E9 231: Digital Array Signal Processing

Scribe: Vinod K Jaiswal
Dept. of ECE
Indian Institute of Science
Bangalore 560 012, India
vinodj@ece.iisc.ernet.in

Class held on: 08 Sept 2008

1 Topic

- Miscellaneous points about Rectangular Array design.
- Circular Arrays.

2 Class Notes

2.1 Miscellaneous points about Rectangular Array design

We can extend results from Linear Array Design.

1. Beam Space Processing.
2. Rectangular Apperture.
3. Beam Pattern design algorithms from chapter 3 of "Optimum array processing".
4. Difference beams.
5. Rectangular array of apertures.(non-isotropic sensors)

2.2 Circular arrays

Circular (Ring) Aperture

In this type of array, elements are placed on a circle(ring) as shown in figure.
Approach to design
1. Study the continuous aperture , sample to get a circular array
2. Start from here and let $\theta_T \Rightarrow 0$ as shown in Figure 1:
Consider a Ring aperture as shown in Figure 2:
The Direction of Arrival (DoA) of the wave is given by the Wavenumber k,

$$k = -2\pi/\lambda \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \tag{1}$$

Position vector of an element on array is given as

$$p_{\phi_1} = R \begin{bmatrix} \cos\phi_1 \\ \sin\phi_1 \\ 0 \end{bmatrix} \tag{2}$$

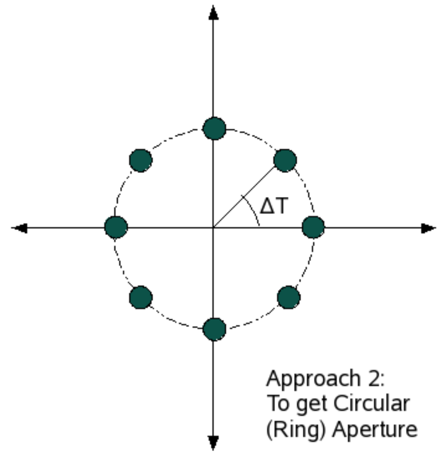

Figure 1: Approach 2: To get Circular (Ring) Aperture

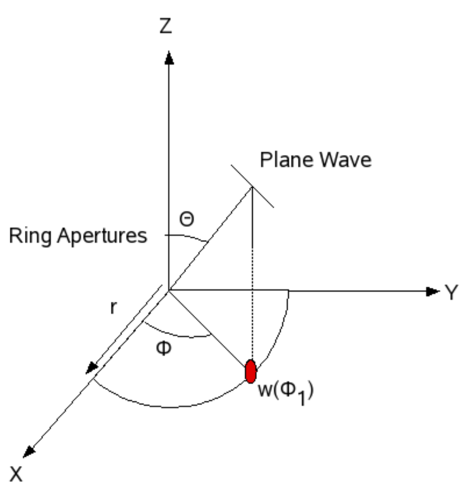

Figure 2: Ring Aperture

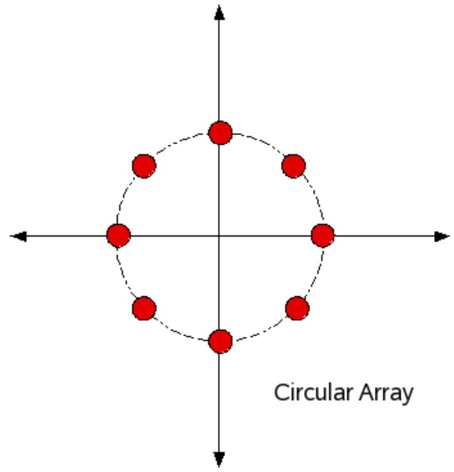

Figure 3: Circular Array

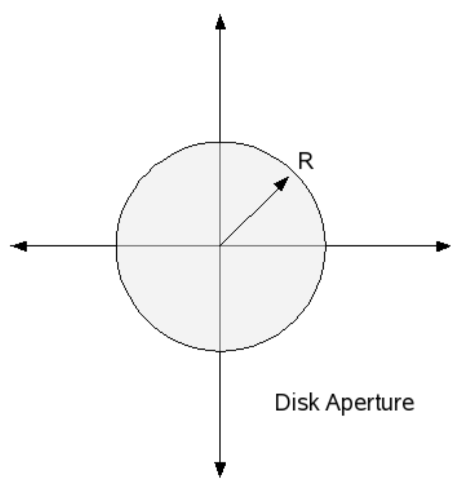

Figure 4: Disk Aperture

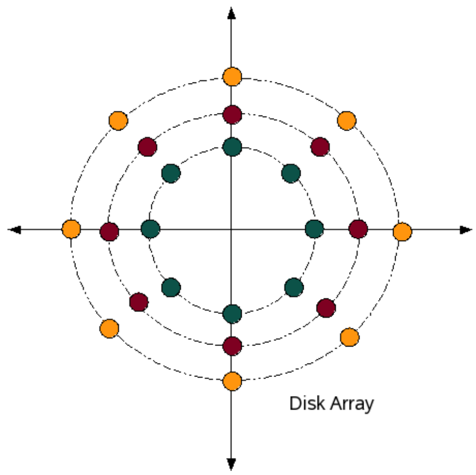

Figure 5: Disk Array

$$k^T p_{\phi_1} = -\frac{2\pi R}{\lambda}[\sin\theta(\cos\phi_1\cos\phi + \sin\phi_1\sin\phi)] \tag{3}$$

$$k^T p_{\phi_1} = \omega\tau_{\phi_1} = -\frac{2\pi R}{\lambda}[\sin\theta cos(\phi - \phi_1)] \tag{4}$$

Assume weight w(ϕ_1) along the ring
So weight function w(ϕ_1).δ(r-R)
The Frequency wave number response $\gamma(\omega, k)$ is given by,

$$\gamma(\omega, k) = \int_{\phi_1}\int_r w(\phi_1)\delta(r - R)e^{j\frac{2\pi R}{\lambda}\sin\theta\cos(\phi-\phi_1)} r dr d\phi_1 \tag{5}$$

Writing it as a beam pattern

$$B(\theta, \phi) = \int_{\phi_1=-\pi}^{\pi} w(\phi_1)e^{j\frac{2\pi R}{\lambda}\sin\theta\cos(\phi-\phi_1)} R d\phi_1 \tag{6}$$

w(ϕ_1) is a periodic function with period of 2π
In Fourier series representation If period = T_0 then fundamental freq = $\frac{2\pi}{T_0}$
and harmonics = k $\frac{2\pi}{T_0}$
In this case

$$w(\phi_1) = \sum_{m=-\infty}^{\infty} w'_m e^{jm\phi_1} \tag{7}$$

where

$$w'_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\phi_1)e^{-jm\phi_1} d\phi_1 \tag{8}$$

each w'_m term is called a ”phase mode”

so after substituting the value of w(ϕ_1) in Beam pattern equation

$$B(\theta, \phi) = 2\pi\sum_{m=-\infty}^{\infty} w'_m R\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\frac{2\pi R}{\lambda}\sin\theta\cos(\phi-\phi_1)} e^{jm\phi_1} d\phi_1 \tag{9}$$

$$B(\theta, \phi) = \sum_{m=-\infty}^{\infty} w_m j^m J_m(\frac{2\pi R\sin\theta}{\lambda})e^{jm\phi} \tag{10}$$

Note that the term $w_m j^m J_m(\frac{2\pi R\sin\theta}{\lambda})$ are the weights.
where $J_m(.)$ is Bessel function of mth order, 1st kind and $w_m = 2\pi\mathrm{R}w'_m$ and the above expression is known as **phase mode excitation**. Here each term is known as the phase mode excitation term. Each of the phase mode excitation terms give rise to a spatial harmonic in the beam pattern.
One can generate a desired beam pattern approximately using M phase mode excitation terms.
Let

$$w_m j^m J_m(\frac{2\pi R\sin\theta}{\lambda})e^{jm\phi} = B_m(\theta,\phi)$$

so

$$B(\theta,\phi) = \sum_{m=-\infty}^{\infty} B_m(\theta,\phi) \simeq \sum_{m=-M}^{M} B_m(\theta,\phi)$$

and $J_m(x)$ is such that x$\epsilon[0, \frac{2\pi R}{\lambda}]$ and is negligible for large m.

Uniform Weighting

In case of uniform weighting

$$w(\phi) = 1$$

so for m = 0

$$w'_m = 1$$

other wise

$$w'_m = 0$$

So

$$B(\theta,\phi) = B_0(\theta,\phi)$$

or

$$B(\theta,\phi) = J_0(\frac{2\pi R\sin\theta}{\lambda})$$

It is uniform in ϕ and main response axis is perpendicular to the surface of aperture

In circular apperature Bessel function does same thing as sinc function does in linear aperature
First side lobe occures at $\frac{2\pi R\sin\theta}{\lambda} = 3.8$ and the height is given as
height = -7.9 db it is not so good as in case of Linear aperture

$$\theta_{null} = \sin^{-1}(\frac{2.4\lambda}{2\pi R})$$

It is clear from above equation that beam width decreases as R increases
A more general case turns out to that,

$$M \simeq \frac{2\pi R}{\lambda}$$

is a good choice, then we will have $(2M+1)$ phase modes.
Circular Array (Ring Array) from Circular Apertures

$$x[n] \leftrightarrow X(e^{j\omega}) = \sum_n x[n]e^{j\omega n}$$

where N is the number of samples and $\omega = \frac{2\pi k}{N}$,

$$X(e^{j\omega}) \leftrightarrow \tilde{x}[n] = \sum_k x[n + kN]$$

Suppose $x[n]$ was of length L, then for no aliasing $N \geq L$
Let say the circular aperture has $(2M + 1)$ phase modes, If we use $N \geq (2M + 1)$, we can get a beam of a corresponding discrete array.
Let say we sample at intervals of ϕ_T where, $N = \frac{2\pi}{\phi_T}$ sampling function is given as

$$S(\phi) = \sum_{n=-\infty}^{\infty} \delta(\phi - n\phi_T)$$

$$= \frac{1}{\phi_T} \sum_{q=-\infty}^{\infty} e^{jqN\phi}$$

So new weighting function is

$$w(\phi)S(\phi)$$

$$S(\phi) = \frac{1}{\phi_T}(1 + \sum_{q=1}^{\infty} e^{jqN\phi} + \sum_{q=1}^{\infty} e^{-jqN\phi})$$

Let

$$w(\phi) = \sum_{m=-\infty}^{\infty} w_m e^{jm\phi}$$

, where $w_m(\phi) = w_m e^{jm\phi}$
so

$$w_m(\phi)S(\phi) = \frac{1}{\phi_T}(e^{jm\phi} + \sum_{q=1}^{\infty} e^{j(m\phi+qN\phi)} + \sum_{q=1}^{\infty} e^{j(m\phi-qN\phi)})$$

$$w_m(\phi)S(\phi) = \frac{1}{\phi_T}(e^{jm\phi} + \sum_{q=1}^{\infty} e^{j(m+qN)\phi} + \sum_{q=1}^{\infty} e^{j(m-qN)\phi})$$

We get

$$B(\theta, \phi) = \sum_{m=-\infty}^{\infty} w_m j^m J_{|m|}(\frac{2\pi R}{\lambda} sin\theta)e^{jm\phi} \quad (11)$$

$$B(\theta, \phi) = \sum_{m=-\infty}^{\infty} B_m(\theta, \phi) \quad (12)$$

$$B_m(\theta, \phi) = w_m j^m J_m(\frac{2\pi R}{\lambda} \sin\theta)e^{jm\phi} + \sum_{q=1}^{\infty} w_g e^{jg\phi} j^{-g} J_g(\frac{2\pi R}{\lambda} \sin\theta) + \sum_{q=1}^{\infty} w_n e^{jh\phi} j^{-h} J_h(\frac{2\pi R}{\lambda} \sin\theta) \quad (13)$$

where $g = (Nq + m)$ and $h = (Nq - m)$,
In above equation last two terms contribute to aliasing and needs to be checked for contributions from these terms to ensure no aliasing
For example: "first" distortion term that is more dominant term

$$J_h(\frac{2\pi R}{\lambda} \sin\theta)$$

with $h = N - m$ will be negligible if $N - m > \frac{2\pi R}{\lambda}$ since

$$m \leq M \leq \frac{2\pi R}{\lambda}$$

so

$$N > m + \frac{2\pi R}{\lambda} \geq \frac{4\pi R}{\lambda}$$

hence we have,

$$N \geq \frac{4\pi R}{\lambda} \tag{14}$$

For Example: Consider a Circular Aperture, let $d_{cir} \leq \frac{\lambda}{2}$, where d_{cir} is the distance between two consecutive sensors at the circumference of the circle. Then we have $N > \frac{4\pi R}{\lambda}$ that means that we must place the sensors by atmost $\frac{\lambda}{2}$ distance along the circumference of the circle.

Reading Assignment: Read chapter 4 especially 4.3 and 4.4 from the textbook.