E9 231: Digital Array Signal Processing

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1 Announcements

1. Class timings: **Monday & Thusday, 4 - 5:30pm**.
2. Final exam: **Nov. 26, 2008**, 4-7pm.

2 Topics

- Window Based LPF Design
- Frequency Sampling Based Designs
- Null Steering

3 Class Notes

Note:
An array of N omnidirectional antennas with weights such that,

$$w_n = \begin{cases} 1 & n = (N-1)/2 \\ 0 & n \neq (N-1)/2 \end{cases} \tag{1}$$

is equivalent to a single antenna with omnidirectional beam and thus sidelobes are of $-\infty$dB. So the array has unit directivity and normalised directivity of $\frac{1}{N}$.

3.1 Array Polynomials And Z Transform

Zeros on the unit circle give sharp nulls which can be used when precise information about the angle of arrival(AoA) of jammer is known. But most often we do not have accurate AoA of the interferer. In such a case the difference beam can be used to find the DoA/AoA. The array with 2^{nd} or 3^{rd} order zeros can be used which will widen the nulls. But this limits the total number of nulls possible for an array of given length.

3.2 Window Based Designs

Window based designing is used to get the weights of an array of finite length whose response is as close as possible to the desired response.
Suppose the desired response is

$$B_d(\Psi) = \begin{cases} 1 & |\Psi| \leq \Psi_c \\ 0 & |\Psi| > \Psi_c \end{cases} \tag{2}$$

Therefore,

$$\begin{aligned} B_d(\Psi) &= \sum_{\tilde{n}=-\infty}^{\infty} w_{d\tilde{n}}^{*} e^{j\tilde{n}\Psi} \\ B(\Psi) &= \sum_{\tilde{n}=-\frac{N-1}{2}}^{\frac{N-1}{2}} w_n^{*} e^{j\tilde{n}\Psi} \qquad n = \tilde{n} + \frac{N-1}{2} \\ w_{d\tilde{n}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} B_d^{*}(\Psi) e^{j\tilde{n}\Psi} d\Psi \qquad -\infty < \tilde{n} < \infty \end{aligned} \tag{3}$$

Let $w_{\tilde{n}}$ be the weights of the antenna array of finite length and $w_{w\tilde{n}}$ be the window function.

$$w_{\tilde{n}} = w_{d\tilde{n}} w_{w\tilde{n}} \tag{4}$$

$$w_{w\tilde{n}} \begin{cases} \neq 0, & -\frac{N-1}{2} \leq \tilde{n} \leq \frac{N-1}{2} \\ = 0 & else \end{cases} \tag{5}$$

For example rectangular window,

$$w_{w\tilde{n}} = \begin{cases} 1, & -\frac{N-1}{2} \leq \tilde{n} \leq \frac{N-1}{2} \\ 0 & else \end{cases} \tag{6}$$

The rectangular window has MMSE property. Let $B(\Psi)$ be the response closest to $B_d(\Psi)$ in MMSE sense. So, $\frac{1}{2\pi}\int_{-\pi}^{\pi} |B(\Psi) - B_d(\Psi)|^2 d\Psi$ must be minimum.
By Parsevals theorem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |B(\Psi) - B_d(\Psi)|^2 d\Psi = \sum_{\tilde{n}=-\infty}^{\infty} |w_d[\tilde{n}] - w[\tilde{n}]|^2 \tag{7}$$

$$= \sum_{\tilde{n}\in\{-\frac{N-1}{2},\ldots,\frac{N-1}{2}\}} |w_d[\tilde{n}] - w[\tilde{n}]|^2 + \sum_{\tilde{n}\notin\{\frac{-N-1}{2},\ldots,\frac{N-1}{2}\}} |w_d[\tilde{n}]|^2 \tag{8}$$

The term $\sum_{\tilde{n}\in\{-\frac{N-1}{2},\ldots,\frac{N-1}{2}\}} |w_d[\tilde{n}] - w[\tilde{n}]|^2 = 0$ when $w[\tilde{n}] = w_d[\tilde{n}]$
It can be seen that a rectangular window would result in MMSE for the given desired response. But it is not optimum in terms of passband(main lobe) and stop band (side lobe) ripples. So alternate windows like Kaiser is a good choice where a parameter β can be used for adjusting the sidelobe levels.
Designing beamformers using windows is easy because window based designs are equivalent to equiripple FIR filter designs $i.e$ design for $\min(\delta_p, \delta_s)$.This results in overdesigning the array.
To overcome this drawback:

- Control δ_p/δ_s independently
- equally distribute the error in pass and stop band

3.3 MINIMAX Design Using Weighted Error Function

$$e(\Psi) \triangleq w_{pm}(\Psi)[B_d(\Psi) - B(\Psi)]$$

F_p : passband $0 \leq \Psi \leq \Psi_p$
F_s : stopband $\Psi_S \leq \Psi \leq \pi$

$$w_{pm}(\Psi) = \begin{cases} \delta_p/\delta_s & \Psi \in F_p \\ 1 & \Psi \in F_s \end{cases} \tag{9}$$

where subscript pm stand for Parks McClellan,
Now the minimax design problem is

$$\min_{W} \max_{\Psi \in F_p \cup F_s} |w_{pm}(\Psi)[B_d(\Psi) - B(\Psi)]| \tag{10}$$

This optimisation problem was solved by Parks McClellan by using polynomial approximation.

3.3.1 Alternation Theorem

Let F denote a closed set consisting of the union of disjoint closed subsets of the real line.
Let $P(x) = \sum_{k=0}^{L} C_k x^k, D_p(x)$and $w_{pm}(x) \geqslant 0$ be a continuous funtion on F

$$\begin{aligned} e(x) &= w_{pm}(x)[D_p(x) - P(x)] \\ \|E\| &= \max_{x \in F} |e(x)| \end{aligned} \tag{11}$$

Then a necessary and sufficient condition for $P(x)$ to be the unique polynomial that minimises $\|E\|$ is that $e(x)$ exhibits atleast $(L + 2)$ alternations i:e $\exists$ atleast $(L + 2)$ values $x_1 < x_2 < ... < x_{L+2}$ such that
$\Rightarrow e(x_i) = -e(x_{i+1}) = \pm\|E\|$ for $1 \leq i \leq L + 1$

3.3.2 Parks-McClellan-Rabiner Algorithm

This algorithm gives a beam pattern with maximum ripple of δ in passband and kδ in stopband. Check: Remez Algorithm in Matlab with required inputs Ψ_p,Ψ_s, k and N and outputs obtained will be δ and w.

3.4 Frequency Sampling Based Designs (Woodward 1946)

The idea is to sample the beam response such that we can express the samples as the fourier transform of weights.
$B_d(\Psi)$ is sampled at $\Psi_l = \dfrac{2\pi}{N} l$

$$\begin{aligned} B(\Psi_l) &= e^{-j\frac{(N-1)}{2}\Psi_l} \textstyle\sum_{n=0}^{N-1} w_n^* e^{jn\Psi_l} \\ B(l) &= B^*(\Psi_l) e^{-j\frac{(N-1)}{2}\Psi_l} \\ b_n &= w_n \\ \therefore B(l) &= \textstyle\sum_{n=0}^{N-1} b_n e^{-jn\frac{2\pi}{N}l} \end{aligned} \tag{12}$$

which is DFT.
So, $\mathbf{b}$ is complex conjugate of the weight vector.
Let, $\mathbf{B} = [B(0) \dots B(N-1)]^T$ & $\mathbf{b} = [b_0 \dots b_{N-1}]^T$, then

$$\mathbf{B} = \mathbf{F}\mathbf{b}$$

where $\mathbf{F}$ is the DFT matrix.

$$F_{kl} = e^{-j\frac{2\pi}{N}Kl}$$

We can find $\mathbf{b}$ via the IDFT as

$$\begin{aligned} b_n &= \tfrac{1}{N} \textstyle\sum_{l=0}^{N-1} B_l e^{\frac{j2\pi kl}{N}} \\ \mathbf{b} &= \mathbf{F}^{-1}\mathbf{B} \\ &= \tfrac{1}{N}\mathbf{F}^H\mathbf{B} \end{aligned} \tag{13}$$

Thus the weights can be directly obtained from the samples of beam response using IDFT matrix.

3.5 Beamformer design with Null Constraints

Suppose in addition to a desired response we have null constraints also, i.e
Distortionless constraint: $B(\Psi_T) = 1$,
Null constraints: $\Psi_1 ... \Psi_{M0}$ such that $B(\Psi_1) = ... = B(\Psi_{M0}) = 0, \quad M0 < N$.
Therefore,

$$\mathbf{W}^H \mathbf{V}(\Psi_1) = ... = \mathbf{W}^H \mathbf{V}(\Psi_{M0}) = 0$$

If $\mathbf{C_0} = [\mathbf{V}(\Psi_1) ... \mathbf{V}(\Psi_{M0})]$ then

$$\mathbf{W}^H \mathbf{C_0} = 0$$

We might also have $\dfrac{dB(\Psi)}{d\Psi} = 0$ at $\Psi = \mathit{\Psi}_1^{'} .... \mathit{\Psi}_{M1}^{'}$. This forces nulls to be wider and thus the mainlobe to be of adequate width. Therefore,

$$\tfrac{dB(\Psi)}{d\Psi} = \mathbf{W}^H \tfrac{d\mathbf{V}(\mathit{\Psi})}{d\mathit{\Psi}}$$

Let $\mathbf{d}(\mathit{\Psi}) = \frac{d\mathbf{V}(\mathit{\Psi})}{d\mathit{\Psi}}$.

$$\therefore \mathbf{W}^H [\mathbf{d}(\mathit{\Psi}_1^{'}) .... \mathbf{d}(\mathit{\Psi}_{M1}^{'})] = 0$$

If $\mathbf{C_1} = [\mathbf{d}(\mathit{\Psi}_1^{'}) .... \mathbf{d}(\mathit{\Psi}_{M1}^{'})]$ then,

$$\begin{aligned} \mathbf{W}^H \mathbf{C_1} &= 0 \quad M1 + M0 < N \\ \mathbf{W}^H [\mathbf{C}_0 | \mathbf{C}_1] &= 0, \\ \mathbf{W}^H \mathbf{C} &= 0 \end{aligned} \tag{14}$$

Now the optimisation problem changes to

$$\min_w \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |B(\mathit{\Psi}) - B_d(\mathit{\Psi})|^2 d\mathit{\Psi} \right) \text{ sub to } \mathbf{W}^H \mathbf{C} = 0. \text{The solution for this optimisation problem is} \tag{15}$$

$$\mathbf{W_0} = [\mathbf{I} - \mathbf{C}(\mathbf{C}^H \mathbf{C})^{-1} \mathbf{C}^H] \mathbf{W}_d$$

Assumption:$(\mathbf{C}^H \mathbf{C})$ is invertible.
Thus, $\mathbf{W_0}$ are the optimum weights of the array of a given length to get the desired response with required null constraints.