E9 231: Digital Array Signal Processing

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Topics to be covered

- Finish Null Steering
- Asymmetric beams
- Beam Space Processing
- Broadband Array

Homework No. 3

- 3.5.5 a, c, e
- 3.7.2
- List atleast six properties of$P_C = C(C^H C)^{-1} C^H$

1 Null Steering Continued ...

We have desired $B_d(\Psi) = \mathbf{W}_d V(\Psi)$
We try to get as close as possible to $B_d(\psi)$, while satisfying additional null constraints.
$P_C = C(C^H C)^{-1} C^H$ Projection Matrix
$C^H \mathbf{W} = 0 \Rightarrow \mathbf{W} \in \mathcal{N}(C^H)$

1.1 Recap of Linear Algebra

$\mathcal{R}(A) = \{\mathbf{y} | \mathbf{y} = A\mathbf{x} \quad \text{for some } \mathbf{x}\}$

$\mathcal{N}(A) = \{\mathbf{x} | A\mathbf{x} = 0\}$

$\mathcal{R}(C) \perp \mathcal{N}(C^H)$

$\mathbf{x}_1 \in \mathcal{N}(C^H)$

$\mathbf{y} = C\mathbf{x}_2$
$\mathbf{y}^H \mathbf{x}_1 = \mathbf{x}_2^H C^H \mathbf{x}_1 = 0$

$P_C \mathbf{x} \in \mathcal{R}(C) \quad \forall \quad \mathbf{x}$

$P_C^{\perp} = I - P_C$

$\mathbf{y}_1 = P_C^{\perp} \mathbf{x}$

$\mathbf{y}_2 = P_C \mathbf{x}$

$\mathbf{x} = \mathbf{y}_1 + \mathbf{y}_2, \quad \mathbf{y}_1^H \mathbf{y}_2 = 0 = \mathbf{x}^H (I - P_C^H) P_C \mathbf{x}$

and $\quad P_C^H P_C = C(C^H C)^{-1} C^H C (C^H C)^{-1} C^H = P_C$

Thus,

$$\begin{aligned} ||\mathbf{W}_d - \mathbf{W}||^2 &= ||P_C(\mathbf{W}_d - \mathbf{W}) + (I - P_C)(\mathbf{W}_d - \mathbf{W})||^2 \\ &= ||P_C(\mathbf{W}_d - \mathbf{W})||^2 + ||P_C^{\perp}(\mathbf{W}_d - \mathbf{W})||^2 \end{aligned}$$

But $C^H \mathbf{W} = 0 \quad \Rightarrow \mathbf{W} \in \mathcal{N}(C^H) \quad \Rightarrow P_C \mathbf{W} = 0$

$$\begin{aligned} ||\mathbf{W}_d - \mathbf{W}||^2 &= ||P_C \mathbf{W}_d||^2 + ||P_C^{\perp} \mathbf{W}_d - P_C^{\perp} \mathbf{W}||^2 \\ &\geq ||P_C \mathbf{W}_d||^2 \quad \because \mathbf{W} \in \mathcal{N}(C^H) \end{aligned}$$

Equality holds when $P_C \mathbf{W}_d = \mathbf{W}$

$\mathbf{W}_{opt} = (I - C(C^H C)^{-1} C^H) \mathbf{W}_d$

2 Asymmetric Beam (mainly for DOA estimation)

Figure 1:

If beam pattern has a non zero slope at steering angle can use it in a closed loop system to find the DOA.

2.1 Difference Beam

Define an asymmetric beam

$$W[n] = \begin{cases} -W[n], & n = -(N-1)/2, -(N-1)/2+1, ..., -1, 1, ..., (N-1)/2 \\ 0, & n = 0 \end{cases},$$

$$B_d = e^{-j(N-1)\psi/2} \sum_{n=0}^{N-1} e^{jn\psi} W_n^*$$

Uniform weighting $W_n = 1/N, \quad n \geq 1$

$B_d(\Psi) = \frac{2j \sin((N+1)\psi/2) \sin((N-1)\psi/4)}{(N \sin(\psi/2))}$

2.2 Properties

- Purely imaginary
- $B_d(\psi) = 0$ for $\psi = 0, \pm(\frac{4\pi}{N+1})$
- $\frac{dB_d(\psi)}{d\psi}|_{\psi=0} = N/4$
- 1980's used for DOA estimation "Monopulse Radar"

3 Minimum Redundancy Array

Non uniform array used.

Obtained by thinning out a ULA.

Define $E\{x(t, id), x^*(t, jd)\} \triangleq R_x((i-j)d)$

If we have *—d—*—d—* , we can obtain $R_x(d), \quad R_x(2d)$

If we have *—d—*——2d——*, we can obtain $R_x(d), \quad R_x(2d), \quad R_x(3d)$

Same as a 4 element ULA.

3 element array: minimum redundancy array.

3.1 4 element example

————.——.——*——.——*

we can get $R_x(d), \quad R_x(2d), \quad R_x(3d), \quad R_x(4d), \quad R_x(5d), \quad R_x(6d)$

'*'–Sensor present
'.'–Sensor absent

same as with a 7 element ULA.

Q: How many distinct R_x values we can have at most with N element array?

A: $\binom{N}{2} = \frac{N(N-1)}{2}$

"Perfect" MRA : Array that allows you to measure $R_x(d), ..., R_x\left(\frac{N(N-1)}{2}d\right)$

Perfect MRA doen't exist for $N > 4$

One way to measure redundancy

$C(\gamma) = \sum_n W_n W_{n+\gamma}$

$$\begin{aligned} W_n &\triangleq 1 \quad \text{if sensor is present at location} \\ W_n &\triangleq 0 \quad \text{if sensor is not present at location} \end{aligned}$$

Four Element Example $W[n] = [1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1]$

If $C(\gamma) = 2$, then there are 2 ways to measure $R_x(\gamma)$

Figure 2:

N-element ULA

Figure 3:

3.2 Possible Approaches

1. Minimum Redundant Array (MRA):
   Maximize the number of lags with minimum redundancy and no gaps

$C(\gamma) \neq 0 \quad \forall \quad \gamma \quad < \quad \gamma_{max}$

But some $C(\gamma)$ may be $\quad > \quad 1$

$$\begin{aligned} N_a &\triangleq \text{autocorrelations that can be computed} \\ &= \frac{N(N-1)}{2} - N_R + N_H \end{aligned}$$

where N_R is the number of redundancies, N_H is the number of holes

For a MRA , $N_H = 0$

So wish to maximize $N_a = \frac{N(N-1)}{2} - N_R$

2. Non Redundant Array $C(\gamma) = 1$ or 0 except at $\gamma = 0 \Rightarrow$ may have "holes". Try to maximize number of lags with $C(\gamma) = 1$ or 0 except at $\gamma = 0$.

   Price: Sidelobe rejection not as good as ULA with same number of lags.

4 Beam Pattern Design Algorithms

1. **Design 1** Maximize Directivity

   $D = \frac{B(u_T)}{0.5 \int_{-1}^{1} |B(u)|^2 du}, \qquad B(u_T) = 1$ - distortionless constraint

$$\begin{aligned} \min_W \quad & 0.5 \int_{-1}^{1} |B(u)|^2 du \\ \text{subject to} \quad & B(u_T) = W^H V(u_T) = 1 \end{aligned}$$

$$\begin{aligned} 0.5 \int_{-1}^{1} |W^H V(u)|^2 du &= W^H [0.5 \int_{-1}^{1} V(u) V^H(u) du] W \\ &= W^H A W \end{aligned}$$

   Solving, $W_{opt} = A^{-1} V(u_T) [V^H(u_T) A^{-1} V(u_T)]^{-1}$

   For standard ULA, $A = I$ and $W_{opt} = \frac{V(u_T)}{N}$ as expected.

2. **Design 2** Add more constraints on the sidelobe level
   Devide u-space into $\Omega_1, ..., \Omega_r$

   Let $B_{di}(u)$ = desired response in sector i $\exists W_{di}$

Assume that $B_{di}(u) = W_{di}^H V(u)$

$$\begin{aligned}
\epsilon_i^2 &= \int\limits_{\Omega i} |B_{di}(u) - B(u)|^2 du \\
&= (W_{di} - W)^H \left(\int_{\Omega i} V(u) V^H(u) du \right) (W_{di} - W) \\
&= (W_{di} - W)^H Q_i (W_{di} - W)
\end{aligned}$$

where $Q_i = \int_{\Omega i} V(u) V^H(u) du$.

New optimization problem

$$\begin{aligned}
\min_W \quad & W^H A W \\
\text{subject to} \quad & W^H V(u_T) = 1 \\
\epsilon_i^2 = (W_{di} - W)^H Q_i (W_{di} - W) \leq \quad & L_i
\end{aligned}$$

Need to use some standard numerical toolboxes to solve the above problem.