Wavenumber-Frequency Space

ECE 6279: Spatial Array Processing
Fall 2013
Lecture 3

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Where We Are in J&D

- **Material drawn from Sec. 2.5**

- **For now, we will skip Sec. 2.6 on random space-time fields (but we will come back to those ideas later)**

Different Definitions of FT Pairs

- **Engineer's style:**

$$S^{eng}(\omega) = \int_{-\infty}^{\infty} s(t)\exp(-j\omega t)\,dt$$

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{eng}(\omega)\exp(j\omega t)\,d\omega$$

- **Mathematician's style:**

$$S^{math}(\omega) = \int_{-\infty}^{\infty} s(t)\exp(+j\omega t)\,dt$$

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{math}(\omega)\exp(-j\omega t)\,d\omega$$

Adapting an Engineer's FT Table

$$S^{math}(\omega) = \int_{-\infty}^{\infty} s(t)\exp(+j\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} s(t)\exp\left[-j(-\omega)t\right]dt = S^{eng}(-\omega)$$

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{math}(\omega)\exp(-j\omega t)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{math}(\omega)\exp\left[j\omega(-t)\right]d\omega$$

$$= F_{eng}^{-1}\{S^{math}\}(-t)$$

Fourier Transforms of a Delta Function

- **Engineer's style:**

$$S(\omega) = \int_{-\infty}^{\infty} \boxed{\delta(t)} \exp(-j\omega t)\, dt = \int_{-\infty}^{\infty} \delta(t) \exp(-j\omega 0)\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- **Mathematician's style:**

$$S(\omega) = \int_{-\infty}^{\infty} \delta(t) \exp(+j\omega t)\, dt = 1$$

Inverse FT of a Delta Function

- **Engineer's style (works for math style too):**

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{\delta(\omega)} \exp(j\omega t)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) \exp(j0t)\, d\omega = \frac{1}{2\pi}$$

$$1 \overset{F}{\Leftrightarrow} 2\pi\delta(\omega)$$

From previous slide: $\delta(t) \overset{F}{\Leftrightarrow} 1$

Time Shift Property: Engineer's Style

- **Engineer's style:**

$$S_{timesh}^{eng}(\omega) = \int_{-\infty}^{\infty} \boxed{s(t - t_0)} \exp(-j\omega t)\, dt$$

substitute $\tilde{t} = t - t_0, \quad t = \tilde{t} + t_0$

$$S_{timesh}^{eng}(\omega) = \int_{-\infty}^{\infty} s(\tilde{t}) \exp\left[-j\omega(\tilde{t} + t_0)\right] dt$$

$$= \exp(-j\omega t_0) \int_{-\infty}^{\infty} s(\tilde{t}) \exp(-j\omega \tilde{t})\, d\tilde{t}$$

$$= \exp(-j\omega t_0) S^{eng}(\omega)$$

Time Shift Prop.: Mathematician's Style

- **Mathematician's style:**

$$S_{timesh}^{math}(\omega) = \int_{-\infty}^{\infty} \boxed{s(t - t_0)} \exp(+j\omega t)\, dt$$

substitute $\tilde{t} = t - t_0, \quad t = \tilde{t} + t_0$

$$S_{timesh}^{math}(\omega) = \int_{-\infty}^{\infty} s(\tilde{t}) \exp\left[+j\omega(\tilde{t} + t_0)\right] dt$$

$$= \exp(+j\omega t_0) \int_{-\infty}^{\infty} s(\tilde{t}) \exp(+j\omega \tilde{t})\, d\tilde{t}$$

$$= \exp(+j\omega t_0) S^{math}(\omega)$$

Freq. Shift Property: Engineer's Style

- **Engineer's style:**

$$s_{freqsh}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{S^{eng}(\omega - \omega_0)} \exp(j\omega t)\, d\omega$$

substitute $\tilde{\omega} = \omega - \omega_0, \quad \omega = \tilde{\omega} + \omega_0$

$$s_{freqsh}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{eng}(\tilde{\omega}) \exp\left[j(\tilde{\omega} + \omega_0)t\right] d\omega$$

$$= \exp(j\omega_0 t)\frac{1}{2\pi}\int_{-\infty}^{\infty} S^{eng}(\tilde{\omega}) \exp(j\tilde{\omega} t)\, d\tilde{\omega}$$

$$= \exp(j\omega_0 t) s(t)$$

Freq. Shift Prop.: Mathematician's Style

- **Mathematician's style:**

$$s_{freqsh}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \boxed{S^{math}(\omega - \omega_0)} \exp(-j\omega t)\, d\omega$$

substitute $\tilde{\omega} = \omega - \omega_0, \quad \omega = \tilde{\omega} + \omega_0$

$$s_{freqsh}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S^{math}(\tilde{\omega}) \exp\left[-j(\tilde{\omega} + \omega_0)t\right] d\omega$$

$$= \exp(-j\omega_0 t)\frac{1}{2\pi}\int_{-\infty}^{\infty} S^{math}(\tilde{\omega}) \exp(-j\tilde{\omega} t)\, d\tilde{\omega}$$

$$= \exp(-j\omega_0 t) s(t)$$

Quick Proofs of Math-Style Shift Props.

- **Time shift:**

$$s(t-t_0) \overset{Feng}{\Leftrightarrow} \exp(-j\omega t_0)S^{eng}(\omega)$$

$$s(t-t_0) \overset{Fmath}{\Leftrightarrow} \exp(+j\omega t_0)S^{eng}(-\omega)$$

$$= \exp(+j\omega t_0)S^{math}(\omega)$$

- **Frequency shift:**

$$\exp(-j\omega_0 t)s(t) \overset{Feng}{\Leftrightarrow} S^{eng}(\omega+\omega_0)$$

$$\exp(-j\omega_0 t)s(t) \overset{Fmath}{\Leftrightarrow} S^{eng}(-\omega+\omega_0)$$

$$= S^{math}(\omega-\omega_0)$$

Special Case: Deltas and Constants

- **Engineer's style:**

$$\delta \longrightarrow s(t-t_0) \overset{Feng}{\Leftrightarrow} \exp(-j\omega t_0)\overbrace{S^{eng}(\omega)}^{1}$$

$$\exp(j\omega_0 t)\underbrace{s(t)}_{1} \overset{Feng}{\Leftrightarrow} S^{eng}(\omega-\omega_0) \longleftarrow 2\pi\delta$$

- **Mathematician's style:**

$$\delta \longrightarrow s(t-t_0) \overset{Fmath}{\Leftrightarrow} \exp(+j\omega t_0)\overbrace{S^{math}(\omega)}^{1}$$

$$\exp(-j\omega_0 t)\underbrace{s(t)}_{1} \overset{Fmath}{\Leftrightarrow} S^{math}(\omega-\omega_0) \longleftarrow 2\pi\delta$$

Transforms of Delta Functions

- **Engineer's style:**

$$\delta(t - t_0) \overset{Feng}{\Leftrightarrow} \exp(-j\omega t_0)$$

$$\exp(j\omega_0 t) \overset{Feng}{\Leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

- **Mathematician's style:**

$$\delta(t - t_0) \overset{Fmath}{\Leftrightarrow} \exp(+j\omega t_0)$$

$$\exp(-j\omega_0 t) \overset{Fmath}{\Leftrightarrow} 2\pi\delta(\omega - \omega_0)$$

Space-Time FT Pair

- **A 4-D S-T Fourier transform**

$$S(\vec{k},\omega) = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}}_{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} s(\vec{x},t)\exp\left\{-j\left(\omega t - \vec{k}\cdot\vec{x}\right)\right\}\underbrace{d\vec{x}}_{dxdydz}\,dt$$

“Engineer’s style” in time

“Mathematician’s style” in space

- **A 4-D S-T Inverse Fourier transform**

$$s(\vec{x},t) = \frac{1}{(2\pi)^4}\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}}_{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}} S(\vec{k},\omega)\exp\left\{j\left(\omega t - \vec{k}\cdot\vec{x}\right)\right\}\underbrace{d\vec{k}}_{dk_x dk_y dk_z}\,d\omega$$

Take Home Message

- **Just like "any" 1-D function can be written as a weighted integral of complex exponentials** $\exp(j\omega t)$ **…**

- **…"any" space-time signal - even nonpropagating ones! - can be written as a weighted integral of propagating plane waves**

$$\exp\left[j\left(\omega t - \vec{k}\cdot\vec{x}\right)\right]$$

Monochromatic Plane Wave

- **What's the 4-D S-T FT of**

$$s(\vec{x},t) = \exp\left\{ j\left(\omega_0 t - \vec{k}^0 \cdot \vec{x}\right)\right\}$$

$$S(\vec{k},\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\vec{x},t)\exp\left\{-j\left(\omega t - \vec{k}\cdot\vec{x}\right)\right\} d\vec{x}dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(j\omega_0 t)\exp(-j\omega t)\exp(-j\vec{k}^0\cdot\vec{x})\exp(j\vec{k}\cdot\vec{x}) d\vec{x}dt$$

$$= (2\pi)^4 \delta(\vec{k}-\vec{k}^0)\delta(\omega-\omega_0) \leftarrow$$

A point in wavenumber-frequency space

where $\delta(\vec{v}) \equiv \delta(v_x)\delta(v_y)\delta(v_z)$

General Plane Wave

- **What's the 4-D S-T FT of**

$$ss(\vec{x},t) = s(t - \vec{\alpha}^0 \cdot \vec{x})$$

Notation borrowed from Chris Barnes

- **Take Eng. FT in time domain first:**

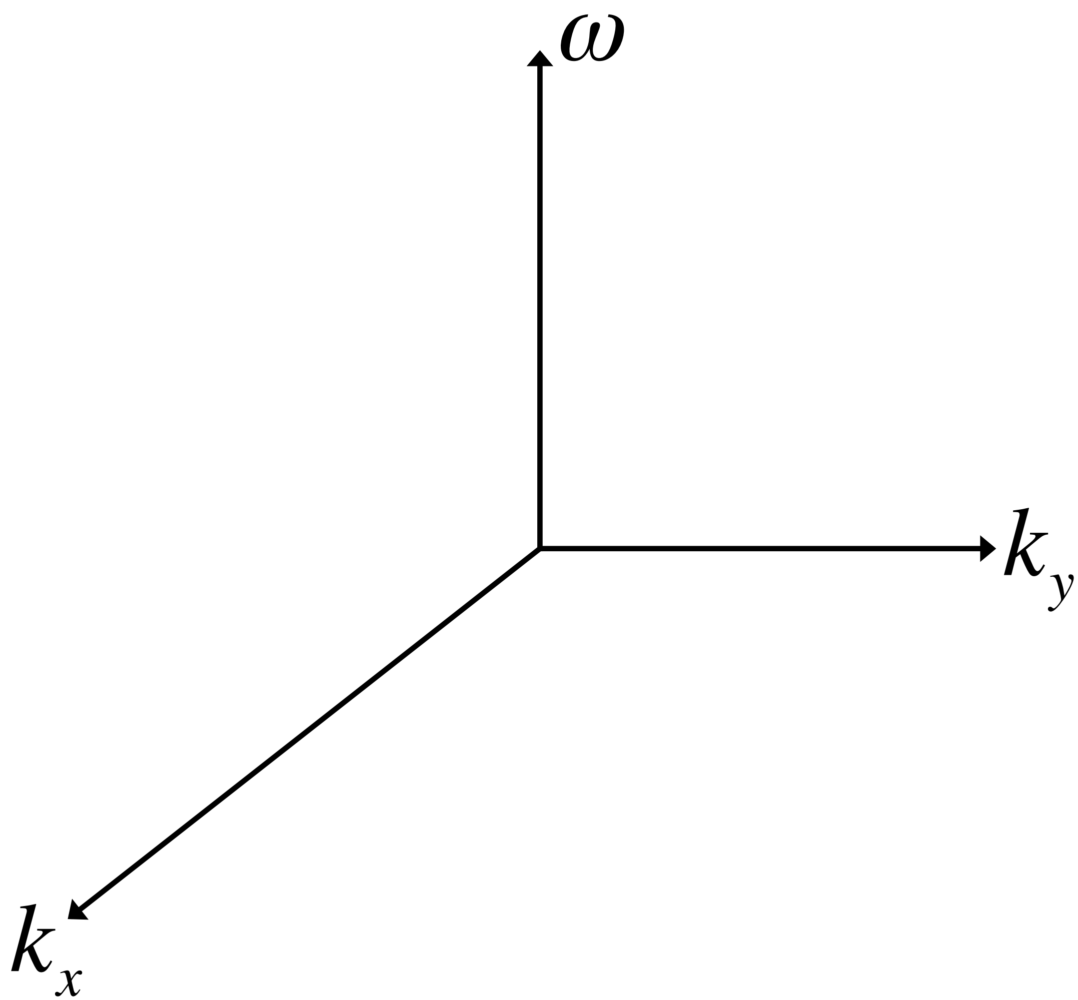
$$sS(\vec{x},\omega) = S(\omega)\exp(-j\omega\vec{\alpha}^0 \cdot \vec{x})$$

- **Then Math. FT in spatial domain:**

$$SS(\vec{k},\omega) = S(\omega)(2\pi)^3\delta(\vec{k} - \omega\vec{\alpha}^0)$$

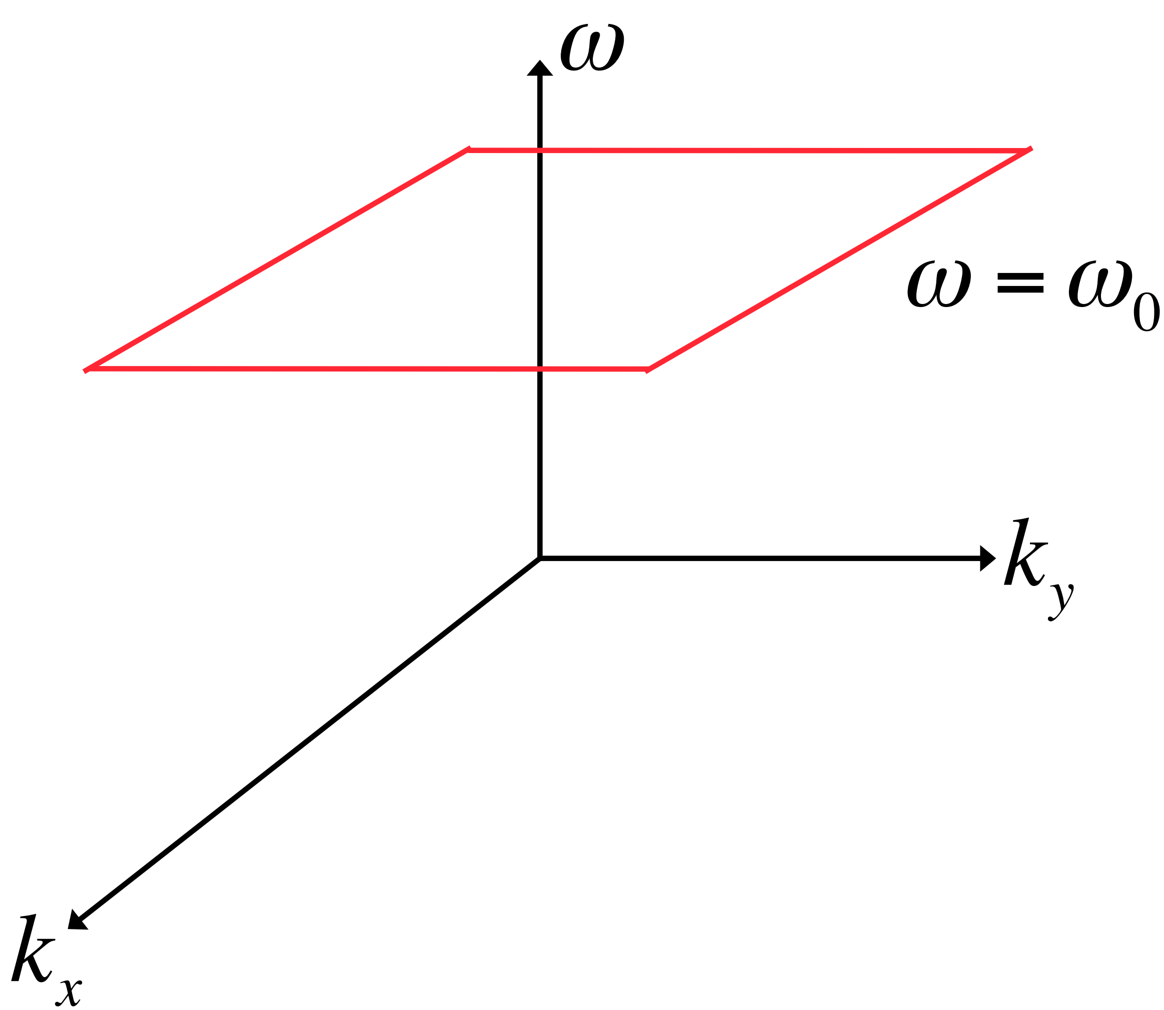
A line in wavenumber-frequency space

Axes for Showing S-T Fourier Support



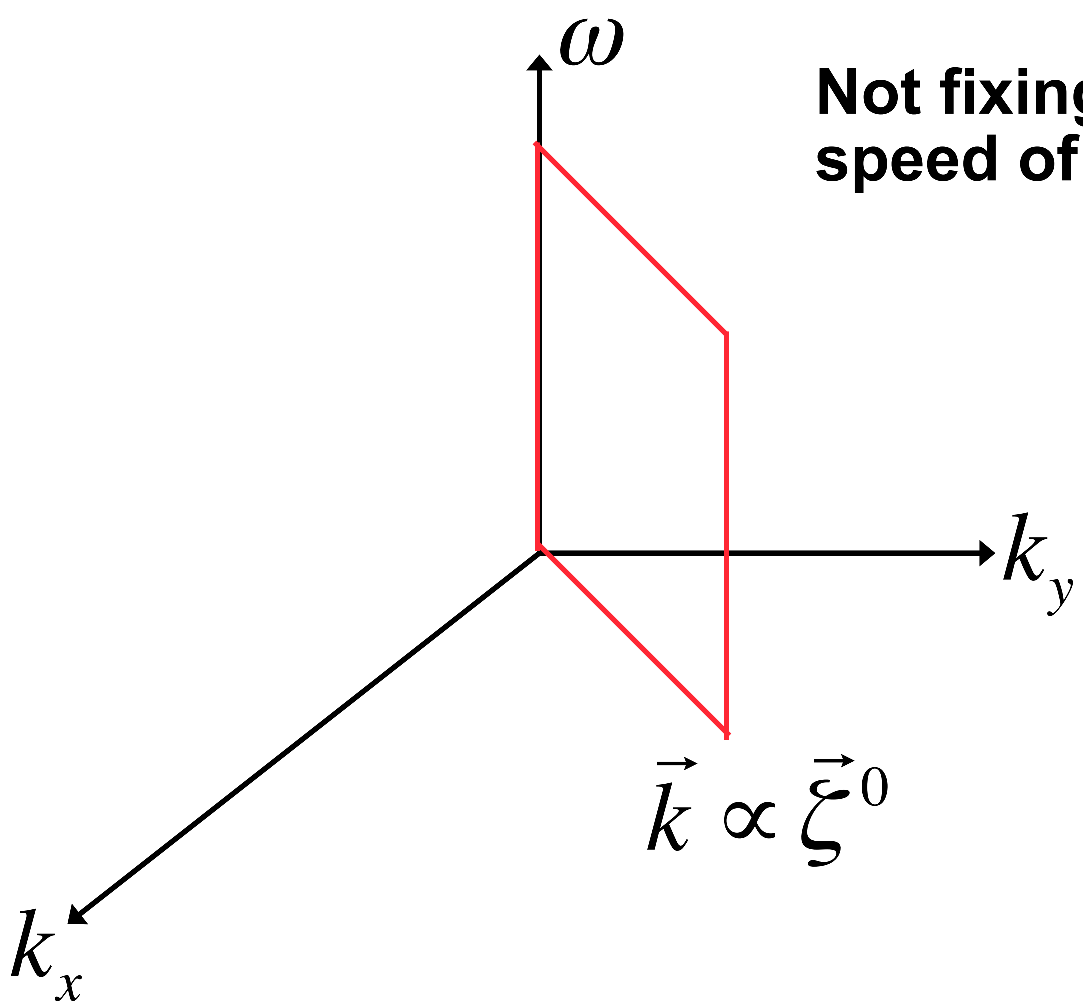
J&D, pp. 42-43

Narrowband, Nonpropagating S-T Signal



J&D, pp. 42-43

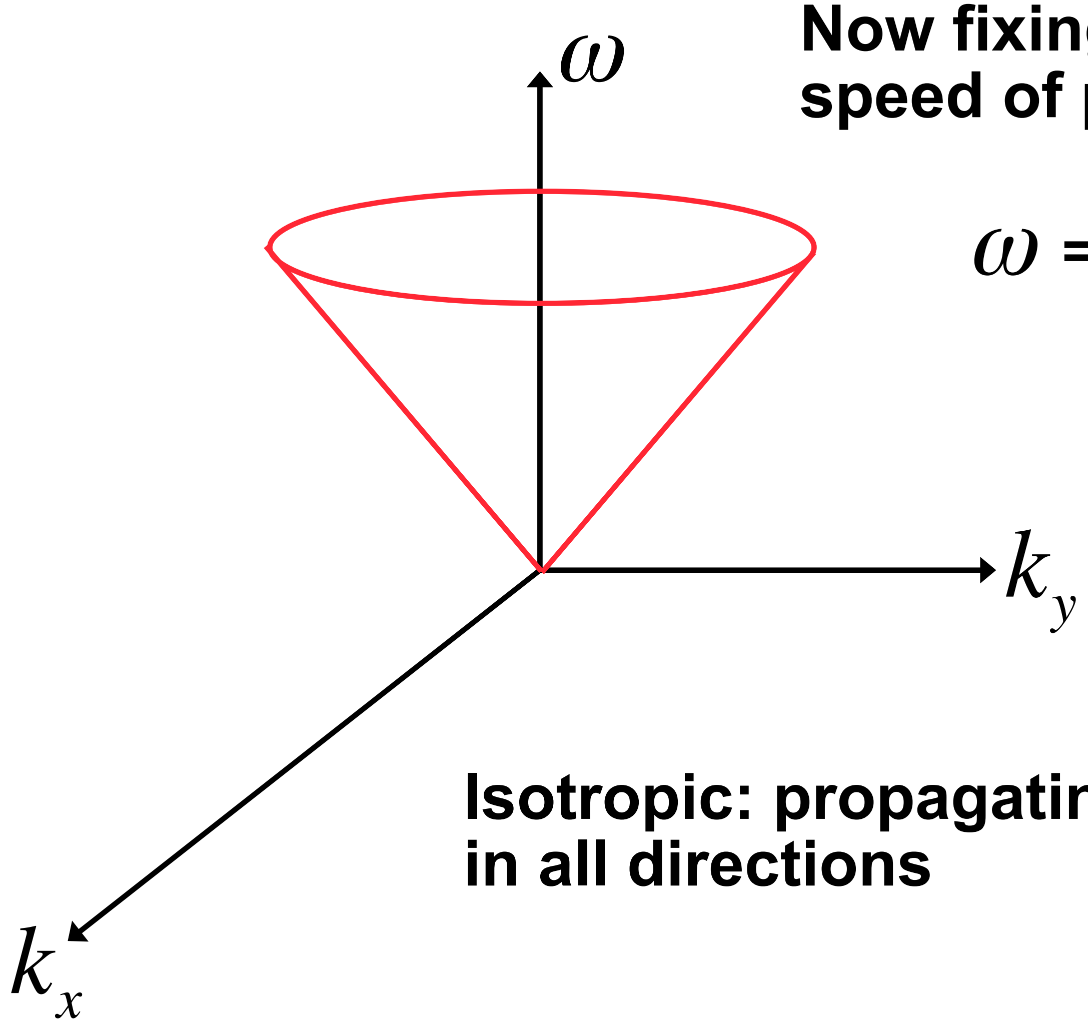
Wideband, Directional S-T Signal



Not fixing a specific speed of propagation!

J&D, pp. 42-43

Wideband, Iso., Fixed-Speed S-T Signal



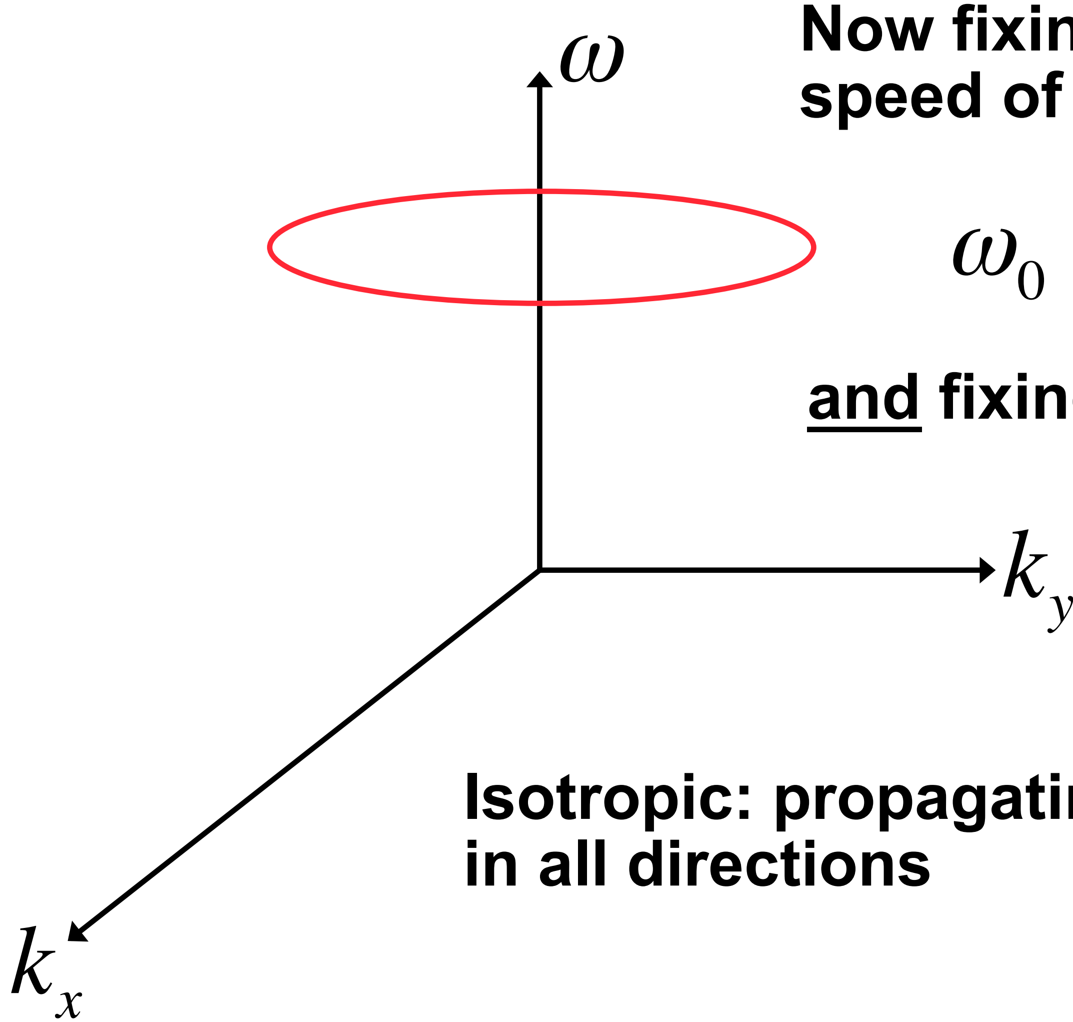
Now fixing a specific speed of propagation

$$\omega = c\left|\vec{k}\right|$$

Isotropic: propagating in all directions

J&D, pp. 42-43

Narrowband, Iso., Fixed-Speed S-T Signal



Now fixing a specific speed of propagation

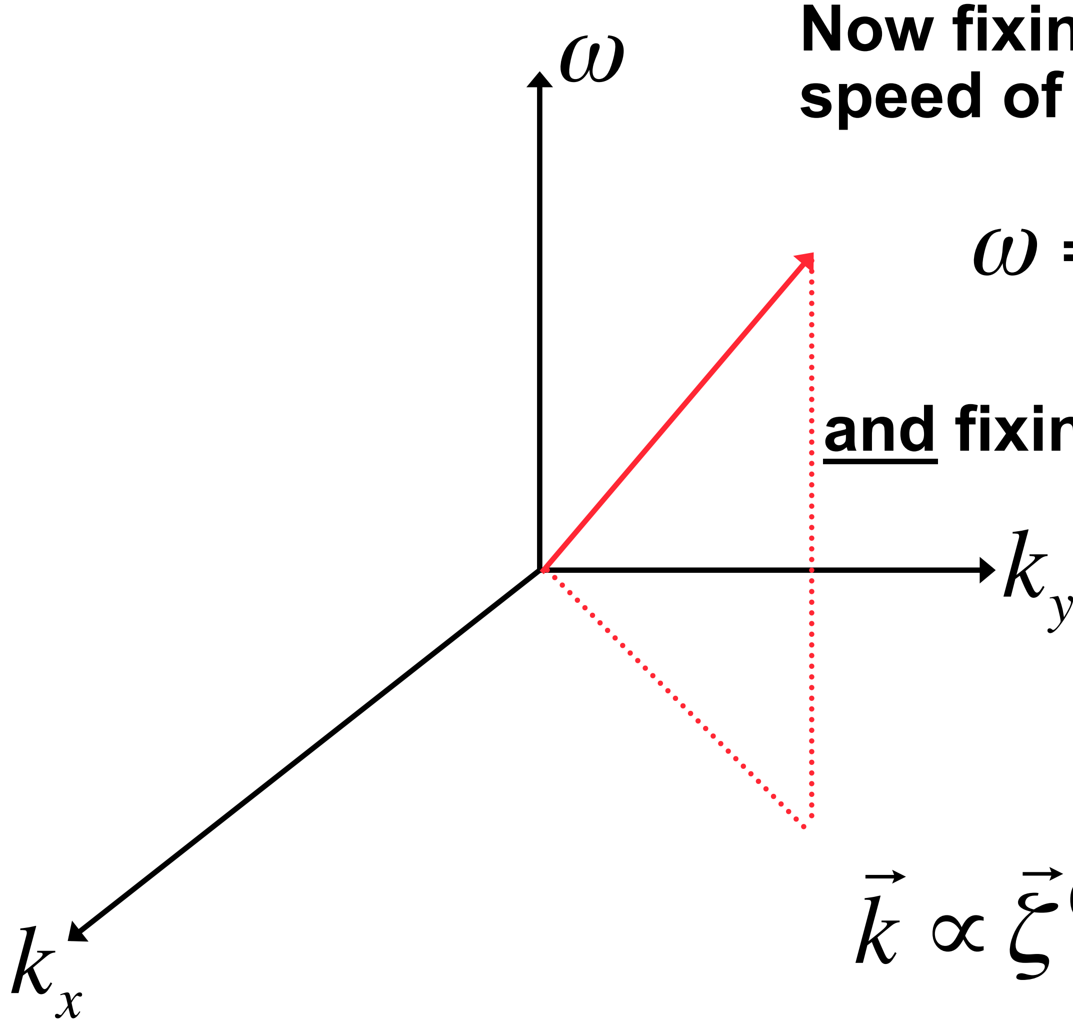
$$\omega_0 = c\left|\vec{k}\right|$$

<u>and</u> fixing frequency

Isotropic: propagating in all directions

J&D, pp. 42-43

Wideband, Dir., Fixed-Speed S-T Signal



Now fixing a specific speed of propagation

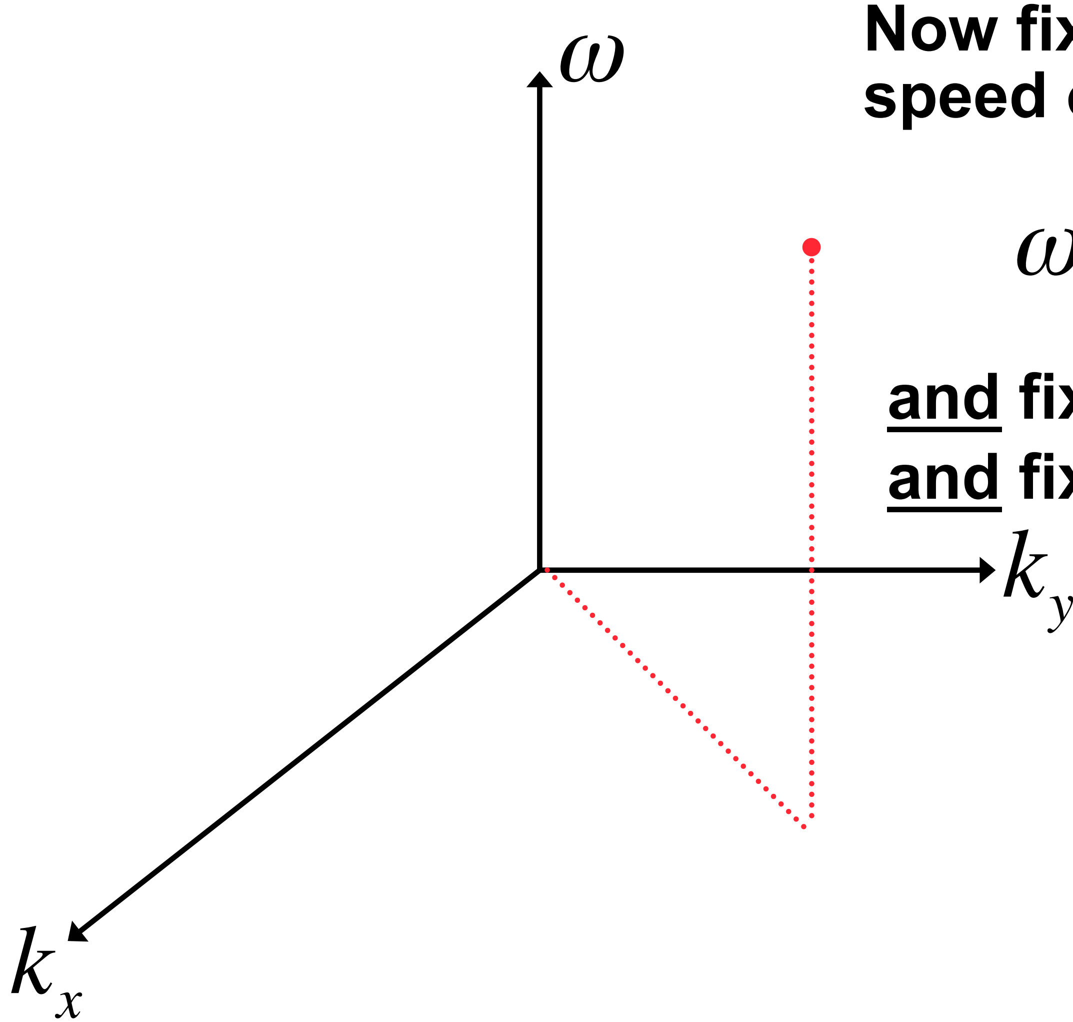
$$\omega = c\left|\vec{k}\right|$$

<u>and</u> fixing direction

$$\vec{k} \propto \vec{\xi}^{\,0}$$

J&D, pp. 42-43

Narrowband, Dir., Fixed-Speed S-T Signal



Now fixing a specific speed of propagation

$$\omega_0 = c\left|\vec{k}\right|$$

<u>and</u> fixing frequency
<u>and</u> fixing direction

J&D, pp. 42-43

Monochromatic Spherical Wave

- **What's the 4-D S-T FT of**

$$s(r,t) = \exp\left\{ j\left(\omega_0 t - k^0 r\right)\right\} / r$$

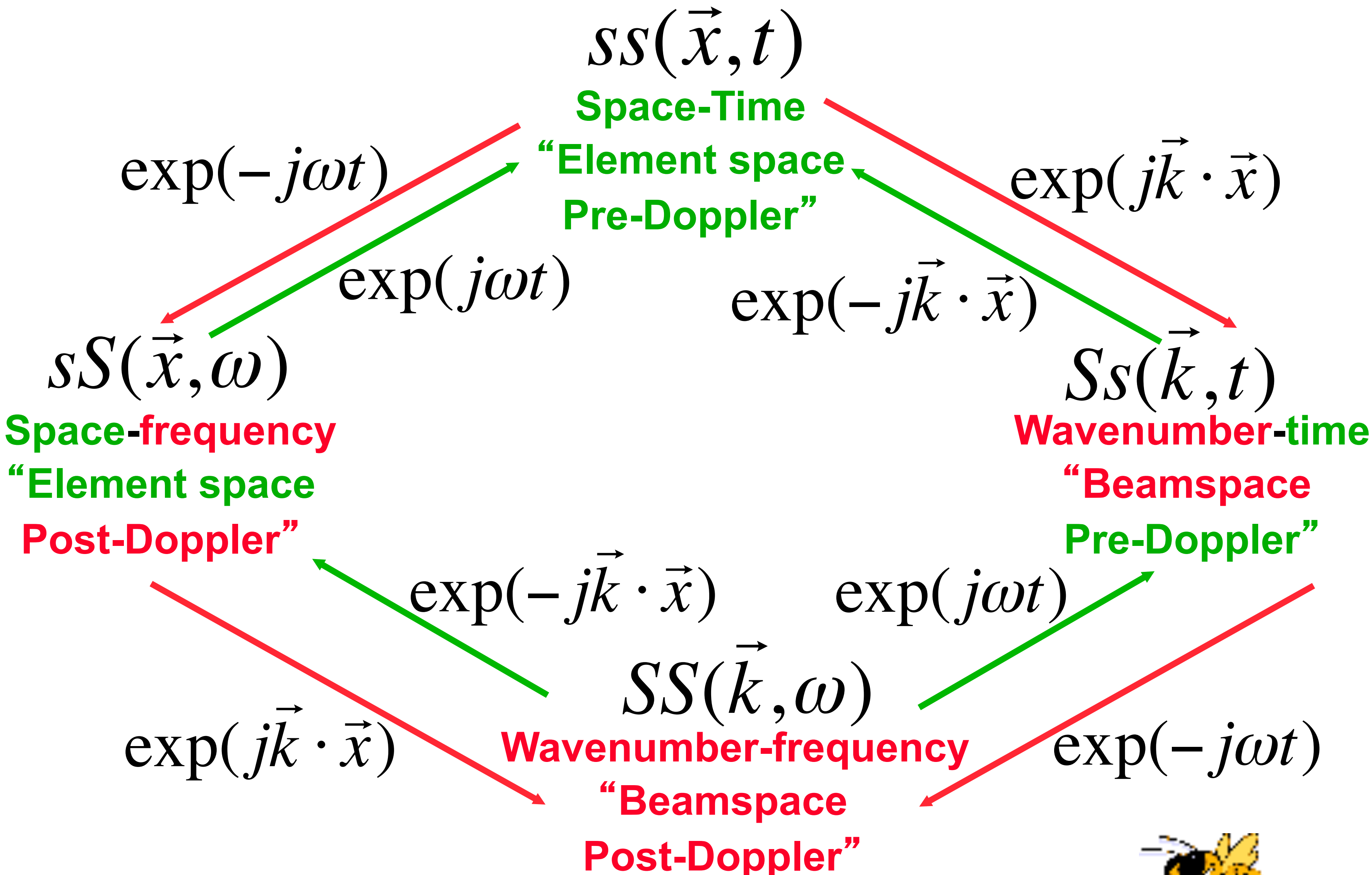
- **With polar wavenumber coordinates:**

$$S(k,\omega) =$$

$$\left[\frac{2\pi^2}{jk^0}\delta(k-k^0) + \frac{4\pi}{k^2-(k^0)^2}\right]\delta(\omega-\omega_0)$$

(at least according to J&D, p. 44)

Doug Williams' Chart



Filtering to Extract Information

- **Filter data in wavenumber-frequency space:**

$$Y(\vec{k},\omega) = H(\vec{k},\omega)S(\vec{k},\omega)$$

- **Ideal examples:**
  - Focus on one frequency

$$H(\vec{k},\omega) = \delta(\omega - \omega_0)$$

  - Focus in one direction

$$H(\vec{k},\omega) = \delta(\vec{k} - \vec{k}^0)$$

Spatiotemporal Convolution

- **Multiplication in Fourier domain…**

$$Y(\vec{k},\omega) = H(\vec{k},\omega)S(\vec{k},\omega)$$

- **Corresponds to convolution in space-time domain:**

$$y(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\vec{x}-\vec{\xi},t-\tau)s(\vec{\xi},\tau)d\vec{\xi}d\tau$$

- **Hence ideal filters on previous slide aren't practical - have infinite extent in space-time**

Spatiotemporal Filter Design Problem

- **Challenge is to find a space-time impulse response $h(\vec{x},t)$ that gets close to the desired $H(\vec{k},\omega)$ under some constraints:**
  - If we want real-time implementation, temporal support must be restricted to $t>0$ (causality)
    - **Tricks from ECE4270 come into play**
  - More importantly, spatial support must be limited to where you can put sensors!
    - **New spin in ECE6279**