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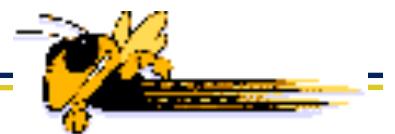
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# ***Introduction to Cramér-Rao Bounds***

**ECE 6279: Spatial Array Processing  
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Lecture 23**

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# Basic Univariate Cramér-Rao Bound

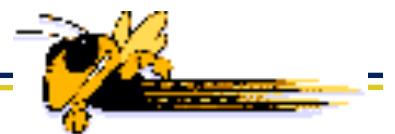
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- For any unbiased estimator  $\hat{\xi}$

$$MSE = \text{var}_{\xi}[\hat{\xi}(\underline{y})] \geq 1 / F(\xi)$$

where the Fisher Information is

$$F(\xi) = E_{\xi} \left\{ \left[ \frac{d}{d\tilde{\xi}} \ln p(\underline{y}; \xi) \right]^2 \Big|_{\tilde{\xi}=\xi} \right\}$$



# Two Ways to Compute the F.I.

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- Under some common conditions,

$$F(\xi) = E_{\xi} \left\{ \left[ \frac{d}{d\tilde{\xi}} \ln p(\underline{y}; \tilde{\xi}) \right]^2 \Big|_{\tilde{\xi}=\xi} \right\}$$
$$= -E_{\xi} \left\{ \frac{d^2}{d\tilde{\xi}^2} \ln p(\underline{y}; \tilde{\xi}) \Big|_{\tilde{\xi}=\xi} \right\}$$



# Gaussian Example, 1<sup>st</sup> Derivative (1)

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- Consider F.I. for one data point for

$$\underline{y} \sim N(f(\xi), \sigma^2)$$

$$\frac{d}{d\xi} \ln p(y; \xi) =$$

$$\frac{d}{d\xi} \ln \left( \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right)$$



## Gaussian Example, 1<sup>st</sup> Derivative (2)

$$\begin{aligned} & \frac{d}{d\xi} \ln \left( \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \right) \\ &= \frac{d}{d\xi} \left\{ -\frac{[y - f(\xi)]^2}{2\sigma^2} \right\} \\ &= \frac{[y - f(\xi)]}{\sigma^2} \frac{df(\xi)}{d\xi} \end{aligned}$$



# Gaussian Example, 2<sup>st</sup> Derivative (1)

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$$\frac{d^2}{d\xi^2} \ln p(y; \xi)$$

$$= \frac{d}{d\xi} \left\{ \frac{[y - f(\xi)]}{\sigma^2} \frac{df(\xi)}{d\xi} \right\}$$

$$= \frac{1}{\sigma^2} \left\{ [y - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[ \frac{df(\xi)}{d\xi} \right]^2 \right\}$$



# Using Derivative-Squared Version (1)

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$$\begin{aligned} F(\xi) &= E_{\xi} \left\{ \left[ \frac{\partial}{\partial \xi} \ln p(\underline{y}; \xi) \right]^2 \right\} \\ &= E_{\xi} \left\{ \left[ \frac{[\underline{y} - f(\xi)]}{\sigma^2} \frac{df(\xi)}{d\xi} \right]^2 \right\} \end{aligned}$$



# Using Derivative-Squared Version (2)

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$$F(\xi) = \left[ \frac{df(\xi)}{d\xi} \right]^2 \frac{E_{\xi}\{[\underline{y} - f(\xi)]^2\}}{(\sigma^2)^2}$$
$$= \left[ \frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$$



# Using Double-Derivative Version (1)

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$$F(\xi) = -E_{\xi} \left\{ \frac{\partial^2}{\partial \xi^2} \ln p(\underline{y}; \xi) \right\}$$

$$= -\frac{1}{\sigma^2} E_{\xi} \left\{ [\underline{y} - f(\xi)] \frac{d^2 f(\xi)}{d \xi^2} - \left[ \frac{df(\xi)}{d \xi} \right]^2 \right\}$$



# Using Double-Derivative Version (2)

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$$\begin{aligned} & -\frac{1}{\sigma^2} \left\{ E_{\xi} [y - f(\xi)] \frac{d^2 f(\xi)}{d\xi^2} - \left[ \frac{df(\xi)}{d\xi} \right]^2 \right\} \\ & = \left[ \frac{df(\xi)}{d\xi} \right]^2 / \sigma^2 \end{aligned}$$



# Independent, Identically Dist. Data

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- If you have  $L$  i.i.d. data points, the F.I. is just  $L$  times the F.I. for one data point:

$$F(\xi) = LF_1(\xi)$$

- For our ex.,  $F(\xi) = L \left[ \frac{df(\xi)}{d\xi} \right]^2 / \sigma^2$

$$\text{var}_{\xi}[\hat{\xi}(\underline{y})] \geq \sigma^2 \left\{ L \left[ \frac{df(\xi)}{d\xi} \right]^2 \right\}$$



# Basic Multivariate Cramér-Rao Bound

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- For any unbiased estimator  $\hat{\xi}$

$$\text{cov}_{\xi}[\hat{\xi}(\underline{y})] \geq F^{-1}(\hat{\xi})$$

( $A \geq B$  means  $A - B$  is nonneg. def.)

where the entries of  
of the F.I. matrix are

$$F_{rc} = E_{\xi} \left\{ \left[ \frac{\partial}{\partial \xi_r} \ln p(\underline{y}; \xi) \right] \left[ \frac{\partial}{\partial \xi_c} \ln p(\underline{y}; \xi) \right] \right\}$$



# Nonnegative Definiteness

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- If  $A$  is nonnegative definite, then
  - $\mathbf{x}^T A \mathbf{x} \geq 0$  for any real vector  $\mathbf{x}$
  - Eigenvalues of  $A$  are nonnegative
- Useful consequences: if  $A \geq B$ ,
  - Diagonals dominated:  $A_{ii} \geq B_{ii}$
  - Does not mean  $A_{rc} \geq B_{rc}$  in general!
  - Total sum property:

$$\sum_{r,c} A_{rc} \geq \sum_{r,c} B_{rc}$$



# Two Ways to Compute the F.I.M.

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- Under some common conditions:

$$F_{rc} = E_{\xi} \left\{ \left[ \frac{\partial}{\partial \tilde{\xi}_r} \ln p(\underline{y}; \tilde{\xi}) \right] \left[ \frac{\partial}{\partial \tilde{\xi}_c} \ln p(\underline{y}; \tilde{\xi}) \right] \right\}$$

$$= -E_{\xi} \left\{ \frac{\partial^2}{\partial \xi_r \partial \xi_c} \ln p(\underline{y}; \xi) \right\}$$



# Efficiency

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- An unbiased estimator that achieves the CR bound with equality is called **efficient**
- If an efficient estimator exists, the maximum-likelihood estimator is it!
- Even if an estimator isn't efficient, it may be **asymptotically efficient**
- ML estimators are **asymptotically efficient**



# CR Bounds for Biased Estimators

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- Both univariate and multivariate versions for biased estimators exist...
- ...but they require taking derivatives of the bias...
- ...which requires you have an analytic form for the bias...
- ...which you almost never have...
- ...so it's rarely used; people usually just compute the CR bound for the unbiased estimator and then handwave

