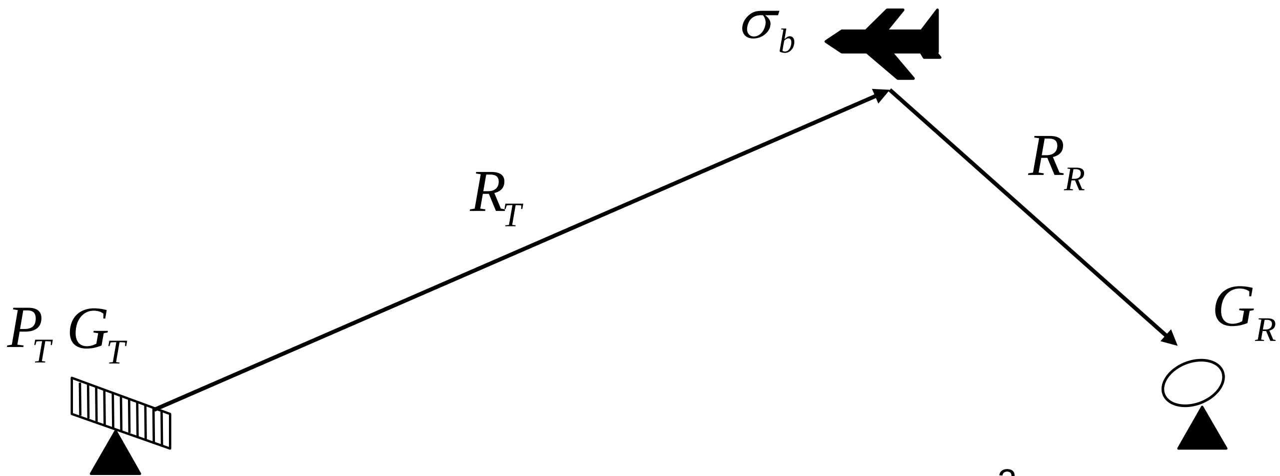
1.5 *Noise and interference*

1. Noise and noise statistics
2. Probability of detection and probability of false alarm
3. Sensor signatures
4. Bistatic radar
5. Low frequency radar
6. Impulse radar

Bistatic radar equation

This is derived in the same way as the monostatic radar equation :



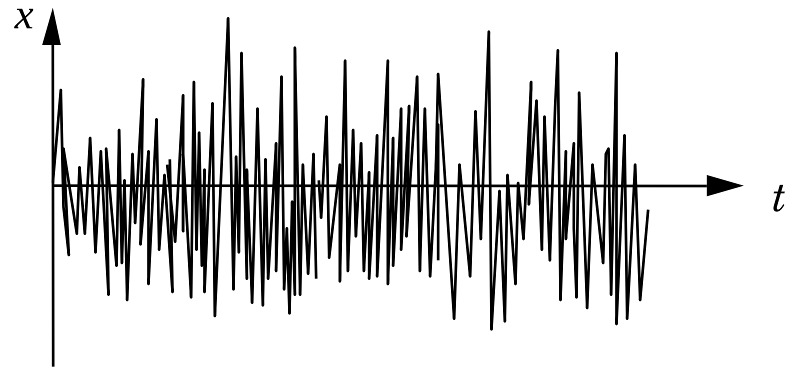
$$\frac{P_R}{P_N} = \frac{P_T G_T}{4\pi R_T^2} . \sigma_b . \frac{1}{4\pi R_R^2} . \frac{G_R \lambda^2}{4\pi} . L_p . \frac{1}{k T_0 B F}$$

$$= \frac{P_T G_T G_R \lambda^2 \sigma_b L_p}{(4\pi)^3 R_T^2 R_R^2 k T_0 B F}$$

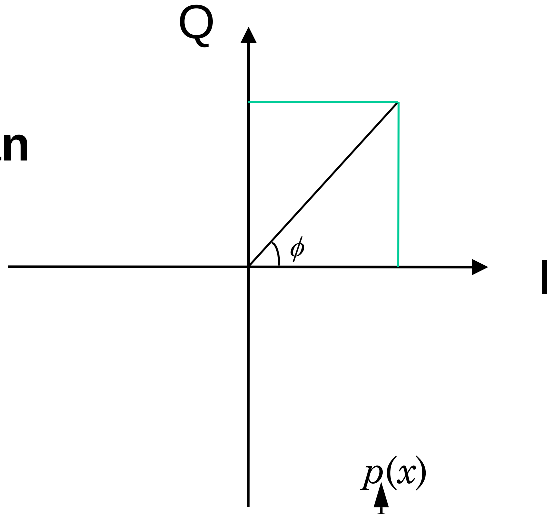
The dynamic range of signals to be handled is reduced, because of the defined minimum range.

Noise

If you were to look at noise on an oscilloscope, you'd see something like :

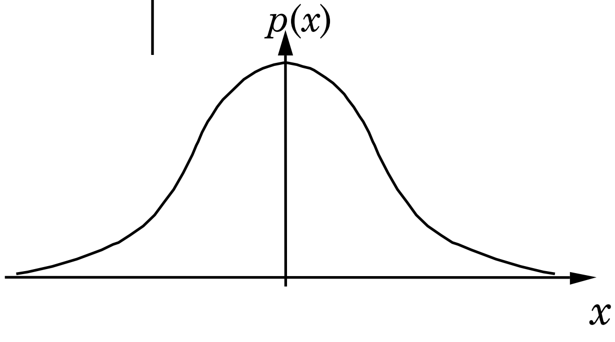


If the conditions of the Central Limit Theorem are satisfied, this means that the I and Q components each have a zero-mean *Gaussian* (normal) pdf :



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

where σ is the standard deviation.



You can verify that $\int_{-\infty}^{\infty} p(x)dx = 1$

Central limit theorem

- a large number of contributions
- phase uniformly distributed over $[0, 2\pi]$
- all of comparable amplitude

Noise

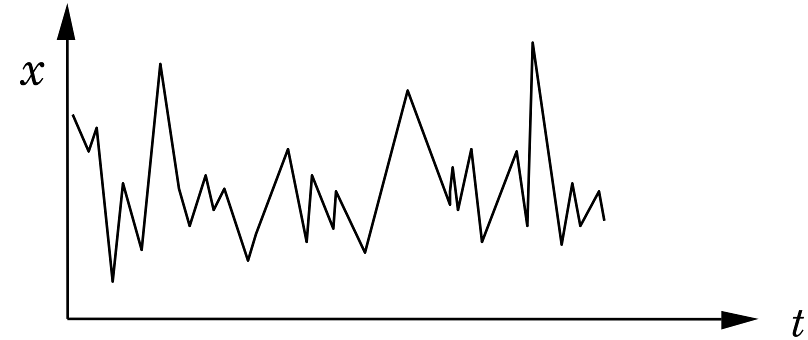
The amplitude of the noise follows a *Rayleigh* distribution :

$$p(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x}{2\sigma^2}^2 \right) \qquad x \geq 0$$

$$\bar{x} = \sqrt{\frac{\pi}{2}}\sigma$$

$$\overline{x^2} = 2\sigma^2$$

$$\overline{x^n} = \sigma^n 2^{n/2} \Gamma(1 + n/2)$$



Noise

and the corresponding distribution of power (as would be measured by a square-law detector) is *negative exponential* :

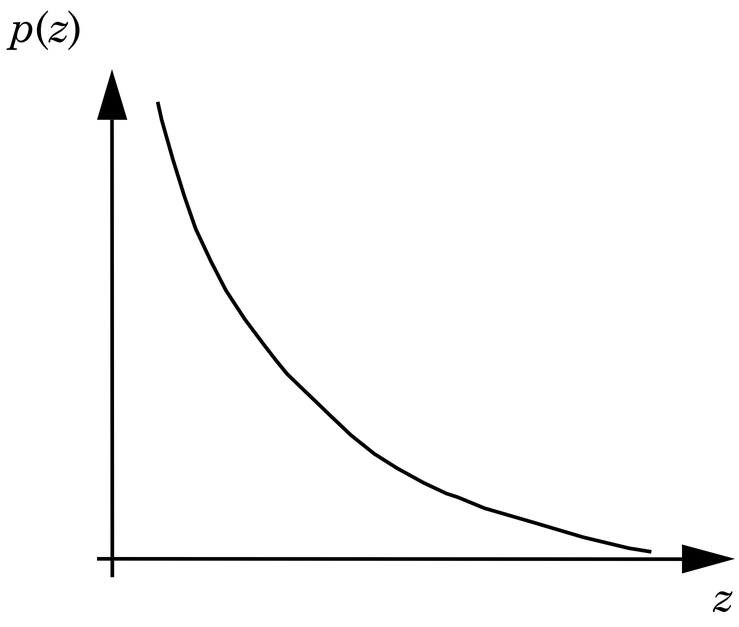
$$p(z) = \frac{1}{2\sigma^2} \exp\left(-\frac{x}{2\sigma^2} \right)$$

for which

$$z = x^2$$

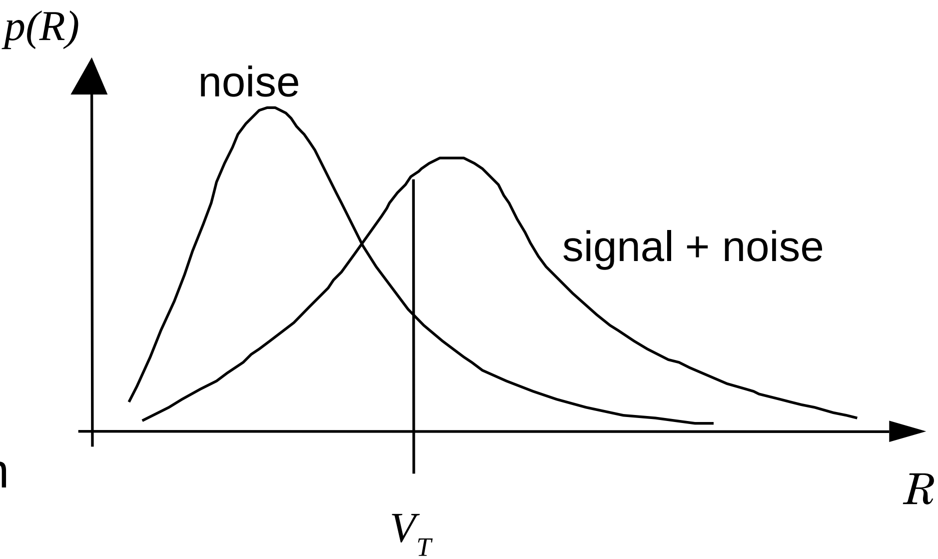
$$\overline{z} = 2\sigma^2$$

$$\overline{z^n} = n!\left(2\sigma^2\right)^n$$



Detection

So detection is a statistical process.



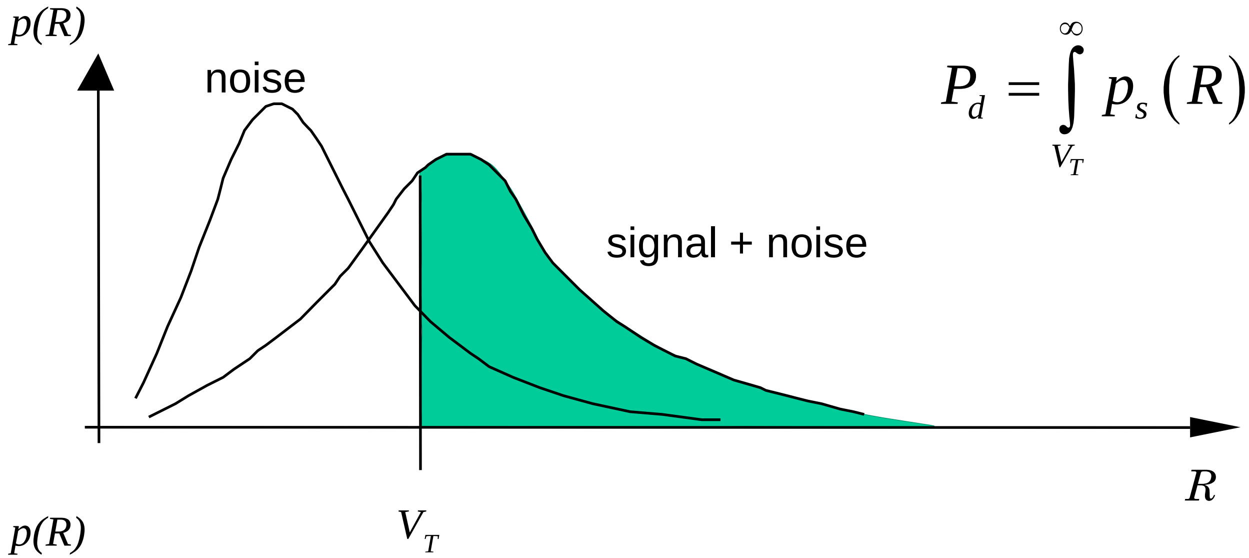
If we set a detection threshold V_T , then there is a finite probability of false alarm on noise alone :

$$P_{FA} = \int_{V_T}^{\infty} p_n\left(R\right)_{\text{noise alone}} dR$$

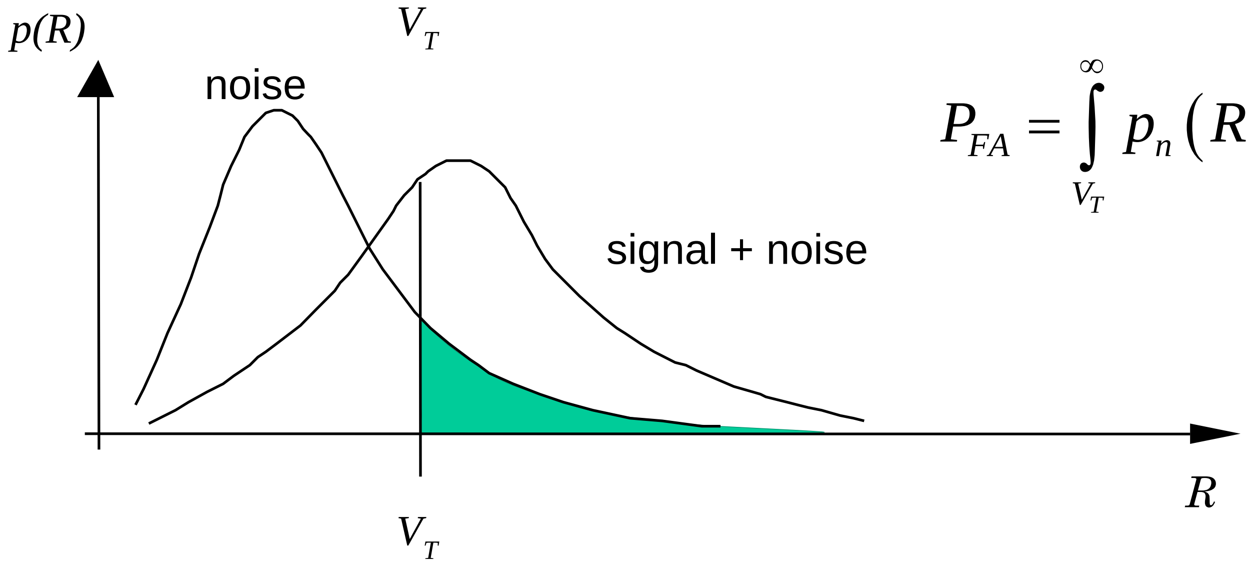
and a finite probability of detection with target present :

$$P_d = \int_{V_T}^{\infty} p_s\left(R\right)_{\text{target+noise}} dR$$

Detection



$$P_d = \int_{V_T}^{\infty} p_s(R)_{\text{target+noise}}\, dR$$



$$P_{FA} = \int_{V_T}^{\infty} p_n(R)_{\text{noise alone}}\, dR$$

Error Probability P_e

- It is desirable to set the threshold in order to minimise the error probability.
- An error occurs when $R > V_T$ **<u>and</u>** there is no target (H_0) or when $R < V_T$ **<u>and</u>** there is a target (H_1).

$$P_e = P\left(R > V_{T,}H_0\right) + P\left(R < V_{T,}H_1\right)$$

Using Bayes' theorem P_e can be expressed as a function of the probability of false alarm and the probability of detection

$$P_e = P(R > V_T|H_0)P(H_0) + [1 - P(R > V_T|H_1)]P(H_1)$$

$$P_e = P_{FA}P(H_0) + (1 - P_d)P(H_1)$$

Bayes probability

- The Bayes detector minimises the error probability P_e
- The decision is made on the quantity $\Lambda(R)$ which is defined as

$$\Lambda(R) = \frac{p(R|H_1)}{p(R|H_0)} = \frac{p_s(R)_{target+noise}}{p_n(R)_{noise\ alone}}$$

- $\Lambda(R)$ is called the **Likelihood ratio**
- It can be demonstrated that the optimal threshold V_T that minimises P_e is

$$V_T = \frac{P(H_0)}{P(H_1)}$$

- The decision is made so that when $\Lambda(R) > V_T$ a target is declared and vice versa when $\Lambda(R) < V_T$ no target is declared
- The Bayes detector relies on prior knowledge of $P(H_0)$ and $P(H_1)$
- These are commonly unknown in practice and other type of detectors need to be used (e.g. Neyman Pearson)

Probability of false alarm P_{FA}

We can evaluate the probability of false alarm on noise alone. Writing the Rayleigh pdf in the form

$$p_n(R) = \frac{R}{\psi_0} \exp\left(-\frac{R^2}{2\psi_0} \right)$$

then :

$$P_{FA} = \int_{V_T}^{\infty} \frac{R}{\psi_0} \exp\left(-\frac{R^2}{2\psi_0} \right) dR \;=\; \exp\left(-\frac{V_T^2}{2\psi_0} \right)$$

Probability of detection P_d

When there is a signal present as well, the output of the envelope detector is represented by the Ricean distribution :

$$p_s(R) = \frac{R}{\psi_0} \exp\left(-\frac{R^2 + A^2}{2\psi_0} \right) I_0\left(\frac{RA}{\psi_0} \right)$$

where $I_0(Z)$ is the modified Bessel function of order zero and argument Z. For Z large, an asymptotic expansion for $I_0(Z)$ is :

$$I_0(Z) \approx \frac{e^Z}{\sqrt{2\pi Z}} \left(1 + \frac{1}{8Z} + \ldots \right)$$

When $A = 0$ this reduces to the pdf for noise alone.

Probability of detection P_d

The probability of detection is then :

$$P_d = \int_{V_T}^{\infty} p_s(R)\; dR \;=\; \int_{V_T}^{\infty} \frac{R}{\psi_0} \exp\left(-\frac{R^2 + A^2}{2\psi_0}\right) I_0\left(\frac{RA}{\psi_0}\right) dR$$

This cannot be evaluated by simple means, and numerical techniques or series approximations must be used. One such approximation gives :

$$P_d = \frac{1}{2}\left(1 - \operatorname{erf} \frac{V_T - A}{\sqrt{2\psi_0}}\right) + \frac{\exp\left[-\left(V_T - A\right)^2 \big/ 2\psi_0\right]}{2\sqrt{2\pi}\left(A\big/\sqrt{\psi_0}\right)} \times \left[1 - \frac{V_T - A}{4A} + \frac{1 + \left(V_T - A\right)^2 \big/ \psi_0}{8A^2 \big/ \psi_0} - \ldots\right]$$

where the error function is defined as :

$$\operatorname{erf} Z = \frac{2}{\sqrt{\pi}} \int_0^Z \exp\left(-u^2\right) du$$

The Marcum Q-function

The probability of detection is then :

$$P_d = \int_{V_T}^{\infty} p_s(R)\, dR = \int_{V_T}^{\infty} \frac{R}{\psi_0} \exp\left(-\frac{R^2 + A^2}{2\psi_0}\right) I_0\left(\frac{RA}{\psi_0}\right) dR$$

This (using the same notation as Mathematica) may be recognised as the *Marcum Q-function* :

$$Q_M(\alpha, \beta) = \frac{1}{Q^{M-1}} \int_{\beta}^{\infty} x^M \exp\left(-\frac{x^2 + \alpha^2}{2}\right) I_{M-1}(\alpha, x)\, dx$$

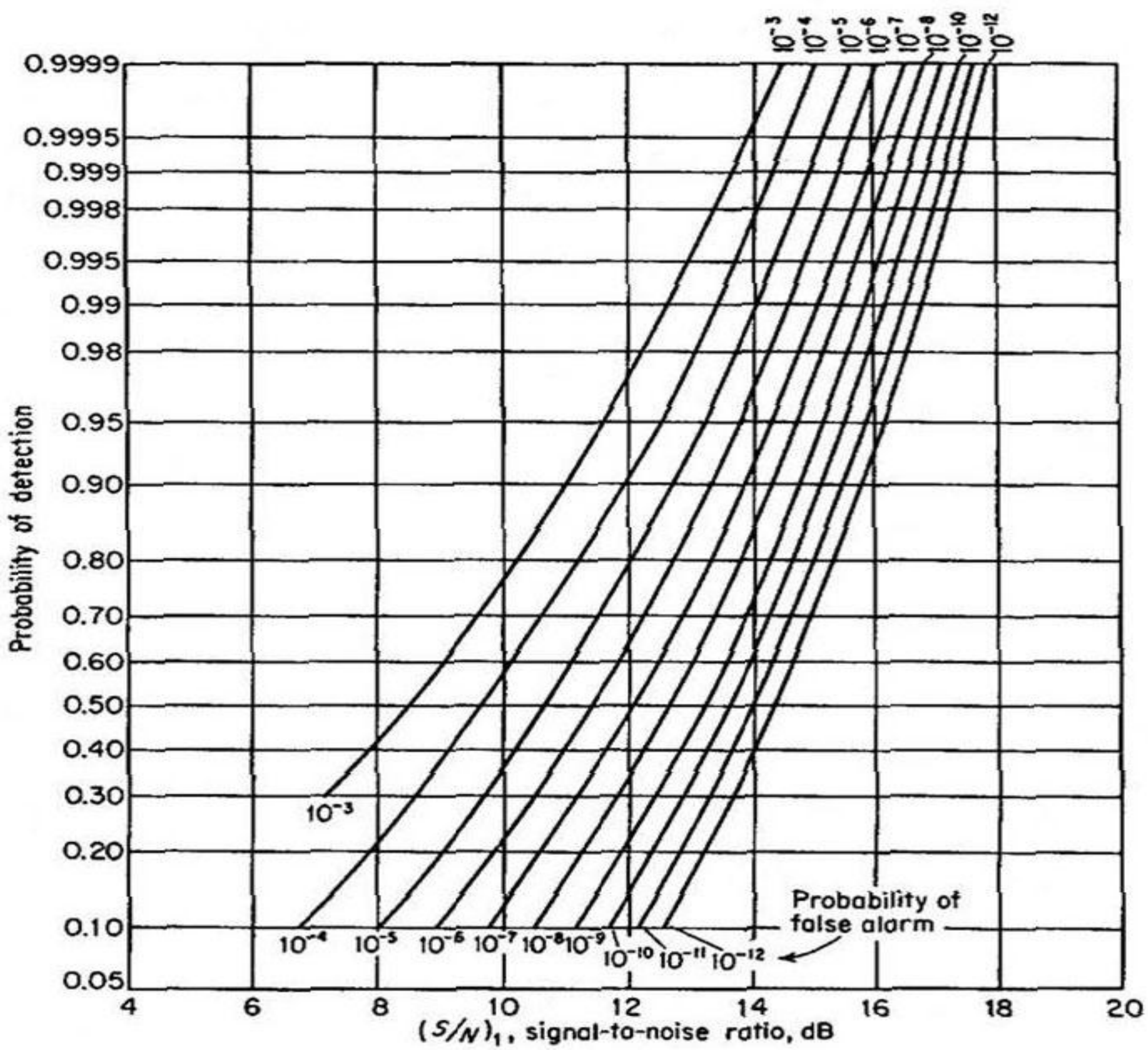
Probability of detection P_d

The signal-to-rms noise voltage ratio is related to signal-to-noise power by :

$$\frac{A}{\sqrt{\psi_0}} = \frac{\text{signal amplitude}}{\text{rms noise voltage}} = \left(\frac{2S}{N}\right)^{1/2}$$

From all of which it is possible to plot curves of probability of detection versus signal-to-noise ratio, for given values of probability of false alarm

P_d vs P_{FA} and SNR



The Neyman-Pearson Criterion

The classical solution to the detection problem is based on hypothesis likelihood testing. There are two hypotheses: H_0 that no target is present, and H_1 that a target is present. The output of the detector processing is a decision, either D_0 that a target is absent, or D_1 that the target is present.

The probability of detection P_d is the conditional probability of the decision D_1 occurring given the situation H_1:

$$P_d = P\left(D_1 \middle| H_1\right) = 1 - P\left(D_0 \middle| H_1\right)$$

where $P\left(D_0 \middle| H_1\right)$ is the probability of missed detection.

The probability of false alarm is therefore given by

$$P_{FA} = P\left(D_1 \middle| H_0\right)$$

The Neyman-Pearson Criterion

The most general criterion for decision thresholding is Bayes' Rule, which assigns costs to the various responses and chooses a decision rule to minimise the total cost.

Rev. Thomas Bayes (1702-1761)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

The Neyman-Pearson criterion sets the maximum allowable false alarm rate and within that constraint maximises the probability of detection.

Formally this amounts to

$$\max\left\{P\left(D_1|H_1\right)\right\} \quad \text{given} \quad P\left(D_1|H_0\right) \le P_{FA}$$

Bayes, Thomas, 'An essay toward solving a problem in the doctrine of chances', *Phil. Trans. Roy. Soc. London*, 53, pp. 370-418 (1763)

The Neyman-Pearson Criterion

The Neyman-Pearson Criterion forms a likelihood ratio test for a given probability of false alarm:

$$\Lambda(y) = \frac{p(\mathbf{y}|H_1)}{p(\mathbf{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

The Neyman-Pearson test requires no assumptions to be made about the forms of the probabilities.

η is calculated as the threshold that provides the given probability of false alarm and it only depends on $p_n(R)_{noise\ alone}$

$$\eta \triangleq \int_{\eta}^{\infty} p_n(R)_{noise\ alone} dR = P^*{}_{FA}$$

In fact it can be shown that the likelihood ratio is identical regardless of the criterion chosen – only the threshold changes if a Bayes or other model is used instead.

Generalised Likelihood Ratio Test

More formally, the problem of detecting an unknown random signal **s** in the presence of an additive disturbance **c** can be formulated in terms of a binary hypothesis test. The two hypotheses are H_0 (signal absent) and H_1 (signal present)

$$H_0: \quad \mathbf{y} = \mathbf{c} \qquad \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \ldots\ldots, N$$

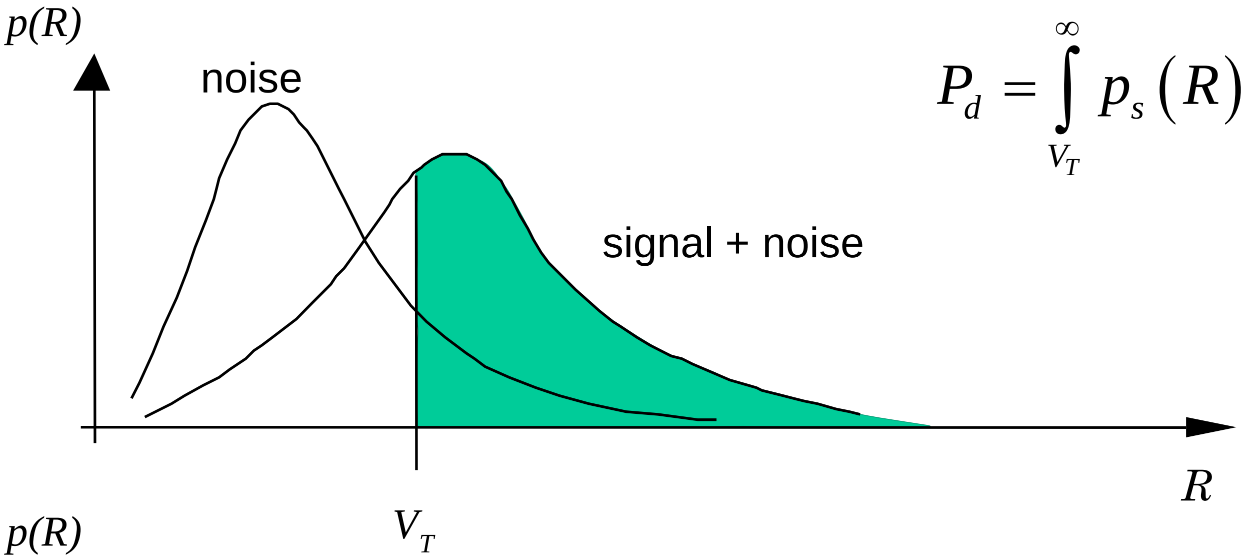
$$H_1: \quad \mathbf{y} = \mathbf{s} + \mathbf{c} \qquad \mathbf{y}_i = \mathbf{c}_i \quad i = 1, \ldots\ldots, N$$

We form the likelihood ratio

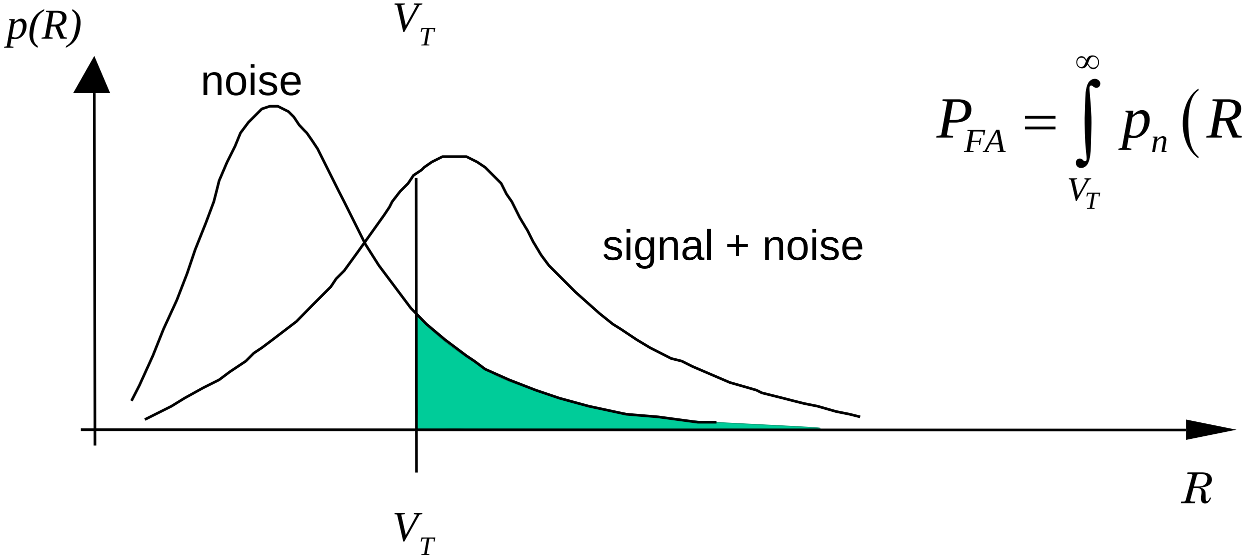
$$\Lambda(y) = \frac{p(\mathbf{y}|H_1)}{p(\mathbf{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

- In this case the probabilities $\mathrm{p(y|H_1)}$ and $\mathrm{p(y|H_0)}$ depend on the unknown signal s
- A solution is to use the maximum likelihood estimate of ***s*** and then calculate the Likelihood ratio (GLRT)

Detection



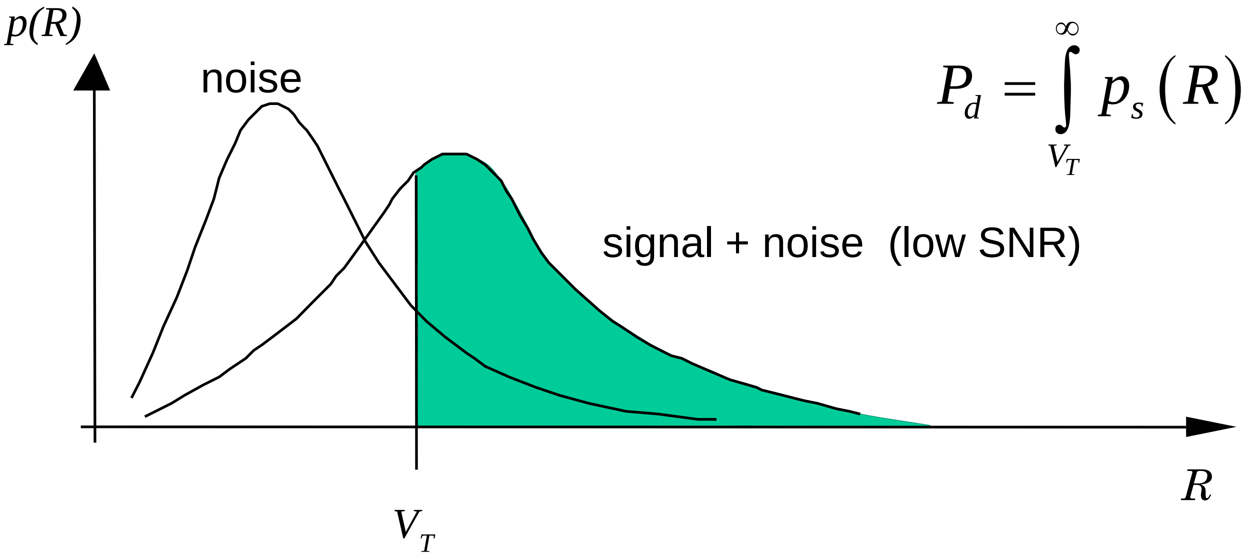
$$P_d = \int_{V_T}^{\infty} p_s\left(R\right)_{\text{target+noise}} dR$$

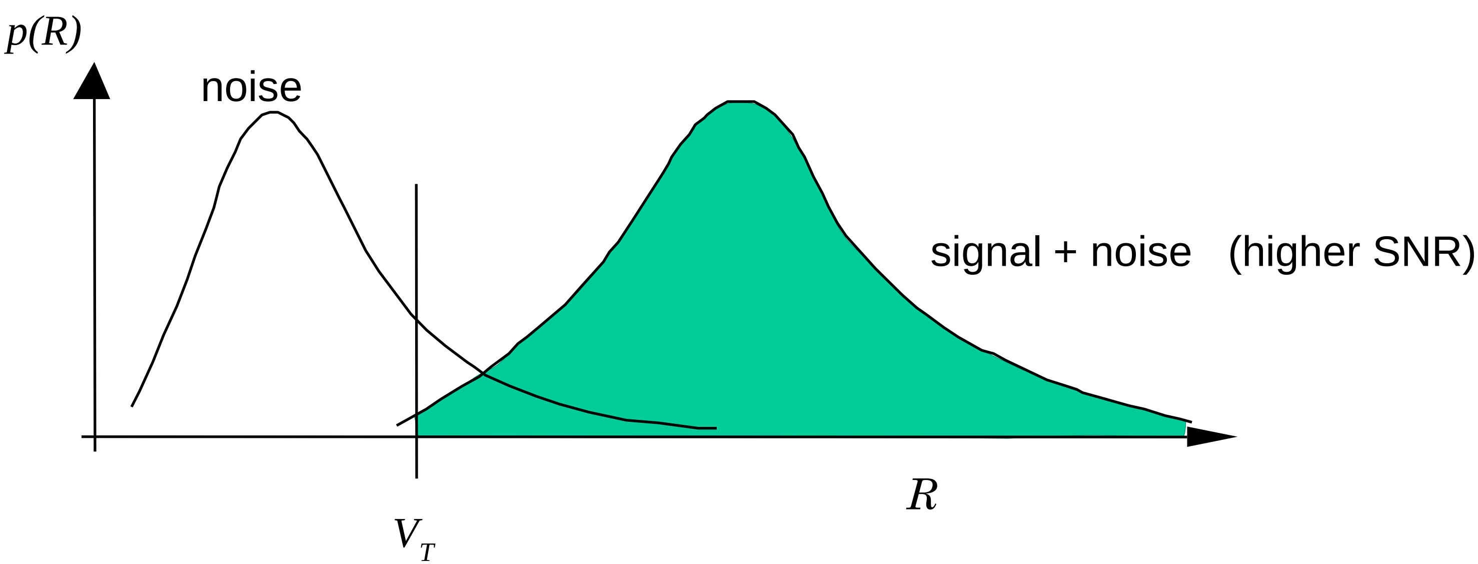


$$P_{FA} = \int_{V_T}^{\infty} p_n\left(R\right)_{\text{noise alone}} dR$$

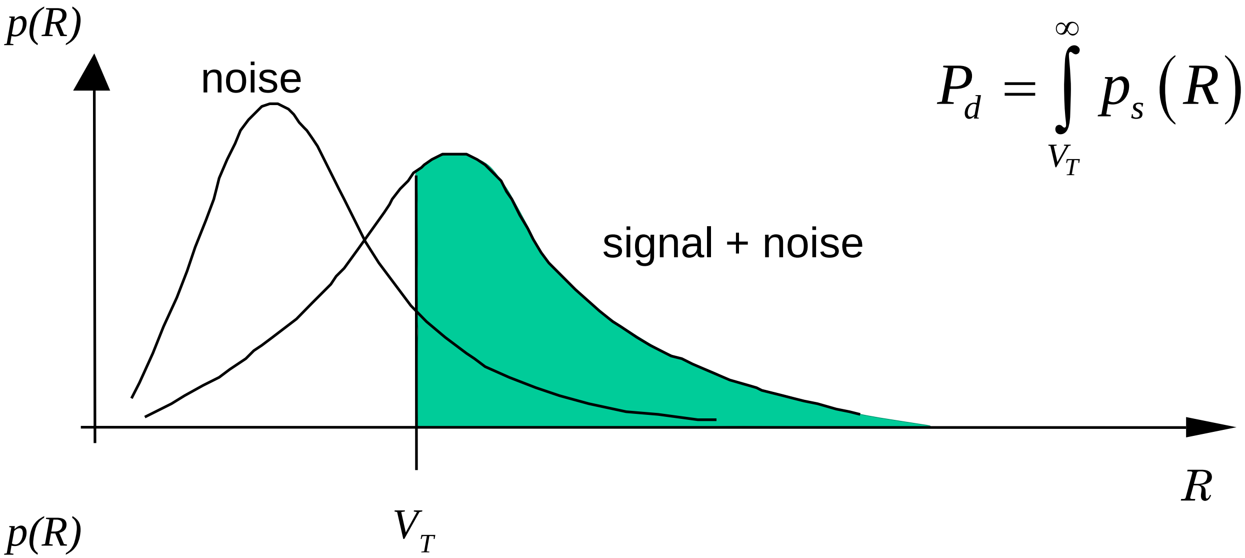
Detection

$$P_d = \int_{V_T}^{\infty} p_s\left(R\right)_{\text{target+noise}} dR$$

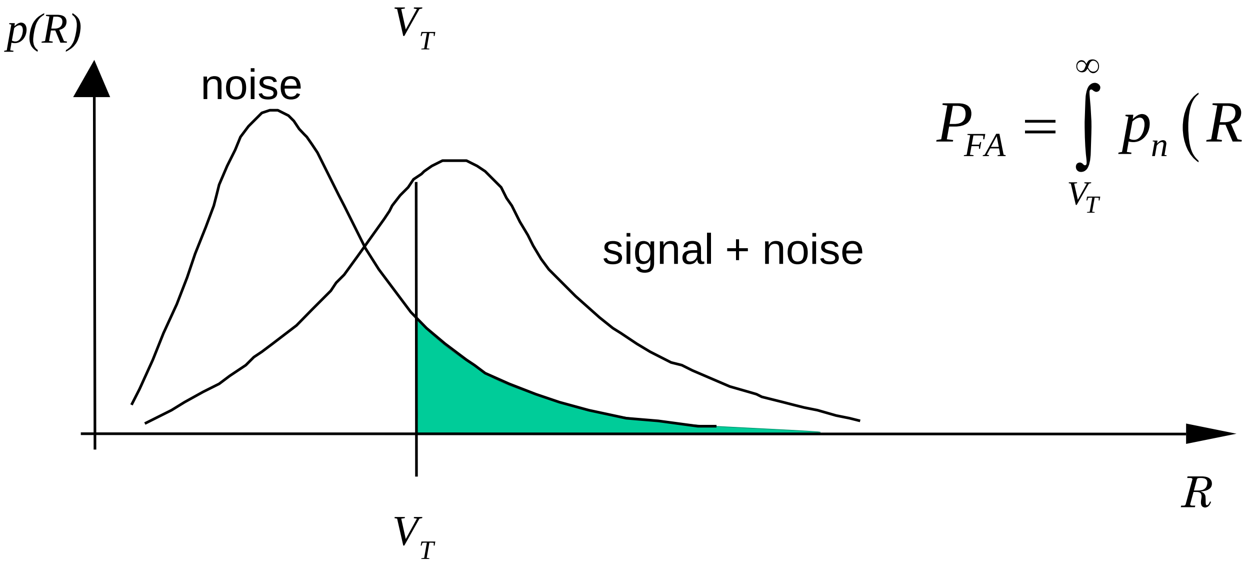




Detection

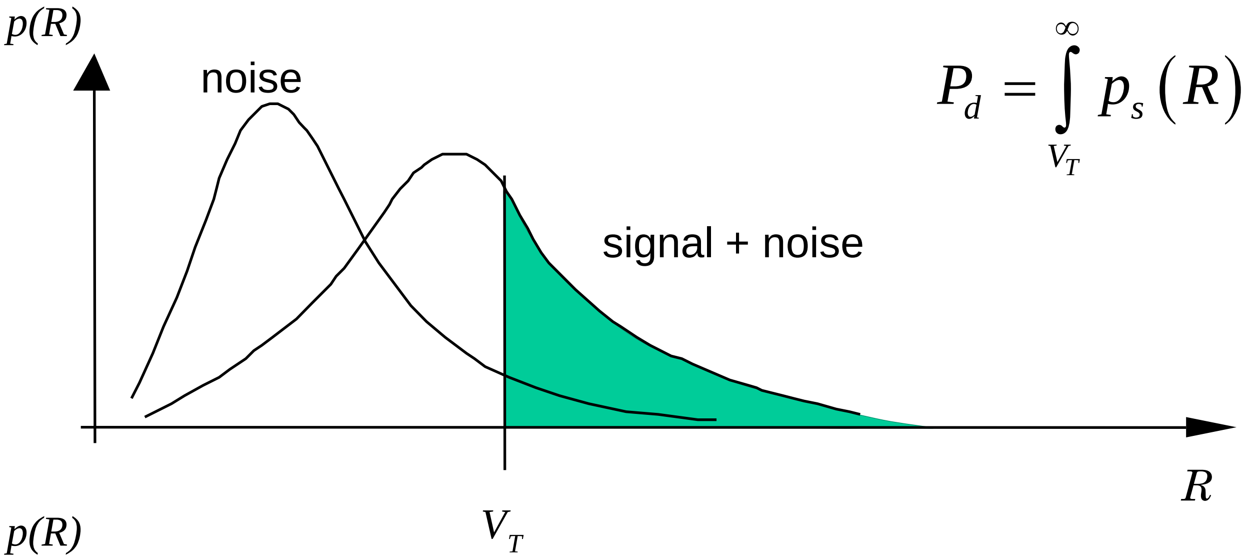


$$P_d = \int_{V_T}^{\infty} p_s(R)_{\text{target+noise}}\, dR$$

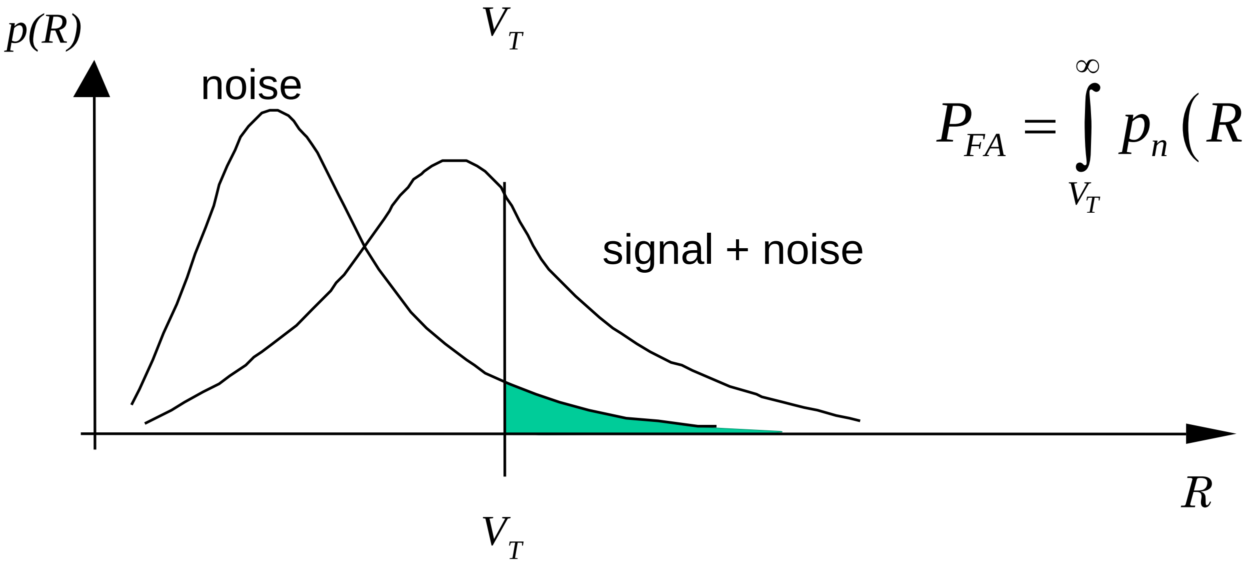


$$P_{FA} = \int_{V_T}^{\infty} p_n(R)_{\text{noise alone}}\, dR$$

Detection



$$P_d = \int_{V_T}^{\infty} p_s\left(R\right)_{\text{target+noise}} dR$$



$$P_{FA} = \int_{V_T}^{\infty} p_n\left(R\right)_{\text{noise alone}} dR$$

Example – Swerling 1 Target

Assume that the Noise and the target are both Gaussian distributed

1) $p(R)|H_0 \in N(0, \sigma^2{}_N)$

2) $p(R)|H_1 \in N(0, \sigma^2{}_N + \sigma^2{}_s)$

The probability of false alarm can be expressed as

$$P_{FA} = \int_{V_T}^{\infty} \frac{1}{2\sigma_N{}^2} e^{-\frac{R}{2\sigma_N{}^2}} dR = e^{-\frac{V_T}{2\sigma_N{}^2}}$$

Similarly the probability of detection can be expressed as

$$P_d = \int_{V_T}^{\infty} \frac{1}{2(\sigma_N{}^2 + \sigma_s{}^2)} e^{-\frac{R}{2(\sigma_N{}^2+\sigma_s{}^2)}} dR = e^{-\frac{V_T}{2(\sigma_N{}^2+\sigma_s{}^2)}}$$

Example – Swerling 1 Target

According to the Neyman-Pearson criterion the threshold η is calculated so to obtain a given $P_{FA} = \overline{P_{FA}}$ and is equal to

$$\eta = -2{\sigma_N}^2 \ln(\overline{P_{FA}})$$

This leads to a corresponding P_d

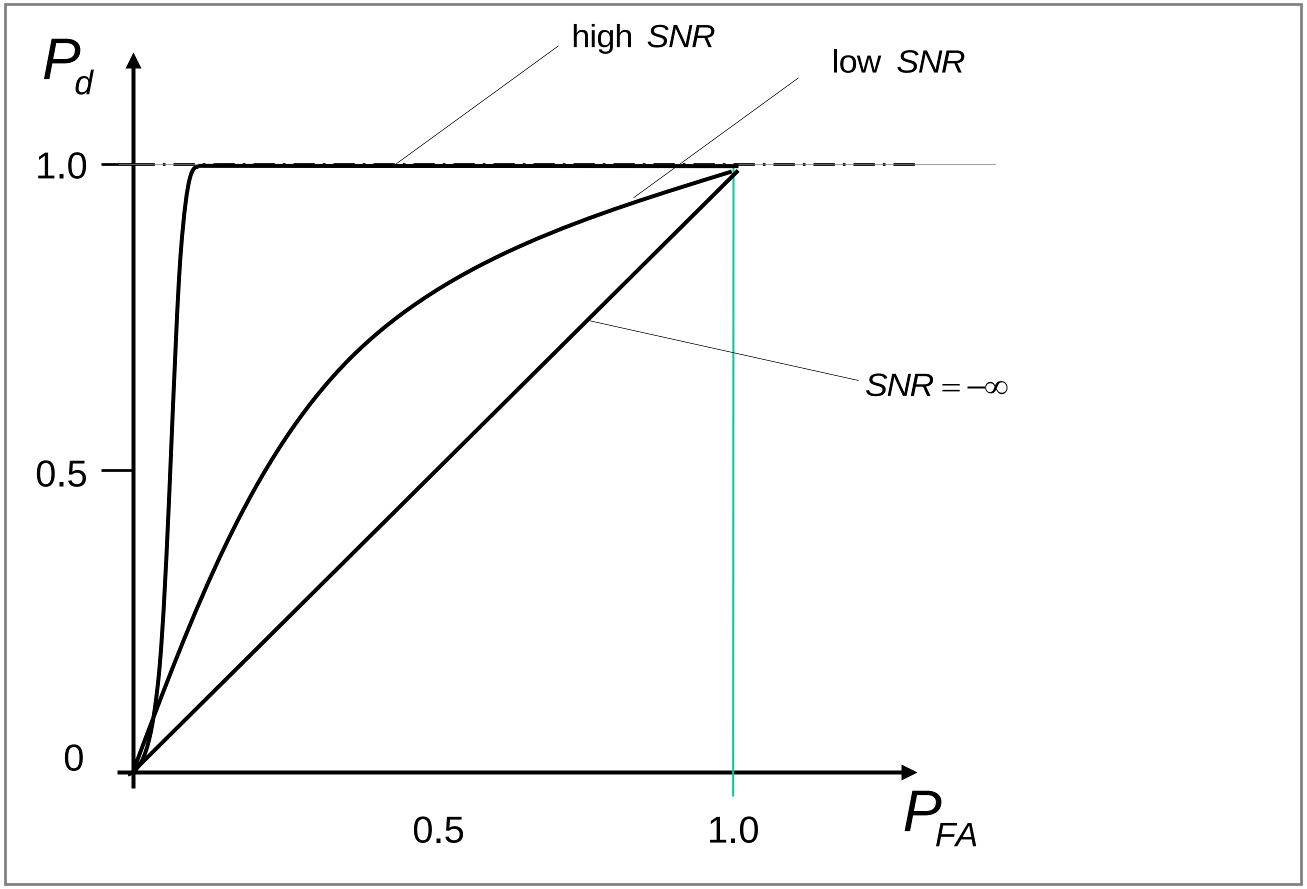
$$P_d = \overline{P_{FA}}^{\frac{{\sigma_N}^2}{{\sigma_N}^2 + {\sigma_s}^2}} = \overline{P_{FA}}^{\frac{1}{1+SNR}}$$

The required SNR to obtain a desired probability of detection P_d given a probability of false alarm $\overline{P_{FA}}$ can be expressed as

$$SNR = \frac{ln\overline{P_{FA}} - lnP_d}{lnP_d}$$

Receiver Operating Curve (ROC)

This is a plot of P_d vs P_{FA}, as a function of signal-to-noise or signal-to-clutter ratio



Integration gain

All of the foregoing has been in terms of a single pulse. If several successive pulses are integrated, we can get an integration gain.

For coherent integration (at IF or I/Q baseband) the gain is n, where n is the number of pulses.

For incoherent integration (after envelope detection), the gain is approximately $n^{0.8}$.

Integration results in a higher SNR and hence in a higher probability of detection.

Detection range is also increased

Summary

- In calculation detection performance against noise and interference, it is important to properly characterise the statistical distribution – and long-tailed (spiky) distributions will have the greatest effect.