Cramér-Rao Bounds for Arrays

ECE 6279: Spatial Array Processing
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Lecture 24

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References

- P. Stoica and A. Nehorai, "MUSIC, Maximum Likelihood, and Cramer-Rao Bound", *IEEE Trans. on Acoustics, Speech, and Signal Proc*., Vol. 37, No. 5, May 1989, pp. 720-741.
- P. Stoica and A. Nehorai, "MUSIC, Maximum Likelihood, and Cramer-Rao Bound: Further Results and Comparisons", *IEEE Trans. on Acoustics, Speech, and Signal Proc*., Vol. 38, No. 12, Dec. 1990, pp. 2140-2150.
- Van Trees *Vol. IV: Optimum Array Processing*

Review of "Deterministic Gaussian" Model

- N_s sources in additive noise

$$\underline{\mathbf{y}}(l) = \sum_{n=1}^{N_s} \mathbf{e}(\theta_n) s_n(l) + \underline{\mathbf{n}}(l)$$

$$\underline{\mathbf{n}} \sim CN(0, \mathbf{K_n})$$

$$\mathbf{s}(l) = \begin{bmatrix} s_1(l) \\ \vdots \\ s_{N_s}(l) \end{bmatrix}$$

Modeled as deterministic parameters we need to estimate

Maximum-Likelihood Procedure

- Luckily, it so happened that if we fix a particular Θ, we could find a closed form solution for the maximizing $s(l)$ and plug that into the likelihood; turns out we want to maximize

$$\mathrm{tr}\left\{\mathbf{D}(\Theta)[\mathbf{D}^H(\Theta)\mathbf{D}(\Theta)]^{-1}\mathbf{D}^H(\Theta)\hat{\mathbf{R}}_{\mathbf{y}}\right\}$$

- No closed form solution

Assumption for This Presentation

$$\mathbf{y}(l) = \mathbf{D}(\boldsymbol{\Theta})\mathbf{s}(l) + \mathbf{n}(l)$$

$$\mathbf{D}(\boldsymbol{\Theta}) = \begin{bmatrix} \mathbf{e}(\phi_1) & \cdots & \mathbf{e}(\phi_{N_s}) \end{bmatrix}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} \phi_1 & \cdots & \phi_{N_s} \end{bmatrix}$$ **(special case)**

- **For this lecture, assume that each signal direction is parameterized by one parameter**

Additional Assumptions for This Lecture

- This lecture will assume that

$$\mathbf{K_n} = \sigma^2\mathbf{I}$$

- We will also assume that σ^2 is unknown
- Turns out we can treat the variance like the signals; ML procedure winds up not changing!

A Wanton Abuse of Notation

$$\mathbf{D}(\Theta) = \left[\mathbf{e}(\phi_1) \ \cdots \ \mathbf{e}(\phi_{N_s}) \right]$$

$$\frac{d\mathbf{D}(\Theta)}{d\Theta} \equiv \left[\frac{d\mathbf{e}(\phi_1)}{d\phi_1} \ \cdots \ \frac{d\mathbf{e}(\phi_{N_s})}{d\phi_{N_s}} \right]$$

Some More Notation

$$\mathbf{S}(l) \equiv \begin{bmatrix} s_1(l) & & \\ & \ddots & \\ & & s_{N_s}(l) \end{bmatrix}$$

F.I.M. for Deterministic Gaussian Model

- **Stoica & Nehorai, M, ML, CRB, Eq. 4.1:**

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} \mathrm{Re}\left\{ \mathbf{S}^H(l) \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \times \right.$$

$$[\mathbf{I} - \mathbf{D}(\Theta)\{\mathbf{D}^H(\Theta)\mathbf{D}(\Theta)\}^{-1}\mathbf{D}^H(\Theta)] \times$$

$$\left. \frac{d\mathbf{D}(\Theta)}{d\Theta} \mathbf{S}(l) \right\}$$

Special Case: One Target, Linear Array

$$\mathbf{D}(\Theta) = \mathbf{e}(\gamma) = \begin{bmatrix} 1 \\ e^{j\gamma} \\ \vdots \\ e^{j(M-1)\gamma} \end{bmatrix}$$

Derivative Vector in the Special Case

$$\frac{d\mathbf{D}(\Theta)}{d\Theta} = \begin{bmatrix} 0 \\ je^{j\gamma} \\ \vdots \\ j(M-1)e^{j(M-1)\gamma} \end{bmatrix}$$

Useful Tidbits (1)

$$\frac{d\mathbf{D}^H(\Theta)}{d\Theta}\frac{d\mathbf{D}(\Theta)}{d\Theta} =$$

$$\sum_{m=0}^{M-1}(-jme^{-jm\gamma})(jme^{jm\gamma}) = \sum_{m=0}^{M-1} m^2$$

$$= \frac{M(M-1)(2M-1)}{6}$$

Useful Tidbits (2)

$$\frac{d\mathbf{D}^H(\Theta)}{d\Theta}\mathbf{D}(\Theta) = \sum_{m=0}^{M-1}(-jme^{-jm\gamma})(e^{jm\gamma})$$

$$= -j\sum_{m=0}^{M-1} m \;= -j\frac{M(M-1)}{2}$$

$$\mathbf{D}^H(\Theta)\frac{d\mathbf{D}(\Theta)}{d\Theta} = j\frac{M(M-1)}{2}$$

Here's That General F.I.M. Again

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} \mathrm{Re}\left\{ \mathbf{S}^H(l) \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \times \right.$$

$$[\mathbf{I} - \mathbf{D}(\Theta)\{\mathbf{D}^H(\Theta)\mathbf{D}(\Theta)\}^{-1}\mathbf{D}^H(\Theta)] \times$$

$$\left. \frac{d\mathbf{D}(\Theta)}{d\Theta} \mathbf{S}(l) \right\}$$

Simplifying for This Special Case

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2 \times$$

$$\mathrm{Re}\left\{ \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \left[\mathbf{I} - \frac{\mathbf{D}(\Theta)\mathbf{D}^H(\Theta)}{M} \right] \frac{d\mathbf{D}(\Theta)}{d\Theta} \right\}$$

Some Minor Rearranging

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2 \times$$

$$\mathrm{Re}\left\{ \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \frac{d\mathbf{D}(\Theta)}{d\Theta} - \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \frac{\mathbf{D}(\Theta)\mathbf{D}^H(\Theta)}{M} \frac{d\mathbf{D}(\Theta)}{d\Theta} \right\}$$

Using the Useful Tidbits

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2 \times$$

$$\mathrm{Re}\left\{ \frac{M(M-1)(2M-1)}{6} \right.$$

$$\left. - \left[-j \frac{M(M-1)}{2} \right] \left[j \frac{M(M-1)}{2} \right] \Big/ M \right\}$$

After an Insane Amount of Algebra

$$F(\Theta) = \frac{M(M^2 - 1)}{6\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2$$

$$SNR = \frac{1}{L} \sum_{l=0}^{L-1} |s(l)|^2 \Big/ \sigma^2$$

$$F(\Theta) \approx \frac{M^3 L}{6} SNR$$

The Big Picture

$$\mathrm{var}_\gamma[\hat{\gamma}(\underline{y})] \geq \frac{1}{F(\gamma)}$$

$$F(\gamma) \approx \frac{M^3 L}{6} SNR$$

$$\mathrm{var}_\gamma[\hat{\gamma}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)}$$