Mathematics / Natural Sciences

Mathematical model and numerical approximation of shallow ice glacier sliding

Project Report
Mathematical Modeling, Simulation and Optimization

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Abstract

Glaciers and ice sheets are important components of Earth's cryosphere, influencing sea level rise and climate dynamics. Mathematical model of glaciers is useful to study the flow and deformation of ice masses, in order to predict their behavior under changing environmental conditions. However, solving the governing equations of such models analytically is often difficult due to their complexity. Numerical methods provide an alternative approach to obtaining approximate solutions.

In this project, we focus on the numerical approximation of the shallow ice model, which is essential for studying the dynamics of glaciers and ice sheets. Our approach employs a finite difference scheme to approximate the numerical solution of the shallow ice model.

The numerical results obtained through our approach is useful to observe how the ice flow behavior evolves over time under varying input parameters, providing valuable insights into the response of glaciers and ice sheets to environmental changes.

Contents

1 Introduction

A glacier or ice sheet is constantly moving because of its own weight, slowly spreading outward over a land surface or flowing down a hill or valley. The effects that these ice sheets have on global sea levels is significant. When ice sheet or glacier melts, the water is flowed into the oceans, inturn causing sea levels to rise.

The ice sheets present in Antarctica and Greenland plays a crucial role in regulating global climate and sea levels. The potential impacts of their continued melting due to ongoing climate change are of significant concern, as the release of freshwater into the oceans could disrupt ocean circulation patterns and contribute to further sea level rise.

Glacier melting in the Himalayan region also holds critical significance due to its role in sustaining water resources for millions of people across the region. The accelerated melting observed in recent years, driven by rising temperatures and climate change, threatens the balance of water supply in the Himalayan region, leading to potential disruptions.

Understanding the dynamics of ice flow within ice sheets is crucial for predicting how they will respond to changing climate conditions. Models that simulate ice sheet behavior take into account factors such as temperature, snowfall rates, ice viscosity, and the underlying bedrock topography to predict how ice sheets will evolve over time.

Typically, the areas near the margins of a glacier or ice sheet experience higher temperatures, resulting in the ice's melting. Moreover, ice can detach from the glacier's edges that enter the sea. This phenomenon of calving plays a vital role in the considerable reduction of ice within the ice sheet and significantly influences the ice sheet's overall mass equilibrium.

Changes in snowfall, temperature, and melt patterns can influence the balance between ice accumulation and loss within the ice sheet. If the rate of snowfall exceeds the rate of melt and calving, the ice sheet will gain mass and grow. Conversely, if melting and calving outpace snowfall, the ice sheet will lose mass and shrink.

In our project, we have studied shallow Ice approximation of ice sheet to find leading equations which can predict the glacier profile as time progresses under certain assumptions, and implemented finite difference numerical scheme to approximate the equations.

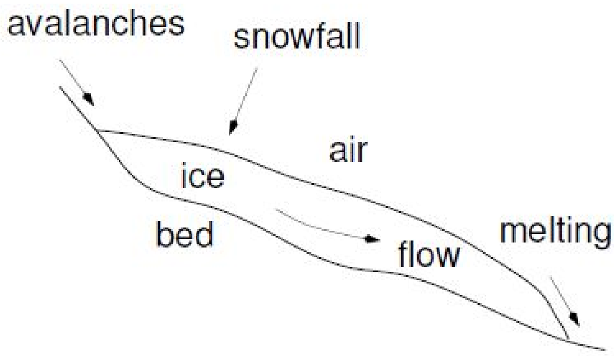

Figure 1.1: 2D representation of a Glacier flow

Finally, the obtained results from numerical methods are plotted for different input conditions to observe and compare the behaviour of glacier profile development with time.

1.1 Shallow Ice Approximation

Typically, Glaciers and ice sheets have higher lateral extent than thickness. Which means the aspect ratio of these typical ice sheets are very small, this fact is exploited to derive simplified equations which can be found later in chapter 4. This approximation is commonly known as shallow ice approximation in glaciology.

The scope of this project is limited to shallow ice approximation model and we have used this approximation as basis to derive for our leading equations in chapter 4 .

2 Related Works

The 2002 doctoral dissertation of Christian Schoof at the University of Oxford [6] offers a thorough investigation of the mathematical models of glacier sliding and drumlin development.

We have studied the classical model for shallow ice sheet development, which has been derived in several glaciological literature, its principle components can be found in Morland and Johnson, 1980 [3]

3 Ice Flow equations

In the following work, we use Cartesian coordinate system in 2D, see figure 3.1, here $D(x,t)$ and $H(x)$ represent height of the glacier and bed respectively. The parameter $a(x,t)$ is called accumulation rate, it represent the rate at which ice accumulates at the surface as a result of snowfall. When $a < 0$, it signifies net mass loss due to melting or sublimation. Generally, a is prescribed externally depending on the climatic conditions.

while ice is a solid, over long time scales and under the influence of its own weight, it can deform and flow much like a viscous fluid. Given this behavior, we can treat ice in glaciers as a moving fluid and describe its motion using the velocity field $V = (u, v)$.

The creep deformation of glaciers, for a simple model, is modelled as slow, incompressible, viscous flow [4] . So, we can introduce the Navier-Stokes equation as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \textit{incompressibility} \tag{3.1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \sigma_{xx}}{\partial x} \quad \textit{x-momentum balance} \tag{3.2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} \quad \textit{y-momentum balance} \tag{3.3}$$

where u, v are the velocity of the glacier in x and y-direction, ρ is the density, p is the pressure, g is the acceleration due to gravity, and τ_{yx} and σ_{xx}, are the shear stress and normal stress along x-axes and τ_{xy} and σ_{yy} are the shear stress and normal stress along y-axes.

Since the glacier ice has a constant density and it's compressibility effects are negligible, the incompressibility assumption remains valid in equation (3.1).

Slow flow means that the forces of inertia are negligible, this also come from the fact that Reynolds number for glacier or ice sheets are typically $< 10^{-13}$ [2], it implies the following for equations (3.2) and (3.3),

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \approx 0$$

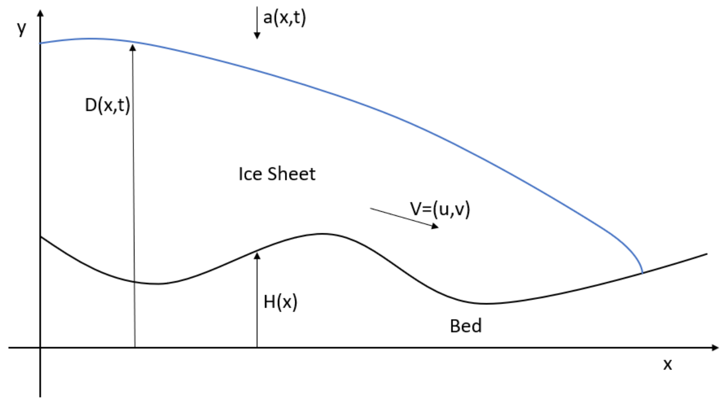

Figure 3.1: 2D representation of a Glacier flow

and

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\approx 0$$

The most commonly used model rheology for ice [4] relates stress tensor $\dot{\tau}$ and strain rate tensor $\dot{e}$ in the following way,

$$\dot{\tau}=2\eta\dot{e} \tag{3.4}$$

where, η is the non-linear viscosity and it takes the following form,

$$\eta=A^{-1/n}\dot{e}^{1/n-1}/2 \tag{3.5}$$

The above expression for η can be found in many literature [1], here A, for isothermal case, is a constant with $n=3$ a popular choice.

If the glacier is frozen to the bed, then we can assume that there is no slip at the base $y=H(x)$,

$$V=0 \quad \text{on} \quad y=H(x) \tag{3.6}$$

At the free surface $y=D(x,t)$, there is no applied traction. So,

$$\dot{\tau}n_D=0 \quad \text{on} \quad y=D(x,t) \tag{3.7}$$

$$\text{where,} \quad n_D = \frac{1}{\left(1 + \left(\frac{\partial D}{\partial x}\right)^2\right)^{1/2}} \begin{pmatrix} -\frac{\partial D}{\partial x} \\ 1 \end{pmatrix} \quad \text{is unit normal to the surface} \tag{3.8}$$

We can define the mass conservation equation as

$$\frac{\partial D}{\partial t} + \frac{\partial Q}{\partial x} = a(x,t) : \quad Q = \int_{H(x)}^{D(x,t)} u \, dy \tag{3.9}$$

where, D is the surface elevation, Q is the ice flux and a is the accumulation rate and t is time.

4 Scaling and Non dimentionilization

We define the following dimensionless variables,

$$x = [L]x^*, \quad y = [D]y^*, \quad t = [t]t^*, \tag{4.1a}$$
$$D(x,t) = [D]D^*(x^*,t^*), \quad H(x) = [D]H^*(x^*), \tag{4.1b}$$
$$u = [u]u^*, \quad v = \epsilon[u]v^*, \quad \dot{e} = [u]/(2[D])\dot{e}^*, \tag{4.1c}$$
$$a = [a]a^*, \quad Q = [u][D]Q^*, \tag{4.1d}$$
$$\tau_{xy} = [\tau]\tau_{xy}^*, \quad \sigma_{xx} = -\sigma_{yy} = \epsilon[\tau]\sigma_{xx}^*, \quad p = \rho g[D](D^* - y^*) + \epsilon[\tau]p^* \tag{4.1e}$$

where, ϵ is the aspect ratio of the ice sheet,

$$\epsilon = \frac{[D]}{[L]} \tag{4.2}$$

For Himalayan glaciers (eg Gangotri glacier [5]), typical values of the scales used are, $[L] = 30 \quad km, \quad [a] = 0.01 \quad ma^{-1} \quad \text{and} \quad [D] = 30m$ approximately.

This gives us the aspect ratio,

$$\epsilon = \frac{[D]}{[L]} \approx 10^{-3} \tag{4.3}$$

Equipped with the scales (4.1) and dropping asterisks on the dimensionless variables, we obtain the following scaled field equations,

$$\frac{\partial \tau_{xy}}{\partial y} + \epsilon^2 \frac{\partial \sigma_{xx}}{\partial x} - \epsilon^2 \frac{\partial p}{\partial x} - \frac{\partial D}{\partial x} = 0, \tag{4.4}$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} - \frac{\partial p}{\partial y} = 0, \tag{4.5}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \tag{4.6}$$

where

$$\tau_{xy} = \dot{e}^{1/n-1}\frac{\partial u}{\partial y} + \epsilon^2 \dot{e}^{1/n-1}\frac{\partial v}{\partial x} \tag{4.7}$$

$$\sigma_{xx} = 2\dot{e}^{1/n-1}\frac{\partial u}{\partial x} = -\sigma_{yy} \tag{4.8}$$

and

$$\dot{e} = \left[\left(\frac{\partial u}{\partial y}\right)^2 + \epsilon^4\left(\frac{\partial v}{\partial x}\right)^2 + 2\epsilon^2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\right]^{1/2} \tag{4.9}$$

Scaled boundary conditions at the base $y = H$ are

$$u = v = 0, \tag{4.10}$$

and at the surface $y = D$,

$$\tau_{xy} + \epsilon^2\frac{\partial D}{\partial x}(p - \sigma_{xx}) = 0 \tag{4.11}$$

$$\sigma_{yy} - p - \frac{\partial D}{\partial x}\tau_{xy} = 0 \tag{4.12}$$

while equation (3.9) remains unchanged.

Dropping the terms $O(\epsilon^2)$ on the basis that $\epsilon << 1$ gives the desired approximations:

$$\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{1/n-1}\frac{\partial u}{\partial y}\right) = \frac{\partial D}{\partial x} \quad \text{on} \quad y \in (H, D), \tag{4.13}$$

$$u = 0 \quad \text{on} \quad y = H, \tag{4.14}$$

$$\left(\left| \frac{\partial u}{\partial y} \right|^{1/n-1} \frac{\partial u}{\partial y} \right) = 0 \quad \text{on} \quad y = D. \tag{4.15}$$

There are further equations arising from (4.5) (4.8) and (4.12) which determine v and p, but these decouple from the leading-order model.

Equations (4.13) - (4.15) can now be integrated to give ice flux:

$$Q = -\frac{1}{n+2}(D-H)^{n+2} \left| \frac{\partial D}{\partial x} \right|^{n-1} \frac{\partial D}{\partial x}, \tag{4.16}$$

Finally, this can be plugged in mass conservation equation (3.9) to give,

$$\frac{\partial D}{\partial t} - \frac{\partial}{\partial x} \left[\frac{1}{n+2}(D-H)^{n+2} \left| \frac{\partial D}{\partial x} \right|^{n-1} \frac{\partial D}{\partial x} \right] = a(x,t) \tag{4.17}$$

Equation (4.17) is a non linear parabolic equation, which is the leading equation to determine D, with prescribed accumulation rate a. Further sections discuss about the numerical approximation of the above equation.

5 Numerical Approximation

Our leading equation (4.17) is highly non linear, parabolic partial differential equation and is analogous to diffusion equation,

$$\frac{\partial D}{\partial t} - \frac{\partial}{\partial x}\left[\kappa(x,t)\frac{\partial D}{\partial x}\right] = a(x,t) \tag{5.1}$$

With source term $a(x,t)$.

Patial differential equations of the form (5.1) can be found in many physical and mathematical applications, for example, heat equation. However, the diffusivity term $\kappa(x,t)$ in equation (5.1) is not constant, but depends non linearly on the ice thickness and the slope of glacier surface.

$$\kappa(x,t) = \frac{1}{5}(D-H)^5\left|\frac{\partial D}{\partial x}\right|^2 \tag{5.2}$$

To approach the numerical approximation of equation (5.1), we first start with the simplest finite difference scheme i.e explicit scheme.

5.1 Explicit scheme

To define the explicit finite difference scheme, we first discretize space and time in following equidistant grid,

$$x_i = ih, \text{ where } i = 0,\dots,n \text{ and step size } h = L/n$$

$$t^k = k\tau, \text{ where } k = 0,\dots,m \text{ and } \tau = T/m$$

Using the above defined discretization, we can approximate the equation (5.1) explicitly in the following manner,

$$\frac{D_i^{k+1} - D_i^k}{\tau} - \frac{\kappa_i^k}{h^2}\left[D_{i-1}^k - 2D_i^k + D_{i+1}^k\right] = a(x,t) \tag{5.3}$$

and,

$$\kappa_i^k = \frac{1}{5}(D_i^k - H_i)^5 \left|\frac{D_{i+1}^k - D_i^k}{h}\right|^2 \tag{5.4}$$

We call the above scheme explicit because it directly computes the height of glacier D_i^{k+1} at time step t^{k+1} from the values of glacier height at old time step t^k

The space-time stencil for the explicit scheme (5.3) is given in Figure 5.1

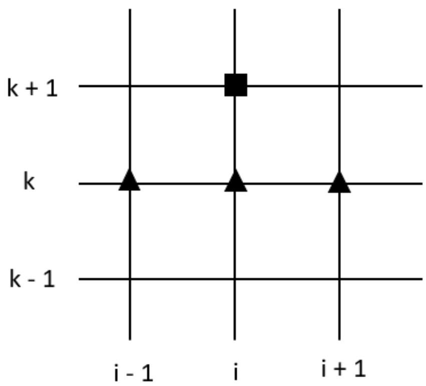

Figure 5.1: space-time stencil for explicit scheme (5.3). Represent evaluation of D at new time step (square) from values at old time step (triangle)

5.1.1 Stability Criteria for Explicit Scheme

Implementing the defined explicit scheme (5.3) does not always give stable results. The unstable values starts to arise from the end points of glacier profile. In order to analyze the stability of the defined scheme, we first revisit the stability analysis of linear parabolic PDE i.e. von-Neumann stability analysis [7]

Explicit approximation of linear diffusion equation with constant diffusivity can produce stable results only if the time steps are sufficiently smaller than space steps, in particular, if it adhere to following rule.

$$\tau \leq h^2/2\kappa(x,t) \tag{5.5}$$

However, in our case, the value of $\kappa(x,t)$ is not constant, but varying with x and t which makes it difficult to limit the time step programmatically and therefore it results in unstable results even for very small time steps.

An approach to tackle the problem of changing diffusivity is to use adaptive time steps, however it makes the algorithms computationally challenging and difficult to implement.

5.2 Implicit Scheme

There is an alternative stability fix instead of adaptivity, namely “implicitness.” Implicit schemes for linear parabolic PDE are unconditionally stable. Another well-known scheme, Crank-Nicolson is also unconditionally stable.

Implicit Schemes are harder to implement than explicit scheme, specifically for non linear PDE, as it produces system of non linear equation which must be solved at each time step, leaving higher computational demand. For this reason, we define the following mixed scheme which is implicit for all derivatives of D but explicit for $\kappa(x,t)$.

$$\frac{D_i^{k+1} - D_i^k}{\tau} - \left[\frac{\kappa_{i+\frac{1}{2}}^k \left(\frac{D_{i+1}^{k+1} - D_i^{k+1}}{h} \right) - \kappa_{i-\frac{1}{2}}^k \left(\frac{D_i^{k+1} - D_{i-1}^{k+1}}{h} \right)}{h} \right] = a(x,t) \tag{5.6}$$

and,

$$\kappa_{i+\frac{1}{2}}^k = \frac{1}{5} \left(\frac{D_{i+1}^k - H_{i+1} + D_i^k - H_i}{2} \right)^5 \left| \frac{D_{i+1}^k - D_i^k}{h} \right|^2 \tag{5.7}$$

$$\kappa_{i-\frac{1}{2}}^k = \frac{1}{5} \left(\frac{D_{i-1}^k - H_{i-1} + D_i^k - H_i}{2} \right)^5 \left| \frac{D_i^k - D_{i-1}^k}{h} \right|^2 \tag{5.8}$$

The above scheme produces system of linear equation at every time step, which is comparatively easier to solve than pure implicit scheme.

The space-time stencil for implicit scheme (5.6) is given in figure 5.2

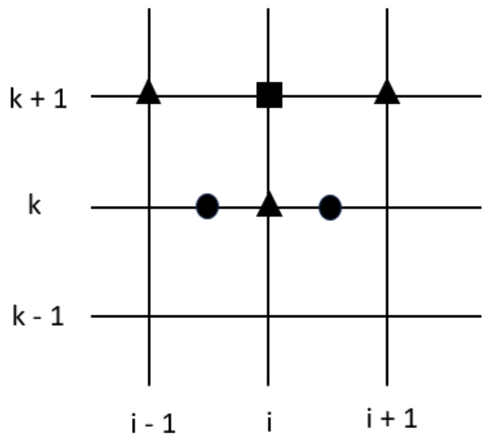

Figure 5.2: space-time stencil for implicit scheme (5.6). Represent evaluation of D at new time step (square) from implicit values (triangle), and κ evaluated at old time step (circle)

We assume the slope of glacier profile to be flat at the center of glacier and so we prescribe Neumann boundary condition at $x = 0$. The ice thickness approaches zero towards the end of glacier profile this automatically makes our equation a free boundary problem towards the right end.

We have implemented the following code for above defined mixed scheme (5.6) in python.

```
def kappa_plus(D, i, h):
    return ((0.5 * (D[i+1] + D[i]))**5) * (((D[i+1] - D[i])/h)**2)*0.2

def kappa_minus(D, i, h):
    return ((0.5 * (D[i-1] + D[i]))**5) * (((D[i] - D[i-1])/h)**2)*0.2

def ImplicitScheme(Initial_val):
    time_step = 0.001
    space_step= 0.01
    a = Get_accumulation()
    transeint_D = Initial_val.reshape(1,100)
    for i in range(1000):
        # Definition of ghost point for neumann condition
        D = np.ones(100+2)
        for i in range(100):
            D[i+1]= transeint_D[-1][i]
        D[0] = transeint_D[-1][1]
        D[-1] = transeint_D[-1][-2]

        H = Bed()
```

```

        mainDiag = np.ones(100 + 2)
        for i in range(1, 102-1):
            mainDiag[i] = 1/time_step + ((kappa_minus(D,i,space_step, H) +  kappa_plus(D,i,space_step,H) )/(space_step**2))

        rightDiag = np.ones(100 + 1)
        for i in range(1, 101):
            rightDiag[i] = -(kappa_plus(D,i,space_step,H)/(space_step**2))

        leftDiag = np.ones(100 + 1)
        for i in range(0, 101-1):
            leftDiag[i] = -(kappa_minus(D,i,space_step,H)/(space_step**2))

        mainDiag[0] = 1
        mainDiag[-1] = 1
        rightDiag[0] = 0
        leftDiag[-1] = 0

        A = np.zeros((102,102), dtype=float)
        for i in range(102):
            A[i][i] = mainDiag[i]
        for i in range(101):
            A[i][i+1] = rightDiag[i]
        for i in range(1,101):
            A[i][i-1] = leftDiag[i]

        A[0][2] = -1 # boundary conditions
        A[-1][-3] = -1 # boundary conditions

        b = np.ones(102)
        for i in range(100):
            b[i+1] = a[i] + transeint_D[-1][i]/time_step
        b[0] = 0
        b[101] = 0

        NewD = np.linalg.solve(A,b)
        NewD = RemoveNegatives(NewD)
        transeint_D = np.append(transeint_D,[NewD[1:-1]],axis=0)

    return transeint_D
```

The above code produces stable results. We have generated results by altering the bed profile and accumulation rate so as to study its impacts on glacier profile. This is further discussed in next section.

6 Results of Numerical simulation

To evaluate results on flat bed and slanted bed by numerical simulation, we need initial condition and meaningful accumulation rate.

We first find solution for equilibrium case i.e $\frac{\partial D}{\partial t} = 0$ for constant accumulation rate, this leads us to second order ordinary differential equation. The solution for equilibrium case is computed numerically using explicit Euler's method and the result obtained serves us as initial condition for problem (5.6). The equilibrium case is plotted in figure 6.1 and figure 6.2

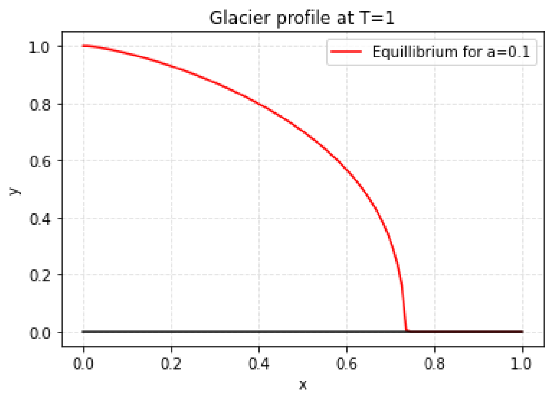

Figure 6.1: Equilibrium case for flat bed

Accumulation rate is generally prescribed externally depending on climatic conditions. Generally, accumulation rate tends to achieve negative values towards the end of glacier and positive value near the center[8]. For simplicity, we have assumed accumulation rate to be linearly decreasing with negative values towards right edge of glacier.

We have computed the glacier profile for three different accumulation rates, described in figure 6.3. The intention is to observe the behaviour of glacier flow when accumulation rates are decreased, mimicking global warming.

From the obtained graphs in figure 6.4 and figure 6.5, we can conclude that the simulation gives meaning results for glacier profile. since glacier sliding is a slow flow process, the results for three different accumulation rates lies close to each other, although it is clear

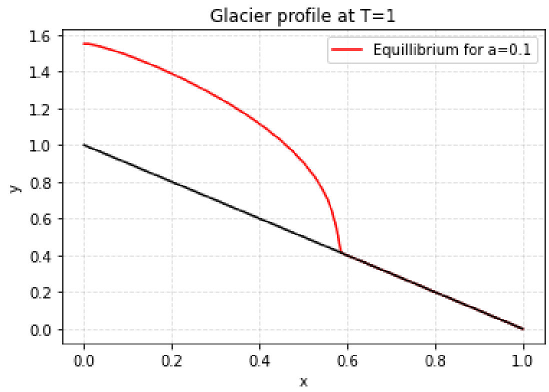

Figure 6.2: Equilibrium case for slanted bed at 45° angle

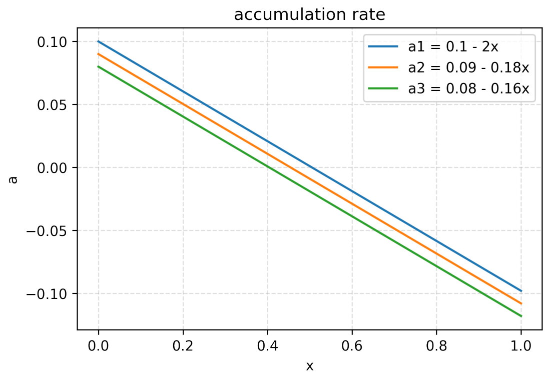

Figure 6.3: accumulation rates

that as accumulation rate is decreased, the thickness of ice is reduced near the center of glacier. The glacier profile has propagated more distance towards right edge for higher accumulation rate.

The observed propagation of glacier seems logical as for higher accumulation rate there is more thickness near the center and thus it pushes the ice to flow forward by its self weight, this leaves more chance for ice to runoff, also it increases melting as it lies in negative accumulation rate zone.

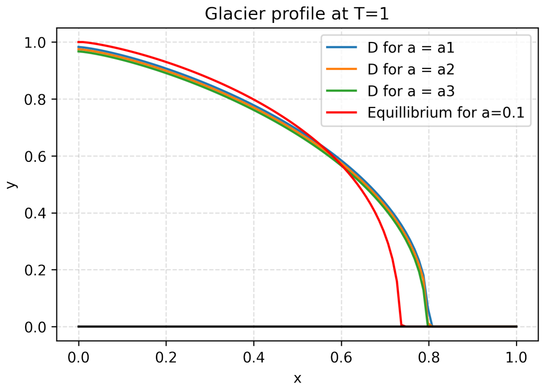

Figure 6.4: Results on flat bed

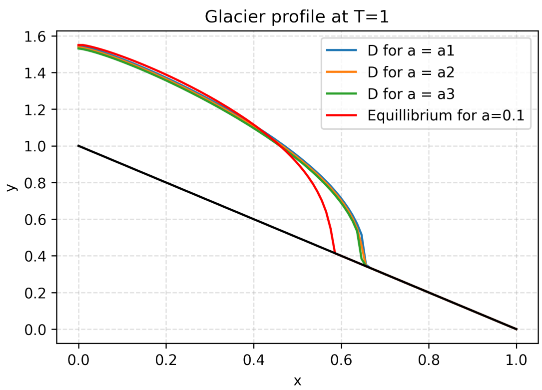

Figure 6.5: Results on slanted bed

7 Conclusion

From the results obtained by numerical simulation, we can conclude the following observation

- In figure 6.3, accumulation rate is decreased by 10% from a1 to a2 and 20% from a1 to a3. The observed decrease in height of glacier at $x = 0$ for flat bed is approximately 0.79% and 1.59% respectively, and for slanted bed is approximately 0.44% and 0.88% respectively.

- It is clear that bed profile can significantly affect the height of glacier and its propagation for same climatic conditions. It could be part of further research to better model the bed profile for more realistic and accurate results.

- The shallow ice approximation model, presented in this report, has been derived numerous times in glaceological literature. It is usually assumed that the bed is able to support shear stresses of any magnitude. However, assumption like this is not always appropriate and more realistic models can be developed with better assumption with slipping between bed and glacier.

- Another topic of further research is to better prescribe realistic accumulation rate. A more realistic accumulation rate would consider the effects of changing environmental conditions and randomness in climatic conditions.

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