Signal to Noise

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Lecture 11

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Where We Are (and Aren't) in J&D

- **Inspired by Section 4.5**

Single Plane Wave, Stochastic Model

To keep notation compact, suppress notation of time variable:

Data: $\underline{\mathbf{y}} = \mathbf{e}(\vec{k}^{\,0})\underline{s} + \underline{\mathbf{n}}$

Beamformer output: $z = \mathbf{e}^{H}(\vec{k})\mathbf{W}\underline{\mathbf{y}}$

$$= \mathbf{e}^{H}(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^{\,0})\underline{s} + \mathbf{e}^{H}(\vec{k})\mathbf{W}\underline{\mathbf{n}}$$

Signal to Noise Ratio (1)

$$SNR = \frac{E\left[\left|\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\underline{s}\right|^2\right]}{E\left[\left|\mathbf{e}^H(\vec{k})\mathbf{W}\underline{\mathbf{n}}\right|^2\right]}$$

$$= \frac{E\left[\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\underline{s}\,\underline{s}^H\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})\right]}{E\left[\mathbf{e}^H(\vec{k})\mathbf{W}\underline{\mathbf{n}}\,\underline{\mathbf{n}}^H\mathbf{W}^H\mathbf{e}(\vec{k})\right]}$$

Signal to Noise Ratio (2)

$$= \frac{\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)E[\underline{s}\,\underline{s}^H]\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k})\mathbf{W}E[\underline{\mathbf{n}}\,\underline{\mathbf{n}}^H]\mathbf{W}^H\mathbf{e}(\vec{k})}$$

$$= \frac{P_s\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})}{\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{K}_\mathbf{n}\mathbf{W}^H\mathbf{e}(\vec{k})}$$

Special Case of the Numerator (1)

Suppose $\vec{k} = \vec{k}^{\,0}$

$$\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^{\,0}) = \mathbf{e}^H(\vec{k}^{\,0})\mathbf{W}\mathbf{e}(\vec{k}^{\,0}) =$$

$$\begin{bmatrix} e^{j\vec{k}^0\cdot\vec{x}_0} & \cdots & e^{j\vec{k}^0\cdot\vec{x}_{M-1}} \end{bmatrix} \begin{bmatrix} w_0 & & \\ & \ddots & \\ & & w_{M-1} \end{bmatrix} \begin{bmatrix} e^{-j\vec{k}^0\cdot\vec{x}_0} \\ \vdots \\ e^{-j\vec{k}^0\cdot\vec{x}_0} \end{bmatrix}$$

$$= \sum_{m=0}^{M-1} w_m$$

Special Case of the Numerator (2)

Suppose $\vec{k} = \vec{k}^{0}$

$$numer = P_s \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k})$$

$$= P_s \mathbf{e}^H(\vec{k}^0)\mathbf{W}\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{W}^H\mathbf{e}(\vec{k}^0)$$

$$= P_s\left(\sum_{m=0}^{M-1} w_m\right)\left(\sum_{m=0}^{M-1} w_m^*\right) = P_s\left|\sum_{m=0}^{M-1} w_m\right|^2$$

Special Case of the Denominator

Special case $\mathbf{K_n} = \sigma_n^2 \mathbf{I}$

$$denom = \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{K_n}\mathbf{W}^H\mathbf{e}(\vec{k})$$

$$= \sigma_n^2 \sum_{m=0}^{M-1} |w_m|^2$$

Both Special Cases

Suppose $\vec{k} = \vec{k}^0$ **and** $\mathbf{K_n} = \sigma_n^2 \mathbf{I}$

$$SNR = \frac{P_s \left| \sum_{m=0}^{M-1} w_m \right|^2}{\sigma_n^2 \left(\sum_{m=1}^{M-1} \left| w_m \right|^2 \right)}$$

If $w_m = w_0 : SNR = \dfrac{P_s M^2 w_0^2}{\sigma_n^2 M w_0^2} = \dfrac{P_s M}{\sigma_n^2}$

The Array Gain

Define the array gain:

$$G = \frac{SNR_{array}}{SNR_{sensor}} = \frac{P_s M / \sigma_n^2}{P_s / \sigma_n^2} = M$$

in this special case

Can We Do Better?

- **Schwartz Inequality:**

$$\left|\sum_{m=0}^{M-1} a_m b_m^*\right|^2 \le \sum_{m=0}^{M-1} |a_m|^2 \sum_{m=0}^{M-1} |b_m|^2$$

with equality iff $a_m = \kappa b_m$

- **Let** $a_m = w_m, \quad b_m = 1$

$$\left|\sum_{m=0}^{M-1} w_m\right|^2 \le M \sum_{m=0}^{M-1} |w_m|^2$$

No, That's the Best We Can Do!

$$\frac{\left|\sum_{m=0}^{M-1} w_m\right|^2}{\sum_{m=0}^{M-1} \left|w_m\right|^2} \leq M$$

So in uniform noise, uniform weights maximizes the SNR when beamforming on true wavenumber vector

A Fresh Approach

To keep notation compact, suppress notation of time variable:

Data: $\underline{\mathbf{y}} = \mathbf{e}(\vec{k}^{\,0})\underline{s} + \underline{\mathbf{n}}$

Beamformer output: $z = \mathbf{a}^H \underline{\mathbf{y}}$

$$= \mathbf{a}^H \mathbf{e}(\vec{k}^{\,0})\underline{s} + \mathbf{a}^H \underline{\mathbf{n}}$$

What choice of $\mathbf{a}$ maximizes SNR?

SNR for Colored Noise

$$SNR = \frac{P_s \left| \mathbf{a}^H \mathbf{e}(\vec{k}^0) \right|^2}{\mathbf{a}^H \mathbf{K_n} \mathbf{a}}$$

- **Want to study case of a general $\mathbf{K_n}$**
- **Substitute $\tilde{\mathbf{a}} = \mathbf{K_n}^{1/2}\mathbf{a}, \quad \mathbf{a} = \mathbf{K_n}^{-1/2}\tilde{\mathbf{a}}$**

$$SNR = \frac{P_s \left| \tilde{\mathbf{a}}^H \mathbf{K_n}^{-1/2} \mathbf{e}(\vec{k}^0) \right|^2}{\tilde{\mathbf{a}}^H \tilde{\mathbf{a}}}$$

Rewriting Schwartz

- **Schwartz Inequality:**

$$\left|\mathbf{f}^H\mathbf{g}\right|^2 \leq (\mathbf{f}^H\mathbf{f})(\mathbf{g}^H\mathbf{g}) \equiv \left\|\mathbf{f}\right\|^2 \left\|\mathbf{g}\right\|^2$$

with equality iff $\mathrm{f} = \kappa\mathbf{g}$

$$\frac{\left|\mathbf{f}^H\mathbf{g}\right|^2}{\left\|\mathbf{f}\right\|^2} \leq \left\|\mathbf{g}\right\|^2$$

Use the Schwartz!

- Let $\mathbf{f} = \tilde{\mathbf{a}}, \; \mathbf{g} = \mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^{\,0})$

$$\frac{\left|\tilde{\mathbf{a}}^{H}\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^{\,0})\right|^{2}}{\left\|\tilde{\mathbf{a}}\right\|^{2}} \leq \left\|\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^{\,0})\right\|^{2}$$

with equality iff

$$\tilde{\mathbf{a}} = \kappa\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^{\,0})$$

Maximizing SNR in Colored Noise

- **Schwartz inequality tells us that to maximize SNR, we can pick**

$$\tilde{\mathbf{a}} = \mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^{\,0})$$

$$\mathbf{a} = \mathbf{K}_{\mathbf{n}}^{-1/2}\tilde{\mathbf{a}} = \mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{K}_{\mathbf{n}}^{-1/2}\mathbf{e}(\vec{k}^{\,0})$$

$$= \mathbf{K}_{\mathbf{n}}^{-1}\mathbf{e}(\vec{k}^{\,0})$$

$$z = \mathbf{a}^H \underline{\mathbf{y}} = \mathbf{e}^H(\vec{k}^{\,0})\mathbf{K}_{\mathbf{n}}^{-1}\underline{\mathbf{y}}$$

Direction Finding in Colored Noise

- **Realistically, sweep $\vec{k}$ as usual:**

$$z = \mathbf{e}^H(\vec{k})\mathbf{K}_{\mathbf{n}}^{-1}\underline{\mathbf{y}}$$