
“Stochastic Signal”

Gaussian Model

ECE 6279: Spatial Array Processing
Fall 2013
Lecture 22

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology

AL: 404-385-2548
[<lanterma@ece.gatech.edu>](mailto:lanterma@ece.gatech.edu)



A Stochastic Gaussian Model

- N sources in additive noise

$$\underline{\mathbf{y}}(l) = \sum_{n=1}^{N_s} \mathbf{e}(\theta_n) \underline{s}_n(l) + \underline{\mathbf{n}}(l)$$

$$\underline{\mathbf{n}} \sim \mathcal{CN}(0, \mathbf{K}_{\mathbf{n}})$$

$$\underline{\mathbf{s}} = \begin{bmatrix} s_1 \\ \vdots \\ s_{N_s} \end{bmatrix} \sim \mathcal{CN}(0, \mathbf{K}_{\mathbf{s}})$$



Uncorrelated Sources

- Sources may be uncorrelated (usual model we've used in class)...

$$\mathbf{K}_s = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{N_s}^2 \end{bmatrix}$$



Correlated Sources

- Or they may be correlated (for instance, due to multipath)

$$\mathbf{K}_s = \begin{bmatrix} \sigma_1^2 & c_{12}\sigma_1\sigma_2 & \cdots & c_{1N_s}\sigma_1\sigma_{N_s} \\ c_{12}^*\sigma_1\sigma_2 & \sigma_2^2 & & c_{2N_s}\sigma_2\sigma_{N_s} \\ \vdots & \ddots & \ddots & \vdots \\ c_{1N_s}^*\sigma_1\sigma_{N_s} & c_{2N_s}^*\sigma_2\sigma_{N_s} & \cdots & \sigma_{N_s}^2 \end{bmatrix}$$



Fun Fact from Probability

- Independent \Rightarrow uncorrelated
- Uncorrelated $\not\Rightarrow$ independent
(in general)
- Uncorrelated+Gaussian
 \Rightarrow independent



Distribution of the Data

- N sources in additive noise

$$\underline{\mathbf{y}} \sim \mathcal{CN}(0, \mathbf{K}_y)$$

$$\mathbf{K}_y = \mathbf{D}(\Theta) \mathbf{K}_s \mathbf{D}^H(\Theta) + \mathbf{K}_n$$

$$\mathbf{D}(\Theta) = [e(\theta_1) \cdots e(\theta_{N_s})]$$

$$\Theta = [\theta_1 \cdots \theta_{N_s}]$$



Data Covariance (1)

$$\underline{\mathbf{y}}(l) = \mathbf{D}(\Theta)\underline{\mathbf{s}}(l) + \underline{\mathbf{n}}(l)$$

$$\mathbf{K}_y = E[\underline{\mathbf{y}}\underline{\mathbf{y}}^H]$$

$$= E[\{\mathbf{D}(\Theta)\underline{\mathbf{s}} + \underline{\mathbf{n}}\}\{\mathbf{D}(\Theta)\underline{\mathbf{s}} + \underline{\mathbf{n}}\}^H]$$

$$= E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

$$+ E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{n}}^H] + E[\underline{\mathbf{n}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)]$$

$$= E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$



Data Covariance (2)

$$\begin{aligned}\mathbf{K}_y &= E[\mathbf{D}(\Theta) \underline{\mathbf{s}} \underline{\mathbf{s}}^H \mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}} \underline{\mathbf{n}}^H] \\ &= \mathbf{D}(\Theta) E[\underline{\mathbf{s}} \underline{\mathbf{s}}^H] \mathbf{D}^H(\Theta) + E[\underline{\mathbf{n}} \underline{\mathbf{n}}^H] \\ &= \mathbf{D}(\Theta) \mathbf{K}_s \mathbf{D}^H(\Theta) + \mathbf{K}_n\end{aligned}$$



Probability Density of the Data

$$\underline{\mathbf{y}} \sim \mathcal{CN}(0, \mathbf{K}_y)$$

- Circular (Goodman) zero-mean Gaussian density for L independent snapshots: $p(\mathbf{y}) =$

$$\frac{1}{(\pi^M \det \mathbf{K}_y)^L} \exp \left[- \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right]$$



Loglikelihood Calculation

- Loglikelihood $\ln p(\mathbf{y}) =$

$$\begin{aligned} & \cancel{-LM \ln \pi} - L \ln \det \mathbf{K}_y \\ & - \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \end{aligned}$$

$$L \text{tr} \left[\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right]$$



Trace Rearrangement Trick

$$\begin{aligned} & \text{tr} \left[\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right] \\ &= \frac{1}{L} \sum_{l=0}^{L-1} \text{tr} \left[\mathbf{y}^H(l) \mathbf{K}_y^{-1} \mathbf{y}(l) \right] \\ &= \text{tr} \left[\underbrace{\frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l)}_{\hat{\mathbf{K}}_y} \mathbf{K}_y^{-1} \right] = \text{tr} \left[\hat{\mathbf{K}}_y \mathbf{K}_y^{-1} \right] \end{aligned}$$



Maximum-Likelihood Estimation

- Goal: maximize

$$-L \ln \det \mathbf{K}_y(\xi) - L \text{tr} [\hat{\mathbf{K}}_y \mathbf{K}_y^{-1}(\xi)]$$

where

$$\mathbf{K}_y(\xi) = \mathbf{D}(\Theta) \mathbf{K}_s \mathbf{D}^H(\Theta) + \mathbf{K}_n$$

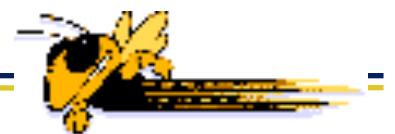
over ξ , which represents all the parameters (angles, signal powers, correlation coefficients)



Important Caveat

- The maximization must be over the feasible set, i.e. $K_y(\xi)$ must be a legitimate covariance matrix (nonnegative definite)
- To make sense at all, we need

$$\dim(\xi) \leq M^2$$



How Can We Maximize?

- No closed form solution in general
- Still an open research problem!
- One approach: try an **expectation-maximization** algorithm (sometimes ECE7251 covers EM algorithms)

M.I. Miller and D.R. Fuhrmann, “Maximum-Likelihood Narrow-Band Direction Finding and the EM Algorithm,” *IEEE Trans. ASSP*, 38(9), Sept. 1990, pp. 1560-1577.



Adding More Detail

- Could extend model with:
 - Unknown noise covariance, for instance

$$\mathbf{K}_n = \mathbf{I} \underbrace{\sigma_n^2}_{\text{Make a parameter}}$$

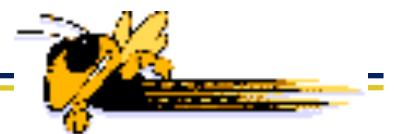
- Uncertainties in sensor positions

$$x = \bar{x} + \underbrace{\delta x}_{\text{Make a parameter}}$$



Asymptotic ML Est. for Stochastic Model

- If you have
 - “Lots of snapshots,”
 - Uncorrelated sources,
 - and I.I.D. receiver noise
- Then VT-IV, pp. 985-991 shows that you can get rid of the power parameters $\sigma_1^2, \dots, \sigma_N^2$ and find ML estimates of angles by maximizing...



Asymptotic ML Procedure

- New goal: maximize

$$-\ln \det \left[\mathbf{P}_D \hat{\mathbf{K}}_y \mathbf{P}_D + \sigma_n^2 \mathbf{P}_D^\perp \right] - \text{tr} \left[\mathbf{P}_D^\perp \hat{\mathbf{K}}_y \right] / \sigma_n^2$$

over Θ

where $\mathbf{P}_D = \mathbf{D}(\mathbf{D}^H \mathbf{D})^{-1} \mathbf{D}^H$

$$\mathbf{P}_D^\perp = \mathbf{I} - \mathbf{P}_D$$



Alternative: Least Squares Hack

- Find feasible parameter ξ that minimizes

$$\left\| \hat{\mathbf{K}}_y - \mathbf{K}_y(\xi) \right\|^2$$

- Nonlinear Least Squares problem
 - Large literature on numerical techniques
 - Can use MATLAB Optimization Toolbox
- Discussed in Sec. 7.1.2 of J&D

