E9 231: Digital Array Signal Processing

Scribe: Nithya V S
Dept. of ECE & EE
Indian Institute of Science
Bangalore 560 012, India
nithya@ee.iisc.ernet.in

Class held on: 28 Oct 2008

Eigenspace Beamformers

One way of identifying the lower dimensional subspace of $\mathbf{S_X}$ is through the eigen decomposition.

$$\mathbf{S_X} = \sum_{i=1}^{N} \lambda_i \mathbf{\Phi}_i \mathbf{\Phi}_i^H = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H \tag{1}$$

where $\mathbf{U}$ is an $N \times N$ matrix of eigenvectors,

$$\mathbf{U} = [\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2 \quad \cdots \quad \mathbf{\Phi}_N]$$

and $\mathbf{\Lambda}$ is the diagonal matrix of the ordered eigenvalues,

$$\mathbf{\Lambda} = \mathrm{diag}\left[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_N\right],$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N > 0$$
$$\mathbf{\Phi}_i^H \mathbf{\Phi}_j = \delta_{ij}$$

First r eigenvalues: Signal + Interference
Remaining eigenvalues: Noise only

$$\begin{aligned} \mathbf{U}_{S+I} &\triangleq [\mathbf{\Phi}_1 \quad \mathbf{\Phi}_2 \quad \cdots \quad \mathbf{\Phi}_r] \Rightarrow \text{eigenspace that spans signal } + \text{ interference} \\ \mathbf{U}_N &\triangleq [\mathbf{\Phi}_{r+1} \quad \mathbf{\Phi}_{r+2} \quad \cdots \quad \mathbf{\Phi}_N] \Rightarrow \text{spans the noise only subspace} \end{aligned}$$

$$\mathbf{S_X} = \mathbf{U}_{S+I}\mathbf{\Lambda}_{S+I}\mathbf{U}_{S+I}^H + \mathbf{U}_N\mathbf{\Lambda}_N\mathbf{U}_N^H \tag{2}$$

$$\begin{aligned} \mathbf{\Lambda}_{S+I} &\triangleq \mathrm{diag}\left[\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_r\right] \\ \mathbf{\Lambda}_N &\triangleq \mathrm{diag}\left[\lambda_{r+1} \quad \lambda_{r+2} \quad \cdots \quad \lambda_N\right] \end{aligned}$$

$$\begin{aligned}
[\mathbf{S_X}]_T &= \mathbf{U}_{S+I}\mathbf{\Lambda}_{S+I}\mathbf{U}_{S+I}^H \\
[\mathbf{S_X}]_T^{-1} &= \mathbf{U}_{S+I}\mathbf{\Lambda}_{S+I}^{-1}\mathbf{U}_{S+I}^H
\end{aligned}$$

The $\mathbf{w}_{mpdr}$ reduces to (if we replace $\mathbf{S_X}$ by $[\mathbf{S_X}]_T$)

$$\mathbf{w}_{mpdr,es} = \gamma_{es}\,[\mathbf{S_X}]_T^{-1}\,\mathbf{v}_m \tag{3}$$

where

$$\gamma_{es} = \frac{1}{\mathbf{v}_m^H\,[\mathbf{S_X}]_T^{-1}\,\mathbf{v}_m}$$

Rationale

Signal + D interfering plane wave model

$$\begin{aligned}
\mathbf{X} &= \mathbf{VF} + \mathbf{N} && (4) \\
\mathbf{V} &= \begin{bmatrix}\mathbf{v}_s & \mathbf{v}_{\mathbf{k}_1} & \ldots & \mathbf{v}_{\mathbf{k}_D}\end{bmatrix}_{N\times(D+1)} \\
\mathbf{F} &= \begin{bmatrix}F_s & F_1 & \ldots & F_D\end{bmatrix}^T_{(D+1)\times 1} \\
\mathbf{S_X} &= \mathbf{VS_fV}^H + \sigma_n^2\mathbf{I} && (5)
\end{aligned}$$

where the noise is assumed to be spatially white, making $\mathbf{S_X}$ positive definite.

Assume $(D+1) < N$.

$$\lambda_{\mathbf{S_X}_i} = \begin{cases} \lambda_{\mathbf{S_f}_i} + \sigma_n^2, & 1 \le i \le D+1 \\ \sigma_n^2, & D+1 < i \le N \end{cases}$$

We can express

$$\mathbf{VS_fV}^H = \begin{bmatrix}\mathbf{U}_S & \mathbf{U}_S^\perp\end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}_S^H \\ \mathbf{U}_S^{\perp H} \end{bmatrix} \tag{6}$$

Subspace method:

$$\mathbf{S_X} = \begin{bmatrix}\mathbf{U}_S & \mathbf{U}_S^\perp\end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_S + \sigma_n^2\mathbf{I}_{D+1} & \mathbf{0} \\ \mathbf{0} & \sigma_n^2\mathbf{I}_{N-(D+1)} \end{bmatrix} \begin{bmatrix} \mathbf{U}_S^H \\ \mathbf{U}_S^{\perp H} \end{bmatrix} \tag{7}$$

- We can look at "how many equal minimum eigenvalues $\mathbf{S_X}$ has" to find $D+1$.
  - Multiplicity of σ_n^2 is $N-D-1$.
  - $\mathcal{R}(\mathbf{U}_S) = \mathcal{R}(\mathbf{V})$: signal + interference subspace
  - $\mathcal{R}(\mathbf{U}_S^\perp) \perp \mathcal{R}(\mathbf{V})$: noise-only subspace
- Dropping $\mathbf{U}_S^\perp$ (or $\sigma_n^2\mathbf{I}_{N-(D+1)}$) part is good because
  - No signal there anyway

– Inverting $[\mathbf{S_X}]_T$ is easier than inverting $\mathbf{S_X}$ as we dropped the smallest eigenvalues.

Note that $\mathbf{S_X}^{-1}\mathbf{v}_s = [\mathbf{S_X}]_T^{-1}\mathbf{v}_s$:

$$\mathbf{S_X}^{-1}\mathbf{v}_s = \left[\mathbf{U}_S\left(\mathbf{\Lambda}_S + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{U}_S^H + \sigma_n^{-2}\mathbf{U}_S^{\perp}\left(\mathbf{U}_S^{\perp}\right)^H\right]\mathbf{v}_s$$

The desired signal direction $\mathbf{v}_s \in \mathcal{R}(\mathbf{V}) = \mathcal{R}(\mathbf{U}_S)$

$$\left(\mathbf{U}_S^{\perp}\right)^H \mathbf{v}_s = \mathbf{0}$$

$$\mathbf{S_X}^{-1}\mathbf{v}_s = \mathbf{U}_S\left(\mathbf{\Lambda}_S + \sigma_n^2\mathbf{I}\right)^{-1}\mathbf{U}_S^H\mathbf{v}_s + \mathbf{0} = [\mathbf{S_X}]_T^{-1}\mathbf{v}_s$$

$$\mathbf{w}_{mpdr,es} = \frac{\sum_{i=1}^{r}\frac{1}{\lambda_i}\left(\mathbf{\Phi}_i^H\mathbf{v}_s\right)\mathbf{\Phi}_i}{\sum_{i=1}^{r}\frac{1}{\lambda_i}\left|\mathbf{\Phi}_i^H\mathbf{v}_s\right|^2} \tag{8}$$

λ_is are the eigenvalues of $[\mathbf{S_X}]_T$.

This beamformer is called *eigenspace beamformer* or *signal + interference subspace beamformer*.

Another Interpretation:

$\mathbf{P}_{\mathbf{U}_{S+I}} \triangleq \mathbf{U}_{S+I}\mathbf{U}_{S+I}^H$ is a projection matrix as $\mathbf{U}_{S+I}^H\mathbf{U}_{S+I} = \mathbf{I}$.

$$\mathbf{v}_p \triangleq \mathbf{P}_{\mathbf{U}_{S+I}}\mathbf{v}_s = \mathbf{U}_{S+I}\mathbf{U}_{S+I}^H\mathbf{v}_s \tag{9}$$

The projection beamformer is given by

$$\begin{aligned}
\mathbf{w}_p &= \gamma_p\mathbf{S_X}^{-1}\mathbf{v}_p \\
&= \gamma_p\mathbf{S_X}^{-1}\mathbf{U}_{S+I}\mathbf{U}_{S+I}^H\mathbf{v}_s \\
&= \gamma_p\left(\mathbf{U}_{S+I}\mathbf{\Lambda}_{S+I}^{-1}\mathbf{U}_{S+I}^H + \mathbf{U}_N\mathbf{\Lambda}_N^{-1}\mathbf{U}_N^H\right)\mathbf{U}_{S+I}\mathbf{U}_{S+I}^H\mathbf{v}_s \\
&= \gamma_p\mathbf{U}_{S+I}\mathbf{\Lambda}_{S+I}^{-1}\mathbf{U}_{S+I}^H\mathbf{v}_s,
\end{aligned} \tag{10}$$

which is the same as $\mathbf{w}_{mpdr,es}$ when $\gamma_p = \gamma_{es}$.

A third interpretation

$$\begin{aligned}
\mathbf{X}_{es} &\triangleq \mathbf{U}_{S+I}^H\mathbf{X} \\
\mathbf{v}_{es} &\triangleq \mathbf{U}_{S+I}^H\mathbf{v}_s
\end{aligned}$$

If we build an MPDR beamformer with $\mathbf{X}_{es}$ replacing $\mathbf{X}$ and $\mathbf{v}_{es}$ replacing $\mathbf{v}_s$,

$$\mathbf{w}_{mpdr} = \frac{\mathbf{\Lambda}_{S+I}^{-1}\mathbf{v}_{es}}{\mathbf{v}_{es}^H\mathbf{\Lambda}_{S+I}^{-1}\mathbf{v}_{es}} \tag{11}$$

With diagonal loading:

$$\mathbf{w}_{es,mpdr,dl}^H = \frac{\mathbf{v}_{es}^H\left(\mathbf{\Lambda}_{S+I} + \sigma_L^2\mathbf{I}\right)^{-1}}{\mathbf{v}_{es}^H\left(\mathbf{\Lambda}_{S+I} + \sigma_L^2\mathbf{I}\right)^{-1}\mathbf{v}_{es}} \tag{12}$$

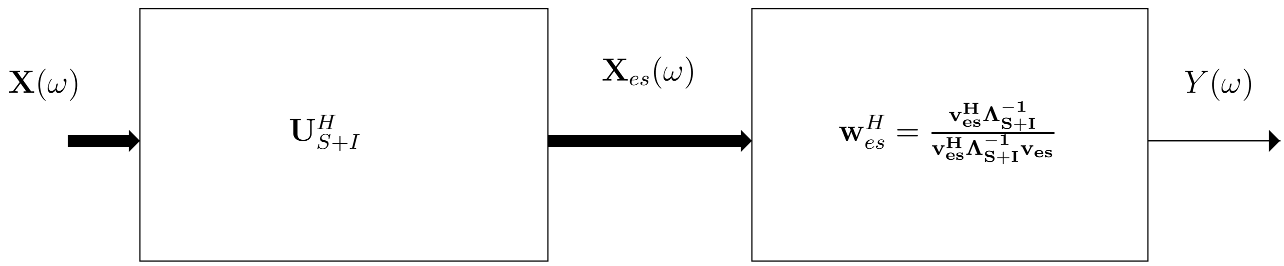

Figure 1: Eigenspace beamformer

LCMP Eigenspace Beamforming

$$\mathbf{C}^H\mathbf{w} = \mathbf{g}^H$$
$$\mathbf{w} = \mathbf{U}_{S+I}\mathbf{w}_{es}$$

Constraint becomes

$$\mathbf{C}^H\mathbf{U}_{S+I}\mathbf{w}_{es} = \mathbf{g}^H$$
$$\Rightarrow \mathbf{C}_{es}^H\mathbf{w}_{es} = \mathbf{g}^H, \text{ where } \mathbf{C}_{es}^H = \mathbf{U}_{S+I}^H\mathbf{C}$$
$$\mathbf{w}_{LCMP,es}^H = \mathbf{g}^H\left(\mathbf{C}_{es}^H\mathbf{\Lambda}_{S+I}^{-1}\mathbf{C}_{es}\right)^{-1}\mathbf{C}_{es}^H\mathbf{\Lambda}_{S+I}^{-1} \tag{13}$$

Performance of Eigenspace Beamformer

1. Critically dependent on knowing dim(S+I).
2. At high SNR (> INR), underestimating dim(S+I) is OK.
3. At low SNR, underestimating dim(S+I) is bad as pushing the signal into the noise-only subspace will result in a drastic reduction of performance.

Dominant Mode Rejection (DMR) Beamformers

Eigen decomposition of $\mathbf{S_X}$ gives

$$\mathbf{S_X} = \sum_{i=1}^{D_m}\lambda_i\mathbf{\Phi}_i\mathbf{\Phi}_i^H + \sum_{i=D_m+1}^{N}\lambda_i\mathbf{\Phi}_i\mathbf{\Phi}_i^H \tag{14}$$

Replace the eigenvalues $\lambda_{D_m+1}, \ldots, \lambda_N$ by $\alpha \triangleq \frac{1}{N-D_m}\sum_{i=D_m+1}^{N}\lambda_i$.

Note 1: $\alpha = \frac{1}{N-D_m}\left(\mathrm{tr}(\mathbf{S_X}) - \sum_{i=1}^{D_m}\lambda_i\right)$.

So it is enough to find $\lambda_1, \ldots, \lambda_{D_m}$.

Note 2: If the noise is spatially white and there are $< D_m + 1$ desired signal + interferers, $\alpha = \sigma_n^2 \Rightarrow$ same as eigenspace decomposition.

Modified $\mathbf{S_X}$:

$$\tilde{\mathbf{S}}_{\mathbf{X}} = \sum_{i=1}^{D_m} \lambda_i \mathbf{\Phi}_i \mathbf{\Phi}_i^H + \alpha \sum_{i=D_m+1}^{N} \mathbf{\Phi}_i \mathbf{\Phi}_i^H \tag{15}$$

Dominant-mode subspace,

$$\mathbf{U}_{dm} \triangleq [\mathbf{\Phi}_1 \quad \cdots \quad \mathbf{\Phi}_{D_m}]$$

Noise subspace,

$$\mathbf{U}_{dm}^{\perp} \triangleq [\mathbf{\Phi}_{D_m+1} \quad \cdots \quad \mathbf{\Phi}_N]$$

$$\tilde{\mathbf{S}}_{\mathbf{X}} = \mathbf{U}_{dm} \mathbf{\Lambda}_{dm} \mathbf{U}_{dm}^H + \alpha \mathbf{U}_{dm}^{\perp} \left(\mathbf{U}_{dm}^{\perp} \right)^H$$

where $\mathbf{\Lambda}_{dm} = \operatorname{diag}[\lambda_1, \cdots, \lambda_{D_m}]$.

$$\begin{aligned} \tilde{\mathbf{S}}_{\mathbf{X}}^{-1} &= \mathbf{U}_{dm} \mathbf{\Lambda}_{dm}^{-1} \mathbf{U}_{dm}^H + \frac{1}{\alpha} \mathbf{U}_{dm}^{\perp} \left(\mathbf{U}_{dm}^{\perp} \right)^H \\ &= \frac{1}{\alpha} \left(\mathbf{U}_{dm} \alpha \mathbf{\Lambda}_{dm}^{-1} \mathbf{U}_{dm}^H + \mathbf{P}_{dm}^{\perp} \right) \end{aligned} \tag{16}$$

$\mathbf{P}_{dm}^{\perp} \triangleq \mathbf{U}_{dm}^{\perp} \left(\mathbf{U}_{dm}^{\perp} \right)^H$ is a projection matrix.

The weight vector becomes

$$\begin{aligned} \mathbf{w}_{dm} &= \frac{\tilde{\mathbf{S}}_{\mathbf{X}}^{-1} \mathbf{v}_m}{\mathbf{v}_m^H \tilde{\mathbf{S}}_{\mathbf{X}}^{-1} \mathbf{v}_m} \\ &= \frac{\alpha \sum_{i=1}^{D_m} \frac{1}{\lambda_i} \left(\mathbf{\Phi}_i^H \mathbf{v}_m \right) \mathbf{\Phi}_i + \mathbf{P}_{dm}^{\perp} \mathbf{v}_m}{\alpha \sum_{i=1}^{D_m} \frac{1}{\lambda_i} \left| \mathbf{\Phi}_i^H \mathbf{v}_m \right|^2 + \mathbf{v}_m^H \mathbf{P}_{dm}^{\perp} \mathbf{v}_m} \end{aligned} \tag{17}$$

When $\mathbf{P}_{dm}^{\perp} \mathbf{v}_m = \mathbf{0}$, i.e., $\mathbf{v}_m \in (S+I)$ subspace,

$$\mathbf{w}_{dm} = \frac{\sum_{i=1}^{D_m} \frac{1}{\lambda_i} \left(\mathbf{\Phi}_i^H \mathbf{v}_m \right) \mathbf{\Phi}_i}{\sum_{i=1}^{D_m} \frac{1}{\lambda_i} \left| \mathbf{\Phi}_i^H \mathbf{v}_m \right|^2} \tag{18}$$

i.e., the same as the eigenspace beamformer.

If the input spectrum matrix was $\lambda_k \mathbf{\Phi}_k \mathbf{\Phi}_k^H$,

$$\begin{aligned} \mathbf{P}_{\mathbf{\Phi}_k} &= \text{output power due to the } k\text{th eigenmode} \\ &= \frac{\frac{1}{\lambda_k} \left| \mathbf{v}_m^H \mathbf{\Phi}_k \right|^2}{\sum_{i=1}^{D_m} \frac{1}{\lambda_i} \left| \mathbf{\Phi}_i^H \mathbf{v}_m \right|^2} \\ \lambda_k &= \lambda_k^s + \alpha, \end{aligned} \tag{19}$$

where λ_k^s is the signal component and α is the noise component.

- If λ_k is large, $\mathbf{P}_{\mathbf{\Phi}_k}$ will be small. Hence the name "dominant mode rejection" beamformer.
- As the projection of $\mathbf{v}_m$ on $\mathbf{\Phi}_k$ increases, the depth of the null on the kth eigenmode decreases.
- As $\mathbf{v}_m$ changes (from $\mathbf{v}_a$), the projection of $\mathbf{v}_m$ in the noise-only subspace increases.
- As the eigenspace beamformer totally discarded the noise-only subspace, there would be no hope of recovering the signal if we underestimated D_m and the signal happened to be in the noise-only subspace. The DMR beamformer works even when D_m is underestimated and the signal lies fully in the noise-only subspace.
- For high SNR (>INR), use eigenspace beamformer. For low SNR, use the DMR beamformer.
- DMR fails when SNR is high and there is a significant mismatch between $\mathbf{v}_m$ and $\mathbf{v}_a$.