E9 231: Digital Array Signal Processing

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I. BEAM PATTERN PARAMETERS

The expression for Beam Pattern is given by

$$B(\psi) = \frac{1}{N} \frac{\sin\left(N\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \tag{1}$$

1. **Half Power Beam Width(HPBW):**It is defined to be a point where

$$|\mathbf{B}(\theta)|^2 = \frac{1}{2} \tag{2}$$

Under a large N assumption ($N \geq 10$), the HPBW is approximately given as

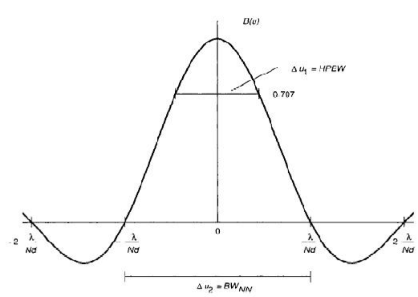
$$\Delta u = \frac{0.891\lambda}{Nd} \tag{3}$$

2. **Distance to the first null:** It is equal to half the BW_{NN}.

3. **Distance to first sidelobe:** It is approximately equal to $\frac{3\pi}{10}$ for $N = 10$

4. **Height of first sidelobe:** For uniform weighting, the sidelobe level is approximately -13dB

5. **Location of nulls:** It is the location of remaining nulls not including the first null.



Main lobe of beam pattern

Figure 1: Mainlobe of beam pattern

6. **Rate of decrease of sidelobes:** For uniform weighting, it is $\frac{1}{(2m+1)}$ (approx.), where 'm' is the sidelobe number.

II. **ARRAY PERFOMANCE MEASURES**

1. **Directivity:**

$$P(\theta,\phi) = |B(\theta,\phi)|^2$$

$$D = \frac{P(\theta_T,\phi_T)}{\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=-\pi}^{\pi} P(\theta,\phi)\sin\theta d\theta d\phi} \tag{4}$$

For Linear array we have,

$$D = \frac{P(u_T)}{\frac{1}{2}\int_{-1}^{1} P(u)du} \tag{5}$$

With $P(\theta_T,\phi_T) = 1$ and $B(\theta_T,\phi_T) = 1$ usually.
Where θ_T is the steering direction or maximum response axis (MRA).

So the expression for the Directivity can be written as

$$D = \{\frac{1}{2}\int_{-1}^{-1} |B(u)|^2 du\}^{-1} \tag{6}$$

For Uniform Linear Array,

$$B(u) = e^{-\frac{N-1}{2}\frac{2\pi d}{\lambda}u}\sum_{n=0}^{N-1} w_n^* e^{\frac{j2\pi d}{\lambda}nu} \tag{7}$$

$$B(u-u_T) = e^{-\frac{N-1}{2}\frac{2\pi d}{\lambda}(u-u_T)}\sum_{n=0}^{N-1} w_n^* e^{\frac{j2\pi d}{\lambda}n(u-u_T)} \tag{8}$$

For distortionless array response, $\sum_{n=0}^{N-1} w_n^* = 1$

Also,

$$|B(u)|^2 = (e^{j\frac{N-1}{2}\frac{2\pi d}{\lambda}(u-u_T)}\sum_{n=0}^{N-1} w_n e^{-\frac{j2\pi d}{\lambda}nu})\ (e^{j\frac{N-1}{2}\frac{2\pi d}{\lambda}(u-u_T)}\sum_{l=0}^{N-1} w_l^* e^{\frac{j2\pi d}{\lambda}lu})$$

which simplifies to,

$$|B(u)|^2 = \sum_{n,l=0}^{N-1} w_n w_l^* (e^{-j\frac{2\pi d}{\lambda}u(n-l)} \tag{9}$$

For a standard Uniform Linear Array i.e., for $d = \frac{\lambda}{2}$

$$\frac{1}{2}\int_{-1}^{-1} |B(u)|^2 du = \sum_{n=0}^{N-1} w_n {w_n}^* = ||w||_2^2$$

From equation (6), we have

$$D = \frac{1}{||w||_2^2} \tag{10}$$

where,

$$||w||_2^2 = (w^H w)^{\frac{1}{2}} = (\sum_{i=1}^{N} |w_i|^2)^{\frac{1}{2}}$$

Notes:
(i) Directivity (D) is independent of $(\theta_T,\, \phi_T)$.
(ii)The above result for D is valid only for $d = \frac{\lambda}{2}$, and if $d \neq \frac{\lambda}{2}$, then the directivity will dependent on $(\theta_T,\, \phi_T)$.

Then for Uniform Weighting, $w_n = \frac{1}{N}$ maximizes the directivity (D) of a standard linear array .

This is an Optimization problem of $max(||w||_2^2)^{-1}$ subject to the constraint $\sum_{i=1}^{N} w_i = 1$ has a solution of the form

$$w_i = \frac{1}{N} \quad (Proof\ is\ left\ as\ an\ exercise) \tag{11}$$

2. **Array Gain Versus Spatially White Noise:**

$$x_n(t) = f(t-\tau) + z_n(t) \tag{12}$$

where,
$z_n(t)$ is a representation of spatial white noise,.
$f(t-\tau)$ is the time shifted replica of $f(t)$.

$$SNR_{in}(\omega) = \frac{S_f(w)}{S_n(w)} \tag{13}$$

$$S_Y(\omega) = H^H(\omega) S_x(\omega) H(\omega) \tag{14}$$

$$S_Y(\omega) = w^H S_x(\omega) w \quad (Output\ Spectrum) \tag{15}$$

The constraint for distortionless array response is given by $w^H v_k(k_s) = 1$, where k_s is the steering number. The input signal vector and its fourier transform are given by

$$\mathbf{f}(t) = \begin{bmatrix} f(t-\tau_0) \\ f(t-\tau_1) \\ . \\ . \\ f(t-\tau_N - 1) \end{bmatrix} \tag{16}$$

$$\mathbf{F}(t) = v_k(k_s) F(\omega) \tag{17}$$

where,

$$v_k(k_s) = \begin{bmatrix} e^{jk_s^T p_0} \\ e^{jk_s^T p_1} \\ . \\ . \\ e^{jk_s^T p_{N-1}} \end{bmatrix} \tag{18}$$

$$\mathbf{S}_f(\omega) = v_k(k_s) S_f(\omega) v_k^H(k_s) \tag{19}$$

Also the noise output power spectral density is expressed as

$$\mathbf{S}_{yn}(\omega) = ||w||^2 S_n(\omega) \tag{20}$$

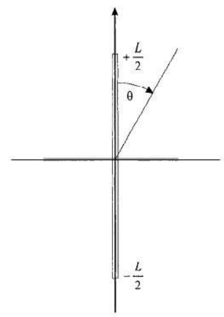


Figure 2: Linear Aperture of length L

Which results in a output signal to noise ratio $(SNR_o(\omega))$ of

$$SNR_o(\omega) = \frac{\mathbf{w}^H v_k(k_s) S_f(\omega) v_k^H(k_s)\mathbf{w}}{||w||^2 S_n(\omega} \tag{21}$$

We have $\mathbf{w}^H v_k(k_s) = v_k^H(k_s)\mathbf{w} = 1$,
So the Array Gain is the ratio of output$(SNR_o(\omega))$ to input$(SNR_{in}(\omega))$ signal to noise ratio

$$A_w = SNR_o(\omega) = \frac{S_f(\omega)}{||w||^2 S_n(\omega)} = \frac{1}{||w||^2} \tag{22}$$

Notes:
(i) The above expression for Array Gain is valid for arbitrary geometry provided that $w^H v_k(k_s) = 1$ is true.
(ii)For a standard linear array $d = \frac{\lambda}{2}$, the Array Gain(A_w) is identical to directivity.

III. **LINEAR APERTURE**

$$\gamma(\omega, k_z) = \int_{-\frac{L}{2}}^{\frac{L}{2}} w_a^*(z) e^{-jk_z z} dz \tag{23}$$

where $w_a^*(z)$ is the aperture weighting function and $e^{-jk_z z}$ is the array manifold function.

Let $w_a^*(z) = 0$ for $|z| > \frac{L}{2}$ then the frequency wavenumber response function can be written as

$$\gamma(\omega, k_z) = \int_{-\infty}^{\infty} w_a^*(z) e^{-jk_z z} dz \tag{24}$$

which represents the fourier transform of $w_a^*(z)$ and the inverse relation is expressed as,

$$w_a^*(z) = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \gamma(\omega, k_z) e^{jk_z z} dk_z \tag{25}$$

The aperture weighting function and the frequency wavenumber response are fourier transform pairs in the 'z'and 'k_z'domains.

For Uniform Aperture Weighting Function $w_a^*(z) = \frac{1}{L}, \quad |z| \leq \frac{L}{2}$

$$\gamma(\omega, k_z) = \frac{\sin(\frac{L}{2}k_z)}{(\frac{L}{2}k_z)} \tag{26}$$

where $k_z = \frac{2\pi}{\lambda}u$. The beam pattern can be expressed as

$$B(u) = \frac{\sin(\frac{L}{2}\frac{2\pi}{\lambda}u)}{\frac{L}{2}\frac{2\pi}{\lambda}u} \tag{27}$$

Comparing this beam pattern with that of uniform linear array i.e., $B(u) = \frac{\sin(\frac{2\pi d}{\lambda}u\frac{N}{2})}{\frac{2\pi d}{\lambda}u\frac{1}{2}}$. Equating the arguments of numerator we get, $(\frac{L}{2}\frac{2\pi}{\lambda}u) = (\frac{N}{2}\frac{2\pi}{\lambda}d)$ which simplifies to

$$L = Nd \tag{28}$$

Which ensures the same BW_{NN} and null spacing for both the linear aperture and uniform linear array.

IV. NON-ISOTROPIC SENSORS

Consider the case of two identical non-isotropic sensors $ULA1$ and $ULA2$, the effective frequency response function is expressed as

$$\gamma(\omega, k) = \gamma_0(\omega, k)w_0^* e^{-jk^T p_0} + \gamma_1(\omega, k)w_1^* e^{-jk^T p_1} \tag{29}$$

$$\gamma(\omega, k) = \gamma_1(\omega, k)(w_0^* e^{-jk^T p_0} + w_1^* e^{-jk^T p_1}) \tag{30}$$

It is as if we had isotropic sensors at p_0 and p_1.

V. DESIGNING UNIFORM LINEAR ARRAYS

1. **Rectangular window**
Uniform weighting, $w_n^* = \frac{1}{N}$. Narrowest mainlobe, but the side lobe levels are high (first sidelobe level is -13dB).

$$B(\psi) = e^{-\frac{N-1}{2}\psi}\sum_{n=0}^{N-1} w_n^* e^{jn\psi} \tag{31}$$

$$w_n^* = \frac{1}{2\pi}\int_{-\pi}^{\pi} B(\psi)e^{j\frac{N-1}{2}\psi}e^{-jn\psi}d\psi \tag{32}$$

2. **Cosine Weighting**

$$w[\tilde{n}] = \sin(\frac{\pi}{2N})\cos(\frac{\pi\tilde{n}}{N}) \tag{33}$$

where $\tilde{n} = n - \frac{N-1}{2}$ and $\frac{-(N-1)}{2} \leq \tilde{n} \leq \frac{(N-1)}{2}$

Beam pattern is given by

$$B(u) = \frac{1}{2}\sin\frac{\pi}{2N}\left[\frac{\sin\frac{N}{2}\pi(u-\frac{1}{N})}{\sin\frac{\pi}{2}(u-\frac{1}{N})} + \frac{\sin\frac{N}{2}\pi(u+\frac{1}{N})}{\sin\frac{\pi}{2}(u+\frac{1}{N})}\right] \tag{34}$$

The usage of windows offers a trade-off between the mainlobe width and sidelobe rejection. In general, the weighting functions other than uniform weights results in smaller sidelobe levels(sidelobe level is -22dB for cosine weighting) at a cost of increase in mainlobe width. [1]

[1] All the figures considered in this lecture notes are taken from *Optimum Array Processing, Part IV of Detection, Estimation And Modulation Theory* by Harry L. Van Trees, 2002. Wiley-InterScience