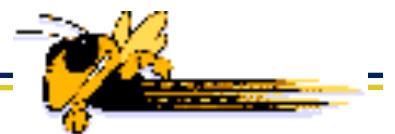

Spatial Averaging and Co-arrays

**ECE 6279: Spatial Array Processing
Fall 2013
Lecture 13**

Prof. Aaron D. Lanterman

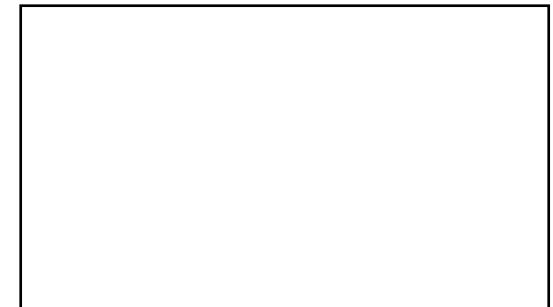
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Where We Are in J&D

- Material from:
 - Sec. 4.9.2 on spatial averaging
 - Sec. 3.3.4 on co-arrays
 - We will use uniform weights
 - Notes by Doug Williams



Why Spatial Averaging?

- Last lecture, we looked at temporal averaging
- Sometimes the source is moving too fast to employ a lot of temporal averaging
- If the SNR is still too low, it helps to have some **redundancy** in the array



Ex: Uniform Linear Array

- For a ULA, with multiple **incoherent sources**, and no noise:

$$\mathbf{R}_y = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{M-1} \\ R_1^* & R_0 & R_1 & \cdots & R_{M-2} \\ R_2^* & R_1^* & R_0 & \cdots & R_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{M-1}^* & R_{M-2}^* & R_{M-3}^* & \cdots & R_0 \end{bmatrix}$$

← Toeplitz structure



Forcing a Toeplitz Structure (1)

$$\hat{\mathbf{R}}_y = \frac{1}{L} \sum_{l=0}^{L-1} \mathbf{y}(l) \mathbf{y}^H(l)$$

$$\hat{\mathbf{R}}_y = \begin{bmatrix} \hat{R}_{0,0} & \hat{R}_{0,1} & \hat{R}_{0,2} & \cdots & \hat{R}_{0,M-1} \\ \hat{R}_{0,1}^* & \hat{R}_{1,1} & \hat{R}_{1,2} & \cdots & \hat{R}_{1,M-1} \\ \hat{R}_{0,2}^* & \hat{R}_{1,2}^* & \hat{R}_{2,2} & \cdots & \hat{R}_{2,M-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{R}_{0,M-1}^* & \hat{R}_{1,M-1}^* & \hat{R}_{2,M-1}^* & \cdots & \hat{R}_{M-1,M-1} \end{bmatrix} \rightarrow \begin{array}{c} \hat{\mathbf{R}}_{M-1} \\ \hat{\mathbf{R}}_1 \\ \hat{\mathbf{R}}_0 \end{array}$$

- One idea: Average along diagonals
- Elements near main diagonal get larger variability reduction



Forcing a Toeplitz Structure (2)

- Now we're guaranteed a Toeplitz structure:

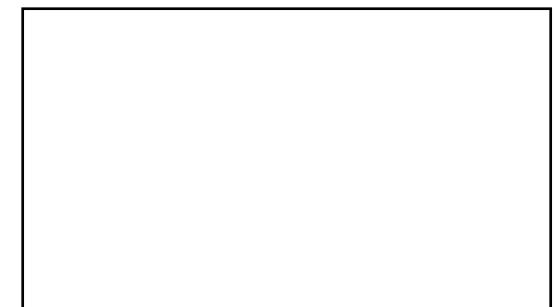
$$\hat{\mathbf{R}}_y = \begin{bmatrix} \hat{\hat{R}}_0 & \hat{\hat{R}}_1 & \hat{\hat{R}}_2 & \cdots & \hat{\hat{R}}_{M-1} \\ \hat{\hat{R}}_1^* & \hat{\hat{R}}_0 & \hat{\hat{R}}_1 & \cdots & \hat{\hat{R}}_{M-2} \\ \hat{\hat{R}}_2^* & \hat{\hat{R}}_1^* & \hat{\hat{R}}_0 & \cdots & \hat{\hat{R}}_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\hat{R}}_{M-1}^* & \hat{\hat{R}}_{M-2}^* & \hat{\hat{R}}_{M-3}^* & \cdots & \hat{\hat{R}}_0 \end{bmatrix}$$

- ...but we're not guaranteed a nonnegative definite one!
 - i.e., at least one eigenvalue may become negative



What Can Go Wrong

- Resulting matrix not guaranteed to be nonnegative definite!
- Also, procedure can introduce some bias into angle estimates
 - Usually only a major problem if coherent signals are present
- One solution: maximum-likelihood structured covariance estimation
 - Expectation-Maximization (EM) algorithm (computationally intensive)
 - M.I. Miller and D.L. Snyder, “The Role of Likelihood and Entropy in Incomplete Data Problems: Application to Estimating Point Process Intensity and Toeplitz Constrained Covariances,” Proc. IEEE, Vol. 75, No. 7, pp. 892-907, July 1987.

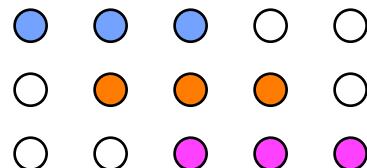


Subaperture Concept

- Suppose we have a 5 element ULA:



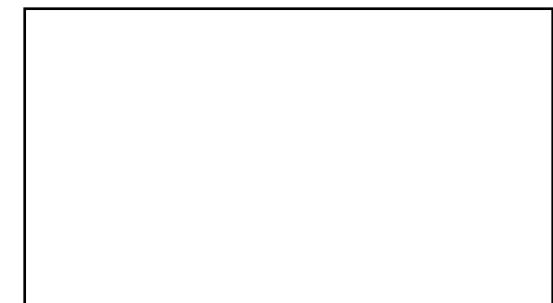
- Form 3 **subarrays**:



$$\hat{\mathbf{R}}_y = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$
$$\frac{1}{3} \sum = \hat{\mathbf{R}}_{sub} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

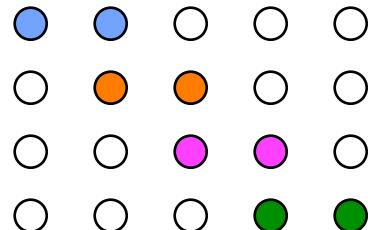
Subaperture Averaging: Pros and Cons

- **Advantages**
 - All elements of estimated spatial correlation matrix (SCM) get the same improvement
 - Resulting SCM estimate is guaranteed to be nonnegative definite
 - Helps reduce problems with coherent sources
 - **Sinusoidal components along diagonals get smoothed out**
- **Disadvantage**
 - Individual subapertures have lower resolution

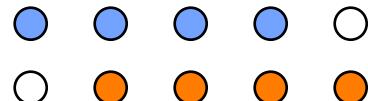


The Tradeoff

- Lower variability in covariance estimate comes at the expense of lower resolution
- 4 small subarrays (bad resolution, low/good variability):



- 2 big subarrays (good resolution, high/bad variability)



Forward/Backward Averaging

- From Prob. 7.11 on p. 418 of J&D
- Consider the data for a **uniform linear array**:

$$\mathbf{y} = [y_0 \quad y_1 \quad \cdots \quad y_{M-2} \quad y_{M-1}]^T$$

- Define:

$$\mathbf{y}_r^* = [y_{M-1}^* \quad y_{M-2}^* \quad \cdots \quad y_1^* \quad y_0^*]^T$$

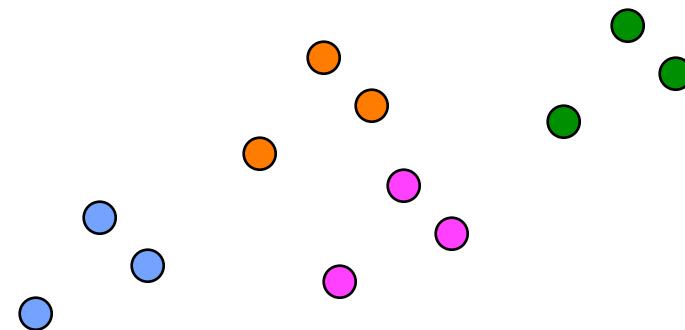
- Describes same signal and noise situation, but coherence effects differ

$$\hat{\mathbf{R}}_{fb} = \frac{1}{2L} \sum_{l=0}^{L-1} \left\{ \mathbf{y}(l) \mathbf{y}^H(l) + \mathbf{y}_r^*(l) [\mathbf{y}_r^*(l)]^H \right\}$$

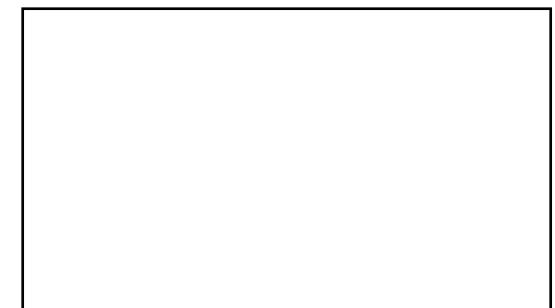
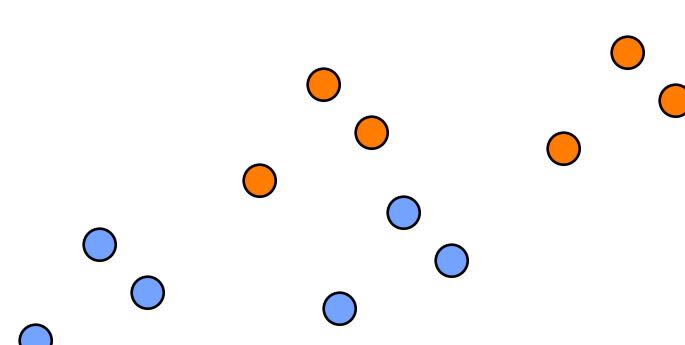


General Subarrays

- 12 element array
- 4, 3-element subarrays



- 2, 6-element subarrays



Interpreting the SCM

- SCM can be viewed as samples of the **correlation function** for the entire field
- Samples are taken at **differences** between sensor locations
- For a single plane wave:

$$[\mathbf{R}]_{m_1, m_2} = P_s \exp\{jk^0 \cdot (\vec{x}_{m_1} - \vec{x}_{m_2})\}$$

- For a general WSS field :

$$[\mathbf{R}]_{m_1, m_2} = R_f(\vec{x}_{m_1} - \vec{x}_{m_2})$$

where $R_f(\chi) = E[f(0)f(\chi)]$



The Coarray

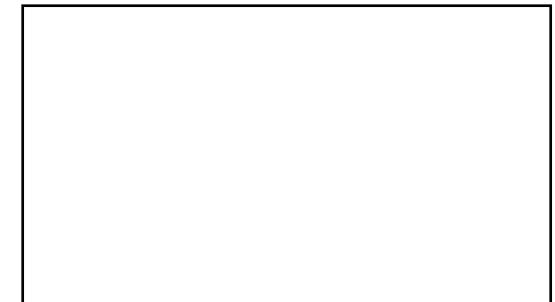
- Set of differences between all pairs of sensor locations is called the **coarray**

$$\bigcup_{m_1, m_2} \{ \vec{x}_{m_1} - \vec{x}_{m_2} \}$$

- Differences are known as **lags**

$$\vec{z}_{m_1, m_2} \equiv \vec{x}_{m_1} - \vec{x}_{m_2}$$

- Elements of the coarray have associated **coarray values**, which is the number of distinct baselines (pairs of actual sensors) with the same vector difference

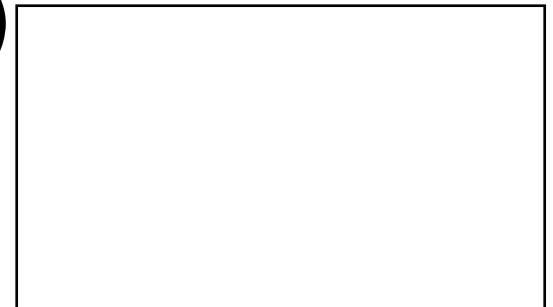


Properties of Co-arrays

- Dimension of coarray is same as dimension of array
 - Linear <-> Linear
 - Planar <-> Planar
- Certain lags must exist

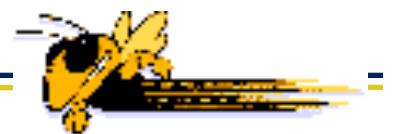
$$\vec{z}_{m,m} = 0 \text{ (coarray value of } M\text{)}$$

$$\vec{z}_{m_1,m_2} = -\vec{z}_{m_2,m_1}$$



Redundancies

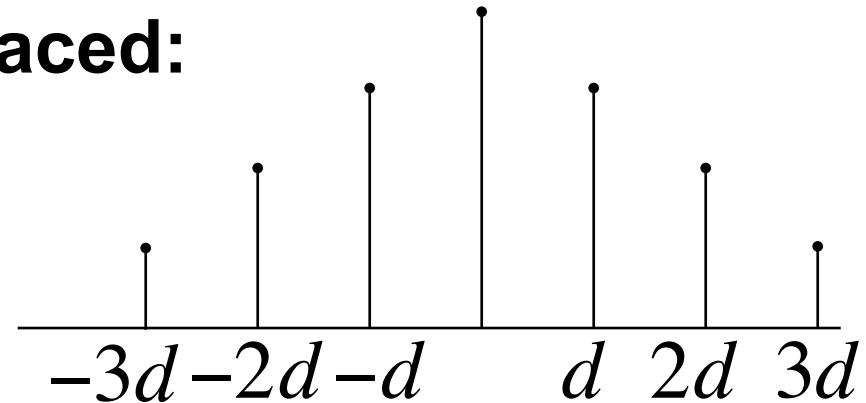
- Repeated lags (i.e. co-array values > 1) are called **redundancies**
- Zero-lag redundancies occur along main diagonal of SCM
 - Ideally equal if only incoherent signals are present
 - Usually not equal due to noise or coherent signals
- Minimum no. of distinct positive lags: M
- Maximum no. of distinct positive lags: $M(M-1)/2$ (not achievable if $M>4$)



Linear Co-array Examples

- 4-sensor uniformly spaced:

• d • d • d •



- 4-sensor “perfect” array:

• d • $3d$ • $2d$ •

