Stochastic Narrowband Models

ECE 6279: Spatial Array Processing
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Lecture 10

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Where We Are (and Aren't) in J&D

- **Most of the lecture drawn from:**
  - Notes from Dan Fuhrmann's class at Washington University
  - "A Note on Generating Complex Gaussian Data," by Tim Barton
  - Typical practice in the literature

- **Last few slides based on J&D, Chapter B, particularly Sec. B.5**

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An Abuse of Notation

- **We will use complex baseband signal representations**

- **We will drop the superscript b:**

$$\underbrace{s^b(t)}_{\text{Old}} \equiv \underbrace{s(t)}_{\text{New}}$$

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Complex Random Variables

- **General complex random process:**

$$\underline{z}(t) = \underline{x}(t) + j\underline{y}(t)$$

- **Mean:**

$$E[\underline{z}(t)] = E[\underline{x}(t)] + jE[\underline{y}(t)] \equiv \mu(t)$$

- **Zero-mean:**

$$\mu(t) = 0$$

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Circular Random Processes

$$\underline{z}(t) = \underline{x}(t) + j\underline{y}(t)$$

Circular or Goodman class:

$$E\begin{bmatrix} \underline{x}(u)\underline{x}(v) & \underline{x}(u)\underline{y}(v) \\ \underline{y}(u)\underline{x}(v) & \underline{y}(u)\underline{y}(v) \end{bmatrix}$$

$$= \begin{cases} \dfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\sigma^2(u), & u = v \\[2ex] \dfrac{1}{2}\begin{bmatrix} \alpha(u,v) & -\beta(u,v) \\ \beta(u,v) & \alpha(u,v) \end{bmatrix}\sigma(u)\sigma(v), & u \neq v \end{cases}$$

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Correlation Function

$$\underline{z}(t) = \underline{x}(t) + j\underline{y}(t)$$

- **Correlation function** **for a circular-complex random process:**

$$R_z(u,v) = E(\underline{z}(u)\underline{z}^*(v))$$

$$= \begin{cases} \sigma^2(u), & u = v \\ [\alpha(u,v) + j\beta(u,v)]\sigma(u)\sigma(v), & u \neq v \end{cases}$$

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WSS Processes

- **Wide sense stationary:**

$$R_z(u,v) = E[\underline{z}(u)\underline{z}^*(v)] = R_z(\underbrace{u-v}_{\tau}) \equiv R_z(\tau)$$

- **Property:** $R_z(-\tau) = R_z^*(\tau)$

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Formulation for *N* Sources

$$\mathbf{y}(t) = \sum_{n=1}^{N} \mathbf{e}(\vec{k}_n) s_n(t) \qquad \vec{\mathbf{k}} = \left[\vec{k}_1, \ldots, \vec{k}_N\right]$$

J&D call this $\mathbf{S}$ which is a horrible idea $\longrightarrow \mathbf{D}(\vec{\mathbf{k}})$

$$= \underbrace{\begin{bmatrix} \mathbf{e}(\vec{k}_1) & \cdots & \mathbf{e}(\vec{k}_N) \end{bmatrix}}_{\mathbf{D}(\vec{\mathbf{k}})} \underbrace{\begin{bmatrix} s_1(t) \\ \vdots \\ s_N(t) \end{bmatrix}}_{\mathbf{s}(t)} = \mathbf{D}(\vec{\mathbf{k}})\mathbf{s}(t)$$

$$\begin{bmatrix} y_0(t) \\ \vdots \\ y_{M-1}(t) \end{bmatrix} = \exp\left(\begin{bmatrix} -j\vec{k}_1 \cdot \vec{x}_0 & \cdots & -j\vec{k}_N \cdot \vec{x}_0 \\ \vdots & & \vdots \\ -j\vec{k}_1 \cdot \vec{x}_{M-1} & \cdots & -j\vec{k}_N \cdot \vec{x}_{M-1} \end{bmatrix} \right) \begin{bmatrix} s_1(t) \\ \vdots \\ s_N(t) \end{bmatrix}$$

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Another Common Notation

$$\mathbf{y}(t) = \sum_{n=1}^{N} \mathbf{e}(\boldsymbol{\theta}_n) s_n(t) \qquad \boldsymbol{\theta}_n = [\phi_n, \theta_n]$$

$$= \underbrace{\begin{bmatrix} \mathbf{e}(\boldsymbol{\theta}_1) & \cdots & \mathbf{e}(\boldsymbol{\theta}_N) \end{bmatrix}}_{\mathbf{D}(\boldsymbol{\Theta})} \underbrace{\begin{bmatrix} s_1(t) \\ \vdots \\ s_N(t) \end{bmatrix}}_{\mathbf{s}(t)} = \mathbf{D}(\boldsymbol{\Theta})\mathbf{s}(t)$$

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\theta}_1 & \cdots & \boldsymbol{\theta}_N \end{bmatrix} = [\phi_1, \theta_1, \ldots \phi_N, \theta_N]$$

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Noise Model

- **Add noise**
  - Background radiation
  - Receiver electronics

$$\underline{\mathbf{y}}(t) = \mathbf{D}(\boldsymbol{\Theta})\mathbf{s}(t) + \underline{\mathbf{n}}(t)$$

- **Usually assume noise is a zero-mean, circular complex, wide-sense-stationary random process**

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Sampled Data (1)

- **Sample signal with sample spacing *T***
- **There is a random process equivalent of Shannon's sampling theorem**
  - **Involves the power spectrum**
  - **Often covered in ECE7251**
  - **Many subtleties that I will gloss over**

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Sampled Data

$$\underline{\mathbf{y}}(t) = \mathbf{D}(\Theta)\mathbf{s}(t) + \underline{\mathbf{n}}(t)$$

$$t = kT$$

$$\underline{\mathbf{y}}(kT) = \mathbf{D}(\Theta)\mathbf{s}(kT) + \underline{\mathbf{n}}(kT)$$

Abuse: $\underline{\mathbf{y}}(k) = \mathbf{D}(\Theta)\mathbf{s}(k) + \underline{\mathbf{n}}(k)$

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Two Signal Models

1. Signals $s_n(t)$ are deterministic, but usually unknown, functions

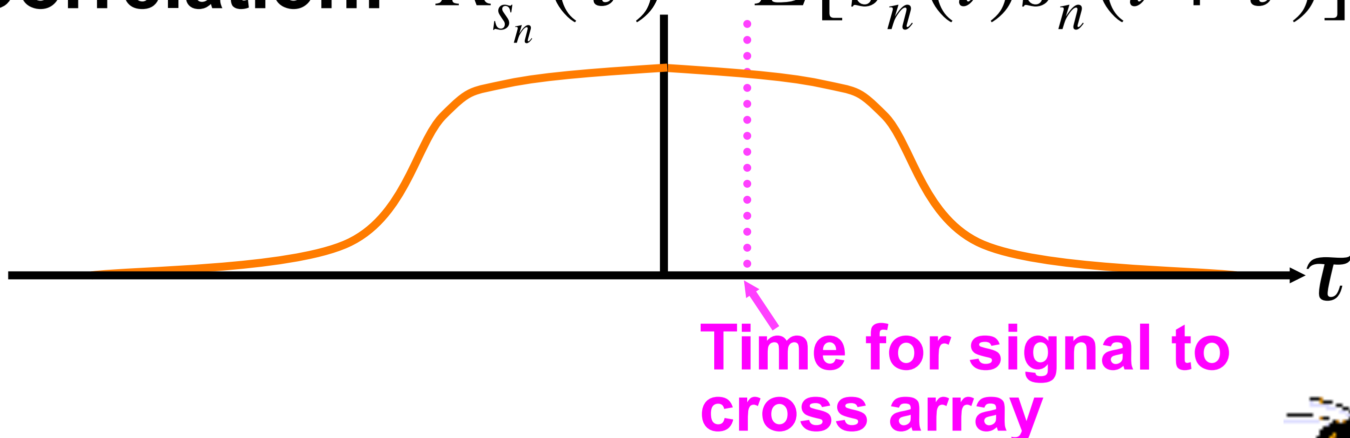
$$\underline{\mathbf{y}}(t) = \mathbf{D}(\boldsymbol{\Theta})\mathbf{s}(t) + \underline{\mathbf{n}}(t)$$

2. Signals $s_n(\mathrm{t})$ are wide sense stationary circular-complex random processes:

$$\underline{\mathbf{y}}(t) = \mathbf{D}(\boldsymbol{\Theta})\underline{\mathbf{s}}(t) + \underline{\mathbf{n}}(t)$$

Signal and noise uncorrelated

Correlation: $R_{s_n}(\tau) = E[s_n(t)s_n^*(t+\tau)]$



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Snapshot Assumptions

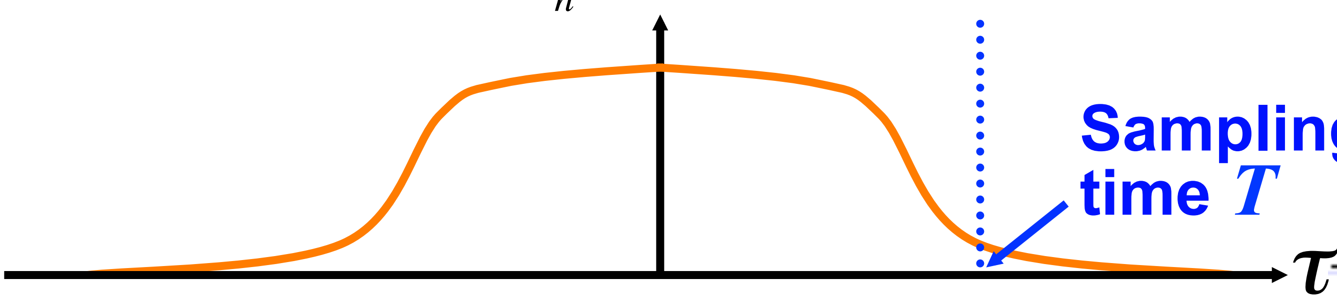
- Usually <u>assume</u> "snapshots" (discrete-time samples) are uncorrelated

Deterministic: $\underline{\mathbf{y}}(k) = \mathbf{D}(\Theta)\mathbf{s}(k) + \underline{\mathbf{n}}(k)$

Stochastic: $\underline{\mathbf{y}}(k) = \mathbf{D}(\Theta)\underline{\mathbf{s}}(k) + \underline{\mathbf{n}}(k)$

- This is usually an approximation; they're rarely truly uncorrelated

Correlation: $R_{s_n}(\tau) = E[s_n(t)s_n^*(t+\tau)]$



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Correlation Matrices

- Usually characterize the noise with a single correlation matrix

$$\mathbf{R}_{\mathbf{n}} = E[\underline{\mathbf{n}}(k)\underline{\mathbf{n}}^H(k)]$$

- In the stochastic signal model, usually characterize the signal with a single correlation matrix

$$\mathbf{R}_{\mathbf{s}} = E[\underline{\mathbf{s}}(k)\underline{\mathbf{s}}^H(k)] = \mathbf{C} \text{ in J\&D}$$

(careful – J&D use $\mathbf{R}_{\mathbf{s}}$ differently)

- Over long time periods, statistics may change, and correlation matrices become a function of time

$$\mathbf{R}_{\mathbf{s}}(k), \quad \mathbf{R}_{\mathbf{n}}(k)$$

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Signal Correlation Matrices

- **Signal correlation matrix is often modeled as diagonal**
- **If not diagonal, usually a symptom of multipath reflections**

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Noise Correlation Matrices

- Usually assume noise is zero mean
- Receiver-noise-only model yields uniform diagonal correlation matrix:

$$\mathbf{R_n} = \mathbf{K_n} = \sigma_n^2 \mathbf{I}$$

- Background radiation may result in more complicated noise correlation with off-diagonal terms

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Data Correlation for Stochastic Signals (1)

$$\underline{\mathbf{y}}(k) = \mathbf{D}(\Theta)\underline{\mathbf{s}}(k) + \underline{\mathbf{n}}(k)$$

$$\begin{aligned}\mathbf{R}_{\mathbf{y}} &= E[\underline{\mathbf{y}}\underline{\mathbf{y}}^H] \\ &= E[\{\mathbf{D}(\Theta)\underline{\mathbf{s}} + \underline{\mathbf{n}}\}\{\mathbf{D}(\Theta)\underline{\mathbf{s}} + \underline{\mathbf{n}}\}^H]\end{aligned}$$

$$= E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$
$$+E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{n}}^H] + E[\underline{\mathbf{n}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)]$$

$$= E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H\mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

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Data Correlation for Stochastic Signals (2)

$$\mathbf{R}_{\mathbf{y}} = E[\mathbf{D}(\Theta)\underline{\mathbf{s}}\underline{\mathbf{s}}^H \mathbf{D}^H(\Theta)] + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

$$= \mathbf{D}(\Theta)E[\underline{\mathbf{s}}\underline{\mathbf{s}}^H]\mathbf{D}^H(\Theta) + E[\underline{\mathbf{n}}\underline{\mathbf{n}}^H]$$

$$= \mathbf{D}(\Theta)\mathbf{R}_{\mathbf{s}}\mathbf{D}^H(\Theta) + \mathbf{R}_{\mathbf{n}}$$

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General Correlation Matrix Properties

- **Hermitian (conjuate) symmetry:**

$$\mathbf{R} = \mathbf{R}^H$$

  - **Implies real eigenvalues**
  - **Implies eigenvectors associated with distinct eigenvalues are orthogonal**

- **Nonnegative definiteness:**

$$\mathbf{a}\mathbf{R}\mathbf{a}^H \geq 0, \quad \forall \mathbf{a} \neq 0$$

  - **Implies nonnegative eigenvalues**

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Eigenvector Matrix

- Eigenvector-eigenvalue relationship:

$$\mathbf{R}\mathbf{v} = \lambda \mathbf{v}$$

- Form eigenvector matrix:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_M \end{bmatrix}$$

- If $\mathbf{R}$ has distinct eigenvalues, then eigenvectors are orthogonal, and hence $\mathbf{V}$ is unitary:

$$\mathbf{V}^H \mathbf{V} = \mathbf{I}$$

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Eigenexpansions

$$\mathbf{V}^H\mathbf{R}\mathbf{V} = diag(\lambda_1, \ldots \lambda_M)$$

- **Since $\mathbf{V}$ is unitary:**

$$\mathbf{R} = \mathbf{V}diag(\lambda_1, \ldots \lambda_M)\mathbf{V}^H = \sum_{m=1}^{M} \lambda_m \mathbf{v}_m \mathbf{v}_m^H$$

- **Note $\mathbf{R}\mathbf{v} = \lambda\mathbf{v}$ means $\frac{1}{\lambda}\mathbf{v} = \mathbf{R}^{-1}\mathbf{v}$**

$$\mathbf{R}^{-1} = \sum_{m=1}^{M} \frac{1}{\lambda_m} \mathbf{v}_m \mathbf{v}_m^H$$

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Quadratic Forms

- **Sometimes this comes in handy:**

$$\mathbf{a}^H \mathbf{R}\mathbf{a} = \mathbf{a}^H \left(\sum_{m=1}^{M} \lambda_m \mathbf{v}_m \mathbf{v}_m^H \right) \mathbf{a}$$

$$= \sum_{m=1}^{M} \lambda_m \left| \mathbf{a}^H \mathbf{v}_m \right|^2$$