EE269
Signal Processing for Machine Learning
Lecture 3

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Recap: Basis

- A **basis** for a class of signals is a collection of M signals in the class that have the property that *any other signal in that class* can be written as a weighted sum of those signals.

$$x[n] = \sum_{k=0}^{M-1} c[k] y^{(k)}[n] \ .$$

Why change basis ?

- If we want to compress, we want $c[k]$ to have more small values or zero values than $x[k]$.
- If we want to classify (e.g., recognize different speakers), we want $c[k]$ to have spikes in different locations for different classes (e.g., different frequencies will have large Fourier coefficients).
- If we want to separate sources (e.g., separate sources of air pollution given measurements across the city), we want $c[k]$ again to have spikes for different k depending on the source (e.g., using a spatial-group sparse basis).
- If we want to reconstruct (e.g., image inside the body from external measurements), we want $c[k]$ to capture the most important aspects of the signal (e.g., outlines of tumors; bases designed for preserving these edges include wavelets and curvelets).

Standard Basis

- The collection of shifted deltas is a basis,

$$x[n] = \sum_{m=0}^{M-1} c[m]\delta[n-m] \; .$$

- *canonical basis* or *the standard basis*.

- **How to check if some vectors form a basis ?**

  For the vector space of length-N signals, if we have N **linearly independent** vectors then they form a basis.

  A collection of N signals $y^{(0)}[n], \ldots, y^{(N-1)}[n]$ is *linearly independent* if the following is true:

$$\sum_{m=0}^{N-1} \beta_m y^{(m)}[n] = 0 \text{ implies that } \beta_m = 0 \text{ for all } m = 0, \ldots, N-1 \; .$$

  If a set of signals is not linearly independent, we call it *linearly dependent*.

- *Orthonormal Bases:* If we have a basis $y^{(0)}[n], \ldots, y^{(N-1)}[n]$ where all the signals are mutually orthogonal:

$$\langle y^{(k)}[n], y^{(\ell)}[n] \rangle = 0 \text{ for all } k \neq \ell$$

  and if all the signals in the basis have norm 1:

$$\|y^{(k)}[n]\| = 1 \text{ for all } k = 1, \ldots, N$$

  then we call it an *orthonormal basis*.
- Delta basis is an orthonormal basis

- *Fourier Basis:* An important orthonormal basis for length-N complex signals is the *normalized Fourier basis* defined as:

$$w_m[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nm} \quad \text{for} \quad \begin{array}{c} n = 0, \ldots, N-1 \\ m = 0, \ldots, N-1 \end{array}$$

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- Orthogonal ?

  $\langle w_k[n],\, w_r[n] \rangle =$

How to change basis?

- Suppose we have a signal $x[n]$ in the standard basis. In order to write a signal in a different basis, as long as that domain is an **orthonormal basis**, we do the following:

  1. Take the inner product of your signal with every element from the orthonormal basis. These are called the *expansion coefficients*
  $$\langle y^{(k)}[n], x[n] \rangle$$
  2. Multiply each basis vector by the corresponding inner product and sum all the scaled basis vectors together.
  $$x[n] = \sum_{k=0}^{N-1} \langle y^{(k)}[n], x[n] \rangle y^{(k)}[n]$$

Example Consider $\mathbb{R}^2$, $x = \begin{bmatrix} 2 & 5 \end{bmatrix}^T$. We can write it in the canonical basis as follows:

$$x = \sum_{k=0}^{1} x[k]\delta[n-k] = x[0]\delta_0 + x[1]\delta_1 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Note that writing it like this actually followed the formula

$$x[n] = \sum_{k=0}^{N-1} \langle y^{(k)}[n], x[n] \rangle y^{(k)}[n]$$

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Consider the orthonormal basis,

$$y^{(0)} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad y^{(1)} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

1. Take the inner product of your signal with every element from the orthonormal basis:
2. Multiply each basis vector by the corresponding inner product and sum all the scaled basis vectors together:

Basis to Orthonormal basis

- Gram-Schmidt Orthogonalization

  Given basis $\phi_1, \phi_2, ...\phi_N$

  1. $\psi_1 := \dfrac{\phi_1}{||\phi_1||}$
  2. $\psi_2' := \phi_2 - \langle \psi_1, \phi_2 \rangle \psi_1$ and $\psi_2 := \dfrac{\psi_2'}{||\psi_2'||}$

  $\vdots$

  k. $\psi_k' := \phi_k - \sum_{i=1}^{k-1} \langle \psi_i, \phi_k \rangle \psi_i$ and $\psi_k := \dfrac{\psi_k'}{||\psi_k'||}$

Changing to the Discrete Fourier Basis: $w_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk}$

1. Take the inner product of your signal with every element from the orthonormal basis:

$$\begin{aligned} \langle w_k[n], x[n] \rangle &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi}{N}nk} \right)^* x[n] \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}} =: X[k] \end{aligned}$$

2. Multiply each basis vector by the corresponding inner product and sum all the scaled basis vectors together.

$$x[n] = \sum_{k=0}^{N-1} X[k] w_k[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] e^{-j\frac{2\pi}{N}nk} .$$

Changing to the Discrete Fourier Basis: $w_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk}$

Changing to the Fourier basis is called the **Discrete Fourier Transform**.

Analysis:

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

Synthesis:

$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] W_N^{-nk}$$

where $W_N := e^{-j\frac{2\pi}{N}}$.

Note that sometimes the normalizing factor $1/\sqrt{N}$ is distributed differently among the two equations.

DFT pairs

$$x[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$x[n] = \delta[n - n_0] \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = e^{-j\frac{2\pi}{N}n_0 k}$$

DFT pairs

$$x[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$x[n] = 1 \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = N\delta[k]$$

DFT pairs

$$x[n] \quad \stackrel{DFT}{\longleftrightarrow} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$x[n] = e^{j\frac{2\pi}{N}k_0 n} \quad \stackrel{DFT}{\longleftrightarrow} \quad X[k] = N\delta[k - k_0]$$

DFT pairs

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$$x[n] = 1 \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = N\delta[k]$$

$$x[n] = e^{j\frac{2\pi}{N}k_0 n} \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = N\delta[k - k_0]$$

Example: Cosine using linearity:

$$\cos\left(\frac{2\pi}{N}Ln + \phi\right) = \frac{1}{2}\left(e^{j\left(\frac{2\pi}{N}Ln+\phi\right)} + e^{-j\left(\frac{2\pi}{N}Ln+\phi\right)}\right) \; .$$

DFT pairs

$$x[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

$$x[n] = \begin{cases} 1 & \text{for } n \leq M-1 \\ 0 & \text{for } M \leq n \leq N-1 \end{cases} \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = \frac{\sin\left(\frac{\pi}{N}Mk\right)}{\sin\left(\frac{\pi}{N}k\right)} e^{-j\frac{\pi}{N}(M-1)k}$$

Continuous frequencies into N discrete bins

$x[n] = x(nT)$ real signal, sampling frequency $f_s = \frac{1}{T}$

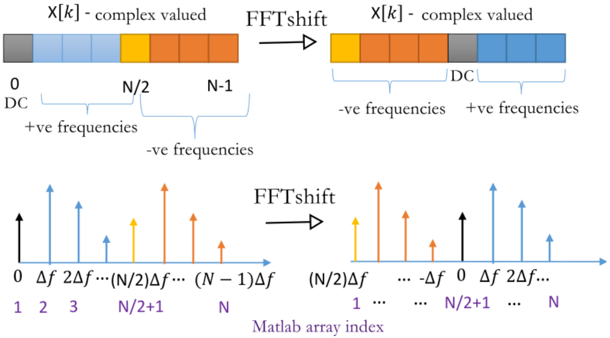
$$e^{-jw} \rightarrow e^{-j2\pi\frac{k}{N}}$$

- k'th bin $\iff$ continuous frequency $k\frac{f_s}{N}$
- Example: $f_s = 44.1kHz$, and $N = 1024$

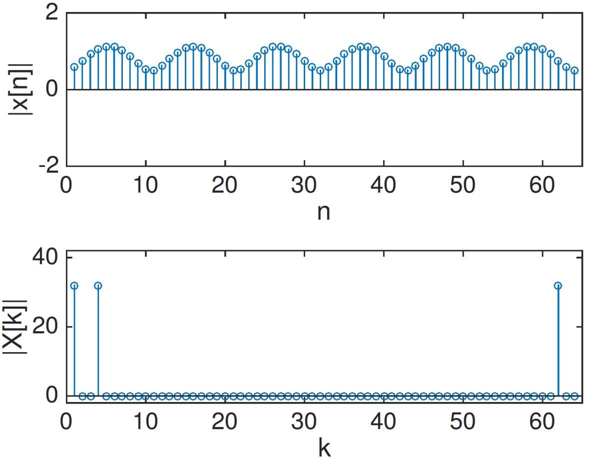
```
  0:   0 * 44100 / 1024 =     0.0 Hz
  1:   1 * 44100 / 1024 =    43.1 Hz
  2:   2 * 44100 / 1024 =    86.1 Hz
  3:   3 * 44100 / 1024 =   129.2 Hz
  4: ...
  5: ...
     ...
511: 511 * 44100 / 1024 = 22006.9 Hz
```

- Nyquist frequency $fs/2 = 22050$ Hz
  $X[0], X[1], X[2]..., X[511], X[512], X[513], ..., X[1023]$
- Conjugate symmetry $X[N-k] =$

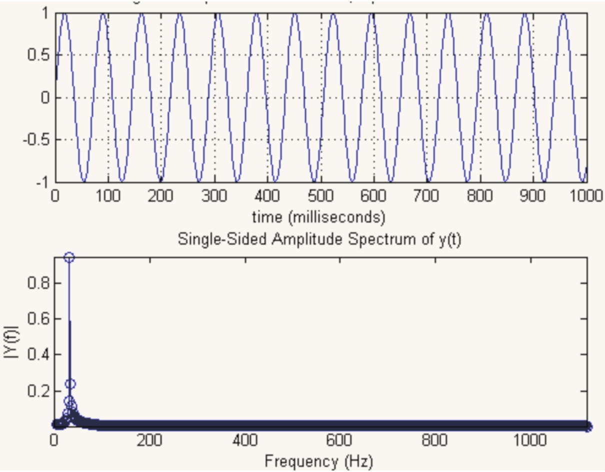
fftshift


X[k] - complex valued
FFTshift
X[k] - complex valued
0
DC
N/2
N-1
+ve frequencies
-ve frequencies
DC
-ve frequencies
+ve frequencies
FFTshift
0 Δf 2Δf ··· (N/2)Δf ··· (N − 1)Δf
1 2 3 N/2+1 N
(N/2)Δf ··· -Δf 0 Δf 2Δf...
1 ··· ··· N/2+1 ··· N
Matlab array index

Spectral Leakage



Spectral Leakage



DFT Applications: Frequency Extraction

DFT properties

- Linearity: Suppose

$$x^{(m)}[n] \overset{DFT}{\longleftrightarrow} X^{(m)}[k] \quad \text{for } m = 0, \ldots, M \; .$$

$$\text{DFT}\left\{\sum_{m=0}^{M} \alpha_m x^{(m)}[n]\right\} = \sum_{m=0}^{M} \alpha_m \text{DFT}\left\{x^{(m)}[n]\right\} = \sum_{m=0}^{M} \alpha_m X^{(m)}[k] \; .$$

DFT properties

- Duality:

$$x[n] = \delta[n - n_0] \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = e^{-j\frac{2\pi}{N}n_0 k}$$

$$x[n] = e^{j\frac{2\pi}{N}k_0 n} \quad \overset{DFT}{\longleftrightarrow} \quad X[k] = N\delta[k - k_0]$$

In general:

DFT properties

- Symmetries:

$$\begin{aligned} x^*[n] &\quad \overset{DFT}{\longleftrightarrow} \quad X^*[-k \mod N] \\ x^*[-n \mod N] &\quad \overset{DFT}{\longleftrightarrow} \quad X^*[k] \\ x[-n \mod N] &\quad \overset{DFT}{\longleftrightarrow} \quad X[-k \mod N] \end{aligned}$$

Proof of the first property:

DFT properties

- *Circular shift* (in MATLAB `circshift`).

$$x[(n-n_0) \mod N] \quad \overset{DFT}{\longleftrightarrow} \quad W_N^{kn_0} X[k]$$

$$W_N^{-nk_0} x[n] \quad \overset{DFT}{\longleftrightarrow} \quad X[(k-k_0) \mod N]$$

DFT properties

- Symmetry of real signals

  Their Fourier transforms are conjugate symmetric, i.e., their magnitude is an even function and their phase is odd.

$$x[n] \text{ real} \quad \stackrel{DFT}{\longleftrightarrow} \quad X[k] = X^*[-k \mod N]$$

$$x[n] \text{ real} \quad \stackrel{DFT}{\longleftrightarrow} \quad |X[k]| = |X[-k \mod N]|$$

$$x[n] \text{ real} \quad \stackrel{DFT}{\longleftrightarrow} \quad \angle X[k] = -\angle X[-k \mod N]$$

$$x[n] \text{ real} \quad \stackrel{DFT}{\longleftrightarrow} \quad Re\{X[k]\} = Re\{X[-k \mod N]\}$$

$$x[n] \text{ real} \quad \stackrel{DFT}{\longleftrightarrow} \quad Im\{X[k]\} = -Im\{X[-k \mod N]\}$$

DFT properties

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$$x[n] \text{ real} \quad \overset{DFT}{\longleftrightarrow} \quad \angle X[k] = -\angle X[-k \mod N]$$

Matrix multiplication (view 1)

$$y = Ax$$

$$y = \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_N \end{array} \right] x$$

$$y[k] = \sum_{k=1}^{N-1} a_n[k] x[n]$$

Matrix multiplication (view 2)

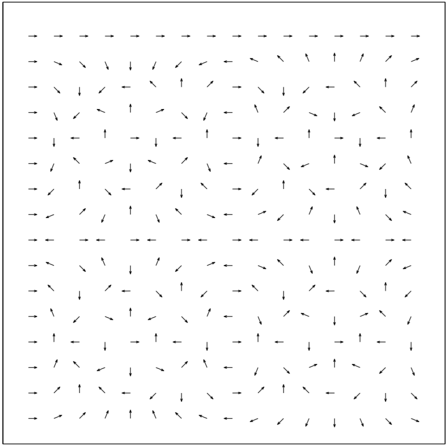
$$y = Bx$$

$$y = \begin{bmatrix} b_1^H \\ b_2^H \\ \dots \\ b_N^H \end{bmatrix} x$$

$$y[k] = \sum_{n=1}^{N-1} b_k^*[n] x[n]$$

DFT as Matrix multiplication

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$



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