E9 231: Digital Array Signal Processing

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1 Topics

- Waveform Estimation (Ch 6)

2 Announcements

- HW: 6.2.4 & 6.3.3.

3 Class Notes

3.1 Optimum Beamformers

Recall:

$$X_{\Delta T}(\omega_m, k) = \begin{bmatrix} X_1(\omega_m, k) \\ X_2(\omega_m, k) \\ \vdots \\ X_N(\omega_m, k) \end{bmatrix} \tag{1}$$

$\omega_m = m^{th}$ frequency $\frac{-M-1}{2} \leq \frac{M-1}{2}$

k = snapshot index $1 \leq k \leq K$

ΔT = snapshot duration

N = number of sensors

$$X_{\Delta T}(\omega_m, k) = F_s(\omega_m, k) * V(\omega_m, \underline{k_s}) + \sum_{l=1}^{D-1} F_l(\omega_m, k) * V(\omega_m, \underline{k_s}) + N(\omega_m, k) \tag{2}$$

Interesting Questions:

1. Waveform estimation i.e estimete $F_s(\omega_m, k)$,
2. Detecting the presence and absence of signal,
3. Spectral estimation i.e estimate the spatial spectrum of x(t), the space-time process impinging on array,

4. Direction of arrival estimation,

<u>Methods</u>

1. MLE,
2. MMSE,
3. Weighted least squares,
4. AdHoc/Heuristic estimation,

<u>Assumptions</u>

1. Known direction of arrival
2. Known statistics (noise or interference)
3. known signal or unknown but random signal and random signal

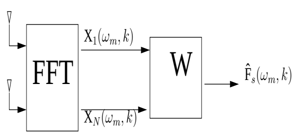
<u>Waveform Estimation</u>

- Estimate $F_s(\omega_m, k)$
- Optimum Beamforming : depends on criterion

3.2 Minimum Variance Distortionless Response (MVDR)

- Optimum under several different criteria
- Criterion independent optimum beamforming

$$\begin{aligned}
X(\omega_m, k) &= F_s(\omega_m, k)V(\omega_m, \underline{k_s}) + N_1(\omega_m, k) \\
N_1(\omega_m, k) &= \textstyle\sum_{l=1}^{D-1} F_l(\omega_m, k)V(\omega_m, \underline{k_s}) + N_A(\omega_m, k) \\
S_n(\omega) &= E(N_1(\omega_m, k)N_1^H(\omega_m, k)) \\
N_1 &= [V_1 \ldots V_{D-1}][F_1 \ldots F_{D-1}] + N_A \\
S_n(\omega_m) &= \mathbf{E}((VF + N_A)(VF + N_A)^H) \\
&= V\mathbf{E}(FF^H)V_H + S_{N_A}
\end{aligned} \tag{3}$$



Assume $\underline{k_s}$ is known.

1. Distortionless

$$W^H V(\omega_m, k_s) = 1 \tag{4}$$

2. Minimum Variance

$$W^H X(\omega_m, k) = F_s(k) + w^H N_1(\omega_m, k) \text{Let} y_n(k) = W^H N_1(\omega_m, k) \tag{5}$$

Since $\mathrm{F_s}$ is nonrandom, minimisation of variance implies

$$\min_W \mathbf{E}(|y_n|^2) \tag{6}$$

$$\mathbf{E}(|y_n|^2) = W^H S_n W \tag{7}$$

So the optimisation problem is

$$\min_W W^H S_n W \text{ st } W^H V(\omega_m, k_s) = 1 \tag{8}$$

Using Legrange's multiplier we get,

$$\begin{aligned} W_{MVDR} &= \bigwedge(\omega_m) S_n^{-1}(\omega_m) V(\omega_m, \underline{k_s}) \\ \bigwedge(\omega_m) &\triangleq \frac{1}{V^H(\omega_m, \underline{k_s}) S_n^{-1}(\omega_m, \underline{k_s})} \end{aligned} \tag{9}$$

Note:

$$\begin{aligned} \text{If } Sn(\omega_m) &= \sigma_n^2 I \\ W_{MVDR} &= \frac{V(\omega_m, \underline{k_s})}{N} \end{aligned} \tag{10}$$

3.2.1 Here is an example why the MVDR BF is criterion independent

Maximum Likelihood Estimation

Assume $N_1(\omega_m, k)$ is circularly guassian.

$$X(\omega_m, k) = F_s(\omega_m, k) V(\omega_m, \underline{k_s}) + N_1(\omega_m, k) \tag{11}$$

The log likelihood function at ω_m,

$$l(\omega_m) = (X - F_s V)^H S_n^{-1} (X - F_s V) \tag{12}$$

$$\hat{F}_{ML} \triangleq \min_{F_s} l(\omega_m) \tag{13}$$

Differentiate $l(\omega_m)$ wrt F_s

$$\hat{F}_{ML} = \frac{S_n^{-1}(\omega_m) X(\omega_m, k)}{V^H(\omega_m, \underline{k_s}) S_n^{-1}(\omega_m, \underline{k_s})} \tag{14}$$

This is exactly the same as the MVDR BF.

Note:

- In the MLE of waveform we does not did not assume linear processing. If signal and noise are jointly guassian the MLE is linear.
- Since the snapshots at different freuencies and time instances are uncorrelated, independent and guassian the total loglikelihood function is sum of loglikelihood functions. So MVDR is optimum across frequency and time.
- Assumed $\underline{k_s}$ and S_n is known.

Unbiased Estimation:

$$\mathbf{E}(\omega^H X(\omega_m, k)) = F_s(\omega_m, k) \text{ for unbiasedness} \tag{15}$$

$$\Rightarrow \min \mathbf{E}(|W^H X - F_s|^2) = \min W^H S_{\underline{n}} W \tag{16}$$

This leads to MVDR beamformer.

So MVDR BF is the BLUE. (Best linear unbiased estimator)

3.3 Array Gain

$$\begin{aligned} Y_(\omega_m) &= \bigwedge(\omega_m) S_n^{-1}(\omega_m) V(\omega_m, \underline{k_s}) X_(\omega_m) \\ \bigwedge(\omega_m) &\triangleq \frac{1}{V^H(\omega_m, \underline{k_s}) S_n^{-1}(\omega_m, \underline{k_s})} \\ S_{y_n}(\omega) &= W_{MVDR}^H S_n^{-1}(\omega) W_{MVDR} = \Lambda(\omega) \end{aligned} \tag{17}$$

Since W_{MVDR} is distortionless

$$\begin{aligned} S_{y_f}(\omega) &= S_f(\omega) \\ A_o(\omega, \underline{k_s}) &= \frac{\underline{S_n}(\omega)}{\Lambda(\omega)} \end{aligned} \tag{18}$$

$$= S_n(\omega) V^H(\omega_m, \underline{k_s}) S_{\underline{n}}^{-1}(\omega_m) V(\omega_m, \underline{k_s}) \tag{19}$$

If we define $\rho_{\underline{n}}$ by

$$\begin{aligned} S_{\underline{n}}(\omega) &= S_n(\omega) \rho_{\underline{n}} \\ A_o(\omega, k_s) &= V^H(\omega_m, \underline{k_s}) \rho_{\underline{n}}^{-1}(\omega_m) V(\omega_m, \underline{k_s}) \end{aligned} \tag{20}$$

<u>Note</u>
Delay and Sum Beamformer:

$$\begin{aligned} W_c^H &= \frac{1}{N} V^H(\omega_m, \underline{k_s}) \\ S_{y_n}(\omega) &= \frac{1}{N^2} V^H(\omega_m, \underline{k_s}) S_{\underline{n}}^{-1}(\omega_m, \underline{k_s}) V(\omega_m, \underline{k_s}) \\ A_c &= \frac{N^2}{V_s^H \rho_{\underline{n}}^{-1} V_s} \end{aligned} \tag{21}$$

If the noise is white, $A_o = A_c = N$
In general, $A_o > A_c$

3.4 Linear Minimum Mean Square error Estimation

Assume F(ω) and N(ω) zero mean and a variance of S$_f(\omega)$ and S$_{\underline{n}}(\omega)$ respectively.

S$_x(\omega) = \mathbf{E}$(X(ω)X$^H(\omega)$) = S$_f(\omega)$V$_s$V$_s^H$+S$_{\underline{n}}(\omega)$

V$_s \triangleq$ V(ω, k_s)

MMSE estimation

$$\begin{aligned} D(\omega) &= \text{Desired signal} \\ &= F(\omega) \\ \hat{D}(\omega) &= \text{estimate of } D(\omega) = H(\omega)X(\omega) \\ \xi &\triangleq \mathbf{E}\{|D(\omega) - \hat{D}(\omega)|^2\} \\ \therefore \xi &= \mathbf{E}\{(D(\omega) - H(\omega)X(\omega))^H (D(\omega) - H(\omega)X(\omega))\} \end{aligned} \tag{22}$$

MMSE is done by minimising ξ wrt H Differentiate wrt $H^H(\omega)$

$$\begin{aligned}
\mathbf{E}(D(\omega)X^H(\omega)) &= H(\omega)\mathbf{E}(X(\omega)X^H(\omega)) \\
\text{Let } S_{dx^H}(\omega) &\triangleq \mathbf{E}(D(\omega)X^H(\omega)) \\
\text{and } S_x(\omega) &\triangleq \mathbf{E}(X(\omega)X^H(\omega)) \\
H_o(\omega) &= S_{dx^H}(\omega)S_x^{-1}(\omega) \\
S_{dx^H}(\omega) &= S_f(\omega)V_s^H \\
\therefore\ H_o(\omega) &= S_f(\omega)V_s^H(S_f(\omega)V_sV_s^H + S_n(\omega))^{-1}
\end{aligned} \tag{23}$$

Using matrix inversion lemma

$$\begin{aligned}
S_x^{-1} &= S_n^{-1} - S_n^{-1}S_fV_s(1 + V_s^HS_n^{-1}S_fV_s)^{-1}V_s^HS_n^{-1} \\
H_o(\omega) &= \frac{S_f}{(S_f + \Lambda)}.\Lambda V_s^HS_n^{-1}
\end{aligned} \tag{24}$$

LMMSE consists of the optimum distortionless beamforming followed by scaling.