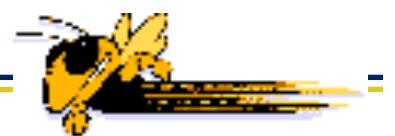

Cramér-Rao Bounds for Arrays

**ECE 6279: Spatial Array Processing
Fall 2013
Lecture 24**

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References

- P. Stoica and A. Nehorai, "MUSIC, Maximum Likelihood, and Cramer-Rao Bound", *IEEE Trans. on Acoustics, Speech, and Signal Proc.*, Vol. 37, No. 5, May 1989, pp. 720-741.
- P. Stoica and A. Nehorai, "MUSIC, Maximum Likelihood, and Cramer-Rao Bound: Further Results and Comparisons", *IEEE Trans. on Acoustics, Speech, and Signal Proc.*, Vol. 38, No. 12, Dec. 1990, pp. 2140-2150.
- Van Trees Vol. IV: *Optimum Array Processing*



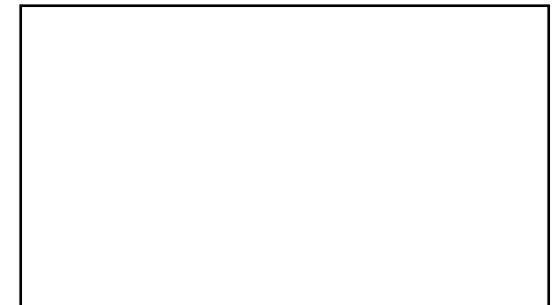
Review of “Deterministic Gaussian” Model

- N_s sources in additive noise

$$\underline{\mathbf{y}}(l) = \sum_{n=1}^{N_s} \mathbf{e}(\theta_n) s_n(l) + \underline{\mathbf{n}}(l)$$
$$\underline{\mathbf{n}} \sim \mathcal{CN}(0, \mathbf{K}_n)$$

$$\mathbf{s}(l) = \begin{bmatrix} s_1(l) \\ \vdots \\ s_{N_s}(l) \end{bmatrix}$$

Modeled as
deterministic
parameters we
need to estimate

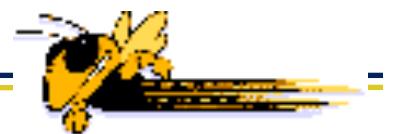


Maximum-Likelihood Procedure

- Luckily, it so happened that if we fix a particular Θ , we could find a closed form solution for the maximizing $s(l)$ and plug that into the likelihood; turns out we want to maximize

$$\text{tr} \left\{ \mathbf{D}(\Theta) [\mathbf{D}^H(\Theta) \mathbf{D}(\Theta)]^{-1} \mathbf{D}^H(\Theta) \hat{\mathbf{R}}_y \right\}$$

- No closed form solution



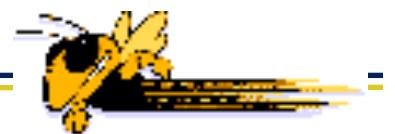
Assumption for This Presentation

$$\mathbf{y}(l) = \mathbf{D}(\Theta)\mathbf{s}(l) + \mathbf{n}(l)$$

$$\mathbf{D}(\Theta) = [\mathbf{e}(\phi_1) \cdots \mathbf{e}(\phi_{N_s})]$$

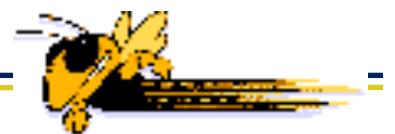
$$\Theta = [\phi_1 \cdots \phi_{N_s}] \text{ (special case)}$$

- For this lecture, assume that each signal direction is parameterized by one parameter



Additional Assumptions for This Lecture

- This lecture will assume that
$$K_n = \sigma^2 I$$
- We will also assume that σ^2 is unknown
- Turns out we can treat the variance like the signals; ML procedure winds up not changing!



A Wanton Abuse of Notation

$$\mathbf{D}(\Theta) = \begin{bmatrix} \mathbf{e}(\phi_1) & \cdots & \mathbf{e}(\phi_{N_s}) \end{bmatrix}$$

$$\frac{d\mathbf{D}(\Theta)}{d\Theta} \equiv \begin{bmatrix} \frac{d\mathbf{e}(\phi_1)}{d\phi_1} & \cdots & \frac{d\mathbf{e}(\phi_{N_s})}{d\phi_{N_s}} \end{bmatrix}$$



Some More Notation

$$\mathbf{S}(l) \equiv \begin{bmatrix} s_1(l) \\ \ddots \\ s_{N_S}(l) \end{bmatrix}$$



F.I.M. for Deterministic Gaussian Model

- Stoica & Nehorai, M, ML, CRB, Eq. 4.1:

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} \text{Re} \left\{ \mathbf{S}^H(l) \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \times \right.$$
$$[\mathbf{I} - \mathbf{D}(\Theta)\{\mathbf{D}^H(\Theta)\mathbf{D}(\Theta)\}^{-1}\mathbf{D}^H(\Theta)] \times$$
$$\left. \frac{d\mathbf{D}(\Theta)}{d\Theta} \mathbf{S}(l) \right\}$$



Special Case: One Target, Linear Array

$$\mathbf{D}(\Theta) = \mathbf{e}(\gamma) = \begin{bmatrix} 1 \\ e^{j\gamma} \\ \vdots \\ e^{j(M-1)\gamma} \end{bmatrix}$$



Derivative Vector in the Special Case

$$\frac{d\mathbf{D}(\Theta)}{d\Theta} = \begin{bmatrix} 0 \\ je^{j\gamma} \\ \vdots \\ j(M-1)e^{j(M-1)\gamma} \end{bmatrix}$$



Useful Tidbits (1)

$$\frac{d\mathbf{D}^H(\Theta)}{d\Theta} \frac{d\mathbf{D}(\Theta)}{d\Theta} =$$

$$\sum_{m=0}^{M-1} (-jme^{-jm\gamma})(jme^{jm\gamma}) = \sum_{m=0}^{M-1} m^2$$
$$= \frac{M(M-1)(2M-1)}{6}$$



Useful Tidbits (2)

$$\frac{d\mathbf{D}^H(\Theta)}{d\Theta} \mathbf{D}(\Theta) = \sum_{m=0}^{M-1} (-jme^{-jmy})(e^{jmy})$$

$$= -j \sum_{m=0}^{M-1} m = -j \frac{M(M-1)}{2}$$

$$\mathbf{D}^H(\Theta) \frac{d\mathbf{D}(\Theta)}{d\Theta} = j \frac{M(M-1)}{2}$$



Here's That General F.I.M. Again

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} \operatorname{Re} \left\{ \mathbf{S}^H(l) \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \times \right.$$
$$[\mathbf{I} - \mathbf{D}(\Theta)\{\mathbf{D}^H(\Theta)\mathbf{D}(\Theta)\}^{-1}\mathbf{D}^H(\Theta)] \times$$
$$\left. \frac{d\mathbf{D}(\Theta)}{d\Theta} \mathbf{S}(l) \right\}$$



Simplifying for This Special Case

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2 \times$$

$$\text{Re} \left\{ \frac{d\mathbf{D}^H(\Theta)}{d\Theta} \left[\mathbf{I} - \frac{\mathbf{D}(\Theta)\mathbf{D}^H(\Theta)}{M} \right] \frac{d\mathbf{D}(\Theta)}{d\Theta} \right\}$$



Some Minor Rearranging

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2 \times$$
$$\text{Re} \left\{ \frac{\frac{d\mathbf{D}^H(\Theta)}{d\Theta} \frac{d\mathbf{D}(\Theta)}{d\Theta}}{d\Theta} - \frac{\frac{d\mathbf{D}^H(\Theta) \mathbf{D}(\Theta) \mathbf{D}^H(\Theta)}{d\Theta} \frac{d\mathbf{D}(\Theta)}{d\Theta}}{M} \right\}$$



Using the Useful Tidbits

$$F(\Theta) = \frac{2}{\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2 \times \text{Re} \left\{ \frac{M(M-1)(2M-1)}{6} \right.$$
$$\left. - \left[-j \frac{M(M-1)}{2} \right] \left[j \frac{M(M-1)}{2} \right] / M \right\}$$



After an Insane Amount of Algebra

$$F(\Theta) = \frac{M(M^2 - 1)}{6\sigma^2} \sum_{l=0}^{L-1} |s(l)|^2$$

$$SNR = \frac{1}{L} \sum_{l=0}^{L-1} |s(l)|^2 \Bigg/ \sigma^2$$

$$F(\Theta) \approx \frac{M^3 L}{6} SNR$$



The Big Picture

$$\text{var}_\gamma[\hat{\gamma}(\underline{y})] \geq \frac{1}{F(\gamma)}$$

$$F(\gamma) \approx \frac{M^3 L}{6} SNR$$

$$\text{var}_\gamma[\hat{\gamma}(\underline{y})] \gtrsim \frac{6}{M^3 L(SNR)}$$

