E9 231: Digital Array Signal Processing

Scribe: Nissar K.E.
ME(Telecommunication)
Dept. of ECE
Indian Institute of Science
Bangalore 560 012, India
nissarke@ece.iisc.ernet.in

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1 Topics

- HW 4: 4.1.5 and 4.1.1.
- Rectangular array Beam pattern
- Visible Region
- Grating lobes and Array steering
- Directivity
- Planar Array Design

1.1 Rectangular Array Beampattern

The rectangular array is placed in X-Y Plane. The position vectors of the sensors in the array is given by,

$$P_{n,m} = \begin{bmatrix} \left[n - \frac{N-1}{2}\right] d_x \\ \left[m - \frac{M-1}{2}\right] d_y \\ 0 \end{bmatrix} = \begin{bmatrix} p_{xn} \\ p_{yn} \\ p_{zn} \end{bmatrix} \tag{1}$$

where N, is the number of sensors in the X- direction and M is the number of sensors in the Y- direction. d_x is the spacing between sensors in X- direction and d_y is the spacing between sensors in Y- direction. The wave number k is given by,

$$k = \frac{-2\pi}{\lambda} \begin{bmatrix} [\sin\theta\cos\phi] \\ [\sin\theta\sin\phi] \\ [\cos\theta] \end{bmatrix} \tag{2}$$

$$0 \leq \theta \leq 90\,\text{deg}, 0 \leq \phi \leq 360\,\text{deg} \tag{3}$$

$$\omega\tau_{n,m} = \frac{-2\pi}{\lambda}\left[\left(n - \frac{N-1}{2}\right) d_x \sin\theta\cos\phi + \left(m - \frac{M-1}{2}\right) d_y \sin\theta\sin\phi\right] \tag{4}$$

$$u = \begin{pmatrix} [\sin\theta\cos\phi] \\ \\ [\sin\theta\sin\phi] \\ \\ [\cos\theta] \end{pmatrix} \tag{5}$$

$$\psi = \begin{pmatrix} \frac{2\pi d_x}{\lambda} u_x \\ \\ \frac{2\pi d_y}{\lambda} u_y \end{pmatrix} \tag{6}$$

Let $w_{n,m}^*$ be the sensor weight given to $[n,m]^{th}$ sensor, the beampattern is given by,

$$B(\psi_x, \psi_y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} w_{n,m}^* \exp(-j\omega\tau_{m,n}) \tag{7}$$

$$B(\psi_x, \psi_y) = \exp^{-j\left(\frac{N-1}{2}\psi_x + \frac{M-1}{2}\psi_y\right)} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} w_{n,m}^* \exp^{j(n\psi_x + m\psi_y)} \tag{8}$$

1.2 Visible Region

$$0 \leq \theta \leq 90\deg, 0 \leq \phi \leq 360\deg \tag{9}$$

$$u_x = \sin\theta\cos\phi \tag{10}$$

$$u_y = \sin\theta\sin\phi \tag{11}$$

$$u_x^2 + u_y^2 = \sin^2\theta \leq 1. \tag{12}$$

In terms of ψ_x and ψ_y,

$$\left[\frac{\psi_x}{d_x}\right]^2 + \left[\frac{\psi_y}{d_y}\right]^2 \leq \left[\frac{2\pi}{\lambda}\right]^2 \tag{13}$$

If $d_x = d_y = \frac{\lambda}{2}$

$$\psi_x^2 + \psi_y^2 \leq \pi^2 \tag{14}$$

1.3 Grating Lobes and Array Steering

When the array is steered to broadside direction, the beampattern will be periodic with a perioid of $\frac{\lambda}{d_x}$ in the X- direction $\frac{\lambda}{d_y}$ in the Y- Direction. The grating lobes will occur at

$$u_x = p\frac{\lambda}{d_x}, p = 1, 2, ... \tag{15}$$

$$u_y = q\frac{\lambda}{d_y}, q = 1, 2, ... \tag{16}$$

If the array is steered towards a direction, say $[u_{x0}, u_{y0}]$ the beampattern in $u-$ space will be

$$B_{ST}(u_x, u_y) = B(u_x - u_{x0}, u_y - u_{y0}) \tag{17}$$

$$u_{x0} = \sin\theta_0\cos\phi_0 \tag{18}$$

$$u_{y0} = \sin\theta_0 \sin\phi_0 \tag{19}$$

Here all the sensors are phase aligned towards $[\theta_0, \phi_0]$. The grating lobe shift along the steering direction. The worst case scenario is when steering along the X- axis and Y- axis. that is, $[\theta_0, \phi_0] = \left[90^0, 90^0\right]$ and $[\theta_0, \phi_0] = \left[90^0, 0^0\right]$ To avoid grating lobes when steered towards $\left[90^0, 0^0\right]$

$$\frac{\lambda}{-d_x} + 1 \leq -1 \tag{20}$$
$$\rightarrow \frac{d_x}{\lambda} \leq \frac{1}{2}$$

similary, to avoid grating lobes when steered towards $\left[90^0, 90^0\right]$

$$\frac{\lambda}{-d_y} + 1 \leq -1 \tag{21}$$
$$\rightarrow \frac{d_y}{\lambda} \leq \frac{1}{2}$$

when $d_x = d_y = \frac{\lambda}{2}$, we call the array a standard uniform rectangular array.

1.4 Separable Weighting

If $w_{n,m} = w_n.w_m$ the beampattern will be

$$B(\psi_x, \psi_y) = B_x(\psi_x).B_y(\psi_y) \tag{22}$$

If the weighting is uniform $\rightarrow w_n = \frac{1}{N} and w_m = \frac{1}{M}$ the beam pattern will be

$$B(\psi_x, \psi_y) = \frac{1}{N}\frac{\sin\frac{N\psi_x}{2}}{\sin\frac{\psi_x}{2}}\frac{1}{M}\frac{\sin\frac{M\psi_y}{2}}{\sin\frac{\psi_y}{2}} \tag{23}$$

1.5 Half Power Beam Width (HPBW)

If we fix θ and plot beampattern along ϕ we get HPBW in ϕ space. The HPBW is a contour in $[\theta, \phi]$ space, where $|B(\theta, \phi)|^2 = \frac{1}{2}$

There are some weightings where the array when steered to broadside ($\theta = 0$) HPBW contour is a circle or an ellipse. all other cases we must evaluate numerically. For a large array (M, N) large, if we steer near broadside, the beam pattern will be

$$\theta_{HPBW} = \sqrt{\frac{1}{\cos\theta_0^2\left(\frac{\cos\phi_0^2}{\theta_{x0}^2} + \frac{\sin\phi_0^2}{\theta_{y0}^2}\right)}} \tag{24}$$

where $\theta_{x0} \rightarrow$ HPBW of broadside linear array having N elements, $\theta_{y0} \rightarrow$ HPBW of broadside linear array having M elements and

$$\phi_{HPBW} = \sqrt{\frac{1}{\left(\frac{\sin\phi_0^2}{\theta_{x0}^2} + \frac{\cos\phi_0^2}{\theta_{y0}^2}\right)}} \tag{25}$$

1.6 Directivity of a Planar Array

Directivity of a uniform rectangular array is given by

$$D = \frac{\left|B\left(\theta_0, \phi_0\right)\right|^2}{\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left|B\left(\theta, \phi\right)\right|^2 \sin\theta d_\theta d_\phi} \tag{26}$$

$$B\left(\theta, \phi\right) = \sum_{n=0}^{\tilde{N}-1} w_n^* \exp j\frac{2\pi}{\lambda}\left(p_{xn}\sin\theta\cos\phi + p_{yn}\sin\theta\sin\phi\right) \tag{27}$$

where $\tilde{N} = NM$ is the total number of elements in the rectangular array and (θ_0, ϕ_0) is the steering direction. Let

$$W = \begin{pmatrix} w_0, w_1, ...w_{\tilde{N}-1} \end{pmatrix}^T \tag{28}$$

and

$$V\left(\theta, \phi\right) = \begin{bmatrix} \exp^{j\frac{2\pi}{\lambda}\left(p_{x0}\sin\theta\cos\phi + p_{y0}\sin\theta\sin\phi\right)} \\ . \\ . \\ . \\ \exp^{j\frac{2\pi}{\lambda}\left(p_{x\tilde{N}-1}\sin\theta\cos\phi + p_{y\tilde{M}-1}\sin\theta\sin\phi\right)} \end{bmatrix} \tag{29}$$

The beam pattern will become,

$$B\left(\theta, \phi\right) = W^H V\left(\theta, \phi\right) \tag{30}$$

the denominator in the directivity equation can be written as

$$Dr = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[W^H V\left(\theta, \phi\right) V^H\left(\theta, \phi\right) W\right]\sin\theta d_\theta d_\phi = W^H B W \tag{31}$$

where

$$B = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[V\left(\theta, \phi\right) V^H\left(\theta, \phi\right)\right]\sin\theta d_\theta d_\phi \tag{32}$$

B is an $\tilde{N} \times \tilde{N}$ matrix depends on the sensor position in the array. Now the directivity will be

$$D = \frac{\left|W^H V\left(\theta_0, \phi_0\right)\right|^2}{W^H B W} \tag{33}$$

If we impose the distortionless response in the array, $\left|W^H V\left(\theta_0, \phi_0\right)\right|^2 = 1$, the directivity will be

$$B = \left[W^H B W\right]^{-1} \tag{34}$$

For maximum directivity,

$$w_{opt} = \alpha B^{-1} V\left(\theta_0, \phi_0\right) \tag{35}$$

$$D_{opt} = \frac{V^H\left(\theta_0, \phi_0\right) B^{-1} V\left(\theta_0, \phi_0\right) V^H\left(\theta_0, \phi_0\right) B^{-1} V\left(\theta_0, \phi_0\right)}{V^H\left(\theta_0, \phi_0\right) B^{-H} B B^{-1} V\left(\theta_0, \phi_0\right)} \tag{36}$$

the optimum value of the directivity will be,

$$D_{opt} = V^H\left(\theta_0, \phi_0\right) B^{-1} V\left(\theta_0, \phi_0\right) \tag{37}$$

1.7 Designing Planar Arrays

If $d_x = d_y = \frac{\lambda}{2}$, The beam pattern will be the fourier transform of the array weighting function. So if $B\left(\psi_x, \psi_y\right)$ is known, we can use IDFT to write

$$w_{nm}^* = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} B\left(\psi_x, \psi_y\right)\exp^{j\left(\frac{N-1}{2}\psi_x + \frac{M-1}{2}\psi_y\right)}\exp^{-j\left(n\psi_x + m\psi_y\right)} \tag{38}$$

As the array will be having only finite elements, we need to truncate the weighting fuction. this leads to window based designs.