E9 231: Digital Array Signal Processing

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1 Topics

- Spatial Smoothing and the Forward Backward method.
- Direction of Arrival Estimation.

2 Class Notes

2.1 Spatial Smoothing

- Applicable to arrays with structure for example:
- Rectangular (uniform) array.
- Uniform Linear Array.

$$S_X = \begin{bmatrix} V_S & V_I \end{bmatrix} \begin{bmatrix} * & & \mathbf{0} & \\ & * & & \\ & & * & \\ \mathbf{0} & & & * \end{bmatrix} \begin{bmatrix} V_S^H \\ V_I^H \end{bmatrix}$$

where

$$V(\psi) = \begin{bmatrix} V_S \; V_I \end{bmatrix}_{N \times (D+1)}$$

Goal: To Remove singularity(improve condition number) in S_X caused due to correlation between signal and interference. If we have N sensors, we pick first $1 : M$ sensors at a time, then we pick next $2 : M + 1$ sensors and continuing this way we get L such blocks where $N = L + M - 1$. Now we define,

$$V_M(\psi) = \begin{bmatrix} 1 \\ e^{j\psi} \\ \vdots \\ e^{j(M-1)\psi} \end{bmatrix} e^{j(\frac{M-1}{2})\psi}$$

$$V_N(\psi) = \begin{bmatrix} e^{-j(\frac{N-1}{2})\psi} \\ \vdots \\ e^{+j(\frac{N-1}{2})\psi} \end{bmatrix}$$

$V_M^{(l)}(\psi)$: pick off the l^{th} length M block

$$V_M^{(1)}(\psi) = \begin{bmatrix} e^{-j(\frac{N-1}{2})\psi} \\ \vdots \\ e^{-j(\frac{N-1}{2}-(M-1))\psi} \end{bmatrix}$$

$$V_M^{(1)}(\psi) = V_M(\psi)e^{-j(\frac{N-1}{2})\psi}e^{j(\frac{M-1}{2})\psi}$$

$$V_M^{(1)}(\psi) = V_M(\psi)e^{-j(\frac{N-1}{2}-\frac{M-1}{2})\psi}$$

$$V_M^{(l)}(\psi) = V_M(\psi)e^{-j(\frac{N-1}{2}-\frac{M-1}{2}-(l-1))\psi}$$

Recall V_S is dependent on ψ_S and V_I is dependent on ψ_I
$V_M^{(l)}(\psi)$: l through $l+M-1$ rows of $V(\psi)$ matrix

$$V_M^{(l)}(\psi) = \left[\begin{array}{c} V_M^{(l)}(\psi_1)\dots V_M^{(l)}(\psi_{D+1}) \end{array} \right]$$

where

$$V_M^{(l)}(\psi_{D+1}) = V_M(\psi_{D+1})e^{-j(\frac{N-1}{2}-\frac{M-1}{2}-(l-1))\psi_{D+1}}$$

$$V_M^{(l)}(\psi) = V_M(\psi)D^{(l)}$$

$$D^{(l)} = \text{diag}\left[\begin{array}{c} e^{-j(\frac{N-1}{2}-\frac{M-1}{2}-(l-1))\psi_1} \; e^{-j(\frac{N-1}{2}-\frac{M-1}{2}-(l-1))\psi_2} \dots e^{-j(\frac{N-1}{2}-\frac{M-1}{2}-(l-1))\psi_{D+1}} \end{array} \right]_{(D+1)\times(D+1)}$$

$$V_M(\psi) = \left[\begin{array}{c} V_M(\psi_1)\; V_M(\psi_2)\dots V_M(\psi_{D+1}) \end{array} \right]$$

Remember our model was

$$X = V(\psi)F_S + N$$
$$X^{(l)} = V_M^{(l)}(\psi)F_S + N^{(l)}$$
$$X^{(l)} = V_M(\psi)D^{(l)}F_S + N^{(l)}$$
$$S_X^{(l)} = V_M(\psi)D^{(l)}S_f D^{(l)H}V_M^H(\psi) + \sigma_n^2 I_{M\times M}$$

Smoothed Subarray: subscript SS stands for spatial smoothing

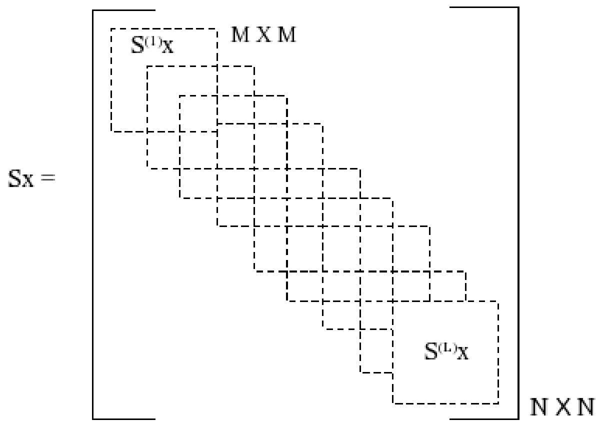
$$S_{SS} = \frac{1}{L}\sum_{l=1}^{L} S_X^{(l)}$$

Claim: The interference in S_{SS} is more uncorrelated with the desired signal, than in S_X

$$S_{SS} = V_M(\psi)(\frac{1}{L}\sum_{l=1}^{L} D^{(l)}S_f D^{(l)H})V_M^H(\psi) + \sigma_n^2 I_{M\times M}$$

can be shown that

- S_{SS} will be close to diagonal.
- The condition number has improved. The condition number of S_{SS} is lower than the condition number of S_X



- If $L \geq \frac{D}{2}$ guarantees that S_{SS} will be non-singular.
- $N = L+M-1$ so for a fixed N if we decrease M, L will increase correspondingly. We can also say that S_{SS} is more diagonal. But output SNR will decrease because of loss of degrees of freedom. Therefore for every S_f there exists an optimum M that maximizes output SNR.

$$V_M(\psi) = \begin{bmatrix} e^{-j(\frac{M-1}{2})\psi} \\ \vdots \\ e^{+j(\frac{M-1}{2})\psi} \end{bmatrix}$$

$$V_M(\psi) = JV_M^*(\psi)$$

J is exchange matrix

$$J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

$$J^2 = I$$

consider

$$JS_{SS}^*J = JV_M^*(\psi)S_{f,SS}^*V_M^T(\psi)J + \sigma_n^2 I$$

Let

$$S_{f,SS} = \frac{1}{L}\sum_{l=1}^{L} D^{(l)}S_f D^{(l)H}$$

therefore

$$JS_{SS}^*J = V_M(\psi)S_{f,SS}^*V_M^H(\psi) + \sigma_n^2 I$$

S_f is supposed to be hermitian symmetric, so we can average them out. Hence, after solving we get,

$$S_{SS,FB} = \frac{1}{2}\left(S_{SS} + JS_{SS}^*J\right)$$

$$S_{SS,FB} = V_M \left(\frac{1}{2} \left(S_{f,SS} + S^*_{f,SS} \right) \right) V_M^H + \sigma_n^2 I$$

$$S_{f,SS,FB} = \frac{1}{2} \left(S_{f,SS} + S^*_{f,SS} \right)$$

Uncorrelated sources: for a ULA S_X is toeplitz, then it has a structure which looks like

$$S_X = \begin{bmatrix} 1 & r & r^2 & \cdots & r^{N-1} \\ r^{-1} & 1 & r & \cdots & r^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{-(N-1)} & r^{-(N-2)} & r^{-(N-3)} & \cdots & 1 \end{bmatrix}$$

Here we can use ideas from min redundant array to pick subsets of elements with unique seperation between elements. If signal were decorrelated, we would have got a toeplitz structure. However, here we can consider a toeplitz approximation. Important thing about toeplitz matrix is that its inversion is computationally very fast.
computations: $M^3 \rightarrow M^2$
Can also do a weighted averaging of submatrices.
Fast forward to chapter (9)

2.2 Frequency wave number spectrum/DOA estimation

Quadratic algorithms:

- The ψ is varied and a quadratic function of $V(\psi)$ is computed. Then we find the $D+1$ peaks of the above and call them.

$$\hat{\psi}_1,\ \hat{\psi}_2,\ \ldots,\ \hat{\psi}_{D+1}$$

- Assume D is known

$$X[k] = V(\psi_s) F_s[k] + N[k]$$

where $k = 1, \ldots, K$
$P(\psi)$ = spatial power spectrum
This is obtained by scanning ψ, forming a beam and measuring the power at output of array.

Approach 1: Beam scan algorithm

$$\hat{P}_B(\psi) = \frac{1}{K} \sum_{k=1}^{K} |V^H(\psi) X[k]|^2 \quad -\pi \leq \psi \leq \pi$$

subscript B stands for Bartlett Beamformer

$$\hat{P}_B(\psi) = \frac{1}{K} \sum_{k=1}^{K} V^H(\psi) X[k] X^H[k] V(\psi)$$

$$\hat{P}_B(\psi) = V^H(\psi) C_X V(\psi)$$

where

$$C_X = \frac{1}{K} \sum_{k=1}^{K} X[k] X^H[k]$$

Ideally C_X should be equal to S_X

- Select D or $D+1$ peaks of $\hat{P}_B(\psi)$ as $\hat{\psi}_1 \dots \hat{\psi}_D$

- Drawback: Resolution cannot exceed $\frac{2}{N}$ for a ULA.

- In case of a ULA, the Bartlett Beamformer reduces to an FFT operation as

$$V(\psi) = C \begin{bmatrix} 1 \\ e^{j\psi} \\ \vdots \\ e^{j(N-1)\psi} \end{bmatrix}$$

$$V^H(\psi)X[k] = C \sum_{l=0}^{N-1} X_K[l]e^{jl\psi}$$

This is FFT of

$$[X_k(0) \ \dots \ X_k(N-1)]$$

It is also known as the Bartlett method of power spectrum estimation in time series analysis.

Approach 2: Apply weighting or the Welch Method

Here we apply window to $X[k]$ to get

$$\begin{bmatrix} W(0)X_k(0) \\ \vdots \\ W(N-1)X_k(N-1) \end{bmatrix}$$

which further can be written as

$$W_{N\times N} \begin{bmatrix} X_k(0) \\ \vdots \\ X_k(N-1) \end{bmatrix}$$

where,

$$W_{N\times N} = \text{diag}\left[W(0), \dots, W(N-1)\right]$$

$$\hat{P}_{BW}(\psi) = \frac{1}{K} \sum_{k=1}^{K} |V^H(\psi)WX[k]|^2$$

$$\hat{P}_{BW}(\psi) = V^H(\psi)C_{X,W}V(\psi)$$

where

$$C_{X,W} = WC_XW^H$$

- We have applied a window (for ex: Kaiser) to data before taking the FT
- Affect the main lobe to side lobe ratios.
- Reject weak interferers in the sidelobe region
- Beam (main lobe) broadens which implies loss in resolution.

Another option

We can improve the performance by FB averaging

- Conjugate Symmetric arrays:

$$V(\psi) = JV^*(\psi)$$

$$\hat{P}_{B,FB}(\psi) = \frac{1}{2K}\sum_{k=1}^{K}|V^H(\psi)(X[k] + JX^*[k])|^2 \quad -\pi \le \psi \le \pi$$

$$\hat{P}_{B,FB}(\psi) = V^H(\psi)C_{X,FB}V(\psi)$$

where

$$C_{X,FB} = \frac{1}{2}(C_X + JC_X^*J)$$