E9 231: Digital Array Signal Processing

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1 Topics

- HW 2: 2.6.2, 2.6.6 and 3.26
- Spectral Weighting
- Array Response and Z-transform

1.1 Different Weight Functions

- Uniform Weighting:
$$w(n) = \frac{1}{N}, \quad 0 \le n \le N-1 \tag{1}$$
- Cosine Weighting:
$$w(\tilde{n}) = \sin\left(\frac{\pi}{2N}\right)\cos\left(\frac{\pi\tilde{n}}{N}\right), \quad -\frac{N-1}{2} \le \tilde{n} \le \frac{N-1}{2} \tag{2}$$
- Raised Cosine Weighting:
$$w(\tilde{n}) = c(p)(p + (1-p)\cos\left(\frac{\pi\tilde{n}}{N}\right), \quad -\frac{N-1}{2} \le \tilde{n} \le \frac{N-1}{2} \tag{3}$$
$$c(p) = \frac{p}{N} + \frac{(1-p)}{2}\sin\left(\frac{\pi}{2N}\right) \tag{4}$$
where c(p) is a normalisation constant chosen so that $B_u(0) = 1$ and $p \in [0 \quad 1]$. This is a mixing of uniform and cosine weighting. As p increases the main lobe gets narrower.
- $Cosine^m$ Weighting:
$$w_m(\tilde{n}) = c_m \cos^m\left(\frac{\pi\tilde{n}}{N}\right), \quad m = 2, 3, 4.... \tag{5}$$
when $m = 2$, it is called Hann weighting.
- Raised $Cosine^2$ Weighting:
$$w_(\tilde{n}) = c_2(p)\Big[p + (1-p)\cos^2\left(\frac{\pi\tilde{n}}{N}\right)\Big] \quad = \frac{c_2(p)}{2}\Big[(1+p) + (1-p)\cos\left(\frac{2\pi\tilde{n}}{N}\right)\Big] \quad -\frac{N-1}{2} \le \tilde{n} \le \frac{N-1}{2} \tag{6}$$

- Hamming weighting:

$$w(\tilde{n}) = 0.54 + 0.46\cos\Big(\frac{2\pi\tilde{n}}{N}\Big), \qquad -\frac{N-1}{2} \leq \tilde{n} \leq \frac{N-1}{2} \tag{7}$$

  This corresponds to Raised cosine -squared weighting with $p = 0.08$. Hamming window sacrifice lower order sidelobes height to ensure that second sidelobe is lower The Hamming window exploits the characteristics of the rectangular pattern and the cosine-squared pattern to place a null at the peak of the first sidelobe.

- Blackmann-Harris weighting:

  The Blackmann-Harris window places nulls at the peak of the first two sidelobes.

$$w(\tilde{n}) = 0.42 + 0.5\cos\Big(\frac{2\pi\tilde{n}}{N}\Big) + 0.08\cos\Big(\frac{4\pi\tilde{n}}{N}\Big), \qquad -\frac{N-1}{2} \leq \tilde{n} \leq \frac{N-1}{2} \tag{8}$$

Table 1: Coparison of different window functions

Weighting	HPBW	BW_{NN}	First sidelobe Ht	$D_N = \frac{D}{N}$
Uniform	$0.89\frac{2}{N}$	$2\frac{2}{N}$	$-13.1dB$	1
Cosine	$1.18\frac{2}{N}$	$3\frac{2}{N}$	$-23.5dB$	0.816
Hann	$1.44\frac{2}{N}$	$4\frac{2}{N}$	$-31.4dB$	0.664
Hamming	$1.31\frac{2}{N}$	$4\frac{2}{N}$	$-39.6dB$	0.730
Blackmann Harris	$1.65\frac{2}{N}$	$6\frac{2}{N}$	$-56.6dB$	0.577

- Prolate Spheroidal weighting:
  The problem of interest is to develop a weighting that will maximize the percentage of the total power that is concentrated in a given angular region.

  The problem is:

$$\max_{\mathbf{w}} \frac{1}{2\pi}\int_{-\psi_0}^{\psi_0} |B_\psi(\psi)|^2 d\psi \quad s.t \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |B_\psi(\psi)|^2 d\psi = 1 \tag{9}$$

$$B_\psi(\psi) = \mathbf{w}^H V_\psi(\psi) \tag{10}$$

$$\frac{1}{2\pi}\int_{-\psi_0}^{\psi_0} |B_\psi(\psi)|^2 d\psi = \frac{1}{2\pi}\mathbf{w}^H [\int_{-\psi_0}^{\psi_0} V_\psi(\psi) V_\psi^H(\psi) d\psi]\mathbf{w} = \frac{1}{2\pi}\mathbf{w}^H A\mathbf{w} \tag{11}$$

  If $\psi_0 = \pi$, $\int_{-\psi_0}^{\psi_0} V_\psi(\psi) V_\psi^H(\psi) d\psi \quad = \quad 2\pi I_{N\times N}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |B_\psi(\psi)|^2 d\psi = 1 \Rightarrow \mathbf{w}^H\mathbf{w} = 1 \tag{12}$$

  Now the optimisation problem is :

$$\max_{\mathbf{w}} \frac{1}{2\pi}\mathbf{w}^H A\mathbf{w} \quad s.t \quad \mathbf{w}^H\mathbf{w} = 1 \tag{13}$$

The optimisation problem is reformulated as:

$$\max_{\mathbf{w}} \quad \frac{\mathbf{w}^H A \mathbf{w}}{2\pi \mathbf{w}^H \mathbf{w}} \tag{14}$$

The solution is $\mathbf{w} \quad st \quad A\mathbf{w} = \mu\mathbf{w}$
μ is the largest eigen value of A and $\mathbf{w}$ is the corresponding eigen vector.

$$\begin{aligned} A_{mn} &= \int_{-\psi_0}^{\psi_0} e^{j(n-m)\psi} d\psi \\ A_{mn} &= \begin{cases} \frac{\sin(n-m)\psi_0}{(n-m)} & n \neq m \\ 2\psi_0 & n = m \end{cases} \\ &= 2\psi_0 sinc\big((n-m)\psi_0\big) \end{aligned} \tag{15}$$

Eigen vectors of A are known as discrete prolate spheroidal sequences (also known as slepian sequences).

- Kaiser Weighting:

$$w(\tilde{n}) = I_0\left(\beta\sqrt{1-\left(\frac{2\tilde{n}}{N}\right)^2}\right) \tag{16}$$

$I_0(u)$ is the modified Bessel function of the zeroth order.

$$I_0(u) = 1 + \sum_{r=1}^{\infty}\left(\frac{\left(u/2\right)^r}{r!}\right)^2 \tag{17}$$

Kaiser weighting is an approximation to prolate spheroidal weighting. β can be used to adjust the sidelobes level.

- Dolph-Chebyshev Weighting:

$$w(\tilde{n}) = \frac{1}{N}\left(\frac{1}{\gamma} + 2\sum_{k=1}^{N-1} T_k\Big(\beta\cos\Big(\frac{k\pi}{N}\Big)\Big)\cos\Big(\frac{2\pi k\tilde{n}}{N}\Big)\right) \tag{18}$$

All sidelobes are of same level. Minimizes mainlobe width for a given sidelobe level. γ is the ratio of the amplitude of sidelobe to mainlobe.

$$\begin{aligned} \beta &\triangleq \cosh\left(\frac{1}{2M}\cosh^{-1}\left(\frac{1}{\gamma}\right)\right) \\ M &\triangleq \frac{N-1}{2}, N \quad odd \end{aligned} \tag{19}$$

$T_l(x)$ is the l^{th} order chebyshev polynomial.

$$T_l(x) = \begin{cases} \cos(l\cos^{-1}x) & , |x| \leq 1 \\ \cosh(l\cosh^{-1}x) & , |x| > 1 \end{cases} \tag{20}$$

1.2 Array Response and Z-transform

Recall definition of $B(\psi)$

$$B(\psi) = e^{-j(\frac{(N-1)}{2})\psi} \sum_{n=0}^{N-1} w_n^* e^{jn\psi} = e^{-j(\frac{(N-1)}{2})\psi} \left(\sum_{n=0}^{N-1} w_n e^{-jn\psi} \right)^* \tag{21}$$

Define $z \triangleq e^{j\psi}$

$$B_z(\psi) \triangleq \sum_{n=0}^{N-1} w_n z^{-n} \tag{22}$$

$$B_\psi(\psi) = z^{-\left(\frac{N-1}{2}\right)} B_z^*(z)|_{z=e^{j\psi}} \tag{23}$$

So we can say that the array is acting exactly as an FIR filter. For Real symmetric weights $w_n = w_n^*$ and $w_n = w_{N-n}$. To compute zeros of the FIR filter:

$$B(z) = z^{-(N-1)} B_z(z^{-1}) = 0 \tag{24}$$

Zeros of B(z) occur in $4's$. If z_1 is a zero, then zeros occur in $z_1, z_1^{-1}, z_1^*, (z_1^{-1})^*$

- Remarks:
  1. Can use this technique to compare behaviour of different weights.
  2. Zeros on the unit circle lead to sharp null in the array response.
  3. $2^{nd}/3^{rd}$ order zeros would broaden the null.