Optimum Coherent Detection in Gaussian Disturbance

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Binary Hypothesis Testing Problem

- The function of a surveillance radar is to ascertain whether targets are present in the data. Given the observed **vector z,** the signal processor must make a decision as to which of the two hypotheses is true:

$$\begin{cases} \mathbf{z} = \mathbf{d} & H_0 : \text{Target absent} \\ \mathbf{z} = \mathbf{s}_t + \mathbf{d} & H_1 : \text{Target present} \end{cases}$$

- The test is implemented on-line for each **range cell under test** (CUT)
- The detection is a **binary hypothesis problem** whereby we decide either hypothesis H_0 is true and only disturbance is present or hypothesis H_1 is true and a target signal is present with disturbance.

Neyman-Pearson Criterion

- According to the **Neyman-Pearson (NP) criterion** (maximize P_D while keeping constant P_{FA}), the optimal decision strategy is a **likelihood ratio test (LRT)**:

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta} \quad \Rightarrow \quad \ln\Lambda(\mathbf{z}) = \ln\left(\frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i)$ is the probability density function (PDF) of the random vector **z** under the hypothesis H_i, i=0,1.

η is the detection threshold (set according to the desired P_{FA}).

The Optimal Detector

- The complex multidimensional PDF of Gaussian disturbance is given by:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = p_{\mathbf{d}}(\mathbf{z}) = \frac{1}{\pi^{NM}|\mathbf{R}|}\exp\left(-\mathbf{z}^H\mathbf{R}^{-1}\mathbf{z}\right)$$

$$\mathbf{z}|H_0 \in \mathcal{CN}\left(\mathbf{0},\mathbf{R}\right)$$

Nx1 data vector

$$|\mathbf{R}| = \det\{\mathbf{R}\}, \quad \mathbf{R} = \mathbf{R}^H \rightarrow \text{Hermitian matrix}$$

- The complex multidimensional PDF of *target* + *disturbance*: $p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = ?$
- It depends on the target signal model: $\mathbf{z} = \mathbf{s}_t + \mathbf{d}$
- If the target vector is deterministic:

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)$$

The Optimal Detector

- If the target vector is random with known PDF:

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = E_{\mathbf{s}_t}\left\{p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)\right\} = \int \underbrace{\frac{1}{\pi^N|\mathbf{R}|}\exp\left(-(\mathbf{z}-\mathbf{s}_t)^H\mathbf{R}^{-1}(\mathbf{z}-\mathbf{s}_t)\right)}_{p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)} \cdot p_{\mathbf{s}_t}(\mathbf{s}_t)d\mathbf{s}_t$$

- Under some assumptions, we have found for the target signal:

$$\underset{N\times 1}{\mathbf{s}_t} = \beta\,\mathbf{p}(f_d) \quad [\beta \text{ is the target complex amplitude}]$$

$\underset{N\times 1}{\mathbf{p}(f_d)}$ **Steering Vector**

$$f_d = \frac{2v_t T_r}{\lambda} \quad \text{Doppler frequency} \Leftrightarrow \text{target radial velocity}$$

Narrowband Slowly-Fluctuating Target Signal Model

$$\underset{N\times 1}{\mathbf{p}(f_d)} = \begin{bmatrix} 1 \\ e^{j2\pi f_d} \\ \vdots \\ e^{j2\pi(N-1)f_d} \end{bmatrix}$$

- **Temporal Steering Vector**

  It has the Vandermonde form because the waveform PRF is uniform and target velocity is constant during the CPI.

- The detection algorithm is optimized for a specific Doppler.
- Since the target velocity is unknown a-priori, **p** is a known function of unknown parameters, so the radar receiver should implement multiple detectors that form a **filter bank** to cover all potential target Doppler frequencies.

Target Signal Models

$$\underset{N\times 1}{\mathbf{s}_t} = \beta \mathbf{p}(f_d)$$

Different models of $\mathbf{s}_t$ have been investigated to take into account different degrees of *a priori* knowledge on the target signal:

(1) $\mathbf{s}_t$ perfectly known

(2) $\mathbf{s}_t = \beta \mathbf{p}$ with $\beta \in \mathcal{CN}(0,\sigma_s^2)$ i.e., Swerling I model, and $\mathbf{p}$ perfectly known;

(3) $\mathbf{s}_t = \beta \mathbf{p}$ with $|\beta|$ deterministic and $\angle\beta$ random, uniformly distributed in $[0,2\pi)$,

i.e. Swerling 0 (or Swerling V) model, and $\mathbf{p}$ perfectly known

(4) $\mathbf{s}_t = \beta \mathbf{p}$ with β unknown deterministic and $\mathbf{p}$ perfectly known;

Perfectly Known Target Signal - Case #1

- The **optimal NP decision strategy** is a **LRT** (or **log-LRT**):

$$l(\mathbf{z}) = \ln \Lambda(\mathbf{z}) = \ln \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \mathbf{z}^H \mathbf{R}^{-1}\mathbf{z} - (\mathbf{z}-\mathbf{s}_t)^H \mathbf{R}^{-1}(\mathbf{z}-\mathbf{s}_t)$$

$$= \mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{z} + \mathbf{z}^H \mathbf{R}^{-1}\mathbf{s}_t - \mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{s}_t = 2\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{z}\right\} - \mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{s}_t \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- It is the so-called **coherent whitening matched filter (CWMF)** detector:

$$\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{z} = \mathbf{s}_t^H \mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\mathbf{z} = \mathbf{s}_t^H \left(\mathbf{R}^{-1/2}\right)^H \mathbf{R}^{-1/2}\mathbf{z} = \left(\mathbf{R}^{-1/2}\mathbf{s}_t\right)^H \mathbf{R}^{-1/2}\mathbf{z} = \overline{\mathbf{s}}_t^H \overline{\mathbf{z}}$$

$$\overline{\mathbf{s}}_t \triangleq \mathbf{R}^{-1/2}\mathbf{s}_t, \quad \overline{\mathbf{z}} \triangleq \mathbf{R}^{-1/2}\mathbf{z}$$

whitening transformation

matched filtering

$$E\left\{\overline{\mathbf{z}}\,\overline{\mathbf{z}}^H\right\} = E\left\{\mathbf{R}^{-1/2}\mathbf{z}\mathbf{z}^H\mathbf{R}^{-1/2}\right\} = \mathbf{R}^{-1/2}E\left\{\mathbf{z}\mathbf{z}^H\right\}\mathbf{R}^{-1/2} = \mathbf{R}^{-1/2}\mathbf{R}\mathbf{R}^{-1/2} = \mathbf{I}$$

Performance of the coherent WMF

$$l(\mathbf{z}) = 2\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{z}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\mathbf{z}\,|\,H_0 \in \mathcal{CN}\left(0,\mathbf{R}\right), \quad \mathbf{z}\,|\,H_1 \in \mathcal{CN}\left(\mathbf{s}_t,\mathbf{R}\right) \quad \Rightarrow \quad l \text{ is a real Gaussian r.v.}$$

$$m_0 = E\left\{l\,|\,H_0\right\} = 2\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} E\left\{\mathbf{z}\,|\,H_0\right\}\right\} = 0$$

$$m_1 = E\left\{l\,|\,H_1\right\} = 2\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} E\left\{\mathbf{z}\,|\,H_1\right\}\right\} = 2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{s}_t$$

- The fact that **d** is a complex circular Gaussian random vector implies that [Ch.13, Kay98]:

$$E\left\{\mathbf{d}\mathbf{d}^H\right\} = \mathbf{R} = \mathbf{R}^H \quad \text{and} \quad E\left\{\mathbf{d}\mathbf{d}^T\right\} = 0$$

[Kay98] S.M. Kay, *Fundamentals of Statistical Signal Processing. Detection Theory*. Prentice Hall, 1998.

Performance of the coherent WMF

$$\sigma_0^2 = \mathrm{var}\left\{l \middle| H_0\right\} = E\left\{\left|l - E\left\{l \middle| H_0\right\}\right|^2 \middle| H_0\right\}$$

$$= E\left\{\left|2\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{z}\right\}\right|^2 \middle| H_0\right\} = E\left\{\left|\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{z} + \mathbf{z}^H \mathbf{R}^{-1}\mathbf{s}_t\right|^2 \middle| H_0\right\}$$

$$= E\left\{\left(\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t\right)\left(\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t\right)^H\right\}$$

$$= E\left\{\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d} + \mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t + \mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t\mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t\right\}$$

$$= E\left\{\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^T \mathbf{R}^{-T}\mathbf{s}_t^* + 2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t + \mathbf{s}_t^T \mathbf{R}^{-1}\mathbf{d}^*\mathbf{d}^H \mathbf{R}^{-T}\mathbf{s}_t\right\}$$

$$= E\left\{2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^T \mathbf{R}^{-T}\mathbf{s}_t^* + 2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^H \mathbf{R}^{-1}\mathbf{s}_t\right\}$$

$$= 2\mathbf{s}_t^H \mathbf{R}^{-1} E\left\{\mathbf{d}\mathbf{d}^T\right\}\mathbf{R}^{-T}\mathbf{s}_t^* + 2\mathbf{s}_t^H \mathbf{R}^{-1} E\left\{\mathbf{d}\mathbf{d}^H\right\}\mathbf{R}^{-1}\mathbf{s}_t$$

$$= 2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{R}\mathbf{R}^{-1}\mathbf{s}_t = 2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{s}_t$$

$$\sigma_1^2 = \mathrm{var}\left\{l \middle| H_1\right\} = E\left\{\left|l - E\left\{l \middle| H_1\right\}\right|^2 \middle| H_1\right\} = 2\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{s}_t$$

we used the fact that $\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d} = \left(\mathbf{s}_t^H \mathbf{R}^{-1}\mathbf{d}\right)^T = \mathbf{d}^T \mathbf{R}^{-T}\mathbf{s}_t^*$ and $E\left\{\mathbf{d}\mathbf{d}^T\right\} = 0$

Performance of the coherent WMF

- The **probability of detection (*P_D*)** and **probability of false alarm (*P_{FA}*)** can be calculated using the statistics we have just derived:

$$P_{FA} = \Pr\{l > \eta | H_0\} = \int_{\eta}^{+\infty} p_{l|H_0}(l|H_0)dl = \int_{\eta}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(l-m_0)^2}{2\sigma_0^2}\right)dl$$

$$\overset{x=\frac{l-m_0}{\sigma_0}}{=} \int_{\frac{\eta-m_0}{\sigma_0}}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)dx = Q\left(\frac{\eta-m_0}{\sigma_0}\right) = Q\left(\frac{\eta}{\sqrt{2\mathbf{s}_t^H\mathbf{R}^{-1}\mathbf{s}_t}}\right)$$

$$P_D = \Pr\{l > \eta | H_1\} = \int_{\eta}^{+\infty} p_{l|H_1}(l|H_1)dl = Q\left(\frac{\eta-m_1}{\sigma_1}\right) = Q\left(\frac{\eta - 2\mathbf{s}_t^H\mathbf{R}^{-1}\mathbf{s}_t}{\sqrt{2\mathbf{s}_t^H\mathbf{R}^{-1}\mathbf{s}_t}}\right)$$

What is the meaning of $2\mathbf{s}_t^H\mathbf{R}^{-1}\mathbf{s}_t$?

Performance of the coherent WMF

- The filter output can be written as : $y = \mathbf{w}^H \mathbf{z} = \mathbf{w}^H (\mathbf{s}_t + \mathbf{d}) = \mathbf{w}^H \mathbf{s}_t + \mathbf{w}^H \mathbf{d}$ ***(H₁)***

- The **Signal-to-Interference plus Noise ratio (SNR)** at the output is:

$$SNR = \frac{E\left\{\left|\mathbf{w}^H \mathbf{s}_t\right|^2\right\}}{E\left\{\left|\mathbf{w}^H \mathbf{d}\right|^2\right\}} = \frac{E\left\{\mathbf{w}^H \mathbf{s}_t \mathbf{s}_t^H \mathbf{w}\right\}}{E\left\{\mathbf{w}^H \mathbf{d}\mathbf{d}^H \mathbf{w}\right\}} = \frac{\mathbf{w}^H E\left\{\mathbf{s}_t \mathbf{s}_t^H\right\} \mathbf{w}}{\mathbf{w}^H E\left\{\mathbf{d}\mathbf{d}^H\right\} \mathbf{w}} = \frac{\mathbf{w}^H \mathbf{s}_t \mathbf{s}_t^H \mathbf{w}}{\mathbf{w}^H \mathbf{R} \mathbf{w}} = \frac{\left|\mathbf{w}^H \mathbf{s}_t\right|^2}{\mathbf{w}^H \mathbf{R} \mathbf{w}}$$

if the target signal is deterministic

- Let's apply this consideration to our case where $\mathbf{w} = \mathbf{R}^{-1}\mathbf{s}_t$ and

$$\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z}\right\} = \Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t\right\} + \Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d}\right\} = \mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t + \Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d}\right\}$$

Performance of the coherent WMF

- Therefore, the SNR at the output of the **coherent WMF** (CWMF) is:

$$SNR_{CWMF} = \frac{\left(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t\right)^2}{E\left\{\left(\Re e\left\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d}\right\}\right)^2\right\}} = \frac{\left(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t\right)^2}{E\left\{\left(\dfrac{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d} + \mathbf{d}^H \mathbf{R}^{-1} \mathbf{s}_t}{2}\right)^2\right\}}$$

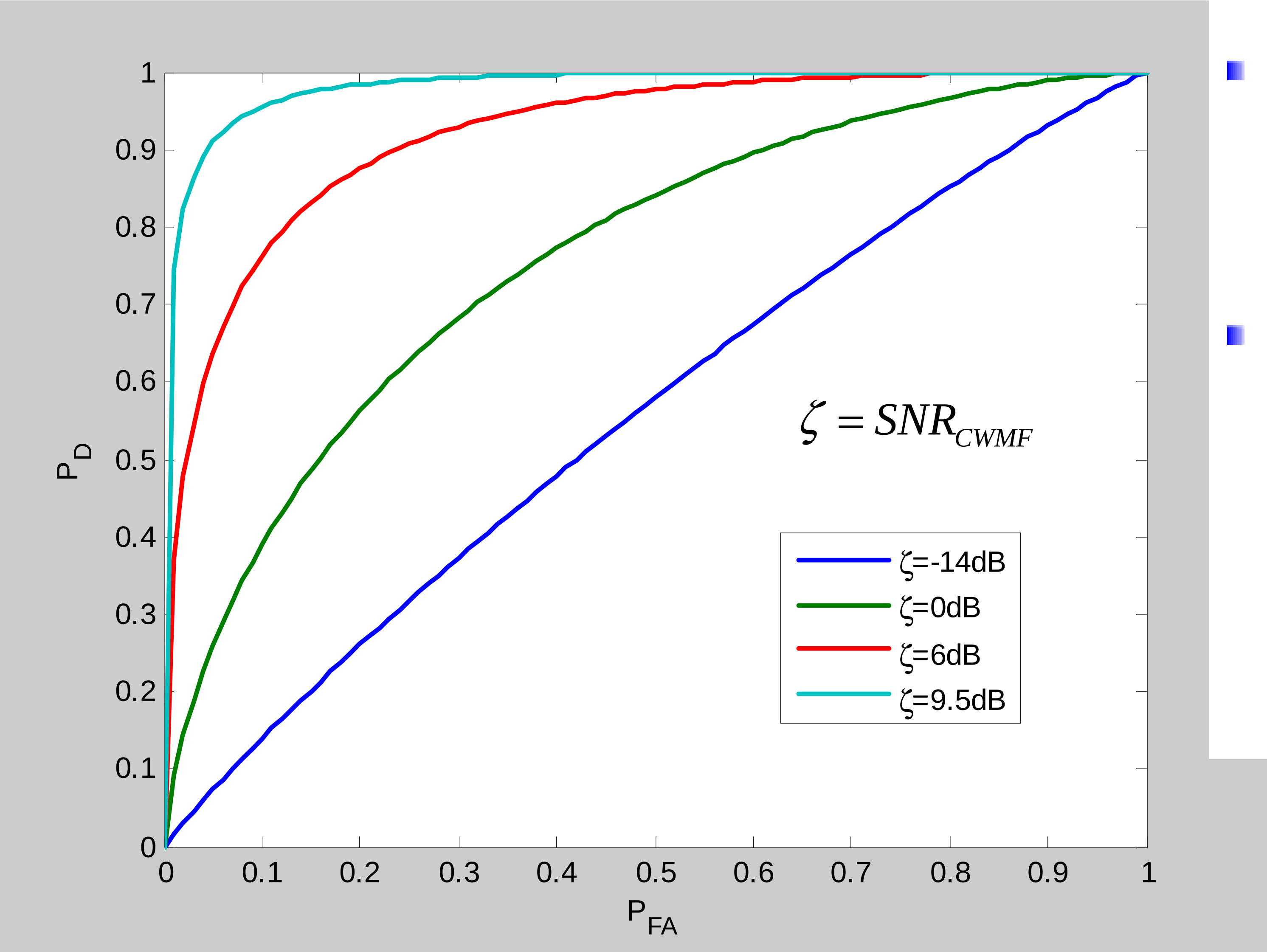
$$= \frac{4\left(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t\right)^2}{2E\left\{\left(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{d}\right)^2\right\}} = \frac{4\left(\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t\right)^2}{2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t} = 2\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{s}_t$$

$$P_{FA} = Q\left(\frac{\eta}{\sqrt{SNR_{CWMF}}}\right) \quad \Rightarrow \quad \eta = \sqrt{SNR_{CWMF}} \cdot Q^{-1}\left(P_{FA}\right)$$

$$P_D = Q\left(\frac{\eta - SNR_{CWMF}}{\sqrt{SNR_{CWMF}}}\right) \quad \Rightarrow \quad P_D = Q\left(Q^{-1}\left(P_{FA}\right) - \sqrt{SNR_{CWMF}}\right)$$

Performance of the coherent WMF

- **Receiver Operating Characteristic (ROC) curves:**



- The ROC is a function of only the SNR and the threshold.
- Here, the threshold is varied to cover the entire range of interest and the ROC curve is plotted for four different SNR values.

Swerling I Target Signal - Case #2

$$\mathbf{s}_t = \beta \mathbf{p}(f_d), \quad \beta \in \mathcal{CN}\left(0, \sigma_s^2\right), \quad \mathbf{p} \text{ perfectly known}$$

$$\beta = |\beta| e^{j\varphi} \in \mathcal{CN}\left(0, \sigma_s^2\right) \quad \Rightarrow \quad |\beta| \text{ is Rayleigh distributed, } E\left\{|\beta|^2\right\} = \sigma_s^2$$

$$\varphi \text{ is uniformly distributed on } [0, 2\pi)$$

- Based on these assumptions:

$$\mathbf{z} | H_0 \in \mathcal{CN}\left(0, \mathbf{R}\right), \quad \mathbf{z} | H_1 \in \mathcal{CN}\left(0, \sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R}\right)$$

$$E\left\{\mathbf{z}\mathbf{z}^H | H_1\right\} = E\left\{(\beta\mathbf{p} + \mathbf{d})(\beta\mathbf{p} + \mathbf{d})^H\right\} = E\left\{|\beta|^2\right\}\mathbf{p}\mathbf{p}^H + \mathbf{R} = \sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R}$$

Swerling I Target Signal - Case #2

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = \frac{1}{\pi^N |\mathbf{R}|} \exp\left(-\mathbf{z}^H \mathbf{R}^{-1} \mathbf{z}\right)$$

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = \frac{1}{\pi^N \left|\sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R}\right|} \exp\left(-\mathbf{z}^H (\sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R})^{-1} \mathbf{z}\right)$$

- The log-likelihood ratio (log-LR or LLR) is given by:

$$\ln \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \ln|\mathbf{R}| - \ln\left|\sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R}\right| + \mathbf{z}^H \mathbf{R} \mathbf{z} - \mathbf{z}^H (\sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R})^{-1} \mathbf{z}$$

- By making use of *Woodbury's identity*:

$$(\sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R})^{-1} = \mathbf{R}^{-1} - \frac{\sigma_s^2 \mathbf{R}^{-1} \mathbf{p}\mathbf{p}^H \mathbf{R}^{-1}}{1 + \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}$$

Swerling I Target Signal - Case #2

$$\ln \Lambda(\mathbf{z}) = \ln\left|\mathbf{R}\right| - \ln\left|\sigma_s^2 \mathbf{p}\mathbf{p}^H + \mathbf{R}\right| + \sigma_s^2 \frac{\mathbf{z}^H \mathbf{R}^{-1}\mathbf{p}\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}}{1+\sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- By incorporating the non-essential terms in the threshold, we derive that in this case the optimal NP decision strategy is:

$$\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- Again, the optimal decision strategy requires calculation of the WMF output, but instead of taking the real part of the output we calculate the modulo of the output: this is due to the fact that the target phase in this case is unknown.
- This is sometimes called the **noncoherent WMF**

Performance of the non-coherent WMF

- Performance of the non-coherent WMF: **Signal-to-Noise power ratio (SNR)**

$$\left|\mathbf{w}^H\mathbf{z}\right|^2 = \left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\mathbf{w} = \kappa\mathbf{R}^{-1}\mathbf{p}$$ optimal weights

$$SNR = \frac{E\left\{\left|\mathbf{w}^H\mathbf{s}_t\right|^2\right\}}{E\left\{\left|\mathbf{w}^H\mathbf{d}\right|^2\right\}} = \frac{E\left\{\left|\mathbf{w}^H\mathbf{s}_t\right|^2\right\}}{\mathbf{w}^H\mathbf{R}\mathbf{w}} \Rightarrow SNR_{WMF} = E\left\{\left|\beta\right|^2\right\}\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}$$

$$SNR_{WMF} = \begin{cases} \sigma_s^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}, & \beta \in \mathcal{CN}\left(0,\sigma_s^2\right) \\ \left|\beta\right|^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}, & \beta \text{ deterministic} \end{cases}$$

Performance of the non-coherent WMF – SW1

- Performance of the non-coherent WMF when the target complex amplitude is a zero-mean circular complex Gaussian r.v. (Swerling I).

$$\mathbf{z}\big|H_0 \in \mathcal{CN}\left(0,\mathbf{R}\right), \quad \mathbf{z}\big|H_1 \in \mathcal{CN}\left(0,\sigma_s^2\mathbf{p}\mathbf{p}^H+\mathbf{R}\right)$$

$$\Rightarrow \quad Y \triangleq \mathbf{p}^H\mathbf{R}^{-1}\mathbf{z} \text{ is a complex circular Gaussian r.v. } \big|H_i$$

$$\Rightarrow \quad Y\big|H_i \in \mathcal{CN}(m_i,\sigma_i^2), i=0,1$$

$$m_0 = E\left\{Y\big|H_0\right\} = \mathbf{p}^H\mathbf{R}^{-1}E\left\{\mathbf{z}\big|H_0\right\} = \mathbf{p}^H\mathbf{R}^{-1}E\left\{\mathbf{d}\right\} = 0$$

$$m_1 = E\left\{Y\big|H_1\right\} = \mathbf{p}^H\mathbf{R}^{-1}E\left\{\mathbf{z}\big|H_1\right\} = \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\cdot E\left\{\beta\right\} + \mathbf{p}^H\mathbf{R}^{-1}E\left\{\mathbf{d}\right\} = 0$$

$$X = \left|Y\right|^2 = Y_R^2 + Y_I^2 \quad \Rightarrow \quad X\big|H_i \in \mathrm{Exp}(\sigma_i^2), i=0,1$$

$$p_{X|H_i}(x|H_i) = \frac{1}{\sigma_i^2}e^{-\frac{x}{\sigma_i^2}}u(x)$$

$$u(x) = \begin{cases} 0 & x<0 \\ 1 & x\ge 0 \end{cases}$$

Performance of the non-coherent WMF – SW1

$$\sigma_0^2 = \mathrm{var}\left\{Y \middle| H_0\right\} = E\left\{\left|Y\right|^2 \middle| H_0\right\} = E\left\{\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2 \middle| H_0\right\} = E\left\{\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{d}\right|^2\right\}$$

$$= E\left\{\mathbf{p}^H\mathbf{R}^{-1}\mathbf{d}\mathbf{d}^H\mathbf{R}^{-1}\mathbf{p}\right\} = \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}$$

$$\sigma_1^2 = \mathrm{var}\left\{Y \middle| H_1\right\} = E\left\{\left|Y\right|^2 \middle| H_1\right\} = E\left\{\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2 \middle| H_1\right\}$$

$$= E\left\{\left|\beta\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p} + \mathbf{p}^H\mathbf{R}^{-1}\mathbf{d}\right|^2\right\} = E\left\{\left|\beta\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\right|^2 + \left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{d}\right|^2\right\}$$

$$= \sigma_s^2\left(\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\right)^2 + \mathbf{p}^H\mathbf{R}^{-1}\mathbf{p} = \left(\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\right)\left(1+\sigma_s^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\right)$$

$$P_{FA} = \int_\eta^{+\infty} p_{X|H_0}(x|H_0)dx = e^{-\frac{\eta}{\sigma_0^2}}$$

$$P_D = \int_\eta^{+\infty} p_{X|H_1}(x|H_1)dx = e^{-\frac{\eta}{\sigma_1^2}}$$

Performance of the non-coherent WMF – SW1

$$P_{FA} = e^{-\frac{\eta}{\sigma_0^2}} \quad \Rightarrow \quad \eta = \sigma_0^2 \ln\left(1/P_{FA}\right)$$

$$P_D = e^{-\frac{\eta}{\sigma_1^2}} = e^{-\frac{\sigma_0^2}{\sigma_1^2}\ln\left(1/P_{FA}\right)} = \left(e^{-\ln\left(1/P_{FA}\right)}\right)^{\frac{\sigma_0^2}{\sigma_1^2}} = \left(P_{FA}\right)^{\frac{\sigma_0^2}{\sigma_1^2}}$$

where $\dfrac{\sigma_1^2}{\sigma_0^2} = 1 + \sigma_s^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p} = 1 + SNR_{WMF}$

- **Receiver Operating Characteristic (ROC) curve**s:

$$P_D = \left(P_{FA}\right)^{\frac{1}{1+SNR_{WMF}}}$$

- P_D is a monotonic increasing function of SNR_{WMF}.

Swerling 0 Target Signal – Case #3

- If the amplitude is non fluctuating and the phase is uniformly distributed:

$$\beta = |\beta| e^{j\varphi} \quad \Rightarrow \quad |\beta| \text{ is deterministic (nonfluctuating)}$$

$$\varphi \text{ is uniformly distributed on } [0,2\pi)$$

$$\mathbf{z}|\varphi, H_1 \in \mathcal{CN}\left(|\beta| e^{j\varphi}\mathbf{p}, \mathbf{R}\right)$$

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z},\varphi|H_1}(\mathbf{z},\varphi|H_1)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z}|\varphi,H_1}(\mathbf{z}|\varphi,H_1)p_{\varphi|H_1}(\varphi|H_1)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}$$

$$= \frac{\int_{-\infty}^{+\infty} p_{\mathbf{z}|\varphi,H_1}(\mathbf{z}|\varphi,H_1)p_{\varphi}(\varphi)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\frac{1}{2\pi}\int_{0}^{2\pi} p_{\mathbf{z}|\varphi,H_1}(\mathbf{z}|\varphi,H_1)d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}$$

Swerling 0 Target Signal

$$\Lambda(\mathbf{z}) = \frac{\frac{1}{2\pi}\int_0^{2\pi} p_{\mathbf{z}|\varphi,H_1}(\mathbf{z}|\varphi,H_1)\,d\varphi}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}$$

$$= \frac{\frac{1}{2\pi}\int_0^{2\pi} \exp\left(-(\mathbf{z}-|\beta|e^{j\varphi}\mathbf{p})^H\mathbf{R}^{-1}(\mathbf{z}-|\beta|e^{j\varphi}\mathbf{p})\right)d\varphi}{\exp\left(-\mathbf{z}^H\mathbf{R}^{-1}\mathbf{z}\right)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\left(|\beta|e^{-j\varphi}\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z} + |\beta|e^{j\varphi}\mathbf{z}^H\mathbf{R}^{-1}\mathbf{p} - |\beta|^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\right)d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\left(2|\beta|\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|\cos(\varphi-\varphi_0) - |\beta|^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}\right)d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\left(2|\beta|\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|\cos(\varphi-\varphi_0)\right)d\varphi \cdot e^{-|\beta|^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}},$$

where $\varphi_0 \triangleq \angle\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}$

Swerling 0 Target Signal – Case #3

$$\Lambda(\mathbf{z}) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left(2|\beta|\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|\cos(\varphi-\varphi_0)\right)d\varphi \cdot e^{-|\beta|^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}}$$

$$= I_0\left(2|\beta|\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|\right)\cdot e^{-|\beta|^2\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

where $I_0(x)$ is the modified Bessel function of the 1st kind:

$$I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi} \exp\left[x(\varphi-\varphi_0)\right]d\varphi$$

- Since $I_0(x)$ is monotonic in x, the LRT reduces to the **noncoherent WMF**:

$$\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right| \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\eta}$$

or equivalentely

$$\left|\mathbf{p}^H\mathbf{R}^{-1}\mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Performance of the non-coherent WMF – SW0

- Performance of the non-coherent WMF when the target amplitude $|\beta|$ is deterministic, i.e. nonfluctuating, and the target phase is uniformly distributed over $[0,2\pi)$ (Swerling 0).

$$\mathbf{z}\big|H_0 \in \mathcal{CN}\left(0,\mathbf{R}\right), \quad \mathbf{z}\big|\varphi, H_1 \in \mathcal{CN}\left(\left|\beta\right|e^{j\varphi}\mathbf{p},\mathbf{R}\right)$$

$$p_{\mathbf{z}|H_1}(\mathbf{z}\big|H_1) = \frac{1}{2\pi}\int_0^{2\pi} p_{\mathbf{z}|\varphi,H_1}(\mathbf{z}\big|\varphi,H_1)d\varphi$$

- The **probability of false alarm** is the same as before, since it does not depend on the target signal model:

$$P_{FA} = e^{-\frac{\eta}{\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}}}$$

Performance of the non-coherent WMF – SW0

- The decision rule can be put in the form:

$$X = |Y|^2 = Y_R^2 + Y_I^2 = \left|\mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- We want to calculate:

$$P_D = \int_{\eta}^{+\infty} p_{X|H_1}(x|H_1)dx = \int_{\sqrt{\eta}}^{+\infty} p_{|Y||H_1}(y|H_1)dy$$

PDF of $|Y|$ under the hypothesis H_1

- Let us calculate first the (conditional) PDF of the complex r.v. Y:

$$Y = Y_R + jY_I = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{z}$$

Performance of the non-coherent WMF – SW0

$\Rightarrow \quad Y\big|\varphi, H_1 = \mathbf{p}^H \mathbf{R}^{-1}\mathbf{z}$ is a complex Gaussian r.v.

$\Rightarrow \quad Y\big|\varphi, H_1 \in \mathcal{CN}(m_1, \sigma_1^2)$

$$m_1 = E\left\{Y\big|\varphi, H_1\right\} = \mathbf{p}^H \mathbf{R}^{-1} E\left\{\mathbf{z}\big|\varphi, H_1\right\} = \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\cdot|\beta| e^{j\varphi}$$

$$\sigma_1^2 = E\left\{\left|Y - m_1\right|^2 \big|\varphi, H_1\right\} = E\left\{\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{z} - \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}\cdot|\beta| e^{j\varphi}\right|^2 \big|\varphi, H_1\right\}$$

$$= E\left\{\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}\right|^2 \big|\varphi, H_1\right\} = E\left\{\left|\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}\right|^2\right\}$$

$$= E\left\{\mathbf{p}^H \mathbf{R}^{-1}\mathbf{d}\mathbf{d}^H \mathbf{R}^{-1}\mathbf{p}\right\} = \mathbf{p}^H \mathbf{R}^{-1} E\left\{\mathbf{d}\mathbf{d}^H\right\}\mathbf{R}^{-1}\mathbf{p} = \mathbf{p}^H \mathbf{R}^{-1}\mathbf{p}$$

$$p_{Y_R, Y_I|H_1}(y_R, y_I\big|H_1) \equiv p_{Y|H_1}(y\big|H_1) = \frac{1}{2\pi}\int_0^{2\pi} p_{Y|\varphi, H_1}(y\big|\varphi, H_1)\, d\varphi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\pi\sigma_1^2}\exp\left(-\frac{1}{\sigma_1^2}\left|y - |\beta| e^{j\varphi}\sigma_1^2\right|^2\right) d\varphi$$

Performance of the non-coherent WMF – SW0

$$p_{Y_R,Y_I|H_1}(y_R, y_I | H_1) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|y|^2 + |\beta|^2 \sigma_1^4 - 2|y||\beta|\sigma_1^2 \cos(\varphi - \angle y)}{\sigma_1^2} \right) d\varphi$$

$$= \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|y|^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) \cdot \frac{1}{2\pi}\int_0^{2\pi} \exp\left(2|y||\beta|\cos(\varphi - \angle y)\right) d\varphi$$

$$= \frac{1}{\pi\sigma_1^2} \exp\left(-\frac{|y|^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0\left(2|y||\beta|\right)$$

- Now, to derive the PDF of $|Y|$, we need to consider the following 2-D transformation of r.v.'s (i.e. from Cartesian to polar coordinates):

$$\begin{cases} R = |Y| = \sqrt{Y_R^2 + Y_I^2} \\ \vartheta = \angle Y = \arctan\left(Y_I / Y_R\right) \end{cases}$$

Performance of the non-coherent WMF – SW0

- From the well-known theorem of r.v. transformations:

$$p_{|Y|,\angle Y|H_1}\left(|y|,\angle y\,|H_1\right) \equiv p_{R,\vartheta|H_1}(r,\vartheta|H_1) = \left.\frac{p_{Y_R,Y_I|H_1}(y_R,y_I\,|H_1)}{|\mathbf{J}|}\right|_{\begin{cases} y_R = r\cos\vartheta \\ y_I = r\sin\vartheta \end{cases}}$$

- Where **J** is the Jacobian of the transformation: $|\mathbf{J}| = 1/r$

$$p_{R,\vartheta|H_1}(r,\vartheta|H_1) = \frac{r}{\pi\sigma_1^2}\exp\left(-\frac{r^2+|\beta|^2\sigma_1^4}{\sigma_1^2}\right) I_0\left(2r|\beta|\right), \quad 0 \le \vartheta < 2\pi,\ r \ge 0$$

$$p_{R|H_1}(r|H_1) = \int_0^{2\pi} p_{R,\vartheta|H_1}(r,\vartheta|H_1)\,d\vartheta$$

$$= \frac{2r}{\sigma_1^2}\exp\left(-\frac{r^2+|\beta|^2\sigma_1^4}{\sigma_1^2}\right) I_0\left(2r|\beta|\right), \quad r \ge 0$$

Performance of the non-coherent WMF – SW0

- The PDF of the envelope $R=|Y|$ under the hypothesis H_1 is a Rician function:

$$p_{R|H_1}(r|H_1) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2 + |\beta|^2 \sigma_1^4}{\sigma_1^2} \right) I_0\left(2r|\beta|\right), \quad r \geq 0,$$

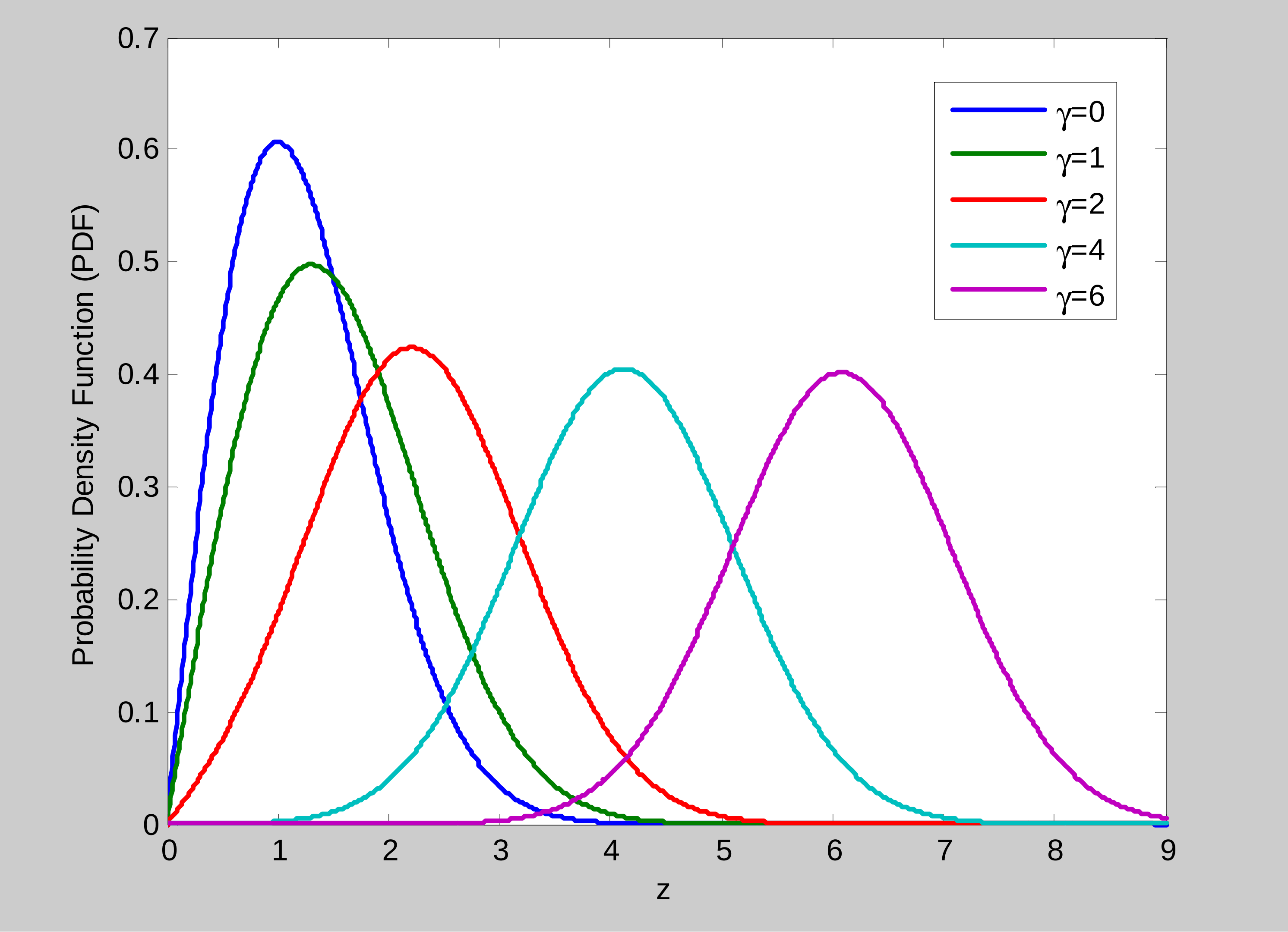
$$SNR_{WMF} = |\beta|^2 \sigma_1^2 = |\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$

- Under the hypothesis H_0, the PDF is a Rayleigh function; it can be obtained from the Rice PDF by setting $\beta=0$:

$$p_{R|H_0}(r|H_0) = \frac{2r}{\sigma_1^2} \exp\left(-\frac{r^2}{\sigma_1^2} \right), \quad r \geq 0, \quad \sigma_1^2 = \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}$$

Performance of the non-coherent WMF – SW0

$$z \triangleq \frac{\sqrt{2}\,r}{\sigma_1},\ \gamma \triangleq \sqrt{2SNR_{WMF}},\quad SNR_{WMF} = |\beta|^2\,\mathbf{p}^H\mathbf{R}^{-1}\mathbf{p}$$



Performance of the non-coherent WMF – SW0

$$p_{R|H_1}(r|H_1) = \frac{2r}{\sigma_1^2}\exp\left(-\frac{r^2+|\beta|^2\sigma_1^4}{\sigma_1^2}\right)I_0\left(2r|\beta|\right), \quad r \geq 0$$

$$P_D = \int_{\sqrt{\eta}}^{+\infty} p_{R|H_1}(r|H_1)dr = \int_{\sqrt{\eta}}^{+\infty} \frac{2r}{\sigma_1^2}\exp\left(-\frac{r^2+|\beta|^2\sigma_1^4}{\sigma_1^2}\right)I_0\left(2r|\beta|\right)dr$$

$$= \int_{\lambda}^{+\infty} z\exp\left(-\frac{z^2+\gamma^2}{2}\right)I_0\left(z\gamma\right)dz = Q_M\left(\gamma,\lambda\right)$$

where we defined $z \triangleq \frac{\sqrt{2}\,r}{\sigma_1}, \gamma \triangleq \sqrt{2}|\beta|\ \sigma_1 \quad \Rightarrow \quad z\gamma = 2r|\beta|, \lambda = \sqrt{2\eta/\sigma_1^2}$

$Q_M\left(\gamma,\lambda\right)$ is the so-called **Marcum Q - function**

Performance of the non-coherent WMF – SW0

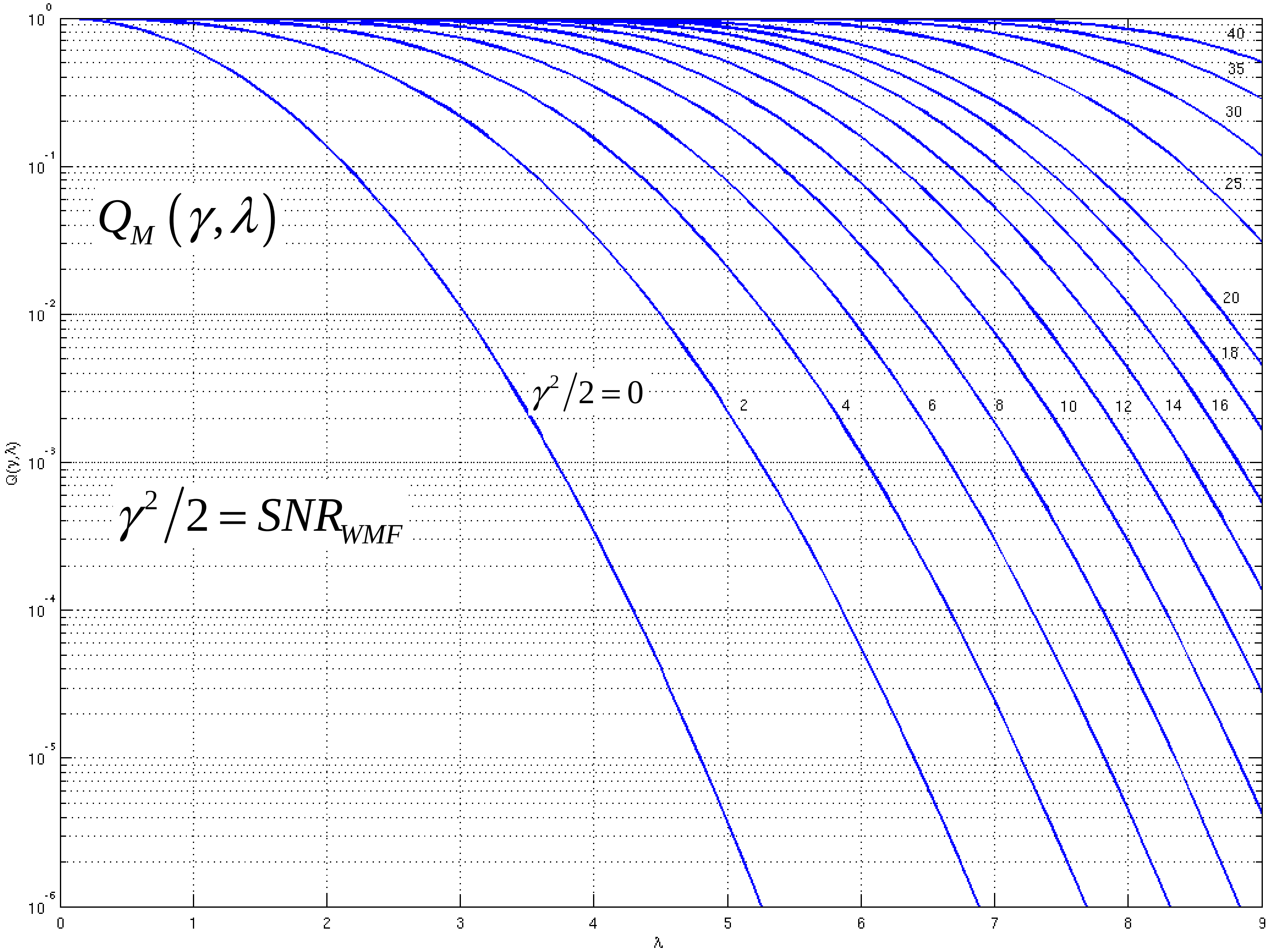
Summarizing:

$$P_{FA} = \exp\left(-\frac{\eta}{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}\right)$$

Note that: $SNR_{WMF} = \dfrac{SNR_{CWMF}}{2}$

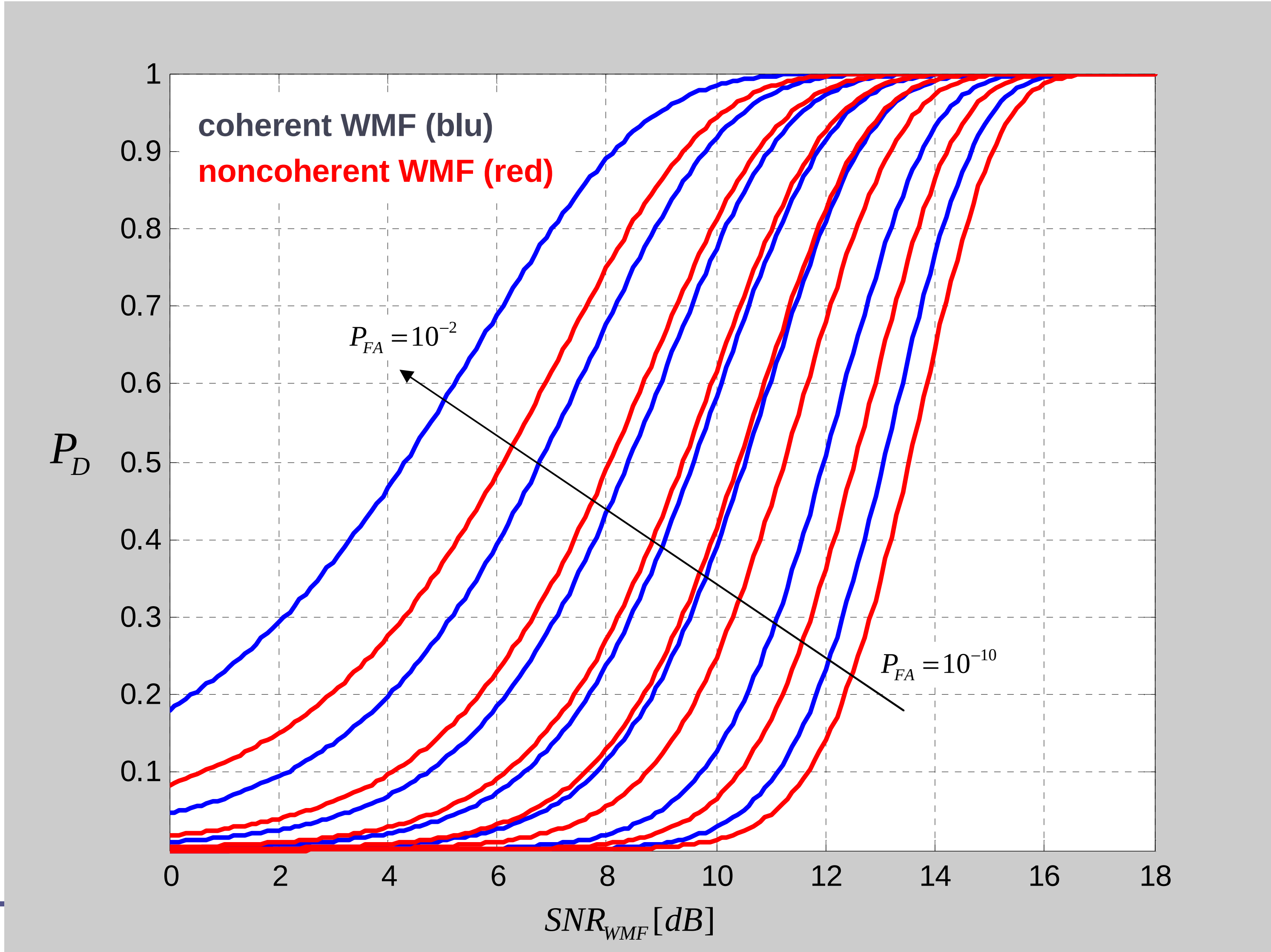
$$P_D = Q_M\left(\sqrt{2|\beta|^2 \mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}, \sqrt{\frac{2\eta}{\mathbf{p}^H \mathbf{R}^{-1} \mathbf{p}}}\right) = Q_M\left(\sqrt{2SNR_{WMF}}, \sqrt{-2\ln(P_{FA})}\right)$$

Performance of the non-coherent WMF – SW0



Performance of the WMF

ROC: the noncoherent WMF (SW-0) vs. coherent WMF.



White Gaussian Noise

- The complex multidimensional PDF of Gaussian disturbance is given by:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = p_{\mathbf{d}}(\mathbf{z}) = \frac{1}{\left(\pi\sigma_d^2\right)^N}\exp\left(-\frac{\mathbf{z}^H\mathbf{z}}{\sigma_d^2}\right)$$

$$\mathbf{z}|H_0 \in \mathcal{CN}\left(\mathbf{0}, \sigma_d^2\mathbf{I}\right)$$

Nx1 data vector

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = E_{\mathbf{s}_t}\left\{p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)\right\} = \int \underbrace{\frac{1}{\left(\pi\sigma_d^2\right)^N}\exp\left(-\frac{(\mathbf{z}-\mathbf{s}_t)^H(\mathbf{z}-\mathbf{s}_t)}{\sigma_d^2}\right)}_{p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)} \cdot p_{\mathbf{s}_t}(\mathbf{s}_t)d\mathbf{s}_t$$

Perfectly Known Target Signal - Case #1

- The **optimal NP detection strategy** is:

$$l(\mathbf{z}) = \frac{2}{\sigma_d^2}\Re e\left\{\mathbf{s}_t^H \mathbf{z}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$P_{FA} = Q\left(\frac{\eta}{\sqrt{SNR_{CWMF}}}\right) \Rightarrow \eta = \sqrt{SNR_{CWMF}} \cdot Q^{-1}\left(P_{FA}\right)$$

$$P_D = Q\left(\frac{\eta - SNR_{CWMF}}{\sqrt{SNR_{CWMF}}}\right) \Rightarrow P_D = Q\left(Q^{-1}\left(P_{FA}\right) - \sqrt{SNR_{CWMF}}\right)$$

where $SNR_{CWMF} = \frac{2}{\sigma_d^2}\mathbf{s}_t^H \mathbf{s}_t$

Swerling 0 Target Signal – Case #3

- If the amplitude is non fluctuating and the phase is uniformly distributed:

$$\beta = |\beta| e^{j\varphi} \quad \Rightarrow \quad |\beta| \text{ is deterministic (nonfluctuating)}$$

$$\varphi \text{ is uniformly distributed on } [0,2\pi)$$

$$\mathbf{z} | \varphi, H_1 \in \mathcal{CN}\left(|\beta| e^{j\varphi} \mathbf{p}, \sigma_d^2 \mathbf{I} \right)$$

$$\Lambda(\mathbf{z}) = \frac{1}{2\pi} \int_0^{2\pi} \exp\left(\frac{2|\beta|}{\sigma_d^2} \left| \mathbf{p}^H \mathbf{z} \right| \cos(\varphi - \varphi_0) \right) d\varphi \cdot e^{-\frac{|\beta|^2}{\sigma_d^2} N}$$

$$= I_0\left(\frac{2|\beta|}{\sigma_d^2} \left| \mathbf{p}^H \mathbf{z} \right| \right) \cdot e^{-\frac{|\beta|^2}{\sigma_d^2} N} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

where $\mathbf{p}^H \mathbf{p} = N$

Swerling 0 Target Signal – Case #3

Since $I_0(x)$ is monotonic in x, the LRT reduces to:

$$\left|\mathbf{p}^H \mathbf{z}\right| \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\eta}$$

or equivalentely

$$\left|\mathbf{p}^H \mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

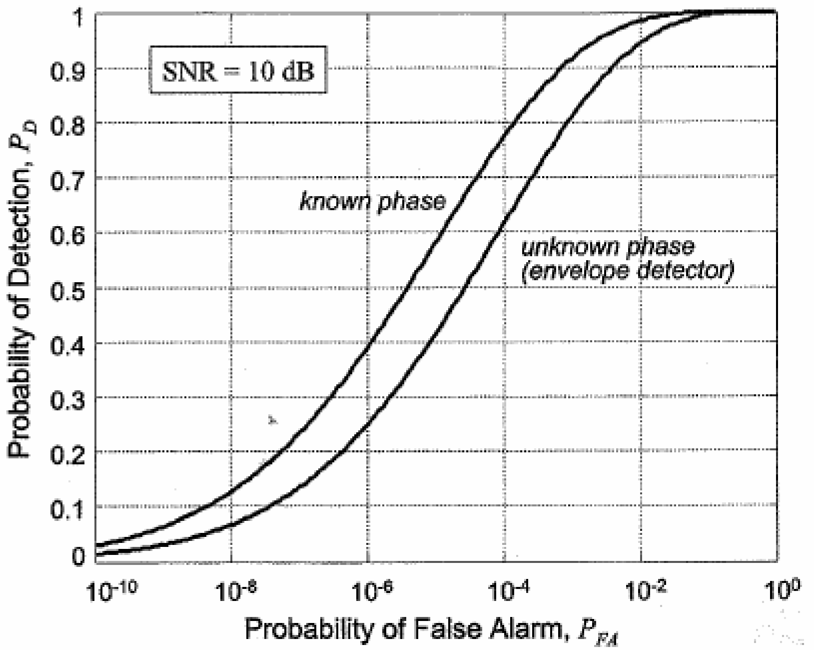
ROC:

$$P_{FA} = \exp\left(-\frac{\eta\sigma_d^2}{N}\right) \quad P_D = Q_M\left(\sqrt{2SNR_{WMF}}, \sqrt{-2\ln(P_{FA})}\right)$$

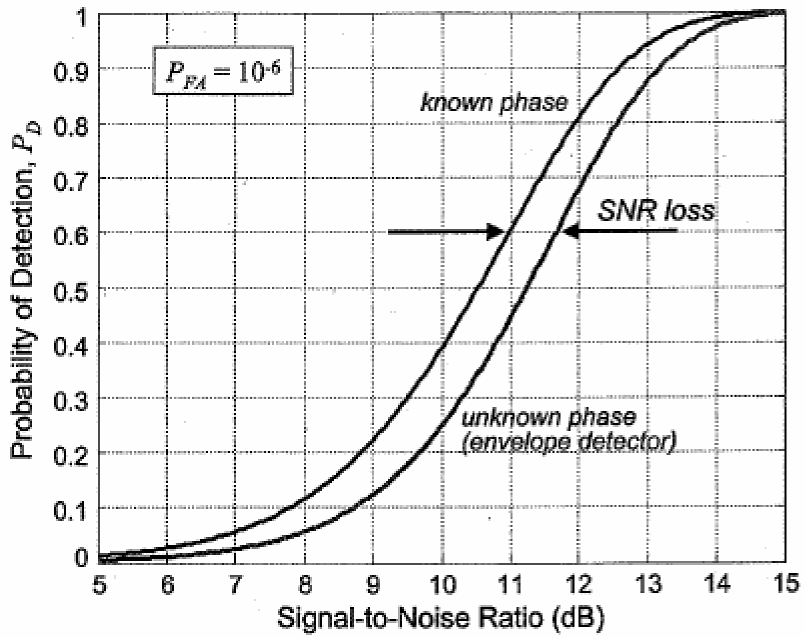
$$\text{where: } SNR_{WMF} = \frac{N\left|\beta\right|^2}{\sigma_d^2}$$

The **integration** in this case is said **coherent**, because all the pulse samples are first weigthed by **p** and then coherently summed up. The modulo is taken after summation. **The gain in the SNR for coherent integration is equal to *N*.**

Perfectly known vs Swerling 0 target



(a)



(b)

Figure 6.7 Performance difference between coherent and envelope detectors for the Gaussian example. (a) Difference in P_D for $\chi = 10$ dB. (b) Difference in P_D for $P_{FA} = 10^{-6}$.

Incoherent pulse train - Incoherent integration

$$\begin{cases} z(n) = d(n) & H_0 : \text{Target absent} \\ z(n) = Ae^{j\phi(n)} + d(n) & H_1 : \text{Target present} \end{cases}$$

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\prod_{n=0}^{N-1} \int_{-\infty}^{+\infty} p(z(n)|\varphi, H_1)\, p_{\varphi|H_1}(\varphi|H_1)\, d\varphi}{\prod_{n=0}^{N-1} p(z(n)|H_0)} = \prod_{n=0}^{N-1} \Lambda\big(z(n)\big)$$

$$\Lambda(z(n)) = I_0\left(\frac{2A|z(n)|}{\sigma_d^2}\right) \cdot e^{-\frac{A^2}{\sigma_d^2}}$$

Incoherent pulse train

Applying the logarithm: $$\ln\Lambda(\mathbf{z}) = \sum_{n=0}^{N-1} \ln\Lambda\big(z(n)\big)$$

$$l(\mathbf{z}) = \sum_{n=0}^{N-1} \ln I_0\left(\frac{2A\left|z(n)\right|}{\sigma_d^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The calculation of I_0 is not easy, so we have to resort to some approximation

$$\ln I_0(x) \cong \frac{x^2}{4} \qquad for \quad x \ll 1$$

$$\ln I_0(x) \cong x \qquad for \quad x \gg 1$$

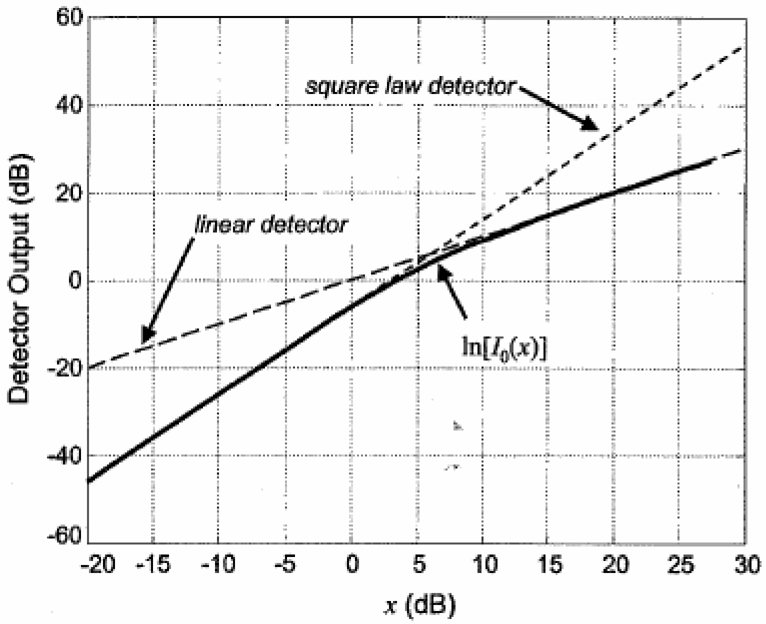


Figure 6.8 Approximation of the $\ln[I_0]$ detector characteristic by the square law detector when its argument is small, and the linear detector when its argument is large.

Incoherent pulse train

Then for x<1 (small SNR) the detector becomes:

$$\sum_{n=0}^{N-1} |z(n)|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

For x>>1 (large SNR) the detector becomes:

$$\sum_{n=0}^{N-1} |z(n)| \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Both detectors perform an incoherent integration, since they first calculate the modulus of each sample, then sum up the results.

The linear approximation fits the ln I_0 very well for x>10dB, the square law detector is an excellent fit for x<5dB.

Incoherent pulse train

The integration gain for coherent processing is N

For the incoherent integration, the calculation is not so easy because in general it depends on N, PD and PFA.

For very large values of N, the gain of the quadratic detector can be approximated (in dB) with $10\log_{10}\sqrt{N}-5.5$. As a rule of thumbs we use $\sqrt{N}$.

For the linear detector

Albersheim's equation is used.

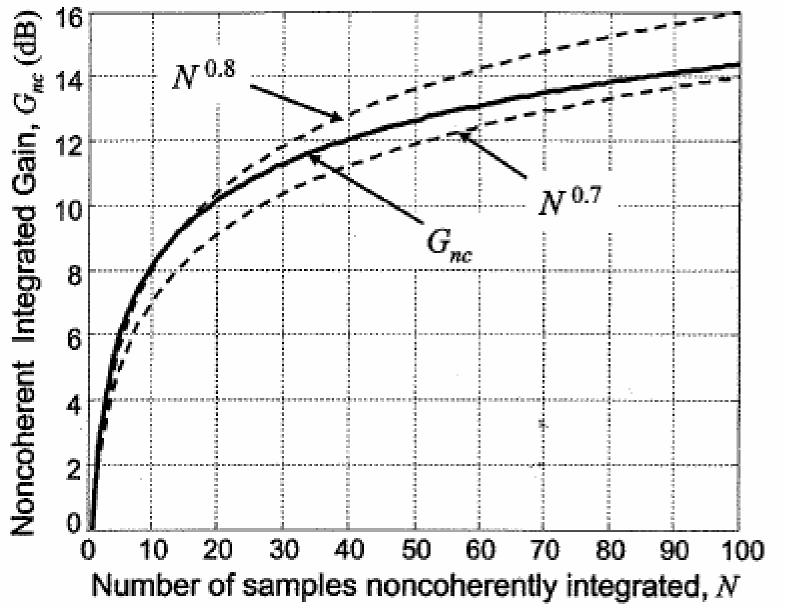


Figure 6.11 Noncoherent integration gain G_{nc} for a nonfluctuating target, as estimated using Albersheim's equation.

Binary integration

- Any coherent or incoherent detection is given by comparing the detection statistic with a threshold, so the output of the detector is binary in the sense that it takes one of only two possible outcomes.
- If the entire detection process is repeated *N* times for a given range, Doppler, or angle cell, *N* binary detections will be available.
- Each decision of "target present" will have some probability P_D of being correct, and prob. P_{FA} of being incorrect. To improve the reliability of the detection decision, the decision rule can require that a target is detected on some number *M* out of *N* decisions before it is finally accepted as a valid target detection.
- This process is called binary integration with a "*M* of *N*" rule.
- With the binary integration both overall P_{FA} and P_D are given by

$$P(M,N) = \sum_{k=M}^{N} \binom{N}{k} p^k (1-p)^{N-k}$$