
MVDR Beamforming

**ECE 6279: Spatial Array Processing
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Lecture 15**

Prof. Aaron D. Lanterman

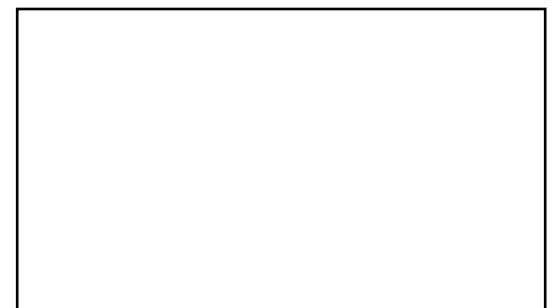
**School of Electrical & Computer Engineering
Georgia Institute of Technology**

AL: 404-385-2548
[<lanterma@ece.gatech.edu>](mailto:lanterma@ece.gatech.edu)



Where We Are in J&D

- **Section 7.2.1 and Appendix B**



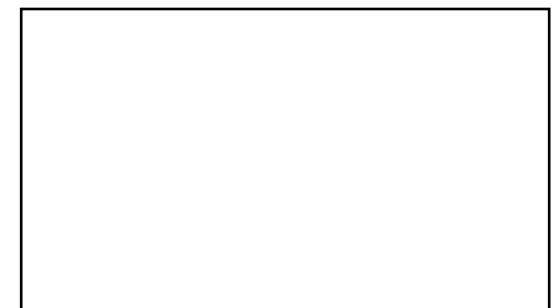
“Minimum Variance”

$$\mathbf{w}_{\diamond} = \arg \min_{\mathbf{w}} \mathbf{w}^H \mathbf{R}_y \mathbf{w} \text{ s.t. } \mathbf{C}\mathbf{w} = \mathbf{c}$$

Solution: $\mathbf{w} = \mathbf{R}_y^{-1} \mathbf{C}^H (\mathbf{C} \mathbf{R}_y^{-1} \mathbf{C}^H)^{-1} \mathbf{c}$

Power of beamformer output:

$$\mathbf{w}^H \mathbf{R}_y \mathbf{w} = \mathbf{c}^H (\mathbf{C} \mathbf{R}_y^{-1} \mathbf{C}^H)^{-1} \mathbf{c}$$



“Distortionless Response”

- For ex., $\mathbf{w}^H \mathbf{e}(\vec{k}) = 1$ assures signal in look direction passes unharmed

$$\mathbf{e}^H(\vec{k})\mathbf{w} = 1$$

$$\mathbf{C}\mathbf{w} = \mathbf{c}$$



MVDR Weights

$$\mathbf{w} = \mathbf{R}_y^{-1} \mathbf{C}^H (\mathbf{C} \mathbf{R}_y^{-1} \mathbf{C}^H)^{-1} \mathbf{c}$$

- Plug in $\mathbf{C} = \mathbf{e}^H(\vec{k}), \mathbf{c} = 1$

$$\mathbf{w}(\vec{k}) = \mathbf{R}_y^{-1} \mathbf{e}(\vec{k}) [\mathbf{e}^H(\vec{k}) \mathbf{R}_y^{-1} \mathbf{e}(\vec{k})]^{-1}$$

$$= \frac{\mathbf{R}_y^{-1} \mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k}) \mathbf{R}_y^{-1} \mathbf{e}(\vec{k})}$$



MVDR Beamforming

$$z = \mathbf{w}^H \mathbf{y}$$

$$= \mathbf{e}^H(\vec{k}) \frac{\mathbf{R}_y^{-1}}{\mathbf{e}^H(\vec{k}) \mathbf{R}_y^{-1} \mathbf{e}(\vec{k})} \mathbf{y}$$

- In practice:

$$z = \mathbf{e}^H(\vec{k}) \frac{\hat{\mathbf{R}}_y^{-1}}{\mathbf{e}^H(\vec{k}) \hat{\mathbf{R}}_y^{-1} \mathbf{e}(\vec{k})} \mathbf{y}$$



MVDR Power in Output Beamformer

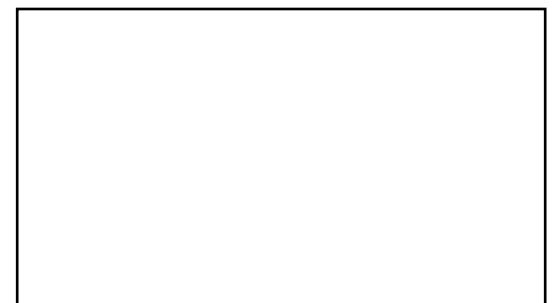
$$\mathbf{w}^H \mathbf{R}_y \mathbf{w} = \mathbf{c}^H (\mathbf{C} \mathbf{R}_y^{-1} \mathbf{C}^H)^{-1} \mathbf{c}$$

- Plug in $\mathbf{C} = \mathbf{e}^H(\vec{k})$, $\mathbf{c} = 1$

$$P^{MV}(\vec{k}) = \frac{1}{\mathbf{e}^H(\vec{k}) \mathbf{R}_y^{-1} \mathbf{e}(\vec{k})}$$

- Compare with:

$$P^{CONV}(\vec{k}) \equiv \mathbf{e}^H(\vec{k}) \mathbf{R}_y \mathbf{e}(\vec{k})$$



Using Empirical Correlation Matrices

- In practice:

$$P^{MV}(\vec{k}) \equiv \frac{1}{\mathbf{e}^H(\vec{k}) \hat{\mathbf{R}}_y^{-1} \mathbf{e}(\vec{k})}$$

$$P^{CONV}(\vec{k}) \equiv \mathbf{e}^H(\vec{k}) \hat{\mathbf{R}}_y \mathbf{e}(\vec{k})$$



Analysis of a Simple Case

- **Correlation matrix for one signal in white noise**

$$\mathbf{R}_y = P_s^2 \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}$$

$$\mathbf{R}_y^{-1} = [P_s^2 \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}]^{-1}$$



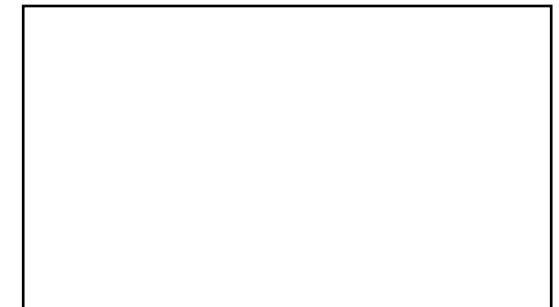
Matrix Inversion Lemma (1)

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

- See p. 489 for other variations
- Special case from p. 490:

$$B = x, \quad C = \mu, \quad D = y^H$$

$$(A + \mu xy^H)^{-1} = A^{-1} - A^{-1}x \left(y^H A^{-1}x + \frac{1}{\mu} \right)^{-1} y^H A^{-1}$$



Matrix Inversion Lemma (2)

$$\begin{aligned} & (\mathbf{A} + \mu \mathbf{x} \mathbf{y}^H)^{-1} \\ &= \mathbf{A}^{-1} - \mathbf{A}^{-1} \frac{\mathbf{x} \mathbf{y}^H}{(1/\mu) + \mathbf{y}^H \mathbf{A}^{-1} \mathbf{x}} \mathbf{A}^{-1} \\ &= \mathbf{A}^{-1} \left(\mathbf{I} - \frac{\mathbf{x} \mathbf{y}^H}{(1/\mu) + \mathbf{y}^H \mathbf{A}^{-1} \mathbf{x}} \mathbf{A}^{-1} \right) \end{aligned}$$

- Why were we doing this again?

$$\mathbf{R}_y^{-1} = \underbrace{[P_s^2 \mathbf{e} \mathbf{e}^H + \underbrace{\sigma_n^2 \mathbf{I}}_{\mathbf{A}}]}_{\mu \mathbf{x} \mathbf{y}^H}^{-1}$$



Use the MIL on the Correlation Matrix (1)

$$\begin{aligned}\mathbf{R}_y^{-1} &= \left[\underbrace{P_s^2 \mathbf{e} \mathbf{e}^H}_{\mu \mathbf{x} \mathbf{y}^H} + \underbrace{\sigma_n^2 \mathbf{I}}_{\mathbf{A}} \right]^{-1} \\ &= \mathbf{A}^{-1} \left(\mathbf{I} - \frac{\mathbf{x} \mathbf{y}^H}{(1/\mu) + \mathbf{y}^H \mathbf{A}^{-1} \mathbf{x}} \mathbf{A}^{-1} \right) \\ &= \frac{1}{\sigma_n^2} \left(\mathbf{I} - \frac{\mathbf{e} \mathbf{e}^H}{(1/A_s^2) + \underbrace{(\mathbf{e}^H \mathbf{e} / \sigma_n^2)}_M \sigma_n^2} \frac{1}{\sigma_n^2} \right)\end{aligned}$$

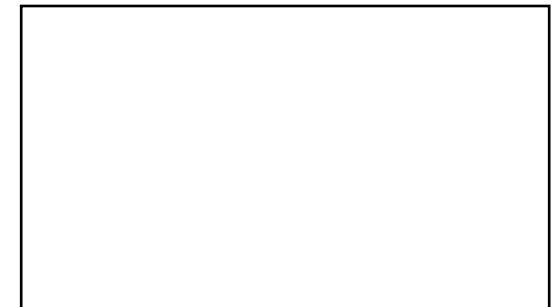


Use the MIL on the Correlation Matrix (2)

$$\mathbf{R}_y^{-1} = \frac{1}{\sigma_n^2} \left(\mathbf{I} - \frac{\mathbf{e}\mathbf{e}^H}{(1/P_s^2) + (M/\sigma_n^2)} \frac{1}{\sigma_n^2} \right)$$

$$= \frac{1}{\sigma_n^2} \left(\mathbf{I} - \frac{P_s^2 \mathbf{e}\mathbf{e}^H}{\sigma_n^2 + MP_s^2} \right)$$

$$= \frac{1}{\sigma_n^2} \left(\mathbf{I} - \frac{P_s^2 \mathbf{e}\mathbf{e}^H}{MP_s^2 + \sigma_n^2} \right)$$



MVDR Power for Simple Special Case (1)

$$P^{MV}(\vec{k}) = [\mathbf{e}^H(\vec{k}) \mathbf{R}_y^{-1} \mathbf{e}(\vec{k})]^{-1}$$

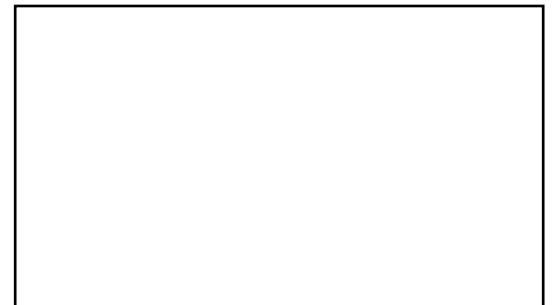
$$= \sigma_n^2 \left[\mathbf{e}^H(\vec{k}) \left(\mathbf{I} - \frac{P_s^2 \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0)}{M P_s^2 + \sigma_n^2} \right) \mathbf{e}(\vec{k}) \right]^{-1}$$

$$\mathbf{R}_y^{-1} = \frac{1}{\sigma_n^2} \left(\mathbf{I} - \frac{P_s^2 \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0)}{M P_s^2 + \sigma_n^2} \right)$$



MVDR Power for Simple Special Case (2)

$$P^{MV}(\vec{k}) =$$
$$= \sigma_n^2 \left[\mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}) - \frac{P_s^2 \mathbf{e}(\vec{k})^H \mathbf{e}(\vec{k}^0) \mathbf{e}^H(\vec{k}^0) \mathbf{e}(\vec{k})}{MP_s^2 + \sigma_n^2} \right]^{-1}$$
$$= \sigma_n^2 \left[M - \frac{P_s^2 |\mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}^0)|^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$



Max. MVDR Power for Special Case (1)

$$P^{MV}(\vec{k}) = \sigma_n^2 \left[M - \frac{P_s^2 |\mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}^0)|^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

- **Maxes out when $k = \vec{k}^0$:**

$$P^{MV}(\vec{k}^0) = \sigma_n^2 \left[M - \frac{M^2 P_s^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$



Max. MVDR Power for Special Case (2)

$$\begin{aligned} P^{MV}(\vec{k}^0) &= \sigma_n^2 \left[\frac{M(MP_s^2 + \sigma_n^2)}{MP_s^2 + \sigma_n^2} - \frac{M^2 P_s^2}{MP_s^2 + \sigma_n^2} \right]^{-1} \\ &= \sigma_n^2 \left[\frac{M\sigma_n^2}{MP_s^2 + \sigma_n^2} \right]^{-1} = \frac{MP_s^2 + \sigma_n^2}{M} \\ &= P_s^2 + \frac{\sigma_n^2}{M} \end{aligned}$$

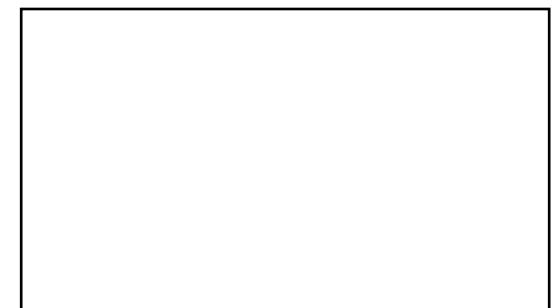
Bias → M



Min. MVDR Power for Special Case

$$P^{MV}(\vec{k}) = \sigma_n^2 \left[M - \frac{P_s^2 \left| \mathbf{e}^H(\vec{k}) \mathbf{e}(\vec{k}^0) \right|^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

- **Smallest possible:** $P^{MV}(\vec{k}) = \sigma_n^2 / M$
when $\left| \mathbf{e}^H(\vec{k}) \mathbf{e}(\vec{k}^0) \right| = 0$



Peak-to-Sidelobe Ratio for Special Case

$$\frac{\max_{\vec{k}} P^{MV}(\vec{k})}{\min_{\vec{k}} P^{MV}(\vec{k})} = \frac{P_s^2 + (\sigma_n^2 / M)}{\sigma_n^2 / M}$$
$$= M \left(\frac{P_s^2}{\sigma_n^2} \right) + 1$$

- **PSR is a function of SNR!**
(This is **not the case with conventional beamforming.)**

