E9 231: Digital Array Signal Processing

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Class held on: 23 Oct 2008

1 Topics

- LCMV Beamformer
- Linear Algebra Approach

1.1 LCMV Beamformer

The LCMV beamformer problem is formulated as

$$\min w^H S_n w \tag{1}$$
$$\text{subject to } C^H w = g \tag{2}$$

The LCMV solution is given by,

$$w_{LCMV} = \frac{S_n^{-1} C g}{C^H S_n^{-1} C} \tag{3}$$

similarly the LCMP problem is formulated as

$$\min w^H S_x w \tag{4}$$
$$\text{subject to } C^H w = g \tag{5}$$

The LCMP solution is given by,

$$w_{LCMP} = \frac{S_x^{-1} C g}{C^H S_x^{-1} C} \tag{6}$$

Now we will discuss the linear algebra approach for solving the LCMV/LCMP beamforming problem.

1.2 Linear Algebra Approach

Here we starts from the constraint equations

$$C^H w = g \tag{7}$$

where $C \in R^{NXM}$. M is the number of constraints $M < N$. w is the beamformer weights. The singular value decompostion(SVD) of C is obtained as

$$C = U \Sigma V^H \tag{8}$$

$U \in R^{NXN}$, $\Sigma \in R^{NXM}$, $V \in R^{MXM}$ and Σ is a diagonal matrix with its diagonal elements are the singular values. U is the left singular vector matrix with $UU^H = I_N$ and V is the right singular vector matrix with $VV^H = I_M$.
Let $rank(C) = r$, $r < M$. since the rank of C is r, there will be r nonzero sigular values in the singular value matrix Σ.

The SVD of C can be written as

$$\begin{bmatrix} U1 & U2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix} \tag{9}$$

Where

The economy form of the SVD is given by,

$$C = U_1 S V_1^H \tag{10}$$

Four fundamental spaces associated with the matrix C are,

Where S is a submatrix of Σ of, $S \in R^{r \times r}$.
$R(C) = y : Cx = y, \forall x \in R^M = R(U1)$.

$N(C) = \forall x \in R^M : Cx = 0 = R(V2)$.

$R(C^H) = p : C^H x = p, \forall x \in R^N = R(V1)$.

$N(C^H) = \forall x \in R^N : C^H x = 0 = R(U2)$.

Since U is an orthonormal basis, we can write w as

$$w = U\beta \tag{11}$$

a linear combination of U. So,

$$w = U_1\beta_1 + U_2\beta_2 \tag{12}$$

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{13}$$

Now

$$C^H w = \begin{bmatrix} V1 & V2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^H \\ U_2^H \end{bmatrix} \begin{bmatrix} U_1\beta_1 + U_2\beta_2 \end{bmatrix} \tag{14}$$

since U_1 and U_2 are hermitian matrices

$$C^H w = V_1 S\beta_1 + 0 = g \tag{15}$$

$$V_1^H V_1 S\beta_1 = V_1^H g \tag{16}$$

$$\beta_1 = S^{-1} V_1^H g \tag{17}$$

$$U_1\beta_1 = U_1 S^{-1} V_1^H g = C^\dagger g \tag{18}$$

This is a minimum norm solution. $C^\dagger$ is the Moore Penrose Pseudo inverse.

$$C^\dagger = C(C^H C)^{-1} \tag{19}$$

$$U_1\beta_1 = C(C^H C)^{-1} g \tag{20}$$

Conclusions :-
(1). w has a fixed part $U_1\beta 1 = w_q = C(C^H C)^{-1} g$, the quiscent solution.
(2). w has a free component, given by $U_2\beta_2$.
Then $\longrightarrow \beta w_a = -U_2\beta_2$
(3). $w = U_1\beta_1 + U_2\beta_2$ is the sum of two orthogonal components because $U_1 U_2^H = 0$.
Then

$$\|w\|^2 = \|\beta_1\|^2 + \|\beta_2\|^2 \tag{21}$$

so if we want to minimize $\|w\|^2$ subject to $C^H w = g$, then it is clear to choose $\beta_2 = 0$. We obtain

$$w = U_1\beta = C^\dagger g = \frac{Cg}{[C^H C]} \tag{22}$$

Also $\|w\|^2$ has another interpretation.

$$\sigma^2 \|w\|^2 = w^H S_n w \tag{23}$$

$$S_n = \sigma^2 \mathtt{I} \tag{24}$$

So w_q is the optimum solution under spatially white noise. Now, There are two approaches to the LCMV/LCMP problem.
Approach-1: Lagrange Multiplier approach and obtain w_{LCMV}/w_{LCMP}
Approach-2: Unconstrained optimization problem over w_a.
The general solution is obtained as

$$w = w_q - \beta w_a \tag{25}$$

here w_q and β are fixed. so choose optimum w_a to minimize the objective function. This solution leads to the generalized sidelobe canceller (GSC). interpretation of LCMV.
For solving the original problem of LCMV, we take the cholesky factorization of S_n and obtain,

$$w^H S_n w = w^H L L^H w = \widetilde{w}^H \widetilde{w} \tag{26}$$

where $\widetilde{w} = L^H w$ is a $(1-1)$ mapping. therefor we have

$$w^H S_n w = \|\widetilde{w}\|^2 \tag{27}$$

$$C^H w = C^H (L^{-H} \widetilde{w}) = g \tag{28}$$

$$\widetilde{C}^H \widetilde{w} = g \tag{29}$$

Where $\tilde{C} = L^{-1}C$
Now the optimization problem is

$$min\|w\|^2$$
$$s.t.\tilde{C}^H\tilde{w} = g$$

The minimum norm solution is obtained as

$$\tilde{w}_{opt} = \frac{\tilde{C}g}{\left[\tilde{C}^H\tilde{C}\right]} \tag{30}$$

Then

$$w_{LCMV} = L^{-H}\tilde{w}_{opt} = \frac{L^{-H}L^{-1}Cg}{[C^H L^{-H}L^{-1}C]} \tag{31}$$

$$w_{LCMV} = \frac{S_n^{-1}Cg}{[C^H S_n^{-1}C]} \tag{32}$$

similarly w_{LCMP} is obtained as

$$w_{LCMP} = \frac{S_x^{-1}Cg}{[C^H S_x^{-1}C]} \tag{33}$$

Now consider the general solution, $w = w_q - \beta w_a$, w_q depends only on C and g. so it can be computed before the data arrives. β depends only on C. so it can also be pre computed. w_a depends on S_n/S_x. using these results we can obtain a useful implementation of the LCMV/LCMP beamformer. This is referred to as the generalized sidelobe canceller as shown in the figure.

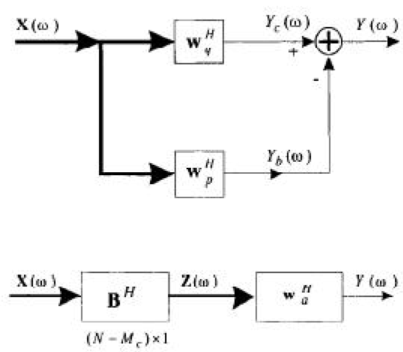

Figure 1: Generalized Sidelobe Canceller

The output Power,

$$P_o = [w_q - \beta w_a]^H S_x [w_q - \beta w_a] \tag{34}$$

Use LMS or RLS algorithm for adaptively updating w_a.
Also for LCMV, the optimization problem,

$$min\left[w^H S_n w\right]$$
$$s.t.C^H w = g$$

is same as the unconstrained optimization,

$$\underbrace{min}_{w_a}[w_q - \beta w_a]^H S_n [w_q - \beta w_a]$$

The above unconstrained optimization probem is solved by finding the gradient with respect to w_a and equate to zero.
Then we obtain

$$\left[w_q^H - w_a^H B^H\right] S_n \beta = 0$$
$$w_a^H = \frac{\beta_q^H S_n \beta}{\beta^H S_n \beta}$$

therefore

$$w_{LCMV} = \frac{S_n^{-1} C g}{C^H S_n^{-1} C} \tag{35}$$

also

$$w_{LCMV} = w_q - \frac{\beta \beta^H S_n w_q}{\beta^{-1} S_n \beta} \tag{36}$$

The above two solutions are identical.
Now, consider a sigle plane wave input.

$$S_x = \sigma_s^2 V_m V_m^H + S_n \tag{37}$$

Where V_m is the model array manifold vector(The array steering vector) and let $V_m = V_a$, where V_a is the Actual array facor in the direction of the signal. Since the first column of C is V_m,we must have the following.

$$V_m \in \mathbf{R}(C)$$
$$V_m^H \beta = 0 \Longrightarrow S_x \beta = S_n \beta$$

Here LCMP and LCMV are identical and w_{LCMP} reduces to w_{LCMV}

1.2.1 Array Gain

The output signal power is given by

$$P_o = \left|w_{LCMV} V_a\right|^2 \sigma_s^2 \tag{38}$$

Let the received data model be

$$X = F V_S + N \tag{39}$$

The output noise power is given by

$$P_n = w_{LCMV}^H S_n w_{LCMV} = \sigma_n^2 w_{LCMV}^H \rho_n w_{LCMV} \tag{40}$$

where ρ_n have ones along the diagonal. The SNR is given by,

$$SNR = \frac{\sigma_s^2}{\sigma_n^2} = \frac{\left|w_{LCMV} V_a\right|^2}{w_{LCMV}^H \rho_n w_{LCMV}} \tag{41}$$

The distortionless constraint ensures $\left|w_{LCMV} V_a\right|^2 = 1$. so,

$$SNR = \frac{\sigma_s^2}{\sigma_n^2} = \frac{1}{w_{LCMV}^H \rho_n w_{LCMV}} \tag{42}$$

Now the array gain

$$A_{LCMV} = \frac{1}{w_{LCMV}^H \rho_n w_{LCMV}}$$
$$A_{LCMV} = \frac{1}{g^H \left[C^H S_n^{-1} C\right]^{-1} \left[C^H S_n^{-1} \frac{S_n}{\sigma_n^2} S_n^{-1}\right] C \left[C^H S_n^{-1} C\right]^{-1} g}$$

$$A_{LCMV} = \frac{1}{g^H \left[C^H \rho_n^{-1} C\right]^{-1} g} \tag{43}$$

The array gain is maximum when the distortionless constraint is the only constraint impossed. As the number of constratint increases, we trade off robustness for array gain performance.
We can derive A_{LCMP} also as discussed above.
Recommended Reading : Examples 6.7.5 – 6.7.9 in the text.