E9 231: Digital Array Signal Processing

Scribe: K. Murali Krishna
Dept. of EE & ECE
Indian Institute of Science
Bangalore 560 012, India
murali@ee.iisc.ernet.in

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1 Topics

- Conditional mean estimator.
- LMMSE revisited.
- Weighted least square estimator.
- Maximising the SNR.
- Sensitivity Analysis.

2 Class Notes

2.1 Conditional Mean Estimator

2.1.1 Weighted Least-Squres Estimator

We would like to minimise,

$$\min_F \qquad ||X - V_sF||^2 \tag{1}$$

$$= \min_F \qquad (X - V_sF)^H(X - V_sF)$$

$$= \min_F \qquad F^H V_s^H V_s F - F^H V_s^H X - V_s F X^H + X^H X \tag{2}$$

The optimum solution to above is given by differentiating wrt F^H and setting it to zero,

$$F_{opt} = (V_s^H V_S)^{-1} V_s^H X \quad \text{(for White Noise)} \tag{3}$$

now for weighted least square we have,

$$\min_F \qquad (X - V_sF)^H Sn^{-1}(X - V_sF) \tag{4}$$

$$\hat{F}_{WLS} = (V_s^H S_n^{-1} V_S)^{-1} V_s^H S_n^{-1} X = \quad W_{MVDR}^H X \tag{5}$$

2.1.2 LMMSE revisited

Recall,

$$H(w) = S_f(w)S_x^{-1}(w)V(w, K_s) = W_{LMMSE} \tag{6}$$

$$S_x(w) = V(w, K_s)S_f(w)V^H(w, K_s) + S_n(w) \tag{7}$$

LMMSE: Assume second order statistics of signal and noise are known, restrict to linear processor and minimize the expected error between signal and it's estimate.

2.2 Minimum Mean Square Error (MMSE) Estimator

MMSE estimator may not necessary be linear,

$$X = FV_s + N \tag{8}$$

Conditional Mean Estimator F/X will logically minimise the error between $\hat{F}$ and F,

$$\hat{F}_{MMSE} = E\{F/X\} \tag{9}$$

Is known as the conditional mean estimator. If we assume both the signal and noise are Guassian processes,

$$P_{X/F}(X/F) = (const)\exp\{(X - FV_s)^H S_n^{-1}(X - FV_s)\} \tag{10}$$

$$P_F(F) = (const')\exp\{-F^H S_f^{-1} F\} \tag{11}$$

$$P_{F/X}(F/X) = \frac{P_{X/F}P_F}{P_X} \tag{12}$$

$$P_{F/X}(F/X) = (const'')\exp\{-(F^* - X^H S_n^{-1} V_s H_s^*)H_s^{-1}(F - H_s V_s^H S_n^{-1} X)\} \tag{13}$$

$$H_s^{-1} = \frac{1}{\Lambda(w_m)} + \frac{1}{S_f(w_m)} \tag{14}$$

Conditional PDF is also Guassian

$$\hat{F}_{CME} = H_s V_s^H S_n^{-1} X \tag{15}$$

$$\hat{F}_{CME} = \frac{S_f}{S_f + \Lambda}\Lambda V_s^H S_n^{-1} X \tag{16}$$

$$\hat{F}_{CME} = \frac{S_f}{S_f + \Lambda} W_{MVDR}^H X \tag{17}$$

$$\tag{18}$$

-Same as the LMMSE
-Linear processor (Guassian signal and noise) **Data model**

$$X = V_s F + N \tag{19}$$

$$W_{MVDR} = W_{ML} = \Lambda S_n^{-1} V_s \tag{20}$$

$$\Lambda = \frac{1}{V_s^H S_n^{-1} V_s} \tag{21}$$

$$W_{LMMSE} = \frac{S_f}{S_f + \Lambda} W_{MVDR} \tag{22}$$

2.3 Maximum SNR Beamformers

$$X = FV_s + N \quad = X_s + N \tag{23}$$

$$S_{Xs} = E[X_s X_s^H] = V_s S_f V_s^H = S_f V_s V_s^H \tag{24}$$

$$W^H X = W^H V_s F + W^H N \tag{25}$$

$$\text{SNR} = \frac{W^H V_s S_f V_s^H W}{W^H S_n W} \tag{26}$$

$$\text{SNR} = \frac{W^H S_{Xs} W}{W^H S_n W} \tag{27}$$

Optimization:

$$W_{SNRmax} = \arg\max_W SNR \tag{28}$$

differentiate wrt W and solve
For such quadratic forms use Cholsky factorization to solve,

$$S_n = LL^H \tag{29}$$

L is lower triangular and invertible, and assume S_n is full rank

$$S_n = LQQ^H L^H$$

where Q is a diagonal matrix with real values and is unique and L is lower triangular and invertible,

$$\text{SNR} = \frac{W^H S_{Xs} W}{W^H L(L^H W)}$$

Let $\gamma \triangleq L^H W$ or $W = L^{-H}\gamma$ then we have,

$$\text{SNR} = \frac{W^H S_{Xs} W}{W^H L(L^H W)} = \frac{(\gamma^H L^{-1}) S_{Xs} (L^{-H}\gamma)}{\gamma^H \gamma} \tag{30}$$

$$\text{SNR} = \frac{\gamma^H \tilde{S}_{Xs} \gamma}{\gamma^H \gamma} \tag{31}$$

Where

$$\tilde{S}_{Xs} = L^{-1} S_{Xs} L^{-H} \tag{32}$$

$$\gamma_o = \arg\max_\gamma SNR \tag{33}$$

where γ_o is the eigenvector of $\tilde{S}_{Xs}$ corresponding to maximum eigenvalue α_{max} of $\tilde{S}_{Xs}$
the γ_o satisfies

$$\tilde{S}_{Xs}\gamma_o = \alpha_{max}\gamma_o \tag{34}$$

$$L^{-1} S_{Xs} L^{-H} \gamma_o = \alpha_{max}\gamma_o \tag{35}$$

$$L^{-H} L^{-1} S_{Xs} W_o = \alpha_{max} W_o \tag{36}$$

$$\therefore \quad S_n^{-1} S_{Xs} W_o = \alpha_{max} W_o \tag{37}$$

find the eigenvector corresponding to maximum eigenvalue of $S_n^{-1} S_{Xs}$

$$S_{Xs} = S_f V_s V_s^H \tag{38}$$

$$S_f S_n^{-1} V_s V_s^H W_o = \alpha_{max} W_o \tag{39}$$

$$W_o = \frac{S_f V_s^H W_o}{\alpha_{max}} S_n^{-1} V_s \qquad = (\text{scaling}) S_n^{-1} V_s \tag{40}$$

can set scaling = 1

$$\therefore W_{opt} = \Lambda S_n^{-1} V_s = W_{MVDR} \tag{41}$$

Which turns out to be an MVDR Beamformer. Now the question arises that,
When can we do better than the MVDR?
-noise is non-guassian
-DOA/Signal/Noise statistics are unknown or in error
-willing to do non-linear processing
One more note about the MVDR Beamformer:

$$W_{MVDR} = \Lambda S_n^{-1} V_s \tag{42}$$

$$S_n = \underbrace{V_I S_I V_I^H}_{\text{interference}} + \underbrace{\sigma_n^2 I}_{\text{AWGN}} \tag{43}$$

Here we can use the matrix inversion lemma to show as $\sigma_{n^2} \to \infty, W_{MVDR} \to W_c$ the Conventional BF. Also as the interference power $\to \infty$ the W_{MVDR} places deeper and deeper nulls in the direction of the interfering signals.

2.4 Minimum Power Distortionless Response(MPDR) Beamformer

(a) Require $W^H V_s = 1$
(b) Minimize output power $E\{|W^H X|^2\} = W^H S_X W$

$$\min_W \qquad W^H S_X W \tag{44}$$
$$\text{subject to } W^H V_s = 1$$

$$\therefore W_{MPDR} = \frac{1}{V_s^H S_X^{-1} V_s} S_X^{-1} V_s \tag{45}$$

For example: We can show by using the matrix inversion lemma that with one source (or uncorrelated sources) $W_{MPDR} = W_{MVDR}$ when $S_x = S_f V_s V s^H + S_n$