EE269
Signal Processing for Machine Learning

Lecture 5

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Systems

$$x[n] \rightarrow \boxed{\text{system}} \rightarrow y[n]$$

- Processing operation performed on the signal

  e.g., a natural system, designed filter, predictor

Systems

$$x[n] \rightarrow \boxed{\text{system}} \rightarrow y[n]$$

- Linear Systems

$$x_1[n] \rightarrow \boxed{\mathcal{T}} \rightarrow y_1[n] \qquad x_2[n] \rightarrow \boxed{\mathcal{T}} \rightarrow y_2[n]$$

$$c_1 x_1[n] + c_2 x_2[n] \rightarrow \boxed{\mathcal{T}} \rightarrow c_1 y_1[n] + c_2 y_2[n]\ .$$

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- **Example.** *Moving average:* average of the last M values of the signal

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k] =: \mathcal{T}_{ma}\{x[n]\} \ .$$

Linear? Suppose that

$$y_1[n] = \mathcal{T}_{ma}\{x_1[n]\} \quad \text{and} \quad y_2[n] = \mathcal{T}_{ma}\{x_2[n]\}$$

Systems

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Systems

$$x[n] \rightarrow \boxed{\text{system}} \rightarrow y[n]$$

- Time invariant Systems

$$x[n] \rightarrow \boxed{\mathcal{T}} \rightarrow y[n] \ ,$$

implies that for all n_0,

$$x[n-n_0] \rightarrow \boxed{\mathcal{T}} \rightarrow y[n-n_0] \ .$$

Systems

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- **Example.** *An accumulator*

$$y[n] = \mathcal{T}_{acc}\{x[n]\} = \sum_{k=-\infty}^{n} x[k] \ .$$

Is $\mathcal{T}_{acc}$ is time-invariant? $y[n-n_0] =? \ \mathcal{T}_{acc}\{x[n-n_0]\}$

Linear Time Invariant (LTI) Systems

- Impulse response

$$\delta[n] \rightarrow \boxed{\mathcal{T}} \rightarrow h[n]$$

- Time invariant $\implies$

$$\delta[n-k] \rightarrow \boxed{\mathcal{T}} \rightarrow h[n-k]$$

Linear Time Invariant (LTI) Systems

- Impulse response

$$\delta[n] \rightarrow \boxed{\mathcal{T}} \rightarrow h[n]$$

- Time invariant systems:

$$\delta[n-k] \rightarrow \boxed{\mathcal{T}} \rightarrow h[n-k]$$

  using the Delta basis:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \rightarrow \boxed{\mathcal{T}} \rightarrow y[n]$$

$$y[n] = \mathcal{T}\left\{ \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \right\} .$$

Linear Time Invariant (LTI) Systems

- Impulse response

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- Linearity

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\mathcal{T}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- LTI systems are entirely described by their impulse response

Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] = x_1[n] * x_2[n] \; .$$

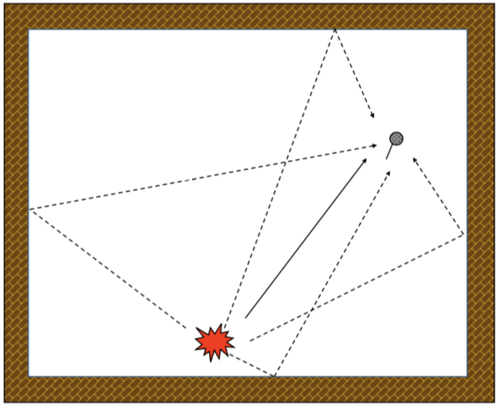
- LTI systems convolve with $h[n]$

$$x[n] \rightarrow \boxed{\mathcal{T}} \rightarrow y[n] \qquad \Longrightarrow \quad y[n] = x[n] * h[n]$$

- In Hilbert spaces

$$x_1[n] * x_2[n] = \langle x_1^*[n-k], x_2[k] \rangle$$

Convolution with acoustic impulse response



Circular convolution in $\mathbb{R}^N$

$$y[n] = \sum_{k=0}^{N-1} x[k]h[(n-k) \mod N] = x[n] \circledast h[n] \ .$$

- Matrix vector notation

$$y = Hx$$

$$H = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \dots & h[2] & h[1] \\ h[1] & h[0] & h[N-1] & \dots & h[3] & h[2] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \dots & h[1] & h[0] \end{bmatrix}$$

Circular convolution in $\mathbb{R}^N$

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$$\text{Fourier basis: } w_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk}$$

$$w_k^H H = H[k] w_k^H$$

$$Fy = FHx = \text{diag}(H[0], ..., H[N-1])Fx$$

Multiplication in DFT domain

$$\text{Fourier basis: } w_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk}$$

$$y[n] = h[n] \circledast x[n]$$

$$Y[k] = H[k]X[k] \quad \forall k$$

Diagonalization of Circulant matrices

$$y[n] = \sum_{k=0}^{N-1} x[k]h[(n-k) \mod N] = x[n] \circledast h[n] \; .$$

$$\text{Fourier basis: } w_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}nk}$$

$$FH = \mathrm{diag}(H[0], ..., H[N-1])F$$
$$H = F^H \mathrm{diag}(H[0], ..., H[N-1])F$$

Deconvolution

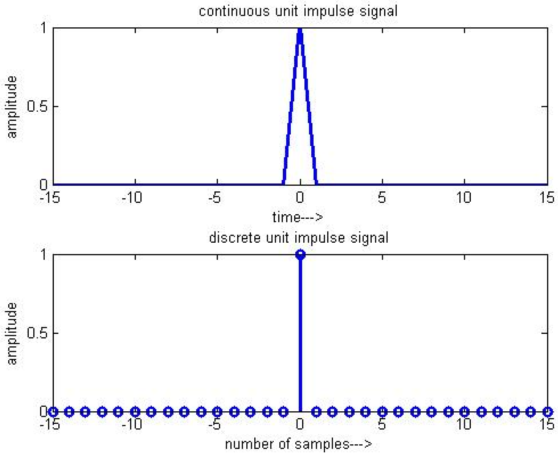
$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$$

$$y = H_2 H_1 x$$

$$\begin{aligned} Fy = FH_2H_1x &= \mathrm{diag}(H_2[0:N-1])FH_1x \\ &= \mathrm{diag}(H_2[0:N-1])\mathrm{diag}(H_1[0:N-1])Fx \end{aligned}$$

Estimating impulse response

- generate an approximation to $\delta[n]$



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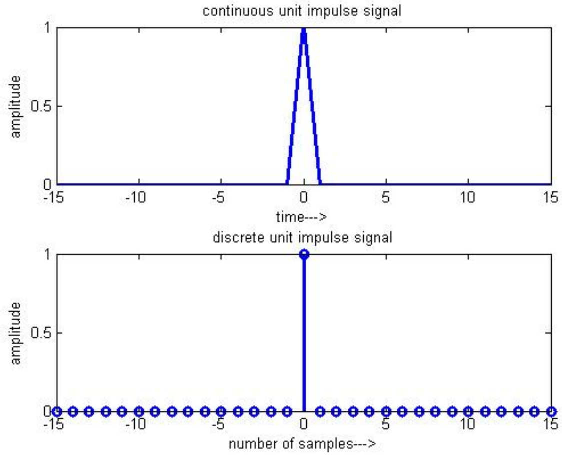

- sweeps the frequency from low to high frequency (chirp signal)