E9 231: Digital Array Signal Processing

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STATISTICAL ARRAY SIGNAL PROCESSING

I. Frequency Domain Snapshot Model

Time Segmentation of the total observation interval,

$$\begin{array}{l} k=1,\ 0 \le t < \Delta T \\ k=1,\ \ \Delta T \le t < 2\Delta T \\ \quad\quad \vdots \\ k=k,\ \ (k-1)\Delta T \le t < k\Delta T \\ \quad\quad \vdots \\ k=K,\ \ (K-1)\Delta T \le t < K\Delta T \end{array}$$

Requirement on ΔT : $\Delta T >>$ Maximum delay between Sensors.
The keypoint is to decompose signal into statistically independent components. The most general form of the transform that decomposes a signal into statistically independent components is KLT (Karhunen-Loeve Transform for which we need to solve Eigen function problem).
For Long snapshot (i.e., $\Delta T \to \infty$), the Fourier transform is a KL transform. Also $B_s \cdot \Delta T$ should be large, typically in the range $B_s \cdot \Delta T \approx 16$. This is the motivation for using the frequency domain snapshot model.
(Stationary process + Long ΔT) $\Rightarrow$ Fourier Transform is KLT $\Rightarrow$ Can process the different frequency bins independently.

Let $x(t)$ be a zero mean Bandpass signal, centred at w_c with a bandwidth B_s. Consider $0 \le t \le \Delta T$

$$X_{\Delta T}(\omega_m) \triangleq \frac{1}{\sqrt{\Delta T}} \int_0^{\Delta T} x(t) e^{-j\omega_m t} \tag{1}$$

where $\omega_m \triangleq \omega_m + m\omega_\Delta$, Centre frequency of the m^{th} bin.
m corresponds to the m^{th} fourier frequency. $\omega_\Delta = \frac{2\pi}{\Delta T},\ \ m \in \{\frac{-(M-1)}{2}, \frac{-(M-1)}{2}+1, ... \frac{(M-1)}{2}\}$ and $M \approx B_s \Delta T$.

We want to derive an expression for the following

$$\begin{aligned} S_{X_{\Delta T}}(m_1, m_2) &\triangleq E\left\{X_{\Delta T}(\omega_{m_1})\ X^H_{\Delta T}(\omega_{m_2})\right\} \\ &= E\left[\frac{1}{\Delta T}\int_0^{\Delta T}\int_0^{\Delta T} dt\ du\ x(t)\ e^{-j\omega_{m_1} t}\ x^H(u)\ e^{j\omega_{m_2} u}\right] \end{aligned} \tag{2}$$

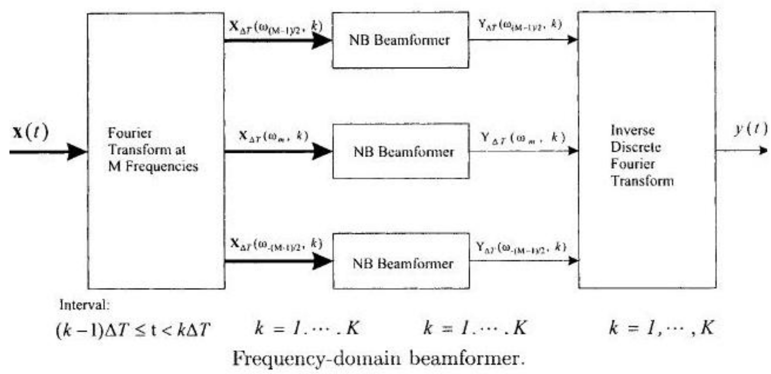

Figure 1: Frequency domain beamformer

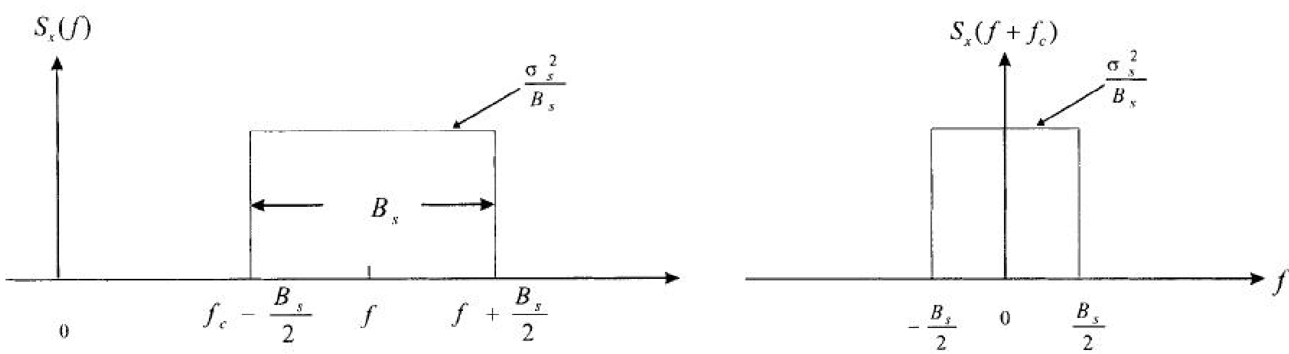

Figure 2: Bandpass Signal Spectrum and corresponding lowpass spectrum

Assume $x(t)$ is stationary i.e., $E\{x(t)x^H(t)\} = R_x(t-u)$, and we know $S_X(\omega) \leftrightarrow R_x(t)$ (Auto-correlation function and Power Spectral Density form a Fourier Transform pair).

$$S_{X_{\Delta T}}(m_1, m_2) = \frac{1}{2\pi\Delta T} \int\limits_{-\infty}^{\infty} \int\limits_{0}^{\Delta T} \int\limits_{0}^{\Delta T} d\omega\ dt\ du\ S_X(\omega)\ e^{(j\omega(t-u))}\ e^{(-j\omega_{m_1}t + j\omega_{m_2}u)}$$

where $\omega_{m_i} = \omega_c + m_i\omega_\Delta$

$$S_{X_{\Delta T}}(m_1, m_2) = \frac{1}{2\pi\Delta T} \int_{-\infty}^{\infty} \int_{0}^{\Delta T} \int_{0}^{\Delta T} d\omega\ dt\ du\ S_X(\omega)\ e^{(j(\omega-\omega_c)(t-u))}\ e^{(-jm_1\omega_\Delta t)}\ e^{(jm_2\omega_\Delta u)}$$

where $\omega_L = \omega - \omega_c$, the expression can be simplified as,

$$S_{X_{\Delta T}}(m_1, m_2) = e^{-j\pi(m_1-m_2)} \int\limits_{-\infty}^{\infty} S_X(\omega_L + \omega_c)\ \frac{1}{\omega_\Delta}\ \text{sinc}\left[\pi\left(\frac{\omega_L}{\omega_\Delta} - m_1\right)\right]\ \text{sinc}\left[\pi\left(\frac{\omega_L}{\omega_\Delta} - m_2\right)\right] d\omega_L \quad (3)$$

Same frequency bin $m_1 = m_2 = m$, the above expression reduces to

$$S_{X_{\Delta T}}(m, m) = \int\limits_{-\infty}^{\infty} S_X(\omega_L + \omega_c)\ \frac{\text{sinc}^2\left[\pi\left(\frac{\omega_L}{\omega_\Delta} - m\right)\right]}{\omega_\Delta}\ d\omega_L \quad (4)$$

Diagonal Element for the n^{th} sensor is expressed as,

$$[S_{X_{\Delta T}}(m, m)]_{n,n} = \int_{-\infty}^{\infty} [S_X(\omega_L + \omega_c)]_{n,n} \frac{\text{sinc}^2\left[\pi\left(\frac{\omega_L}{\omega_\Delta} - m\right)\right]}{\omega_\Delta}\ d\omega_L \quad (5)$$

$B_s\Delta T \approx 1$ or greater than 1, and $\lim\limits_{\Delta T\to\infty} [S_{X_{\Delta T}}(m, m)]_{n,n} = S_X(\omega_c + m\omega_\Delta)$,

$$\lim_{\Delta T\to\infty} [S_{X_{\Delta T}}(m, m)] = S_X(\omega_c + m\omega_\Delta) \quad (6)$$

This expression is known as Wiener-Khinchine theorem $\Rightarrow$ Covariance(FFT) = FFT(Covariance)

For $B_s\Delta T << 1$ (Narrowband case), $\text{sinc}^2(\cdot) \approx 1$, we have

$$[S_{X_\Delta T}(0, 0)]_{nn} = \Delta T[S_X(\omega_c)]_{nn} \quad (7)$$

Also $[S_{X_{\Delta T}}(m, m)]_{nn} \approx 0$, for $m \neq 0$. The beamformer processes one frequency bin at a time (Narrow-band frequency snapshot model).

Some conclusions:

1. $E[x_{\Delta T}(\omega_m, k)\ x^H_{\Delta T}(\omega_l, p)] = 0$, $l \neq m$ even if $k = p$. If $\omega_m \neq \omega_l$, then two ouptuts will be uncorrelated as $\Delta T \to \infty$.
2. $E[x_{\Delta T}(\omega_m, k)\ x^H_{\Delta T}(\omega_m, l)] = 0$, $k \neq l$.
3. $E[x_{\Delta T}(\omega_m, k)\ x^H_{\Delta T}(\omega_m, k)] = S_{\Delta T}(\omega_m)$, which is the Spatial Covariance matrix.

Replace a continuous random process $\mathbf{x}(t)$ by a vector sequence of uncorrelated random variables. $S_{\Delta T}(\omega_m) = \frac{1}{K}\sum\limits_{k=1}^{K} X_{\Delta T}(\omega_m, k)\ X^H_{\Delta T}(\omega_m, k)$. So we can assume we have only one frequency bin from now on.

Gaussian Random Processes:

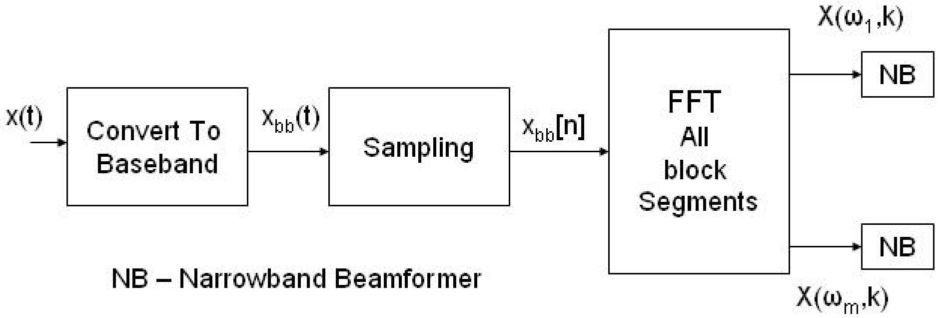

Figure 3: Narrowband beamformer processing

Circular Gaussian Random Vector (Zero Mean) : $E\{XX^H\} = R_{XX},\ E\{XX^T\} = 0$

$$f_X(x) = \frac{1}{\pi^N \det(C)} e^{\{-x^H C^{-1} x\}} \tag{8}$$

Cases of Interest: Consider $x(t) = x_s(t) + n(t)$, the following two cases are of interest

Case1: Single Plane wave from direction $\mathbf{k}_s$

$$X(\omega_m, k) = X_s(\omega_m, k) + N(\omega_m, k) \tag{9}$$

$$X(\omega_m, k) = F(\omega_m, k)V(\omega_m, k_s) + N(\omega_m, k) \tag{10}$$

where $V(\omega_m, k_s)$, is the Fourier Transform of the array manifold vector at frequency ω_m.

Case2: Multiple Plane waves

The model considered is that of input of multiple plane waves with a desired signal + (D-1) Interferers.

$$X_N(\omega_m, k) = F(\omega_m, k)V(\omega_m, k_s) + \sum_{l=1}^{D-1} F_l(\omega_m, k)V(\omega_m, k_l) + N(\omega_m, k) \tag{11}$$

The following are some of the assumptions on $F(\omega_m, k)$:
1. Desired signal is deterministic (either it could be known or unknown).
2. The interfering signals may be deterministic but unknown or random with zero mean.

Further,

$$X(\omega_m, k) = F(\omega_m, k)V(\omega_m, k_s) + [V(\omega_m, k1), ...V(\omega_m, k1)]\ [F_1(\omega_m, k), ..., F_{D-1}(\omega_m, k)]^T + N(\omega_m, k) \tag{12}$$

which can be written as,

$$X(\omega_m, k) = F(\omega_m, k)V(\omega_m, k_s) + V_I F_I + N \tag{13}$$

where $V_I = [V(\omega_m, k1), ...V(\omega_m, k1)]$, and $F_I = [F_1(\omega_m, k), ..., F_{D-1}(\omega_m, k)]^T$

Also,

$$E[X(\omega_m, k)\ X^H(\omega_m, k)] = |F(\omega_m, k)|^2 V(\omega_m, k_s)V^H(\omega_m, k_s) + V_I E[F_I F_I^H]V_I^H + E[NN^H] \tag{14}$$

where the Interference Matrix $S_{IN}(\omega)$ is given by the following expression,

$$S_{IN}(\omega) = V_I E[F_I F_I^H]V_I^H + E[NN^H]^1 \tag{15}$$

[1] All the figures considered in this lecture notes(except figure 3) are taken from *Optimum Array Processing, Part IV of Detection, Estimation And Modulation Theory* by Harry L. Van Trees, 2002. Wiley-InterScience