E9 231: Digital Array Signal Processing

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1 Minutes of Last Lecture

1. MVDR beamformer (maximum likelihood)
2. Bayesian framework LMMSE beamformer
3. MMSE conditional mean estimator
   -goes back to (2) for gaussian
4. max. SNR (goes back to (1))
5. MPDR beamformer ($\mathbf{S_x}$ instead of $\mathbf{S_n}$)

2 Today:

- sensitivity analysis
- LCMV/LCMP beamformers

3 Mismatched MVDR/MPDR

Recall:

$$\mathbf{W}_{MVDR} = \Lambda S_{\mathbf{n}}^{-1}\mathbf{V}_m$$
$$\text{where } \Lambda = (\mathbf{V}_m^H \mathbf{S}_{\mathbf{n}}^{-1}\mathbf{V}_m)^{-1}$$
$$\mathbf{W}_{MPDR} = \Lambda_1 \mathbf{S}_{\mathbf{x}}^{-1}\mathbf{V}_m$$
$$\text{where } \Lambda_1 = (\mathbf{V}_m^H \mathbf{S}_{\mathbf{x}}^{-1}\mathbf{V}_m)^{-1}$$

Also,

$$\mathbf{W}_c = \frac{1}{N}\mathbf{V}_m$$

4 Types of errors

1. DOA mismatch

$$\begin{aligned} \mathbf{V}_a &= \text{“actual” DOA} \\ \mathbf{V}_m &= \text{“model” DOA} \\ \mathbf{V}_m &\neq V_a \end{aligned}$$

2. Array perturbation:

$$\mathbf{V}_m(\mathbf{k}_d) \neq \mathbf{V}_a(\mathbf{k}_d)$$

3. Errors in spatial spectral estimate

$$\hat{\mathbf{S}}_\mathbf{n} \neq \mathbf{S}_\mathbf{n}$$
$$\hat{\mathbf{S}}_\mathbf{x} \neq \mathbf{S}_\mathbf{x}$$

5 DOA mismatch

$$\mathbf{X} = \mathbf{V}_a F + \mathbf{N}$$

$$\begin{aligned}
\mathbf{V}_a &= \text{array manifold corresponding to the actual DOA} \\
&= \mathbf{V}_a(\omega, \mathbf{k}_a) \neq \mathbf{V}_m \\
\mathbf{S}_\mathbf{n} &= s_n \rho_\mathbf{n}
\end{aligned}$$

$$\begin{aligned}
P_o &= \mid \mathbf{W}^H \mathbf{X} \mid^2 \\
&= \sigma_s^2 \mid \mathbf{W}^H \mathbf{V}_a \mid^2 + \mathbf{W}^H \mathbf{S}_\mathbf{n} \mathbf{W} \\
&= P_s + P_n \\
SNR_O = \frac{P_s}{P_n} &= \frac{\sigma_s^2 \mid \mathbf{W}^H \mathbf{V}_a \mid^2}{\mathbf{W}^H \mathbf{S}_\mathbf{n} \mathbf{W}} \\
\text{array gain} &= \frac{\mid \mathbf{W}^H \mathbf{V}_a \mid^2}{\mathbf{W}^H \rho_\mathbf{n} \mathbf{W}}
\end{aligned}$$

5.1 Conventional beamformer

Recall from chapter 2 :-

$$\begin{aligned}
B(\mathbf{V}_m : \mathbf{V}_a) &= \frac{1}{N} \mathbf{V}_m^H \mathbf{V}_a \\
SNR_o = \frac{P_s}{P_n} &= \frac{\sigma_s^2 \mid \mathbf{W}_c^H \mathbf{V}_a \mid^2}{\mathbf{W}_c^H \mathbf{S}_\mathbf{n} \mathbf{W}_c}; \text{ where } \mathbf{W}_c = \frac{1}{N} \mathbf{V}_m \\
&= \frac{\sigma_s^2 \mid \mathbf{V}_m^H \mathbf{V}_a \mid^2}{\mathbf{V}_m^H \mathbf{S}_\mathbf{n} \mathbf{V}_m} \\
&= \frac{\sigma_s^2 N^2 \mid B_c(\mathbf{V}_m : \mathbf{V}_a) \mid^2}{\mathbf{V}_m^H \mathbf{S}_\mathbf{n} \mathbf{V}_m} \\
A_c &= \frac{N^2 \mid B_c(\mathbf{V}_m : \mathbf{V}_a) \mid^2}{\mathbf{V}_m^H \rho_\mathbf{n} \mathbf{V}_m} \\
\frac{A_c(V_a)}{A_c(V_m)} &= \mid B_c(\mathbf{V}_m : \mathbf{V}_a) \mid^2
\end{aligned}$$

5.2 MVDR beamformer

generalized cosine:

$$\begin{aligned}
\cos^2(\mathbf{V}_m, \mathbf{V}_a : \rho_{\mathbf{n}}^{-1}) &\triangleq \frac{\mid \mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a \mid^2}{(\mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_m)(\mathbf{V}_a^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a)} \\
\mathbf{W}_{MVDR} &= \frac{\mathbf{S}_{\mathbf{n}}^{-1} \mathbf{V}_m}{\mathbf{V}_m^H \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{V}_m} \\
P_s &= \sigma_s^2 \mid \mathbf{W}_{MVDR}^H \mathbf{V}_a \mid^2 \\
&= \frac{\sigma_s^2 \mid \mathbf{V}_m^H \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{V}_a \mid^2}{\left(\mathbf{V}_m^H \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{V}_m\right)^2} \\
&= \frac{\sigma_s^2 \left(\mathbf{V}_a^H \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{V}_a\right) \cos^2(\mathbf{V}_m, \mathbf{V}_a : \mathbf{S}_{\mathbf{n}}^{-1})}{\left(\mathbf{V}_m^H \mathbf{S}_{\mathbf{n}}^{-1} \mathbf{V}_m\right)} \\
&= \frac{\sigma_s^2 \left(\mathbf{V}_a^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a\right) \cos^2(\mathbf{V}_m, \mathbf{V}_a : \rho_{\mathbf{n}}^{-1})}{\left(\mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_m\right)}
\end{aligned}$$

$$SNR_o = \frac{P_s}{P_n} = \frac{\frac{\sigma_s^2 \left(\mathbf{V}_a^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a\right) \cos^2(\mathbf{V}_m, \mathbf{V}_a : \rho_{\mathbf{n}}^{-1})}{\left(\mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_m\right)}}{s_n \frac{\mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \rho_{\mathbf{n}} \rho_{\mathbf{n}}^{-1} \mathbf{V}_m}{\left(\mathbf{V}_m \rho_{\mathbf{n}}^{-1} \mathbf{V}_m\right)^2}}$$

$$\begin{aligned}
SNR_o &= \frac{\sigma_s^2}{s_n} \mathbf{V}_a^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a \cos^2(\mathbf{V}_m, \mathbf{V}_a : \rho_{\mathbf{n}}^{-1}) \\
A_{MVDR}(\mathbf{V}_a) &= \mathbf{V}_a^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a \cos^2(\mathbf{V}_m, \mathbf{V}_a : \rho_{\mathbf{n}}^{-1}) \\
\frac{A_{MVDR} \mid_{\mathbf{V}_m \neq \mathbf{V}_a}}{A_{MVDR} \mid_{\mathbf{V}_m = \mathbf{V}_a}} &= \frac{\mid \mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a \mid^2}{\left(\mathbf{V}_m^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_m\right) \left(\mathbf{V}_a^H \rho_{\mathbf{n}}^{-1} \mathbf{V}_a\right)} \\
&= \cos^2(\mathbf{V}_a, \mathbf{V}_m : \rho_{\mathbf{n}}^{-1})
\end{aligned}$$

Ramark: As SNR $\uparrow$, perfomance significantly degrades with DOA mismatch.

6 Condition number related problems

$$\mathbf{W}_{MPDR} = \Lambda_1 \mathbf{S}_{\mathbf{x}}^{-1} \mathbf{V}_m$$

Potential problem:
condition # of $\mathbf{S}_{\mathbf{x}}^{-1}$

$$\triangleq \frac{\lambda_{max}}{\lambda_{min}}$$

If the condition # is large, small errors in $\mathbf{S}_{\mathbf{x}}$ (or even $\mathbf{V}_a$) could be amplified

$$\begin{aligned}
\mathbf{S}_{\mathbf{x}} &= \sum_{i=1}^{N} \lambda_i e_i e_i^H \\
\mathbf{S}_{\mathbf{x}}^{-1} &= \sum_{i=1}^{N} \frac{1}{\lambda_i} e_i e_i^H \\
\mathbf{S}_{\mathbf{x}}^{-1} \mathbf{V}_a &= \sum_{i=1}^{N} \left(\frac{e_i^H \mathbf{V}_a}{\lambda_i}\right) e_i
\end{aligned}$$

could end up becoming very large if $\frac{\lambda_{max}}{\lambda_{min}}$ is large,

$$\begin{aligned}\lambda_{max} &= 10 \longrightarrow \lambda_{max} \pm \epsilon \\ \lambda_{min} &= 0.01 \longrightarrow \lambda_{min} \pm \epsilon\end{aligned}$$

Another way, $\| \mathbf{W}_{MVDR} \|^2$ could end up very large
This motivates including a norm constraint in beamformer design

$$\| \mathbf{W}_{MVDR} \|^2 = \Lambda_1^2 \| \mathbf{S_x}^{-1} \mathbf{V}_a \|^2 = \frac{\mathbf{V}_a^H \mathbf{S_x}^{-2} \mathbf{V}_a}{(\mathbf{V}_a^H \mathbf{S_x}^{-1} \mathbf{V}_a)^2}$$

7 Robust beamformers

$$\text{constraint } \mathbf{W}^H \mathbf{V}_a = 1$$

$$\min_{\mathbf{W}} \mathbf{W}^H \mathbf{S_n} \mathbf{W} + \beta \| \mathbf{W} \|^2, \qquad \text{where } \beta \text{ is the regularization term}$$

$$\begin{aligned}J &= \mathbf{W}^H \mathbf{S_n} \mathbf{W} + \mathbf{W}^H \beta \mathbf{I} \mathbf{W} \\ &= \mathbf{W}^H (\mathbf{S_n} + \beta \mathbf{I}) \mathbf{W}\end{aligned}$$

This is called "diagonal loading".

$$\mathbf{W}_{reg} = \frac{(\mathbf{S_n} + \beta \mathbf{I})^{-1} \mathbf{V}_a}{\mathbf{V}_a^H (\mathbf{S_n} + \beta \mathbf{I})^{-1} \mathbf{V}_a}$$

As $\beta \longrightarrow \infty \quad \mathbf{W}_{reg} \longrightarrow \mathbf{W}_c$
As $\beta \longrightarrow 0 \quad \mathbf{W}_{reg} \longrightarrow \mathbf{W}_{MVDR}$
Finding the appropriate β is difficult.
condition # of $\mathbf{S_n}$ is improved:

$$\begin{aligned}\mathbf{S_n} : &\quad \lambda_{max} &&, \lambda_{min} \\ (\mathbf{S_n} + \beta \mathbf{I}) : &\quad \lambda_{max} + \beta &&, \lambda_{min} + \beta\end{aligned}$$

$$\text{condition } \# \ (\mathbf{S_n} + \beta \mathbf{I}) = \frac{\lambda_{max} + \beta}{\lambda_{min} + \beta}$$

7.1 Alternative approach for improving a norm constraint

MPDR formulation

$$\min_{\mathbf{W}} \mathbf{W}^H \mathbf{S_x} \mathbf{W}$$

$$\begin{aligned}\text{s.t. } \mathbf{W}^H \mathbf{V}_a &= 1 \\ \mathbf{W}^H \mathbf{W} &= T_o\end{aligned}$$

$$J = \mathbf{W}^H \mathbf{S_x} \mathbf{W} + \lambda_1 (\mathbf{W}^H \mathbf{W} - T_o) + \lambda_2 (\mathbf{W}^H \mathbf{V}_a - 1) + \lambda_2^* (\mathbf{V}_a^H \mathbf{W} - 1)$$

$\Longrightarrow$ differentiate w.r.t. $\mathbf{W}^H$:

$$\mathbf{W}^H = \frac{\mathbf{V}_a^H (\mathbf{S_n} + \lambda_1 \mathbf{I})^{-1}}{\mathbf{V}_a^H (\mathbf{S_n} + \lambda_1 \mathbf{I})^{-1} \mathbf{V}_a}$$

Often set λ_1 to a fixed value and use this as our beamformer vector:

$$\mathbf{W}_{MPDR,dl}^H = \frac{\mathbf{V}_a^H \left[\mathbf{I} + \frac{\mathbf{S_x}}{\sigma_L^2} \right]^{-1}}{\mathbf{V}_a^H \left[\mathbf{I} + \frac{\mathbf{S_x}}{\sigma_L^2} \right]^{-1} \mathbf{V}_a}$$

8 Using more linear constraints:

- Linearly Constrained Minimum Variance (LCMV) Beamformer
- Linearly Constrained Minimum Power (LCMP) Beamformer

- Distortionless constraint
- Null constraints
- Derivative constraints

$$\mathbf{W}^H_{1\times N}\mathbf{C}_{N\times M} = \mathbf{g}^H_{1\times M}$$

M < N, otherwise there will not be enough degrees of freedom.
columns of $\mathbf{C}$ are linearly independent

$$\mathbf{C}^H\mathbf{W} = \mathbf{g}$$

Optimization problem:

$$\begin{array}{cccc} & \text{LCMV} & & \text{LCMP} \\ \min & \mathbf{W}^H\mathbf{S_n}\mathbf{W} & \text{or} & \mathbf{W}^H\mathbf{S_x}\mathbf{W} \\ s.t. & \mathbf{C}^H\mathbf{W} = \mathbf{g} & & \end{array}$$

$$\mathbf{Sol}^n:$$

$$\mathbf{W}_{LCMV} = \mathbf{S_n}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{S_n}^{-1}\mathbf{C})^{-1}\mathbf{g}$$
$$\mathbf{W}_{LCMP} = \mathbf{S_x}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{S_x}^{-1}\mathbf{C})^{-1}\mathbf{g}$$

If we set $\mathbf{C} = \mathbf{V}_s$ & $\mathbf{g} = 1$, $\mathbf{W}_{LCMV} \to \mathbf{W}_{MVDR}$

$$J = \mathbf{W}^H\mathbf{S_n}\mathbf{W} + (\mathbf{W}^H\mathbf{C} - \mathbf{g}^H)\lambda + \lambda^H(\mathbf{C}^H\mathbf{W} - \mathbf{g})$$

Differentiate with respect to $\mathbf{W}^H$:

$$\mathbf{S_n}\mathbf{W} + \mathbf{C}\lambda = 0$$
$$\mathbf{W} = -\mathbf{S_n}^{-1}\mathbf{C}\lambda$$
$$\lambda = -(\mathbf{C}^H\mathbf{S_n}^{-1}\mathbf{C})^{-1}\mathbf{g}$$

$$\mathbf{W}_{LCMV} = \mathbf{S_n}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{S_n}^{-1}\mathbf{C})^{-1}\mathbf{g}$$