Delay-and-Sum Beamforming for Plane Waves

ECE 6279: Spatial Array Processing
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Lecture 6

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Where We Are in J&D

- **Lecture material drawn from:**
  - Secs. 4.1, 4.1.2, 4.2.1 (up to but not including "Point Focusing" part on p. 123), 4.2.3
- **Next lecture:**
  - Secs. 4.1.1, 4.1.3, 4.2.1 ("Point Focusing" part on p. 123)

Integrating Across Apertures

- **Here's one way aperture smoothing functions show up**
- **Typically integrate across the aperture**

$$z(t) = \int_{-\infty}^{\infty} w(\vec{x}) f(\vec{x}, t) d\vec{x}$$

- **"Input" a monochromatic plane wave to the "system"**

$$f(\vec{x}, t) = \exp\{ j(\omega^0 t - \vec{k}^0 \cdot \vec{x}) \}$$

$$z(t) = \exp(j\omega^0 t) \underbrace{\int_{-\infty}^{\infty} w(\vec{x}) \exp(-j\vec{k}^0 \cdot \vec{x}) d\vec{x}}_{W(-\vec{k}^0)}$$

Delay-and-Sum Beamforming

- Array of M sensors at positions $\vec{x}_0 \ldots \vec{x}_{M-1}$
- For convenience, put the phase center at the origin

$$\sum_{m=0}^{M-1} \vec{x}_m = \vec{0}$$

- Delay-and-sum beamforming

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

Beamforming for Plane Waves

$$f(\vec{x},t) = s(t - \vec{\alpha}^0 \cdot \vec{x})$$

$$\vec{\alpha}^0 = \vec{\zeta}^0 / c$$

$$y_m(t) = s(t - \vec{\alpha}^0 \cdot \vec{x}_m)$$

$$z(t) = \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

$$= \sum_{m=0}^{M-1} w_m s(t - \Delta_m - \vec{\alpha}^0 \cdot \vec{x}_m)$$

When Things Line Up

$$z(t) = \sum_{m=0}^{M-1} w_m s(t - \Delta_m - \vec{\alpha}^0 \cdot \vec{x}_m)$$

- **If we pick**

$$\Delta_m = -\vec{\alpha}^0 \cdot \vec{x}_m = \frac{-\vec{\zeta}^0 \cdot \vec{x}_m}{c}$$

then we get the signal back!

$$z(t) = \sum_{m=0}^{M-1} w_m s(t) = s(t) \cdot \left[\sum_{m=0}^{M-1} w_m \right]$$

When They Don't

$$z(t) = \sum_{m=0}^{M-1} w_m s(t - \Delta_m - \vec{\alpha}^0 \cdot \vec{x}_m)$$

- **More generally, if we pick**

$$\Delta_m = -\vec{\alpha} \cdot \vec{x}_m = \frac{-\vec{\zeta} \cdot \vec{x}_m}{c}$$

then we get a degraded version of the signal

$$z(t) = \sum_{m=0}^{M-1} w_m s(t + (\vec{\alpha} - \vec{\alpha}^0) \cdot \vec{x}_m)$$

Strategy for Parameter Estimation

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m (t - \Delta_m)$$

$$\Delta_m = -\vec{\alpha} \cdot \vec{x}_m = \frac{-\vec{\xi} \cdot \vec{x}_m}{c}$$

- **Find parameter that maximizes energy in $z(t)$**
  - Radar and sonar: If you know c, sweep $\vec{\xi}$ to find direction of arrival
  - Seismology: If you know $\vec{\xi}$, sweep c to find wave speed (determines material properties)

Monochromatic Plane Waves (1)

$$f(\vec{x},t) = \exp\{j\omega^0(t-\vec{\alpha}^0\cdot\vec{x})\}$$

$$= s(t-\alpha^0\cdot\vec{x})$$

where $s(t) = \exp(j\omega^0 t)$

- **Plane wave delay-and-sum beamformer response**

$$z(t) = \sum_{m=0}^{M-1} w_m s(t+(\vec{\alpha}-\vec{\alpha}^0)\cdot\vec{x}_m)$$

$$= \sum_{m=0}^{M-1} w_m \exp(j\omega^0[t+(\vec{\alpha}-\vec{\alpha}^0)\cdot\vec{x}_m])$$

Monochromatic Plane Waves (2)

$$z(t)=\sum_{m=0}^{M-1} w_m \exp(j\omega^0[t+(\vec{\alpha}-\vec{\alpha}^0)\cdot\vec{x}_m])$$

$$=\left[\sum_{m=0}^{M-1} w_m \exp(j\omega^0(\vec{\alpha}-\vec{\alpha}^0)\cdot\vec{x}_m)\right]\exp(j\omega^0 t)$$

Recall $\vec{k}^0=\omega^0\vec{\alpha}^0$

$$=\left[\sum_{m=0}^{M-1} w_m \exp(j(\omega^0\vec{\alpha}-\vec{k}^0)\cdot\vec{x}_m)\right]\exp(j\omega^0 t)$$

Monochromatic Plane Waves (3)

$$z(t) = \left[\sum_{m=0}^{M-1} w_m \exp(j(\omega^0 \vec{\alpha} - \vec{k}^0) \cdot \vec{x}_m) \right] \exp(j\omega^0 t)$$

$$= W(\omega^0 \vec{\alpha} - \vec{k}^0) \exp(j\omega^0 t)$$

where the aperture smoothing function is

$$W(\vec{k}) = \sum_{m=0}^{M-1} w_m \exp(j\vec{k} \cdot \vec{x}_m)$$

Also called the array pattern

General Wavefields

$$f(\vec{x},t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\vec{k},\omega) \exp\{j(\omega t - \vec{k}\cdot\vec{x})\} \, d\vec{k} \, d\omega$$

Delay-and-sum beamformer focused on $\vec{\alpha}$

$$z(t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\vec{k},\omega) W(\omega\vec{\alpha} - \vec{k}) \exp(j\omega t) \, d\vec{k} \, d\omega$$

General Plane Waves (1)

$$f(\vec{x},t) = s(t - \vec{\alpha}^0 \cdot \vec{x})$$

$$F(\vec{k},\omega) = S(\omega)(2\pi)^3 \delta(\vec{k} - \omega\vec{\alpha}^0)$$

$$z(t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\vec{k},\omega) W(\omega\vec{\alpha} - \vec{k}) \exp(j\omega t)\, d\vec{k}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) W(\omega[\vec{\alpha} - \vec{\alpha}^0]) \exp(j\omega t)\, d\omega$$

$$Z(\omega) = S(\omega) W(\omega[\vec{\alpha} - \vec{\alpha}^0])$$

General Plane Waves (2)

$$Z(\omega) = S(\omega) W(\omega[\vec{\alpha} - \vec{\alpha}^0])$$

- **If we pick** $\vec{\alpha} = \vec{\alpha}^0$

$$Z(\omega) = S(\omega) W(0)$$

$$z(t) = s(t) W(0)$$

we get the original signal back!

- **If we pick** $\vec{\alpha} \neq \vec{\alpha}^0$ **we get a filtered version**

Uniform Linear Array (1)

- **From earlier slide, the response of delay-and-sum beamformer (tuned to $\vec{\alpha}$) to a monochromatic plane wave is**

$$z(t) = W(\omega^0\vec{\alpha} - \vec{k}^0)\exp(j\omega^0 t)$$

- **For a linear uniform array from the last lecture**

$$W(\vec{k}) = \frac{\sin(Mk_x d/2)}{\sin(k_x d/2)}$$

$$W(\omega^0\vec{\alpha} - \vec{k}^0) = \frac{\sin(M[\omega^0\alpha_x - k_x^0]d/2)}{\sin([\omega^0\alpha_x - k_x^0]d/2)}$$

Uniform Linear Array (2)

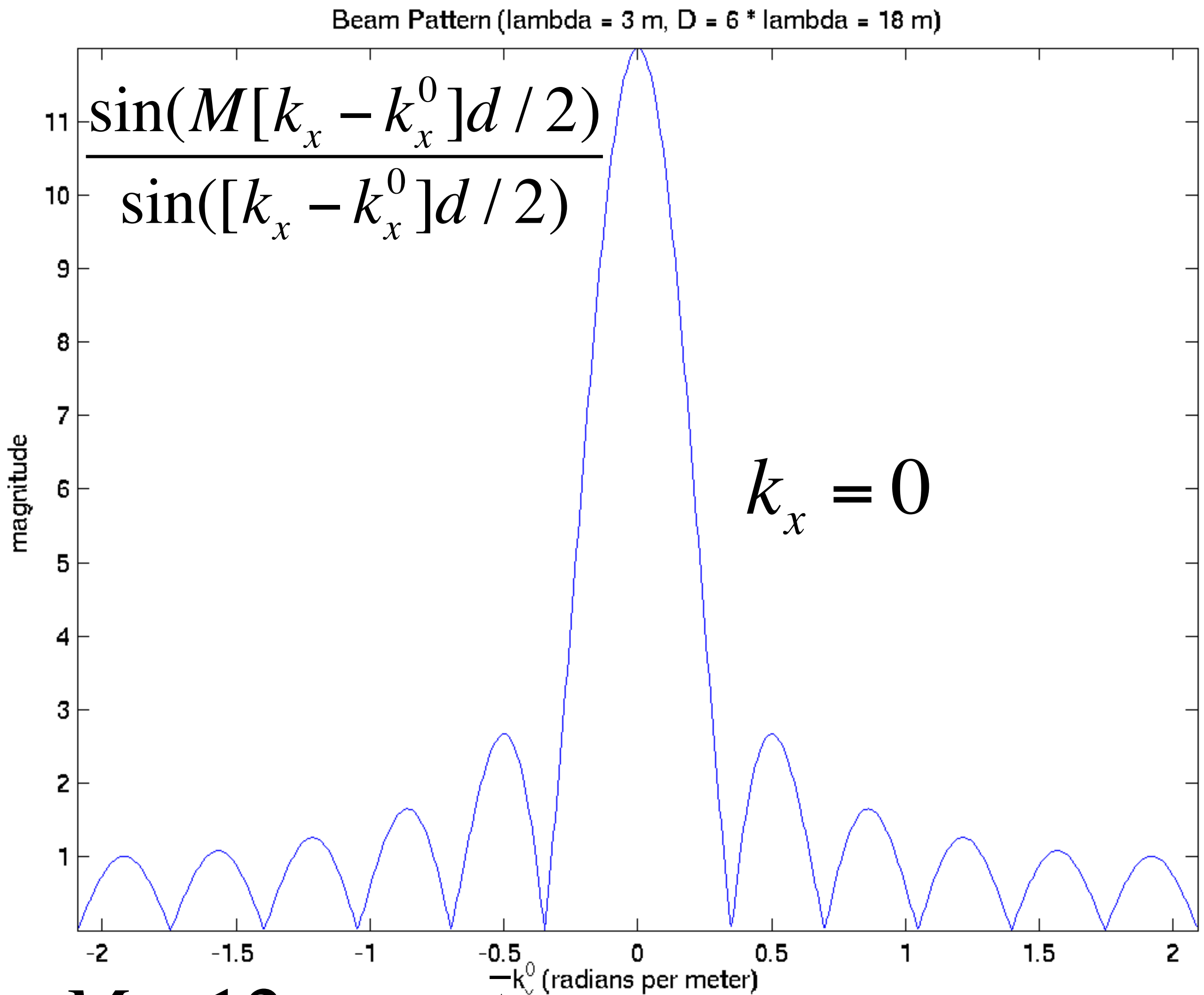
- **Using** $k_x = \omega^0 \alpha_x$

$$W(k_x - k_x^0) = \frac{\sin(M[k_x - k_x^0]d/2)}{\sin([k_x - k_x^0]d/2)}$$

- **In terms of angles, let** $k_x = -(2\pi/\lambda)\sin(\phi)$

$$W(k_x - k_x^0) = \frac{\sin\left(M\dfrac{\pi}{\lambda}[\sin\phi^0 - \sin\phi]d\right)}{\sin\left(\dfrac{\pi}{\lambda}[\sin\phi^0 - \sin\phi]d\right)}$$

Beam Pattern (Boresight)



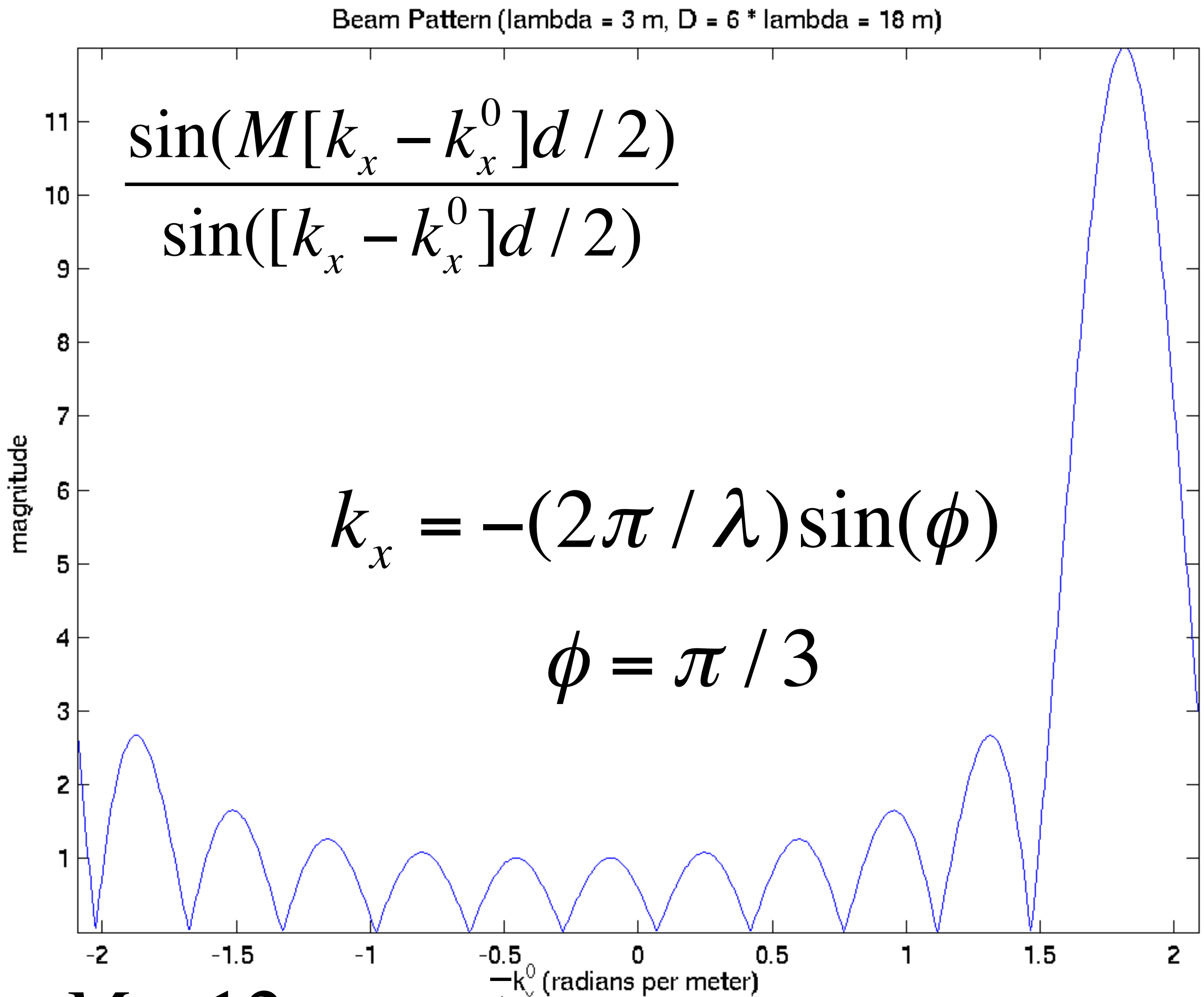
$$\frac{\sin(M[k_x - k_x^0]d/2)}{\sin([k_x - k_x^0]d/2)}$$

$$k_x = 0$$

$$M = 12$$

(notice the negative sign)

Beam Pattern (60°)



$$\frac{\sin(M[k_x - k_x^0]d/2)}{\sin([k_x - k_x^0]d/2)}$$

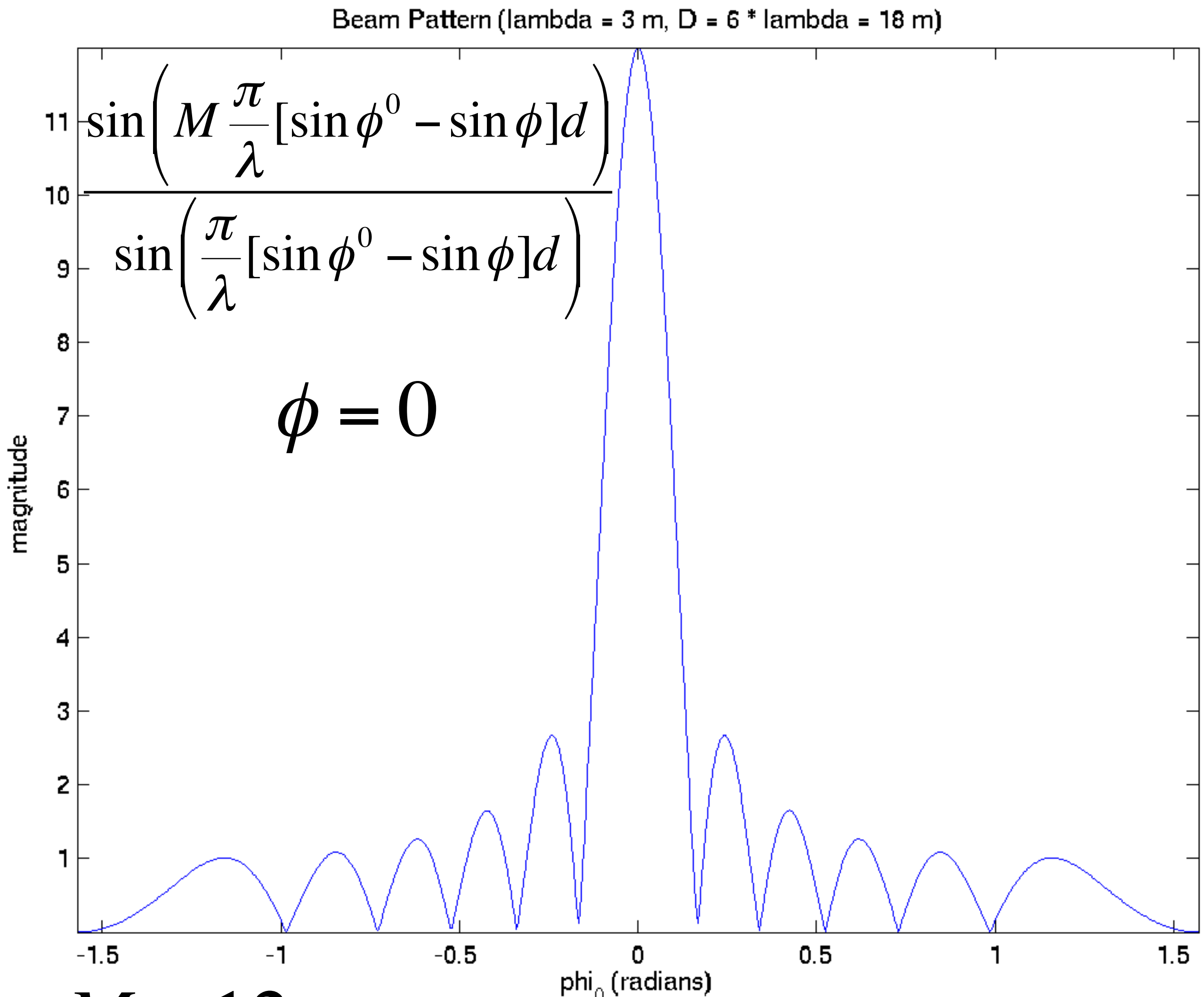
$$k_x = -(2\pi/\lambda)\sin(\phi)$$

$$\phi = \pi/3$$

$$M = 12$$

(notice the negative sign)

Beam Pattern (Boresight)

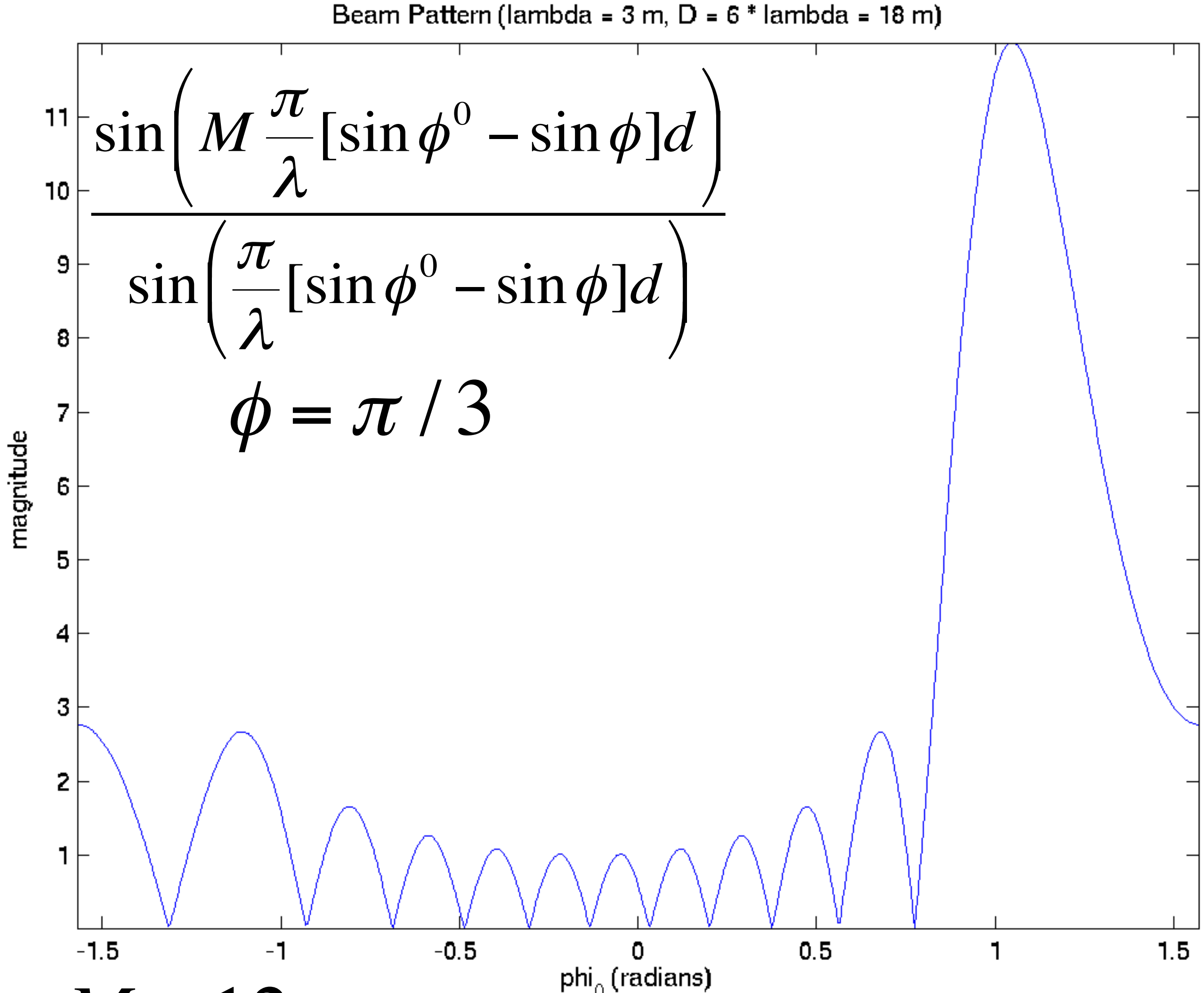


$$\frac{\sin\left(M\frac{\pi}{\lambda}[\sin\phi^0 - \sin\phi]d\right)}{\sin\left(\frac{\pi}{\lambda}[\sin\phi^0 - \sin\phi]d\right)}$$

$$\phi = 0$$

$$M = 12$$

Beam Pattern (60°)



$$\frac{\sin\left(M\frac{\pi}{\lambda}[\sin\phi^0-\sin\phi]d\right)}{\sin\left(\frac{\pi}{\lambda}[\sin\phi^0-\sin\phi]d\right)}$$

$$\phi=\pi/3$$

$$M=12$$

Terminology

Beampattern: fix $\vec{\alpha} = \vec{k} / \omega = k\vec{\xi} / \omega, \quad k \equiv \left|\vec{k}\right|$

$$func(\omega^0, \vec{k}^0) = W(\boxed{\omega^0}\vec{\alpha} - \boxed{\vec{k}^0}) = W\left(\omega^0 \frac{\vec{k}}{\omega} - \vec{k}^0\right)$$

If $\vec{k} = k^0\vec{\xi}, \quad \omega = \omega^0 : \quad func(\vec{\xi}^0) = W(k^0[\vec{\xi} - \boxed{\vec{\xi}^0}])$

Steered response: fix $\omega^0, \vec{k}^0$

$$func(\vec{\alpha}) = W(\omega^0\boxed{\vec{\alpha}} - \vec{k}^0) = W\left(\omega^0 \frac{\vec{k}}{\omega} - \vec{k}^0\right)$$

If $\vec{k} = k^0\vec{\xi}, \quad \omega = \omega^0 : \quad func(\vec{\xi}) = W(k^0[\boxed{\vec{\xi}} - \vec{\xi}^0])$