MVDR Beamforming

ECE 6279: Spatial Array Processing
Spring 2013
Lecture 15

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Where We Are in J&D

- **Section 7.2.1 and Appendix B**

"Minimum Variance"

$$\mathbf{w}_{\langle\rangle} = \arg\min_{\mathbf{w}} \mathbf{w}^H \mathbf{R}_{\mathbf{y}} \mathbf{w} \ \text{ s.t. } \mathbf{C}\mathbf{w} = \mathbf{c}$$

Solution: $\mathbf{w} = \mathbf{R}_{\mathbf{y}}^{-1}\mathbf{C}^H(\mathbf{C}\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{C}^H)^{-1}\mathbf{c}$

Power of beamformer output:

$$\mathbf{w}^H \mathbf{R}_{\mathbf{y}} \mathbf{w} = \mathbf{c}^H(\mathbf{C}\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{C}^H)^{-1}\mathbf{c}$$

"Distortionless Reponse"

- For ex., $\mathbf{w}^H \mathbf{e}(\vec{k}) = 1$ assures signal in look direction passes unharmed

$$\mathbf{e}^H(\vec{k})\mathbf{w} = 1$$

$$\mathbf{C}\mathbf{w} = \mathbf{c}$$

MVDR Weights

$$\mathbf{w} = \mathbf{R}_{\mathbf{y}}^{-1}\mathbf{C}^{H}(\mathbf{C}\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{C}^{H})^{-1}\mathbf{c}$$

- Plug in $\mathbf{C} = \mathbf{e}^{H}(\vec{k}),\ \mathbf{c} = 1$

$$\mathbf{w}(\vec{k}) = \mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})[\mathbf{e}^{H}(\vec{k})\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})]^{-1}$$

$$= \frac{\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})}{\mathbf{e}^{H}(\vec{k})\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})}$$

MVDR Beamforming

$$z = \mathbf{w}^H \mathbf{y}$$

$$= \mathbf{e}^H(\vec{k}) \frac{\mathbf{R}_{\mathbf{y}}^{-1}}{\mathbf{e}^H(\vec{k})\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})} \mathbf{y}$$

- **In practice:**

$$z = \mathbf{e}^H(\vec{k}) \frac{\hat{\mathbf{R}}_{\mathbf{y}}^{-1}}{\mathbf{e}^H(\vec{k})\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})} \mathbf{y}$$

MVDR Power in Output Beamformer

$$\mathbf{w}^H \mathbf{R}_\mathbf{y} \mathbf{w} = \mathbf{c}^H (\mathbf{C}\mathbf{R}_\mathbf{y}^{-1}\mathbf{C}^H)^{-1}\mathbf{c}$$

- Plug in $\mathbf{C} = \mathbf{e}^H(\vec{k}),\ \mathbf{c} = 1$

$$P^{MV}(\vec{k}) \equiv \frac{1}{\mathbf{e}^H(\vec{k})\mathbf{R}_\mathbf{y}^{-1}\mathbf{e}(\vec{k})}$$

- Compare with:

$$P^{CONV}(\vec{k}) \equiv \mathbf{e}^H(\vec{k})\mathbf{R}_\mathbf{y}\mathbf{e}(\vec{k})$$

Using Empirical Correlation Matrices

- **In practice:**

$$P^{MV}(\vec{k}) \equiv \frac{1}{\mathbf{e}^H(\vec{k})\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})}$$

$$P^{CONV}(\vec{k}) \equiv \mathbf{e}^H(\vec{k})\hat{\mathbf{R}}_{\mathbf{y}}\mathbf{e}(\vec{k})$$

Analysis of a Simple Case

- **Correlation matrix for one signal in white noise**

$$\mathbf{R}_{\mathbf{y}} = P_s^2 \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}$$

$$\mathbf{R}_{\mathbf{y}}^{-1} = [P_s^2 \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0) + \sigma_n^2 \mathbf{I}]^{-1}$$

Matrix Inversion Lemma (1)

$$(\mathbf{A}+\mathbf{BCD})^{-1} =$$
$$\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B}+\mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}$$

- See p. 489 for other variations
- Special case from p. 490:

$$\mathbf{B}=\mathbf{x},\ \mathbf{C}=\mu,\ \mathbf{D}=\mathbf{y}^H$$

$$(\mathbf{A}+\mu\mathbf{x}\mathbf{y}^H)^{-1} = \mathbf{A}^{-1} -$$
$$\mathbf{A}^{-1}\mathbf{x}\left(\mathbf{y}^H\mathbf{A}^{-1}\mathbf{x}+\frac{1}{\mu}\right)^{-1}\mathbf{y}^H\mathbf{A}^{-1}$$

Matrix Inversion Lemma (2)

$$(\mathbf{A} + \mu \mathbf{x}\mathbf{y}^H)^{-1}$$

$$= \mathbf{A}^{-1} - \mathbf{A}^{-1} \frac{\mathbf{x}\mathbf{y}^H}{(1/\mu) + \mathbf{y}^H \mathbf{A}^{-1} \mathbf{x}} \mathbf{A}^{-1}$$

$$= \mathbf{A}^{-1} \left(\mathbf{I} - \frac{\mathbf{x}\mathbf{y}^H}{(1/\mu) + \mathbf{y}^H \mathbf{A}^{-1} \mathbf{x}} \mathbf{A}^{-1} \right)$$

- Why were we doing this again?

$$\mathbf{R}_{\mathbf{y}}^{-1} = [\underbrace{P_s^2 \mathbf{e}\mathbf{e}^H}_{\mu \mathbf{x}\mathbf{y}^H} + \underbrace{\sigma_n^2 \mathbf{I}}_{\mathbf{A}}]^{-1}$$

Use the MIL on the Correlation Matrix (1)

$$\mathbf{R}_{\mathbf{y}}^{-1} = [\underbrace{P_s^2 \mathbf{e}\mathbf{e}^H}_{\mu \mathbf{x}\mathbf{y}^H} + \underbrace{\sigma_n^2 \mathbf{I}}_{\mathbf{A}}]^{-1}$$

$$= \mathbf{A}^{-1}\left(\mathbf{I} - \frac{\mathbf{x}\mathbf{y}^H}{(1/\mu) + \mathbf{y}^H \mathbf{A}^{-1}\mathbf{x}}\mathbf{A}^{-1}\right)$$

$$= \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\mathbf{e}\mathbf{e}^H}{(1/A_s^2) + (\underbrace{\mathbf{e}^H\mathbf{e}}_{M}/\sigma_n^2)}\frac{1}{\sigma_n^2}\right)$$

Use the MIL on the Correlation Matrix (2)

$$\mathbf{R}_{\mathbf{y}}^{-1} = \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\mathbf{e}\mathbf{e}^H}{(1/P_s^2) + (M/\sigma_n^2)}\frac{1}{\sigma_n^2}\right)$$

$$= \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{P_s^2\mathbf{e}\mathbf{e}^H}{\sigma_n^2 + MP_s^2}\right)$$

$$= \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{P_s^2\mathbf{e}\mathbf{e}^H}{MP_s^2 + \sigma_n^2}\right)$$

MVDR Power for Simple Special Case (1)

$$P^{MV}(\vec{k}) = [\mathbf{e}^H(\vec{k})\mathbf{R}_{\mathbf{y}}^{-1}\mathbf{e}(\vec{k})]^{-1}$$

$$= \sigma_n^2 \left[\mathbf{e}^H(\vec{k}) \left(\mathbf{I} - \frac{P_s^2 \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)}{MP_s^2 + \sigma_n^2} \right) \mathbf{e}(\vec{k}) \right]^{-1}$$

$$\mathbf{R}_{\mathbf{y}}^{-1} = \frac{1}{\sigma_n^2} \left(\mathbf{I} - \frac{P_s^2 \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)}{MP_s^2 + \sigma_n^2} \right)$$

MVDR Power for Simple Special Case (2)

$$P^{MV}(\vec{k}) =$$

$$= \sigma_n^2 \left[\mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}) - \frac{P_s^2 \mathbf{e}(\vec{k})^H \mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)\mathbf{e}(\vec{k})}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

$$= \sigma_n^2 \left[M - \frac{P_s^2 \left|\mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}^0)\right|^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

Max. MVDR Power for Special Case (1)

$$P^{MV}(\vec{k}) = \sigma_n^2 \left[M - \frac{P_s^2 \left| \mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}^0) \right|^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

- **Maxes out when** $k = \vec{k}^0$**:**

$$P^{MV}(\vec{k}^0) = \sigma_n^2 \left[M - \frac{M^2 P_s^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

Max. MVDR Power for Special Case (2)

$$P^{MV}(\vec{k}^0) = \sigma_n^2 \left[\frac{M(MP_s^2 + \sigma_n^2)}{MP_s^2 + \sigma_n^2} - \frac{M^2 P_s^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

$$= \sigma_n^2 \left[\frac{M\sigma_n^2}{MP_s^2 + \sigma_n^2} \right]^{-1} = \frac{MP_s^2 + \sigma_n^2}{M}$$

$$= P_s^2 + \underbrace{\frac{\sigma_n^2}{M}}_{\textbf{Bias}}$$

Min. MVDR Power for Special Case

$$P^{MV}(\vec{k}) = \sigma_n^2 \left[M - \frac{P_s^2 \left| \mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}^0) \right|^2}{MP_s^2 + \sigma_n^2} \right]^{-1}$$

- **Smallest possible:** $P^{MV}(\vec{k}) = \sigma_n^2 / M$

  when $\left| \mathbf{e}^H(\vec{k})\mathbf{e}(\vec{k}^0) \right| = 0$

Peak-to-Sidelobe Ratio for Special Case

$$\frac{\max_{\vec{k}} P^{MV}(\vec{k})}{\min_{\vec{k}} P^{MV}(\vec{k})} = \frac{P_s^2 + (\sigma_n^2 / M)}{\sigma_n^2 / M}$$

$$= M\left(\frac{P_s^2}{\sigma_n^2}\right) + 1$$

- **PSR is a function of SNR! (This is not the case with conventional beamforming.)**