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Coherent Radar Target Detection in Heavy-Tailed Compound-Gaussian Clutter

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Outline of the talk

❖ Introduction

❖ Radar clutter modeling

❖ Optimum and suboptimum coherent radar detection in compound-Gaussian clutter

❖ Adaptive detection

❖ Concluding remarks

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❖ **Introduction**

❖ Radar clutter modeling

❖ Optimum and suboptimum coherent radar detection in compound-Gaussian clutter

❖ Adaptive detection

❖Concluding remarks

- **Radar systems** detect targets by examining reflected energy, or returns, from objects
- Along with target echoes, returns come from the sea surface, land masses, buildings, rainstorms, and other sources
- Much of this **clutter** is far stronger than signals received from targets of interest
- The main challenge to radar systems is discriminating these weaker target echoes from clutter
- **Coherent signal processing** techniques are used to this end

- **The IEEE Standard Radar Definitions** (Std 686-1990) defines **coherent signal processing** as echo integration, filtering, or detection using the amplitude of the received signals and its phase referred to that of a reference oscillator or to the transmitted signal.

What is the clutter?

Clutter refers to radio frequency (RF) echoes returned from targets which are uninteresting to the radar operators and interfere with the observation of useful signals.

Such targets include natural objects such as ground, sea, precipitations (rain, snow or hail), sand storms, animals (especially birds), atmospheric turbulence, and other atmospheric effects, such as ionosphere reflections and meteor trails.

Clutter may also be returned from man-made objects such as buildings and, intentionally, by radar countermeasures such as chaff.

Radar clutter

➢ **Radar clutter** is defined as unwanted echoes, typically from the ground, sea, rain or other atmospheric phenomena.

➢ These unwanted returns may affect the radar performance and can even obscure the target of interest.

➢ Hence clutter returns must be taken into account in designing a radar system.

Towards this goal, a clutter **model assumption** is necessary!

The function of the **clutter model** is to define a consistent theory whereby a physical model results in an **analytical model** which can be used for **radar design** and **performance analysis**.

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- ❖ Introduction
- ❖ **Radar clutter modeling**
- ❖ Optimum and suboptimum coherent radar detection in compound-Gaussian clutter
- ❖ Adaptive detection
- ❖ Concluding remarks

Radar clutter modeling

•In early studies, the resolution capabilities of radar systems were relatively low, and the scattered return from clutter was thought to comprise a large number of scatterers

•From the **Central Limit Theorem (CLT)**, researchers in the field were led to conclude that the appropriate statistical model for clutter was the **Gaussian** model (for the I & Q components), i.e., the amplitude *R* is **Rayleigh** distributed)

$$Z_I, Z_Q \in N(0,\sigma^2), \text{ I.I.D.}$$

$$Z = Z_I + jZ_Q$$

$$p_Z(\mathrm{z}) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|\mathrm{z}|^2}{2\sigma^2}\right)$$

$$R = |\mathrm{Z}| = \sqrt{Z_I^2 + Z_Q^2}$$

$$p_R(r) = \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)u(r)$$

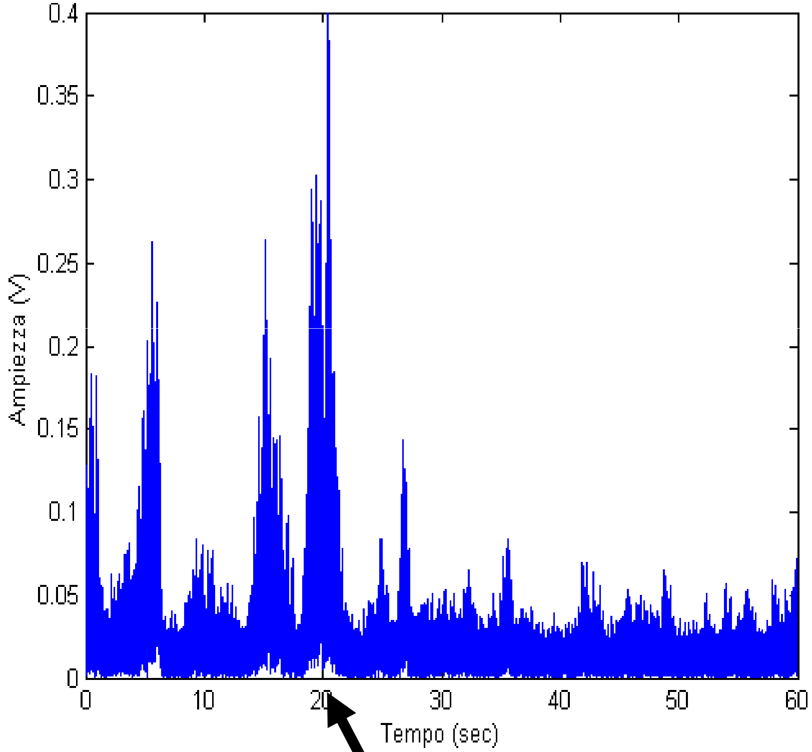
Radar clutter modeling

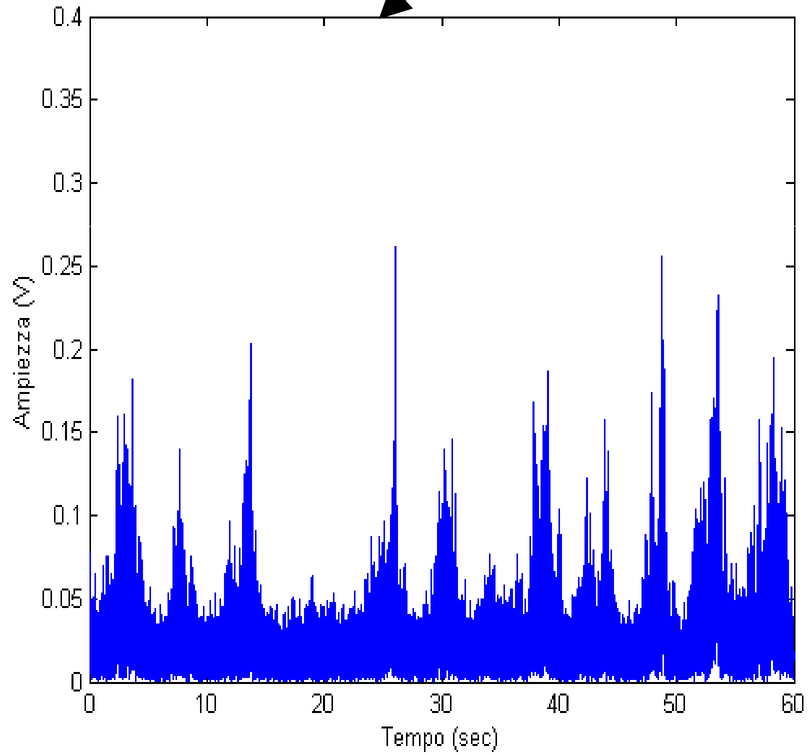
- In the quest for better performance, the resolution capabilities of radar systems have been improved
- For detection performance, the belief originally was that a higher resolution radar system would intercept less clutter than a lower resolution system, thereby increasing detection performance
- However, as resolution has increased, the clutter statistics have no longer been observed to be Gaussian, and the detection performance has not improved directly
- The radar system is now plagued by target-like “spikes” that give rise to non-Gaussian observations
- These spikes are passed by the detector as targets at a much higher false alarm rate (FAR) than the system is designed to tolerate
- The reason for the poor performance can be traced to the fact that the traditional radar detector is designed to operate against Gaussian noise
- New clutter models and new detection strategies are required to reduce the effects of the spikes and to improve detection performance

Sea clutter temporal behaviour (30 m)

The spikes have different behaviour in the two like-polarizations (HH and VV)

The vertically polarized returns appear to be a bit broader but less spiky

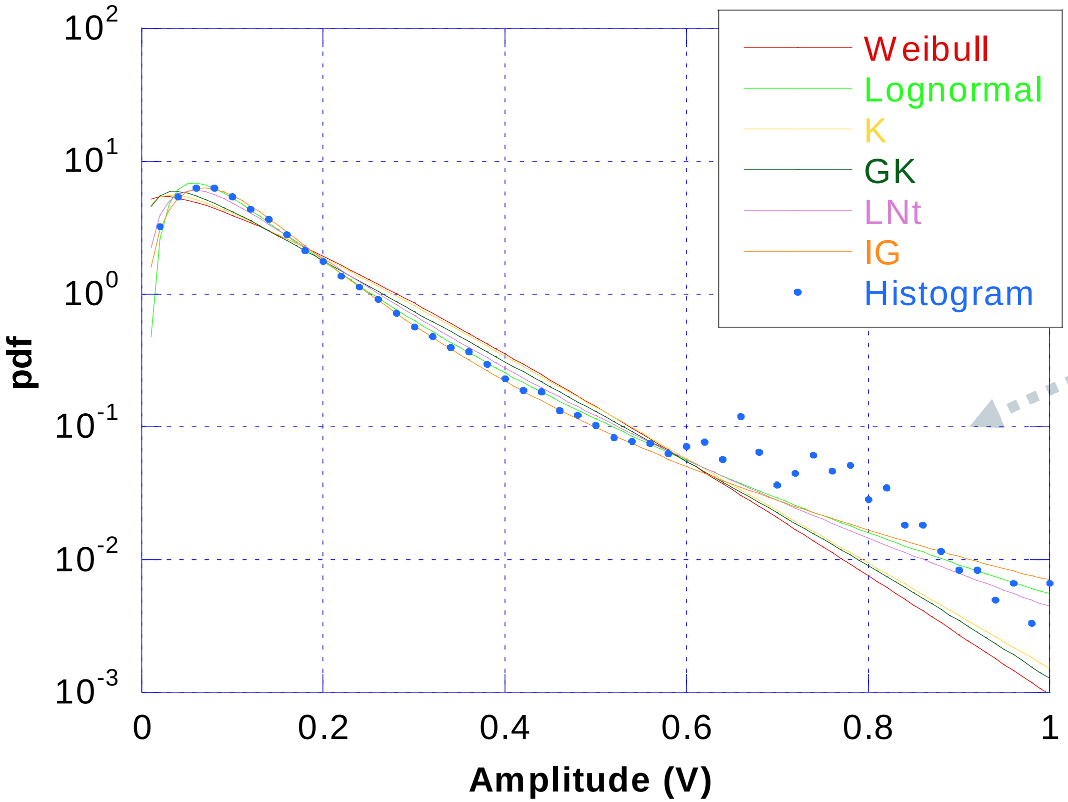




The dominant spikes on the HH record persist for about 1-3 s.

Empirically observed models

Empirical studies have produced several candidate models for spiky non-Gaussian clutter, the most popular being the Weibull distribution, the K distribution, the log-normal, the generalized K, the Student-t, etc.



Measured sea clutter data (IPIX database)

The APDF parameters have been obtained through the **Method of Moments (MoM)**

the Weibull , K, log-normal etc. have heavier tails than the Rayleigh

The Gaussian model

The scattered clutter can be written as the vector sum from N random scatterers

$$z = \sum_{\mathbf{N}} \sqrt{\sigma_{\mathrm{i}}}\, \exp[\, j\phi_i\,]$$

RCS of a single scatterer

phase term

With low resolution radars, N is deterministic and very high in each illuminated cell. Through the application of the central limit theorem (CLT) the clutter returns z can be considered as Gaussian distributed, the amplitude r =|z| is Rayleigh distributed and the most important characteristic is the radar cross section.

$$\boxed{p(r) = \frac{r}{\sigma^2} \exp\left[-\frac{r^2}{2\sigma^2}\right] u(r)}$$

The compound-Gaussian model

This is not true with high resolution systems. With reduced cell size, the number of scatterers cannot be longer considered constant but random, then improved resolution reduces the average RCS per spatial resolution cell, but it increases the standard deviation of clutter amplitude versus range and cross-range and, in the case of sea clutter, versus time as well.

A modification of the CLT to include random fluctuations of the number *N* of scatterers can give rise to the K distribution (for APDF):

$$Z = \frac{1}{\sqrt{\overline{N}}} \sum_{i=1}^{N} a_i e^{j\varphi_i} \xrightarrow{\overline{N} \to \infty} R = |Z|$$

2-D random walk

K distributed if *N* is a negative binomial r.v. (Gaussian distributed if *N* is deterministic, Poisson, or binomial)

$$\overline{N} = E\{N\}, \{a_i\}\ i.i.d., \{\varphi_i\}\ i.i.d.$$

The compound-Gaussian model

In general, taking into account the variability of the local power τ, that becomes itself a random variable, we obtain the so-called **compound-Gaussian** model, then

$$p(r \mid \tau) = \frac{2r}{\tau} \exp\left[-\frac{r^2}{\tau}\right] u(r)$$

$$p(r) = \int_0^{\infty} p(r \mid \tau)\, p(\tau)\, d\tau; \quad 0 \le r \le \infty$$

According to the CG model:

$$z(n) = \sqrt{\tau(n)}\, x(n)$$

$$x(n) = x_I(n) + j x_Q(n)$$

Texture: non negative random process, takes into account the local mean power

Speckle: complex Gaussian process, takes into account the local backscattering

The compound-Gaussian model

Particular cases of CG model (amplitude PDF):

K (Gamma texture)

$$p_R(r) = \frac{\sqrt{4\nu/\mu}}{2^{\nu-1}\Gamma(\nu)}\left(\sqrt{\frac{4\nu}{\mu}}r\right)^{\nu} K_{\nu-1}\left(\sqrt{\frac{4\nu}{\mu}}r\right)u(r)$$

GK (Generalized Gamma texture)

$$p_R(r) = \frac{2br}{\Gamma(\nu)}\left(\frac{\nu}{\mu}\right)^{\nu b}\int_0^{\infty}\tau^{\nu b-2}\exp\left[-\frac{r^2}{\tau}-\left(\frac{\nu}{\mu}\tau\right)^{b}\right]d\tau$$

LNT (log-normal texture)

$$p_R(r) = \frac{r}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}\frac{2}{\tau^2}\exp\left[-\frac{r^2}{\tau}-\frac{1}{2\sigma^2}\left[\ln(\tau/\delta)\right]^2\right]d\tau$$

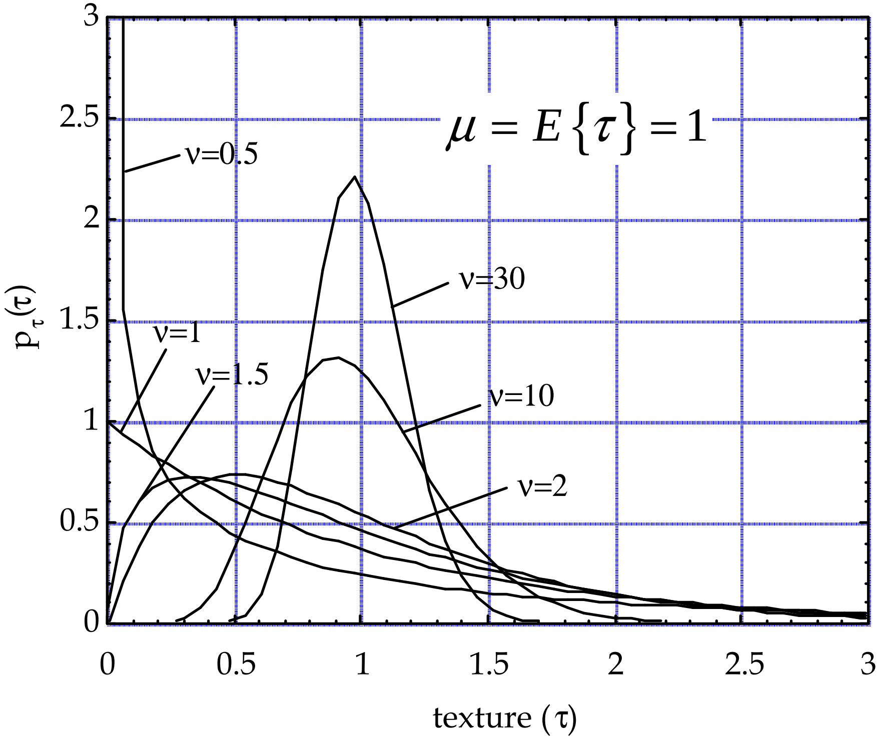
W, Weibull

$$p_R(r) = \frac{c}{b}\left(\frac{r}{b}\right)^{c-1}\exp\left[-(r/b)^c\right]u(r)$$

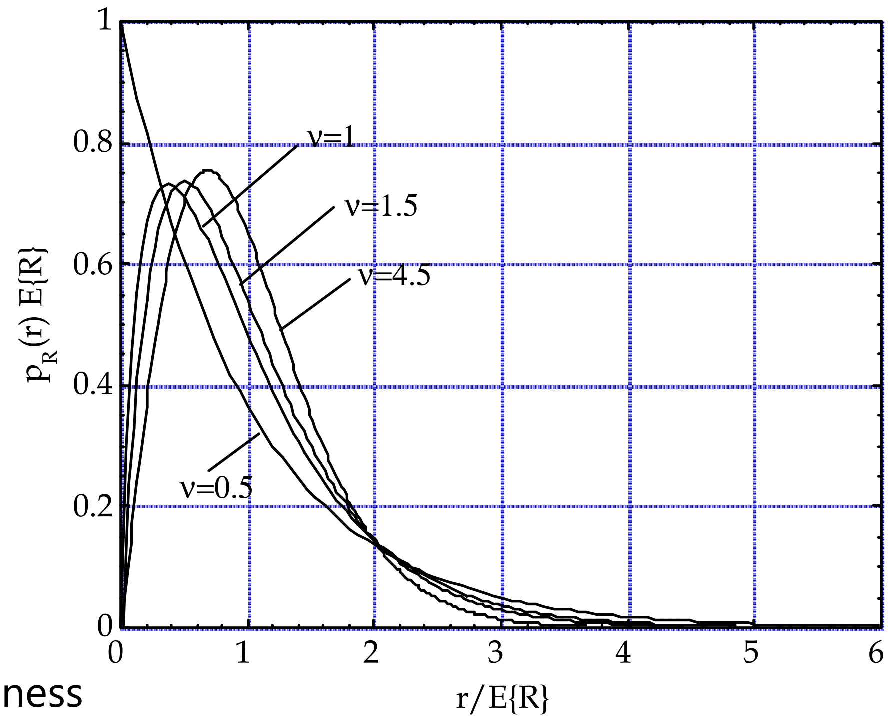
The K model

The K model is a special case of
the CG model:
N = negative binomial r.v.
τ (local clutter power) = Gamma distributed
Amplitude R = K distributed

Gamma-PDF (texture PDF)



K-PDF (amplitude PDF)



The order parameter ν controls clutter spikiness
The clutter becomes spikier as ν decreases
It becomes Gaussian when ν goes to infinity

The multidimensional CG model

• In practice, radars process M pulses at time, thus, to determine the optimal radar processor we need the ***M*-dimensional joint PDF**
• Since radar clutter is generally highly correlated, the joint PDF cannot be derived by simply taking the product of the marginal PDFs
• The appropriate multidimensional non-Gaussian model for use in radar detection studies must incorporate the following features:

1) it must account for the measured first-order statistics (i.e., the APDF should fit the experimental data)
2) it must incorporate pulse-to-pulse correlation between data samples
3) it must be chosen according to some criterion that clearly distinguishes it from the multitude of multidimensional non-Gaussian models, satisfying 1) and 2)

The multidimensional CG model

• If the Time-on-Target (ToT) is short, we can consider the texture as constant for the entire ToT, then the compound-Gaussian model degenerates into the spherically invariant random process **(SIRP)** proposed for modeling the radar sea clutter.

•By sampling a SIRP we obtain a spherically invariant random vector **(SIRV)** whose PDF is given by

$$p_Z(\mathbf{z}) = \int_0^\infty \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left(-\frac{\mathbf{z}^H \mathbf{M}^{-1}\mathbf{z}}{\tau}\right) p_\tau(\tau)\, d\tau$$

where $\mathbf{z}=[z_1 \ z_2 \ . . . \ z_M]^T$ is the M-dimensional complex vector representing the observed data.

• A random process that gives rise to such a multidimensional PDF can be physically interpreted in terms of a locally Gaussian process whose power level τ is random.

• The PDF of the local power τ is determined by the fluctuation model of the number N of scatterers.

Properties of a SIRV

The PDF of a SIRV is a function of a non negative quadratic form:

$$q(\mathbf{z}) = (\mathbf{z} - \mathbf{m}_z)^H \mathbf{M}^{-1} (\mathbf{z} - \mathbf{m}_z)$$

A SIRV is a random vector whose PDF is uniquely determined by the specification of a mean vector $\mathbf{m}_z$, a covariance matrix **M**, and a characteristic first-order PDF $p_t(t)$:

$$\boxed{p_Z(\mathbf{z}) = \frac{1}{(\pi)^M |\mathbf{M}|} h_M(q(\mathbf{z}))}$$

$h_M(q)$ must be positive and monotonically decreasing

$$h_M(q) = \int_0^\infty \tau^{-M} \exp\left(-\frac{q}{\tau}\right) p_\tau(\tau) d\tau$$

First-order amplitude PDF: $$p_R(r) = \frac{r}{\sigma^2} h_1\left(\frac{r^2}{\sigma^2}\right), \quad \sigma^2 = E\{R^2\} = E\{|z|^2\}$$

A SIRV is invariant under a linear transformation: if **z** is a SIRV with characteristic PDF $p_\tau(\tau)$, then **y**=**Az**+**b** is a SIRV with the same characteristic PDF $p_\tau(\tau)$.

Properties of a SIRV

Many known APDFs belong to the SIRV family:

Gaussian, Generalized Gaussian, Contaminated normal, Laplace, Generalized Laplace, Chauchy, Generalized Chauchy, K, Student-t, Chi, Generalized Rayleigh, Weibull, Rician, Nakagami-m. The log-normal can **not** be represented as a SIRV

for some of them $p_t(t)$ is not known in closed form

- The assumption that, during the time that the m radar pulses are scattered, the number N of scatterers remains fixed, implies that the texture t is constant during the coherent processing interval (CPI), i.e., **completely correlated texture**
- A more general model is given by $\longrightarrow$ $z[n] = \sqrt{\tau[n]}\, x[n], n = 1, 2, \cdots, m.$
- Extensions to describe the clutter process (instead of the clutter vector), investigated the **cyclostationary** properties of the texture process $t[n]$

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Coherent Radar Detection

- The general radar detection problem
- Optimum coherent detection in compound-Gaussian clutter
  - the likelihood ratio test (LRT)
  - the estimator-correlator
  - the whitening matched filter (WMF) and data-dependent threshold (DDT)
- Suboptimum detection in compound-Gaussian clutter
  (based on the three interpretations of the optimum detector)
- Performance analysis - design trade-offs

The detection problem: the radar transmits a coherent train of *m* pulses and the receiver properly demodulates, filters and samples the incoming narrowband waveform. The samples of the baseband complex signal (in-phase and quadrature components) are:

Observed data: $\mathbf{z} = \mathbf{z}_I + j\mathbf{z}_Q = [z[1] \cdots z[m]]^T$

Binary hypothesis test:

d = clutter vector

$$\begin{cases} H_0: & \mathbf{z} = \mathbf{d} \\ H_1: & \mathbf{z} = \mathbf{s} + \mathbf{d} \end{cases}$$

Target samples:

$$s[n] = A[n]e^{j\vartheta[n]}p[n]$$

p is the"steering vector"

$$\mathbf{p} = [p[1]\ p[2] \cdots p[m]]^T$$

s = target signal vector

- Perfectly known;
- Unknown:
  - ✓ deterministic (unknown parameters, e.g., amplitude, initial phase, Doppler frequency, Doppler rate, DOA, etc.)
  - ✓ random (rank-one waveform, multi-dimensional waveform)

Coherent detection in compound-Gaussian clutter

The optimum N-P detector is the LLRT:

$$\ln \Lambda(\mathbf{z}) = \ln \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

$$p_{\mathbf{z}}(\mathbf{z}|H_0) = p_{\mathbf{d}}(\mathbf{z}) = \int_0^{+\infty} \frac{1}{(\pi\tau)^m |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right] p_\tau(\tau) d\tau, \qquad p_{\mathbf{z}}(\mathbf{z}|H_1) = ?$$

M is the normalized clutter (speckle) covariance matrix

$$q_0(\mathbf{z}) = \mathbf{z}^H \mathbf{M}^{-1} \mathbf{z}$$

It depends on the target signal model

$$p_{\mathbf{z}}(\mathbf{z}|H_1) = E_{\mathbf{s}}\{p_{\mathbf{z}}(\mathbf{z}-\mathbf{s}|H_0)\}$$

(1) **s** = perfectly known

(2) **s**=β**p** with β unknown deterministic and **p** perfectly known

(3) **s**=β**p** with $\beta \in CN\left(0, \sigma_S^2\right)$ i.e., Swerling-I target model, and **p** perfectly known

(4) **s**=β**p** with $\beta \in CN\left(0, \sigma_S^2\right)$ and **p** unknown (known function of unknown parameters)

(5) **s** = Gaussian distributed random vector (known to belong to a subspace of dim. $r \leq m$)

Coherent detection in compound-Gaussian clutter

Case 2). β unknown deterministic. A UMP test does not exist.
A suboptimal approach is the **Generalized LRT (GLRT):**

$$\max_{\beta} \Lambda(\mathbf{z};\beta) = \Lambda(\mathbf{z};\hat{\beta}_{ML}) = \frac{p_{\mathbf{z}}(\mathbf{z}-\hat{\beta}_{ML}\mathbf{p}|H_0)}{p_{\mathbf{z}}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^{T}$$

The test statistic is given by the LR for known β, in which the unknown parameter has been replaced by its **maximum likelihood (ML)** estimate

When the number *m* of integrated samples increases $\hat{\beta}_{ML} \to \beta$, we expect the GLRT performance to approach that of the N-P detector for known signal

$$\int_0^{+\infty} \frac{1}{\tau^m}\left[\exp\left(-\frac{q_1(\mathbf{z})}{\tau}\right) - \exp\left(T - \frac{q_0(\mathbf{z})}{\tau}\right)\right] p_{\tau}(\tau)\,d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$q_1(\mathbf{z}) = (\mathbf{z}-\hat{\beta}_{ML}\mathbf{p})^H \mathbf{M}^{-1}(\mathbf{z}-\hat{\beta}_{ML}\mathbf{p}) = \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z} - \frac{|\mathbf{p}^H\mathbf{M}^{-1}\mathbf{z}|^2}{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}} \quad \text{where} \quad \hat{\beta}_{ML} = \frac{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{z}}{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}}$$

Coherent detection in compound-Gaussian clutter

Case 3). Swerling-I target. The N-P detector is given by

$$\Lambda(\mathbf{z}) = \frac{E_\beta\left\{p_{\mathbf{z}}(\mathbf{z}-\beta\mathbf{p}|H_0)\right\}}{p_{\mathbf{z}}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^T \qquad SCR(\tau) = \frac{\sigma_S^2}{\tau}$$

$$\int_0^{+\infty} \frac{1}{\tau^m}\left[\frac{1}{\left(1+SCR(\tau)\cdot\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}\right)}\exp\left(-\frac{q_1(\mathbf{z})}{\tau}\right)-\exp\left(T-\frac{q_0(\mathbf{z})}{\tau}\right)\right]p_\tau(\tau)d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

Interestingly, the structure is similar to the previous one, but now

$$q_1(\mathbf{z}) = \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z} - \frac{SCR(\tau)\cdot\left|\mathbf{p}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}{1+SCR(\tau)\cdot\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}}$$

Alternative formulation of the OD: the Estimator-Correlator

• The N-P **optimum detector (OD)** is difficult to implement in the LRT form, since it requires a computational heavy numerical integration!
• The LRT does not give insight that might be used to develop good suboptimum approximations to the OD
• To understand better the operation of the OD, reparametrize the conditional Gaussian PDF by setting

$$\alpha = \frac{1}{\tau}$$

α is the reciprocal of the local clutter power in the range cell under test (CUT)

• The key to understanding the operation of the OD is to express it as a function of the **mmse estimate** of α:

$$\ln \Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} E\left\{\alpha \middle| x\right\} dx \underset{H_0}{\overset{H_1}{\gtrless}} T$$

$$\hat{\alpha}_{MMSE} = E\{\alpha | q_i(\mathbf{z})\}$$

mmse estimate of α under the hypothesis H_i (i=0,1)

Alternative formulation of the OD: the Estimator-Correlator

- In Gaussian disturbance with power σ_G^2 we have $\alpha = 1/\sigma_G^2$

$SCR = \sigma_S^2/\sigma_G^2$

$$\ln\Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} \frac{1}{\sigma_G^2} dx = \frac{q_0(\mathbf{z}) - q_1(\mathbf{z})}{\sigma_G^2} = \frac{SCR \cdot \left|\mathbf{p}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\sigma_G^2 (1 + SCR \cdot \mathbf{p}^H \mathbf{M}^{-1}\mathbf{p})} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

•The structure of the OD in compound-Gaussian clutter is the basic detection structure of the OD in Gaussian disturbance with the quantity $\alpha = 1/\sigma_G^2$, which is known in the case of Gaussian noise, replaced by the mmse estimate of the unknown random α

•This structure is of the form of an estimator-correlator

•The quantity to be estimated is not the local clutter power τ, but its inverse

•This formulation is also difficult to implement, but it is very important because it suggests that sub-optimum detectors may be obtained by replacing the optimum mmse estimator with sub-optimum estimators that may be simpler to implement (e.g., MAP or ML)

Alternative formulation of the OD: the Data-Dependent Threshold

First step: express the PDFs under the two hypotheses as:

$$p_{\mathbf{z}}(\mathbf{z}|H_i) = \frac{1}{\pi^m |\mathbf{M}|} h_m(q_i(\mathbf{z})),\ i = 0,1$$

where $\boldsymbol{h_m(q)}$ is the nonlinear monotonic decreasing function:

$$h_m(q) = \int_0^\infty \frac{1}{\tau^m} \exp\left(-\frac{q}{\tau}\right) p_\tau(\tau) d\tau$$

The LRT can be recast in the form:

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0, T)$$

$\boldsymbol{f(q_0,T)}$ is the DDT, that depends on the data only by means of the quadratic statistic

$$f_{opt}(q_0, T) = q_0 - h_m^{-1}(e^T h_m(q_0))$$

$$q_0(\mathbf{z}) = \mathbf{z}^H \mathbf{M}^{-1} \mathbf{z}$$

Alternative formulation of the OD: the Data-Dependent Threshold

$$\text{Gaussian clutter:} \quad q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma_G^2 T$$

$$\text{C-G clutter:} \quad q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0, T)$$

In this formulation the LRT for CG clutter has a similar structure of the OD in Gaussian disturbance, but now the test threshold is not constant but it depends on the data through $\boldsymbol{q_0}$

Perfectly known signal s (case 1): the OD can be interpreted as the classical whitening-matched filter (WMF) compared to a **data-dependent threshold (DDT)**

$$\underbrace{2\,\mathrm{Re}\left\{\mathbf{s}^H \mathbf{M}^{-1}\mathbf{z}\right\}}_{WMF} \underset{H_0}{\overset{H_1}{\gtrless}} \mathbf{s}^H \mathbf{M}^{-1}\mathbf{s} + f_{opt}(q_0, T)$$

Alternative formulation of the OD: the Data-Dependent Threshold

Signal s with unknown complex amplitude (Case 2): the GLRT again can be interpreted as the classical whitening-matched filter (WMF) compared to the same **DDT**

$$\underbrace{\left|\mathbf{p}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}_{WMF} \underset{H_0}{\overset{H_1}{\gtrless}} \mathbf{p}^H \mathbf{M}^{-1}\mathbf{p} \cdot f_{opt}(q_0, T)$$

Similar results does not hold for the NP detector for Case 3 (Swerling I target signal)!

***Example*: K-distributed clutter.**

In this case the texture is modelled as a Gamma random variable with mean value m and order parameter n. For n-*m*=0.5 we have

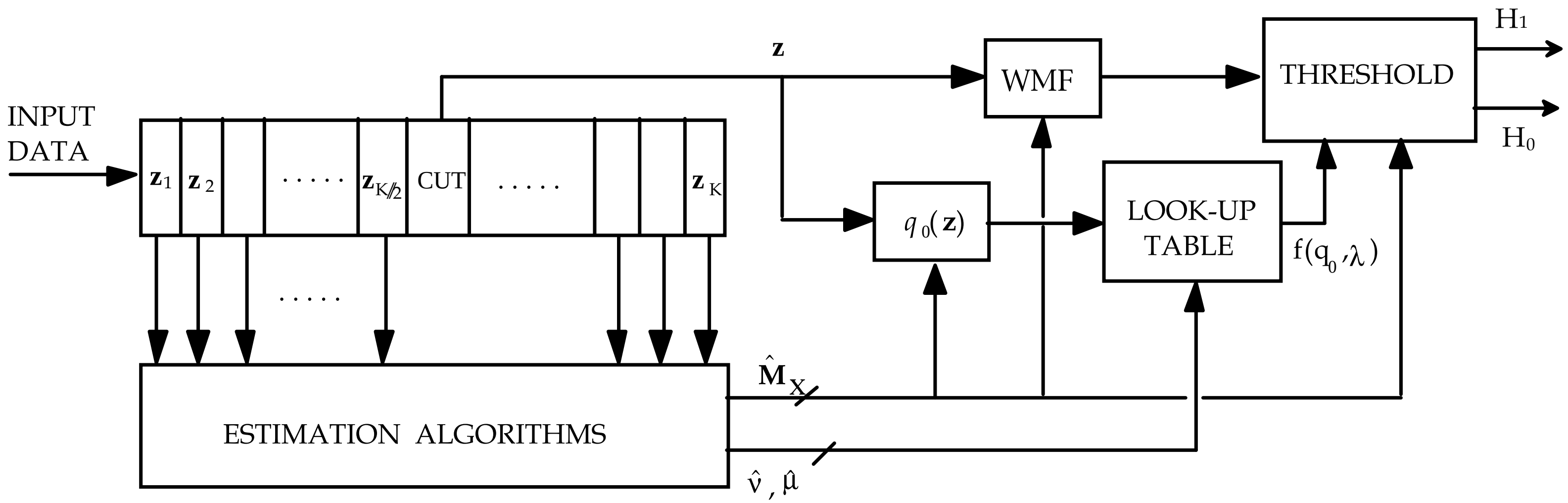
$$f_{opt}(q_0, T) = q_0 - \left(\sqrt{q_0} - T\sqrt{\frac{\mu}{4\nu}}\right)^2 u\left(\sqrt{q_0} - T\sqrt{\frac{\mu}{4\nu}}\right)$$

In general, it is not possible to find a closed-form expression for the DDT, so it must be calculated numerically.

Canonical structure of the optimum detector

- This **canonical structure** suggests a practical way to implement the OD/GLRT
- The DDT can be a priori tabulated, with l set according to the prefixed P_{FA},

and the generated look-up table saved in a memory.



- This approach is highly time-saving, it is canonical for every SIRV, and is useful both for practical implementation of the detector and for performance analysis by means of Monte Carlo simulation
- This formulation provides a deeper insight into the operation of the OD/GLRT and suggests an approach for deriving good suboptimum detectors

Suboptimum detection structures

Suboptimum approximations to the likelihood ratio (LR)

• From a physical point of view, the difficulty in utilizing the LR arises from the fact that the power level τ, associated with the conditionally Gaussian clutter is unknown and randomly varying: we have to resort to numerical integration

•The **idea** is: replace the unknown power level τ with an estimate inside the LR

$$\ln \hat{\Lambda}(\mathbf{z}) = m \ln\left(\frac{\hat{\tau}_0}{\hat{\tau}_1}\right) + \frac{q_0(\mathbf{z})}{2\hat{\tau}_0} - \frac{q_1(\mathbf{z})}{2\hat{\tau}_1} \underset{H_0}{\overset{H_1}{\gtrless}} T$$

Candidate estimation techniques:

MMSE, MAP, ML
The simplest is the ML

$$\hat{\tau}_{ML,i} = \frac{q_i(\mathbf{z})}{m}, i = 0,1$$

$$\left|\mathbf{p}^H \mathbf{M}^{-1} \mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \mathbf{p}^H \mathbf{M}^{-1} \mathbf{p} \cdot f_{ML}(q_0, T)$$

It has been called the NMF

It is in the canonical form, with the adaptive threshold that is a linear function of q_0

$$f_{ML}(q_0, \lambda) = q_0\left(1 - e^{-T/m}\right)$$

Suboptimum approximations to the Likelihood Ratio

The Normalized Matched Filter (NMF) or GLRT-LQ

$$\frac{\left|\mathbf{p}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}} \underset{H_0}{\overset{H_1}{\gtrless}} \left(1-e^{-T/m}\right)\cdot\left(\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}\right)$$

$$P_{FA} = e^{-\frac{T(m-1)}{m}}$$

- This detector is very simple to implement
- It has the constant false alarm rate (CFAR) property with respect to the clutter PDF

$$P_D = \int_0^{+\infty}\left(1+\frac{\tau\left(e^{T/m}-1\right)}{\tau+m\mu\bar{\gamma}}\right)^{-(m-1)} p_\tau(\tau)d\tau$$

$$\mu = E\{\tau\}$$

$$\bar{\gamma} = \frac{\sigma_S^2}{\mu}\cdot\frac{\mathbf{p}^H\mathbf{M}^{-1}\mathbf{p}}{m}$$ is the SCR at the output of the WMF, divided by *m*

Suboptimum approximations to the "Estimator-Correlator" structure

The **mmse estimator** of *a*=1/*t* may be difficult to implement in a practical detector.

e.g. for K-distributed clutter:

$$\hat{\alpha}_{MMSE,i} \neq \frac{1}{\hat{\tau}_{MMSE,i}}$$

$$\hat{\alpha}_{MMSE,i} = \sqrt{\frac{\nu}{\mu q_i(\mathbf{z})}} \cdot \frac{K_{\nu-m-1}\left(\frac{4\nu q_i(\mathbf{z})}{\mu}\right)}{K_{\nu-m}\left(\frac{4\nu q_i(\mathbf{z})}{\mu}\right)}, \quad i=0,1$$

Suboptimum detectors may be obtained by replacing the **mmse estimator** with a **suboptimal estimator** (e.g., MAP or ML):

$$\hat{\alpha}_{MAP,i} \neq \frac{1}{\hat{\tau}_{MAP,i}}$$

$$\hat{\alpha}_{MAP,i} = \frac{m-\nu-1+\sqrt{(m-\nu-1)^2+\frac{4\nu q_i(\mathbf{z})}{\mu}}}{2q_i(\mathbf{z})}, \quad i=0,1$$

As the number ***m*** of samples becomes asymptotically large, the NMF becomes equivalent to the optimum DDT:

$$\hat{\alpha}_{ML,i} = \frac{1}{\hat{\tau}_{ML,i}} \rightarrow NMF$$

$$\hat{\alpha}_{MMSE,i}, \hat{\alpha}_{MAP,i} \xrightarrow{m \gg 1} = \frac{m}{q_i(\mathbf{z})} = \hat{\alpha}_{ML,i} = \frac{1}{\hat{\tau}_{ML,i}}, \quad i=0,1$$

Suboptimum approximations to the DDT structure

• The canonical structure in the form of a WMF compared to a DDT is given by:

$$\underbrace{\left|\mathbf{p}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}_{WMF} \underset{H_0}{\overset{H_1}{\gtrless}} \mathbf{p}^H \mathbf{M}^{-1}\mathbf{p} \cdot f_{opt}(q_0, T)$$

the threshold $f(q_0,T)$ depends in a complicated non linear fashion on the quadratic statistic $q_0(\mathbf{z})$

- The **idea** is to find a good approximation of $f(q_0,T)$ easy to implement. In this way, we avoid the need of saving a look-up table in the receiver memory
- The approximation has to be good only for values of $q_0(\mathbf{z})$ that have a high probability of occurrence
- We looked for the best ***k-th*** **order polynomial approximation** in the MMSE sense

$$f_K(q_0, T) = \sum_{i=0}^{K} c_i q_0^i$$

$f_K(q_0,T)$ is easy to compute from $q_0(\mathbf{z})$

$$\min_{\{c_i\}} \left\{ \left| f_{opt}(q_0, T) - \sum_{i=0}^{K} c_i q_0^i \right|^2 \right\}$$

Suboptimum approximations to the DDT structure

***Example*: <u>K-distributed clutter</u>.** The solution can be derived in closed-form

First order (linear) approximation: $$f_1(q_0,T) = c_0 + c_1 q_0$$

The solution is obtained by solving a (K+1)-*th* order linear system. For K=1:

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & E\{q_0\} \\ E\{q_0\} & E\{q_0^2\} \end{bmatrix}^{-1} \begin{bmatrix} E\{f_{opt}(q_0,T)\} \\ E\{q_0 f_{opt}(q_0,T)\} \end{bmatrix}$$

For n-m=0.5, the MMSE solution is:

$$c_0 = \frac{T\mu}{4\nu}\left(\frac{8\nu^2-2}{4\nu+1} - T\right),\ c_1 = \frac{T}{4\nu+1}$$

- Note that when $\nu \to \infty$ (Gaussian noise), we have c_1=0, so the threshold becomes constant and we get the conventional WMF
- The NMF is obtained as a special case of $f_1(q_0,T)$ for $c_0 = 0,\ c_1 = 1 - e^{-T/m}$

For the 1st (linear) and 2nd-order (quadratic) approximations: ⇒ $c_0 \propto \mu,\ c_1$ independent of $\mu,\ c_2 \propto 1/\mu$

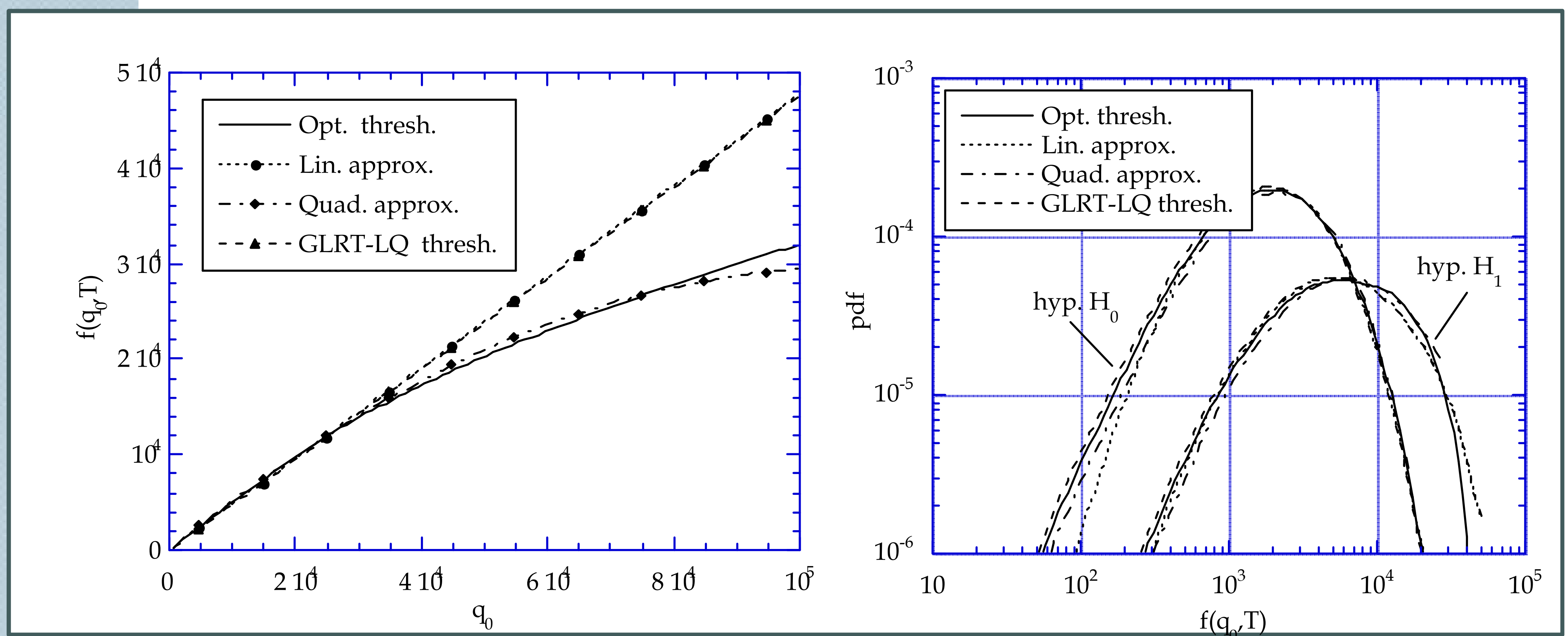
all c_i's are independent of **M**

Suboptimum approximations to the DDT structure

- It is important to know the spread in values of $q_0(\mathbf{z})$ that we expect under H_0 and H_1
- If we want to approximate $f(q_0,T)$, we need to know the range of values over which we require a good approximation
- The approximation must be good under both the hypotheses

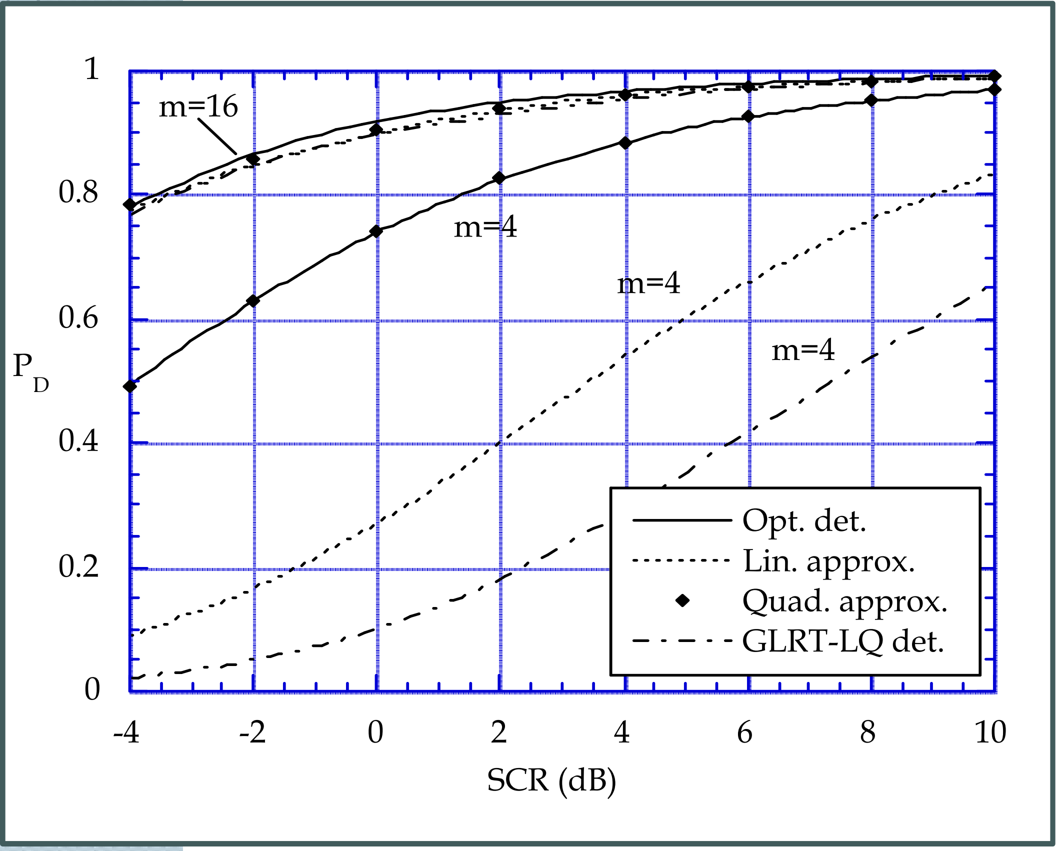
$$SCR = \frac{\sigma_S^2}{E\{\tau\}}$$

$$P_{FA} = 10^{-5}, m = 4, \nu = 4.5, \mu = 10^3, SCR = -4dB, \text{SW-I}$$



Performance Analysis: Sw-I target, K-distributed clutter

In all the cases we examined, the suboptimum detector based on the quadratic (2^{nd}-order) approximation has performance (i.e., P_D) almost indistinguishable from the optimal

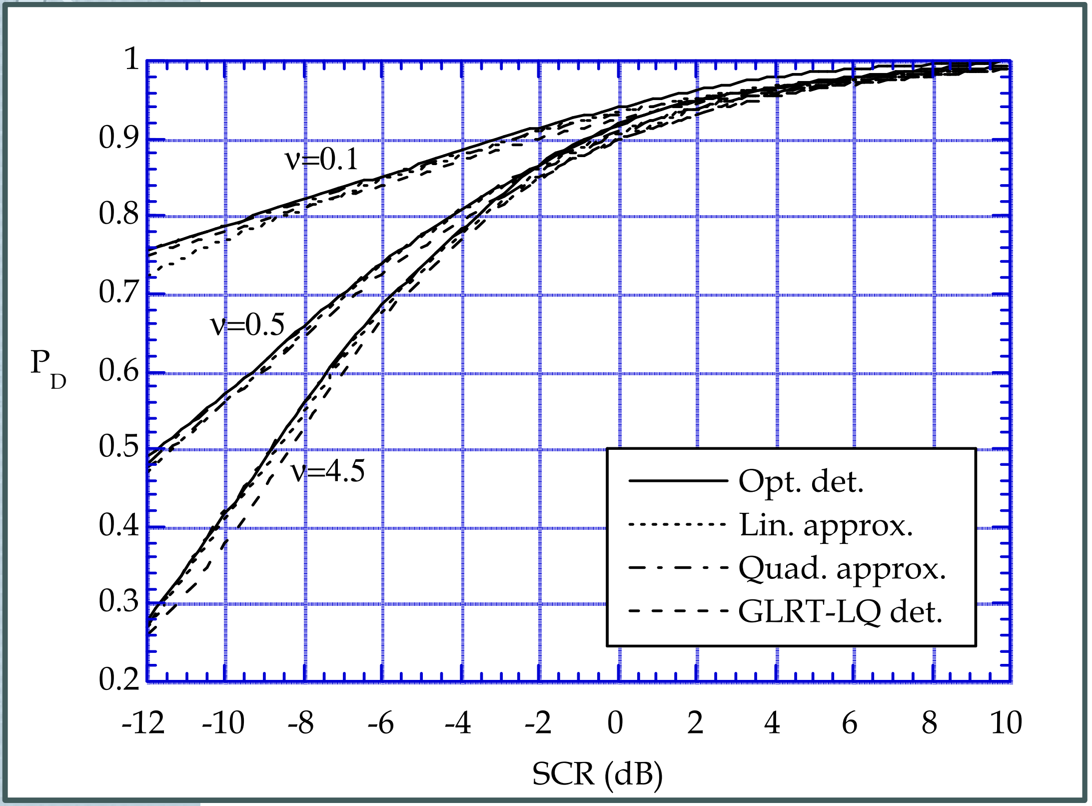


- The detector based on 2^{nd}-order approximation represents a good trade-off between performance and ease of implementation
- It requires knowledge of the clutter APDF parameters (n and m)
- As the number *m* of integrated pulses increases, the detection performance of the GLRT-LQ approaches the optimal performance
- The GLRT-LQ does not require knowledge of n and m
- It is also CFAR w.r.t. texture PDF

$$P_{FA} = 10^{-5},\ f_D = 0.5, \nu = 4.5, \mu = 10^3,$$
$$\rho_X = 0.9\ AR(1)$$

Performance Analysis: Sw-I target, K-distributed clutter

Swerling-I target: it was observed that in this case, P_D increases much more slowly as a function of SCR than for the case of **known target signal**



- Clutter spikiness heavily affects detection performance
- ν=0.1 means very spiky clutter (heavy tailed)
- ν=4.5 means almost Gaussian clutter
- Up to high values of SCR the best detection performance is obtained for spiky clutter (small values of ν): it is more difficult to detect weak targets in Gaussian clutter rather than in spiky K-distributed clutter, provided that the proper decision strategy is adopted

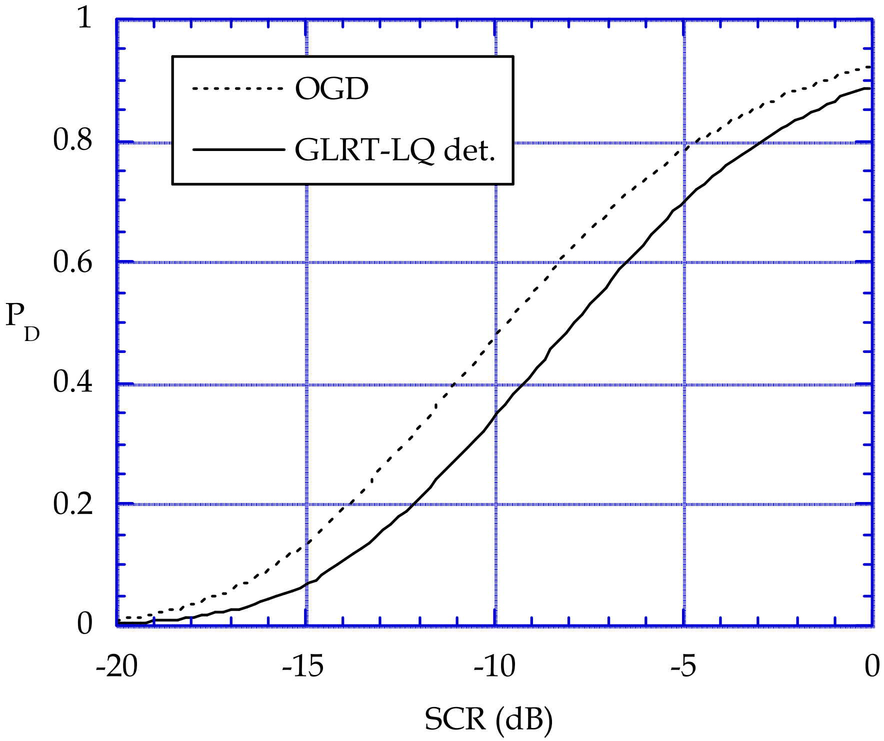
$$P_{FA} = 10^{-5},\ f_D = 0.5,\ m = 16,\ \mu = 10^3,$$
$$\rho_X = 0.9\ AR(1)$$

Performance Analysis: Sw-I target, K-distributed clutter

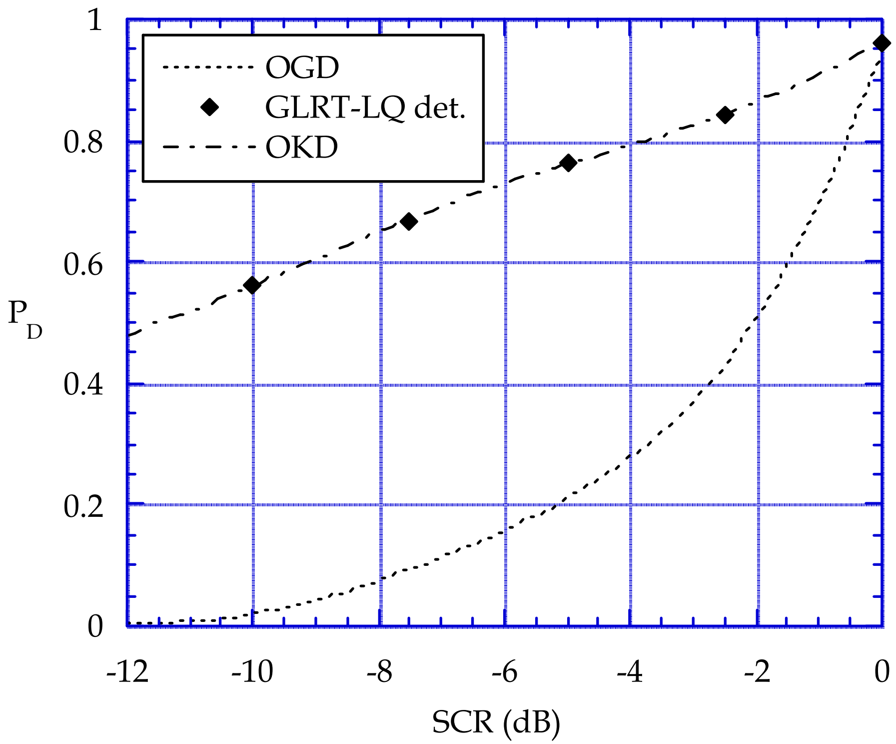
Optimum Detector in K-clutter (OKD)

Optimum Detector in Gaussian clutter (OGD) = whitening matched filter (WMF)

$\nu \to \infty$ (Gaussian clutter)



$\nu = 0.5$ (spiky clutter)



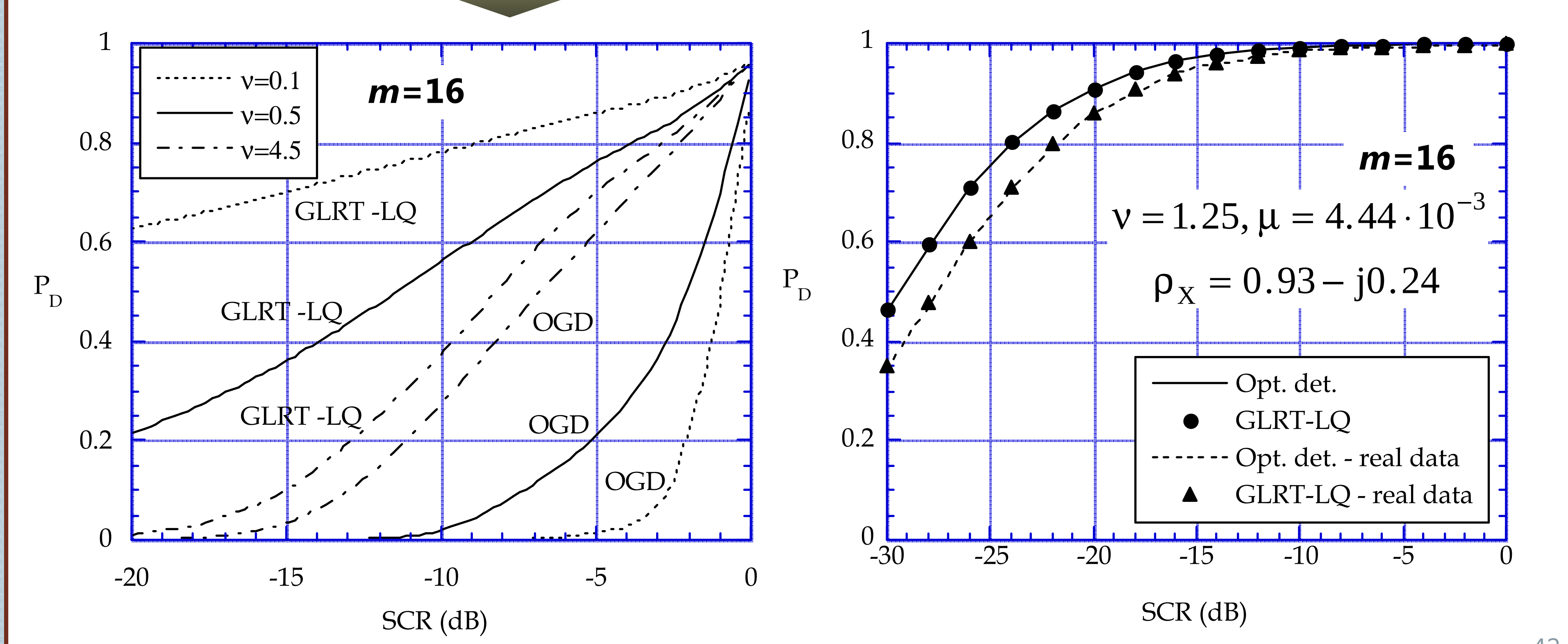
$$P_{FA} = 10^{-5},\ f_D = 0.5,\ m = 16,\ \mu = 10^3,$$
$$\rho_X = 0.9\ AR(1)$$

The 2nd figure shows how a wrong assumption on clutter model (model mismatching) affects detection performance

Performance Analysis: Swerling-I, K-distributed clutter, real sea clutter data

- The gain of the GLRT-LQ over the mismatched OGD increases with clutter spikiness (decreasing values of n)

- Performance prediction have been checked with **real sea clutter data**
- The detectors make use of the knowledge of m, n, and **M** (obtained from the entire set of data)



The DDT structure: New results

- The LRT (and GLRT) can be recast in the form:

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0, T)$$

The left-hand term depends on the target signal model

$$f_{opt}(q_0, T) = q_0 - h_m^{-1}(e^T h_m(q_0))$$

$h_m(q)$ is a nonlinear monotonic decreasing function, which depends on the texture PDF:

$$h_m(q) = \int_0^\infty \frac{1}{\tau^m} \exp\left(-\frac{q}{\tau}\right) p_\tau(\tau) d\tau = \int_0^\infty \alpha^m \exp(-\alpha q) p_\alpha(\alpha) d\alpha$$

Reparametrize in terms of the inverse of the local clutter power:

$$\alpha = \frac{1}{\tau} \Rightarrow p_\alpha(\alpha) = \frac{1}{\alpha^2} p_\tau\left(\frac{1}{\alpha}\right)$$

The Linear-Threshold Detector

The simplest extension beyond the Gaussian case would appear to be a linear form for $f(q_0,T)$:

$$f_{opt}(q_0, T) = a + bq_0$$

we will call a detector whit this form of the DDT: a **linear-threshold detector (LTD)**.

The LTD was originally explored as an approximation to $f(q_0,T)$ for the case of K-distributed compound-Gaussian clutter. Here, we ask a much different question about the LTD:

Is there a compound-Gaussian model for which the linear-threshold detector is the optimum detector?

The Linear-Threshold Detector

If the answer is yes, the multivariate compound-Gaussian clutter model, which is defined through the function $h_m(q_0)$ must satisfy:

$$q_0 - h_m^{-1}(e^T h_m(q_0)) = a + bq_0$$

This problem can be reformulated as an **Abel problem** which has a unique solution.
The solution provides a Gamma PDF for α and as a consequence the texture τ has an **inverse-Gamma PDF**:

$$p_\tau(\tau) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{\eta}\right)^{\lambda} \frac{1}{\tau^{\lambda+1}} e^{-\frac{\lambda}{\eta\tau}}, \qquad \tau \geq 0$$

$\lambda>0$ is the shape parameter and $\eta>0$ is the scale parameter.

Consequently, the **complex multivariate compound-Gaussian clutter model** is:

$$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i) = \frac{1}{\pi^m |\mathbf{M}|} \cdot \frac{\Gamma(m+\lambda)}{\Gamma(\lambda)} \cdot \left(\frac{\lambda}{\eta}\right)^{\lambda} \cdot \left(\frac{\lambda}{\eta} + q_0(\mathbf{z})\right)^{-(m+\lambda)}$$

The Linear-Threshold Detector

The optimum detector in compound-Gaussian clutter with IG texture is given by:

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \left(1 - e^{-\frac{T}{m+\lambda}}\right) \cdot \left(\frac{\lambda}{\eta} + q_0(\mathbf{z})\right)$$

The clutter amplitude PDF that arises from the compound-Gaussian clutter model with IG texture:

$$p_R(r) = 2\eta r \left(1 + \frac{\eta}{\lambda} r^2\right)^{-(\lambda+1)} u(r)$$

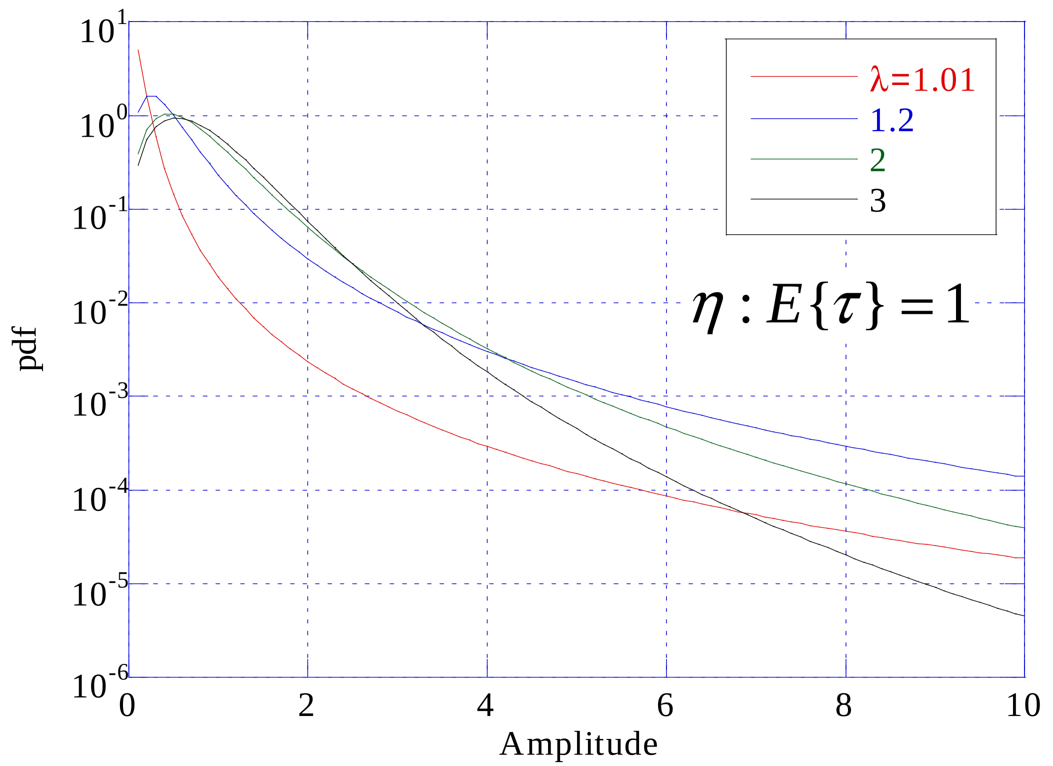
$$\lim_{\lambda \to \infty} p_R(r) = 2\eta r e^{-r^2 \eta} u(r) \quad \text{(Rayleigh PDF)}$$

$$\lim_{\lambda \to 0} p_R(r) = \frac{2\lambda}{r} u(r)$$

the clutter amplitude statistics are extremely heavy-tailed in this limit

The clutter amplitude model

Thus the multivariate clutter model with IG texture varies parametrically between Gaussian clutter (when λ goes to infinity) and extremely heavy-tailed clutter (when λ goes to zero)



$$E\{R^k\} = \left(\frac{\lambda}{\eta}\right)^{\frac{k}{2}} \frac{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\lambda-\frac{k}{2}\right)}{\Gamma(\lambda)}$$

if $\lambda > k/2$

$$E\{\tau\} = E\{R^2\} = \frac{\lambda}{\eta(\lambda-1)}$$

The clutter power $E\{\tau\}$ is finite only for λ>1

Implications of the optimum detector

When the shape parameter λ goes to 0:

$$\frac{T_{m,\lambda}}{\hat{\alpha}_{opt}} \xrightarrow{\lambda \to 0} \left(1 - e^{-\frac{T}{m}}\right) q_0(\mathbf{z}) = f_{ML}(q_0, T)$$

which is the GLRT-LQ (or NMF) detector previously explored, known to be the optimal detector as *m* goes to infinity and also known to be a CFAR detector.

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \left(1 - e^{-\frac{T}{m}}\right) q_0(\mathbf{z})$$

The remarkable observation here is that the GLRT is the optimum detector for any *m* for the CG clutter model with IG texture, in the limit of extremely heavy-tailed clutter as l goes to 0

The Linear-Threshold Detector

When the complex amplitude is deterministic unknown and we resort to the GLRT approach (i.e. we replace *b* with its ML estimate in the LRT):

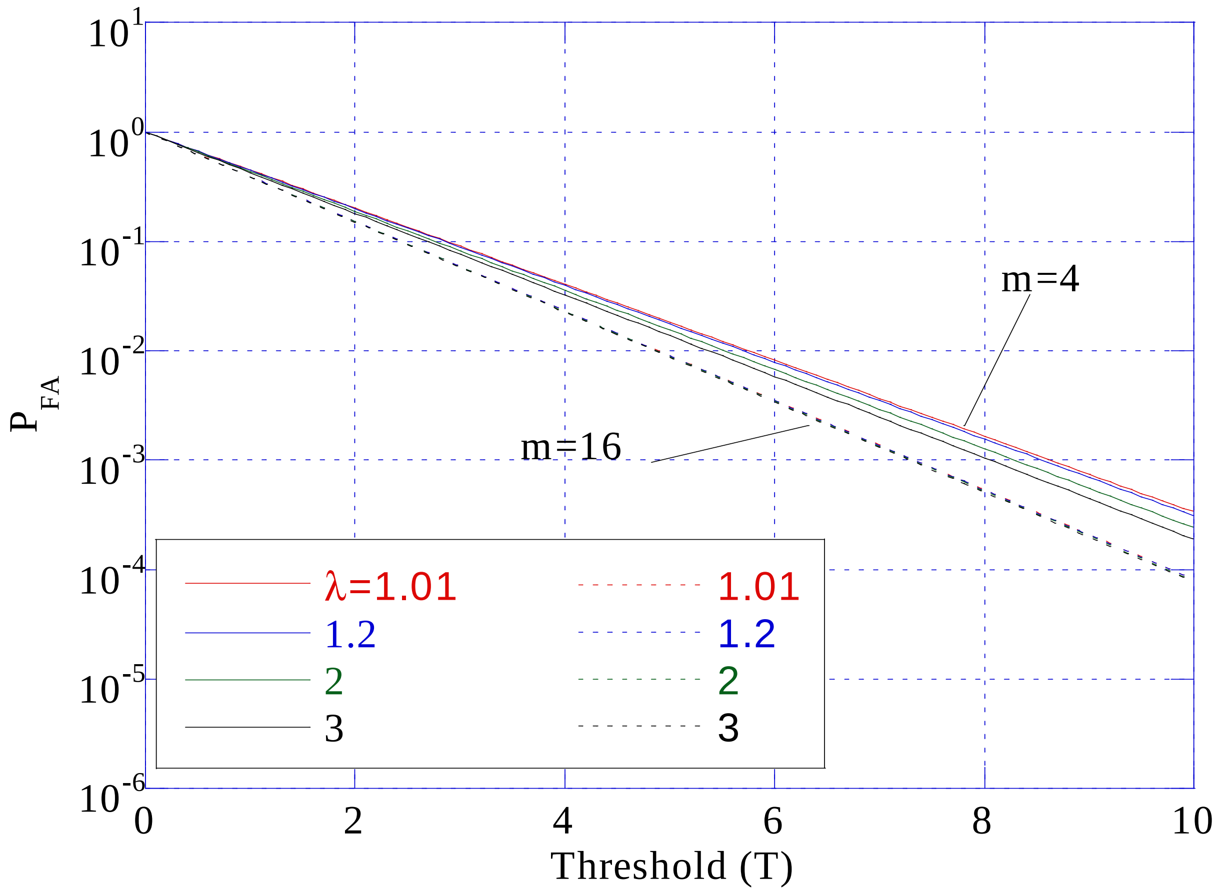
$$q_0(\mathbf{z}) - q_1(\mathbf{z}) = \frac{\left|\mathbf{p}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{p}^H \mathbf{M}^{-1}\mathbf{p}}$$

$$\left|\mathbf{p}^H \mathbf{M}^{-1}\mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \left(\mathbf{p}^H \mathbf{M}^{-1}\mathbf{p}\right) \cdot \left(1 - e^{-\frac{T}{m+\lambda}}\right) \cdot \left(\frac{\lambda}{\eta} + q_0(\mathbf{z})\right)$$

$$P_{FA} = \exp\left[-T\left(\frac{m+\lambda-1}{m+\lambda}\right)\right]$$

$$P_{FA}\Big|_{\lambda=0} = e^{-\frac{T(m-1)}{m}} = P_{FA,GLRT-LQ}$$

The Linear-Threshold Detector



$$P_{FA} = \exp\left[-T \left(\frac{m+\lambda-1}{m+\lambda} \right) \right]$$

$$m \gg 1 \text{ or } \lambda \gg 1$$

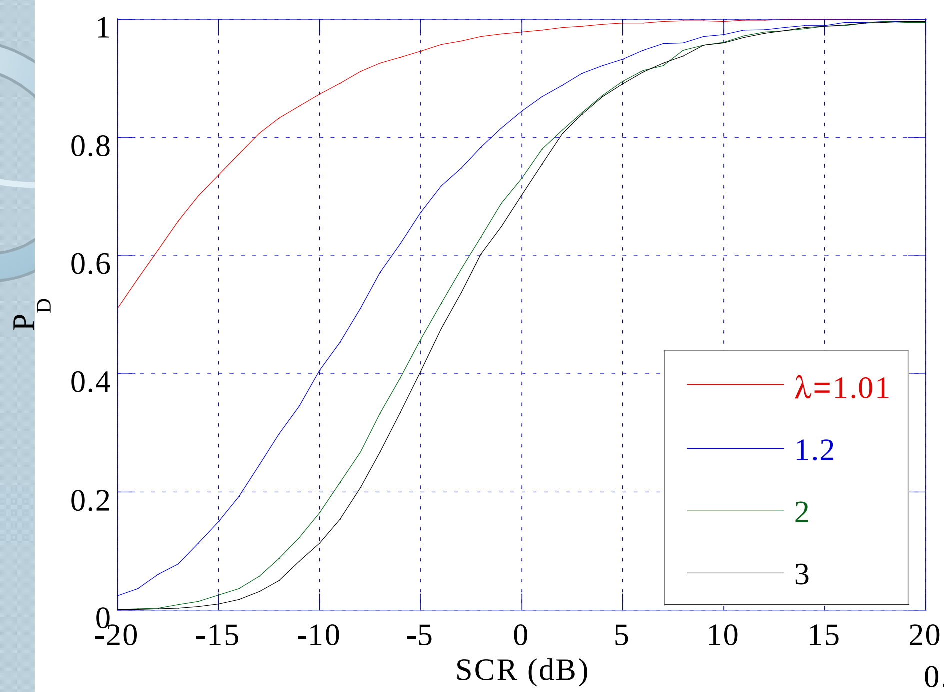
$$\Downarrow$$

$$P_{FA} \cong \exp[-T]$$

the Gaussian case

Thus, the GLRT-LTD detector is in practice CFAR with respect to both λ and η (the texture parameters)

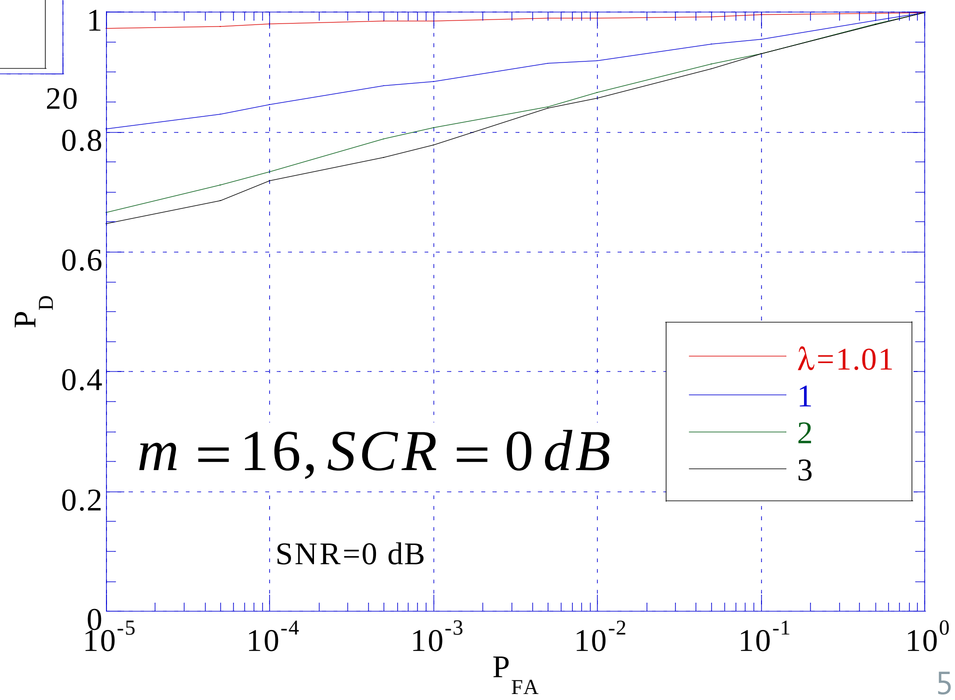
The Linear-Threshold Detector



$$m = 16,\ P_{FA} = 10^{-4}$$

$$SCR = \frac{\sigma_S^2}{E\{\tau\}}$$

Detection performance gets better when the clutter becomes spikier



Summarizing….

- We have examined the problem of optimal and sub-optimal target detection in compound-Gaussian clutter
- Three interpretations of the optimum detector (OD) have been provided
  - **the likelihood ratio test (LRT)**
  - **the estimator-correlator (EC)**
  - **the whitening matched filter (WMF) and data-dependent threshold**
- With these reformulations of the OD, the problem of obtaining sub-optimal detectors may be approached by either:

  (i) approximating the LRT directly,
  (ii) utilizing a sub-optimal estimate in the estimator-correlator structure,
  (iii) utilizing a sub-optimal function to model the data-dependent threshold in the WMF interpretation
- Numerical results suggest that the NMF or GLRT-LQ and the sub-optimal detector based on quadratic approximation of the data-dependent threshold **represent the best trade-off between implementation complexity and detection performance**

Summarizing….

- The GLRT-LQ, which is a popular suboptimum detector due to its CFAR property, is an optimum detector for the CG clutter model with IG texture, in the limit as the tails get extremely heavy (small l)

- The CG clutter model with **inverse Gamma (IG) texture** varies parametrically from the Gaussian clutter model to a clutter model whose tails are evidently heavier than any K model

- The **LTD** is conceptually the simplest extension beyond the optimum Gaussian detector (constant threshold). Although a quadratic-threshold detector may be obtained, such a detector will always be suboptimum for any compound-Gaussian model

Outline of the talk

❖ Introduction

❖ Radar clutter modeling

❖ Optimum and suboptimum coherent radar detection in compound-Gaussian clutter

❖ **Adaptive detection**

❖ Concluding remarks

The adaptive detection problem in CG disturbance

- The Optimum and Suboptimum Detectors in previous section have been obtained supposing that the disturbance covariance matrix is a priori known. Most often this is not true and it must be estimated using K secondary data surrounding the CUT.

- We suppose homogeneous environment:

$\mathbf{z}|H_0$ and $\{\mathbf{z}_k\}_{k=1}^{K}$ are independent and identically distributed (IID)

$$\mathbf{R} = \sigma^2\mathbf{M} \triangleq E\{\mathbf{z}\mathbf{z}^H|H_0\} = E\{\mathbf{z}_k\mathbf{z}_k^H\} = E\{\tau_k\}E\{\mathbf{x}_k\mathbf{x}_k^H\}, k=1,2,\cdots,K$$

$$\mathbf{R}|\tau_k = \tau_k\mathbf{M} \triangleq E\{\mathbf{z}_k\mathbf{z}_k^H|\tau_k\} = \tau_k E\{\mathbf{x}_k\mathbf{x}_k^H\}, k=1,2,\cdots,K$$

The adaptive detection problem in CG disturbance

- We could resort to the Maximum Likelihood approach where the unknowns are the normalized covariance matrix M, the vector of the textures in the secondary and primary vectors $\Theta_\tau = [\tau \quad \tau_1 \quad \ldots \quad \tau_K]^T$ and the amplitude α of the target $\mathbf{s} = \alpha \mathbf{p}$

$$\frac{\max_{\mathbf{M},\boldsymbol{\Theta}_\tau,\alpha} f\left(Z \middle| H_1\right)}{\max_{\mathbf{M},\boldsymbol{\Theta}_\tau} f\left(Z \middle| H_0\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

- This approach leads to an infeasible multidimensional non-linear maximization problem for which no closed form seems to exist.

- An alternative approach is to consider the disturbance matrix as known, derive the GLRT and then replace M with an appropriate estimate, resorting to

$$\frac{\left|\mathbf{p}^H \hat{\mathbf{M}}^{-1} \mathbf{z}\right|^2}{\left(\mathbf{z}^H \hat{\mathbf{M}}^{-1} \mathbf{z}\right)\left(\mathbf{p}^H \hat{\mathbf{M}}^{-1} \mathbf{p}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

Disturbance matrix estimation in CG disturbance

How to estimate the disturbance covariance matrix?

It is useful to observe that, when R is known, the following equation is valid

$$\frac{\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}\right|^2}{\left(\mathbf{z}^H \mathbf{R}^{-1}\mathbf{z}\right)\left(\mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\right)} = \frac{\left|\mathbf{p}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\left(\mathbf{z}^H \mathbf{M}^{-1}\mathbf{z}\right)\left(\mathbf{p}^H \mathbf{M}^{-1}\mathbf{p}\right)}$$

One possible solution is to use the SCM estimator of the covariance matrix

$\mathbf{R} = \sigma^2\mathbf{M}$ we normally apply with homogeneous Gaussian disturbance

$$\boxed{\hat{\mathbf{R}}_{SCM} = \frac{1}{K}\sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H}$$

and replace it in the detector

$$\frac{\left|\mathbf{p}^H \hat{\mathbf{R}}_{SCM}^{-1}\mathbf{z}\right|^2}{\left(\mathbf{z}^H \hat{\mathbf{R}}_{SCM}^{-1}\mathbf{z}\right)\left(\mathbf{p}^H \hat{\mathbf{R}}_{SCM}^{-1}\mathbf{p}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

SCM estimation in CG disturbance

Even for compound-Gaussian disturbance, this estimate is unbiased and consistent.

$$E\left\{\hat{\mathbf{R}}_{SCM}\right\} = \frac{1}{K}\sum_{k=1}^{K} E\left\{\mathbf{z}_k \mathbf{z}_k^H\right\} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{R} = \mathbf{R}$$

$$\lim_{K\to\infty} \hat{\mathbf{R}}_{SCM} = \mathbf{R} \quad \text{(convergence in mean square sense)}$$

$$i.e. \quad \lim_{K\to\infty} E\left\{\left|\mathbf{R}_{i,k} - \hat{\mathbf{R}}_{SCM_{i,k}}\right|^2\right\} = 0$$

Unfortunately, with the SCM estimator, the ANMF is CFAR with respect to the true covariance matrix **R**, but depends on the probability density function of the texture.

NSCM estimation in non-Gaussian disturbance

Another possible approach is to estimate the normalized covariance matrix M as

$$\hat{\mathbf{M}}_{NSCM} = \frac{M}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\mathbf{z}_k^H\mathbf{z}_k}$$

that is, by normalizing the matrix estimate for each secondary vector $\mathbf{z}_k\mathbf{z}_k^H$ by the sample estimate of its power $\hat{\tau}_k = \mathbf{z}_k^H\mathbf{z}_k / M$. This is the Normalized Sample Covariance Matrix (NSCM)

We can easily verify that the NSCM does not depend on the textures.

$$\hat{\mathbf{M}}_{NSCM} = \frac{M}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\mathbf{z}_k^H\mathbf{z}_k} = \frac{M}{K}\sum_{k=1}^{K}\frac{\tau_k\mathbf{x}_k\mathbf{x}_k^H}{\tau_k\mathbf{x}_k^H\mathbf{x}_k} = \frac{M}{K}\sum_{k=1}^{K}\frac{\mathbf{x}_k\mathbf{x}_k^H}{\mathbf{x}_k^H\mathbf{x}_k}$$

For compound-Gaussian disturbance, if the eigenvalues of the matrix M are distinct, the NSCM estimator is biased and it is **not consistent**.

Plugging the NSCM in the GLRT we obtain a test that is CFAR with respect to the values of the textures, but, due to the terms $\hat{\tau}_k = \mathbf{z}_k^H\mathbf{z}_k / M$ it is dependent on the true normalized covariance matrix **M**.

ML matrix estimation in non-Gaussian disturbance

A third possible estimator is the maximum likelihood (ML) one. To derive it we start from the joint pdf of the K secondary vectors

$$p_Z(\mathbf{z}_1,\ \mathbf{z}_2,\ \ldots \mathbf{z}_K) = \prod_{k=1}^{K} p_z(\mathbf{z}_k) = \prod_{k=1}^{K} \int_0^{\infty} p_z(\mathbf{z}_k | \tau_k) p_\tau(\tau_k) d\tau_k$$

$$= \prod_{k=1}^{K} \int_0^{+\infty} \frac{1}{(\pi\tau_k)^M |\mathbf{M}|} \exp\left(- \frac{\mathbf{z}_k^H \mathbf{M}^{-1} \mathbf{z}_k}{\tau_k} \right) p_\tau(\tau_k) d\tau_k$$

Defining the function $$h_M(q) \triangleq \int_0^{+\infty} \frac{1}{\tau^M} \exp(-q/\tau) d\tau$$

$$\Rightarrow \quad p_Z(\mathbf{z}_1,\ \mathbf{z}_2,\ \ldots \mathbf{z}_K) = \pi^{-KM} |\mathbf{M}|^{-K} \prod_{k=1}^{K} h_M(\mathbf{z}_k^H \mathbf{M}^{-1} \mathbf{z}_k)$$

ML matrix estimation in non-Gaussian disturbance

Now derivate the likelihood function with respect to each element of M and equal the derivatives to 0

$$-K\frac{\partial \ln|\mathbf{M}|}{\partial \mathbf{M}}+\sum_{k=1}^{K}\frac{g_M(\mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k)}{h_M(\mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k)}\cdot\frac{\partial\ \ \mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k}{\partial \mathbf{M}}=0$$

where $g_M(x)\triangleq\partial h_M(x)/\partial x$

With some mathematical manipulation we obtain

$$\hat{\mathbf{M}}_{ML}=\frac{1}{K}\sum_{k=1}^{K}\frac{h_{M+1}(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}\mathbf{z}_k)}{h_M(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}\mathbf{z}_k)}\cdot\mathbf{z}_k\mathbf{z}_k^H=\frac{1}{K}\sum_{k=1}^{K}c_M\left(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}\mathbf{z}_k\right)\cdot\mathbf{z}_k\mathbf{z}_k^H$$

where $c_M(x)\triangleq h_{M+1}(x)/h_M(x)$

ML matrix estimation in non-Gaussian disturbance

The ML estimator is the solution (if it exists) of a trascendental equation. We can solve it iteratively:

$$\hat{\mathbf{M}}_{ML}(i+1) = \frac{1}{K}\sum_{k=1}^{K} c_M\left(\mathbf{z}_k^H \hat{\mathbf{M}}_{ML}^{-1}(i)\mathbf{z}_k\right)\cdot \mathbf{z}_k \mathbf{z}_k^H$$

Problems:

- Calculation of the K data-dependent coefficients $c_M(\cdot)$ requires knowledge of the texture PDF.
- Even when the texture pdf is perfectly known, the calculation of these coefficients can be too computationally heavy for real time operation.
- The choice of a good starting point to prevent convergence to local maxima.

Approximate ML matrix estimation

Alternative approach:

1st step: Suppose that the textures in the secondary vectors are known and derivate the conditional pdf of the secondary data w.r.t. M:

$$\frac{\partial \ln p_Z(\mathbf{z}_1,\ \mathbf{z}_2,\ \cdots,\mathbf{z}_K|\tau_1,\ \tau_2,\ \cdots,\tau_K)}{\partial \mathbf{M}} = \sum_{k=1}^{K} \frac{\partial \ln p_{z|\tau}(\mathbf{z}_k|\tau_k)}{\partial \mathbf{M}} = 0$$

The solution is $\hat{\mathbf{M}} = \frac{1}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\tau_k}$

2nd step: Suppose M is known and derive the ML estimates of the textures:

$$\hat{\tau}_k = \arg\max_{\tau_k} p_{z|\tau}(\mathbf{z}_k|\tau_k) = \frac{\mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k}{M}, \qquad k = 1, 2, \cdots, K$$

3rd step: Put together the two estimators:

$$\hat{\mathbf{M}}_{AML} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{z}_k^H\hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k}\right)\cdot\mathbf{z}_k\mathbf{z}_k^H = \frac{1}{K}\sum_{k=1}^{K} d_M\left(\mathbf{z}_k^H\hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k\right)\cdot\mathbf{z}_k\mathbf{z}_k^H$$

Approximate ML matrix estimation

The approximate maximum likelihood estimator (AML) has the same structure as the true ML estimator but the coefficients do no depend on the texture pdf and are much easier to calculate:

$$\hat{\mathbf{M}}_{AML} = \frac{1}{K}\sum_{k=1}^{K} d_M\left(\mathbf{z}_k^H \hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k\right)\cdot \mathbf{z}_k\mathbf{z}_k^H$$

The equation can be solved iteratively and as initialization we can use the NSCM estimator:

$$\hat{\mathbf{M}}_{NSCM} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{z}_k^H\mathbf{z}_k}\right)\mathbf{z}_k\mathbf{z}_k^H$$

Matlab implementation shows that only few iterations (3-4) are necessary for convergence.

The AML estimate does not depend on the textures:

$$\hat{\mathbf{M}}_{AML} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{z}_k^H\hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k}\right)\cdot \mathbf{z}_k\mathbf{z}_k^H = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{x}_k^H\hat{\mathbf{M}}_{AML}^{-1}\mathbf{x}_k}\right)\cdot \mathbf{x}_k\mathbf{x}_k^H$$

The ANMF detector with estimated covariance

Let's now plug the disturbance matrix estimators in the ANMF making this detector adaptive to the covariance matrix.

It is important to verify if the new ANMF is CFAR with respect to the true covariance matrix **M**.

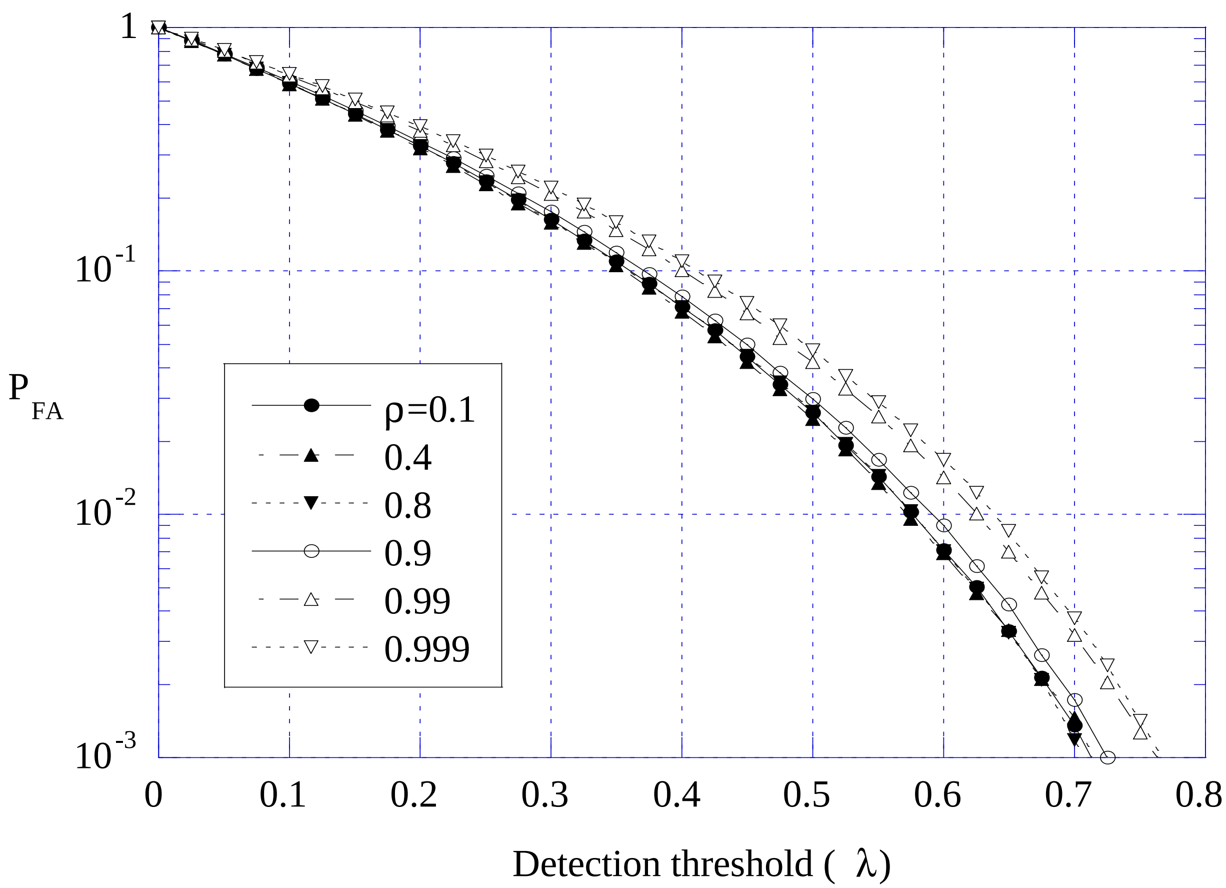
For this purpose, we simulate a K-distributed clutter with an AR(1) speckle correlation function $R_x(l) = \rho^{|l|}$ and shape parameter ν=0.5.

Changing the one-lag correlation coefficient ρ we change the shape of the clutter PSD.

In this simulation the number of integrated pulses in M=8 and the number of secondary vectors is K=3M=24

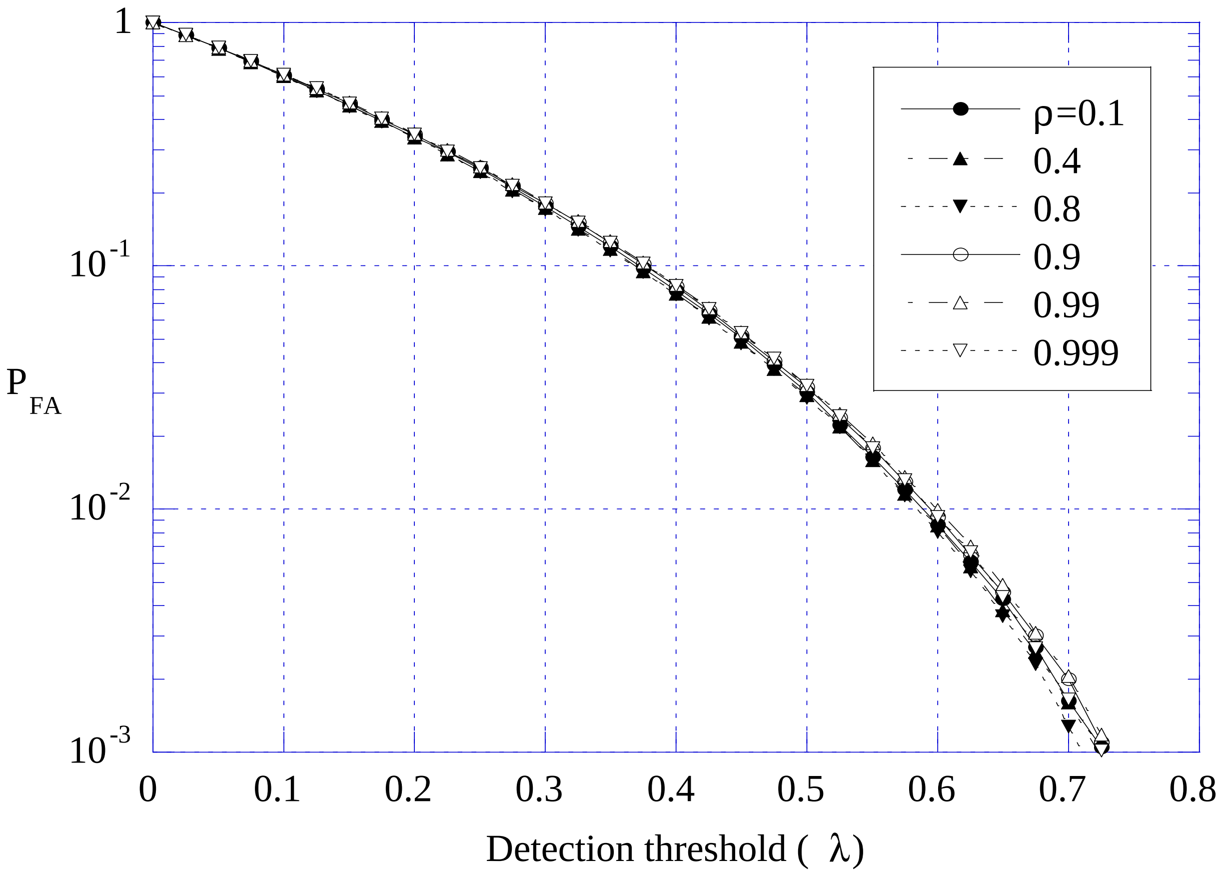
The Doppler frequency of the target is ν_d=0.15.

ANMF-NSCM



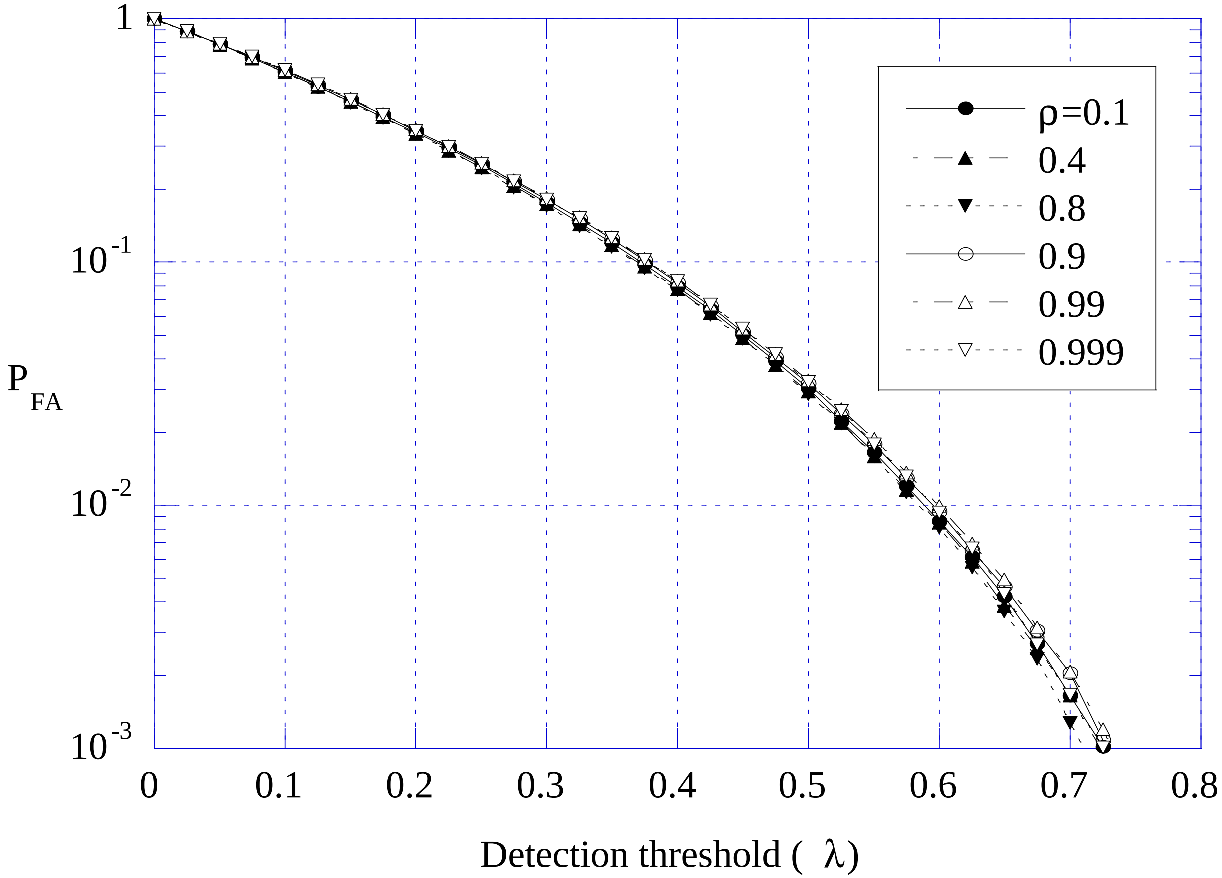
As expected, the ANMF-NSCM is not CFAR with respect to the matrix M (but it is CFAR w.r.t. the texture PDF).

ANMF-ML



The ANMF-ML is very robust (practically CFAR) with respect to the matrix M.

ANNMF-AML



The ANMF-AML is very robust (practically CFAR) with respect to the matrix M and its performance are very similar to that of the ANMF-ML.

Concluding Remarks

- NMF with different disturbance matrix estimates
- Performance of ANMF-ML close to ANMF-AML performance
- Open problem:
  - Impact of non-stationarity of the clutter, particularly for sea surfaces
  - Reduced number of secondary vectors

Thanks for your attention

QUESTIONS?