ESPRIT

ECE 6279: Spatial Array Processing
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Lecture 18

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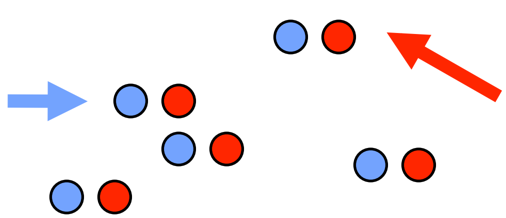
Sources

- ESPRIT: "Estimation of Signal Prameters via Rotational Invariance Techniques"
- J&D, p. 419, Problem 7.22
- Journal papers (linked on class website)
- Lecture notes from Dan Fuhrmann
- On total least squares: see Section 12.3 in Matrix Computations by Golub & Van Loan

Setup

- **Need two identical subarrays displaced (not rotated) by a known displacement vector $\vec{\Delta}$ with magnitude Δ**
- **For simplicity, let's do displacement along the x dimension in these slides**

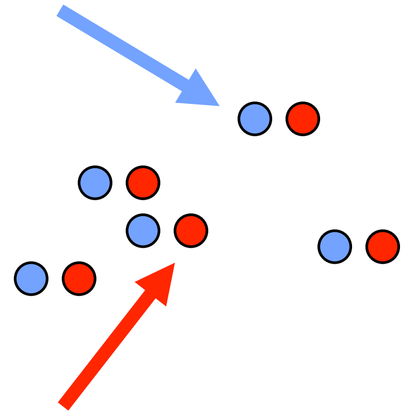
$$\mathbf{e}(\phi)$$



$$\mathbf{e}(\phi)\exp\left(j\frac{2\pi}{\lambda}\Delta\sin(\phi)\right)$$

Notation (1)

$$\mathbf{y}^{(0)}(l) = \underbrace{\mathbf{D}}_{[\mathbf{e}(\phi_1)\ \cdots\ \mathbf{e}(\phi_{N_s})]}\mathbf{s}(l) + \mathbf{n}^{(0)}(l)$$



$$\mathbf{y}^{(1)}(l) = \mathbf{D}\underbrace{\Phi}_{\begin{bmatrix} \exp(j\gamma_1) & & \\ & \ddots & \\ & & \exp(j\gamma_{N_s}) \end{bmatrix}}\mathbf{s}(l) + \mathbf{n}^{(1)}(l)$$

$$\gamma_i = \frac{2\pi}{\lambda}\Delta \sin(\phi_i)$$

Notation (2)

$$\mathbf{y}(l) = \begin{bmatrix} \mathbf{y}^{(0)}(l) \\ \mathbf{y}^{(1)}(l) \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}\Phi \end{bmatrix} \mathbf{s}(l) + \begin{bmatrix} \mathbf{n}^{(0)}(l) \\ \mathbf{n}^{(1)}(l) \end{bmatrix}$$

$$= \bar{\mathbf{D}}\mathbf{s}(l) + \mathbf{n}(l)$$

- **Goal: Exploit structure of $\bar{\mathbf{D}}$ to estimate diagonal elements of Φ without needing to know $\mathbf{D}$**

Ideal Covariance of the Data

- **Ideally:**

$$\mathbf{R_y} = \bar{\mathbf{D}}\underbrace{\mathbf{R_s}}_{\begin{bmatrix} A_1^2 & & \\ & \ddots & \\ & & A_{N_s}^2 \end{bmatrix}}\bar{\mathbf{D}}^H + \sigma_n^2\mathbf{I}$$

Splitting the Signal+Noise Subspace

- Do eigendecomposition of $\mathbf{R}_{\mathbf{y}}$

$$\mathbf{V}_{s+n} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{N_s} \end{bmatrix}$$

- Since $\mathbf{V}_{s+n}$ and $\bar{\mathbf{D}}$ span the same subspace, there exists a $\exists \mathbf{T}$ s.t.

$$\mathbf{V}_{s+n} = \bar{\mathbf{D}}\mathbf{T}$$

$$\mathbf{V}_{s+n} \equiv \begin{bmatrix} \mathbf{E}_0 \\ \mathbf{E}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D}\mathbf{\Phi} \end{bmatrix} \mathbf{T}$$

Some Trickery

$$\mathbf{E}_0 = \mathbf{D}\mathbf{T}$$

$$\mathbf{E}_0\mathbf{T}^{-1} = \mathbf{D}$$

$$\mathbf{E}_1 = \mathbf{D}\Phi\mathbf{T} = \mathbf{E}_0\underbrace{\mathbf{T}^{-1}\Phi\mathbf{T}}_{\Psi}$$

- **Fact from linear algebra: Ψ and Φ have the same eigenvalues**

Ideal ESPRIT Procedure (1)

- Solve $\mathbf{E}_1 = \mathbf{E}_0 \Psi$ for Ψ
- Find eigenvalues of Ψ; these are the diagonal elements of

$$\Phi = \begin{bmatrix} \exp(j\gamma_1) & & \\ & \ddots & \\ & & \exp(j\gamma_{N_s}) \end{bmatrix}$$

(possibly reordered)

Ideal ESPRIT Procedure (2)

$$\gamma_i = \frac{2\pi}{\lambda}\Delta\sin(\phi_i)$$

$$\phi_i = \sin^{-1}\left(\gamma_i \frac{\lambda}{2\pi\Delta}\right)$$

$$= \sin^{-1}\left(\arg(\lambda_i^{(\Psi)})\frac{\lambda}{2\pi\Delta}\right)$$

ESPRIT Procedure in Reality

- **In practice, compute eigendecomposition from empirical covariance matrix**
- **"Solve"** $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_0 \Psi$

$$\hat{\phi}_i = \sin^{-1}\left(\arg(\hat{\lambda}_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$

- **Trouble:** $\hat{\mathbf{E}}_0$ **and** $\hat{\mathbf{E}}_1$ **may not span the same subspace**

Total Least Squares for Practical ESPIRT

- Goal: "Solve" $\hat{\mathbf{E}}_1 = \hat{\mathbf{E}}_0 \Psi$ when both sides have have errors
- Formulation:

$$(\hat{\mathbf{E}}_1 + \Delta\mathbf{E}_1) = (\hat{\mathbf{E}}_0 + \Delta\mathbf{E}_0)\Psi$$

- Find $\hat{\mathbf{E}}_0$, $\hat{\mathbf{E}}_1$, and Ψ that minimizes

$$\left\| [\Delta\mathbf{E}_0 \;\; \Delta\mathbf{E}_1 \right\|_F^2 \equiv \mathrm{tr}([\Delta\mathbf{E}_0 \;\; \Delta\mathbf{E}_1][\Delta\mathbf{E}_0 \;\; \Delta\mathbf{E}_1]^H)$$

The Null Subspace - Ideally

- Ideally $\mathbf{E}_1 = \mathbf{E}_0 \Psi$, so $\mathbf{E}_0$ and $\mathbf{E}_1$ share the same subspace

$$\Rightarrow \overbrace{\begin{bmatrix} \mathbf{E}_0 & \mathbf{E}_1 \end{bmatrix}}^{2N_s} \text{ has rank } N_s$$

$$\Rightarrow \exists \mathbf{F} = \left.\begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix}\right\} 2N_s \qquad \begin{aligned} 0 &= \begin{bmatrix} \mathbf{E}_0 & \mathbf{E}_1 \end{bmatrix} \mathbf{F} \\ &= \mathbf{E}_0 \mathbf{F}_0 + \mathbf{E}_1 \mathbf{F}_1 \end{aligned}$$

spans null subspace of $\begin{bmatrix} \mathbf{E}_0 & \mathbf{E}_1 \end{bmatrix}$

Exploiting the Null Subspace

- Ideally $\mathbf{E}_1 = \mathbf{E}_0 \Psi$, so $\mathbf{E}_0$ and $\mathbf{E}_1$ share the same subspace

$$0 = \mathbf{E}_0 \mathbf{F}_0 + \mathbf{E}_1 \mathbf{F}_1$$

$$\mathbf{E}_1 \mathbf{F}_1 = -\mathbf{E}_0 \mathbf{F}_0$$

$$\mathbf{E}_1 = \mathbf{E}_0 (-\mathbf{F}_0 \mathbf{F}_1^{-1})$$

$$\Longrightarrow \quad \Psi = -\mathbf{F}_0 \mathbf{F}_1^{-1}$$

The Null Subspace – in Reality

- In reality, won't have $\left[\hat{\mathbf{E}}_0 \;\; \hat{\mathbf{E}}_1\right]\mathbf{F} = 0$
- It "is easily shown" we should replace it with

$$\hat{\mathbf{E}}_{01}\mathbf{F} = \left[\Delta\mathbf{E}_0 \;\; \Delta\mathbf{E}_1\right]$$

where $\hat{\mathbf{E}}_{01} = \left[\hat{\mathbf{E}}_0 \;\; \hat{\mathbf{E}}_1\right]$

- Seek $\mathbf{F}$ that minimizes $\left\|\hat{\mathbf{E}}_{01}\mathbf{F}\right\|_F$ s.t. $\mathbf{F}^H\mathbf{F} = \mathbf{I}$

Magic

- "applying standard Lagrange techniques leads to a solution"
- Compute eigenvectors of $\hat{\mathbf{E}}_{01}^{H}\hat{\mathbf{E}}_{01}$

$$\begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \end{bmatrix} = \mathbf{F} = \underbrace{\begin{bmatrix} \tilde{\mathbf{v}}_{N_s+1} & \cdots & \tilde{\mathbf{v}}_{2N_s} \end{bmatrix}}_{N_s \text{ smallest eigenvectors of } \hat{\mathbf{E}}_{01}^{H}\hat{\mathbf{E}}_{01}}$$

- In general cases, total least squares uses a SVD

Practical ESPRIT Algorithm

- Do eigendecomposition of $\hat{\mathbf{R}}_{\mathbf{y}}$

$$\hat{\mathbf{V}}_{s+n} = \left[\hat{\mathbf{v}}_1 \ \cdots \ \hat{\mathbf{v}}_{N_s} \right] \equiv \begin{bmatrix} \hat{\mathbf{E}}_0 \\ \hat{\mathbf{E}}_1 \end{bmatrix}$$

- Let $\mathbf{F} = N_s$ smallest eigvecs of

$$\left[\hat{\mathbf{E}}_0 \ \hat{\mathbf{E}}_1 \right]^H \left[\hat{\mathbf{E}}_0 \ \hat{\mathbf{E}}_1 \right]$$

$$\Psi = -\mathbf{F}_0 \mathbf{F}_1^{-1} \qquad \phi_i = \sin^{-1}\left(\arg(\lambda_i^{(\Psi)}) \frac{\lambda}{2\pi\Delta} \right)$$