Apertures, Part I

ECE 6279: Spatial Array Processing
Spring 2013
Lecture 4

Prof. Aaron D. Lanterman

School of Electrical & Computer Engineering
Georgia Institute of Technology
AL: 404-385-2548
<lanterma@ece.gatech.edu>

Where We Are in J&D

- **Lecture material drawn from:**
  - Sec. 3.1
  - 3.1.1 (through p. 63 only)
  - The last part of Sec. 3.1.3 (starting with "The wavenumber vector…" on p. 71 – don't worry about the first part of that section)

- **Please read example on p. 94 on circular apertures**
  - Won't cover in class, but please know about it

Aperture Functions

- **Want to observe a space-time field**

$$f(\vec{x},t)$$

- **Our sensors can only gather energy over a finite area, indicated by the (spatial) aperture function** $w(\vec{x})$

$$w(\vec{x}) = \begin{cases} \neq 0 & \text{inside aperture} \\ = 0 & \text{outside aperture} \end{cases}$$

Aperture Weighting

- **For places where $w(\vec{x}) \neq 0$, we sometimes get to pick $w(\vec{x})$**
  - Called aperture weighting
  - Also called shading, tapering or apodization, depending on context

- **Subject of future lectures**
  - Essence of beamforming

Smoothing Function

- **Space-time field through aperture:**

$$z(\vec{x},t) = w(\vec{x}) f(\vec{x},t)$$

- **Multiplication in spatial domain is convolution in wavenumber domain**

$$Z(\vec{k},\omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \underbrace{W(\vec{k}-\vec{l})}_{\text{Spatial FT of } w} \underbrace{F(\vec{l},\omega)}_{\text{Spatiotemporal FT of } f} d\vec{l}$$

aperture <u>smoothing</u> function

Mathematician's FT

$$W(\vec{k}) = \int_{-\infty}^{\infty} w(\vec{k}) \exp\{+ j\vec{k}\cdot\vec{x}\} d\vec{x}$$

Plane Wave (1)

- **General plane wave**

$$f(\vec{x},t) = s(t - \vec{\alpha}^0 \cdot \vec{x})$$

$$F(\vec{k},\omega) = S(\omega)(2\pi)^3 \delta(\vec{k} - \omega\vec{\alpha}^0)$$

- **Smoothed in wavenumber space**

$$Z(\vec{k},\omega) = \frac{1}{(2\pi)^3}(W *_{\vec{x}} F)(\vec{k},\omega)$$

$$= S(\omega)W(\vec{k} - \omega\vec{\alpha}^0)$$

Plane Wave (2)

$$Z(\vec{k},\omega) = S(\omega)W(\vec{k} - \omega\vec{\alpha}^0)$$

- **Consider** $\vec{k} = \omega\vec{\alpha}^0$

$$Z(\omega\vec{\alpha}^0,\omega) = S(\omega)W(0)$$

  – Information in signal $s(t)$ is preserved!

- **For** $\vec{k} \neq \omega\vec{\alpha}^0$

  – Information in signal $s(t)$ gets filtered

Multiple Plane Waves

$$f(\vec{x},t) = \sum_i s_i(t - \vec{\alpha}_i^0 \cdot \vec{x})$$

$$F(\vec{k},\omega) = \sum_i S_i(\omega)(2\pi)^3 \delta(\vec{k} - \omega\vec{\alpha}_i^0)$$

$$Z(\vec{k},\omega) = \sum_i S_i(\omega) W(\vec{k} - \omega\vec{\alpha}_i^0)$$

Information Along Line in k-ω space

$$Z(\vec{k},\omega) = \sum_i S_i(\omega) W(\vec{k} - \omega\vec{\alpha}_i^0)$$

- **For** $\vec{k} = \omega\vec{\alpha}_j^0$

$$Z(\omega\vec{\alpha}_j^0,\omega) = \sum_i S_i(\omega) W(\omega\vec{\alpha}_j^0 - \omega\vec{\alpha}_i^0)$$

Spatial Filtering

$$Z(\omega\vec{\alpha}_j^0,\omega) = \sum_i S_i(\omega) W(\omega\vec{\alpha}_j^0 - \omega\vec{\alpha}_i^0)$$

$$= S_j(\omega) W(0)$$

$$+ \sum_{i \neq j} S_i(\omega) W(\omega[\vec{\alpha}_j^0 - \vec{\alpha}_i^0])$$

- **If we can design w so that**

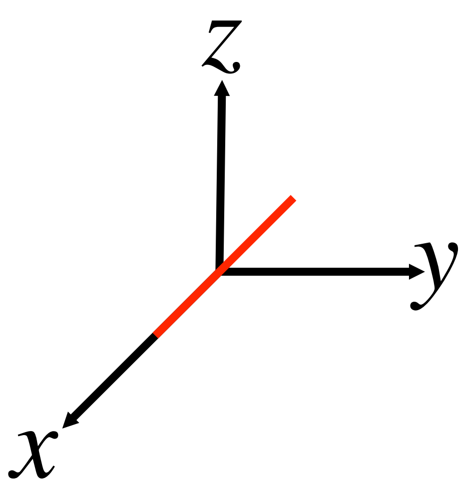
$$W(\omega[\vec{\alpha}_j^0 - \vec{\alpha}_i^0]) << W(0) \text{ for } i \neq j$$

we have a spatial filter
for direction $\vec{\alpha}_j$

Filled Linear Aperture

$$b(x) = \begin{cases} 1, & |x| \le D/2 \\ 0, & \text{otherwise} \end{cases}$$

$$w(\vec{x}) = b(x)\delta(y)\delta(z)$$



Linear Aperture

Mathematician's FT

$$W(k_x) = \int_{-\infty}^{\infty} b(x) \exp(+jk_x x)\, dx$$

$$= \int_{-D/2}^{D/2} \exp(jk_x x)\, dx = \frac{1}{jk_x} \exp(jk_x x) \Big|_{x=-D/2}^{x=D/2}$$

$$= \frac{1}{jk_x} \left[\exp\left(\frac{jk_x D}{2} \right) - \exp\left(-\frac{jk_x D}{2} \right) \right]$$

Our Old Friend, The Sinc

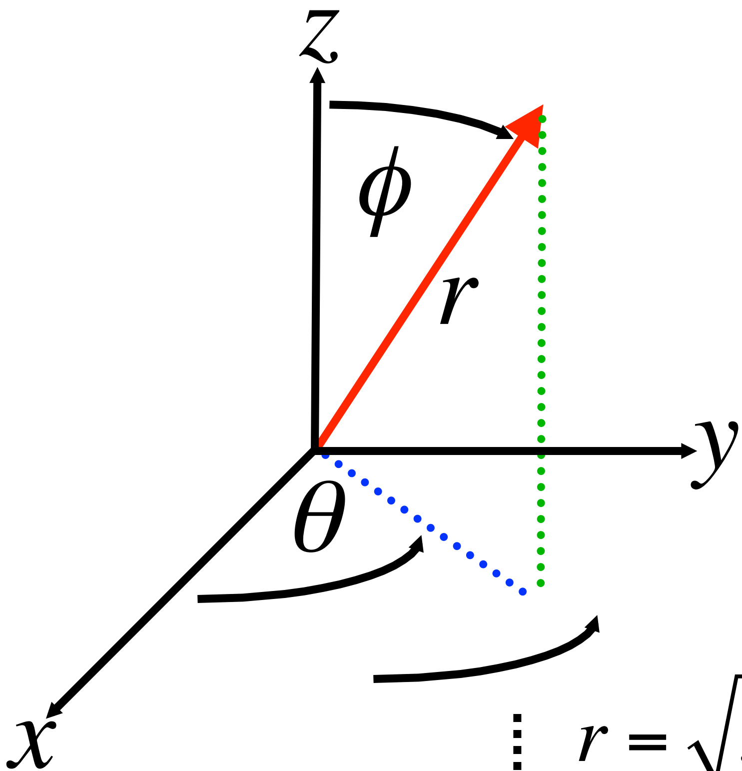
$$W(k_x) = \frac{\sin(k_x D/2)}{k_x/2}$$

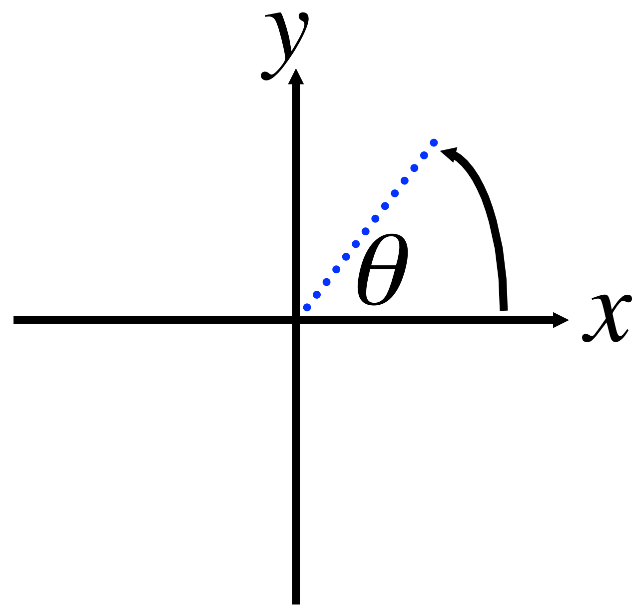
$$W(0) \equiv \lim_{k_x \to 0} \frac{\sin(k_x D/2)}{k_x/2}$$

$$= \frac{\lim_{k_x \to 0} (D/2)\cos(k_x D/2)}{\lim_{k_x \to 0} 1/2}$$

$$= D$$

Spherical Spatial Coordinates

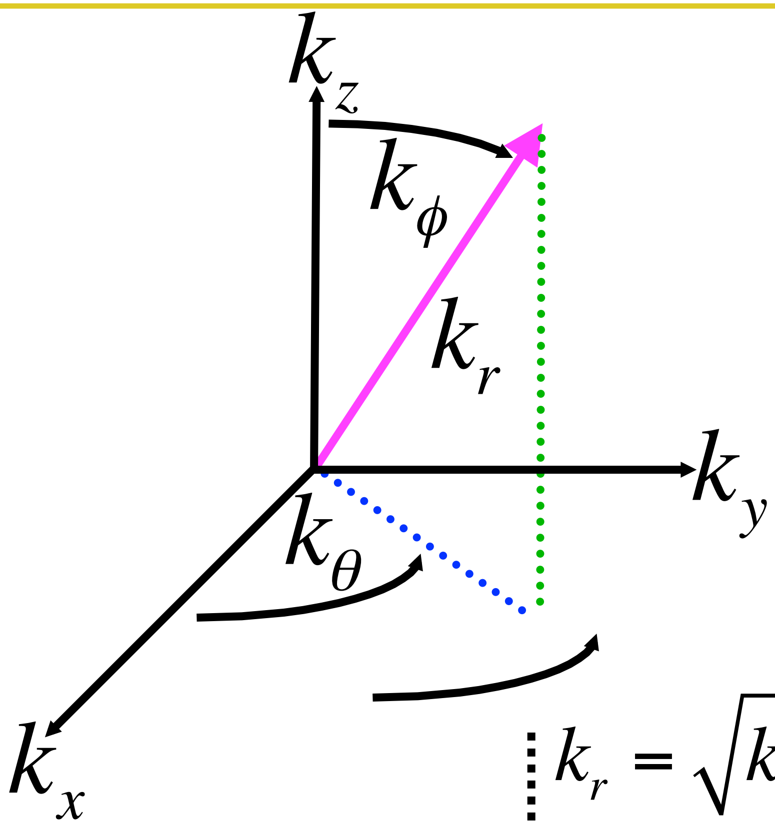


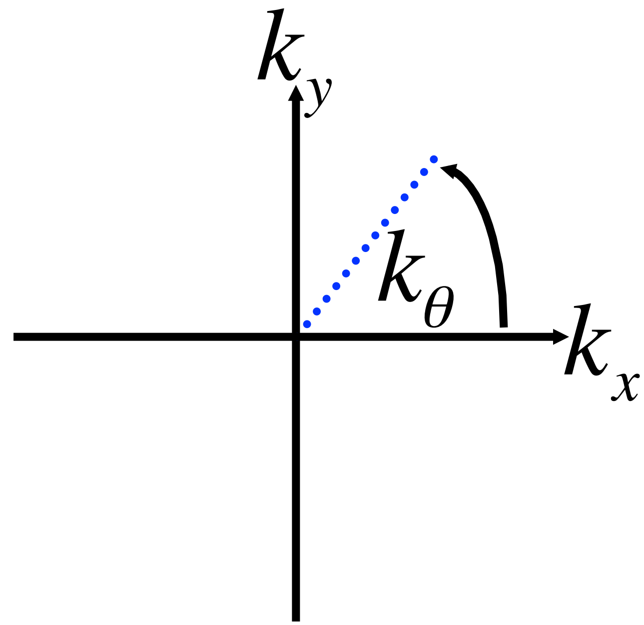


$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) = \sin^{-1}\left(\frac{y}{\sqrt{x^2 + y^2}}\right)$$
$$\phi = \cos^{-1}(z / r)$$

Spherical Wavenumber Coordinates





$$k_x = k_r \sin k_\phi \cos k_\theta$$

$$k_y = k_r \sin k_\phi \sin k_\theta$$

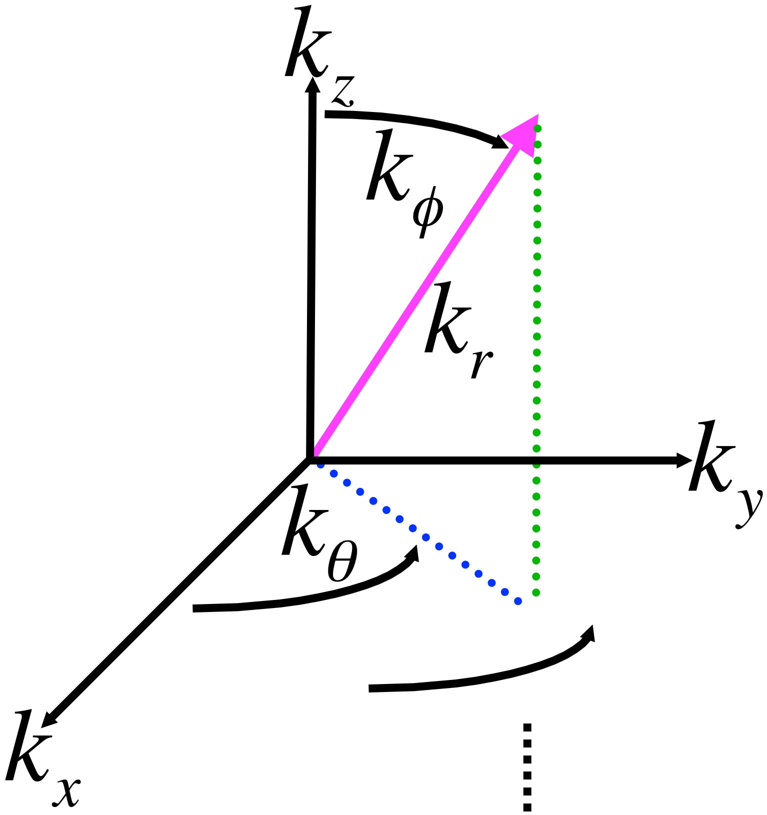
$$k_z = k_r \cos k_\phi$$

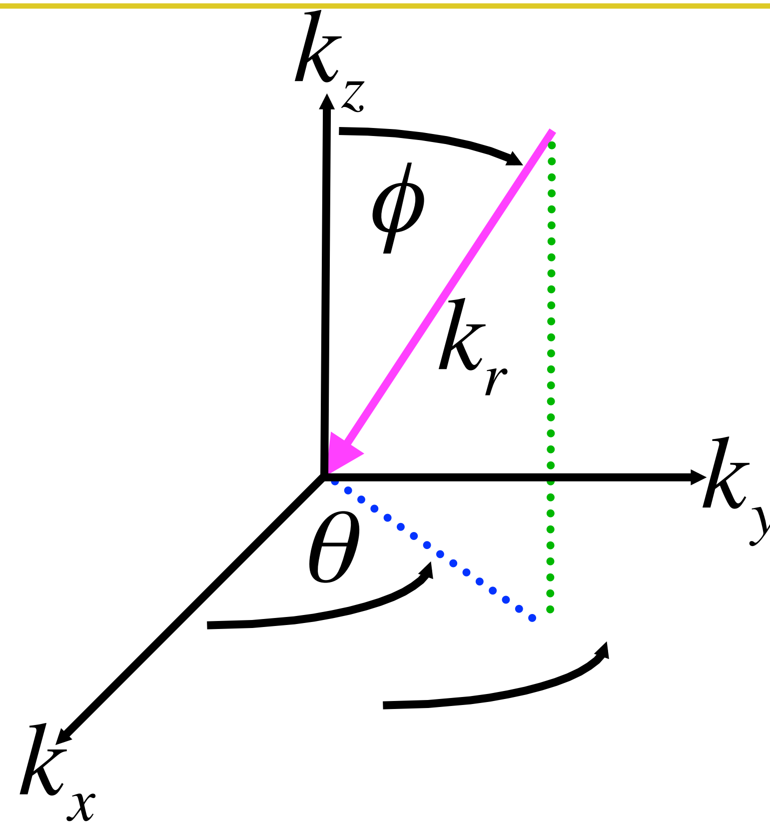
$$k_r = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$k_\theta = \cos^{-1}\left(\frac{k_x}{\sqrt{k_x^2 + k_y^2}}\right) = \sin^{-1}\left(\frac{k_y}{\sqrt{k_x^2 + k_y^2}}\right)$$

$$k_\phi = \cos^{-1}(k_z / k_r)$$

Alternate Wavenumber Coordinates





$$k_x = k_r \sin k_\phi \cos k_\theta$$
$$k_y = k_r \sin k_\phi \sin k_\theta$$
$$k_z = k_r \cos k_\phi$$

$$k_x = -k_r \sin\phi \cos\theta$$
$$k_y = -k_r \sin\phi \sin\theta$$
$$k_z = -k_r \cos\phi$$

Abuse of notation

Using Spherical Coordinates

$$k_x = -k_r \sin\phi\cos\theta = -\frac{2\pi}{\lambda}\sin\phi\cos\theta$$

$$W(k_r,\theta,\phi) = \frac{\sin(k_x D/2)}{k_x/2} = \frac{\sin\left(\frac{2\pi}{\lambda}\frac{D}{2}\sin\phi\cos\theta\right)}{\frac{2\pi}{\lambda}\frac{1}{2}\sin\phi\cos\theta}$$

$$\vec{\zeta}^{\,0} = \frac{\vec{k}}{|\vec{k}|} \qquad \vec{k} = |\vec{k}|\vec{\zeta}^{\,0} = \frac{\omega_0}{c}\vec{\zeta}^{\,0} = \frac{2\pi}{\lambda}\vec{\zeta}^{\,0}$$

$$k \equiv k_r = |\vec{k}| = \frac{\omega_0}{c} = \frac{2\pi}{\lambda}$$

Interpretation of k_x

$$k_x = -\frac{2\pi}{\lambda}\sin\phi\cos\theta$$

- **Note that**

$$-1 \le \sin\phi\cos\theta \le 1$$

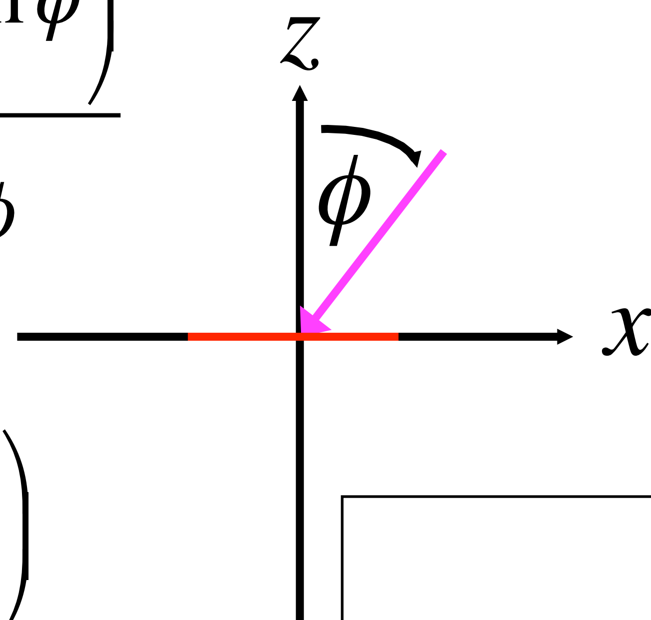
- **Hence, for a fixed λ, the only meaningful k_x lie between**

$$-\frac{2\pi}{\lambda} \le k_x \le \frac{2\pi}{\lambda}$$

Restricting to the Plane

- **Suppose** $\theta = 0$

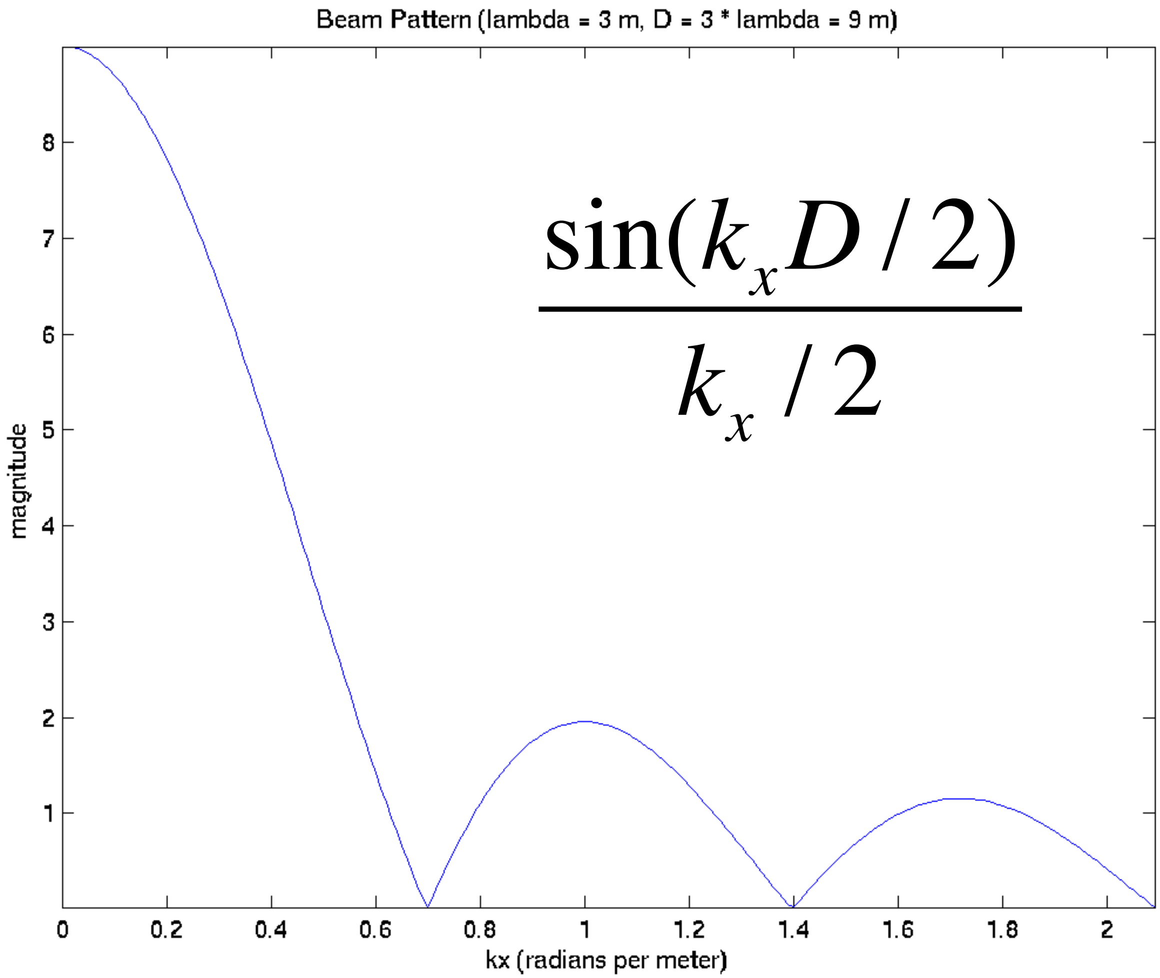
$$W\left(\frac{2\pi}{\lambda},\phi,0\right) = \frac{\sin\left(\frac{2\pi}{\lambda}\frac{D}{2}\sin\phi\right)}{\frac{2\pi}{\lambda}\frac{1}{2}\sin\phi}$$



- **Write as**

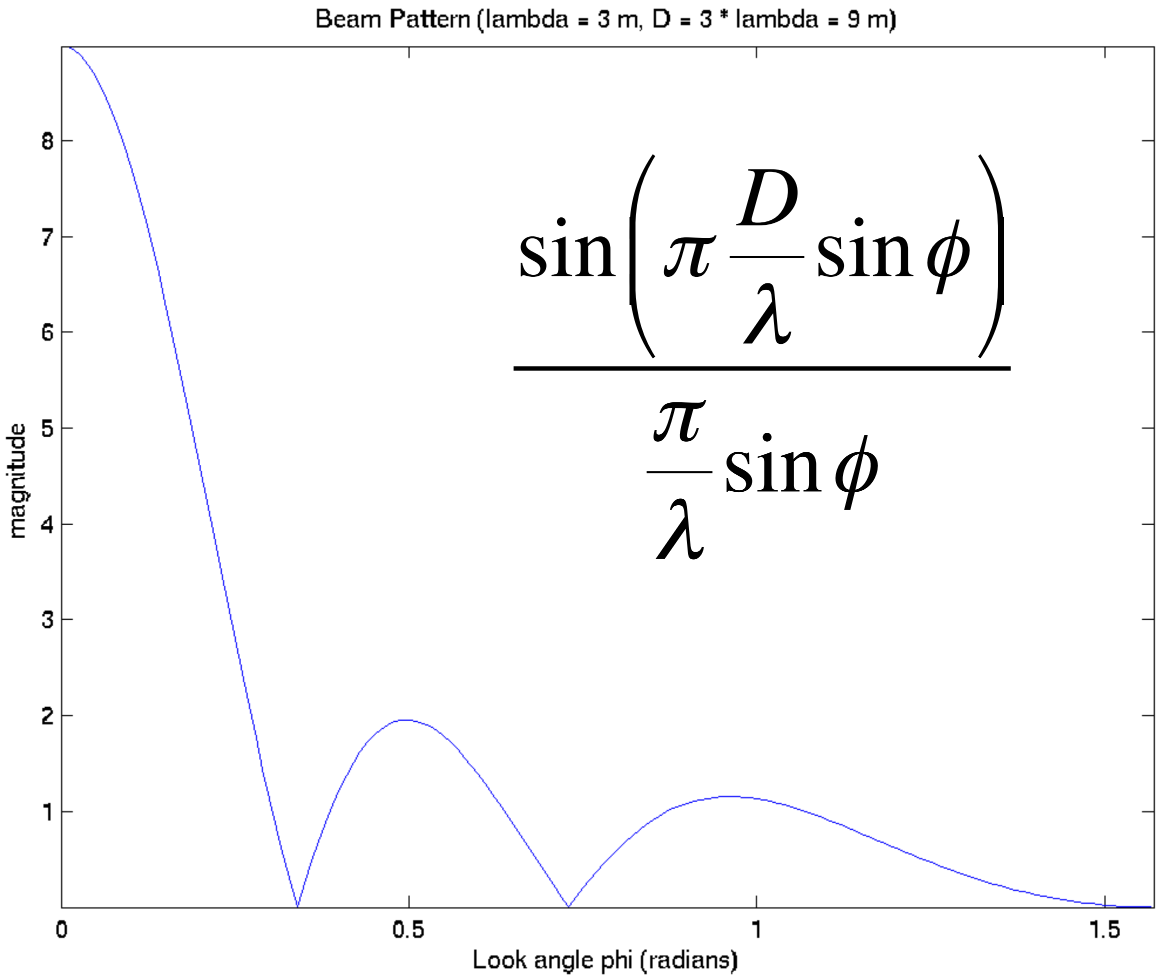
$$W(\phi, D/\lambda) \equiv \frac{\sin\left(\pi\frac{D}{\lambda}\sin\phi\right)}{\frac{\pi}{\lambda}\sin\phi}$$

Small Antenna Ex. - Wavenumber



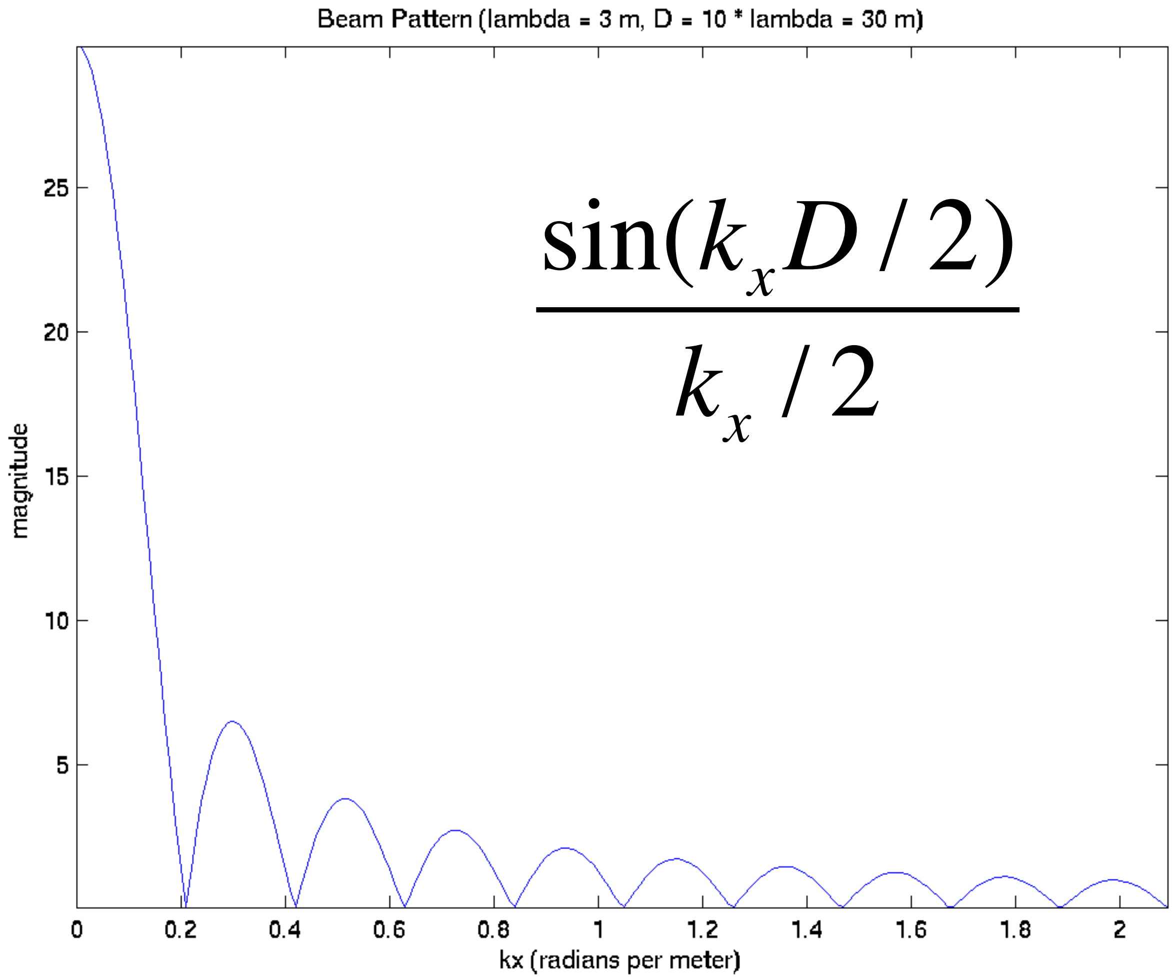
$$\frac{\sin(k_x D/2)}{k_x/2}$$

Small Antenna Ex. - Angle



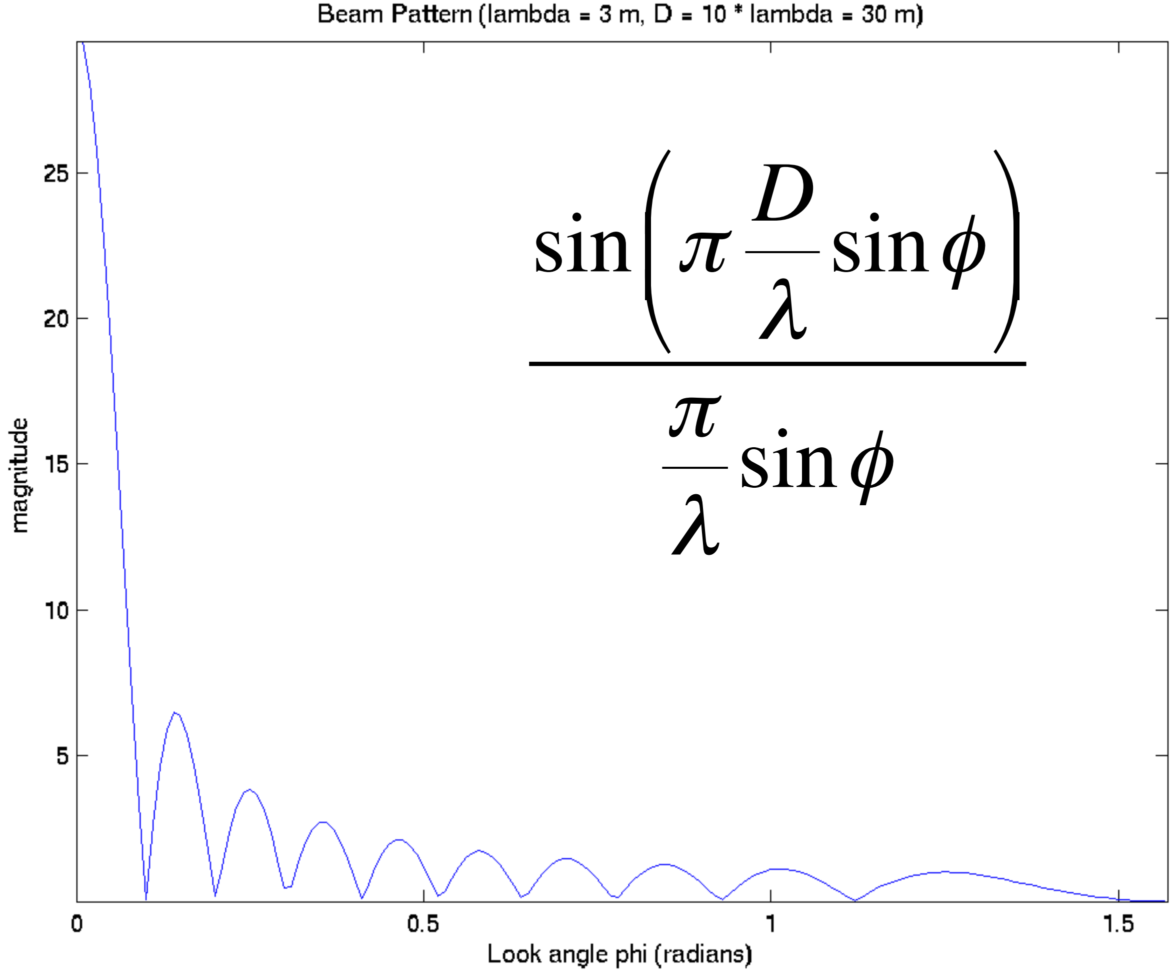
$$\frac{\sin\left(\pi \frac{D}{\lambda} \sin\phi\right)}{\frac{\pi}{\lambda}\sin\phi}$$

Large Antenna Ex. - Wavenumber



$$\frac{\sin(k_x D/2)}{k_x/2}$$

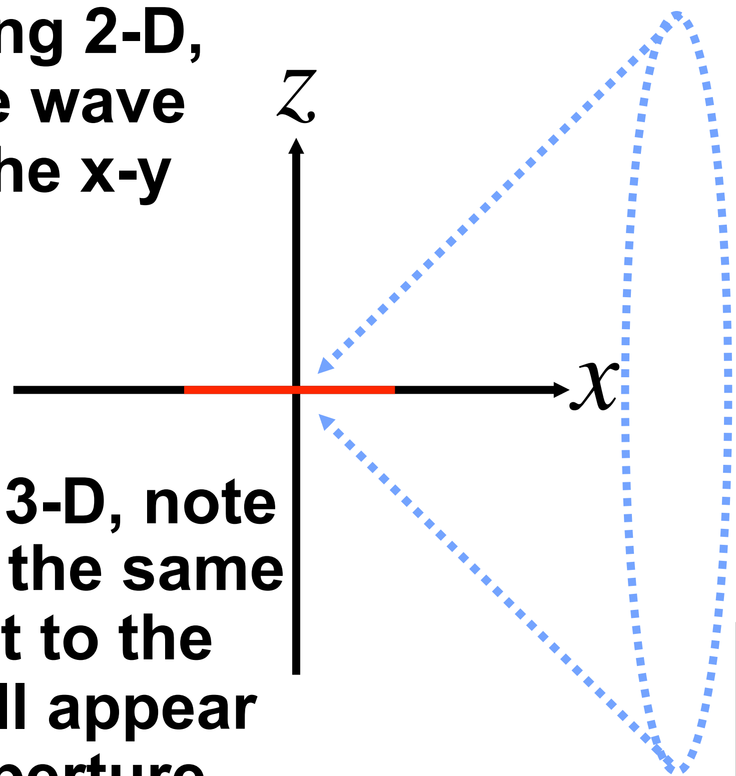
Large Antenna Ex. - Angle



$$\frac{\sin\left(\pi \frac{D}{\lambda} \sin\phi\right)}{\frac{\pi}{\lambda}\sin\phi}$$

Cone of Ambiguity

- **Previous slides thought of everything as being 2-D, and supposed the wave direction was in the x-y plane**



- **When we think in 3-D, note any wave making the same angle with respect to the linear aperture will appear the same to the aperture**