
Filter-and-Sum Beamforming

ECE 6279: Spatial Array Processing
Fall 2013
Lecture 8

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Where We Are in J&D

- **Lecture material drawn from:**
 - Sec. 4.3 (all)



Temporal Filtering

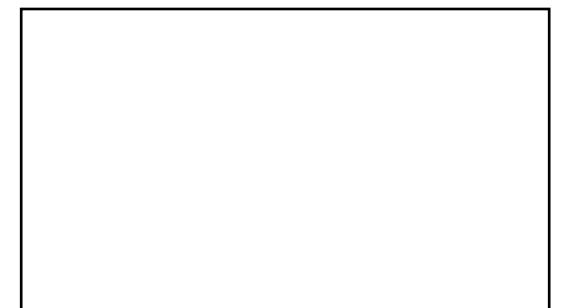
- Filter the signal collected by each sensor in time

$$\begin{aligned} y_m(t) &= \{h_m(\cdot) *_t f(\vec{x}_m, \cdot)\}(t) \\ &= \int_{-\infty}^{\infty} h_m(\tau) f(\vec{x}_m, t - \tau) d\tau \end{aligned}$$

- Then do delay-and-sum beamforming on the filtered signals, using techniques from past two lectures

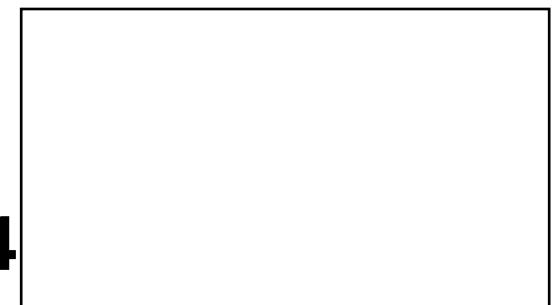
$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

- Choosing $h_m(t) = \delta(t)$ gives straight delay-and-sum beamforming



Why Filter in Time?

- Focus on frequency range of interest
- Reduce noise and interference in unwanted frequency bands
- In digital implementations, need antialiasing filter before A/D converters
- If you have a statistical model for the signal and the noise, you can design a Wiener filter (see ECE6254 class and textbook)



Temporally Filtered Signal at Sensor

The usual: try a monochromatic plane wave!

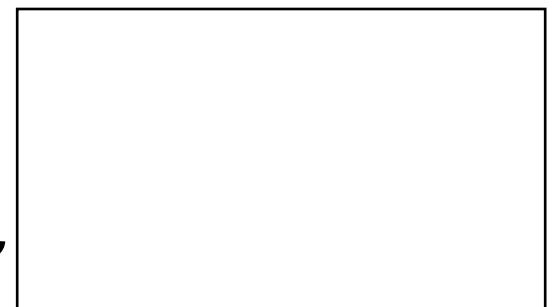
$$f(\vec{x}_m, t) = \exp\{j(\omega^0 t - \vec{k}^0 \cdot \vec{x}_m)\}$$

$$y_m(t) = \int_{-\infty}^{\infty} h_m(\tau) f(\vec{x}_m, t - \tau) d\tau$$

$$= \int_{-\infty}^{\infty} h_m(\tau) \exp\{j[\omega^0(t - \tau) - \vec{k}^0 \cdot \vec{x}_m]\} d\tau$$

$$= \exp\{j(\omega^0 t - \vec{k}^0 \cdot \vec{x}_m)\} \times$$

$$\underbrace{\int_{-\infty}^{\infty} h_m(\tau) \exp(-j\omega^0 \tau) d\tau}_{H_m(\omega^0)}$$



(Temporal Filter)-and-Sum Beamforming

$$\begin{aligned} y_m(t) &= H_m(\omega^0) \exp\{j(\omega^0 t - \vec{k}^0 \cdot \vec{x}_m)\} \\ z(t) &\equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m) \\ &= \sum_{m=0}^{M-1} w_m H_m(\omega^0) \exp\{j(\omega^0 [t - \Delta_m] - \vec{k}^0 \cdot \vec{x}_m)\} \\ &= e^{j\omega^0 t} \underbrace{\sum_{m=0}^{M-1} w_m H_m(\omega^0) \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}}_{\text{Wavenumber-frequency response} \rightarrow H(\vec{k}^0, \omega^0)} \end{aligned}$$



Identical Temporal Filters (1)

$$z(t) = \exp(j\omega^0 t) H(\vec{k}^0, \omega^0)$$

where $H(\vec{k}^0, \omega^0) =$

$$\sum_{m=0}^{M-1} w_m H_m(\omega^0) \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}$$

- **Notice if** $H_m(\omega) = H_0(\omega)$

then $H(\vec{k}^0, \omega^0) =$

$$H_0(\omega^0) \sum_{m=0}^{M-1} w_m \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}$$



Identical Temporal Filters (2)

- If $H_m(\omega) = H_0(\omega)$ and we beamform on a planewave with slowness vector $\vec{\alpha}$

$$\begin{aligned} H(\vec{k}^0, \omega^0) &= H_0(\omega^0) \sum_{m=0}^{M-1} w_m \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\} \\ &= H_0(\omega^0) \sum_{m=0}^{M-1} w_m \exp\{j(\omega^0 \vec{\alpha} - \vec{k}^0) \cdot \vec{x}_m\} \\ &= H_0(\omega^0) W(\omega^0 \vec{\alpha} - \vec{k}^0) \end{aligned}$$

$\Delta_m = -\vec{\alpha} \cdot \vec{x}_m$



Identical Temporal Filters (3)

- **If $H_m(\omega) = H_0(\omega)$ and we beamform on a planewave with slowness vector $\vec{\alpha}$**

$$H(\vec{k}^0, \omega^0) = H_0(\omega^0)W(\omega^0 \vec{\alpha} - \vec{k}^0)$$

- **If $H_0(\omega^0) = 1$, i.e. no filtering, we get**

$$H(\vec{k}^0, \omega^0) = W(\omega^0 \vec{\alpha} - \vec{k}^0)$$

**which is the old
delay-and-sum result**



Quick Question and Answer

- **Question: To avoid needing m filters, why not just beamform first, then apply the $H(\omega)$ filter?**

$$H(\vec{k}^0, \omega^0) = H_0(\omega^0)W(\omega^0 \vec{\alpha} - \vec{k}^0)$$

- **Answer: filter first to avoid unwanted signals eating up the dynamic range of your A/D converters**
 - Temporally filter first
 - Then beamform



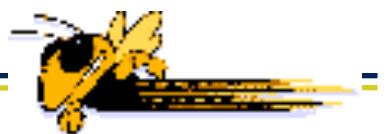
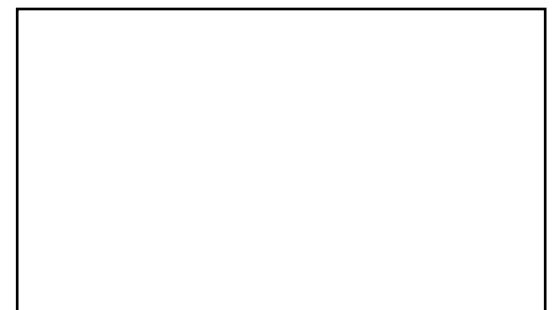
Spatial Filtering

- Suppose now we filter in space

$$\begin{aligned} y_m(t) &= \{h_m(\cdot) *_{\vec{x}} f(\cdot, t)\}(\vec{x}_m) \\ &= \int_{-\infty}^{\infty} h_m(\vec{\chi}) f(\vec{x}_m - \vec{\chi}, t) d\vec{\chi} \end{aligned}$$

- Usually we don't do this deliberately - occurs due to the nature of the given sensors
- Let's study the filter-and-sum beamformer with spatial filtering:

$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$



Spatially Filtered Signal at Sensor

$$\begin{aligned} f(\vec{x}_m, t) &= \exp\{j(\omega^0 t - \vec{k}^0 \cdot \vec{x}_m)\} \\ y_m(t) &= \int_{-\infty}^{\infty} h_m(\vec{\chi}) \overbrace{f(\vec{x}_m - \vec{\chi}, t)} d\vec{\chi} \\ &= \int_{-\infty}^{\infty} h_m(\vec{\chi}) \exp\{j(\omega^0 t - \vec{k}^0 \cdot [\vec{x}_m - \vec{\chi}])\} d\vec{\chi} \\ &= \underbrace{e^{j(\omega^0 t - \vec{k}^0 \cdot \vec{x}_m)} \int_{-\infty}^{\infty} h_m(\vec{\chi}) \exp(j\vec{k}^0 \cdot \vec{\chi}) d\vec{\chi}}_{H_m(\vec{k}^0)} \end{aligned}$$



(Spatial Filter)-and-Sum Beamforming

$$\begin{aligned} y_m(t) &= H_m(\vec{k}^0) \exp\{j(\omega^0 t - \vec{k}^0 \cdot \vec{x}_m)\} \\ z(t) &\equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m) \\ &= \sum_{m=0}^{M-1} w_m H_m(\vec{k}^0) \exp\{j(\omega^0 [t - \Delta_m] - \vec{k}^0 \cdot \vec{x}_m)\} \\ &= e^{j\omega^0 t} \underbrace{\sum_{m=0}^{M-1} w_m H_m(\vec{k}^0) \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}}_{\text{Wavenumber-frequency response} \rightarrow H(\vec{k}^0, \omega^0)} \end{aligned}$$



Identical Spatial Filters (1)

$$z(t) = \exp(j\omega^0 t) H(\vec{k}^0, \omega^0)$$

where $H(\vec{k}^0, \omega^0) =$

$$\sum_{m=0}^{M-1} w_m H_m(\vec{k}^0) \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}$$

- **Notice if** $H_m(\vec{k}) = H_0(\vec{k})$

then $H(\vec{k}^0, \omega^0) =$

$$H_0(\vec{k}^0) \sum_{m=0}^{M-1} w_m \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}$$



Identical Spatial Filters (2)

- If $H_m(\vec{k}) = H_0(\vec{k})$ and we beamform on a planewave with slowness vector $\vec{\alpha}$

$$\begin{aligned} H(\vec{k}^0, \omega^0) &= H_0(\vec{k}^0) \sum_{m=0}^{M-1} w_m \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\} \\ &= H_0(\vec{k}^0) \sum_{m=0}^{M-1} w_m \exp\{j(\omega^0 \vec{\alpha} - \vec{k}^0) \cdot \vec{x}_m\} \\ &= H_0(\vec{k}^0) W(\omega^0 \vec{\alpha} - \vec{k}^0) \end{aligned}$$

$\Delta_m = -\vec{\alpha} \cdot \vec{x}_m$



Identical Spatial Filters (3)

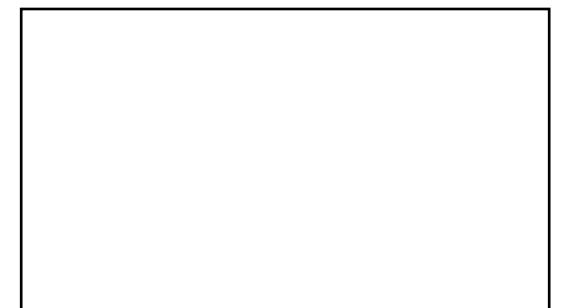
- **If $H_m(\vec{k}) = H_0(\vec{k})$ and we beamform on a planewave with slowness vector $\vec{\alpha}$**

$$H(\vec{k}^0, \omega^0) = H_0(\vec{k}^0)W(\omega^0 \vec{\alpha} - \vec{k}^0)$$

- **If $H_0(\vec{k}^0) = 1$, i.e. no filtering, we get**

$$H(\vec{k}^0, \omega^0) = W(\omega^0 \vec{\alpha} - \vec{k}^0)$$

**which is the old
delay-and-sum result**



Spatiotemporal Filtering

- Might filter in both space and time!

$$y_m(t) = \{h_m * f\}(\vec{x}_m, t)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_m(\vec{\chi}, \tau) f(\vec{x}_m - \vec{\chi}, t - \tau) d\vec{\chi} d\tau$$

- By now we can guess the-filter-and-sum beamformer output will look like

$$z(t) = \exp(j\omega^0 t) H(\vec{k}^0, \omega^0)$$

where $H(\vec{k}^0, \omega^0) =$

$$\sum_{m=0}^{M-1} w_m H_m(\vec{k}^0, \omega^0) \exp\{-j(\omega^0 \Delta_m + \vec{k}^0 \cdot \vec{x}_m)\}$$



Identical Spatiotemporal Filters

- If $H_m(\vec{k}, \omega) = H_0(\vec{k}, \omega)$ and we beamform on a planewave with slowness vector $\vec{\alpha}$

$$H(\vec{k}^0, \omega^0) = H_0(\vec{k}^0, \omega^0)W(\omega^0 \vec{\alpha} - \vec{k}^0)$$

Waveno-freq response of individual sensor

Waveno-freq response of beamformed array

