Propagating Waves

ECE 6279: Spatial Array Processing
Fall 2013
Lecture 2

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Where We Are in J&D

- **Material drawn from Secs. 2.2, 2.2.1-2.2.2, and 2.2.4 of J&D**
- **We will <u>not cover</u>**
  - Sec. 2.2.3 on the Doppler effect
  - Sec 2.3 on dispersion and attenuation
  - Sec 2.4 on refraction and diffraction

Scalar Wave Equation

- **Lossless version of the scalar wave equation**
  - No attenuation or dispersion

$$\nabla^2 s \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$$

Laplacian

  - s above is really $s(x,y,z,t)$, but it's customary shorthand to suppress the (x,y,z,t) part

- **Treat as a mathematical abstraction for a moment**
- **Will relate to the physics of real problems later**

Complex Exponential Solution (1)

- **Let's guess a possible solution, namely a monochromatic (single-frequency) plane wave**

$$s(\vec{x},t) = \exp\left\{ j\left(\omega_0 t - k_x x - k_y y - k_z z\right)\right\}$$

$$= \exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\} = \exp\left\{ j\omega_0 \left(t - \frac{\vec{k}}{\omega_0}\cdot\vec{x}\right)\right\}$$

- **Plugging in:**

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

$$\frac{\partial}{\partial x}(-jk_x)\exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\} + \cdots = \frac{1}{c^2} j\omega_0 \frac{\partial}{\partial t}\exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

$$\left(-jk_x\right)^2 \exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\} + \cdots = \frac{1}{c^2}(j\omega_0)^2 \exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

Complex Exponential Solution (2)

$$(-jk_x)^2 \exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\} + \cdots = \frac{1}{c^2}(j\omega_0)^2 \exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

$$-k_x^2 \exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\} + \cdots = -\frac{1}{c^2}\omega_0^2 \exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

$$\left(k_x^2 + k_y^2 + k_z^2\right)\exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\} = \frac{\omega_0^2}{c^2}\exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega_0^2}{c^2}$$

- **So** $s(\vec{x},t) = \exp\left\{j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$ **is a solution if**

$$\left|\vec{k}\right| = \frac{\omega_0}{c}$$

Why the Name "Monochromatic?"

$$s(\vec{x},t) = \exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

- **Monochromatic** refers to ***temporal behavior***
  - – Fix some position, say (x,y,z)=(0,0,0)

$$s(0,0,0,t) = \exp\left\{ j\left(\omega_0 t\right)\right\} = \cos\left(\omega_0 t\right) + j\sin\left(\omega_0 t\right)$$

Why the Name "Plane Wave?"

$$s(\vec{x},t) = \exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

- **Plane wave refers to *spatial behavior***
  - Fix some time, say t=0

$s(\vec{x},0) = \exp\left\{ j\left(-\vec{k}\cdot\vec{x}\right)\right\}$ **is constant for all points on a plane**

$$\vec{k}\cdot\vec{x} = k_1 x + k_2 y + k_3 z = C$$

These planes of constant phase are perpendicular to $\vec{k}$

Speed of Propagation

- The planes of constant phase move by an amount $\delta\vec{x}$ in time δt

$$s(x+\delta\vec{x}, t+\delta t) = s(x,t)$$

$$\omega_0(t+\delta t) - \vec{k}\cdot(\vec{x}+\delta\vec{x}) = \omega_0 t - \vec{k}\cdot\vec{x}$$

$$\omega_0 \delta t - \vec{k}\cdot\delta\vec{x} = 0$$

- Pick $\delta\vec{x}$ to be in the same direction as $\vec{k}$ to give the smallest magnitude for $\delta\vec{x}$

$$\omega_0 \delta t - \left|\vec{k}\right|\left|\delta\vec{x}\right| = 0$$

$$\left|\vec{k}\right| = \frac{\omega_0}{c}$$

from previous slide

Speed of the wave ⟶ $\dfrac{\left|\delta\vec{x}\right|}{\delta t} = \dfrac{\omega_0}{\left|\vec{k}\right|} = c$ ⟵ Speed of propagation

Terminology

- **Unit vector** $\vec{\xi}^{\,0} = \frac{\vec{k}}{\left|\vec{k}\right|}$ **is the direction of propagation**
- $\vec{k}$ **is the wavenumber vector**
  - $\left|\vec{k}\right|$ is the number of cycles (in radians) per meter in direction of propagation

wavenumber $\longrightarrow k \equiv \left|\vec{k}\right| = \frac{\omega_0}{c} = \frac{2\pi}{\lambda} \longleftarrow$ **wavelength in meters/cycle**

$$s(\vec{x},t) = \exp\left\{ j\left(\omega_0 t - \vec{k}\cdot\vec{x}\right)\right\}$$

$$\lambda = \frac{c}{f_0} = \frac{2\pi c}{\omega_0}$$

temporal frequency variable in radians

3-D spatial frequency variable in radians

temporal frequency in cycles/sec (Hertz): $f_0 = \omega_0 / (2\pi)$

spatial frequency in cycles/meter: $? = k / (2\pi) = 1 / \lambda$

Slowness Vector

$$s(\vec{x},t) = \exp\left\{ j\omega_0 \left(t - \frac{\vec{k}}{\omega_0} \cdot \vec{x} \right) \right\} = \exp\left\{ j\omega_0 \left(t - \vec{\alpha} \cdot \vec{x} \right) \right\}$$

slowness vector → $\vec{\alpha} \equiv \dfrac{\vec{k}}{\omega_0}$

$$\left|\vec{\alpha}\right| = \frac{\left|\vec{k}\right|}{\omega_0} = \frac{2\pi}{\omega_0 \lambda} = \frac{1}{f_0 \lambda} = \frac{1}{c}$$

- **The slowness vector** $\vec{\alpha}$
  - – Points in direction of propagation
  - – Has units of reciprocal velocity (sec/meter)

Periodic Nonchromatic Plane Waves

- **A useful abuse of notation**

$$s(\vec{x},t) = \exp\left\{ j\omega_0 \left(t - \vec{\alpha}\cdot\vec{x} \right) \right\} = s(t - \vec{\alpha}\cdot\vec{x})$$

where $s(u) \equiv \exp(j\omega_0 u)$

- **The wave equation is <u>linear</u>, i.e.**

if $s_1(\vec{x},t)$ **and** $s_2(\vec{x},t)$ **solve the wave equation**

then $as_1(\vec{x},t) + bs_2(\vec{x},t)$ **solves the wave equation**

- **Example:**

Fourier series!

$$s(\vec{x},t) = s(t - \vec{\alpha}\cdot\vec{x}) = \sum_{n=-\infty}^{\infty} S_n \exp\left\{ jn\omega_0 \left(t - \vec{\alpha}\cdot\vec{x} \right) \right\}$$

Fourier series coefficients $\longrightarrow$ $S_n = \dfrac{1}{T}\displaystyle\int_0^T s(u)\exp(-jn\omega_0 u)\,du$

General Nonchromatic Plane Waves

- **Fourier series generalizes to Fourier integral**

$$s(\vec{x},t) = s(t - \vec{\alpha} \cdot \vec{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \exp\left\{ j\omega\left(t - \vec{\alpha} \cdot \vec{x}\right)\right\} d\omega$$

Fourier transform ⟶ $S(\omega) = \int_{-\infty}^{\infty} s(u) \exp(-j\omega u) du$

- **Take home message: any space-time signal of the form $s(\vec{x},t) = s(t - \vec{\alpha} \cdot \vec{x})$ satisfies the wave equation, whatever the $s(u)$**
  - Shape of plane wave is preserved as it propagates
- **Linearity of wave equation implies plane waves in many directions can exist simultaneously:**

$$s(\vec{x},t) = s_1(t - \vec{\alpha}_1 \cdot \vec{x}) + s_2(t - \vec{\alpha}_2 \cdot \vec{x})$$

Monochromatic Spherical Waves

- **Wave equation can be rewritten in spherical coordinates (J&D, p. 16)**
- **One solution is:**

$$s(r,t) = \frac{1}{r}\exp\left\{j\left(\omega_0 t - kr\right)\right\} \quad = \frac{1}{r}\exp\left\{j\omega_0\left(t - \frac{r}{c}\right)\right\}$$

where $k = \dfrac{\omega_0}{c}$

interpreted as propagating away from origin

- **Another solution is:**

$$s(r,t) = \frac{1}{r}\exp\left\{j\left(\omega_0 t + kr\right)\right\} = \frac{1}{r}\exp\left\{j\omega_0\left(t + \frac{r}{c}\right)\right\}$$

interpreted as propagating toward origin

General Spherical Waves

- Abuse notation again: for any univariate function $s(u)$, easy to show that:

$$s(r,t) = \frac{s(t - r/c)}{r}$$

satisfies the wave equation

- Could write as a superposition of monochromatic spherical waves

$$s(r,t) = \frac{1}{2\pi r}\int_{-\infty}^{\infty} S(\omega)\exp\left\{ j\omega\left(t - \frac{r}{c}\right)\right\} d\omega$$

- Can make same arguments for inward-going waves, waves with a different origin, etc.

Other Notes on Spherical Waves

- **Due to space-invariance, can shift origins to wherever you want:**

$$s(\vec{x},t)=\frac{1}{\left|\vec{x}-\vec{x}_0\right|}\exp\left\{j\left(\omega_0 t-k\left|\vec{x}-\vec{x}_0\right|\right)\right\}+\frac{1}{\left|\vec{x}-\vec{x}_1\right|}\exp\left\{j\left(\omega_0 t+k\left|\vec{x}-\vec{x}_1\right|\right)\right\}$$

- **Consider the real monochromatic spherical wave:**

$$s(r,t)=\frac{1}{r}\cos(\omega_0 t-kr)$$

  - Amplitude decreases with increasing r
  - Distance between zero-crossings is constant with increasing r
  - …but distance between maxima isn't!
    - **Gets longer with increasing r**

Wave Equation in Physics (1)

- In electromagnetics, we have Maxwell’s equations
- General equations are in terms of vectors, yielding a more general vector wave equation
  - Won’t worry about in ECE6279
- For a transverse electromagnetic wave, any particular component of the electric or magnetic fields satisfies the 3-D scalar wave equation

Wave Equation in Physics (2)

- **In acoustics, 3-D wave equation gives**
  - Sound pressure at a point in space and time
  - In fluids, have compressional waves
  - Solids have both compressional and transverse waves
  - By linearity, compressional and transverse waves propagate separately, each its own velocity c

- **2-D wave equation gives vertical displacement of waves in shallow water**