Conventional Narrowband Beamforming

ECE 6279: Spatial Array Processing
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Lecture 9

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Where We Are in J&D, Sort Of

- **Most of the lecture material drawn from:**
  - Sec. 4.4, sort of…
  - Book covers full wideband case
  - We'll just cover the narrowband case this semester

- **Some of the lecture is based on what is usually seen in the literature**

Inspiration

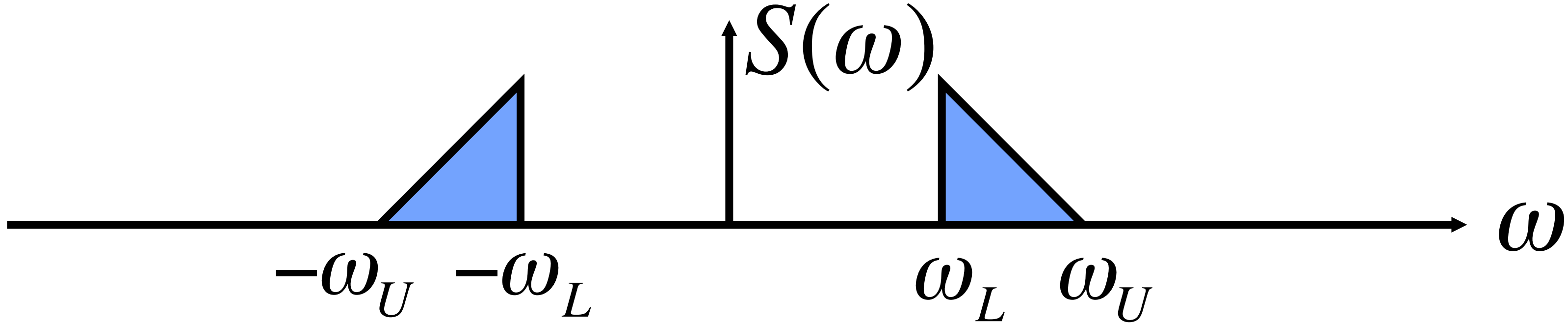
- **Recall basic delay-and-sum beamformer:**

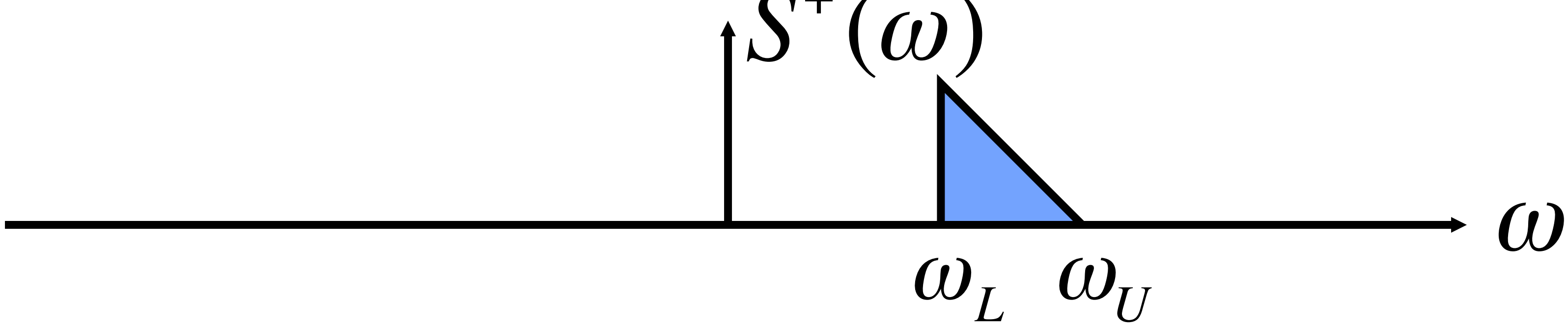
$$z(t) \equiv \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

- **Taking FT on both sides, we see the delay corresponds to a phase shift in Fourier space:**

$$Z(\omega) = \sum_{m=0}^{M-1} w_m Y_m(\omega) \exp(-j\omega\Delta_m)$$

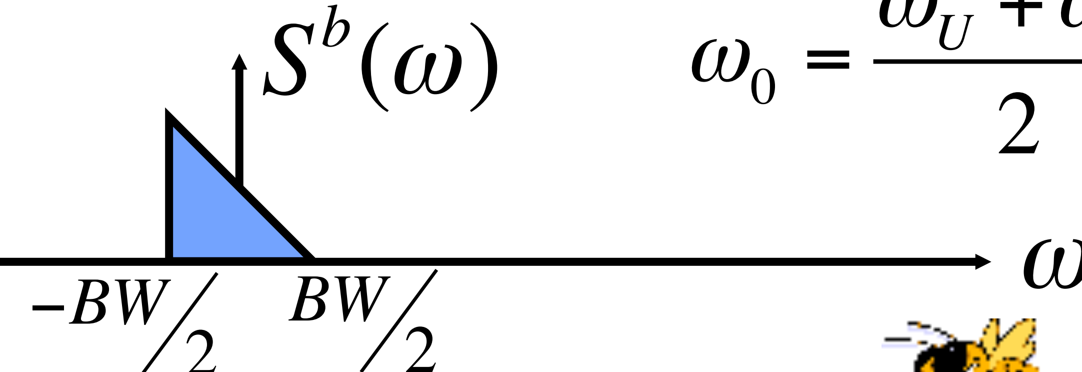
Analytic Representations (Fourier)





$$BW = \omega_U - \omega_L$$

$$\omega_0 = \frac{\omega_U + \omega_L}{2}$$



Analytic Representations (Time)

- **Easiest to represent a real narrowband signal in terms of the analytic signal:**

$$s(t) = \mathrm{Re}\{s^{+}(t)\}$$

- **The analytic signal is:**

$$s^{+}(t) = s_b(t)\exp(j\omega^{0}t)$$

The Narrowband Approximation

$$s^+(t) = s_b(t)\exp(j\omega^0 t)$$

- **Narrowband assumption means that:**

$$s_b(t-\tau) \approx s_b(t)$$ **for largest possible τ**

- **Consider plane waves:**

$$y_m^+(t) \equiv f^+(\vec{x}_m, t) = s^+(t - \vec{\alpha}^0 \cdot \vec{x}_m)$$

$$= s_b(t - \vec{\alpha}^0 \cdot \vec{x}_m)\exp(-j\omega^0 \vec{\alpha}^0 \cdot \vec{x}_m)\exp(j\omega^0 t)$$

$$\approx \exp(-j\vec{k}^0 \cdot \vec{x}_m)\underbrace{s_b(t)\exp(j\omega^0 t)}_{s^+(t)}$$

Data in Baseband Representation

$$y_m^+(t) \approx \exp(-j\vec{k}^0 \cdot \vec{x}_m) s_b(t) \exp(j\omega^0 t)$$

- **We'll work with the baseband signal at the sensors**

$$y_m^b(t) = \exp(-j\vec{k}^0 \cdot \vec{x}_m) s_b(t)$$

remember this is really an approximation, but people usually write an equals sign

Narrowband Beamforming

$$y_m^b(t) = \exp(-j\vec{k}^0 \cdot \vec{x}_m) s_b(t)$$

$$z(t) = \sum_{m=0}^{M-1} w_m \exp(j\vec{k} \cdot \vec{x}_m) y_m^b(t)$$

$$= \sum_{m=0}^{M-1} w_m \exp(j\vec{k} \cdot \vec{x}_m) \exp(-j\vec{k}^0 \cdot \vec{x}_m) s_b(t)$$

$$= \sum_{m=0}^{M-1} w_m \exp\{j(\vec{k} - \vec{k}^0) \cdot \vec{x}_m\} s_b(t)$$

Vector Notation for the Data

$$y_m^b(t) = \exp(-j\vec{k}^0 \cdot \vec{x}_m) s_b(t)$$

$$\underbrace{\begin{bmatrix} y_0^b(t) \\ \vdots \\ y_{M-1}^b(t) \end{bmatrix}}_{\mathbf{y}(t)} = \underbrace{\begin{bmatrix} \exp(-j\vec{k}^0 \cdot \vec{x}_0) \\ \vdots \\ \exp(-j\vec{k}^0 \cdot \vec{x}_{M-1}) \end{bmatrix}}_{\mathbf{e}(\vec{k}^0)} s_b(t)$$

$$\mathbf{y}(t) = \mathbf{e}(\vec{k}^0) s_b(t)$$

steering vector

Vector Notation for Beamforming (1)

$$z(t) = \sum_{m=0}^{M-1} w_m \exp(j\vec{k}\cdot\vec{x}_m) y_m^b(t)$$

$$= \underbrace{\begin{bmatrix} \exp(j\vec{k}\cdot\vec{x}_0) & \cdots & \exp(j\vec{k}\cdot\vec{x}_{M-1}) \end{bmatrix}}_{\mathbf{e}^H(\vec{k})} \begin{bmatrix} w_0 y_0(t) \\ \vdots \\ w_{M-1} y_{M-1}(t) \end{bmatrix}$$

$$\mathbf{e}(\vec{k}) = \begin{bmatrix} \exp(-j\vec{k}\cdot\vec{x}_0) \\ \vdots \\ \exp(-j\vec{k}\cdot\vec{x}_{M-1}) \end{bmatrix}$$

Vector Notation for Beamforming (2)

$$\begin{bmatrix} w_0 y_0^b(t) \\ \vdots \\ w_{M-1} y_{M-1}^b(t) \end{bmatrix} = \underbrace{\begin{bmatrix} w_0 & & 0 \\ & \ddots & \\ 0 & & w_{M-1} \end{bmatrix}}_{\mathbf{W} = diag(w_0, \ldots, w_{M-1})} \begin{bmatrix} y_0^b(t) \\ \vdots \\ y_{M-1}^b(t) \end{bmatrix}$$

$$= \mathbf{W}\mathbf{y}(t)$$

Vector Notation for Beamformer Output

$$\mathbf{y}(t) = \mathbf{e}(\vec{k}^0)s_b(t)$$

$$z(t) = \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(t)$$

$$= \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{e}(\vec{k}^0)s_b(t)$$

Another Frequent Notation

$$\mathbf{e}(\vec{k}) = \begin{bmatrix} \exp(-j\vec{k}\cdot\vec{x}_0) \\ \vdots \\ \exp(-j\vec{k}\cdot\vec{x}_{M-1}) \end{bmatrix}$$

Look direction vector

$$\vec{\iota}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

$$\mathbf{e}(\phi,\theta,\lambda) = \begin{bmatrix} \exp\{jk\vec{\iota}(\theta,\phi)\cdot\vec{x}_0\} \\ \vdots \\ \exp\{jk\vec{\iota}(\theta,\phi)\cdot\vec{x}_{M-1}\} \end{bmatrix}$$

$2\pi/\lambda$

Linear Array (Phi Version)

$$\mathbf{e}(\phi,0,\lambda) = \begin{bmatrix} \exp\left\{-jk\sin(\phi)\dfrac{M-1}{2}d\right\} \\ \vdots \\ \exp\left\{jk\sin(\phi)\dfrac{M-1}{2}d\right\} \end{bmatrix}$$

$$= \exp\left\{-jk\sin(\phi)\frac{M-1}{2}d\right\} \begin{bmatrix} 1 \\ \exp\{jkd\sin(\phi)\} \\ \vdots \\ \exp\{jkd\sin(\phi)(M-1)\} \end{bmatrix}$$

Vandermonde Structure (Phi)

$$\exp\left\{-jk\sin(\phi)\frac{M-1}{2}d\right\}\begin{bmatrix} 1 \\ \exp\{jkd\sin(\phi)\} \\ \vdots \\ \exp\{\underbrace{jkd\sin(\phi)}_{\gamma}(M-1)\} \end{bmatrix}$$

"electrical angle"

$$= const\begin{bmatrix} (e^{j\gamma})^0 \\ (e^{j\gamma})^1 \\ \vdots \\ (e^{j\gamma})^{M-1} \end{bmatrix} = const\begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^{M-1} \end{bmatrix}$$

Vandermonde structure

Linear Array (Theta Version)

- **That ϕ version is popular with textbook authors, and what we used in previous lectures**
  - Elevation, sort of
- **More common in practice to analyze linear arrays in terms of θ: Let $\phi = \pi/2$**
  - Azimuth

$$\mathbf{e}(\pi/2,\theta,\lambda) = \begin{bmatrix} \exp\left\{-jk\cos(\theta)\dfrac{M-1}{2}d\right\} \\ \vdots \\ \exp\left\{jk\cos(\theta)\dfrac{M-1}{2}d\right\} \end{bmatrix}$$

Vandermonde Structure (Theta)

$$\exp\left\{-jk\cos(\theta)\frac{M-1}{2}d\right\}\begin{bmatrix} 1 \\ \exp\{jkd\cos(\theta)\} \\ \vdots \\ \exp\{\underbrace{jkd\cos(\theta)}_{\gamma}(M-1)\} \end{bmatrix}$$

"electrical angle"

$$= const\begin{bmatrix} (e^{j\gamma})^0 \\ (e^{j\gamma})^1 \\ \vdots \\ (e^{j\gamma})^{M-1} \end{bmatrix} = const\begin{bmatrix} z^0 \\ z^1 \\ \vdots \\ z^{M-1} \end{bmatrix}$$

Vandermonde structure

Power in the Beamformed Signal

$$|z(t)|^2 = \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(t)[\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(t)]^H$$

$$= \mathbf{e}^H(\vec{k})\mathbf{W}\underbrace{\mathbf{y}(t)\mathbf{y}^H(t)}_{\mathbf{R}(t)}\mathbf{W}^H\mathbf{e}(\vec{k})$$

spatial correlation matrix → $\mathbf{R}(t)$

$$= \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{R}(t)\mathbf{W}^H\mathbf{e}(\vec{k})$$

Reviewing the Strategy

- **To find the direction of arrival of a single source at time t, find the parameters that maximize the power in the beamformed signal:**

$$\left|z(t)\right|^2 = \left|\mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{y}(t)\right|^2 = \left|\mathbf{e}^H(\phi,\theta,\lambda)\mathbf{W}\mathbf{y}(t)\right|^2$$

- **If the source isn't moving, we can average over time**
  - Well revisit this issue in detail in a later lecture
- **If multiple targets, we can find multiple peaks…**
  - …but this isn't really "optimal" anymore
  - We'll revisit this too

SCM for a Plane Wave

- **Spatial correlation matrix for a single plane wave:**

$$\mathbf{R}(t) = \mathbf{y}(t)\mathbf{y}^H(t) \approx \mathbf{e}(\vec{k}^0)s_b(t)s_b^H(t)\mathbf{e}^H(\vec{k}^0)$$

$$= \underbrace{\mathbf{e}(\vec{k}^0)\mathbf{e}^H(\vec{k}^0)}_{\begin{bmatrix} \exp(j\vec{k}^0 \cdot \vec{x}_0) \\ \vdots \\ \exp(j\vec{k}^0 \cdot \vec{x}_{M-1}) \end{bmatrix} \begin{bmatrix} \exp(-j\vec{k}^0 \cdot \vec{x}_0) & \cdots & \exp(-j\vec{k}^0 \cdot \vec{x}_{M-1}) \end{bmatrix}} \left|s_b(t)\right|^2$$

$$\left[\mathbf{R}(t)\right]_{m_1, m_2} = \left|s_b(t)\right|^2 \exp\{j\vec{k}^0 \cdot (\vec{x}_{m_1} - \vec{x}_{m_2})\}$$

SCM for a Plane Wave/Linear Array

$$\left[\mathbf{R}(t)\right]_{m_1,m_2} = \left|s_b(t)\right|^2 \cdot \exp\{j\vec{k}^0(\vec{x}_{m_1} - \vec{x}_{m_2})\}$$

- **For a linear array with spacing d:**

$$\left[\mathbf{R}(t)\right]_{m_1,m_2} = \left|s_b(t)\right|^2 \cdot \exp\{j\vec{k}^0(m_1 - m_2)d\}$$

$$\mathbf{R}(t) = \begin{bmatrix} R_0 & R_1 & R_2 & \cdots & R_{M-1} \\ R_1^* & R_0 & R_1 & \cdots & R_{M-2} \\ R_2^* & R_1^* & R_0 & \cdots & R_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{M-1}^* & R_{M-2}^* & R_{M-3}^* & \cdots & R_0 \end{bmatrix}$$

← **Toeplitz structure**

Unweighted Case

- **Suppose $\mathbf{W}=\mathbf{I}$:**

$$|z(t)|^2 = \mathbf{e}^H(\vec{k})\mathbf{W}\mathbf{R}(t)\mathbf{W}^H\mathbf{e}(\vec{k})$$

$$= \mathbf{e}^H(\vec{k})\mathbf{R}(t)\mathbf{e}(\vec{k})$$