E9 231: Digital Array Signal Processing

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1 Topics

- HW 6: 5.2.2, 5.2.3 and 5.2.4
- Frequency Domain Snapshot Models(contd...)
- Time Domain Snapshot Models

1.1 Frequency Domain Snap shot Model(Contd..)

We are interested in models with large $B_s \triangle T$, so that we can process each frequency bin independently and

$$S_{X_{\triangle T}}(m_1, m_2) = \begin{cases} S_X(w_m) & m_1 = m_2 \\ 0 & otherwise \end{cases} \tag{1}$$

$$M = \lfloor B_s \triangle T \rfloor + 1 \qquad\qquad B_s \triangle T = 2, 3.... \tag{2}$$

$$S_{X_{\triangle T}}(m_1, m_2) = e^{-j(m_1-m_2)\pi} \int_{-\infty}^{\infty} S_X(w_L + w_c)\ sinc\left(\pi\left(\frac{w_L}{w_\triangle} - m_1\right)\right)\ sinc\left(\pi\left(\frac{w_L}{w_\triangle} - m_2\right)\right) dw_L \tag{3}$$

If $m_1 \neq m_2$ $sinc()_{m_1}$ $sinc()_{m_2}$ are orthogonal to each other
If $S_X(w)$ is relatively constant over an interval $\pm w_\triangle$ then $S_{X_{\triangle T}}(m_1, m_2) = 0$ for $m_1 \neq m_2$
Consider the k^{th} interval $(k-1)\triangle T \leq t < k\triangle T$. Stationarity of $x(t)$ implies $S_{X_{\triangle T}}(m_1, m_2)$ is independent of k. We can show that

$$\mathbf{E}\{X_{\triangle T}(w_{m_1}, k) X^H_{\triangle T}(w_{m_2}, l)\} \approx 0 \qquad if \qquad k \neq l \tag{4}$$

Thus the frequency domain snapshot model generates a sequence of N dimensional complex vectors at M different frequency that are uncorrelated across time and frequency. Here after , we will assume $S_{X_{\triangle T}}(m, m) \approx S_X(w_m)$ and the snapshot for different time (k) or frequency (m) are uncorrelated.

For a Real Gaussian Random Process , $X_{\triangle T}(w_m, k)$ are joint circular complex gaussian random vectors. For a circular complex gaussian random vector the probability density can be expressed in terms of a single correlation matrix $S_{X_{\triangle T}}(w_m) = \mathbf{E}\{X_{\triangle T}(w_m, k) X^H_{\triangle T}(w_m, k)\}$

The probability density for zero mean complex gaussian random variable is

$$p_{x_{\triangle T}} X_{\triangle T}(w_m, k) \triangleq \frac{1}{\pi^N |S_{X_{\triangle T}}(w_m)|} exp\{-[X^H_{\triangle T}(w_m, k) S^{-1}_{X_{\triangle T}} X_{\triangle T}(w_m, k)]\} \tag{5}$$

2 Plane wave Snapshot model

Case1: Desired signal is deterministic or unknown but non-random. The output of the array is an N x1 complex vector,

$$x(t) = x_s(t) + n(t) \tag{6}$$

where $x_s(t)$ is deterministic and $n(t)$ is zero mean complex circular gaussian random process.

$$X_{\triangle T}(w_m, k) = X_{s,\triangle T}(w_m, k) + N_{\triangle T}(w_m, k) \tag{7}$$

Example 1: Single Plane wave desired signal

$$X_{s,\triangle T}(w_m, k) = V(w_m, k_s)F_{s,\triangle T}(w_m, k) \tag{8}$$

$F_{s,\triangle T}(w_m, k)$ known- model used for radar.
$F_{s,\triangle T}(w_m, k)$ unknown but non random - model used for passive sonar and direction finding applications.

Example 2: Single plane wave desired signal and $D-1$ interfering plane wave signal

$$X_{s,\triangle T}(w_m, k) = V(w_m, k_s)F_{s,\triangle T}(w_m, k) + \sum_{i=1}^{D-1} V(w_m, k_i)F_{i,\triangle T}(w_m, k) + N_{\triangle T}(w_m, k) \tag{9}$$

Define

$$V(w_m, k) \triangleq \Bigg[V(w_m, k_s)V(w_m, k_1).............V(w_m, k_{D-1})\Bigg] \tag{10}$$

$$F_{\triangle T}(w_m, k) \triangleq \Bigg[F_{s,\triangle T}(w_m, k)F_{1,\triangle T}(w_m, k).............F_{D-1,\triangle T}(w_m, k))\Bigg] \tag{11}$$

$$X_{\triangle T}(w_m, k) = V(w_m, k)F_{\triangle T}(w_m, k) + N_{\triangle T}(w_m, k) \tag{12}$$

Case 2:

$$x(t) = x_s(t) + n(t) \tag{13}$$

Both $x_s(t)$ and $n(t)$ are sample functions of a gaussian random process.

$$X_{\triangle T}(w_m, k) = X_{s,\triangle T}(w_m, k) + N_{\triangle T}(w_m, k) \tag{14}$$

Single plane wave desired signal, (D-1) plane wave interfering signals , and additive noise. The signal plus interference covariance matrix is given by

$$S_{SI}(w_m) \triangleq \mathbf{E}\{F_{\triangle T}(w_m F^H_{\triangle T}(w_m\} \approx S_f(w_m) \tag{15}$$

Total snapshot spectral matrix is given as

$$S_{X_{\triangle T}}(w_m) = \mathbf{E}\{X_{\triangle T}(w_m X^H_{\triangle T}(w_m\} \tag{16}$$

$$= V(w_m)S_{SI,\triangle T}(w_m)V^H(w_m) + S_{N,\triangle T}(w_m) \approx S_n(w_m) \tag{17}$$

$$S_X \triangleq V(w_m)S_f(w_m)V^H(w_m) + S_n(w_m) \tag{18}$$

The probability function is defined as:

$$p_{x_{\triangle T}} X_{\triangle T}(w_m, k) \triangleq \frac{1}{\pi^N |S_X|} exp\{-[X^H_{\triangle T}(w_m, k)S_X^{-1}X_{\triangle T}(w_m, k)]\} \tag{19}$$

3 Narrow band Time domain Snapshot models

Consider the case of a single plane wave input. The input at the reference sensor located at the origin is a real bandpass signal,

$$f(t) = \sqrt{2}Re\left\{\tilde{f}(t)e^{jw_c t}\right\} \tag{20}$$

The input at the n^{th} sensor is given by

$$f_n(t) = f(t-\tau_n) = \sqrt{2}Re\left\{\tilde{f}(t-\tau_n)e^{jw_c t-\tau_n}\right\} = \sqrt{2}Re\left\{\tilde{f}(t)e^{-jw_c\tau_n}e^{jw_c t}\right\} \tag{21}$$

The output of the quadrature demodulator at the n^{th} sensor is given by

$$\tilde{f}_n(t) \approx \tilde{f}(t)e^{-jw_c\tau_n} \tag{22}$$

$$\tilde{\mathbf{f}}(t) = \tilde{f}(t).V(k) \tag{23}$$

The Covariance matrix at time t

$$\mathbf{R}_{\tilde{\mathbf{f}}}(0) = E\left\{\tilde{\mathbf{f}}(t)\tilde{\mathbf{f}}^H(t)\right\} = V(k)R_f(0)V^H(k) \tag{24}$$

However $R_f(0)$ is just the power in the signal process, So

$$\mathbf{R}_{\tilde{\mathbf{f}}}(0) = \sigma_s^2 V(k))V^H(k) \tag{25}$$

Generalizing the result to the case of D signals and adding a white noise component gives

$$\mathbf{R}_{\tilde{\mathbf{x}}}(0) = \mathbf{V}(k)\mathbf{R}_{\mathbf{f}}(0)\mathbf{V}^H(k) + \sigma_w^2 B_s I \tag{26}$$

If $\tilde{f}(t)$ is band limited to $\frac{-B_s}{2} \leq f \leq \frac{B_s}{2}$, then we sample $\mathbf{x}(t)$ every $1/B_s$ to obtain a time domain snapshot model.

We denote the snapshots as

$$\tilde{x}(k), k = 1, 2..........K \tag{27}$$

If the components of $\tilde{f}(t)$ have a flat spectrum

$$\mathbf{E}\left[\tilde{X}(k_1)\tilde{X}(k_2)^H\right] = 0 \quad for \quad k_1 \neq k_2 \tag{28}$$

Thus under gaussian case , the narrow band time domain snapshot model generates a sequence of statistically independent zero mean gaussian random vectors with covariance matrix given by

$$\mathbf{R}_{\tilde{x}}(k) = \mathbf{S}_{X_{\triangle T}}(0).B_s \tag{29}$$

The spatial characteristics are same in the narrowband time -domain and narrow-band frequency domain model.