Var(\rho) = \frac{(K+2-N)(N-1)}{(K+1)^2(K+2)} \quad (28)$$

$$E(SINR_{SMI}) = \frac{K+2-N}{K+1} SINR_{mvdr} \quad (29)$$

If we desire $E(SINR_{SMI}) = \alpha SINR_{mvdr}$

$$K = \frac{N-2+\alpha}{1-\alpha} \approx \frac{N}{1-\alpha} \quad (30)$$

Ex: $K = 2N - 3$

$E(SINR_{SMI})$ is $\approx 3dB$ below $SINR_{mvdr}$ To achieve $\alpha = 0.95$ we need $K = 20N$. As a rule of thumb,

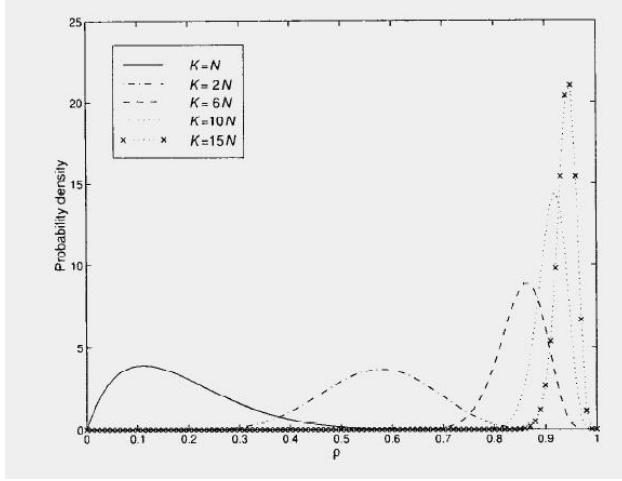


Figure 2: Probability density of ρ

$K = 2N$ is acceptable.

Another approach is, pick K s.t $Pr\{\rho < 1 - \delta\} < \epsilon$, for some δ, ϵ . It can be shown that,

$$Pr\{\rho < 1 - \delta\} = \sum_{m=0}^{N-2} KC_m \delta^m (1 - \delta)^{K-m} \quad (31)$$

Ex: $N > 3$ and $K \geq 3N$, $Pr\{\rho < 0.5\} < 0.0196 \approx 2\%$

and for $K \geq 4N$, $Pr\{\rho < 0.5\} < 0.0032$

For MPDR case, , we can derive pdf of ,

$$\eta \triangleq \frac{SINR_{mpdr,SMI}}{SINR_{mpdr}} \quad (32)$$

$$E\{\eta\} = \frac{K - N + 2}{K + 1} \frac{1}{1 + SINR_{mpdr} \left(\frac{N-1}{K+1} \right)} \quad (33)$$

If $K \approx 2N$ and $\frac{SINR_{mpdr}}{N}$ is small compared to 1, $E\{\eta\} \approx \frac{1}{2}$ which is 3dB below $SINR_{mpdr}$

- Diagonal loading: Add a scaled Identity matrix to \hat{S}_X or \hat{S}_N
 1. provides robustness against noise especially in a weak interference environment.
 2. can have $K < N$
 3. choosing the diagonal loading depends on prior knowledge about signal/noise levels.