E9 231: Digital Array Signal Processing

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Examples

Probelm 2.4.6 of textbook:
Consider the non-uniform 4-element linear array whose sensor separations are d, $3d$, $2d$ where $d = \lambda/2$. The sensor outputs are weighted uniformly.

1. Compute the beam pattern and BW_{NN}.
2. Compare the results in part 1 with a uniform 7-element array with $d = \lambda/2$. Discuss the behavior of the main lobe and the sidelobes.

Solution:

1.

$$\mathbf{V_k}(\mathbf{k}) = \begin{bmatrix} e^{-j\frac{2\pi}{\lambda}3d\cos\theta} \\ e^{-j\frac{2\pi}{\lambda}2d\cos\theta} \\ e^{j\frac{2\pi}{\lambda}d\cos\theta} \\ e^{j\frac{2\pi}{\lambda}3d\cos\theta} \end{bmatrix} = \begin{bmatrix} e^{-j3\pi\cos\theta} \\ e^{-j2\pi\cos\theta} \\ e^{j\pi\cos\theta} \\ e^{j3\pi\cos\theta} \end{bmatrix}$$

$$\mathbf{w} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]^T$$

$$B(\theta, \phi) = \mathbf{w}^H \mathbf{V_k}(\mathbf{k}) = \frac{1}{4}\left[e^{-j3\pi\cos\theta} + e^{-j2\pi\cos\theta} + e^{j\pi\cos\theta} + e^{j3\pi\cos\theta}\right]$$

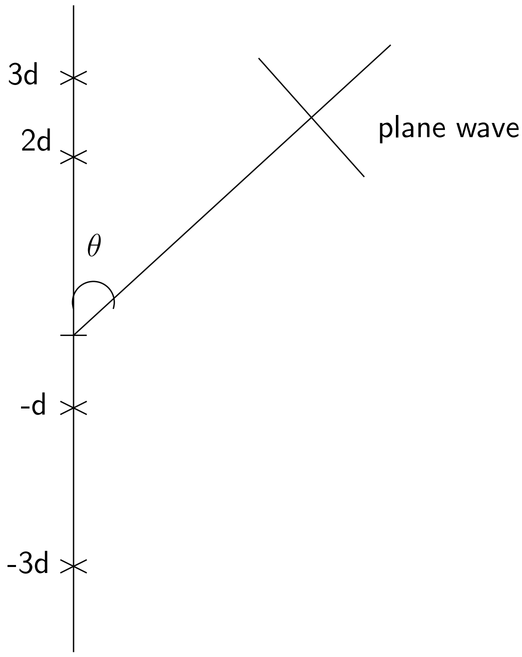

Figure 1: Figure for problem 2.4.6

Using MATLAB, we can find BW_{NN} in the u-space ($u = \cos\theta$) to be the following:

$$BW_{NN} = 0.441$$

2. For a 7-element SLA with uniform weighting, we have

$$\begin{aligned} B_\theta(\theta) &= \frac{\sin\left(\frac{2\pi d}{\lambda}\cos\theta\cdot\frac{7}{2}\right)}{7\sin\left(\frac{2\pi d}{\lambda}\cos\theta\cdot\frac{1}{2}\right)} = \frac{\sin\left(\frac{7\pi}{2}\cos\theta\right)}{7\sin\left(\frac{\pi}{2}\cos\theta\right)} \\ B_u(u) &= \frac{\sin\left(\frac{7\pi}{2}u\right)}{7\sin\left(\frac{\pi}{2}u\right)} \end{aligned}$$

$B_u(u) = 0$ for $u = \frac{2}{7}m, m = 1, \ldots, 6$, and thus we have

$$BW_{NN} = \frac{4}{7} = 0.57$$

Properties of $P_C \triangleq C(C^HC)^{-1}C^H$

Property 1 $P_C^H = \left(C(C^HC)^{-1}C^H\right)^H = C(C^HC)^{-1}C^H = P_C$.

Property 2 $P_C^2 = C(C^HC)^{-1}C^HC(C^HC)^{-1}C^H = C(C^HC)^{-1}C^H = P_C$. *By induction, we have* $P_C^k = P_C, k > 0, k$ *is an integer.*

Property 3 *If* $\mathbf{y} = P_C\mathbf{x}$*, then* $P_C\mathbf{y} = \mathbf{y}$*.*

Property 4 *If* $\mathbf{y} = P_C\mathbf{x}$*, then* $(\mathbf{y} - \mathbf{x}) \perp \mathbf{y}$*.*
Proof:

$$\begin{aligned} P_C\mathbf{x} - \mathbf{x} &= (P_C - I)\mathbf{x} \\ (\mathbf{y} - \mathbf{x})^H y &= \mathbf{x}^H(P_C - I)^H P_C\mathbf{x} \\ &= \mathbf{x}^H(P_C^2 - P_C)\mathbf{x} \\ &= \mathbf{x}^H(P_C - P_C)\mathbf{x} \quad (\textit{by property 2}) \\ &= \mathbf{0} \end{aligned} \tag{1}$$

Property 5 Spectral decomposition: *P_C has all eigenvalues equal to* 0 *or* 1*.*
Proof: *If* $\mathbf{e}$ *is a nonzero eigenvector of* P_C *with corresponding eigenvalue* λ*, then we have*

$$\begin{aligned} P_C\mathbf{e} &= \lambda\mathbf{e} \\ P_C^2\mathbf{e} &= \lambda P_C\mathbf{e} \\ P_C\mathbf{e} &= \lambda^2\mathbf{e} \quad (\textit{by property 2}) \\ \lambda\mathbf{e} &= \lambda^2\mathbf{e} \\ \mathbf{e} \neq \mathbf{0} &\Rightarrow \lambda = 0 or 1 only. \end{aligned} \tag{2}$$

Property 6 *If* $rank(P_C) = r$ *and the eigenvectors are* $\mathbf{e}_i, i = 1, \ldots, r$*, then* $P_C = \sum_{i=1}^{r} \mathbf{e}_i\mathbf{e}_i^H$*.*

Property 7 $\mathbf{e}_i$*'s form an orthonormal basis for the column space of* P_C*. For example,*

$$\begin{aligned} P_C\mathbf{x} &= \sum_{i=1}^{r} \mathbf{e}_i\mathbf{e}_i^H\mathbf{x} \\ &= \sum_{i=1}^{r} \left(\mathbf{e}_i^H\mathbf{x}\right)\mathbf{e}_i \\ &= \sum_{i=1}^{r} x_i\mathbf{e}_i \end{aligned}$$

Property 8 *Decomposition of the identity matrix* I*:*

$$\begin{aligned} &I = P_C + (I - P_C) \\ &P_C = P_C + (I - P_C)P_C \\ &\Rightarrow (I - P_C)P_C = O \\ &\Rightarrow \mathbf{x}^H(I - P_C)P_C\mathbf{x} = 0 \\ &\Rightarrow P_C\mathbf{x} \perp (I - P_C)\mathbf{x} \end{aligned} \tag{3}$$

i.e., the column space of P_C *and the column space of* $(I - P_C)$ *are orthogonal subspaces.* $(I - P_C)$ *is also a projection matrix,* $rank(I - P_C) = n - r$*, where* r *is the rank of* P_C*.*

STATISTICAL SIGNAL PROCESSING

Bandpass Signal

A real signal $f(t)$ is a bandpass signal if $\exists$ real, low-pass signals $a(t)$ and $b(t)$ such that

$$f(t) = a(t)\cos\omega_c t + b(t)\sin\omega_c t,$$

where ω_c is the center frequency of $f(t)$.

$$\tilde{f}(t) \triangleq a(t) + jb(t),$$

and $a(t)$ and $b(t)$ are called the in-phase and quadrature components of $f(t)$ respectively.

$$f(t) = Re\{\tilde{f}(t)e^{-j\omega_c t}\}$$

$a(t)$ and $b(t)$ are assumed to be of zero means henceforth. If $f(t)$ is *wide-sense stationary* (WSS), then $a(t)$ and $b(t)$ are also WSS, and

$$\begin{aligned} R_{aa}(\tau) &= R_{bb}(\tau) \\ R_{ab}(\tau) &= -R_{ba}(\tau) \end{aligned}$$

where $R_{aa}(\tau) \triangleq E[a(t)a(t+\tau)]$, $R_{bb}(\tau) \triangleq E[b(t)b(t+\tau)]$, $R_{ab}(\tau) \triangleq E[a(t)b(t+\tau)]$, and $R_{ba}(\tau) \triangleq E[b(t)a(t+\tau)]$. Often, we let $R_{ab}(\tau) = 0$, i.e., $a(t)$ and $b(t)$ are uncorrelated.

If $f(t)$ is Gaussian, then

- $a(t)$ and $b(t)$ are jointly Gaussian.
- If $R_{ab}(\tau) = 0$, then $a(t)$ and $b(t)$ are independent.
- If $a(t)$ and $b(t)$ are independent and have the same variance, then $\tilde{f}(t)$ is circularly symmetric, complex Gaussian.