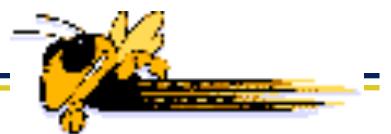

Propagating Waves

**ECE 6279: Spatial Array Processing
Fall 2013
Lecture 2**

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Where We Are in J&D

- Material drawn from Secs. 2.2, 2.2.1-2.2.2, and 2.2.4 of J&D
- We will not cover
 - Sec. 2.2.3 on the Doppler effect
 - Sec 2.3 on dispersion and attenuation
 - Sec 2.4 on refraction and diffraction



Scalar Wave Equation

- **Lossless version of the scalar wave equation**
 - No attenuation or dispersion

$$\nabla^2 s \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$$

↑
Laplacian

- s above is really $s(x,y,z,t)$, but it's customary shorthand to suppress the (x,y,z,t) part
- **Treat as a mathematical abstraction for a moment**
- **Will relate to the physics of real problems later**



Complex Exponential Solution (1)

- Let's guess a possible solution, namely a **monochromatic (single-frequency) plane wave**

$$\begin{aligned}s(\vec{x}, t) &= \exp\left\{j\left(\omega_0 t - k_x x - k_y y - k_z z\right)\right\} \\&= \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\} = \exp\left\{j\omega_0\left(t - \frac{\vec{k}}{\omega_0} \cdot \vec{x}\right)\right\}\end{aligned}$$

- Plugging in:

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$$

$$\frac{\partial}{\partial x}(-jk_x) \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\} + \dots = \frac{1}{c^2} j\omega_0 \frac{\partial}{\partial t} \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\}$$

$$(-jk_x)^2 \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\} + \dots = \frac{1}{c^2} (j\omega_0)^2 \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\}$$



Complex Exponential Solution (2)

$$(-jk_x)^2 \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\} + \dots = \frac{1}{c^2} (j\omega_0)^2 \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\}$$

$$-k_x^2 \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\} + \dots = -\frac{1}{c^2} \omega_0^2 \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\}$$

$$(k_x^2 + k_y^2 + k_z^2) \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\} = \frac{\omega_0^2}{c^2} \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\}$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega_0^2}{c^2}$$

- So $s(\vec{x}, t) = \exp\left\{j(\omega_0 t - \vec{k} \cdot \vec{x})\right\}$ is a solution if

$$|\vec{k}| = \frac{\omega_0}{c}$$



Why the Name “Monochromatic?”

$$s(\vec{x}, t) = \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\}$$

- **Monochromatic** refers to *temporal behavior*
 - Fix some position, say $(x, y, z) = (0, 0, 0)$

$$s(0, 0, 0, t) = \exp\left\{j\left(\omega_0 t\right)\right\} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$



Why the Name “Plane Wave?”

$$s(\vec{x}, t) = \exp\left\{j\left(\omega_0 t - \vec{k} \cdot \vec{x}\right)\right\}$$

- **Plane wave** refers to *spatial behavior*
 - Fix some time, say $t=0$

$s(\vec{x}, 0) = \exp\left\{j\left(-\vec{k} \cdot \vec{x}\right)\right\}$ is constant
for all points on a plane

$$\vec{k} \cdot \vec{x} = k_1 x + k_2 y + k_3 z = C$$

These planes of constant phase are
perpendicular to \vec{k}



Speed of Propagation

- The planes of constant phase move by an amount $\delta\vec{x}$ in time δt

$$s(x + \delta\vec{x}, t + \delta t) = s(x, t)$$

$$\omega_0(t + \delta t) - \vec{k} \cdot (\vec{x} + \delta\vec{x}) = \omega_0 t - \vec{k} \cdot \vec{x}$$

$$\omega_0 \delta t - \vec{k} \cdot \delta\vec{x} = 0$$

- Pick $\delta\vec{x}$ to be in the same direction as \vec{k} to give the smallest magnitude for $\delta\vec{x}$

$$\omega_0 \delta t - |\vec{k}| |\delta\vec{x}| = 0$$

$$|\vec{k}| = \frac{\omega_0}{c} \quad \text{from previous slide}$$

Speed of the wave $\rightarrow \frac{|\delta\vec{x}|}{\delta t} = \frac{\omega_0}{|\vec{k}|} = c$

Speed of propagation



Terminology

- **Unit vector** $\vec{\zeta}^0 = \frac{\vec{k}}{|\vec{k}|}$ **is the direction of propagation**
- \vec{k} **is the wavenumber vector**
 - $|\vec{k}|$ is the number of cycles (in radians) per meter in direction of propagation

$$\text{wavenumber} \rightarrow k \equiv |\vec{k}| = \frac{\omega_0}{c} = \frac{2\pi}{\lambda} \quad \text{wavelength in meters/cycle}$$

$$s(\vec{x}, t) = \exp \left\{ j \left(\omega_0 t - \vec{k} \cdot \vec{x} \right) \right\}$$

temporal frequency variable in radians **3-D spatial frequency variable in radians**

temporal frequency in cycles/sec (Hertz): $f_0 = \omega_0 / (2\pi)$

spatial frequency in cycles/meter: $? = k / (2\pi) = 1 / \lambda$



Slowness Vector

$$s(\vec{x}, t) = \exp \left\{ j\omega_0 \left(t - \frac{\vec{k}}{\omega_0} \cdot \vec{x} \right) \right\} = \exp \left\{ j\omega_0 (t - \vec{\alpha} \cdot \vec{x}) \right\}$$

slowness vector $\longrightarrow \vec{\alpha} \equiv \frac{\vec{k}}{\omega_0}$

$$|\vec{\alpha}| = \frac{|\vec{k}|}{\omega_0} = \frac{2\pi}{\omega_0 \lambda} = \frac{1}{f_0 \lambda} = \frac{1}{c}$$

- **The slowness vector** $\vec{\alpha}$
 - Points in direction of propagation
 - Has units of reciprocal velocity (sec/meter)



Periodic Nonchromatic Plane Waves

- A useful abuse of notation

$$s(\vec{x}, t) = \exp\left\{j\omega_0(t - \vec{\alpha} \cdot \vec{x})\right\} = s(t - \vec{\alpha} \cdot \vec{x})$$

where $s(u) \equiv \exp(j\omega_0 u)$

- The wave equation is linear, i.e.

if $s_1(\vec{x}, t)$ and $s_2(\vec{x}, t)$ solve the wave equation

then $as_1(\vec{x}, t) + bs_2(\vec{x}, t)$ solves the wave equation

- Example:

$$s(\vec{x}, t) = s(t - \vec{\alpha} \cdot \vec{x}) = \sum_{n=-\infty}^{\infty} S_n \exp\left\{jn\omega_0(t - \vec{\alpha} \cdot \vec{x})\right\}$$

Fourier series coefficients $\longrightarrow S_n = \frac{1}{T} \int_0^T s(u) \exp(-jn\omega_0 u) du$

Fourier series!



General Nonchromatic Plane Waves

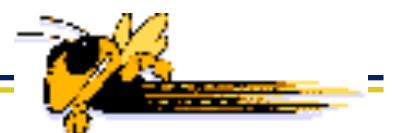
- Fourier series generalizes to Fourier integral

$$s(\vec{x}, t) = s(t - \vec{\alpha} \cdot \vec{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \exp\left\{j\omega(t - \vec{\alpha} \cdot \vec{x})\right\} d\omega$$

Fourier transform $\longrightarrow S(\omega) = \int_{-\infty}^{\infty} s(u) \exp(-j\omega u) du$

- Take home message: any space-time signal of the form $s(\vec{x}, t) = s(t - \vec{\alpha} \cdot \vec{x})$ satisfies the wave equation, whatever the $s(u)$
 - Shape of plane wave is preserved as it propagates
- Linearity of wave equation implies plane waves in many directions can exist simultaneously:

$$s(\vec{x}, t) = s_1(t - \vec{\alpha}_1 \cdot \vec{x}) + s_2(t - \vec{\alpha}_2 \cdot \vec{x})$$



Monochromatic Spherical Waves

- Wave equation can be rewritten in spherical coordinates (J&D, p. 16)
- One solution is:

$$s(r,t) = \frac{1}{r} \exp\left\{ j(\omega_0 t - kr) \right\} = \frac{1}{r} \exp\left\{ j\omega_0 \left(t - \frac{r}{c} \right) \right\}$$

where $k = \frac{\omega_0}{c}$

interpreted as propagating away from origin

- Another solution is:

$$s(r,t) = \frac{1}{r} \exp\left\{ j(\omega_0 t + kr) \right\} = \frac{1}{r} \exp\left\{ j\omega_0 \left(t + \frac{r}{c} \right) \right\}$$

interpreted as propagating toward origin



General Spherical Waves

- Abuse notation again: for any univariate function $s(u)$, easy to show that:

$$s(r,t) = \frac{s(t - r/c)}{r}$$

satisfies the wave equation

- Could write as a superposition of monochromatic spherical waves

$$s(r,t) = \frac{1}{2\pi r} \int_{-\infty}^{\infty} S(\omega) \exp\left\{j\omega\left(t - \frac{r}{c}\right)\right\} d\omega$$

- Can make same arguments for inward-going waves, waves with a different origin, etc.



Other Notes on Spherical Waves

- Due to space-invariance, can shift origins to wherever you want:

$$s(\vec{x}, t) = \frac{1}{|\vec{x} - \vec{x}_0|} \exp\left\{j(\omega_0 t - k|\vec{x} - \vec{x}_0|)\right\} + \frac{1}{|\vec{x} - \vec{x}_1|} \exp\left\{j(\omega_0 t + k|\vec{x} - \vec{x}_1|)\right\}$$

- Consider the real monochromatic spherical wave:

$$s(r, t) = \frac{1}{r} \cos(\omega_0 t - kr)$$

- Amplitude decreases with increasing r
- Distance between zero-crossings is constant with increasing r
- ...but distance between maxima isn't!
 - Gets longer with increasing r



Wave Equation in Physics (1)

- In electromagnetics, we have Maxwell's equations
- General equations are in terms of vectors, yielding a more general **vector wave equation**
 - Won't worry about in ECE6279
- For a transverse electromagnetic wave, any particular component of the electric or magnetic fields satisfies the 3-D scalar wave equation



Wave Equation in Physics (2)

- In acoustics, 3-D wave equation gives
 - Sound pressure at a point in space and time
 - In fluids, have **compressional** waves
 - Solids have both **compressional** and **transverse** waves
 - By linearity, compressional and transverse waves propagate separately, each its own velocity c
- **2-D wave equation** gives vertical displacement of waves in shallow water

