

# E9 231: Digital Array Signal Processing

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Class held on: 04 Aug 2008

## ARRAY CHARACTERISATION USING FREQUENCY WAVENUMBER RESPONSE

Frequency response is a temporal aspect where as wavenumber response is directional. Array is a set of  $N$  isotropic sensors distributed in space.  $\{\mathbf{p}_i; i = 0, \dots, N-1\}$  is the position vector of sensor  $i$ . The received signal is a function of time and position and is written as

$$\mathbf{f}(t, \mathbf{p}_i) = \begin{bmatrix} f(t, \mathbf{p}_0) \\ f(t, \mathbf{p}_1) \\ \vdots \\ f(t, \mathbf{p}_{N-1}) \end{bmatrix} \quad (1)$$

Consider a simple processing unit as shown in Fig.1. The impulse response of LTI systems are given in vector form as

$$\mathbf{h}(t) = \begin{bmatrix} h_0(t) \\ h_1(t) \\ \vdots \\ h_{N-1}(t) \end{bmatrix} \quad (2)$$

The output signal of the processing unit is given by

$$y(t) = \sum_{i=0}^{N-1} \int_{\tau} \mathbf{f}(\tau, \mathbf{p}_i) h_i(t - \tau) d\tau = \int_{\tau} \mathbf{f}(\tau, \mathbf{p}) \mathbf{h}(t - \tau) d\tau \quad (3)$$

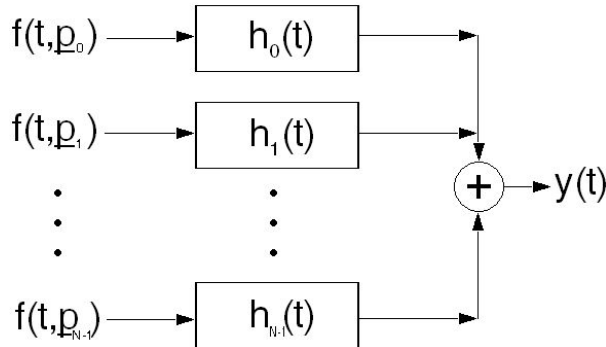


Figure 1: The received signals  $f(t, \mathbf{p}_i)$  are filtered using LTI systems with impulse response  $h_i(t)$ .

Taking Fourier transform of (3), we get,

$$Y(\omega) = \mathbf{F}^T(\omega, \mathbf{p})\mathbf{H}(\omega) \quad (4)$$

where,  $\mathbf{H}(\omega) = \int_t \mathbf{h}(t)e^{-j\omega t}dt$ . With a slight abuse of notation, we write  $\mathbf{F}(\omega, \mathbf{p})$  as  $\mathbf{F}(\omega)$ .

**Need for a model for  $f(t, \mathbf{p}_i)$  :** We consider a plane - wave model, where all sensors get the same signal with a delay of  $\tau_i$ . If  $f(t)$  is the signal received at origin,

$$\tau_i = \frac{\mathbf{a}^T \mathbf{p}_i}{c},$$

where the numerator gives the actual distance of the sensor with respect to the direction of  $\mathbf{a}$ , from the origin with  $\|\mathbf{a}\| = 1$ .

With the plane - wave model,

$$\mathbf{f}(t, \mathbf{p}_i) = \begin{bmatrix} f(t, \mathbf{p}_0) \\ f(t, \mathbf{p}_1) \\ \vdots \\ f(t, \mathbf{p}_{N-1}) \end{bmatrix}$$

becomes

$$\mathbf{f}(t, \mathbf{p}_i) = \begin{bmatrix} f(t - \tau_0) \\ f(t - \tau_1) \\ \vdots \\ f(t - \tau_{N-1}) \end{bmatrix}$$

Taking Fourier transforms, we get

$$\mathbf{F}(\omega) = \begin{bmatrix} e^{-j\omega\tau_0} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix} \quad (5)$$

We also define a direction vector,

$$\mathbf{u} = -\mathbf{a} = - \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} \quad (6)$$

The received signal  $Y(\omega) = \mathbf{H}^T(\omega)\mathbf{F}(\omega)$ .

$$Y(\omega) = \mathbf{H}^T(\omega) \begin{bmatrix} e^{-j\omega\tau_0} \\ \vdots \\ e^{-j\omega\tau_{N-1}} \end{bmatrix} \mathbf{F}(\omega) \quad (7)$$

Here,

$$\omega\tau_n = \frac{\omega \mathbf{a}^T \mathbf{p}_n}{c} = \frac{2\pi}{\lambda} \mathbf{a}^T \mathbf{p}_n = -\frac{2\pi}{\lambda} \mathbf{u}^T \mathbf{p}_n = \mathbf{k}^T \mathbf{p}_n. \quad (8)$$

where  $\mathbf{k}^T \triangleq \frac{2\pi}{\lambda} \mathbf{a}$  is called the wavenumber. Further,  $\|\mathbf{k}\| = \frac{2\pi}{\lambda}$ . Therefore, magnitude of  $\mathbf{k}$  remains constant and only the direction varies.

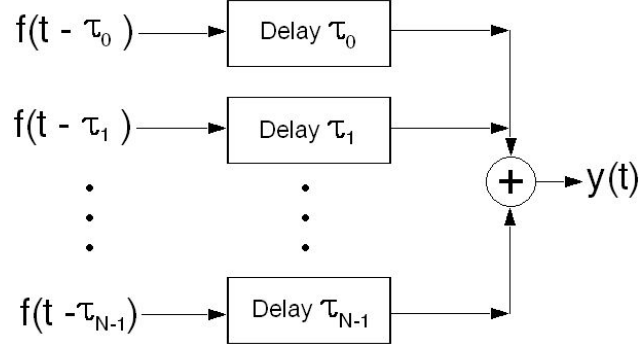


Figure 2: The received signals  $f(t, \mathbf{p}_i)$  are filtered using LTI systems with impulse response  $h_i(t)$ .

We can, therefore, write

$$Y(\omega) = \mathbf{H}^T(\omega) \begin{bmatrix} e^{-j\mathbf{k}^T \mathbf{p}_0} \\ \vdots \\ e^{-j\mathbf{k}^T \mathbf{p}_{N-1}} \end{bmatrix} \mathbf{F}(\omega) \quad (9)$$

$Y(\omega, \mathbf{k})$  is called the frequency - wavenumber response of the array.

$$V_k(\mathbf{k}) = \begin{bmatrix} e^{-j\mathbf{k}^T \mathbf{p}_0} \\ \vdots \\ e^{-j\mathbf{k}^T \mathbf{p}_{N-1}} \end{bmatrix} \quad (10)$$

is called the array manifold. Array manifold depends on:

- wavenumber,
- direction of arrival and
- position of sensors.

To maximize gain  $\gamma(\omega, \mathbf{k})$ , a good choice of  $\mathbf{H}(\omega)$  is

$$\mathbf{H}(\omega) = \begin{bmatrix} e^{j\omega\tau_0} \\ \vdots \\ e^{j\omega\tau_{N-1}} \end{bmatrix} \quad (11)$$

This acts as a delay and hence this type of a beamformer is called a delay- and-sum beamformer and is shown in Fig.2.

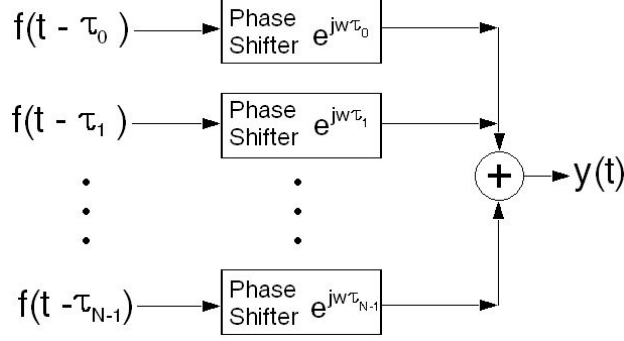


Figure 3: The signals  $f(t - \tau_n)$  are being phase-shifted by  $e^{j\omega\tau_n}$ .

### Beam Pattern:

$$B(\omega, \theta, \phi) = \gamma(\omega, \mathbf{k})|_{k=\frac{2\pi}{\lambda}a(\theta, \phi)} \quad (12)$$

We make the narrowband assumption. Any passband signal can be represented by its baseband envelope.

$$f(t) = \sqrt{2} \operatorname{Re}\{\tilde{f}(t)e^{j\omega_c t}\}, \quad (13)$$

$$f(t - \tau) = \sqrt{2} \operatorname{Re}\{\tilde{f}(t - \tau)e^{j\omega_c(t - \tau)}\} \quad (14)$$

Assuming  $\tilde{f}(t) \approx \tilde{f}(t - \tau)$ , we get

$$f(t - \tau) = \sqrt{2} \operatorname{Re}\{\tilde{f}(t)e^{j\omega_c(t - \tau)}\} \quad (15)$$

which is a phase-shifted version of the original signal. If  $\tau_{max} \ll \frac{1}{2B}$ , then the above approximation is valid. Fig.3 summarizes the above discussion. In case we apply a gain and a phase-shift instead of just a phase-shift, the arrangement is called as “linear array”.

### Uniform Linear Array (ULA):

In a ULA, linear array arrangement is used with the antennas placed along a straight line, with equal separation.

$$p_n = \begin{bmatrix} 0 \\ 0 \\ p_{z_n} \end{bmatrix}$$

We have

$$\omega\tau_n = \frac{\omega \mathbf{a}^T \mathbf{p}_n}{c} = -\frac{\omega \mathbf{u}^T \mathbf{p}_n}{c} = -\frac{\omega p_{z_n} \cos \theta}{c}$$

It should be noted that this is independent of the azimuth. In a ULA,

$$p_{z_n} = \left(n - \frac{N-1}{2}\right) d; \quad n = 0, \dots, N-1$$

$$\omega\tau_n = -\omega \left(n - \frac{N-1}{2}\right) \frac{d}{c} \cos \theta = -\left(n - \frac{N-1}{2}\right) \frac{2\pi d}{\lambda} \cos \theta$$

Define  $K_z = \frac{2\pi}{\lambda} \cos \theta$ . The array manifold vector

$$\mathbf{V}_k(\mathbf{K}) = \begin{bmatrix} e^{-jK^T p_0} \\ \vdots \\ e^{-jK^T p_{N-1}} \end{bmatrix} = \begin{bmatrix} e^{jK_z \left(\frac{N-1}{2}\right) d} \\ e^{jK_z \left(\frac{N-1}{2} - 1\right) d} \\ \vdots \\ e^{-jK_z \left(\frac{N-1}{2}\right) d} \end{bmatrix} \quad (16)$$

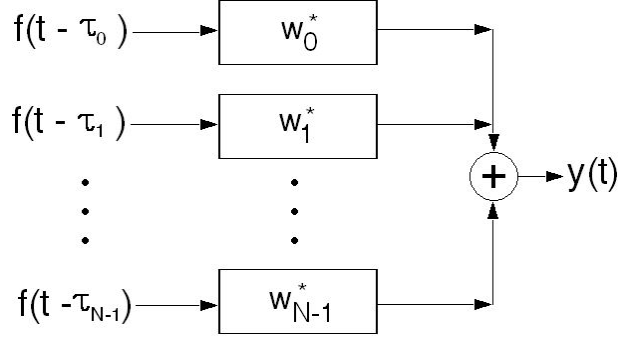


Figure 4: Schematic of Uniform Linear Array.

Define  $\psi = 2\pi \frac{d}{\lambda} \cos \theta$ . Recall that  $u_z = \cos \theta$ .

$$B(\omega, \psi) = \sum_{l=0}^{N-1} \omega_l^* e^{j\omega\tau_l} = \sum_{l=0}^{N-1} \omega_l^* e^{j(l - \frac{N-1}{2})\omega} = \gamma(\omega, k) = e^{-j\frac{N-1}{2}\psi} \left[ \sum_{l=0}^{N-1} \omega_l e^{-jl\psi} \right]^* \quad (17)$$

$\left[ \sum_{l=0}^{N-1} \omega_l e^{-jl\psi} \right]^* \Rightarrow$  DFT of weight sequence  $w_l$ .

$$B(\omega, u_z) = e^{-j\frac{N-1}{2} \frac{2\pi d}{\lambda} u_z} \left[ \sum_{l=0}^{N-1} \omega_l e^{-jl \frac{2\pi}{d} w_z} \right]^* \quad (18)$$

$$B(\omega, \theta) = e^{-j\frac{N-1}{2} \frac{2\pi d}{\lambda} \cos \theta} \left[ \sum_{l=0}^{N-1} \omega_l e^{-jl \frac{2\pi}{d} \cos \theta} \right]^* \quad (19)$$

#### Uniformly Weighted ULA:

Same as ULA, with  $\omega_n = \frac{1}{N}$  (rectangular window) and its Fourier transform being a sinc function.

$$B(\psi) = \sum_{l=0}^{N-1} \frac{1}{N} e^{j(l - \frac{N-1}{2})\psi} = \frac{1}{N} e^{j(\frac{N-1}{2})\psi} \sum_{l=0}^{N-1} e^{jl\psi} = \frac{\sin\left(\frac{N\psi}{2}\right)}{N \sin\left(\frac{\psi}{2}\right)} \quad (20)$$

or

$$B(\theta) = \frac{\sin\left(\frac{N \frac{2\pi d}{\lambda} \cos \theta}{2}\right)}{N \sin\left(\frac{\frac{2\pi d}{\lambda} \cos \theta}{2}\right)} \quad (21)$$