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Coherent Radar Detection in Compound-Gaussian Clutter

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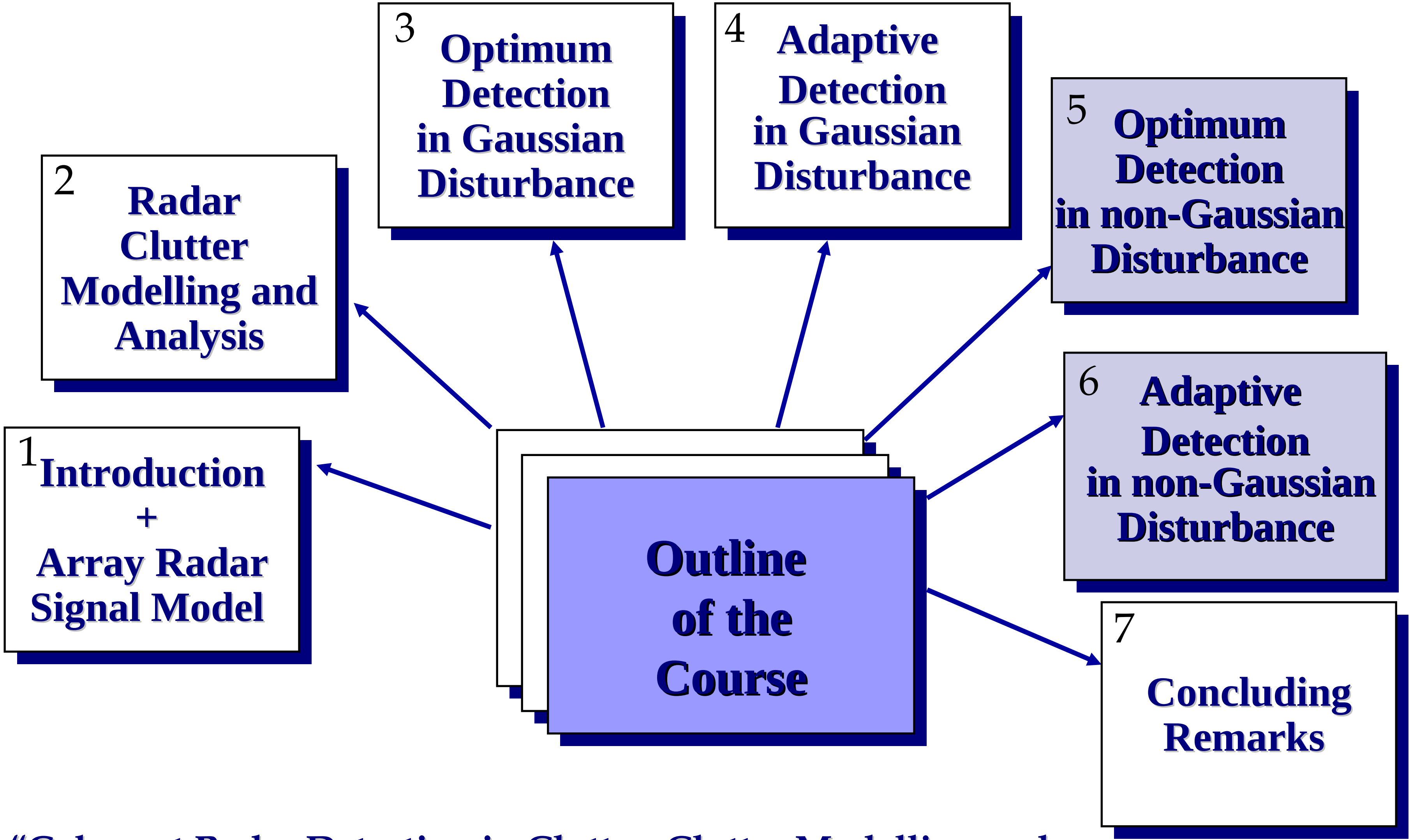

"Coherent Radar Detection in Clutter: Clutter Modelling and Analysis + Adaptive Array Radar Signal Processing"

Outline of the Section on "Optimum Detection"

- **The binary hypothesis testing problem**
- Target models and summary of detectors in Gaussian clutter
- Optimum coherent detection in compound-Gaussian clutter
  - The Likelihood Ratio Test (LRT)
  - The Estimator-Correlator (EC)
  - The Whitening Matched Filter (WMF) compared to a Data-Dependent Threshold (DDT)
- Suboptimum detection in compound-Gaussian clutter
- Performance analysis in K-distributed clutter
- The Linear-Threshold Detector (LTD) and the IG-texture model
- Subspace detectors in compound-Gaussian clutter

Binary Hypothesis Testing Problem

- The binary hypothesis testing problem is formulated as follows: we decide either hypothesis H_0 is true and only disturbance is present (decision D_0) or hypothesis H_1 is true and a target signal is present with disturbance (decision D_1).

$$\begin{cases} \mathbf{z} = \mathbf{d} & H_0 : \text{Target absent} \\ \mathbf{z} = \mathbf{s}_t + \mathbf{d} & H_1 : \text{Target present} \end{cases}$$

- **z** is an MNx1 vector, where N is the number of receiving channels and M is the number of temporal samples (spaced by the Pulse Repetition Interval PRI=1/PRF).

- Disturbance **d** = clutter **c** + white Gaussian noise **w** + jammer **j**
- The test is implemented on-line for each **range cell under test** (CUT), i.e. each slice of the data "cube".
- Performance measures are the Probability of False Alarm (P_{FA}) and the Probability of Detection (P_D):

$$P_{FA} = \Pr\{D_1 | H_0\}, \qquad P_D = \Pr\{D_1 | H_1\}.$$

Neyman-Pearson (or GLRT) Criterion

- According to the **Neyman-Pearson (NP) criterion** (maximize P_D while keeping constant P_{FA}), the optimal decision strategy is a **likelihood ratio test (LRT)**:

$$\Lambda(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta} \quad \Rightarrow \quad \ln\Lambda(\mathbf{z}) = \ln\left(\frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i)$ is the probability density function (PDF) of the data vector **z** under the hypothesis H_i, i=0,1.

η is the detection threshold, set according to the desired P_{FA}.

- When the data PDF (under H_0 and/or H_1) depends on some deterministic unknown parameters, we usually resort to the **generalized likelihood ratio test (GLRT)**, where the unknown parameters are replaced by their **maximum likelihood (ML)** estimates under that hypothesis.

The Gaussian Clutter Model

- In the following, for sake of simplicity, we focus our attention on one receiving channel (i.e. one sensor) or one beam (e.g. the sum beam) $\rightarrow$ *N*=1.
- *M* is the number of temporal samples spaced by PRI=1/PRF.
- In this talk, we assume that (1) no jammer is present, and (2) the noise is negligible w.r.t. the clutter $\rightarrow$ large Clutter-to-Noise power Ratio (*CNR*).
- The complex multidimensional PDF for **Gaussian** clutter:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = p_{\mathbf{c}}(\mathbf{z}) = \frac{1}{\pi^M |\mathbf{R}|} \exp\left(-\mathbf{z}^H \mathbf{R}^{-1} \mathbf{z}\right)$$

R is the clutter covariance matrix

$$\mathbf{R} = \sigma^2 \mathbf{M}$$

$$\sigma^2 = \text{clutter power}$$

*M*x1 *data vector*

$$|\mathbf{R}| = \det\{\mathbf{R}\}, \quad \mathbf{R} = \mathbf{R}^H \rightarrow \text{Hermitian matrix}$$

- In some scenarios, we have seen that the **compound-Gaussian model** is a better model for radar clutter (e.g. high resolution, low grazing angles, etc.)

The Compound-Gaussian Clutter Model

- **Clutter with <u>completely correlated texture</u>:** $\tau[n]=\tau$ during the Coherent Processing Interval (CPI) $\rightarrow$ the non-Gaussian clutter vector is a **Spherically Invariant Random Vector (SIRV)**:

$$\mathbf{z}\big|H_0 = \mathbf{c} = \sqrt{\tau}\,\mathbf{x}$$

Texture

$$p_\tau(\tau),\ \mu = E\{\tau\}$$

Speckle

$$\mathbf{x} \in \mathcal{CN}\left(0,\mathbf{M}\right), \quad [\mathbf{M}]_{i,i} = 1$$

- The PDF of **z** under the hypothesis H_0 is obtained by averaging the PDF of **z**$|\tau$ with respect to τ

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = E_\tau\left\{p_{\mathbf{c}|\tau}(\mathbf{z}|\tau)\right\} = \int_0^\infty \frac{1}{(\pi\tau)^M\,|\mathbf{M}|}\exp\left(-\frac{\mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}}{\tau}\right)p_\tau(\tau)\,d\tau$$

where $\quad |\tau\mathbf{M}| = \tau^M\,|\mathbf{M}|$

The Optimum Detector (known M)

- The multidimensional PDF of *target+clutter* depends on the target signal model.
- Various target models have been proposed in radar applications, which reflect different degrees of a-priori knowledge on the signal backscattered by the target.
- If the target vector is deterministic and **perfectly a-priori known**:

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)$$

- If the target vector is random with known PDF: $p_{\mathbf{z}|\mathbf{s}_t,H_1}(\mathbf{z}|\mathbf{s}_t,H_1) = p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)$

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = E_{\mathbf{s}_t}\left\{p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)\right\}$$

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = \int \underbrace{\int_0^\infty \frac{1}{(\pi\tau)^M|\mathbf{M}|}\exp\left(-\frac{(\mathbf{z}-\mathbf{s}_t)^H\mathbf{M}^{-1}(\mathbf{z}-\mathbf{s}_t)}{\tau}\right)p_\tau(\tau)d\tau}_{p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0)} \cdot p_{\mathbf{s}_t}(\mathbf{s}_t)d\mathbf{s}_t$$

The Optimum Neyman-Pearson Detector

- Define now the following **quadratic statistic** of the data: $q_0(\mathbf{z}) \triangleq \mathbf{z}^H \mathbf{M}^{-1} \mathbf{z}$

- PDF of a **compound-Gaussian clutter** vector:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = \int_0^\infty \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right] p_\tau(\tau) d\tau, \qquad p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = ?$$

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = E_{\mathbf{s}_t}\left\{ p_{\mathbf{z}|H_0}(\mathbf{z} - \mathbf{s}_t | H_0) \right\}$$

It depends on the target signal model.

- The **optimum NP detector** is the **LRT**: $\Lambda(\mathbf{z}) = \dfrac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^\eta$

Target Signal Models

- Different models of $\mathbf{s}_t$ have been investigated to take into account for different degrees of *a-priori* knowledge on the target signal:

(1) $\mathbf{s}_t$ perfectly known $\rightarrow$ **coherent whitening matched filter** (CWMF)

(2) $\mathbf{s}_t = \beta\mathbf{v}$ with $\beta \in \mathcal{CN}(0, \sigma_s^2)$, i.e., Swerling I model, and **v** perfectly known

(3) $\mathbf{s}_t = \beta\mathbf{v}$ with β unknown deterministic and **v** perfectly known

(4) $\mathbf{s}_t = \beta\mathbf{v}$ with β unknown (deterministic or random) and **v** known function of unknown parameters: DOA, Doppler frequency, Doppler rate, etc.

(5) $\mathbf{s}_t$ complex Gaussian random vector (known to belong to a subspace of dim. *r*):

$$\mathbf{s}_t \in \mathcal{CN}\left(0, \sigma_s^2 \mathbf{R}_s\right), \quad r = rank\left(\mathbf{R}_s\right) \leq M, \quad \left[\mathbf{R}_s\right]_{i,i} = 1\,\forall i$$

$\mathbf{R}_s = \mathbf{v}\mathbf{v}^H \quad \Rightarrow \quad rank\left(\mathbf{R}_s\right) = 1$ ■ Swerling I target model $\rightarrow$ **noncoherent WMF**

$\mathbf{R}_s = \mathbf{I} \quad \Rightarrow \quad rank\left(\mathbf{R}_s\right) = M$ ■ Swerling II target model $\rightarrow$ **energy detector** (ED)

(6) $\mathbf{s}_t$ completely unknown deterministic $\rightarrow$ **energy detector** (ED)

1-D Target Model

- In the following we focus on the **1-D target signal model**:

$$\underset{NM\times 1}{\mathbf{s}_t} = \beta \mathbf{v}(\nu_d, \nu_s)$$

$$\underset{NM\times 1}{\mathbf{v}(\nu_d, \nu_s)} = \underset{M\times 1}{\mathbf{s}(\nu_d)} \otimes \underset{N\times 1}{\mathbf{a}(\nu_s)}$$

- Space-Time Steering Vector

$$\nu_s = \frac{d}{\lambda}\cos(\theta_t)\sin(\phi_t) \quad \text{spatial frequency} \Leftrightarrow \text{DOA}$$

$$\nu_d = \frac{2v_t T_r}{\lambda} \quad \text{normalized Doppler frequency} \Leftrightarrow \text{target radial velocity}$$

- Since we assume here *N*=1:

$$\underset{M\times 1}{\mathbf{s}_t} = \beta \mathbf{v}(\nu_d)$$

- Complex amplitude β can be modeled random (usually Gaussian) or unknown deterministic.

Narrowband Slowly-Fluctuating Target Signal Model

$$\underset{M\times 1}{\mathbf{s}(\nu_d)} = \begin{bmatrix} 1 \\ e^{j2\pi\nu_d} \\ \vdots \\ e^{j2\pi(M-1)\nu_d} \end{bmatrix}$$

- **Temporal Steering Vector**

  It has the Vandermonde form because the waveform PRF is uniform and target velocity is constant during the CPI.

$$\underset{N\times 1}{\mathbf{a}(\nu_s)} = \begin{bmatrix} 1 \\ e^{j2\pi\nu_s} \\ \vdots \\ e^{j2\pi(N-1)\nu_s} \end{bmatrix}$$

- **Spatial Steering Vector**

  It has the Vandermonde form because of the ULA geometry and the assumption of identical element patterns.

- The detection algorithm is optimized for a specific angle and Doppler.
- The steering vector should be known (calibrated array).
- Since the target DOA and velocity are unknown a-priori, **v** is a known function of unknown parameters, so the radar receiver should implement multiple detectors that form a **filter bank** to cover all potential target angles and Doppler frequencies.

Gaussian Clutter: The Whitening Matched Filter (WMF)

- It is worth briefly summarizing what can be found for **Gaussian clutter**:

$$\mathbf{R} = \sigma^2 \mathbf{M}$$

$$\sigma^2 = E\left\{\left|c[i]\right|^2\right\} \quad \text{clutter power}$$

$$\mathbf{M} = \text{normalized clutter covairance matrix}$$

$$\mathbf{M}_{i,i} = 1, \; i = 1, 2, \cdots, M$$

$$\mathbf{M}_{i,j} = \text{corr. coeff. between } c[i] \text{ and } c[j]$$

- **<u>Case 1</u>. Perfectly known target signal**:

$$\mathrm{Re}\{\mathbf{s}_t^H \mathbf{R}^{-1} \mathbf{z}\} = \frac{1}{\sigma^2} \mathrm{Re}\{\beta^* \cdot \mathbf{v}^H \mathbf{M}^{-1} \mathbf{z}\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\mathbf{s}_t = \beta \mathbf{v}$$

$$\mathbf{v}_n = e^{j 2\pi \nu_d n}, \quad n = 0, 1, 2, \cdots, M-1$$

$$\nu_d = \text{normalized Doppler frequency}$$

- The statistic of the test is the real part of the so-called **Whitening Matched Filter (WMF)** output (multiplied by the known complex amplitude β).

Gaussian Clutter: The Whitening Matched Filter (WMF)

- It is called **coherent whitening matched filter (CWMF)** detector:

$$\mathbf{s}_t^H\mathbf{R}^{-1}\mathbf{z} = \mathbf{s}_t^H\mathbf{R}^{-1/2}\mathbf{R}^{-1/2}\mathbf{z} = \mathbf{s}_t^H\left(\mathbf{R}^{-1/2}\right)^H\mathbf{R}^{-1/2}\mathbf{z} = \left(\mathbf{R}^{-1/2}\mathbf{s}_t\right)^H\mathbf{R}^{-1/2}\mathbf{z} = \overline{\mathbf{s}}_t^H\overline{\mathbf{z}}$$

$$\overline{\mathbf{s}}_t \triangleq \mathbf{R}^{-1/2}\mathbf{s}_t, \quad \overline{\mathbf{z}} \triangleq \mathbf{R}^{-1/2}\mathbf{z}$$

whitening transformation

matched filtering

$$E\left\{\overline{\mathbf{z}}\,\overline{\mathbf{z}}^H\right\} = E\left\{\mathbf{R}^{-1/2}\mathbf{z}\mathbf{z}^H\mathbf{R}^{-1/2}\right\} = \mathbf{R}^{-1/2}E\left\{\mathbf{z}\mathbf{z}^H\right\}\mathbf{R}^{-1/2} = \mathbf{R}^{-1/2}\mathbf{R}\mathbf{R}^{-1/2} = \mathbf{I}$$

- The **WMF** is a linear FIR filter: $\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z} = \mathbf{w}^H\mathbf{z} = \sum_{n=1}^{M} w_n^* z_n$
- **w** is the optimal weighting vector: $\mathbf{w}^H = \mathbf{v}^H\mathbf{M}^{-1}\mathbf{z} \Rightarrow \mathbf{w} = \mathbf{M}^{-H}\mathbf{v} = \mathbf{M}^{-1}\mathbf{v}$

- **Case 3.**

Unknown complex amplitude:
β unknown deterministic

GLRT approach

$$\hat{\mathbf{s}}_{t,ML} = \hat{\beta}_{ML}\mathbf{v}$$

[in this case a Uniformly Most Powerful (**UMP**) test does not exist]

$$\hat{\beta}_{ML} = \frac{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}}{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}}$$ is the ML estimate of β

- **Case 2/3** (intermediate between 2 and 3).

Unknown complex amplitude:
$|\beta|$ deterministic, $\angle\beta$ r.v. uniformly distributed in $[0,2\pi)$

$\rightarrow$ Swerling 0 (or Swerling V) target model

- **Case 2.**

Unknown complex amplitude:
β random, complex Gaussian r.v.

$\rightarrow$ Swerling I target model

LRT

$$\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The statistic of the test is the modulo squared of the WMF output: it is called the **noncoherent WMF detector**.

Case 4

Unknown complex amplitude and Doppler frequency

GLRT approach

$\rightarrow$ A bank of WMFs, each one "tuned" to a specific Doppler frequency. Then, we select the filter with the max output:

$$\max_{\nu_d} \left| \mathbf{v}^H(\nu_d)\mathbf{M}^{-1}\mathbf{z} \right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Grid search: $\nu_d = k/M, \quad k = -M/2, \cdots, M/2-1$

Case 5 – Matched Subspace Detector (MSD)

$$\mathbf{z}^H\mathbf{Q}_2\mathbf{z} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\mathbf{Q}_2 = \mathbf{M}^{-1}\mathbf{H}(\mathbf{H}^H\mathbf{M}^{-1}\mathbf{H})^{-1}\mathbf{H}^H\mathbf{M}^{-1}$$

$\mathbf{H}: \quad \mathbf{h}_k$ eigenvectors of $\mathbf{R}_s$

if $\mathrm{rank}(\mathbf{R}_s) = 1 \Rightarrow \mathbf{H} = \mathbf{v} \Rightarrow \mathrm{MSD} = |\mathrm{WMF}|^2$

Case 6 – Energy Detector

$$\mathbf{z}^H\mathbf{z} = \|\mathbf{z}\|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

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- **Optimum coherent detection in compound-Gaussian clutter**
  - **The Likelihood Ratio Test (LRT)**
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- **1. <u>The Likelihood Ratio Test (LRT) formulation of the optimum detector</u>**

- Compound-Gaussian clutter vector PDF:

$$p_{\mathbf{z}|H_0}(\mathbf{z}|H_0) = \int_0^{+\infty} \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right] p_\tau(\tau)d\tau, \quad \text{where } q_0(\mathbf{z}) \triangleq \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z}$$

- **<u>Case 1</u>.**

Perfectly a-priori known target signal: $\mathbf{s}_t = \beta\mathbf{v}$

(β and **v** a-priori known deterministic)

$$p_{\mathbf{z}|H_1}(\mathbf{z}|H_1) = p_{\mathbf{z}|H_0}(\mathbf{z}-\mathbf{s}_t|H_0) = \int_0^{+\infty} \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z}-\mathbf{s}_t)}{\tau}\right] p_\tau(\tau)d\tau$$

- The **optimal NP detector (OD)** is a **Likelihood Ratio Test (LRT)**:

$$\ln\Lambda_{NP}(\mathbf{z}) = \ln\frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Optimum Detection in Compound-Gaussian Clutter

$$\Lambda_{NP}(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\displaystyle\int_0^{+\infty} \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z}-\mathbf{s}_t)}{\tau}\right] p_\tau(\tau)d\tau}{\displaystyle\int_0^{+\infty} \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right] p_\tau(\tau)d\tau} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta}$$

$$\int_0^{\infty} \frac{1}{\tau^M}\left[\exp\left(-\frac{q_1(\mathbf{z})}{\tau}\right) - \exp\left(\eta - \frac{q_0(\mathbf{z})}{\tau}\right)\right] p_\tau(\tau)d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$q_0(\mathbf{z}) \triangleq \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}, \qquad q_1(\mathbf{z}) \triangleq q_0(\mathbf{z}-\mathbf{s}_t) = \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z} - 2\,\mathrm{Re}\left\{\mathbf{s}_t^H\mathbf{M}^{-1}\mathbf{z}\right\} + \mathbf{s}_t^H\mathbf{M}^{-1}\mathbf{s}_t$$

- Implementation of the OD requires the solution of an integral by numerical integration $\rightarrow$ High computational complexity.

- 1. **The Likelihood Ratio Test (LRT) formulation of the optimum detector**

- **Case 3.**

Unknown complex amplitude: β unknown deterministic

$$p_{\mathbf{z}|H_1}(\mathbf{z};\beta|H_1) = p_{\mathbf{z}|H_0}(\mathbf{z}-\beta\mathbf{v}|H_0) = \int_0^{+\infty} \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z}-\beta\mathbf{v})}{\tau}\right] p_\tau(\tau)d\tau$$

- A UMP test does not exist. We resort to the **generalized likelihood ratio test (GLRT) approach**, which is not guaranteed to provide the most powerful test:

$$\Lambda_{GLRT}(\mathbf{z}) = \max_{\beta} \Lambda(\mathbf{z};\beta) = \max_{\beta} \frac{p_{\mathbf{z}|H_1}(\mathbf{z};\beta|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{p_{\mathbf{z}|H_1}(\mathbf{z};\hat{\beta}_{ML}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \Lambda(\mathbf{z};\hat{\beta}_{ML}) \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta}$$

- The test statistic is given by the LR for known β, in which the unknown parameters have been replaced by their **maximum likelihood (ML) estimates**:

$$\Lambda_{GLRT}(\mathbf{z}) = \Lambda(\mathbf{z};\hat{\beta}_{ML}) = \frac{p_{\mathbf{z}|H_0}(\mathbf{z}-\hat{\beta}_{ML}\mathbf{v}\,|\,H_0)}{p_{\mathbf{z}|H_0}(\mathbf{z}\,|\,H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta}$$

$$\begin{aligned}\hat{\beta}_{ML} &= \arg\max_{\beta} p_{\mathbf{z}|H_1}(\mathbf{z};\beta\,|\,H_1) = \arg\max_{\beta} p_{\mathbf{z}|H_0}(\mathbf{z}-\beta\mathbf{v}\,|\,H_0) \\ &= \arg\max_{\beta} \int_0^{+\infty} \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z}-\beta\mathbf{v})}{\tau}\right] p_\tau(\tau)\,d\tau\end{aligned}$$

$p_{\mathbf{z}|H_1}(\mathbf{z};\beta\,|\,H_1)$ is a decreasing monotonic function of the quadratic term $q_0(\cdot)$

$$\Rightarrow \quad \hat{\beta}_{ML} = \arg\min_{\beta} q_0(\mathbf{z}-\beta\mathbf{v}) = \arg\min_{\beta} (\mathbf{z}-\beta\mathbf{v})^H \mathbf{M}^{-1}(\mathbf{z}-\beta\mathbf{v})$$

$$\hat{\beta}_{ML} = \arg\min_{\beta} (\mathbf{z} - \beta\mathbf{v})^H \mathbf{M}^{-1} (\mathbf{z} - \beta\mathbf{v})$$

$$= \arg\min_{\beta} (\mathbf{M}^{-1/2}\mathbf{z} - \beta\mathbf{M}^{-1/2}\mathbf{v})^H (\mathbf{M}^{-1/2}\mathbf{z} - \beta\mathbf{M}^{-1/2}\mathbf{v})$$

$$= \arg\min_{\beta} \left\| \mathbf{M}^{-1/2}\mathbf{z} - \beta\mathbf{M}^{-1/2}\mathbf{v} \right\|^2$$

$$\Rightarrow \quad \hat{\beta}_{ML} = \left(\mathbf{M}^{-1/2}\mathbf{v}\right)^{\#} \mathbf{M}^{-1/2}\mathbf{z} = \left[\left(\mathbf{M}^{-1/2}\mathbf{v}\right)^H \left(\mathbf{M}^{-1/2}\mathbf{v}\right)\right]^{-1} \left(\mathbf{M}^{-1/2}\mathbf{v}\right)^H \mathbf{M}^{-1/2}\mathbf{z}$$

$$\boxed{\hat{\beta}_{ML} = \frac{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}}{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v}}}$$

$\mathbf{A}^{\#} =$ Moore-Penrose pseudoinverse of $\mathbf{A}$

$\mathbf{A}^{\#} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$

- Now define the quadratic term $q_1(\mathbf{z})$:

$$q_1(\mathbf{z}) \triangleq q_0(\mathbf{z} - \hat{\beta}_{ML}\mathbf{v}) = (\mathbf{z} - \hat{\beta}_{ML}\mathbf{v})^H \mathbf{M}^{-1} (\mathbf{z} - \hat{\beta}_{ML}\mathbf{v}) = \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z} - \frac{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v}}$$

Optimum Detection in Compound-Gaussian Clutter

$$\Lambda_{GLRT}(\mathbf{z}) = \Lambda(\mathbf{z};\hat{\beta}_{ML}) = \frac{p_{\mathbf{z}|H_0}(\mathbf{z}-\hat{\beta}_{ML}\mathbf{v}|H_0)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{\int_0^\infty \frac{1}{(\pi\tau)^M|\mathbf{M}|}\exp\left[-\frac{q_1(\mathbf{z})}{\tau}\right]p_\tau(\tau)d\tau}{\int_0^\infty \frac{1}{(\pi\tau)^M|\mathbf{M}|}\exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right]p_\tau(\tau)d\tau} \underset{H_0}{\overset{H_1}{\gtrless}} e^\eta$$

$$\int_0^\infty \frac{1}{\tau^M}\left[\exp\left(-\frac{q_1(\mathbf{z})}{\tau}\right) - \exp\left(\eta - \frac{q_0(\mathbf{z})}{\tau}\right)\right]p_\tau(\tau)d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$q_0(\mathbf{z}) \triangleq \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}$$

$$q_1(\mathbf{z}) \triangleq \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z} - \frac{|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}|^2}{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}}$$

- Implementation of the GLRT requires the solution of an integral by numerical integration.
- When the number *M* of integrated samples increases $\hat{\beta}_{ML} \rightarrow \beta$. Hence, the performance of the GLRT approaches that of the NP detector for perfectly known signal.

- **<u>Case 2</u>. Unknown complex amplitude**:

  β complex Gaussian r.v. $\rightarrow$ Swerling I target signal

$$\beta \in \mathcal{CN}(0, \sigma_s^2)$$

- The **optimal Neyman-Pearson (NP) detector** is a **LRT**:

$$\Lambda_{NP}(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{E_\beta\left\{p_{\mathbf{z}|H_0}(\mathbf{z}-\beta\mathbf{v}|H_0)\right\}}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = E_\beta\{\Lambda(\mathbf{z};\beta)\} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta}$$

$$SCR(\tau) \triangleq \frac{\sigma_S^2}{\tau}$$

"Local" Signal-to-Clutter Power Ratio

Interestingly, the structure is similar to the previous one, but now $q_1(\mathbf{z})$ is defined in a different way.

$$\int_0^\infty \frac{1}{\tau^M}\left[\frac{1}{\left(1+SCR(\tau)\cdot\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}\right)}\exp\left(-\frac{q_1(\mathbf{z};\tau)}{\tau}\right)-\exp\left(\eta-\frac{q_0(\mathbf{z})}{\tau}\right)\right]p_\tau(\tau)d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

Optimum Detection in Compound-Gaussian Clutter

$$\int_0^\infty \frac{1}{\tau^M}\left[\frac{1}{\left(1+SCR(\tau)\cdot \mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}\right)}\exp\left(-\frac{q_1(\mathbf{z};\tau)}{\tau}\right)-\exp\left(\eta-\frac{q_0(\mathbf{z})}{\tau}\right)\right]p_\tau(\tau)d\tau \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$q_1(\mathbf{z};\tau) \triangleq \mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}-\frac{SCR(\tau)\cdot\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}{1+SCR(\tau)\cdot\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}}, \quad \text{where } SCR(\tau)\triangleq\frac{\sigma_S^2}{\tau}$$

Remarks:

- In Gaussian disturbance, the GLRT (case 3) and the NP (case 2) detectors for 1-D target signals coincide $\rightarrow$ **noncoherent WMF**.
- In compound-Gaussian clutter, the two approaches provide different detection strategies; both are nonlinear and requires heavy numerical integration.
- When $M\rightarrow\infty$, the GLRT tends to the NP for perfectly known target signal.

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2. <u>The Estimator-Correlator (EC) formulation of the optimum detector</u>

- The **optimum NP detector (OD)** and the **GLRT** are difficult to implement in the LRT form, since they require a computational heavy numerical integration.

- The LRT does not give insight that might be used to develop good suboptimum approximations to the OD and GLRT.

- To understand better the operation of the OD and GLRT, reparametrize the conditional Gaussian PDF by setting:

$$\alpha = \frac{1}{\tau}$$

α is the reciprocal of the local clutter power in the range cell under test (CUT).

$$\alpha = \frac{1}{\tau} \quad \Rightarrow \quad p_\alpha(\alpha) = \frac{1}{\alpha^2} p_\tau\left(\frac{1}{\alpha}\right)$$

The Estimator-Correlator Interpretation

- As an example, if the texture τ has an **inverse-Gamma PDF**, α follows a **Gamma PDF** (and viceversa):

$$p_\tau(\tau) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{\eta}\right)^{\lambda}\frac{1}{\tau^{\lambda+1}}e^{-\frac{\lambda}{\eta\tau}}, \quad \tau \geq 0$$

$$\Downarrow$$

$$p_\alpha(\alpha) = \frac{1}{\alpha^2}p_\tau\left(\frac{1}{\alpha}\right) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{\eta}\right)^{\lambda}\alpha^{\lambda-1}e^{-\frac{\lambda}{\eta}\alpha}, \quad \alpha \geq 0$$

λ>0 is the shape parameter and η>0 is the scale parameter.

The Estimator-Correlator Interpretation

- The key to understanding the operation of the OD (and GLRT) is to express it as a function of the **MMSE estimate** of α, instead of τ:

$$\ln \Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} E\{\alpha | x\} dx \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\hat{\alpha}_{MMSE} = E\{\alpha | q_i(\mathbf{z})\}$$

MMSE estimate of α under the hypothesis H_i (i=0,1)

- The proof of this relation can be found in [San94], (see also [San99]).

[San94] Sangston, K. J., and Gerlach, K. Coherent detection of radar targets in a non-Gaussian background. *IEEE Transactions on Aerospace and Electronic Systems,* Vol.30, No.2, Apr. 1994, pp. 330–340.

[San99] Sangston, K. J., Gini, F., Greco, M. V., and Farina, A. Structures for radar detection in compound-Gaussian clutter. *IEEE Transactions on Aerospace and Electronic Systems*, Vol.35, No.2 Apr. 1999, pp. 445–458.

The Estimator-Correlator Interpretation

- In **Gaussian** clutter with power σ_g^2 we have $\alpha = 1/\sigma_g^2$ (deterministic quantity):

$$\ln \Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} \frac{1}{\sigma_g^2} dx = \frac{q_0(\mathbf{z}) - q_1(\mathbf{z})}{\sigma_g^2} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- **Case 1 - Gaussian clutter.**

Perfectly known target signal: $\mathbf{s}_t = \beta\mathbf{v}$ (β and $\mathbf{v}$ a-priori known deterministic)

$$q_0(\mathbf{z}) \triangleq \mathbf{z}^H \mathbf{M}^{-1} \mathbf{z}, \qquad q_1(\mathbf{z}) \triangleq q_0(\mathbf{z} - \mathbf{s}_t) = \mathbf{z}^H \mathbf{M}^{-1} \mathbf{z} - 2\,\mathrm{Re}\left\{\mathbf{s}_t^H \mathbf{M}^{-1} \mathbf{z}\right\} + \mathbf{s}_t^H \mathbf{M}^{-1} \mathbf{s}_t$$

$$\frac{q_0(\mathbf{z}) - q_1(\mathbf{z})}{\sigma_g^2} = \frac{1}{\sigma_g^2} \cdot \left(2\,\mathrm{Re}\left\{\mathbf{s}_t^H \mathbf{M}^{-1} \mathbf{z}\right\} - \mathbf{s}_t^H \mathbf{M}^{-1} \mathbf{s}_t\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \quad \mathrm{Re}\left\{\mathbf{s}_t^H \mathbf{M}^{-1} \mathbf{z}\right\} = \mathrm{Re}\left\{\beta^* \cdot \mathbf{v}^H \mathbf{M}^{-1} \mathbf{z}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- Hence, we obtain the optimal **Coherent Whitening Matched Filter (CWMF)** detector.

- **Case 3 - Gaussian clutter.**

Unknown complex amplitude: β unknown deterministic

$$q_0(\mathbf{z}) \triangleq \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z}, \qquad q_1(\mathbf{z}) \triangleq q_0(\mathbf{z}-\hat{\beta}_{ML}\mathbf{v}) = \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z} - \frac{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v}}$$

$$\frac{q_0(\mathbf{z}) - q_1(\mathbf{z})}{\sigma_g^2} = \frac{1}{\sigma_g^2} \cdot \frac{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- Hence, we obtain that the **GLRT detector** for this problem is the **noncoherent WMF** [which is also optimal in the NP sense when β is a complex Gaussian r.v. (Swerling I target model) and when $|\beta|$ deterministic and $\angle\beta$ uniformly distributed in [0,2π) (Swerling 0/Swerling V target model).]

The Estimator-Correlator Interpretation

- When the clutter is **compound-Gaussian** distributed, the **MMSE estimator** of α may be difficult to implement in a practical detector, e.g. for K-distributed clutter with scale parameter μ and shape parameter ν:

$$\hat{\alpha}_{MMSE,i} = E\{\alpha | q_i(\mathbf{z})\} = \sqrt{\frac{\nu}{\mu q_i(\mathbf{z})}} \cdot \frac{K_{\nu-M-1}\left(\sqrt{\frac{4\nu q_i(\mathbf{z})}{\mu}}\right)}{K_{\nu-M}\left(\sqrt{\frac{4\nu q_i(\mathbf{z})}{\mu}}\right)}, i = 0,1. \qquad \hat{\alpha}_{MMSE,i} \neq \frac{1}{\hat{\tau}_{MMSE,i}}$$

$$\ln \Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} \sqrt{\frac{\nu}{\mu x}} \cdot \frac{K_{\nu-M-1}\left(\sqrt{\frac{4\nu x}{\mu}}\right)}{K_{\nu-M}\left(\sqrt{\frac{4\nu x}{\mu}}\right)} dx \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The Estimator-Correlator Interpretation

$$\ln \Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} E\{\alpha | x\} dx \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- This structure is of the form of an **estimator-correlator** [Schwartz, *IEEE-IT*, 1975].
- It shows that the structure of the OD in compound-Gaussian clutter is the basic detection structure of the OD in Gaussian clutter with the quantity α=1/σ^2, which is known deterministic in the case of Gaussian clutter, replaced by the **MMSE estimate** of the unknown random α.
- Interestingly, the random quantity to be estimated is not the local clutter power τ, but its reciprocal α =1/τ.
- This formulation is also difficult to implement, but theoretically it is very important, also because it suggests that **<u>suboptimum detectors</u>** can be obtained by replacing the **optimum MMSE estimator** with **suboptimum estimators** that are simpler to implement, e.g., the **Maximum A Posteriori (MAP)** or **Maximum Likelihood (ML)**.

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- **3. The Data-Dependent Threshold formulation of the optimum detector**

- **1st step -** Express the PDFs under the two hypotheses as:

$$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i) = \frac{1}{\pi^M |\mathbf{M}|} h_M(q_i(\mathbf{z})),\ i = 0,1$$

where $h_M(q)$ is the nonlinear monotonic decreasing function:

$$h_M(q) \triangleq \int_0^\infty \frac{1}{\tau^M} \exp\left(-\frac{q}{\tau}\right) p_\tau(\tau) d\tau$$

$$\Lambda_{NP}(\mathbf{z}) = \frac{p_{\mathbf{z}|H_1}(\mathbf{z}|H_1)}{p_{\mathbf{z}|H_0}(\mathbf{z}|H_0)} = \frac{h_M(q_1(\mathbf{z}))}{h_M(q_0(\mathbf{z}))} \underset{H_0}{\overset{H_1}{\gtrless}} e^\eta$$

The Data-Dependent Threshold Interpretation

- Note that not always the PDF in the hypothesis H_1 can be expressed in this form, e.g. the optimal NP detector for Swerling I target signal in compound-Gaussian clutter does not allow this formulation, since q_1 is a function of both **z** and τ.

- **2nd step -** Manipolate the LRT as follows.

$$\frac{h_M(q_1(\mathbf{z}))}{h_M(q_0(\mathbf{z}))} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta} \quad \Rightarrow \quad h_M(q_1(\mathbf{z})) \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta} \cdot h_M(q_0(\mathbf{z}))$$

monotonic decreasing function of *q*

$$\Rightarrow \quad q_1(\mathbf{z}) \underset{H_1}{\overset{H_0}{\gtrless}} h_M^{-1}(e^{\eta} \cdot h_M(q_0(\mathbf{z}))), \quad \text{where } h_M^{-1}(\cdot) \text{ is the inverse function of } h_M(\cdot)$$

- **3rd step -** Define the following **data-dependent threshold (DDT)**:

$$f_{opt}(q,\eta) \triangleq q - h_M^{-1}(e^{\eta} h_M(q))$$

$$\Rightarrow \quad q_1(\mathbf{z}) \underset{H_1}{\overset{H_0}{\gtrless}} h_M^{-1}(e^{\eta} \cdot h_M(q_0(\mathbf{z}))) = q_0(\mathbf{z}) - f_{opt}(q_0(\mathbf{z}),\eta)$$

- **4th step -** Finally, we have that the LRT can be recast in the form :

$$\boxed{q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0(\mathbf{z}),\eta)}$$

$$f_{opt}(q,\eta) \triangleq q - h_M^{-1}(e^{\eta} h_M(q))$$

- $f_{opt}(q_0,\eta)$ is the data-dependent threshold, that depends on the data only by means of the quadratic statistic $q_0(\mathbf{z})$.

The Data-Dependent Threshold Interpretation

- In this formulation, the LRT for compound-Gaussian clutter has a similar structure as the LRT in Gaussian disturbance, with the significant difference that now the detection **threshold** is not constant, but it depends on the data through q_0.

$$\text{Gaussian clutter:} \quad q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma_g^2 \eta$$

$$\text{CG clutter:} \quad q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0(\mathbf{z}), \eta)$$

Case 1: Perfectly known target signal $\mathbf{s}_t$. The OD can be interpreted as the coherent WMF (optimal in Gaussian clutter) compared to a **data-dependent threshold (DDT).**

$$2\,\mathrm{Re}\left\{\mathbf{s}_t^H \mathbf{M}^{-1}\mathbf{z}\right\} - \mathbf{s}_t^H \mathbf{M}^{-1}\mathbf{s}_t \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0(\mathbf{z}), \eta)$$

$$\underbrace{\mathrm{Re}\left\{\mathbf{s}_t^H \mathbf{M}^{-1}\mathbf{z}\right\}}_{CWMF} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\frac{1}{2}\left(\mathbf{s}_t^H \mathbf{M}^{-1}\mathbf{s}_t + f_{opt}(q_0(\mathbf{z}), \eta)\right)}_{DDT}$$

The Data-Dependent Threshold Interpretation

Case 3: Target signal with unknown deterministic complex amplitude β. The GLRT detector can be interpreted as the noncoherent WMF (which is the GLRT in Gaussian clutter) compared to the data-dependent threshold.

$$\frac{\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}} \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}\left(q_0(\mathbf{z}),\eta\right)$$

$$\underbrace{\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}_{WMF} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}\cdot f_{opt}\left(q_0(\mathbf{z}),\eta\right)}_{DDT}$$

- ***Example***: **<u>K-distributed clutter</u>**

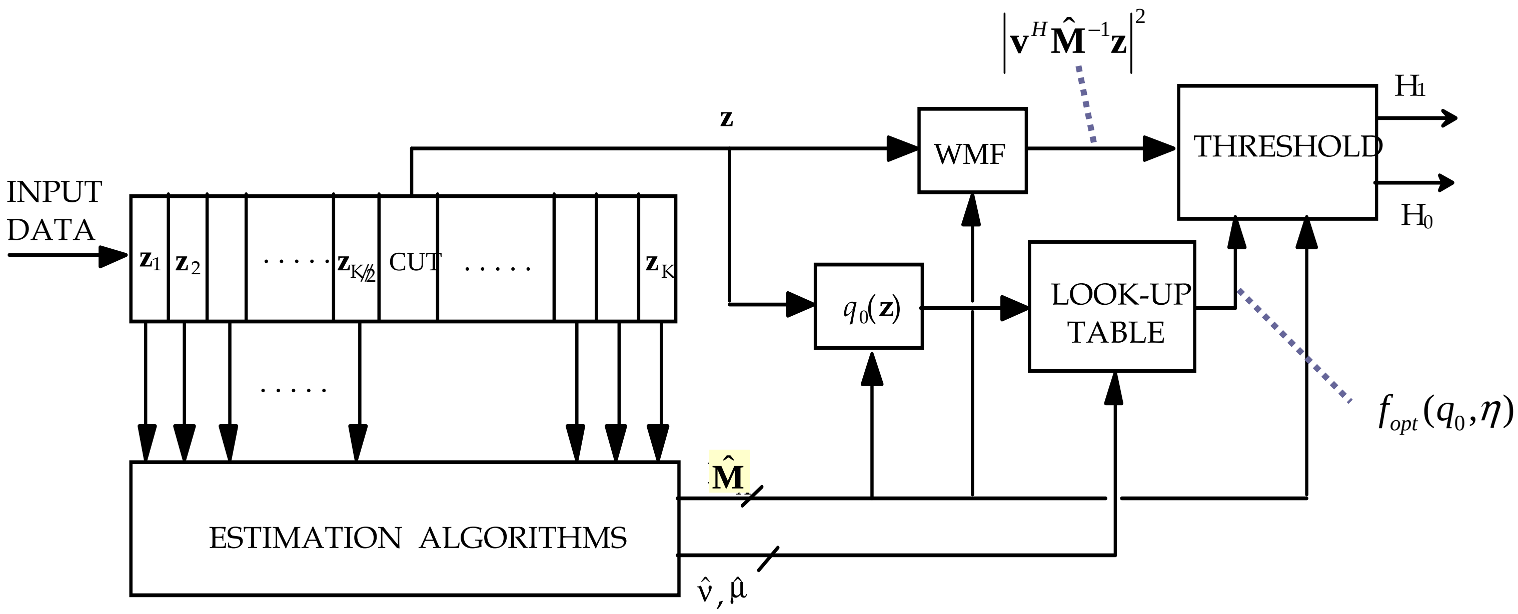
- The texture is modelled as a Gamma r.v. with mean value μ and order parameter ν. For ν–μ=0.5 we have:

$$f_{opt}(q_0,\eta) = q_0 - \left(\sqrt{q_0} - \eta\sqrt{\frac{\mu}{4\nu}}\right)^2 \cdot u\left(\sqrt{q_0} - \eta\sqrt{\frac{\mu}{4\nu}}\right)$$

- In general, it is not possible to find a closed-form expression for the DDT, so it must be calculated numerically.

Canonical Structure of the Optimum Detector

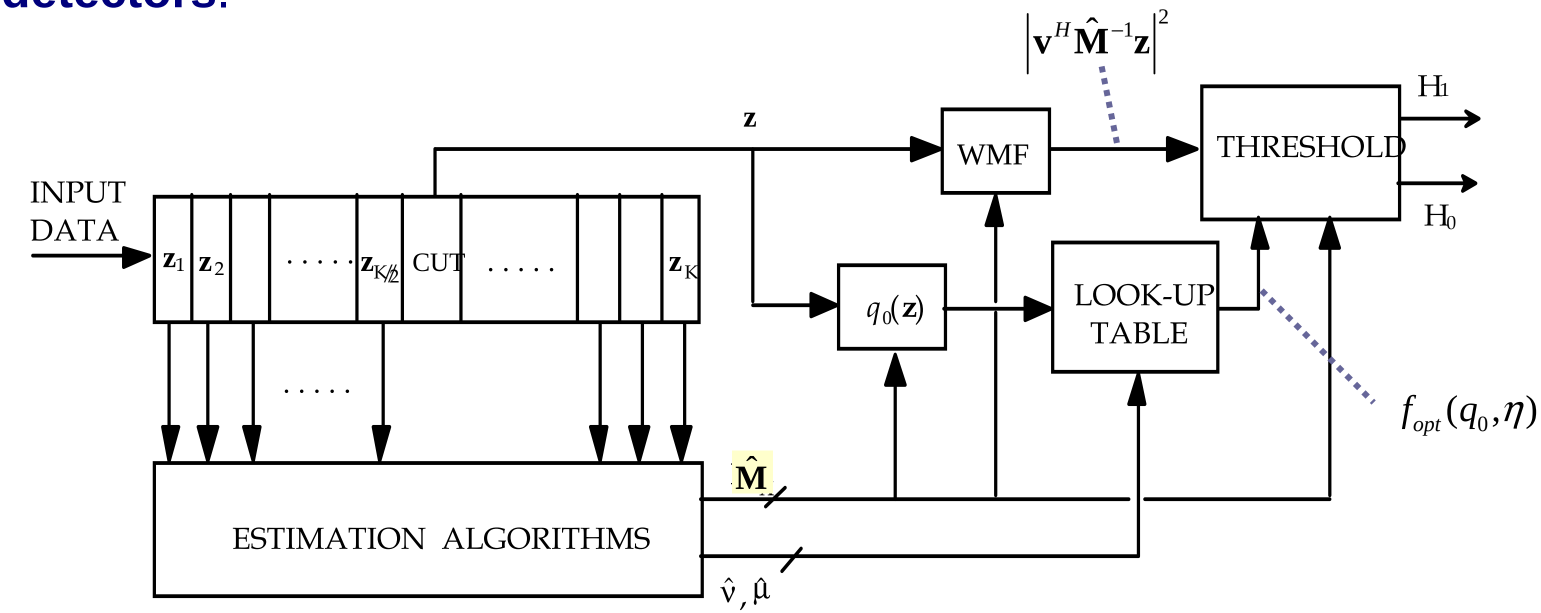
- The DDT formulation of the OD and GLRT is valid for any clutter PDF belonging to the compound-Gaussian family, i.e. it is **canonical**.
- This **canonical structure** suggests a practical way to implement the detector.
- The DDT can be a priori tabulated, with the threshold η set according to the prefixed P_{FA}, and the generated look-up table saved in a memory.



Canonical Structure of the Optimum Detector

- This approach is highly time-saving, it is canonical for every clutter model of the compound-Gaussian family, and it is useful both for practical implementation of the detector and performance analysis by means of Monte Carlo simulation.

- This formulation provides a deeper insight into the operation of the OD and GLRT, and it suggests another approach for deriving good **suboptimum detectors**.



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Suboptimum Detection Structures

1. Suboptimum approximations to the likelihood ratio test (LRT)

- From a physical point of view, the difficulty in utilizing the LRT arises from the fact that the power level τ, associated with the conditionally Gaussian clutter, is unknown and randomly varying: we have to resort to numerical integration.

$$\Lambda_{NP}(\mathbf{z}) = \frac{\int_0^\infty \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_1(\mathbf{z})}{\tau}\right] p_\tau(\tau) d\tau}{\int_0^\infty \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right] p_\tau(\tau) d\tau} = \frac{E_\tau\left\{\frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_1(\mathbf{z})}{\tau}\right]\right\}}{E_\tau\left\{\frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right]\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} e^\eta$$

- The idea is to replace the unknown power level τ with an estimate of it inside the LRT.
- Candidate estimation techniques: MMSE, MAP, ML.
- In general, the simplest to derive and to implement is the ML estimate.

Suboptimum Approximations of the LRT

$$\Lambda_{NP}(\mathbf{z}) = \frac{E_\tau\left\{\frac{1}{(\pi\tau)^M|\mathbf{M}|}\exp\left[-\frac{q_1(\mathbf{z})}{\tau}\right]\right\}}{E_\tau\left\{\frac{1}{(\pi\tau)^M|\mathbf{M}|}\exp\left[-\frac{q_0(\mathbf{z})}{\tau}\right]\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} e^\eta$$

$$\hat{\Lambda}(\mathbf{z}) = \frac{\frac{1}{(\pi\hat{\tau}_1)^M|\mathbf{M}|}\exp\left[-\frac{q_1(\mathbf{z})}{\hat{\tau}_1}\right]}{\frac{1}{(\pi\hat{\tau}_0)^M|\mathbf{M}|}\exp\left[-\frac{q_0(\mathbf{z})}{\hat{\tau}_0}\right]} \underset{H_0}{\overset{H_1}{\gtrless}} e^\eta$$

$$\ln\hat{\Lambda}(\mathbf{z}) = M\ln\left(\frac{\hat{\tau}_0}{\hat{\tau}_1}\right) + \frac{q_0(\mathbf{z})}{\hat{\tau}_0} - \frac{q_1(\mathbf{z})}{\hat{\tau}_1} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- $\hat{\tau}_i$ is an estimate of the local clutter power τ in the CUT under the hypothesis H_i.
- Candidate estimation techniques: MMSE, MAP, ML.
- The simplest is usually the ML.
- The MMSE and MAP estimates of τ were derived in [Sangston, IEEE-AES,1999].

Suboptimum Approximations of the LRT

- The **MMSE estimator** of τ may be difficult to implement in a practical detector, e.g. for K-distributed clutter [Sangston et alii, IEEE-AES, 1999]:

$$\hat{\tau}_{MMSE,i} = E\{\tau | q_i(\mathbf{z})\} = E\{\tau | \mathbf{z}, H_i\} = \int_0^\infty \tau p_{\tau|\mathbf{z},H_i}(\tau|\mathbf{z},H_i)d\tau = \int_0^\infty \tau \frac{p_{\tau,\mathbf{z}|H_i}(\tau,\mathbf{z}|H_i)}{p_{\mathbf{z}|H_i}(\mathbf{z}|H_i)}d\tau$$

$$= \frac{1}{p_{\mathbf{z}|H_i}(\mathbf{z}|H_i)}\int_0^\infty \tau p_{\tau,\mathbf{z}|H_i}(\tau,\mathbf{z}|H_i)d\tau = \frac{1}{p_{\mathbf{z}|H_i}(\mathbf{z}|H_i)}\int_0^\infty \tau p_{\mathbf{z}|\tau,H_i}(\mathbf{z}|\tau,H_i)p_\tau(\tau)d\tau$$

$$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i) = \int_0^\infty \underbrace{\frac{1}{(\pi\tau)^M|\mathbf{M}|}\exp\left[-\frac{q_i(\mathbf{z})}{\tau}\right]}_{p_{\mathbf{z}|\tau,H_i}(\mathbf{z}|\tau,H_i)} p_\tau(\tau)d\tau$$

$$\hat{\tau}_{MMSE,i} = \sqrt{\frac{\mu q_i(\mathbf{z})}{\nu}} \cdot \frac{K_{\nu-M+1}\left(\sqrt{\frac{4\nu q_i(\mathbf{z})}{\mu}}\right)}{K_{\nu-M}\left(\sqrt{\frac{4\nu q_i(\mathbf{z})}{\mu}}\right)}, i = 0,1$$

- The **ML estimator** of τ is easier to derive and simpler to implement:

$$\hat{\tau}_{ML,i} = \arg\max_{\tau} p_{\mathbf{z}|\tau,H_i}(\mathbf{z}|\tau,H_i) = \arg\max_{\tau}\left\{\ln p_{\mathbf{z}|\tau,H_i}(\mathbf{z}|\tau,H_i)\right\}$$

$$= \arg\max_{\tau}\left\{-M\ln\pi - \ln|\mathbf{M}| - M\ln\tau - \frac{q_i(\mathbf{z})}{\tau}\right\}, \quad i=0,1$$

$$\frac{\partial}{\partial\tau}\left\{-M\ln\tau - \frac{q_i(\mathbf{z})}{\tau}\right\} = -\frac{M}{\tau} + \frac{q_i(\mathbf{z})}{\tau^2} = 0 \quad \Rightarrow \quad \hat{\tau}_{ML,i} = \frac{q_i(\mathbf{z})}{M}, \; i=0,1$$

$$\hat{\tau}_{ML,0} = \frac{\mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}}{M}$$

$$\ln\hat{\Lambda}(\mathbf{z}) = M\ln\left(\frac{\hat{\tau}_0}{\hat{\tau}_1}\right) + \frac{q_0(\mathbf{z})}{\hat{\tau}_0} - \frac{q_1(\mathbf{z})}{\hat{\tau}_1} = M\ln\left(\frac{q_0(\mathbf{z})}{q_1(\mathbf{z})}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\frac{q_0(\mathbf{z})}{q_1(\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta/M} \quad \Rightarrow \quad \boxed{q_0(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} q_1(\mathbf{z})e^{\eta/M}}$$

Suboptimum Approximations of the LRT

- **Case 3.**

Unknown complex amplitude: β unknown deterministic

$$q_0(\mathbf{z}) \triangleq \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z}, \qquad q_1(\mathbf{z}) \triangleq q_0(\mathbf{z} - \hat{\beta}_{ML}\mathbf{v}) = \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z} - \frac{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v}}$$

$$\mathbf{z}^H \mathbf{M}^{-1}\mathbf{z} \underset{H_0}{\overset{H_1}{\gtrless}} \mathbf{z}^H \mathbf{M}^{-1}\mathbf{z} \cdot e^{\eta/M} - \frac{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v}} e^{\eta/M}$$

- It was called the GLRT-LQ (or NMF) [Gini, IEE-F, 1997]

$$\frac{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}{(\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v})(\mathbf{z}^H \mathbf{M}^{-1}\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \left(1 - e^{-\eta/M}\right) \quad \Rightarrow \quad \underbrace{\left|\mathbf{v}^H \mathbf{M}^{-1}\mathbf{z}\right|^2}_{WMF} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\mathbf{v}^H \mathbf{M}^{-1}\mathbf{v} \cdot f_{ML}(q_0(\mathbf{z}), \eta)}_{DDT}$$

where $\quad f_{ML}(q_0, \eta) \triangleq q_0\left(1 - e^{-\eta/M}\right) \neq f_{opt}(q_0, \eta)$

- It is in the canonical form, with the adaptive threshold that is a **linear** function of q_0.

Suboptimum Approximations of the LRT

Linear-Quadratic GLRT (GLRT-LQ), a.k.a. Normalized Matched Filter (NMF)

$$\frac{\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}{(\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v})(\mathbf{z}^H\mathbf{M}^{-1}\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} \left(1-e^{-\eta/M}\right)$$

$$P_{FA} = e^{-\frac{\eta(M-1)}{M}}$$

$$P_D = \int_0^\infty \left(1+\frac{\tau\left(e^{\eta/M}-1\right)}{\tau+M\mu\bar{\gamma}}\right)^{-(M-1)} p_\tau(\tau)d\tau$$

$$\mu = E\{\tau\}$$

$$\bar{\gamma} = \frac{\sigma_S^2}{\mu}\cdot\frac{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}}{M}$$ is the *SCR* at the output of the WMF, divided by *M*.

- This detector is quite simple to implement.
- It has the **constant false alarm rate (CFAR)** property with respect to the clutter PDF, i.e. w.r.t. the texture PDF, $p_\tau(\tau)$.
- The same decision strategy was also obtained, under different assumptions, by Korado [Kor68], Picinbono and Vezzosi [Pic70], Scharf and Lytle [Sch71], Conte *et al.* [Con95].

Suboptimum approximations to the EC structure

$$\ln \Lambda(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} E\{\alpha | x\} dx \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The **MMSE estimator** of $a=1/\tau$ is also usually difficult to implement in a practical detector, e.g. for K-distributed clutter [San99]:

$$\hat{\alpha}_{MMSE,i} = E\{\alpha | q_i(\mathbf{z})\} = \sqrt{\frac{\nu}{\mu q_i(\mathbf{z})}} \cdot \frac{K_{\nu-M-1}\left(\sqrt{\frac{4\nu q_i(\mathbf{z})}{\mu}}\right)}{K_{\nu-M}\left(\sqrt{\frac{4\nu q_i(\mathbf{z})}{\mu}}\right)}, \ i = 0,1$$

$$\hat{\alpha}_{MMSE,i} \neq \frac{1}{\hat{\tau}_{MMSE,i}}$$

[San99] Sangston K.J., Gini F., Greco M.V., Farina A., "Structures for radar detection in compound-Gaussian clutter," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 35, No. 2, pp. 445-458, April 1999.

Suboptimum approximations to the EC structure

- Suboptimum detectors may be obtained by replacing the **MMSE estimator** of α with a **suboptimal estimate** (**MAP** or **ML**), e.g. for K-distributed clutter :

$$\hat{\alpha}_{MAP,i} = \frac{M-\nu-1+\sqrt{(M-\nu-1)^2+\dfrac{4\nu q_i(\mathbf{z})}{\mu}}}{2q_i(\mathbf{z})} \qquad \hat{\alpha}_{MAP,i} \neq \frac{1}{\hat{\tau}_{MAP,i}}$$

$$\hat{\alpha}_{ML,i} = \frac{M}{q_i(\mathbf{z})} = \frac{1}{\hat{\tau}_{ML,i}}, \qquad i=0,1$$

$$\ln\hat{\Lambda}(\mathbf{z}) = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} \hat{\alpha}_{ML}(x)dx = \int_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} \frac{M}{x}dx = M\ln(x)\Big|_{q_1(\mathbf{z})}^{q_0(\mathbf{z})} = M\ln\left(\frac{q_0(\mathbf{z})}{q_1(\mathbf{z})}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$\Rightarrow \quad \frac{q_0(\mathbf{z})}{q_1(\mathbf{z})} \underset{H_0}{\overset{H_1}{\gtrless}} e^{\eta/M} \quad \Rightarrow \quad q_0(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} q_1(\mathbf{z})e^{\eta/M} \quad \Rightarrow \quad \text{NMF}$$

Suboptimum approximations to the EC structure

- As the number *M* of temporal samples becomes asymptoticly large, the ML, MAP and MMSE tend to coincide.

$$\hat{\alpha}_{MMSE,i} \xrightarrow{M \gg 1} \frac{M-\nu}{q_i(\mathbf{z})} \cong \frac{M}{q_i(\mathbf{z})} = \hat{\alpha}_{ML,i} = \frac{1}{\hat{\tau}_{ML,i}}, \; i = 0,1$$

$$\hat{\alpha}_{MAP,i} \xrightarrow{M \gg 1} \frac{M-\nu-1}{q_i(\mathbf{z})} + \frac{\nu}{\mu(M-\nu-1)} \cong \frac{M}{q_i(\mathbf{z})} = \hat{\alpha}_{ML,i}$$

$$or \quad \lim_{M\to\infty} \frac{\hat{\alpha}_{ML,i}}{\hat{\alpha}_{MMSE,i}} = \lim_{M\to\infty} \frac{\hat{\alpha}_{ML,i}}{\hat{\alpha}_{MAP,i}} = 1$$

- Hence, the **NMF** is **asymptotically optimum**, whatever the clutter PDF is, provided it belongs to the family of compound-Gaussian distributions.

Suboptimum approximations to the DDT structure

- The canonical structure of the GLRT (unknown target complex amplitude) is in the form of a noncoherent WMF compared to a data dependent threshold:

$$\underbrace{\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}_{WMF} \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}\cdot f_{opt}(q_0(\mathbf{z}),\eta)}_{DDT}$$

$$\left.\begin{aligned} f_{opt}(q,\eta) &\triangleq q - h_M^{-1}(e^{\eta}h_M(q)) \\ h_M(q) &\triangleq \int_0^{\infty}\frac{1}{\tau^M}\exp\left(-\frac{q}{\tau}\right)p_\tau(\tau)d\tau \end{aligned}\right\}$$

The threshold $f_{opt}(q_0,\eta)$ depends in a complicated **nonlinear** fashion on the quadratic statistic $q_0(\mathbf{z})$.

- In the NMF the optimal nonlinear threshold is replaced by a suboptimal threshold that is a linear function of $q_0(\mathbf{z})$:

$$f_{ML}(q_0,\eta) \triangleq q_0\left(1-e^{-\eta/M}\right) \neq f_{opt}(q_0,\eta)$$

Suboptimum approximations to the DDT structure

- The **idea** is to find a good approximation of the otpimal threshold $f_{opt}(q_0,\eta)$ that is easy to implement. In this way, we avoid the need of saving a look-up table in the receiver memory.

- The approximation need to be good only for those values of $q_0(\mathbf{z})$ that have significant probability of occurrence.

- We look for the best *K-th* order **polynomial** approximation in the MMSE sense:

$$f_K(q_0,\eta)=\sum_{k=0}^{K} c_k q_0^k$$

$f_K(q_0,\eta)$ is easy to compute from $q_0(\mathbf{z})$

$$\min_{\{c_k\}} E\left\{\left| f_{opt}(q_0,\eta)-\sum_{k=0}^{K} c_k q_0^k \right|^2\right\}$$

- Note that the problem is linear w.r.t. to the K+1 unknown coefficients c_k, hence easy to solve.

Suboptimum approximations to the DDT structure

***Example*: K-distributed clutter.** The solution can be derived in closed-form

- First order (linear) approximation: $f_1(q_0,\eta) = c_0 + c_1 q_0$
- The solution is obtained by solving a (K+1)-*th* order linear system.
- For K=1:

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 & E\{q_0\} \\ E\{q_0\} & E\{q_0^2\} \end{bmatrix}^{-1} \begin{bmatrix} E\{f_{opt}(q_0,\eta)\} \\ E\{q_0 f_{opt}(q_0,\eta)\} \end{bmatrix}$$

For ν-M=0.5, the MMSE solution is:

$$c_0 = \frac{\eta\mu}{4\nu}\left(\frac{8\nu^2-2}{4\nu+1} - \eta\right), \qquad c_1 = \frac{\eta}{4\nu+1}$$

- Note that when $\nu \to \infty$ (Gaussian clutter), we have c_1=0, so the threshold becomes constant and we get the conventional WMF.

- The NMF is obtained as a special case of $f_1(q_0, \eta)$ for c_0=0 and $c_1=1-e^{-\eta/M}$.

- For the 1st (linear) and 2nd-order (quadratic) approximations:

$$c_0 \propto \mu, c_1 \text{ independent of } \mu, c_2 \propto 1/\mu$$

all c_k's are independent of the normalized clutter covariance matrix **M**.

More details in:

[Gini99] Gini F., Greco M.V., Farina A., "Clairvoyant and adaptive signal detection in non-Gaussian clutter: A data-dependent threshold interpretation," *IEEE Trans. on Signal Processing*, Vol. 47, No. 6, pp. 1522-1531, June 1999.

Suboptimum approximations to the DDT structure

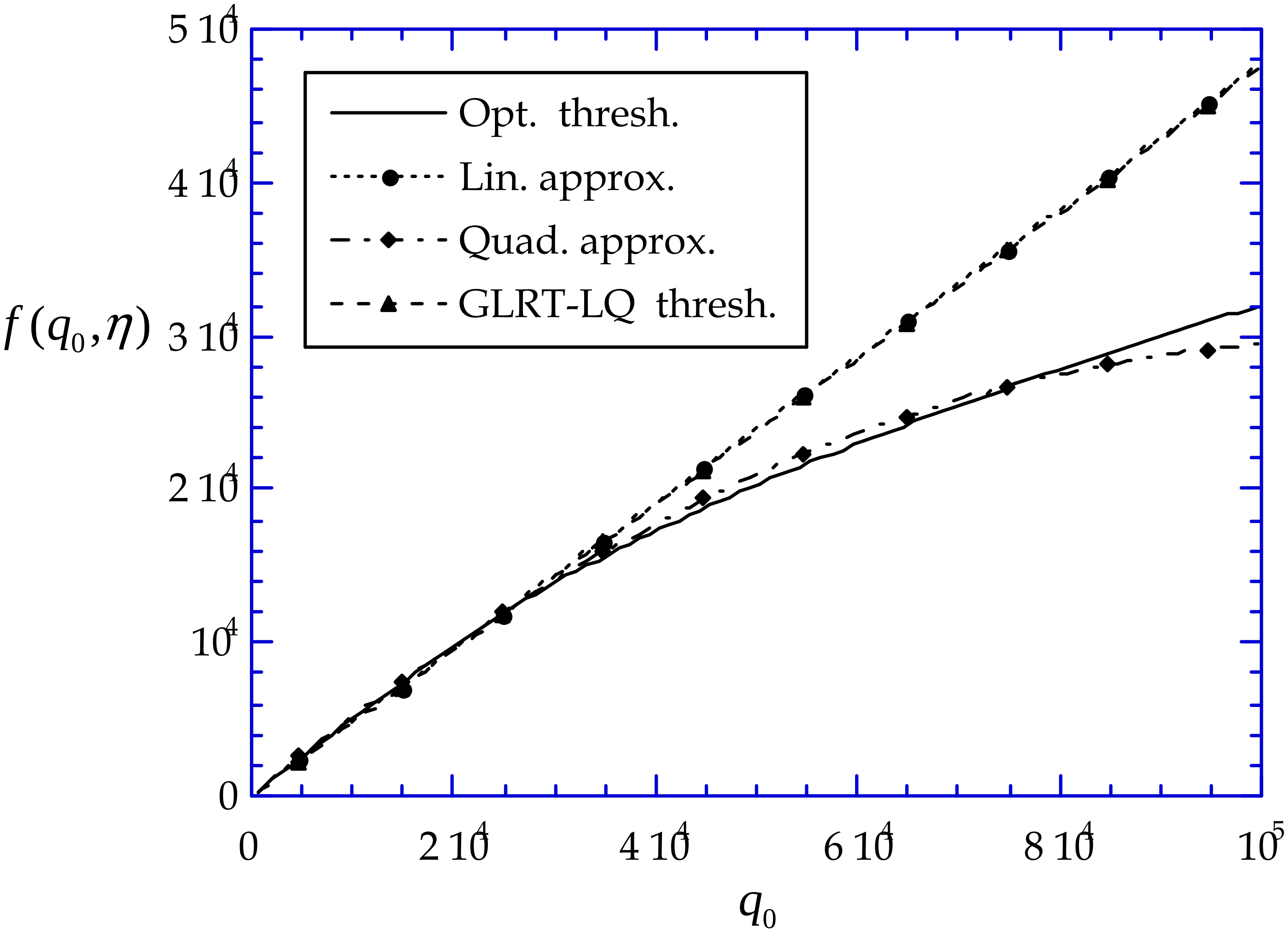
- It is important to know the spread in values of $q_0(\mathbf{z})$ that we expect under H_0 and H_1.
- If we want to approximate $f(q_0, \eta)$, we need to know the range of values over which we require a good approximation.
- The approximation must be good under both the hypotheses.

***Example*: <u>K-distributed clutter</u>**

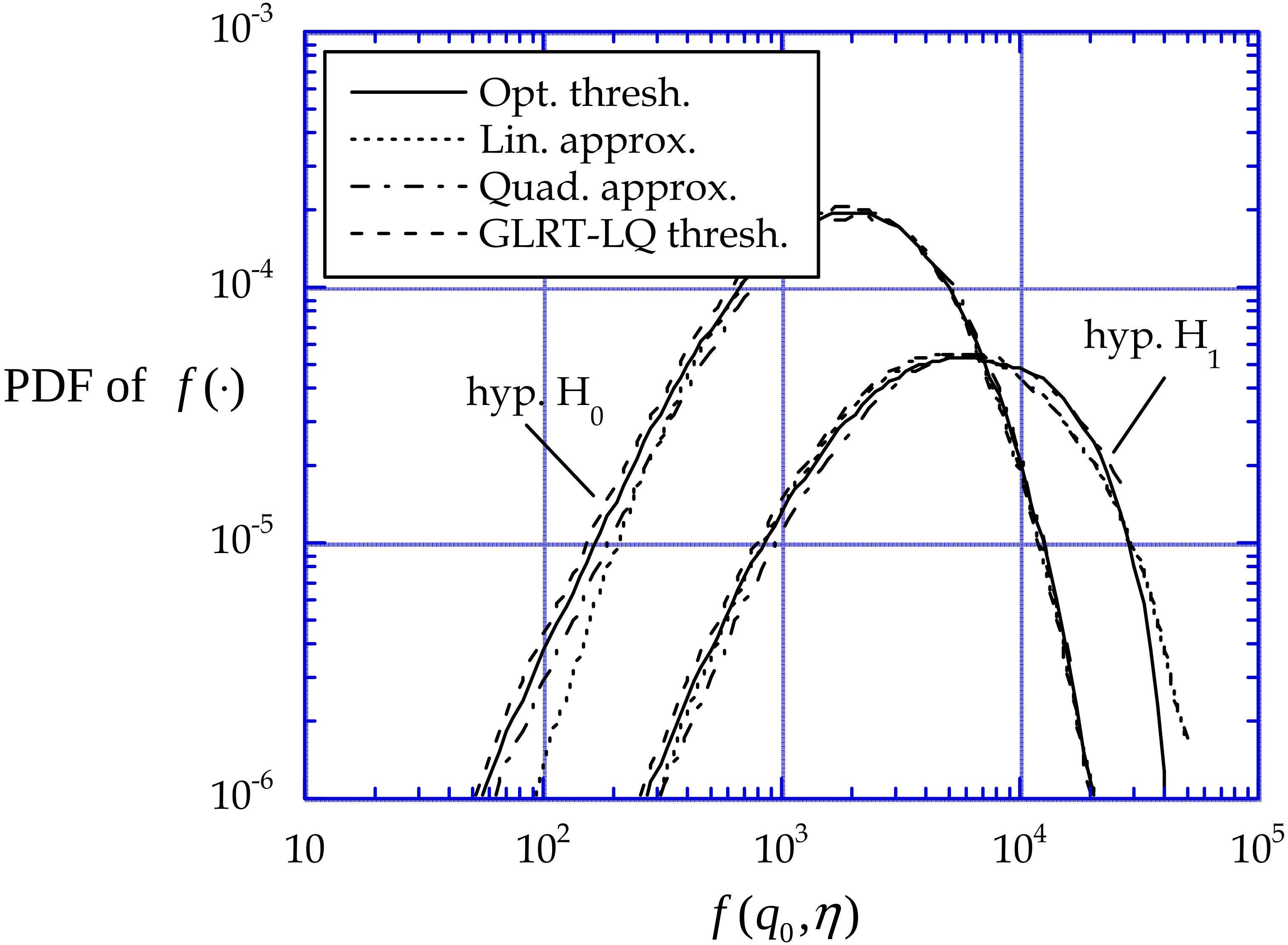
$$P_{FA} = 10^{-5}, M = 4, \nu = 4.5, \mu = 10^3, SCR = -4dB, \text{SW-I target}$$

$$\text{Signal-to-Clutter Power Ratio: } SCR = \frac{\sigma_S^2}{E\{\tau\}} = \frac{\sigma_S^2}{\mu}$$

Suboptimum approximations to the DDT structure



Suboptimum approximations to the DDT structure



Outline of the Section on "Optimum Detection"

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- Target models and detectors in Gaussian clutter
- Optimum coherent detection in compound-Gaussian clutter
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- **Performance analysis in K-distributed clutter**
- The Linear-Threshold Detector (LTD) and the IG-texture model
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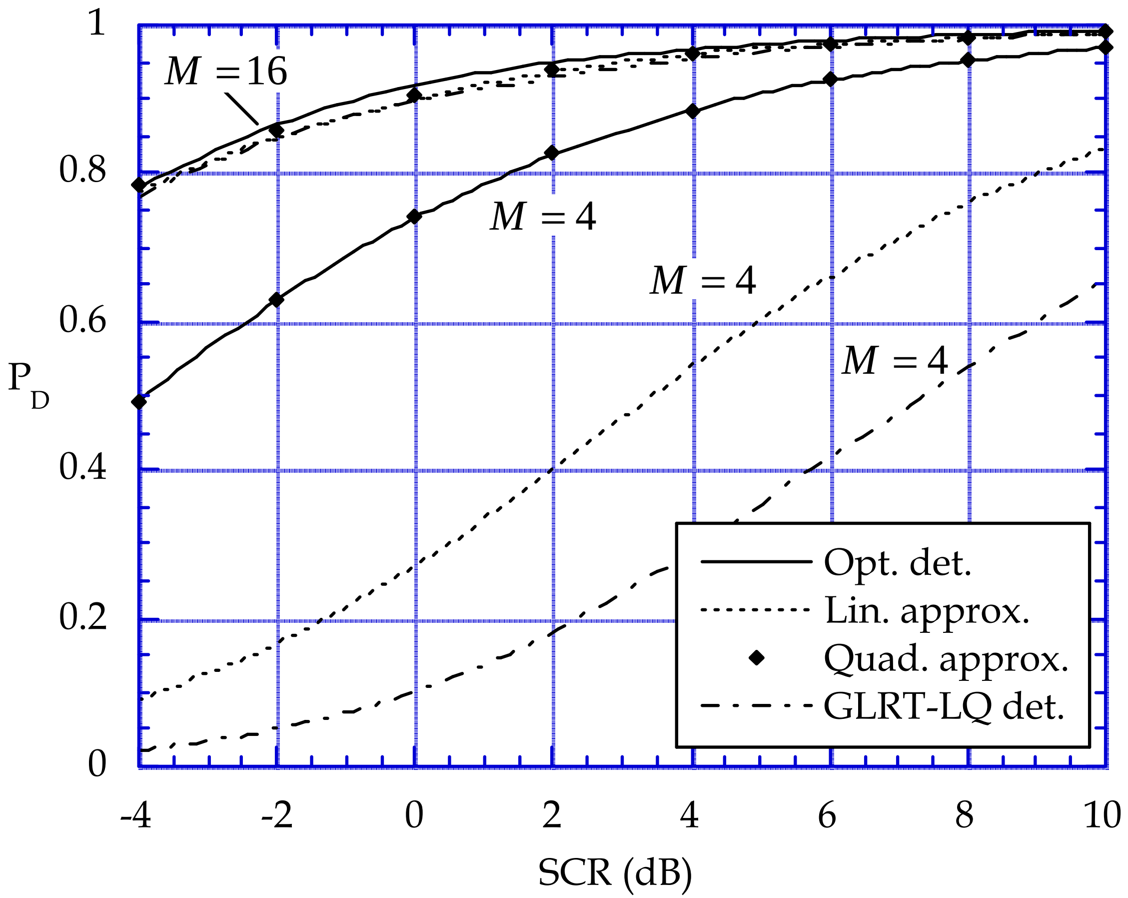
Performance Analysis

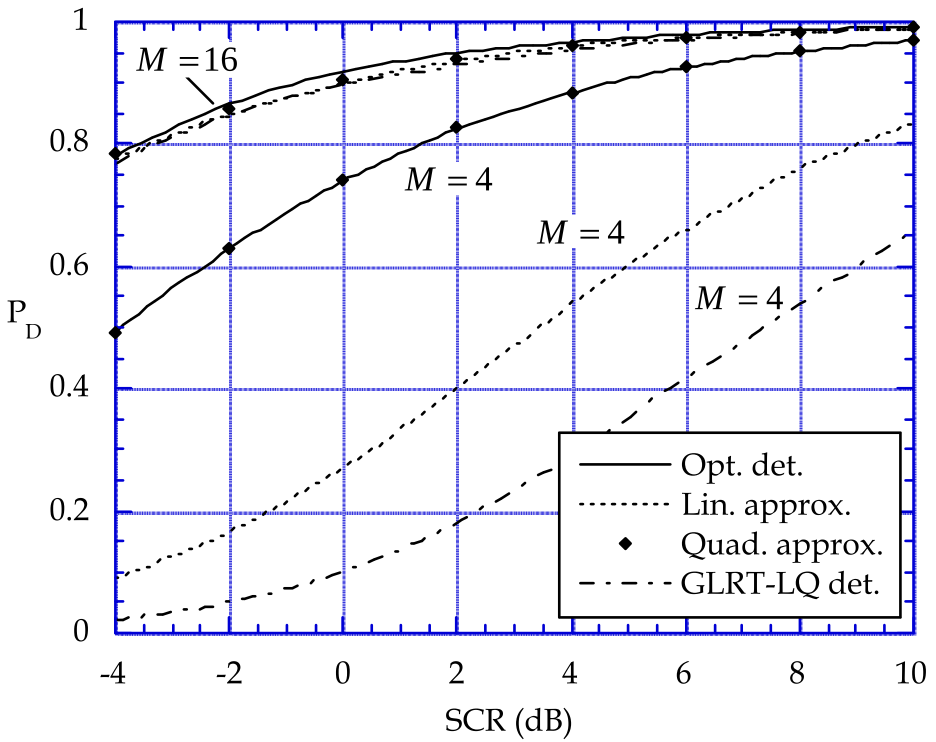
Performance Analysis: Swerling-I target, K-distributed clutter

$$P_{FA} = 10^{-5}, \nu_d = 0.5, \nu = 4.5, \mu = 10^3, \rho_x = 0.9\,[\text{Clutter} \equiv AR(1)]$$



- In all the cases we examined, the suboptimum detector based on the quadratic (2nd-order) approximation has performance (P_D) almost indistinguishable from that of the optimal NP detector.

Performance Analysis in K-Distributed Clutter

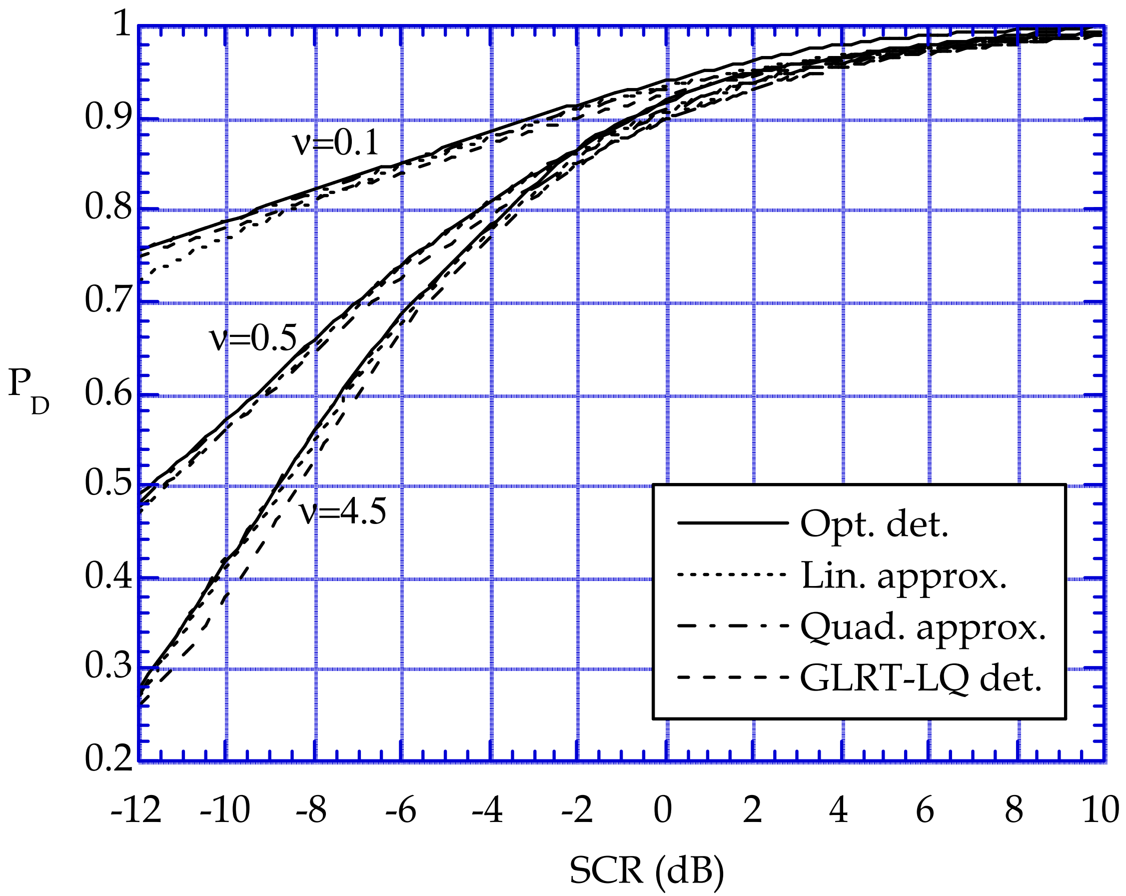


$$P_{FA} = 10^{-5}, \nu_d = 0.5, \nu = 4.5,$$
$$\mu = 10^3, \rho_x = 0.9\,[\text{Clutter} \equiv AR(1)]$$

- The detector based on 2nd-order approximation represents a good trade-off between performance and computational complexity.
- It requires knowledge of the clutter texture parameters (ν and μ).
- As the number *M* of integrated pulses increases, the detection performance of the GLRT-LQ (NMF) approaches that of the optimal NP detector.
- The NMF does not require a priori knowledge of the texture PDF, i.e. of its parameters ν and μ.
- The NMF is CFAR w.r.t. texture PDF.

Performance Analysis in K-Distributed Clutter

Swerling-I target: it was observed that in this case, P_D increases much more slowly as a function of SCR than in the case of **known target signal.**

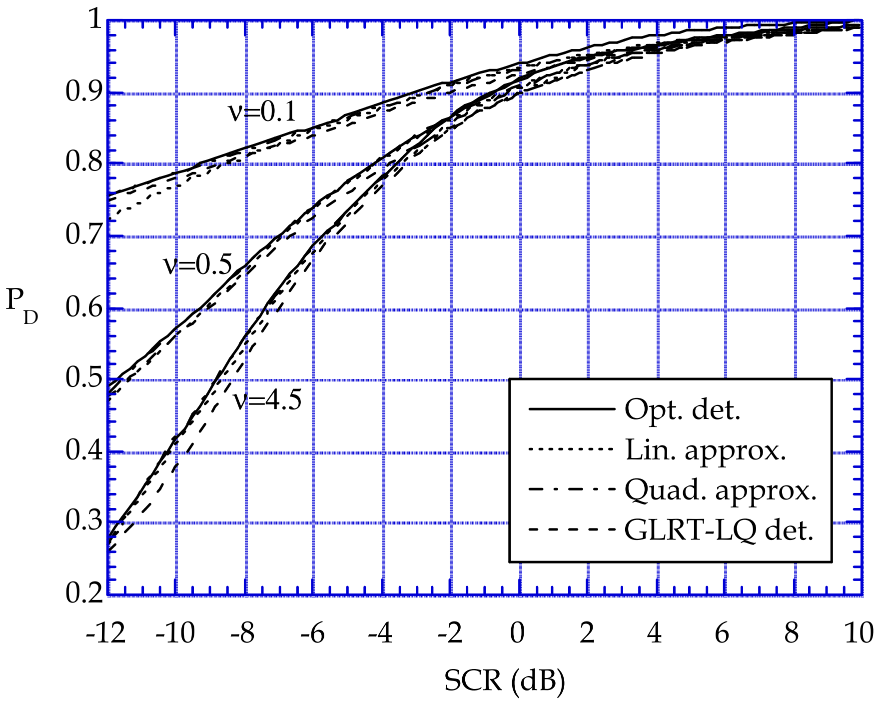


$P_{FA} = 10^{-5}, \nu_d = 0.5,$

$M = 16, \mu = 10^3,$

$\rho_x = 0.9 \; AR(1)$

Performance Analysis in K-Distributed Clutter

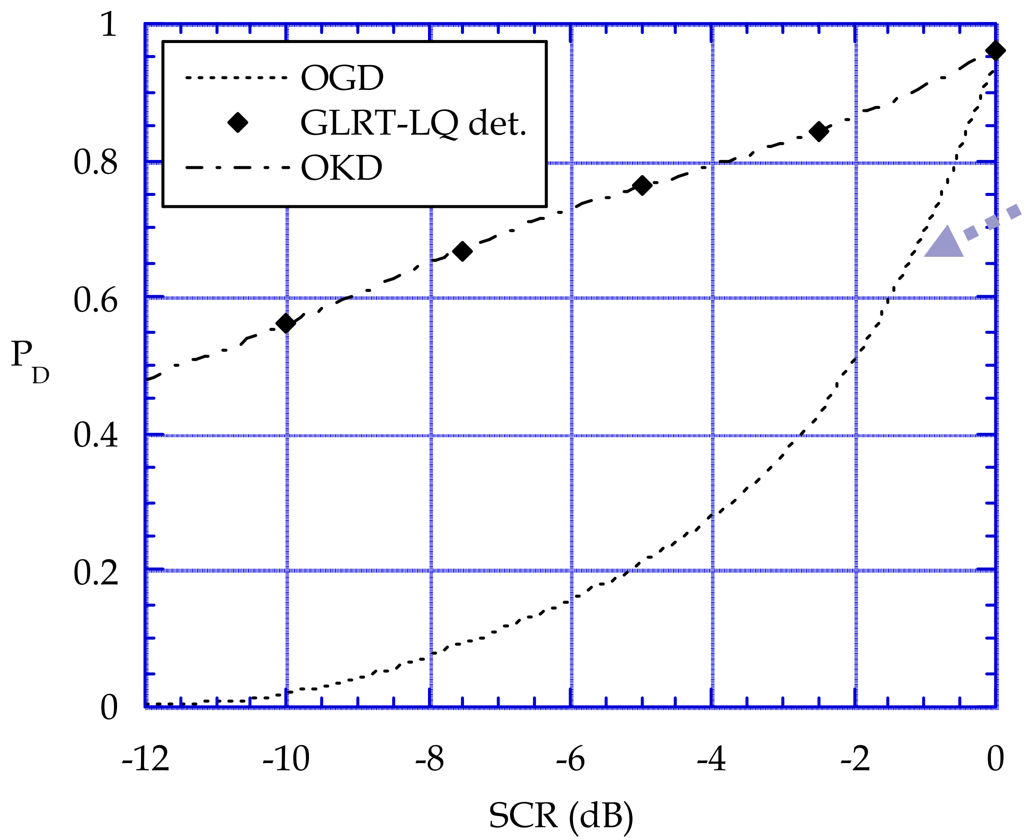


$$P_{FA} = 10^{-5}, \nu_d = 0.5,$$
$$M = 16, \mu = 10^3,$$
$$\rho_x = 0.9 \; AR(1)$$

- Clutter spikiness (modeleld by ν) heavily affects detection performance.
- ν=0.1 means very spiky clutter (heavy tailed clutter).
- ν=4.5 means almost Gaussian clutter.
- Up to high values of *SCR*, the best detection performance is obtained for spiky clutter (small values of ν), i.e. it is more difficult to detect weak targets in Gaussian clutter rather than in spiky *K*-distributed clutter, (provided that the optimal decision strategy is adopted).

Performance Analysis in K-Distributed Clutter

- Optimum Detector in K-clutter (OKD).
- Optimum Detector in Gaussian clutter (OGD) = Whitening Matched Filter (WMF).
- GLRT-LQ = NMF



This figure shows how a wrong assumption on clutter model (model mismatching) affects detection performance

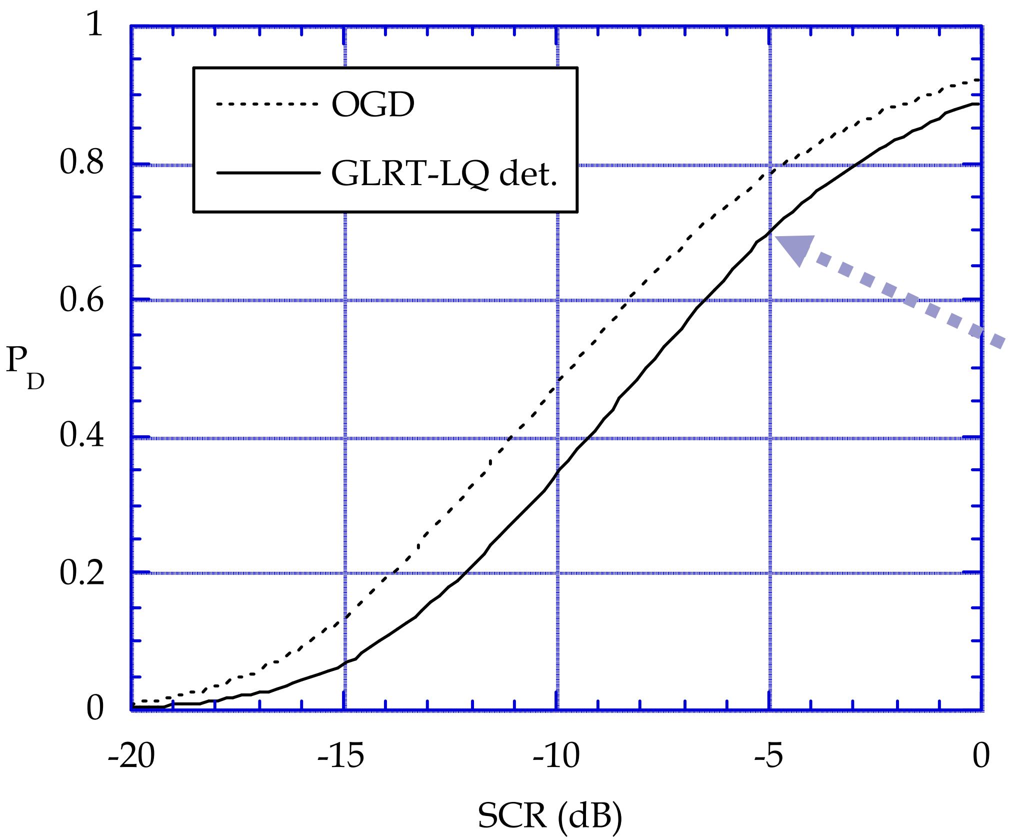
$P_{FA} = 10^{-5}, \nu_d = 0.5, M = 16,$

$\mu = 10^3, \rho_x = 0.9\ AR(1)$

$\nu = 0.5$ [spiky clutter]

Performance Analysis in K-Distributed Clutter

- Swerling I target.
- GLRT Detector in compound-Gaussian clutter (GLRT-LQ) = NMF
- Optimum Detector in Gaussian clutter (OGD) = Whitening Matched Filter (WMF).

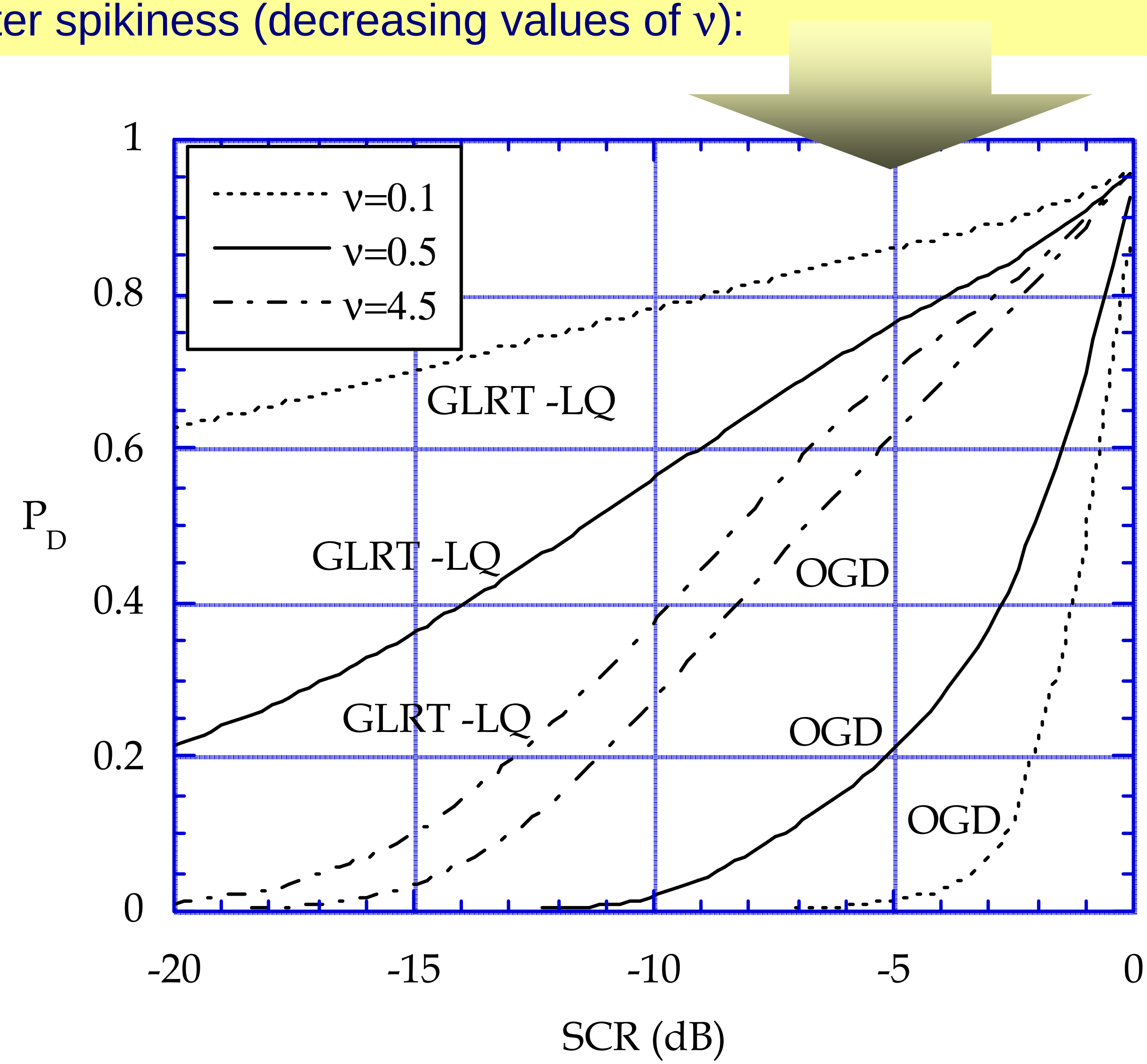


$$P_{FA} = 10^{-5}, \nu_d = 0.5, M = 16,$$
$$\mu = 10^3, \rho_x = 0.9 \; AR(1)$$
$$\nu \to \infty \text{ [Gaussian clutter]}$$

- This figure shows the loss of the NMF w.r.t. to the optimum WMF.
- However, it is worth to remember that the NMF is CFAR, whereas the WMF is not. So, basically the figure shows the CFAR loss of the NMF in Gaussian clutter.

Performance Analysis in K-Distributed Clutter

- The gain of the NMF over the mismatched OGD increases with clutter spikiness (decreasing values of ν):

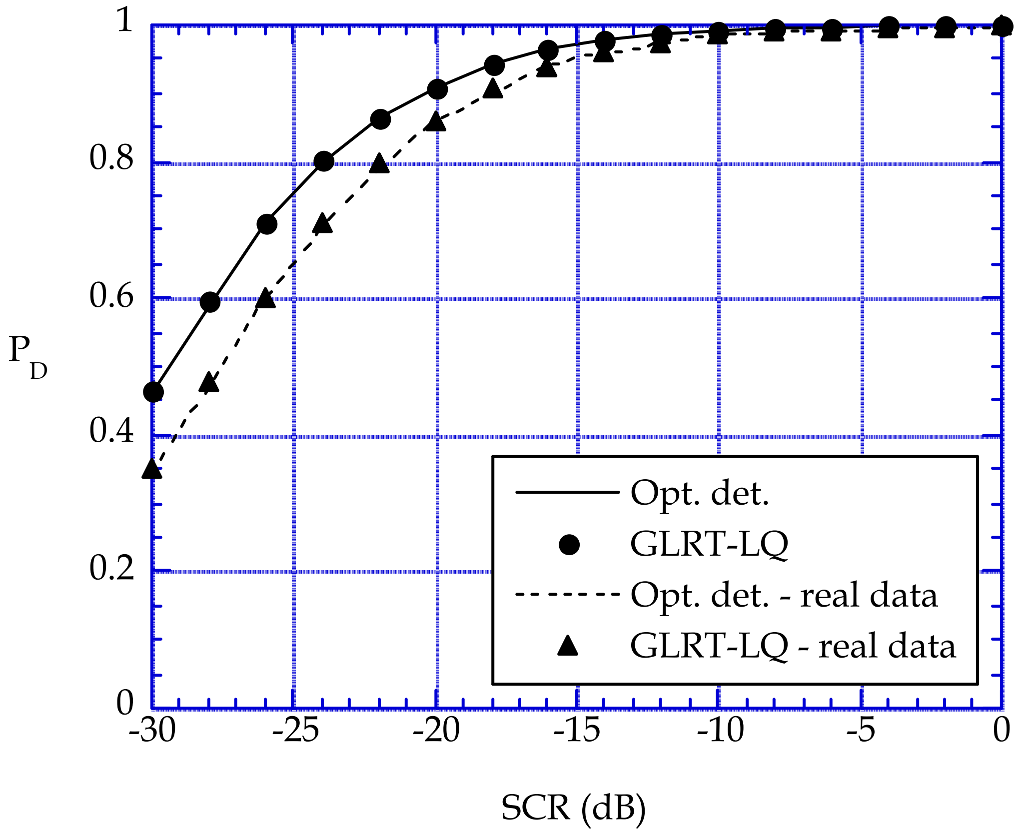


$P_{FA} = 10^{-5}$,
$\nu_d = 0.5$,
$M = 16$,
$\mu = 10^3$,
$\rho_x = 0.9$

Performance Analysis in Real Sea Clutter

- Performance prediction have been checked with real sea clutter data.

- The detectors make use of the knowledge of μ, ν, and **M** (obtained from the entire set of data, adaptive version of these detectors will be derived and investigated later on).



$$P_{FA} = 10^{-5}, \nu_d = 0.5, M = 16,$$
$$\bar{\mu} = 4.44 \cdot 10^{-3}, \bar{\nu} = 1.25,$$
$$\bar{\rho}_x = 0.93 - j0.24$$

- Differences between performance prediction based on simulated data and based on real data are very likely due to clutter non-homogeneity.

Performance Analysis: Some Remarks

- We have examined the problem of optimal and suboptimal target detection in Gaussian clutter and non-Gaussian clutter modelled by the **compound-Gaussian** distribution.
- Three interpretations of the optimum detector (OD) have been provided:
  - the likelihood ratio test (LRT),
  - the estimator-correlator (EC),
  - the whitening matched filter (WMF) + data-dependent threshold.
- With these reformulations of the OD, the problem of obtaining suboptimal detectors has been approached by either:
- (1) approximating the LRT directly, utilizing an estimate of τ,
- (2) utilizing a suboptimal estimate of $\alpha = 1/\tau$ in the estimator-correlator structure,
- (3) utilizing a suboptimal function to model the data-dependent threshold in the WMF interpretation.
- Numerical results suggest that the NMF [(1),(2)] and the suboptimal detector based on quadratic approximation of the data-dependent threshold [(3)] <u>represent the best trade-off between implementation complexity and detection performance.</u>

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More on the Data-Dependent Threshold Structure

- We have seen that the LRT can be recast in the form:

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} f_{opt}(q_0,\eta)$$

The left-hand term depends on the target signal model

$$f_{opt}(q_0,\eta) = q_0 - h_M^{-1}(e^{\eta} h_M(q_0))$$

$f_{opt}(q_0,\eta)$ is the data-dependent threshold, that depends on the data only by means of the quadratic statistic:

$$q_0(\mathbf{z}) = \mathbf{z}^H \mathbf{M}^{-1} \mathbf{z}$$

$h_M(q)$ is a nonlinear monotonic decreasing function, which depends on the texture PDF:

$$h_M(q) = \int_0^\infty \frac{1}{\tau^M} \exp\left(-\frac{q}{\tau}\right) p_\tau(\tau) d\tau = \int_0^\infty \alpha^M \exp(-\alpha q)\, p_\alpha(\alpha) d\alpha$$

Reparametrize in terms of the inverse of the local clutter power:

$$\alpha = \frac{1}{\tau} \Rightarrow p_\alpha(\alpha) = \frac{1}{\alpha^2} p_\tau\left(\frac{1}{\alpha}\right)$$

The DDT structure: Gaussian Clutter

- In Gaussian disturbance with power σ_g^2 we have $\alpha = 1/\sigma_g^2$ (deterministic):

$$p_\alpha(\alpha) = \delta\left(\alpha - \frac{1}{\sigma_g^2}\right) \Rightarrow f_{opt}(q_0, \eta) = \sigma_g^2 \eta$$

$$\text{Gaussian clutter:} \quad q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma_g^2 \eta$$

$$\mathbf{s}_t = \beta \mathbf{v}$$

Case 1. Perfectly known signal,
β a-priori known: $q_0(\mathbf{z}) - q_1(\mathbf{z}) \equiv \mathrm{Re}\{\mathbf{s}_t^H \mathbf{M}^{-1} \mathbf{z}\}$

Case 2. Constant complex amplitude,
β complex Gaussian r.v., Swerling-I target: $q_0(\mathbf{z}) - q_1(\mathbf{z}) \equiv \left|\mathbf{v}^H \mathbf{M}^{-1} \mathbf{z}\right|^2$

Case 3. Constant complex amplitude,
β unknown deterministic: $q_0(\mathbf{z}) - q_1(\mathbf{z}) \equiv \left|\mathbf{v}^H \mathbf{M}^{-1} \mathbf{z}\right|^2$

The Linear-Threshold Detector

- The simplest extension beyond the Gaussian case would appear to be a linear form for $f_{opt}(q_0,\eta)$:

$$f_{opt}(q_0,\eta) = c_0 + c_1 \cdot q_0$$

- A detector with this form of the DDT is called **linear-threshold detector (LTD)**.

- The LTD was originally explored in [Sangston,1999] as an approximation to $f_{opt}(q_0,\eta)$: for the case of K-distributed compound-Gaussian clutter, and then further investigated in [Gini,1999] and [Gini,2002]. We described two different suboptimal linear approximations.

- Now, we ask a different question about the LTD:

Is there a compound-Gaussian model for which the linear-threshold detector is the optimum detector?

The Linear-Threshold Detector

- If the answer is yes, the multivariate compound-Gaussian clutter model, which is defined through the function $h_M(q_0)$ must satisfy:

$$q_0 - h_M^{-1}(e^{\eta} h_M(q_0)) = c_0 + c_1 \cdot q_0$$

- This problem can be reformulated as an **Abel problem,** which has a unique solution:

$$c_0 = \frac{\lambda}{\mu}\left(1 - e^{-\frac{\eta}{M+\lambda}}\right), \quad c_1 = 1 - e^{-\frac{\eta}{M+\lambda}}$$

- The solution provides a Gamma PDF for α:

$$p_\alpha(\alpha) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{\mu}\right)^{\lambda} \alpha^{\lambda-1} e^{-\frac{\lambda}{\mu}\alpha}, \quad \alpha \geq 0$$

- λ>0 is the shape parameter and μ>0 is the scale parameter.

The Linear-Threshold Detector

- When α has Gamma PDF, the texture $\tau = 1/\alpha$ has **inverse-Gamma (IG)** PDF:

$$p_\tau(\tau) = \frac{1}{\tau^2} p_\alpha\left(\frac{1}{\tau}\right) = \frac{1}{\Gamma(\lambda)}\left(\frac{\lambda}{\mu}\right)^\lambda \frac{1}{\tau^{\lambda+1}} e^{-\frac{\lambda}{\mu\tau}}, \quad \tau \geq 0$$

- λ>0 is the shape parameter and μ>0 is the scale parameter.

- The **complex multivariate compound-Gaussian clutter model** is found by solving the following integral:

$$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i) = \frac{1}{\pi^M |\mathbf{M}|} h_M(q_i(\mathbf{z})) = \int_0^\infty \frac{1}{(\pi\tau)^M |\mathbf{M}|} \exp\left[-\frac{q_i(\mathbf{z})}{\tau}\right] p_\tau(\tau) d\tau$$

- Consequently, the **complex multivariate compound-Gaussian clutter model** is a complex *M*-variate ***t* distribution** with λ degrees of freedom:

$$p_{\mathbf{z}|H_i}(\mathbf{z}|H_i) = \frac{1}{\pi^M |\mathbf{M}|} \cdot \frac{\Gamma(M+\lambda)}{\Gamma(\lambda)} \cdot \left(\frac{\lambda}{\mu}\right)^{\lambda} \cdot \left(\frac{\lambda}{\mu} + q_i(\mathbf{z})\right)^{-(M+\lambda)}, \quad i = 0,1$$

More details in:

[San2012] Sangston K.J., Gini F., Greco M.S., "Coherent Radar Target Detection in Heavy-Tailed Compound-Gaussian Clutter," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 48, No. 1, January 2012, pp. 64-77.

The Linear-Threshold Detector

- The optimum detector in compound-Gaussian clutter with IG texture is given by:

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \left(1 - e^{-\frac{\eta}{M+\lambda}}\right) \cdot \left(\frac{\lambda}{\mu} + q_0(\mathbf{z})\right)$$

Gaussian clutter: $$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma_g^2 \eta = \frac{\eta}{1/\sigma_g^2} = \frac{\eta}{\alpha}$$

Compound-Gaussian clutter: $$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\eta_{M,\lambda}}{\hat{\alpha}_{MMSE}}$$

$$\eta_{M,\lambda} = (M+\lambda)\left(1 - e^{-\frac{\eta}{M+\lambda}}\right), \qquad \hat{\alpha}_{MMSE} = E[\alpha | q_0] = \frac{M+\lambda}{\frac{\lambda}{\mu} + q_0}$$

The Clutter Amplitude Model

- The PDF of the clutter amplitude $R=|c|=\sqrt{\tau}\,|x|$ that arises from the compound-Gaussian clutter model with IG texture:

$$p_R(r)=\int_0^{\infty} 2\alpha r e^{-r^2\alpha} p_\alpha(\alpha)d\alpha = 2\mu r\left(1+\frac{\mu}{\lambda}r^2\right)^{-(\lambda+1)}, \quad r\geq 0$$

$$\lim_{\lambda\to\infty} p_R(r)=2\mu r e^{-\mu r^2}, \quad r\geq 0 \quad \text{(Rayleigh PDF)}$$

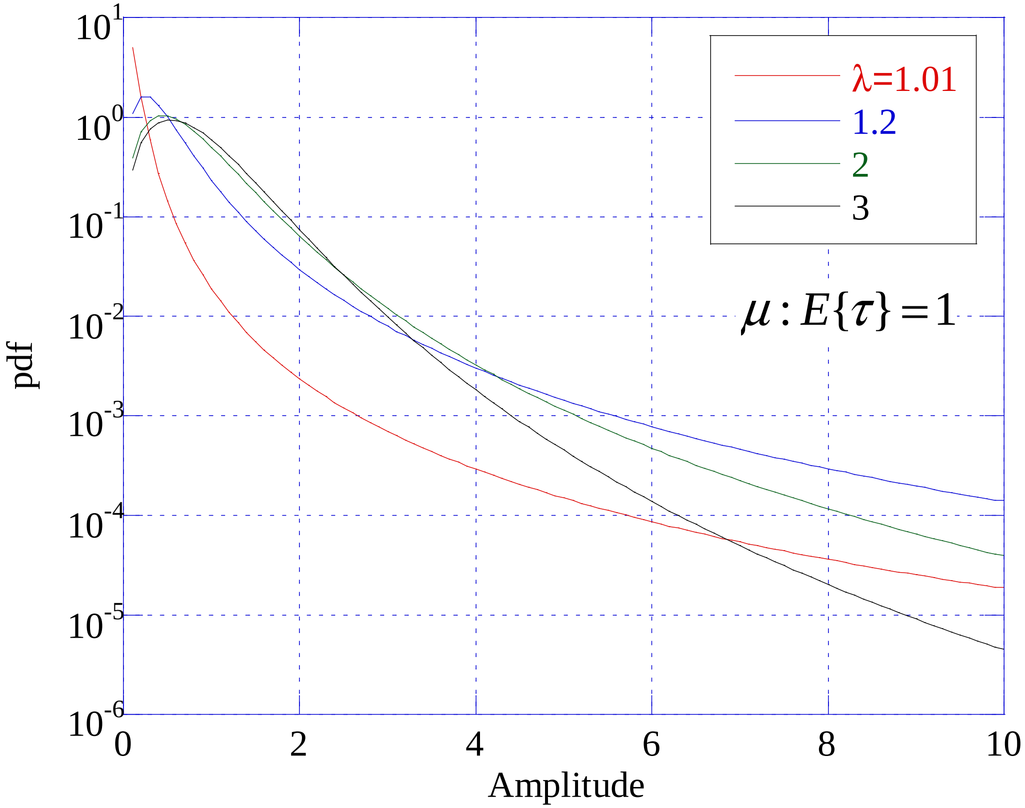
$$\lim_{\lambda\to 0} p_R(r)=\frac{2\lambda}{r}u(r), \quad r\geq 0$$

Interestingly, the PDF of the clutter intensity, $W=R^2$, is a **generalized Pareto distribution.**

The clutter amplitude statistics are extremely heavy-tailed in this limit.

The Clutter Amplitude Model

- Thus, the multivariate clutter model with IG texture varies parametrically between Gaussian clutter (when λ goes to infinity) and extremely heavy-tailed clutter (when λ goes to zero).



$$E\{R^k\} = \left(\frac{\lambda}{\mu}\right)^{\frac{k}{2}} \frac{\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\lambda - \frac{k}{2}\right)}{\Gamma(\lambda)}$$

if $\lambda > k/2$.

$$E\{\tau\} = E\{R^2\} = \frac{\lambda}{\mu(\lambda - 1)}$$

The clutter power $E\{\tau\}$ is finite only for λ>1.

- When the shape parameter λ goes to zero:

$$\frac{\eta_{M,\lambda}}{\hat{\alpha}_{opt}} \xrightarrow{\lambda\to 0} \left(1-e^{-\frac{\eta}{M}}\right) q_0(\mathbf{z}) = f_{ML}(q_0,\eta)$$

- … which is the adaptive threshold of the NMF detector previously explored [Gini, IEE-F, 1997], known to approach the optimal NP detector as $M\to\infty$ [Conte, IEEE-AES, 1995] and also known to be CFAR with respect to the clutter texture PDF.

$$q_0(\mathbf{z}) - q_1(\mathbf{z}) \underset{H_0}{\overset{H_1}{\gtrless}} \left(1-e^{-\frac{\eta}{M}}\right) q_0(\mathbf{z})$$

- The remarkable observation here is that the NMF is the optimum detector <u>for any *M*</u> for compound-Gaussian clutter with Inverse-Gamma distributed texture, in the limit of extremely heavy-tailed clutter, i.e. as $\lambda\to 0$.

The GLRT Linear Threshold Detector (LTD)

- When the complex amplitude is deterministic unknown and we resort to the GLRT approach (i.e. we replace β with its ML estimate in the LRT):

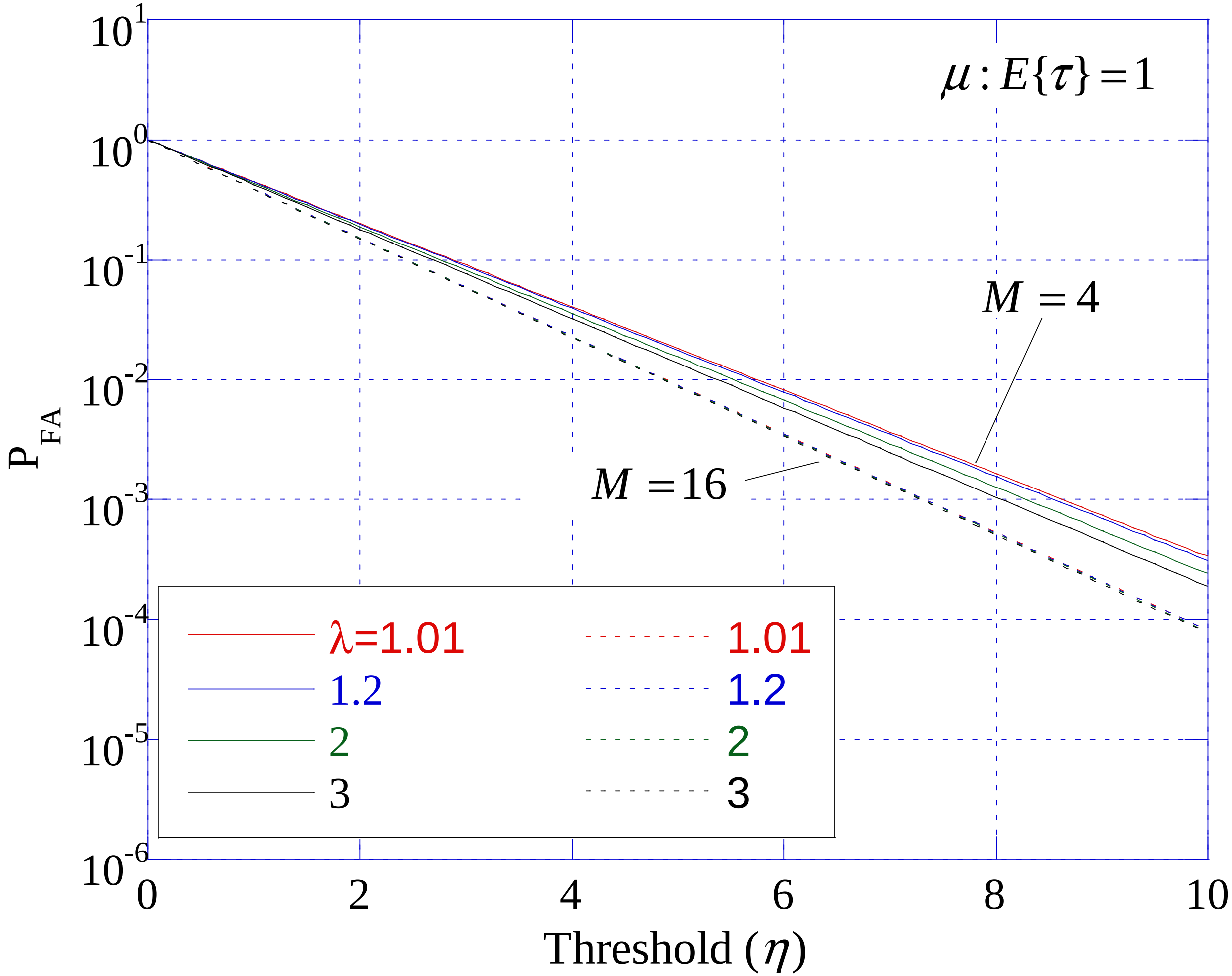
$$q_0(\mathbf{z}) - q_1(\mathbf{z}) = \frac{\left|\mathbf{v}^H \mathbf{M}^{-1} \mathbf{z}\right|^2}{\mathbf{v}^H \mathbf{M}^{-1} \mathbf{v}}$$

$$\left|\mathbf{v}^H \mathbf{M}^{-1} \mathbf{z}\right|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \left(\mathbf{v}^H \mathbf{M}^{-1} \mathbf{v}\right) \cdot \left(1 - e^{-\frac{\eta}{M+\lambda}}\right) \cdot \left(\frac{\lambda}{\mu} + q_0(\mathbf{z})\right)$$

$$P_{FA} = \exp\left[-\eta\left(\frac{M+\lambda-1}{M+\lambda}\right)\right] \Rightarrow \left.P_{FA}\right|_{\lambda=0} = e^{-\frac{\eta(M-1)}{M}} = P_{FA,NMF}$$

Performance of the GLRT-LTD

$$P_{FA} = \exp\left[-\eta\left(\frac{M+\lambda-1}{M+\lambda}\right)\right]$$



$$M + \lambda \gg 1$$

$$(M \gg 1 \text{ or } \lambda \gg 1)$$

$$\Downarrow$$

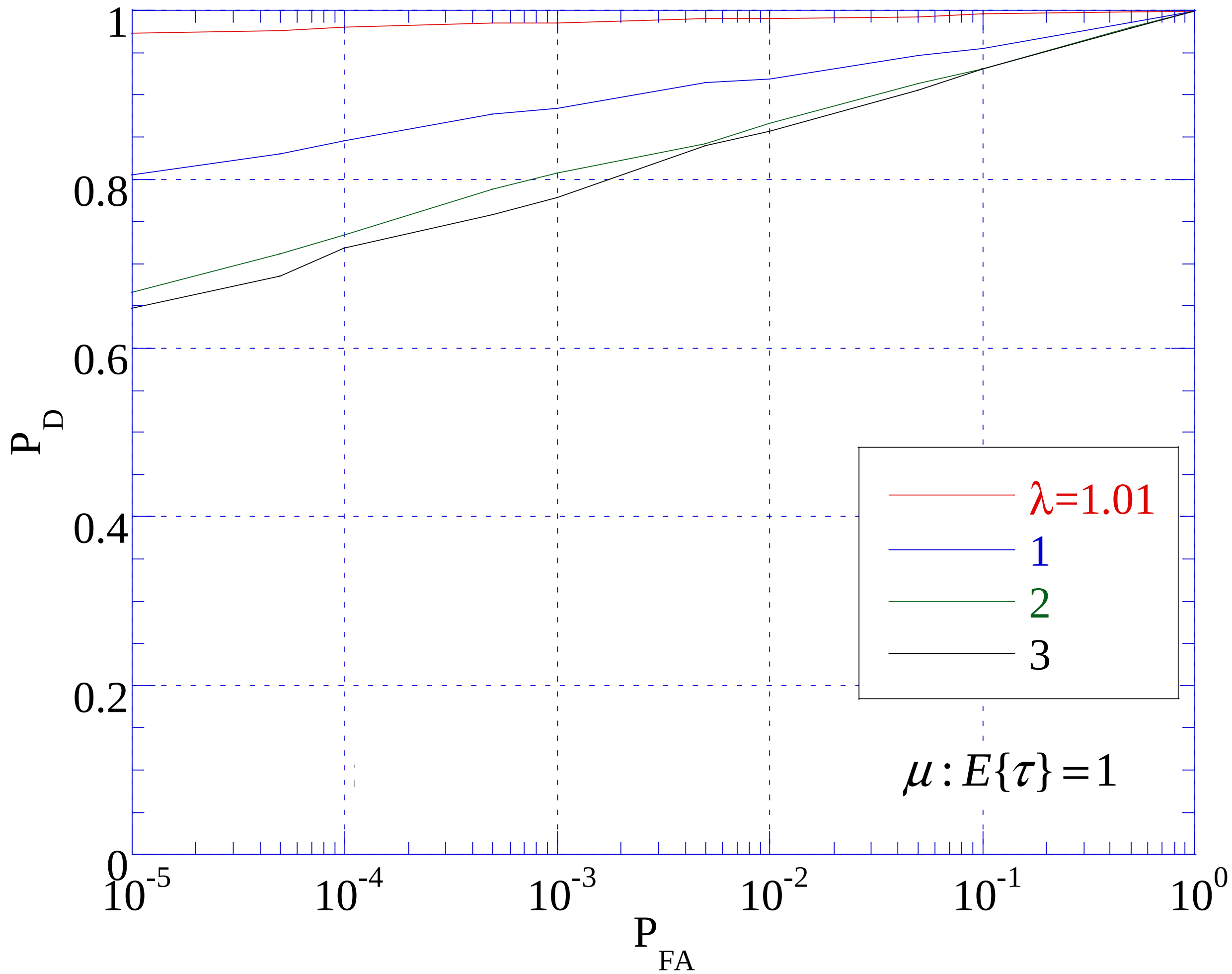
$$P_{FA} \cong \exp[-\eta]$$

(the Gaussian case)

Thus, for large M and/or large λ, the GLRT-LTD detector is CFAR with respect to both λ and μ (the texture parameters).

Performance of the GLRT-LTD

Receiver Operating Characteristic (ROC)

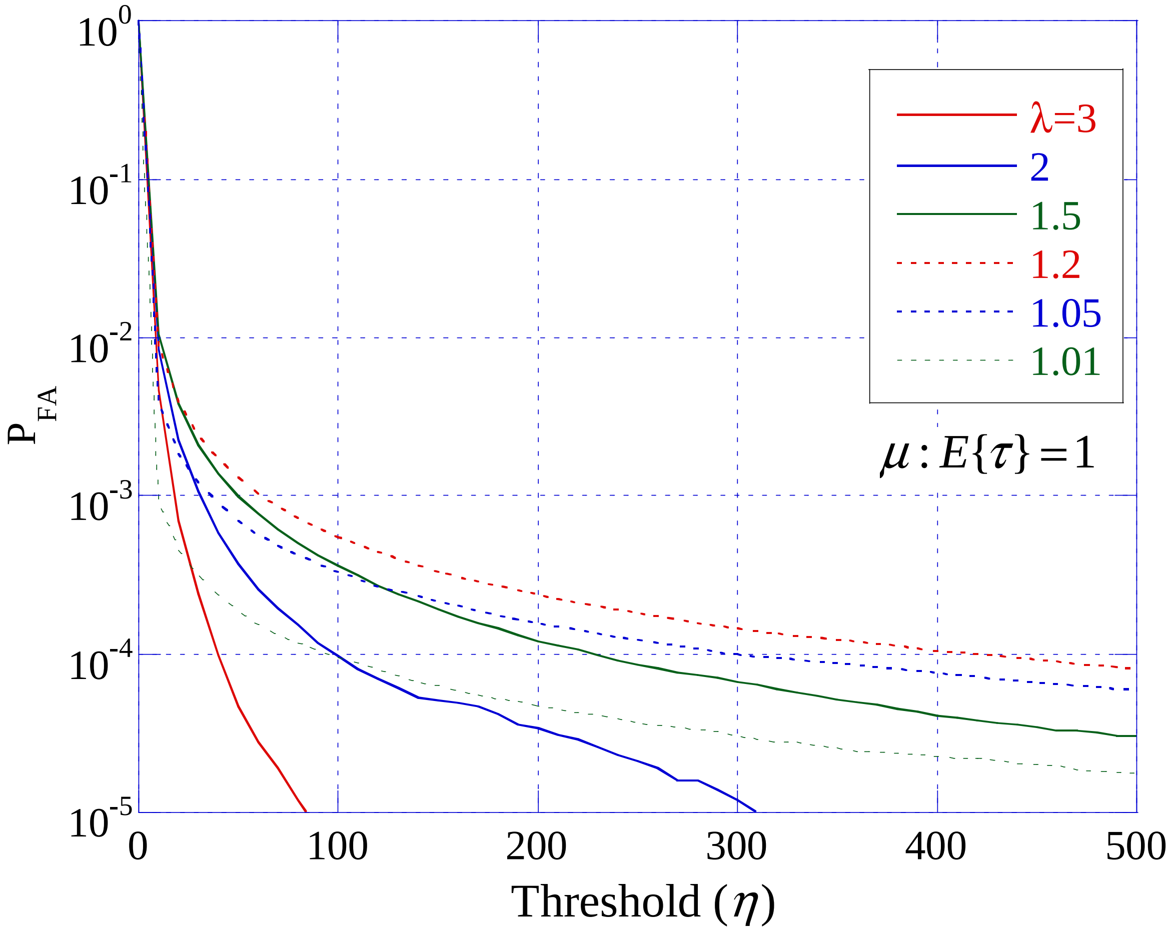


$$M = 16, \quad P_{FA} = 10^{-4}$$

$$SCR = \frac{\sigma_S^2}{E\{\tau\}} = 0\ dB$$

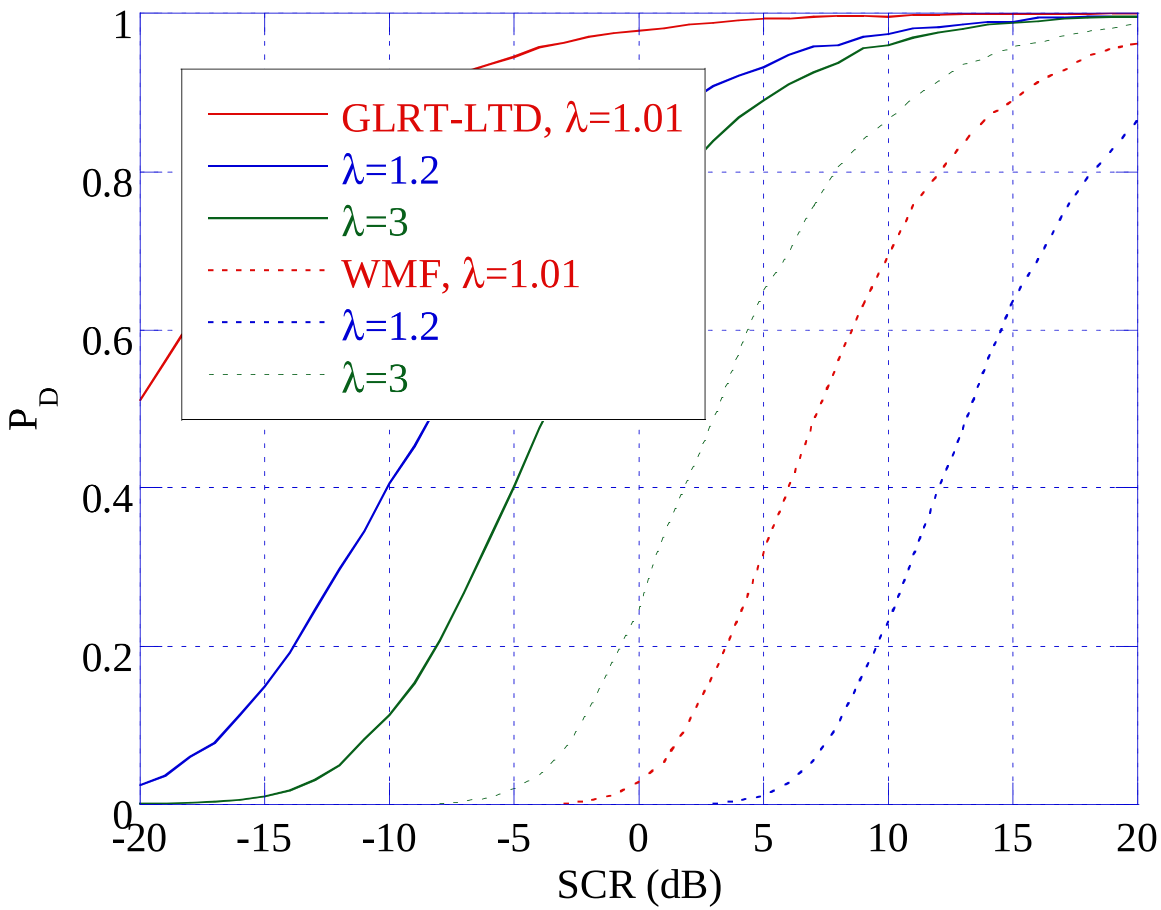
Detection performance of the GLRT-LTD gets better when the clutter becomes spikier (lower λ).

Performance of the GLRT-LTD



For low P_{FA}, the False Alarm Rate (FAR) increases when the clutter becomes spikier (lower λ).

Performance of the GLRT-LTD



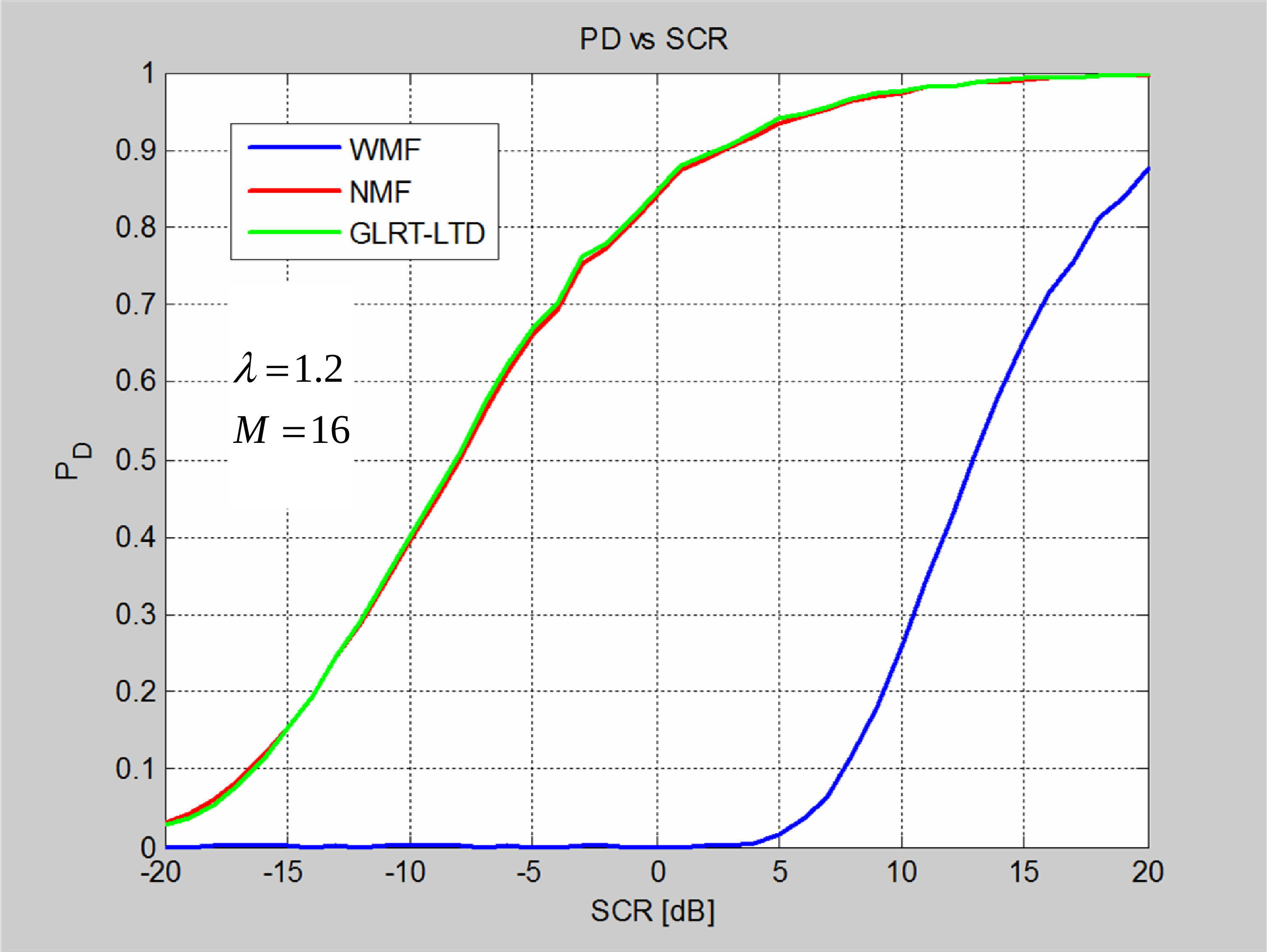
In heavy-tailed clutter (small λ): the GLRT-LTD largely outperforms the (mismatched) WMF, which is optimal only in Gaussian clutter.

$$M = 16,$$

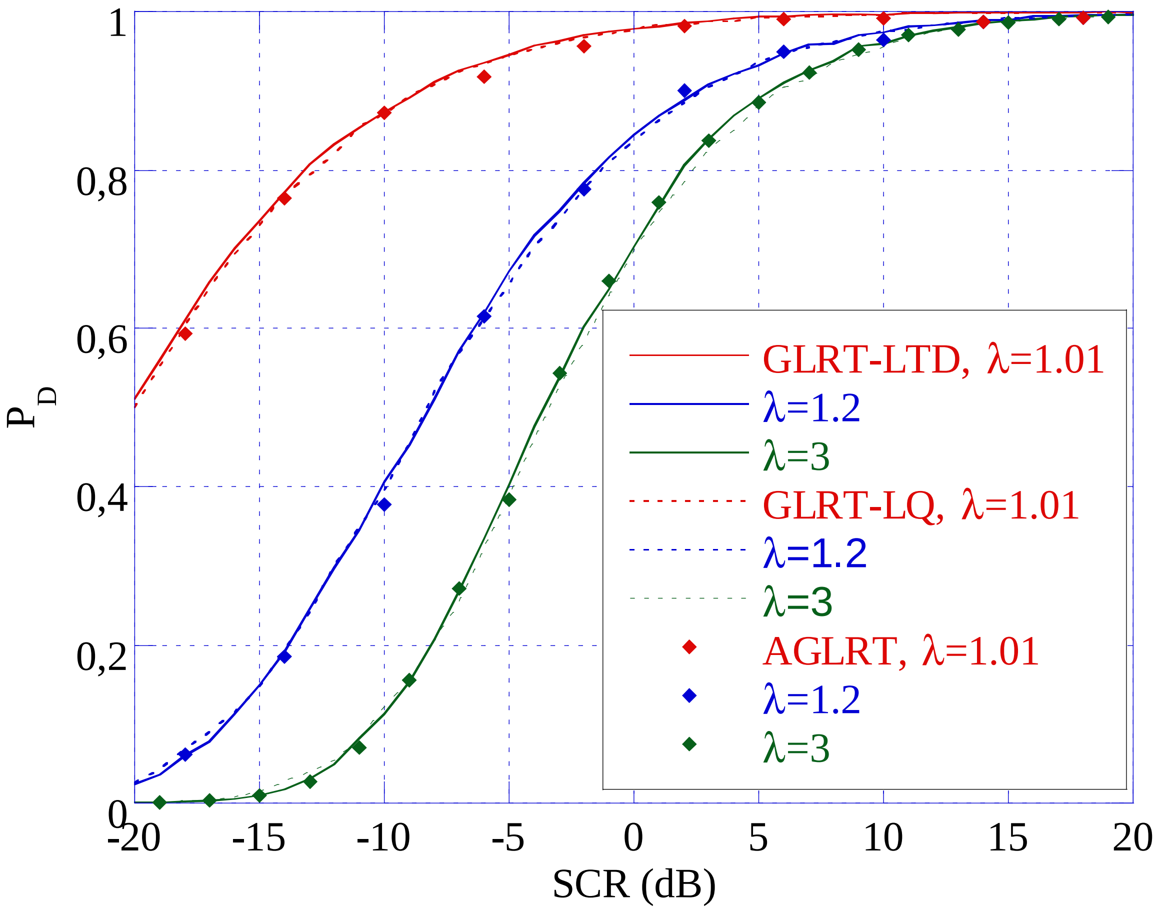
$$P_{FA} = 10^{-4},$$

$$SCR = \frac{\sigma_S^2}{E\{\tau\}}$$

Performance of the GLRT-LTD, NMF and WMF



Performance of the GLRT-LTD



The detection performance of the GLRT-LTD gets better when the clutter becomes spikier.

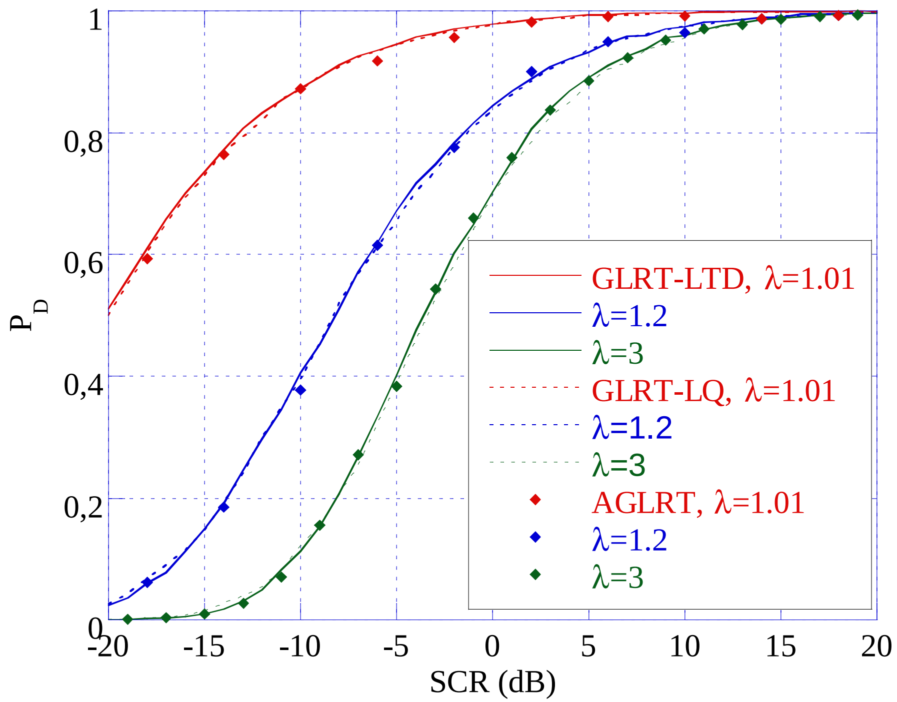
When the number of integrated pulses *M* is large (say *M*>10), the suboptimal NMF and the optimal GLRT-LTD have basically the same performance.

$$M = 16,$$

$$P_{FA} = 10^{-4},$$

$$SCR = \frac{\sigma_S^2}{E\{\tau\}}$$

Performance of the GLRT-LTD



The AGLRT performances are very close to the other two detectors, in particular very close to the clairvoyant GLRT-LDT.

- **Adaptive GLRT (AGLRT):**
- This detector is obtained by replacing in the GLRT-LTD the true values of the clutter parameters λ and μ with their estimates (in this sense this detector is adaptive).
- To estimate the clutter texture parameters λ and μ we used the method of moments (MoM); in particular we resorted to the moments of order ½ and 1 for $\lambda<2$, and to the moments of order 2 and 4 for $\lambda>2$)
- We processed K=24 M-dimensional independent vectors (with M=16) surrounding the cell under test (CUT) to estimate the moments.

Concluding Remarks

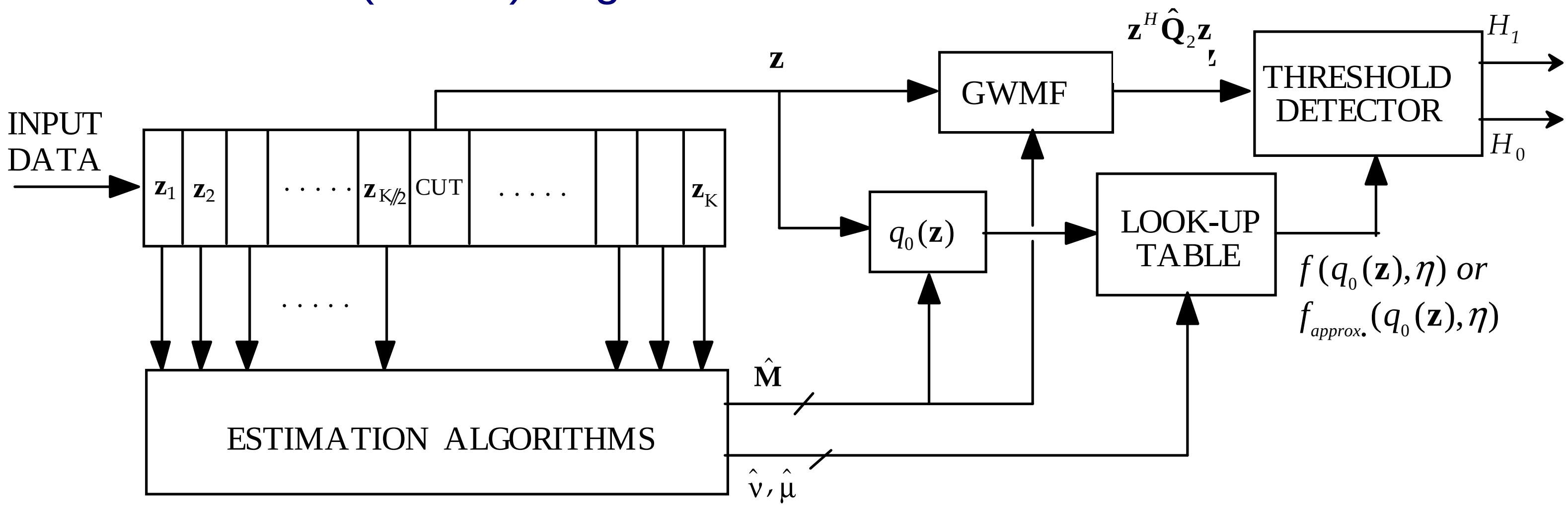
- Based on the interpretation of the optimum detector (OD) as a whitening matched filter (WMF) compared to a data-dependent threshold (DDT), we have derived a compound-Gaussian (CG) clutter model whose OD is the optimum Gaussian WMF compared to a DDT that varies linearly with $q_0(\mathbf{z})$.
- The CG clutter model with inverse Gamma (IG) texture varies parametrically from the Gaussian clutter model to a clutter model whose tails are evidently heavier than any *K* model.
- The NMF, which is a popular suboptimum detector due to its CFAR property, is an OD for the CG clutter model with IG texture, in the limit as the tails get extremely heavy (small λ), and the OD for <u>any</u> CG model for large *M.*
- The LTD is conceptually the simplest extension beyond the optimum Gaussian detector (constant threshold). Although a quadratic-threshold detector may be obtained [Gini, IEEE-SP, 1999], such a detector will always be suboptimum for any compound-Gaussian model.

Outline of the Section on “Optimum Detection”

- The binary hypothesis testing problem
- Target models and detectors in Gaussian clutter
- Optimum coherent detection in compound-Gaussian clutter
  - The Likelihood Ratio Test (LRT)
  - The Estimator-Correlator (EC)
  - The Whitening Matched Filter (WMF) compared to a Data-Dependent Threshold (DDT)
- Suboptimum detection in compound-Gaussian clutter
- Performance analysis in K-distributed clutter
- The Linear-Threshold Detector (LTD) and the IG-texture model
- **Subspace detectors in compound-Gaussian clutter**

Subspace Detectors in Compound-Gaussian Clutter

- The general structure for subspace targets (target signal rank>1) is the same as in the case of 1-D (rank-1) targets.



$$\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2 \leftarrow \mathbf{z}^H\mathbf{Q}_2\mathbf{z}$$

- The **WMF** is replaced by the **Matched Subspace Detector (MSD)**, sometimes called the **Generalized WMF (GMF)**, which is the GLRT detector for a subspace target signal.
- The DDT is the same whatever it is the signal rank *r*.

Outline of the Section on “Adaptive Detection”

- **Adaptive detection in compound-Gaussian clutter**
- The Sample Covariance Matrix (SCM) estimator
- The Normalized Sample Covariance Matrix (NSCM) estimator
- The Maximum Likelihood (ML) Covariance Matrix estimator
- The Approximate ML (AML) Covariance Matrix estimator
- The Adaptive Normalized Matched Filter (ANMF)
- ROC calculation on synthetic and real data
- Clutter non-stationarity effect

Adaptive detection in compound-Gaussian clutter

- The **optimum and suboptimum detectors** in previous section have been obtained assuming that the (normalized) clutter covariance matrix **M** is a-priori known, e.g. we know that **M**=**I**, i.e. temporal samples uncorrelated.

- Most often this is not true and it must be estimated using *K* secondary data, from adjacent range cells, surrounding the CUT.

- We assume homogeneous environment:

$\mathbf{z}|H_0$ and $\{\mathbf{z}_k\}_{k=1}^{K}$ are Independent and Identically Distributed (IID)

$$\mathbf{R} = \sigma^2\mathbf{M} \triangleq E\left\{\mathbf{z}\mathbf{z}^H \middle| H_0\right\} = E\left\{\mathbf{z}_k\mathbf{z}_k^H\right\} = E\left\{\tau_k\right\} E\left\{\mathbf{x}_k\mathbf{x}_k^H\right\}, \quad k = 1, 2, \cdots, K$$

$$\mathbf{M} \triangleq E\left\{\mathbf{x}\ \mathbf{x}^H\right\} = E\left\{\mathbf{x}_k\mathbf{x}_k^H\right\}, \quad k = 1, 2, \cdots, K$$

Adaptive detection in compound-Gaussian clutter

- We could resort to the Maximum Likelihood (ML) approach where the unknowns are the normalized covariance matrix **M** and the parameters of the texture PDF's in the secondary and primary data $\Theta_\tau = [\mu \quad \nu \quad \mu_1 \quad \nu_1 \quad \ldots \quad \mu_K \quad \nu_K]^T$ and the target complex amplitude β:

$$\Lambda_{GLRT}(\mathbf{Z}) = \frac{\max_{\mathbf{M},\boldsymbol{\Theta}_\tau,\beta} f\left(\mathbf{Z}\middle|H_1\right)}{\max_{\mathbf{M},\boldsymbol{\Theta}_\tau} f\left(\mathbf{Z}\middle|H_0\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta, \qquad \mathbf{Z} = [\mathbf{z} \quad \mathbf{z}_1 \quad \cdots \quad \mathbf{z}_K]$$

- This approach leads to an infeasible multidimensional non-linear maximization problem for which no closed form seems to exist.

- An alternative approach is to consider the matrix **M** as known, derive the OD or GLRT detector for known **M**, and then replace **M** with an appropriate estimate:

$$\text{NMF} \quad \rightarrow \quad \frac{\left|\mathbf{v}^H \hat{\mathbf{M}}^{-1} \mathbf{z}\right|^2}{\left(\mathbf{z}^H \hat{\mathbf{M}}^{-1} \mathbf{z}\right)\left(\mathbf{v}^H \hat{\mathbf{M}}^{-1} \mathbf{v}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Adaptive detection in compound-Gaussian clutter

- **How to estimate the (normalized) clutter covariance matrix?**

It is useful to observe that, since **R**=σ^2**M**, the following equation is valid:

$$\frac{\left|\mathbf{v}^H\mathbf{R}^{-1}\mathbf{z}\right|^2}{\left(\mathbf{z}^H\mathbf{R}^{-1}\mathbf{z}\right)\left(\mathbf{v}^H\mathbf{R}^{-1}\mathbf{v}\right)}=\frac{\left|\mathbf{v}^H\mathbf{M}^{-1}\mathbf{z}\right|^2}{\left(\mathbf{z}^H\mathbf{M}^{-1}\mathbf{z}\right)\left(\mathbf{v}^H\mathbf{M}^{-1}\mathbf{v}\right)}$$

- To estimate **R**, one possible solution is to use the well-known **Sample Covariance Matrix (SCM)** estimator, that has been derived as the ML estimator of **R** for homogeneous Gaussian clutter.
- Then, we plug it in the NMF detector $\rightarrow$ **Adaptive NMF (ANMF)**

$$\hat{\mathbf{R}}_{SCM}=\frac{1}{K}\sum_{k=1}^{K}\mathbf{z}_k\mathbf{z}_k^H \qquad \rightarrow \qquad \frac{\left|\mathbf{v}^H\hat{\mathbf{R}}_{SCM}^{-1}\mathbf{z}\right|^2}{\left(\mathbf{z}^H\hat{\mathbf{R}}_{SCM}^{-1}\mathbf{z}\right)\left(\mathbf{v}^H\hat{\mathbf{R}}_{SCM}^{-1}\mathbf{v}\right)}\underset{H_0}{\overset{H_1}{\gtrless}}\eta$$

SCM estimate in compound-Gaussian clutter

- Even for compound-Gaussian clutter, this estimate is **unbiased** and **consistent**:

$$E\left\{\hat{\mathbf{R}}_{SCM}\right\} = \frac{1}{K}\sum_{k=1}^{K} E\left\{\mathbf{z}_k \mathbf{z}_k^H\right\} = \frac{1}{K}\sum_{k=1}^{K} E\left\{\tau_k \mathbf{x}_k \mathbf{x}_k^H\right\}$$

$$= \frac{1}{K}\sum_{k=1}^{K} E\left\{\tau_k\right\} E\left\{\mathbf{x}_k \mathbf{x}_k^H\right\} = \frac{1}{K}\sum_{k=1}^{K} \sigma^2 \mathbf{M} = \sigma^2 \mathbf{M} = \mathbf{R}$$

$$\lim_{K\to\infty} \hat{\mathbf{R}}_{SCM} = \mathbf{R} \quad \text{(convergence in mean square sense)}$$

$$i.e. \quad \lim_{K\to\infty} E\left\{\left|\mathbf{R}_{i,k} - \hat{\mathbf{R}}_{SCM\; i,k}\right|^2\right\} = 0$$

- The adaptive NMF that makes use of the SCM estimator is CFAR with respect to the true covariance matrix **R**, but unfortunately its P_{FA} heavily depends on the PDF of the texture, i.e. on clutter spikiness.

ML estimate in compound-Gaussian clutter

- The "optimal" estimator for **M** is the **Maximum Likelihood (ML)** estimator.
- To derive it we start from the joint PDF of the *K* secondary vectors:

$$p_{\mathbf{Z}}(\mathbf{z}_1,\ \mathbf{z}_2,\ \ldots\mathbf{z}_K) = \prod_{k=1}^{K} p_z(\mathbf{z}_k) = \prod_{k=1}^{K} \int_0^{\infty} p_z(\mathbf{z}_k | \tau_k) p_\tau(\tau_k) d\tau_k$$

$$= \prod_{k=1}^{K} \int_0^{+\infty} \frac{1}{(\pi\tau_k)^M |\mathbf{M}|} \exp\left(-\frac{\mathbf{z}_k^H \mathbf{M}^{-1} \mathbf{z}_k}{\tau_k} \right) p_\tau(\tau_k) d\tau_k$$

Defining the function $h_M(q) \triangleq \int_0^{+\infty} \frac{1}{\tau^M} \exp(-q/\tau) d\tau$

$$\Rightarrow \quad p_{\mathbf{Z}}(\mathbf{z}_1,\ \mathbf{z}_2,\ \ldots\mathbf{z}_K) = \pi^{-KM} |\mathbf{M}|^{-K} \prod_{k=1}^{K} h_M(\mathbf{z}_k^H \mathbf{M}^{-1} \mathbf{z}_k)$$

ML estimate in compound-Gaussian clutter

- Now derivate the likelihood function with respect to each element of **M** and set the derivatives equal to 0:

$$-K\frac{\partial \ln|\mathbf{M}|}{\partial \mathbf{M}}+\sum_{k=1}^{K}\frac{g_M(\mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k)}{h_M(\mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k)}\cdot\frac{\partial \;\; \mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k}{\partial \mathbf{M}}=0$$

where $g_M(x)\triangleq\partial h_M(x)/\partial x$

- With some mathematical manipulation we obtain:

$$\hat{\mathbf{M}}_{ML}=\frac{1}{K}\sum_{k=1}^{K}\frac{h_{M+1}(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}\mathbf{z}_k)}{h_M(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}\mathbf{z}_k)}\cdot\mathbf{z}_k\mathbf{z}_k^H=\frac{1}{K}\sum_{k=1}^{K}c_M\left(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}\mathbf{z}_k\right)\cdot\mathbf{z}_k\mathbf{z}_k^H$$

where $c_M(x)\triangleq h_{M+1}(x)/h_M(x)$

ML estimate in compound-Gaussian clutter

- The **ML estimator** is the solution (if it exists) of a trascendental equation. We can solve it iteratively:

$$\hat{\mathbf{M}}_{ML}(i+1)=\frac{1}{K}\sum_{k=1}^{K}c_M\left(\mathbf{z}_k^H\hat{\mathbf{M}}_{ML}^{-1}(i)\mathbf{z}_k\right)\cdot\mathbf{z}_k\mathbf{z}_k^H,\quad i=0,1,2,\cdots$$

Problems:

- Calculation of the *K* data-dependent coefficients $c_M(.)$ requires knowledge of the texture PDF, i.e. of the clutter shape and scale parameters.
- Even when the texture PDF parameters are perfectly known or accurately estimated, the calculation of these coefficients can be too computationally heavy for real time operation.
- The choice of a good starting point to prevent convergence to local maxima.

NSCM estimate in compound-Gaussian clutter

Alternative approach:

- **1st step**: Suppose that the textures in the secondary data are known and calculate the derivative of the conditional PDF of the secondary data w.r.t. **M**:

$$\frac{\partial \ln p_{\mathbf{Z}}(\mathbf{z}_1,\ \mathbf{z}_2,\ \cdots,\mathbf{z}_K \,|\, \tau_1,\ \tau_2,\ \cdots,\tau_K)}{\partial \mathbf{M}} = \sum_{k=1}^{K} \frac{\partial \ln p_{\mathbf{z}|\tau}(\mathbf{z}_k \,|\, \tau_k)}{\partial \mathbf{M}} = 0$$

The solution is $\hat{\mathbf{M}} = \frac{1}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\tau_k}$

- **2nd step**: Replace the unknown textures (local clutter powers) with their estimates. The simplest estimates are obtained by the Method of Moments (MoM), i.e. we calculate the sample powers:

$$\hat{\tau}_k = \frac{\mathbf{z}_k^H\mathbf{z}_k}{M} = \frac{1}{M}\sum_{n=1}^{M}\left|z_{k,n}\right|^2, \quad k = 1, 2,\ \cdots, K$$

NSCM estimate in compound-Gaussian clutter

- **3rd step**: We get an estimate of the normalized covariance matrix **M** as:

$$\hat{\mathbf{M}}_{NSCM} = \frac{1}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\hat{\tau}_k} = \frac{1}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{(\mathbf{z}_k^H\mathbf{z}_k/M)} = \frac{M}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\mathbf{z}_k^H\mathbf{z}_k}$$

  that is, we normalize the matrix estimate for each secondary vector $\mathbf{z}_k\mathbf{z}_k^H$ by the sample estimate of its power $\hat{\tau}_k = \mathbf{z}_k^H\mathbf{z}_k/M$.

- This is called the **Normalized Sample Covariance Matrix (NSCM)** estimator.

- We can easily verify that the NSCM estimate does not depend on the textures:

$$\hat{\mathbf{M}}_{NSCM} = \frac{M}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\mathbf{z}_k^H\mathbf{z}_k} = \frac{M}{K}\sum_{k=1}^{K}\frac{\tau_k\mathbf{x}_k\mathbf{x}_k^H}{\tau_k\mathbf{x}_k^H\mathbf{x}_k} = \frac{M}{K}\sum_{k=1}^{K}\frac{\mathbf{x}_k\mathbf{x}_k^H}{\mathbf{x}_k^H\mathbf{x}_k}$$

NSCM estimate in compound-Gaussian clutter

- For compound-Gaussian clutter, if the eigenvalues of the matrix **M** are distinct, the NSCM estimator is **biased** and it is not **consistent**, as verified in:

  [Pas08] F. Pascal, P. Forster, J.-P. Ovarlez, P. Larzabal "Performance Analysis of Covariance Matrix Estimates in Impulsive Noise" *IEEE Trans. on Signal Processing*, Vol. 56, No.6, June 2008, pp.2206-2217.

- Plugging the NSCM in the NMF we obtain a test that is CFAR with respect to the texture PDF, but, due to the terms $\hat{\tau}_k = \mathbf{z}_k^H \mathbf{z}_k \big/ M$, it is not CFAR w.r.t. the actual normalized covariance matrix **M**.

- Note that both the SCM and the NSCM estimators do not require knowledge of the texture PDF to implement the estimator.

Approximate ML estimate in compound-Gaussian clutter

- **1st step**: ML estimate of the normalized covariance matrix **M** for known textures:

$$\hat{\mathbf{M}} = \frac{1}{K}\sum_{k=1}^{K}\frac{\mathbf{z}_k\mathbf{z}_k^H}{\tau_k}$$

- **2nd step**: Yet another approach with improved texture estimation: Suppose **M** is known and derive the ML estimates of the textures

$$\hat{\tau}_k = \arg\max_{\tau_k} p_{\mathbf{z}|\tau}(\mathbf{z}_k \mid \tau_k) = \frac{\mathbf{z}_k^H\mathbf{M}^{-1}\mathbf{z}_k}{M}, \quad k = 1, 2, \cdots, K$$

- **3rd step**: Replace the textures with their ML estimates

$$\hat{\mathbf{M}}_{AML} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{z}_k^H\hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k}\right)\cdot\mathbf{z}_k\mathbf{z}_k^H = \frac{1}{K}\sum_{k=1}^{K} d_M\left(\mathbf{z}_k^H\hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k\right)\cdot\mathbf{z}_k\mathbf{z}_k^H$$

Approximate ML estimate in compound-Gaussian clutter

- The **Approximate Maximum Likelihood (AML)** estimator has the same structure as the true ML estimator, but the coefficients $d_M(.)$ do no depend on the texture PDF and are much easier to calculate than the $c_M(.)$:

$$\hat{\mathbf{M}}_{AML} = \frac{1}{K}\sum_{k=1}^{K} d_M\left(\mathbf{z}_k^H \hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k\right)\cdot \mathbf{z}_k\mathbf{z}_k^H$$

The equation can be solved iteratively and initialized by using the NSCM estimator:

$$\begin{cases} \hat{\mathbf{M}}_{AML}(i+1) = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K}\left(\dfrac{M}{\mathbf{z}_k^H \hat{\mathbf{M}}_{AML}^{-1}(i)\mathbf{z}_k}\right)\cdot \mathbf{z}_k\mathbf{z}_k^H, \quad i = 0,1,2,\cdots \\ \hat{\mathbf{M}}_{AML}(0) = \hat{\mathbf{M}}_{NSCM} \end{cases}$$

We found that only few iterations (3-4) are necessary for convergence.

Approximate ML estimate in compound-Gaussian clutter

- To increase the speed of convergence of the iterative procedure, the estimate at each step can be normalized to have the same trace as the true **M:**

$$tr\left\{\hat{\mathbf{M}}_{AML}(i)\right\} = tr\{\mathbf{M}\} = M$$

$$\hat{\mathbf{M}}_{AML}(i) \triangleq \frac{M}{Tr\{\hat{\mathbf{M}}_{AML}(i)\}} \cdot \hat{\mathbf{M}}_{AML}(i).$$

- The AML estimate does not depend on the textures:

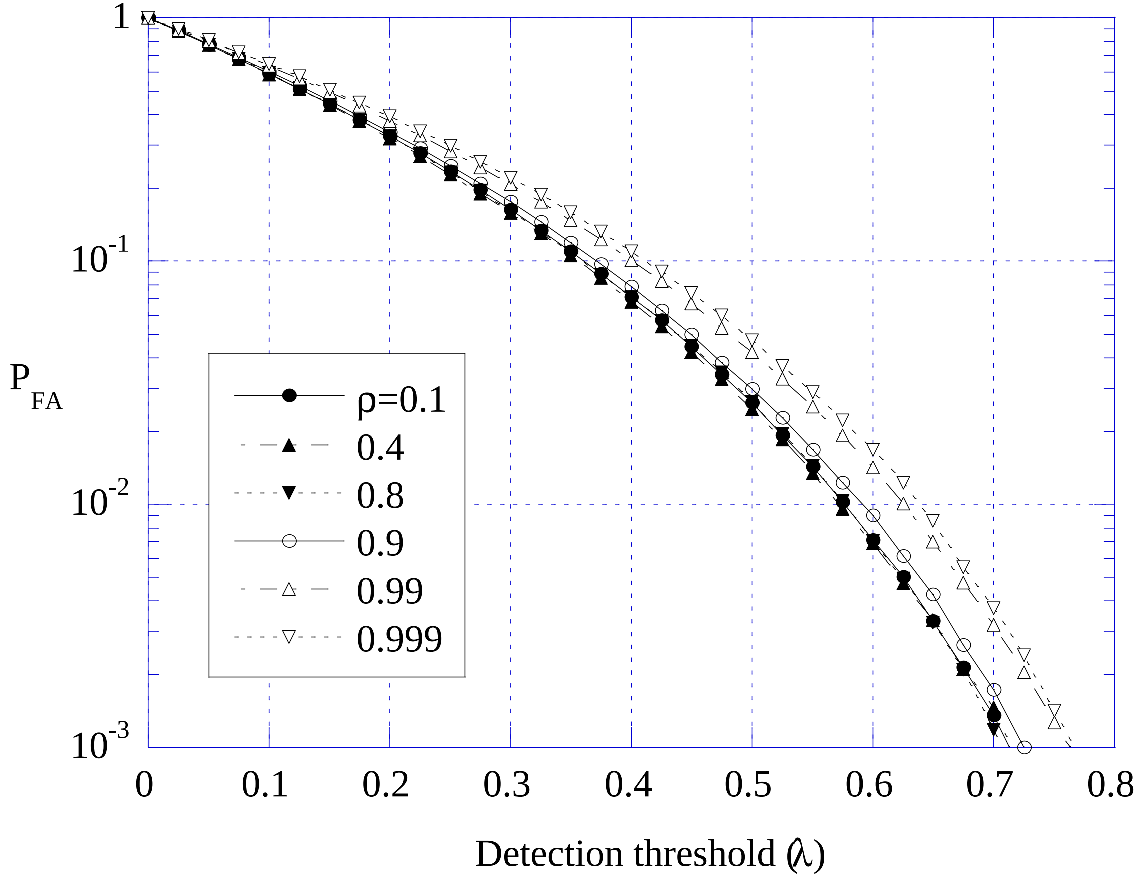
$$\hat{\mathbf{M}}_{AML} = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{z}_k^H \hat{\mathbf{M}}_{AML}^{-1}\mathbf{z}_k}\right)\cdot \mathbf{z}_k\mathbf{z}_k^H = \frac{1}{K}\sum_{k=1}^{K}\left(\frac{M}{\mathbf{x}_k^H \hat{\mathbf{M}}_{AML}^{-1}\mathbf{x}_k}\right)\cdot \mathbf{x}_k\mathbf{x}_k^H$$

ANMF in K-distributed clutter

- Plugging the clutter covariance matrix estimate in the NMF we get the adaptive NMF detector $\rightarrow$ ANMF.
- The P_{FA} of the ANMF which uses the ML or AML estimator does not depend on the textures PDF $\rightarrow$ CFARness w.r.t. texture parameters.
- It is important to verify if this ANMF is CFAR with respect to the actual covariance matrix **M** $\rightarrow$ we resort to Monte Carlo simulation.

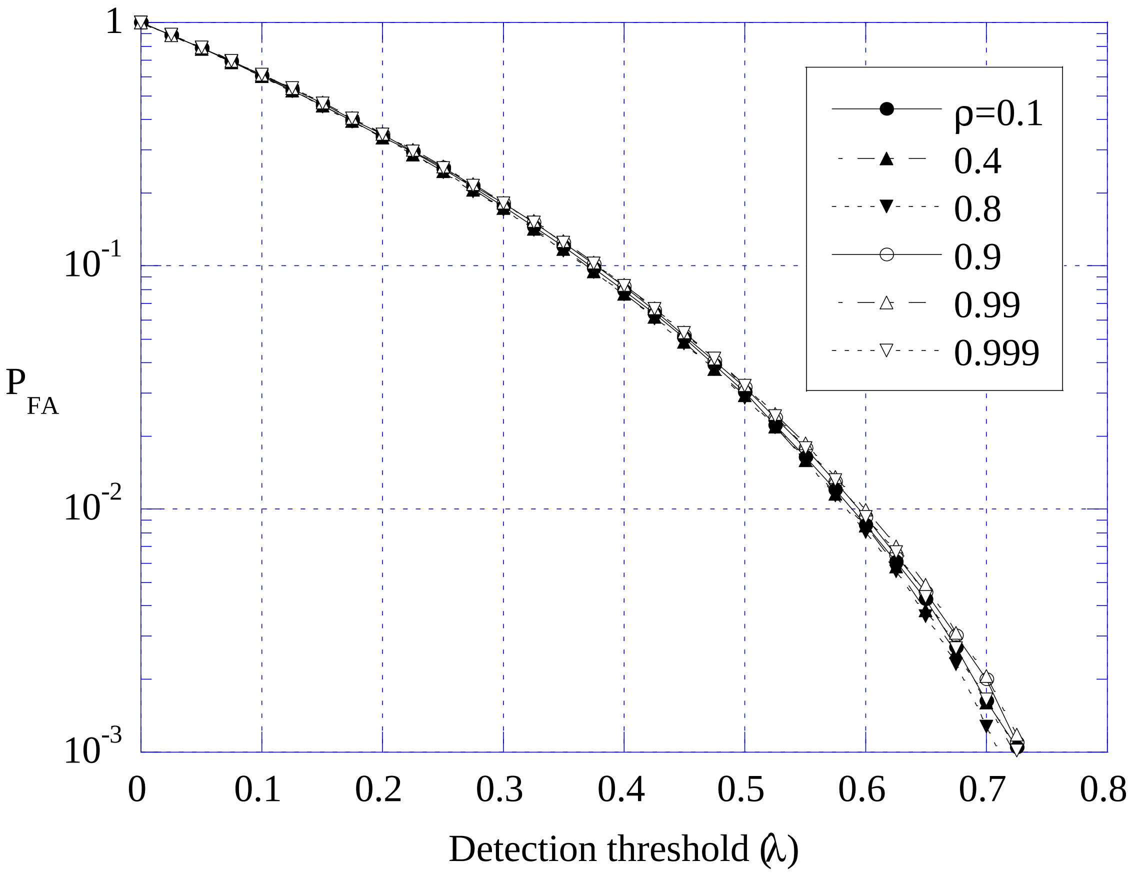
- For this purpose, we simulate K-distributed clutter with an AR(1) speckle correlation function $R_X[m]=\rho^{|m|}$ and shape parameter ν=0.5.
- Changing the one-lag correlation coefficient ρ we change the shape of the clutter power spectral density (PSD).
- Number of temporal pulses is M=8 and the number of secondary vectors is K=3M=24.
- The normalized Doppler frequency of the target is ν_d=0.15.

ANMF in K-distributed clutter



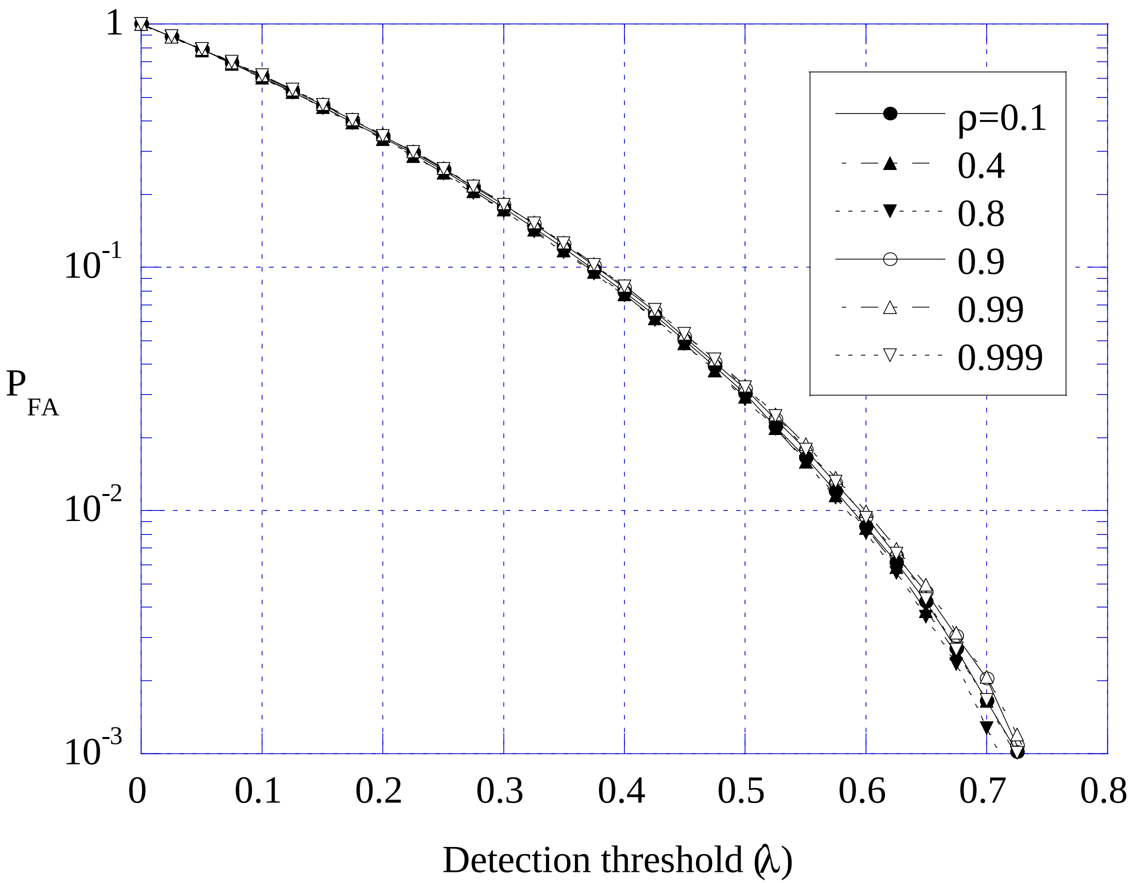
- As expected, the ANMF-NSCM is not CFAR with respect to the matrix **M** (but it is CFAR w.r.t. the texture PDF).

ANMF in K-distributed clutter



- The ANMF-ML is very robust (in practice CFAR) with respect to the matrix **M**.

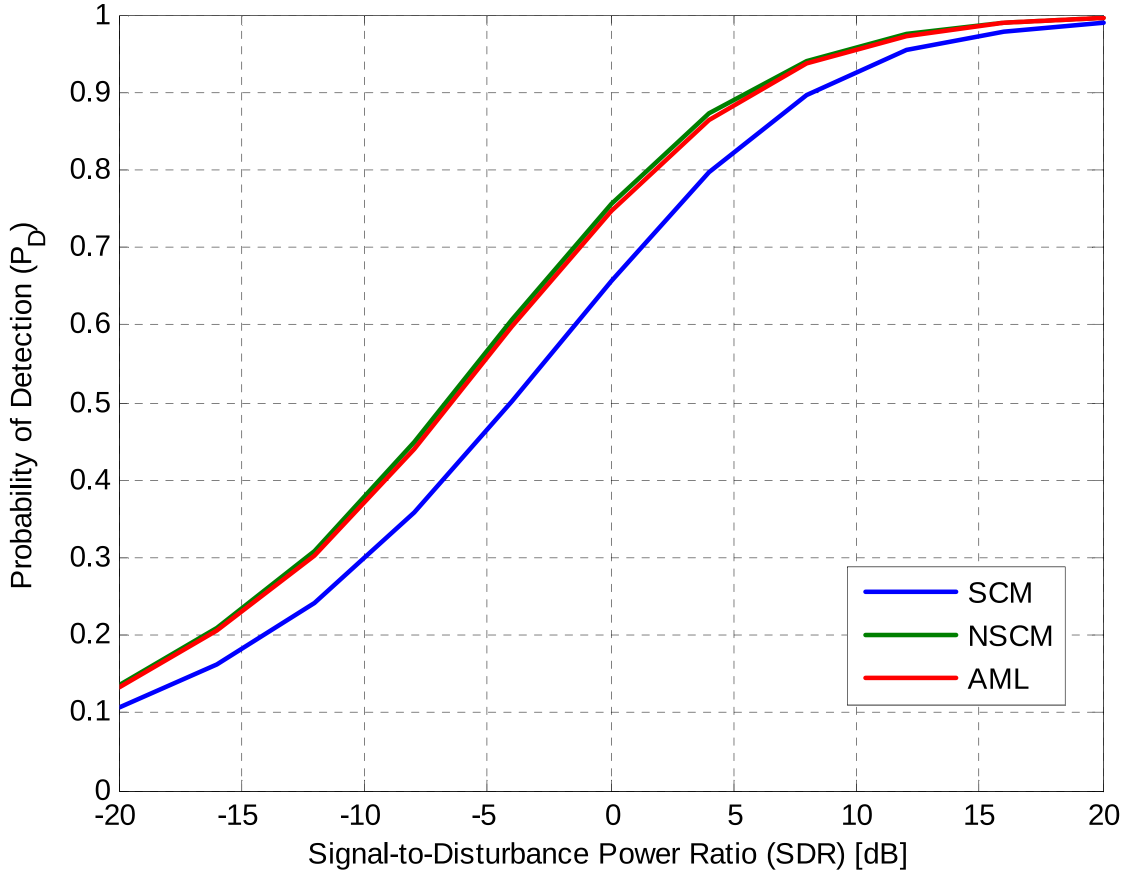
ANMF in K-distributed clutter



- The ANMF-AML is also very robust (in practice CFAR) with respect to the matrix **M** and its performance are very similar to that of the ANMF-ML.

Estimation of the Detection Probability

- Number of Monte Carlo runs: MC=10^4
- Probability of False Alarm: $P_{FA}=10^{-3}$



- M=8 temporal samples
- $K=2M=16$ secondary data vectors
- Swerling I Target: normalized Doppler frequency ν_d=0.25
- Clutter: correlated, K-distributed, with ν=0.5 and ρ=0.9

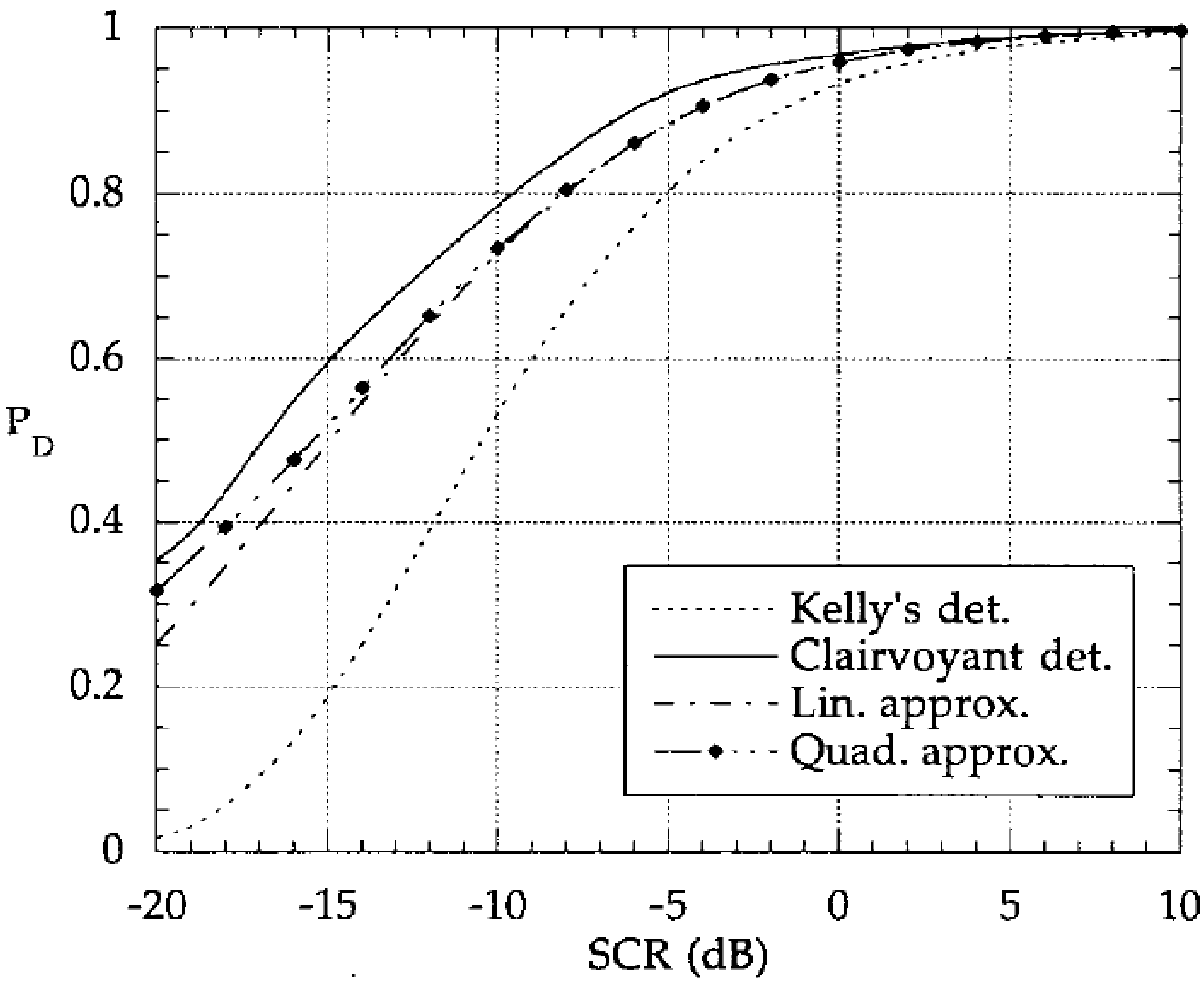


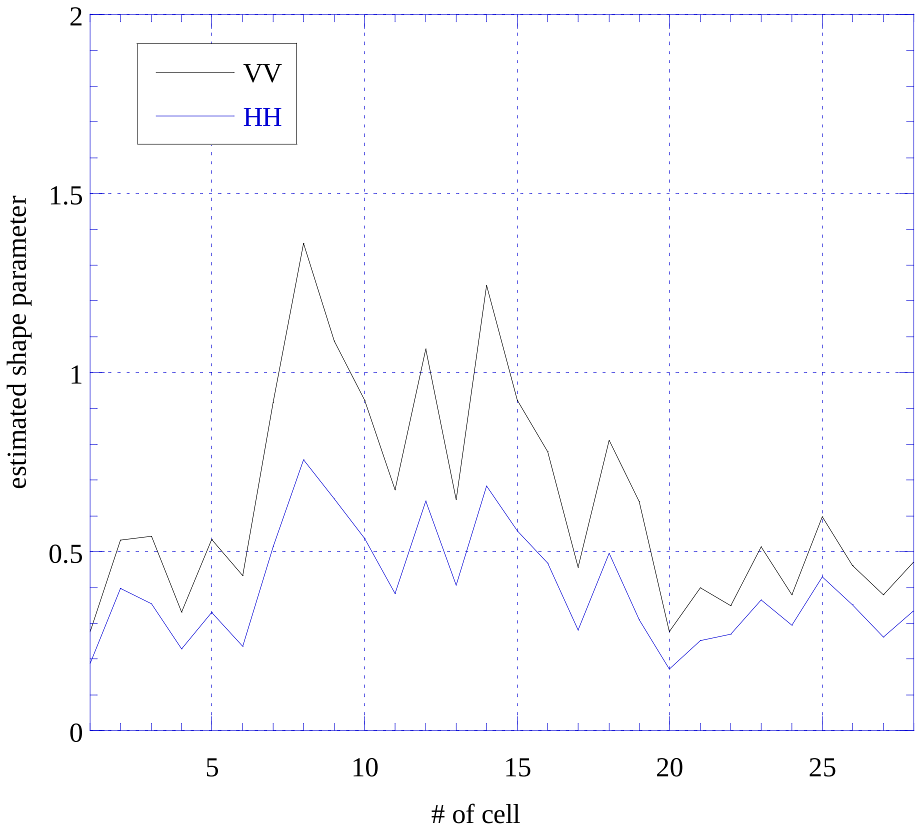
Fig. 9. P_D as a function of SCR for the *clairvoyant* optimum detector and the adaptive detectors; $P_{FA} = 10^{-4}$, $m = 16$, $K = 24$, $\nu = 0.5$, $\rho = 0.9$.

Sea clutter non-stationarity

Problem: if the number *K* of secondary data vectors is too large the assumption that the secondary data are homogeneous is usually violated.

A statistical analysis of real data has been performed:

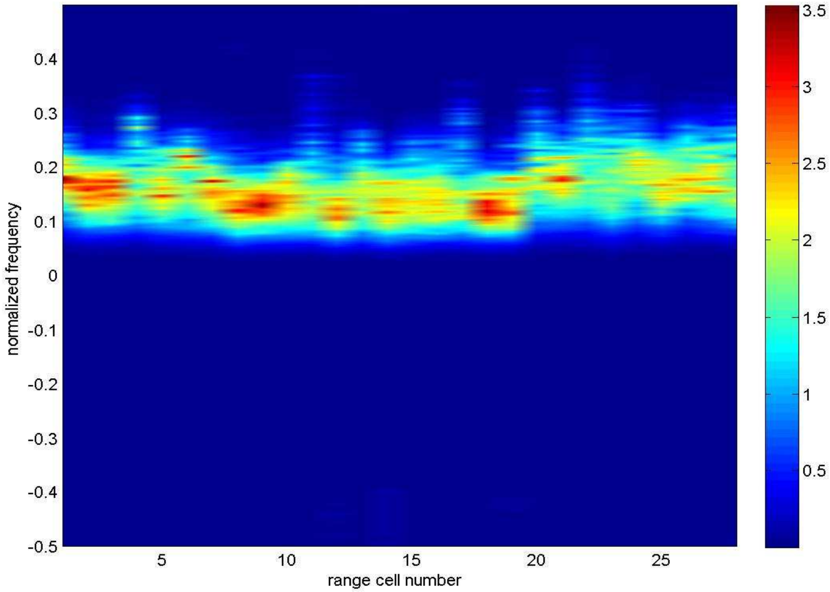
Reasonable fit of the data to the K-model, but the shape parameter ν is not constant on all the range cells.

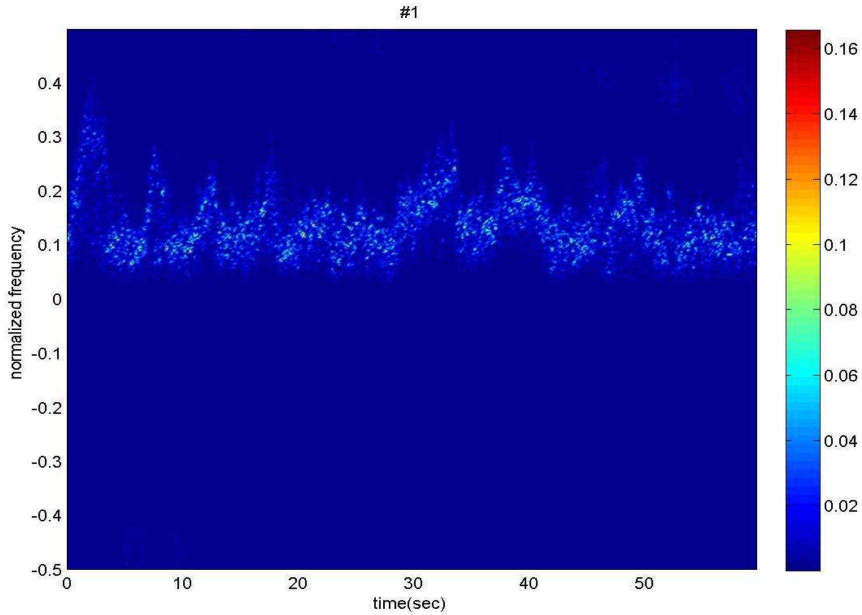


The values of ν for the HH data are always lower than those for the VV data. The ratio between the parameters of the VV data and the HH data is close to 0.6 for each cell and each analyzed file.

Sea clutter non-stationarity

VV data





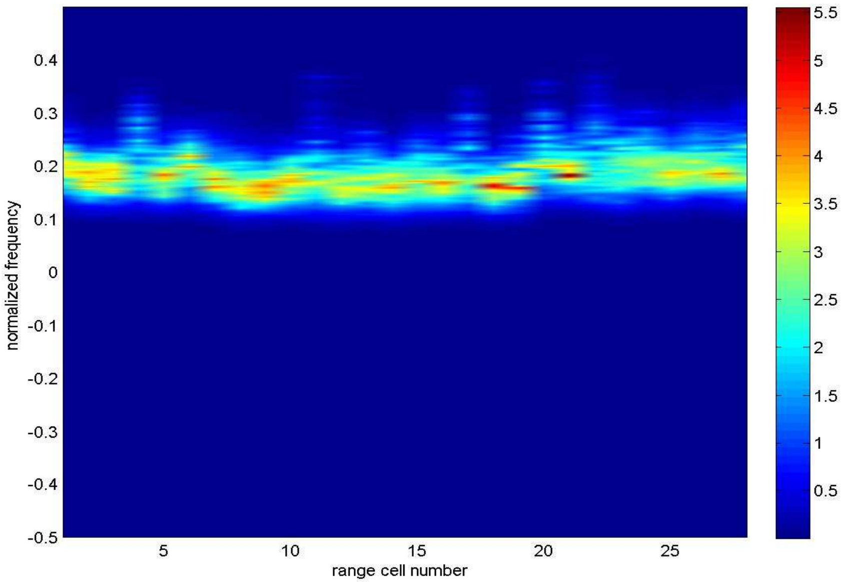
PSD of sea clutter changes from cell to cell:
Spatial non-stationarity

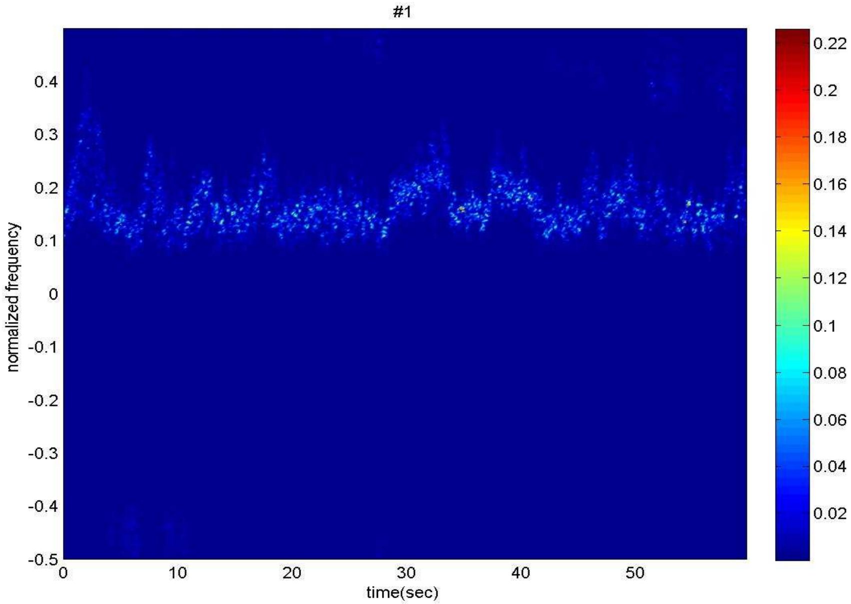
PSD of sea clutter changes with time:
Temporal non-stationarity

The peak of the PSD is in the range [0.1, 0.2] but changes with space and time.

Sea clutter non-stationarity

HH data





PSD of sea clutter changes from cell to cell:
Spatial non-stationarity

Temporal non-stationarity
The PSD exhibits a periodic behavior due to the contribution of the long waves.

Sea clutter non-stationarity

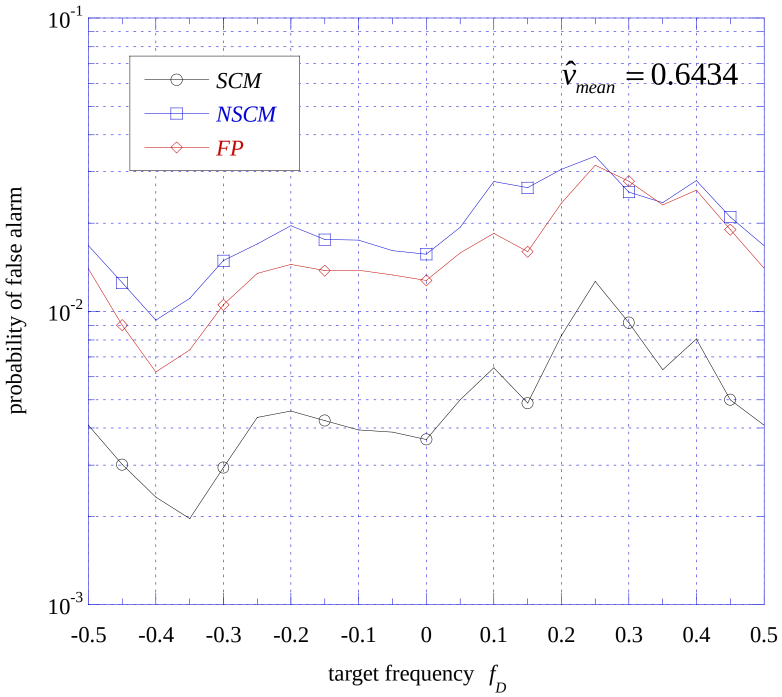
- The sea clutter shows a good fit to K-model but the shape parameter of the PDF changes from cell to cell.
- The speckle PSD is not constant in time and space. The clutter is not spatially and temporally stationary (not homogenous).
- The spectrogram evidences some temporal periodicity in spectrum PSD behavior.
- Cell under test and secondary data vectors do not share exactly the same same covariance structure.

Performance analysis

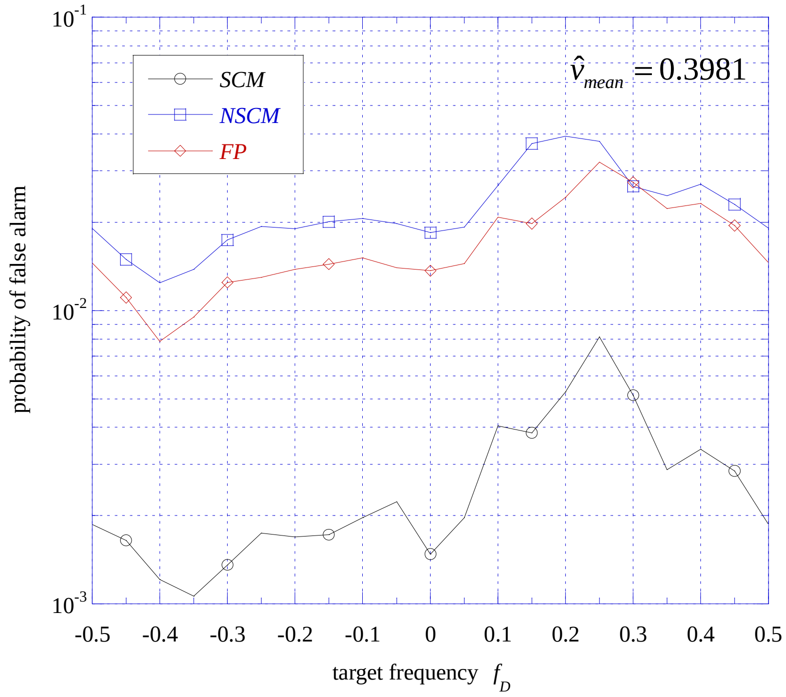
- We generated K-distributed clutter with:
  - covariance matrix equal to the average covariance matrix of the measured data $\mathbf{R}_{\text{mean}}$
  - shape parameter ν equal to the average shape parameter of the measured data ν_{mean}
- We set the threshold for a nominal $P_{\text{FA}}=10^{-2}$ and 10^{-3} in the ANMF with each of the three covariance matrix estimators: SCM, NSCM, AML, a.k.a. Fixed Point (FP) estimator.
- We fed the ANMF with the real data and plot the “true” P_{FA}

Performance comparison: P_{FA}

VV data



HH data



Performance comparison: P_{FA}

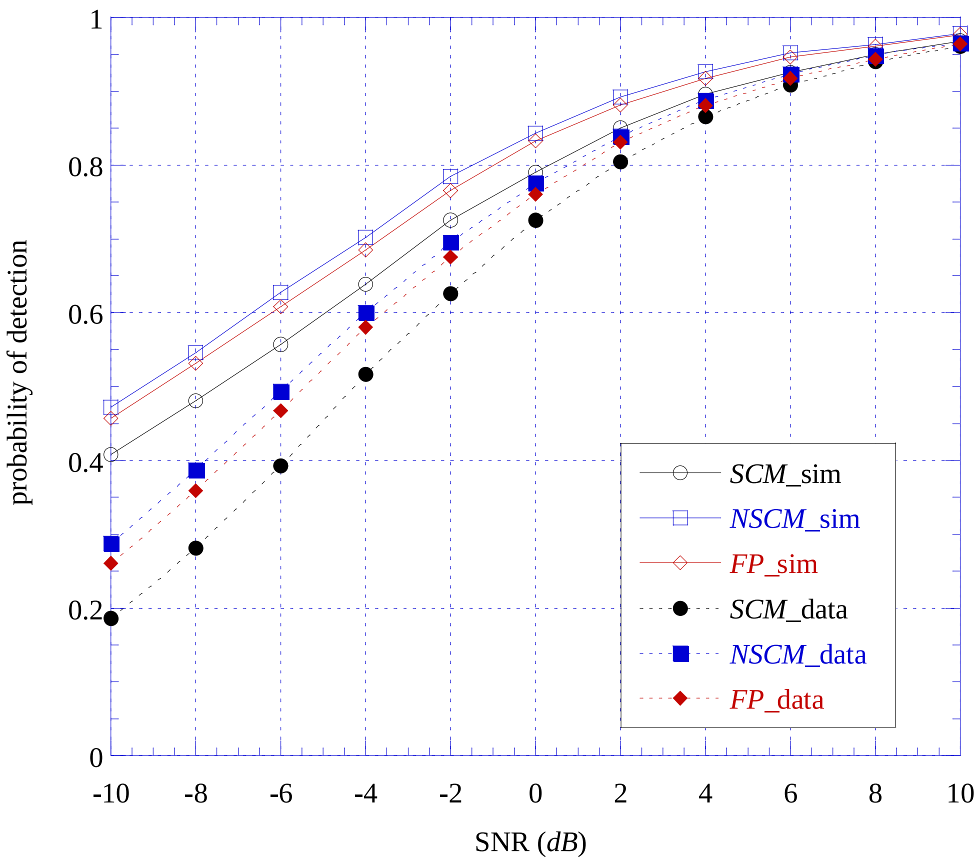
- The actual P_{FA} of ANMF-NSCM and ANMF-AML are close to the nominal one for f_D in the noise floor. For f_D close to the PSD peak, the real P_{FA} is higher.

- The actual P_{FA} of the ANMF-SCM is almost always lower than the nominal one.

- The deviations from nominal value are higher where the spectrum variations due to clutter non-stationarity are greater (it is the peak of the PSD or the Doppler centroid that moves with the long waves originating the almost periodic behavior of the spectrogram).

- The differences in the ANMF-SCM can be mostly due to the non-stationarity of the shape parameter ν, more than to the non-stationarity of the covariance matrix. The ANMF-SCM is particularly sensitive to the clutter texture PDF.

Performance comparison: P_D



VV data

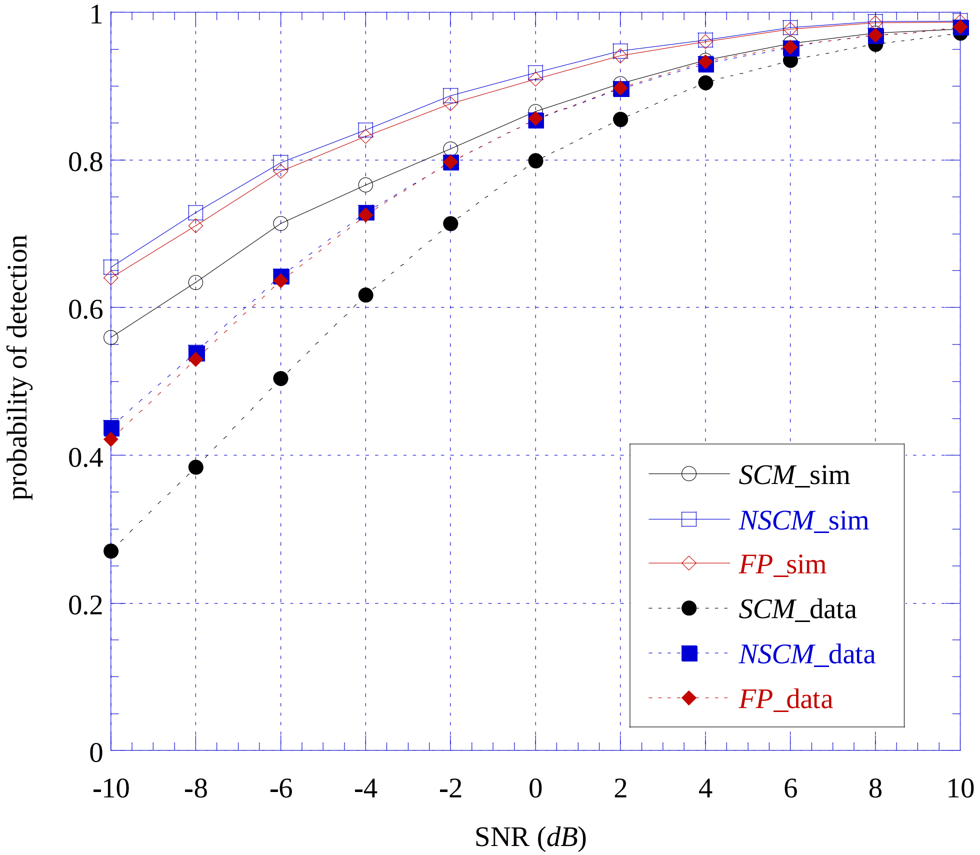
M=8, K=16

Nominal P_{FA}=10^{-2}

ν_d=0

The actual P_D is always lower that the nominal one for each detector and matrix estimator. This is always true also in cases where the actual P_{FA} is higher than the nominal one.

Performance comparison: P_D



HH data

M=8, K=16

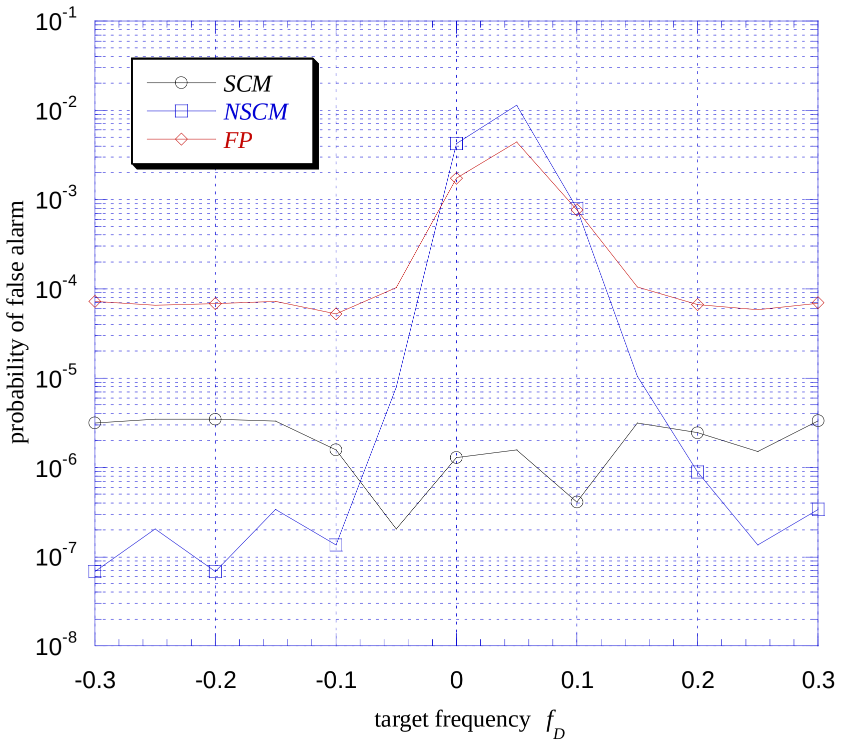
Nominal P_{FA}=10^{-2}

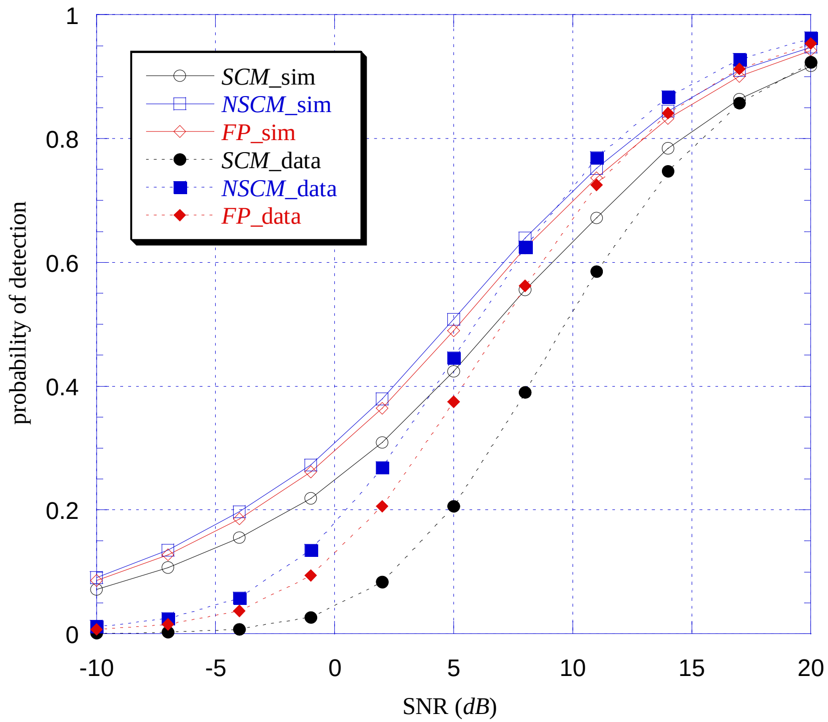
ν_d=0

The non-stationarity of the clutter also influences the P_D of the ANMF.

Performance comparison

VV data, M=8, *K*=16, ν_d=0, nominal P_{FA}=10^{-3}





- The impact of the non-stationarity is increasingly stronger with decreasing P_{FA}.

Final Remarks

- Different non-parametric covariance estimates (SCM, NSCM, AML) have been derived and tested with real data on ANMF detector.
- Differences in P_{FA} and P_D between real and simulated data. Most of these differences are very likely due to the spatial and temporal non-stationarity of real clutter data.
- Impact of non-stationarity increases with clutter spikiness and decreasing P_{FA}.
- When clutter is non-homogenous the number K of secondary data should be low.
- Instead of non-parametric methods, we can adopt model-based methods (typically based on AR modeling of the clutter process) to estimate **M**.
- Performance can be improved by integrating prior knowledge on the scenario, e.g. Digital Terrain Mapping (DTM), in a Bayesian estimator.
- Promising improved performance: cognitive radar, and adaptive waveform design.

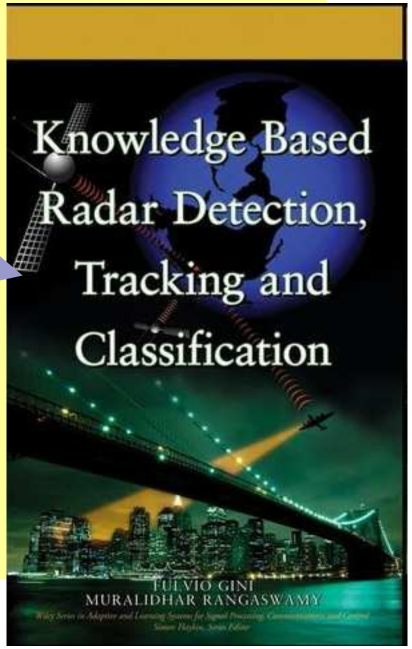
To learn more ….

- Intelligent Adaptive Signal Processing

→ Knowledge Based Systems (KBS) for Radar

F. Gini (Guest Editor), Special Issue on “Knowledge based systems for adaptive radar,” *IEEE Signal Processing Magazine, Vol. 23, No. 1, pp. 14-17, January 2006.*

Knowledge based radar detection, tracking and classification, F. Gini and M. Rangaswamy editors, John Wiley & Sons, Inc., Hoboken, New Jersey, 2008.



To learn more ….

- Adaptivity of Transmission $\rightarrow$ Waveform Diversity and Design (WDD)

  A. Nehorai, F. Gini, M. Greco, A. P. Suppappola, M. Rangaswamy. Special Issue on "Adaptive waveform design for agile sensing and communication," *IEEE Journal of Selected Topics in Signal Processing, Vol. 1, No. 1, pp. 2-5, June 2007.*

  Waveform Design and Diversity for Advanced Radar Systems, F. Gini, A. De Maio, L. K. Patton editors, IET, Radar Sonar and Navigation Series 22, 2012.