E9 231: Digital Array Signal Processing

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1 Topics

- MVDR (Capon Beamformer).
- Subspace Methods:
  1. MUSIC
  2. min Norm
  3. ESPRIT.

2 Class Notes

2.1 MVDR Beamformer (Capon Beamformer)

$$\widehat{P}_B(\psi) = \frac{1}{K}\sum_{k=1}^{K} |V^H(\psi)X_k|^2, \qquad -\pi \le \psi \le \pi. \tag{1}$$

$$\widehat{P}_{mvdr}(\psi) = \frac{1}{K}\sum_{k=1}^{K} |W^H_{mvdr}(\psi)X_k|^2, \qquad -\pi \le \psi \le \pi. \tag{2}$$

$$W_{mvdr} = \frac{C_X^{-1}V(\psi)}{V^H(\psi)C_X^{-1}V(\psi)} \tag{3}$$

Then $\widehat{P}_{mvdr}(\psi)$ can be written as,

$$\widehat{P}_{mvdr}(\psi) = \frac{1}{K}\sum_{k=1}^{K} \frac{|V^H(\psi)C_X^{-1}X_k|^2}{(V^H(\psi)C_X^{-1}V(\psi))^2} \tag{4}$$

The numerator $|V^H(\psi)C_X^{-1}X_k|^2$ can be expressed as $V^H(\psi)C_X^{-1}X_kX_k^HC_X^{-1}V(\psi)$,

$$\widehat{P}_{mvdr}(\psi) = \frac{V^H(\psi)C_X^{-1}(\frac{1}{K}\sum_{k=1}^{K} X_kX_k^H)C_X^{-1}V(\psi)}{(V^H(\psi)C_X^{-1}V(\psi))^2} \tag{5}$$

which simplifies to,

$$\widehat{P}_{mvdr}(\psi) = \frac{1}{V^H(\psi)C_X^{-1}V(\psi)} \tag{6}$$

Now find D peaks of $\widehat{P}_{mvdr}(\psi)$. Otherwise, consider $Q_{mvdr}(\psi) = V^H(\psi)C_X^{-1}V(\psi)$ and look for D minimas, where $Q_{mvdr}(\psi)$, is the null spectrum.

The disadvantages of these algorithms are that they do not assume anything about the signal structure and all of them depend on how well S_x estimation is done.

2.2 Subspace Algorithms

The Signal model is,

$$S_X = VS_fV^H + \sigma_n^2 I \tag{7}$$

Where $V = [V(\psi_1)|V(\psi_2)|....|V(\psi_D)]$, Assume that D is known.
Then compute Eigen decomposition of S_X,

$$S_X = \sum_{i=1}^{N} \lambda_i \Phi_i \Phi_i^H = \Phi \Lambda \Phi^H \tag{8}$$

Where the columns of Φ are $\Phi_1, \Phi_2 ... \Phi_n$ (the Eigen-vectors) and λ_i's are Eigen-values

Also $\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_D > \lambda_{D+1} = \lambda_{D+2}.. = \lambda_N = \sigma_n^2$. The Eigen-vectors $\Phi_1, \Phi_2, ...\Phi_D$ span the Signal + Noise Subspace. Whereas, the Eigen-vectors $\Phi_{D+1}, \Phi_{D+2}, ...\Phi_N$ span the Noise only Subspace. So we can represent them as $U_S \triangleq [\Phi_1, \Phi_2, ...\Phi_D]$ and $U_N \triangleq [\Phi_{D+1}, \Phi_{D+2}, ...\Phi_N]$. Also we have the following relations $V(\psi_l) \in R(U_S),\ R(V) = R(U_S),\ V(\psi_l) \perp R(U_N)$ where $l = 1, \ldots, D$.

$V(\psi_l) \perp R(U_N)$, motivates the following procedure to find the D values:

1) Pick e $\in R(U_N)$

2) Scan ψ and compute $|V^H(\psi)e|^2$

3) Select D minima/nulls.

Candidates for doing this are (i) MUSIC, (ii) Minimum Norm, (iii)ESPRIT.

2.2.1 MUSIC (Multiple Signal Classifications)

$$\widehat{Q}_{MU}(\psi) = V^H(\psi)(\sum_{i=D+1}^{N} \widehat{\Phi}_i \widehat{\Phi}_i^H)V(\psi) \tag{9}$$

Some observations are as given below:

1) Look for D Nulls to find the DOA.

2) $\widehat{Q}_{MU}(\psi) = V^H(\psi)P_N V(\psi)$, where $P_N = \widehat{U_N}\hat{U_N^H},\ \hat{U_N} = [\widehat{\Phi}_{D+1}, \widehat{\Phi}_{D+2}, ...\widehat{\Phi}_N]$

3) Equal Weights to all projections.

4) "Averages" the noise Eigen-vectors.

5) Averaging provides Statistical robustness.

6) $P_N = I - U_S U_S^H$.

7) Uses all the vectors in the noise subspace.

8) Standard Linear Array $V(\psi) = [1\ z\ z^2....\ z^{(N-1)}]$. It becomes a root finding algorithm (called Root-MUSIC Algorithm)

2.2.2 Min-Norm Algorithm

Uses one vector in the noise only subspace. The claim is greater accuracy in estimating $\widehat{\psi}_l$ and less likely to generate false positive sources.

Consider $\underline{e} = [1\ d_2\ .\ .\ .\ d_N]^T$ and $\underline{d} = [d_2\ .\ .\ .\ d_N]^T$

If $\underline{e}$ lies in the noise only subspace then $V^H(\psi_i)\underline{e} = 0,\ i = 1,\ 2,\ .\ .\ .\ D$

$$D(z) = \sum_{i=1}^{N} d_i z^{i-1} \tag{10}$$

$D(z)$ will have D zeros corresponding to ψ_l, $z_k = e^{j\psi_l}$

The Procedure to find the zeros is given below:

1) Find the $(N-D)$ dimensional noise only Subspace.

2) Find $\underline{e}$ as a linear combination of the $(N-D)$ noise only Eigen-vectors.

3) Find D zeros of $D(z)$ close to the unit circle (For a SLA the zeros lie on the unit circle).
Now we describe how to execute step 2,
Since $\underline{e} \in R(U_N)$, $U_S^H\underline{e} = 0$, Find the minimum 2-norm solution subject to $e_1 = 1$. But in real time we have to work with $\widehat{U}_S$, not U_S
Let us write,

$$\widehat{U}_S = \begin{bmatrix} g \\ --- \\ U_S^{'} \end{bmatrix}.$$

where g represents the first row and $U_S^{'}$ represents the remaining rows of $\widehat{U}_S$
We have $\widehat{U}_S\underline{e} = 0$, which implies $\quad g^H + U_S^{'}d = 0$

$$d_{min} = -(U_S^{'})^+ g^H = -U_S^{'}(U_S^{'^H} U_S^{'})^{-1} g^H \tag{11}$$

Where $(U_S^{'})^+$ is the pseudo-inverse of $U_S^{'}$. Also $\widehat{U}_S^H\ \widehat{U}_S = I \implies U_S^{'^H} U_S^{'} + g^H g = I \implies U_S^{'^H} U_S^{'} = I - g^H g$,
Using matrix inversion lemma we have,

$$d_{min} = \frac{-U_S^{'} g^H}{1 - g g^H} \tag{12}$$

$(U_S^{'^H} U_S^{'})^{-1} = I + \frac{g^H g}{1 - g g^H}$

$$e_{min} = \begin{bmatrix} 1 \\ --- \\ d_{min} \end{bmatrix}.$$

where e_{min} is given by,

$$e_{min} = -U_S^{'^H}(g^H + \frac{g^H}{1 - g g^H}) \tag{13}$$

The min-norm algorithm can be interpreted as a Weighted MUSIC algorithm.

$$Q_{MN}(\psi) \triangleq V^H(\psi)\widehat{U}_N W \widehat{U}_N^H V(\psi) \tag{14}$$

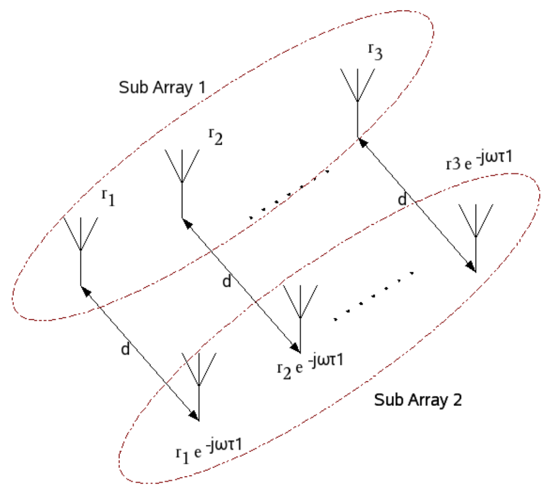

Figure 1: ESPRIT : Selecting Identical subarrays

where $W = \frac{C^H C}{\|C\|_2{}^H}$ and U_N is represented as (with C as the first row)

$$U_N = \begin{bmatrix} C \\ --- \\ U_N' \end{bmatrix}.$$

The min-norm algorithm finds the D minima of $\widehat{Q}_{MN}(\psi)$

2.2.3 ESPRIT - Estimation of Signal Parameters via Rotation Invariant Techniques

Evolved to address the robustness issue with other subspace methods (min-norm etc.) It starts by choosing the identical sub-arrays as shown in Figure 1:

$V_s^{(1)}(\psi_l)$ is the Array Manifold vector at the first Sub-array.
$V_s^{(2)}(\psi_l)$ is the Array Manifold vector at the second Sub-array.

$$V_s^{(2)}(\psi_l) = V_s^{(1)}(\psi_l)\ e^{-j\omega\tau_l} \tag{15}$$

$$V_s(\psi_l) = \begin{bmatrix} V_s^{(1)}(\psi_l) \\ V_s^{(2)}(\psi_l) \end{bmatrix}.$$

$$V_s(\psi_l) = \begin{bmatrix} V_s^{(1)}(\psi_l) \\ V_s^{(1)}(\psi_l)\ e^{-j\omega\tau_l} \end{bmatrix}$$

We know that $X_k = \underline{V}(\psi)F_s(k) + N_k$, where $\underline{V}(\psi) = [V(\psi_1)|V(\psi_2)|\ldots|V(\psi_D)]$

$$V(\psi) = \begin{bmatrix} V_s^{(1)}(\psi_1)\ldots V_s^{(1)}(\psi_D) \\ V_s^{(1)}(\psi_1)e^{-j\omega\tau_l}\ldots V_s^{(1)}(\psi_D)e^{-j\omega\tau_D} \end{bmatrix}.$$

Let us denote $\beta_k = e^{-j\omega\tau_k}$. The goal is to identify (or estimate) $\beta_1, \beta_2 \ldots \beta_D$. We know that
$V^{(1)}(\psi) = [V_s^{(1)}(\psi_1)|V_s^{(1)}(\psi_2)|\ldots|V_s^{(1)}(\psi_D)]$

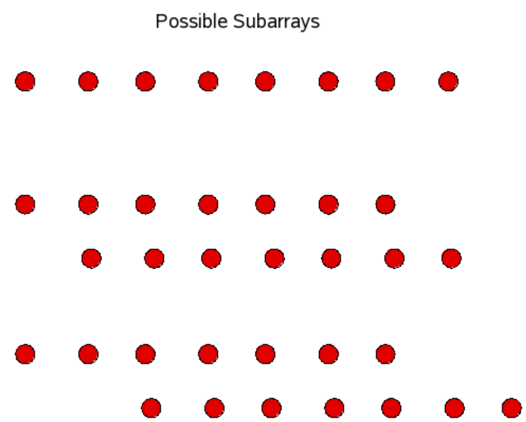

Figure 2: ESPRIT : Possible Subarrays

$$V(\psi) = \begin{bmatrix} V_s^{(1)}(\psi) \\ V_s^{(1)}(\psi)A \end{bmatrix}$$

Where A is a diagonal matrix

$$A = \begin{bmatrix} \beta_1 & 0 & 0 & . & 0 \\ 0 & \beta_2 & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & . & \beta_D \end{bmatrix}$$

$S_X = V(\psi)S_f V^H(\psi) + \sigma_n^2 I$ and $\widehat{U}_S = [\widehat{\phi}_1, \widehat{\phi}_2 . . . \widehat{\phi}_D]$ (Computed from C_X, using Eigen-Value Decomposition), with $R(U_S) = R(V(\psi))$. There exists a non-singular T such that $U_S = V(\psi)T$

So we have the following expression,

$$U_S = \begin{bmatrix} U_S^{(1)} \\ U_S^{(2)} \end{bmatrix}$$

$$U_S = \begin{bmatrix} V_s^{(1)}(\psi)T \\ V_s^{(1)}(\psi)AT \end{bmatrix}$$

where $U_S^{(1)} = V_s^{(1)}(\psi)T$ and $U_S^{(2)} = V_s^{(1)}(\psi)AT$

$$U_S = \begin{bmatrix} V_s^{(1)}(\psi)T \\ V_s^{(1)}(\psi)TT^{-1}AT \end{bmatrix}$$

$T^{-1}AT = \widetilde{A}$ and $\widetilde{V}_s(\psi) = V_s^1(\psi)T$ Eigen values of A are $\beta_1, \beta_2 . . . \beta_D$ and $T^{-1}AT$ is a similarity transformation.

We have,

$$U_S^{(1)}\widetilde{A} = U_S^{(2)} \tag{16}$$

$$\widetilde{A} = (U_S^{(1)})^+ U_S^{(2)} \tag{17}$$

Where $(U_S^{(1)})^+$ is the pseudo-inverse of $U_S^{(1)}$.
Now just find the eigen vectors of $\widetilde{A}$ and we are done. But actually we replace $U_S^{(1)}$ by $\widehat{U}_S^{(1)}$ and $U_S^{(2)}$ by $\widehat{U}_S^{(2)}$ in practical implementation.

Advantages of ESPRIT:

1) Since $\tilde{A}$ is a $D \times D$ matrix so we get exactly D estimates.

2) Don't need to scan $V(\psi)$ for all ψ.

3) Can be used with arbitrary array geometries just need to find a sub-array with shift-invariance property.

Algorithm for ESPRIT:

1) First perform Eigen-value Decomposition of C_X to find $\widehat{U}_S$.

2) Find $\widehat{U}_S^{(1)}$ and $\widehat{U}_S^{(2)}$ by picking appropriate rows.

3) Find $\tilde{A} = (\tilde{U}_S^{(1)})^+ \tilde{U}_S^{(2)}$, where $(\tilde{U}_S^{(1)})^+$ is the pseudo-inverse of $U_S^{(1)}$.

4) Find β_l, the Eigen-values of $\widetilde{A}$.
5) $\beta_l = e^{-j\omega\tau_l} \implies \tau_l$ estimated $\implies \psi_l$ estimated.