Subspace Methods: Eigenvalue Method and MUSIC

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Lecture 16

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Where We Are in J&D

- **Section 7.3**

Eigenexpansions

- **Write correlation matrix in terms of its eigensystem:**

$$\mathbf{R}_{\mathbf{y}} = \sum_{m=1}^{M} \lambda_m \mathbf{v}_m \mathbf{v}_m^H$$

$$\mathbf{R}_{\mathbf{y}}^{-1} = \sum_{m=1}^{M} \lambda_m^{-1} \mathbf{v}_m \mathbf{v}_m^H$$

Eigensystem for a One-Signal Case (1)

- **Think about one signal in white noise:**

$$\mathbf{R}_{\mathbf{y}} = A_s^2 \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}$$

- **Let** $\mathbf{v}_1 = \mathbf{e}(\vec{k}^0)/\sqrt{M}$
- **Let** $\mathbf{v}_2, \ldots, \mathbf{v}_M$ **be any set of orthogonal vectors orthogonal to** $\mathbf{v}_1$

$$\mathbf{R}_{\mathbf{y}} = MA_s^2 \mathbf{v}_1 \mathbf{v}_1^H + \sum_{i=1}^{M} \sigma_n^2 \mathbf{v}_i \mathbf{v}_i^H$$

$$\lambda_1 = MA_s^2 + \sigma_n^2,\ \lambda_2, \ldots, \lambda_M = \sigma_n^2$$

Eigensystem for a One-Signal Case (2)

$$\mathbf{R}_{\mathbf{y}} = A_s^2 \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}$$

$$= MA_s^2 \mathbf{v}_1 \mathbf{v}_1^H + \sum_{m=1}^{M} \sigma_n^2 \mathbf{v}_i \mathbf{v}_i^H$$

$$\lambda_1 = MA_s^2 + \sigma_n^2,\ \lambda_2, \ldots, \lambda_M = \sigma_n^2$$

- **If we had R_{y} exactly, we could find $\mathrm{e}(\vec{k}^0)$ without knowing anything about the array!**

Eigensystem for a Two-Signal Case

- Two signals in white noise: $\mathbf{R_y} =$

$$A_1^2 \mathbf{e}(\vec{k}_1^0)\mathbf{e}^H(\vec{k}_1^0) + A_2^2 \mathbf{e}(\vec{k}_2^0)\mathbf{e}^H(\vec{k}_2^0) + \sigma_n^2 \mathbf{I}$$

- Can express steering vectors as linear combinations of the two largest eigenvalues $\mathbf{v}_1, \mathbf{v}_2$

$$\mathbf{e}(\vec{k}_1^0) = t_{11}\mathbf{v}_1 + t_{21}\mathbf{v}_2$$

$$\mathbf{e}(\vec{k}_2^0) = t_{12}\mathbf{v}_1 + t_{22}\mathbf{v}_2$$

Subspace Decompositions

$$\begin{bmatrix} \mathbf{e}(\vec{k}_1^0) & \mathbf{e}(\vec{k}_2^0) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}}_{\mathbf{V}_{\mathbf{s+n}}} \underbrace{\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}}_{\mathbf{T}}$$

For $N_s < M$ signals, we have

$\mathbf{v}_1, \ldots \mathbf{v}_{N_s}$ **signal+noise subspace**

$\mathbf{v}_{N_s+1}, \ldots \mathbf{v}_M$ **noise subspace**

Can never really know
T, so focus on noise subspace

Decomposing MVDR

$$P^{MV}(\vec{k}) \equiv \left[\mathbf{e}^H(\vec{k})\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})\right]^{-1}$$

$$= \underbrace{\left[\sum_{i=1}^{M} \lambda_i^{-1} \left|\mathbf{e}^H(\vec{k})\mathbf{v}_i\right|^2\right]}$$

$$\overbrace{\underbrace{\sum_{i=1}^{N_s} \lambda_i^{-1} \left|\mathbf{e}^H(\vec{k})\mathbf{v}_i\right|^2}_{\textbf{signal+noise terms}} + \underbrace{\sum_{i=N_s+1}^{M} \lambda_i^{-1} \left|\mathbf{e}^H(\vec{k})\mathbf{v}_i\right|^2}_{\textbf{noise terms}}}$$

Considering the Signal+Noise Terms

- How can we get more “dramatic” peaks from the MVDR spectrum?

$$\sum_{i=1}^{N_s} \lambda_i^{-1} \boxed{\left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2} + \sum_{i=N_s+1}^{M} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2$$

- When $\mathbf{e}(\vec{k})$ corresponds to an actual signal, inner products of it with signal +subspace eigenvectors are large
- This makes the term above not as small as it could be, so MVDR power (reciprocal above term) is not as big as it could be

Considering the Noise Terms

- How can we get more "dramatic" peaks from the MVDR spectrum?

$$\sum_{i=1}^{N_s} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2 + \boxed{\sum_{i=N_s+1}^{M} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2}$$

- When $\mathbf{e}(\vec{k})$ corresponds to an actual signal, inner products of it with noise eigenvectors are small (ideally zero)
- Reciprocals of small numbers are big!

Eigenvalue Method

$$\cancel{\sum_{i=1}^{N_s} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2} + \sum_{i=N_s+1}^{M} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2$$

$$P^{EV}(\vec{k}) \equiv$$

$$\left[\sum_{i=N_s+1}^{M} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k})\mathbf{v}_i \right|^2 \right]^{-1} = \left[\mathbf{e}^H(\vec{k})\mathbf{R}_{EV}^{-1}\mathbf{e}(\vec{k}) \right]^{-1}$$

where $\mathbf{R}_{EV}^{-1} = \sum_{i=N_s+1}^{M} \lambda_i^{-1} \mathbf{v}_i \mathbf{v}_i^H$

MUSIC (1)

$$P^{EV}(\vec{k}) \equiv \left[\sum_{i=N_s+1}^{M} \lambda_i^{-1} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1}$$

- **If $\mathbf{K_n} = \sigma_n^2 \mathbf{I}$, noise subspace eigenvalues are theoretically σ_n^2**

$$P^{MUSIC}(\vec{k}) \equiv \left[\sum_{i=N_s+1}^{M} \left| \mathbf{e}^H(\vec{k}) \mathbf{v}_i \right|^2 \right]^{-1}$$

MUSIC (2)

$$P^{MUSIC}(\vec{k}) = \left[\mathbf{e}^H(\vec{k})\mathbf{R}_{MUSIC}^{-1}\mathbf{e}(\vec{k})\right]^{-1}$$

where $\mathbf{R}_{MUSIC}^{-1} = \sum_{i=N_s+1}^{M} \mathbf{v}_i \mathbf{v}_i^H$

- **"Whitens" noise subspace**
- **"Flattens" background**
- **"Pulls" signals out of noisy background**

Special Cases (1)

- EV doesn't change the eigenvalues it keeps

- Hence, it reduces to MVDR if we take the number of sources to be zero

Special Cases (2)

- The Pisarenko method says:

  Look for valleys in: $\left|\mathbf{e}^H(\vec{k})\mathbf{v}_{\min}\right|$

  Look for peaks in: $\left|\mathbf{e}^H(\vec{k})\mathbf{v}_{\min}\right|^{-1}$

- Like MUSIC method with $N_s = M - 1$

$$P^{MUSIC}(\vec{k}) \equiv \left[\sum_{i=N_s+1}^{M} \left|\mathbf{e}^H(\vec{k})\mathbf{v}_i\right|^2\right]^{-1}$$

Good Properties of EV and Music

- **Good resolution properties overall**
- **In white Gaussian noise:**
  - Resolution increases without limit with increasing SNR
  - Peak locations "asymptotically unbiased"
- **In non-white noise: Resolution still good, but has limits**
  - Peak locations usually biased

Drawbacks of EV and MUSIC

- **Neither EV or MUSIC are very robust to mismatches w.r.t. modeling assumptions**
  - Why “conventional BF” is still popular
- **MUSIC very sensitive to assumption of number of sources due to flattening the eigenvalues**
  - EV also sensitive, but not as sensitive
- **Coherent sources badly mess up subspace methods (see p. 389)**