Exosystem Modeling for Mission Simulation and Survey Analysis

Final Public Oral

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Why Study Exoplanets?

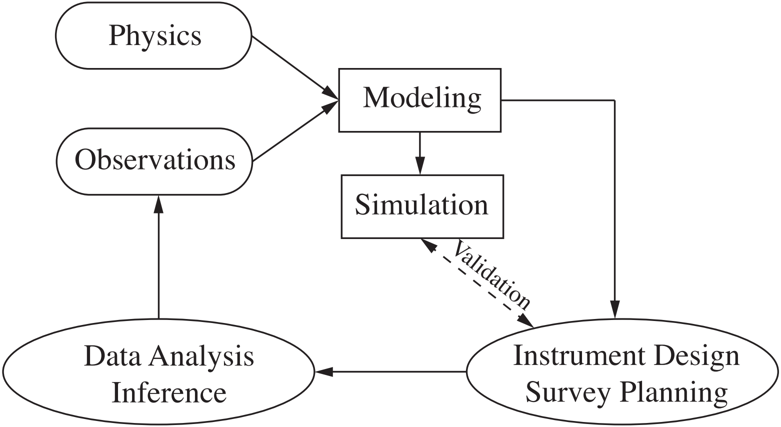
- **Without other examples, we have only one sample of planetary systems**
- **Need exoplanet observations to formulate and test planet formation and development models**
- **Observing systems at other points in their development will help explain our own solar system**
- **We may find life**

The Problem

- Planet observations occur at the absolute boundaries of existing instrument capabilities
- Dedicated planet-finding instruments are large, complicated, and very expensive
- Even a perfectly functioning exoplanet observatory may not find anything

We need to model both exosystems and instruments to ensure that useful data will be collected and the maximum amount of information extracted.

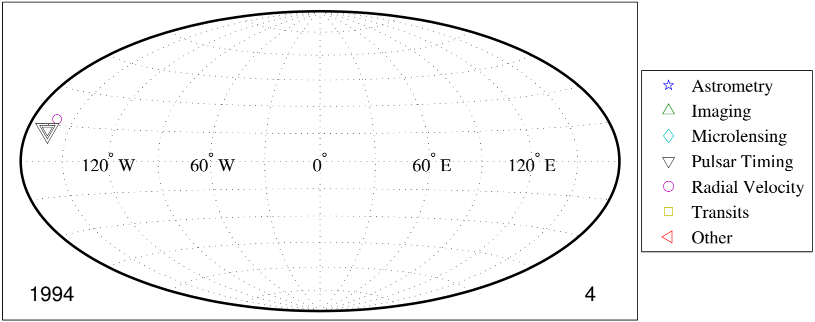
Goal


Physics
Observations
Modeling
Simulation
Validation
Data Analysis
Inference
Instrument Design
Survey Planning

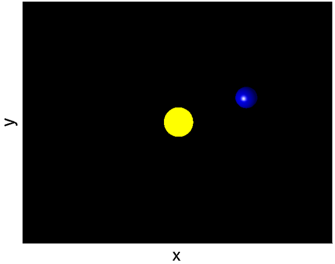
Outline

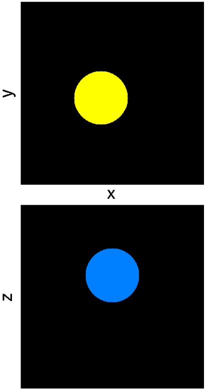
1. Introduction and Background
   - Known Exoplanets
   - Planet Finding Methods
2. Modeling and Simulation
   - Describing Planets and Orbits
   - Constructing Exosystems
3. Applications
   - Detection Statistics
   - Mission Analysis
   - Data Analysis

A Brief History of Exoplanet Exploration



Planet-Finding Techniques Illustrated





Exoplanet Data Sets

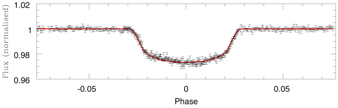


Figure : [Gillon et al., 2006]

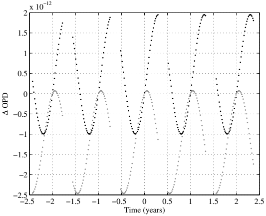


Figure : Simulation

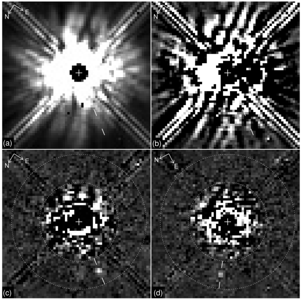


Figure : [Lafrenière et al., 2009]

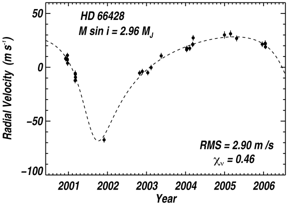


Figure : [Butler et al., 2006]

Imaging Constraints

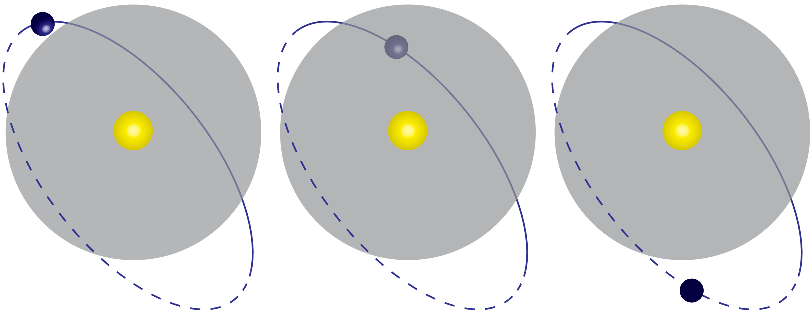


Figure : Schematic of projected exosystem. Planet is sufficiently illuminated for detection on the solid part of the orbit, and observable outside the gray circle.

All imaging systems have an inner working angle - the smallest observable separation between star and planet, and a limiting planet brightness.

Approach

1. Define a parameter set suitable for encoding whole exosystems
2. Find mappings between observation methods and parameter set
3. Develop a statistical description of planet observations
4. Develop capability to simulate populations of exosystems
5. Use these tools to analyze mission concepts and data sets

D. Savransky and N. J. Kasdin, *Simulation and analysis of sub-μas precision astrometric data for planet finding.* ApJ 2010.

D. Savransky, N. J. Kasdin, and E. Cady, *Analyzing the designs of planet finding missions.* PASP 2011.

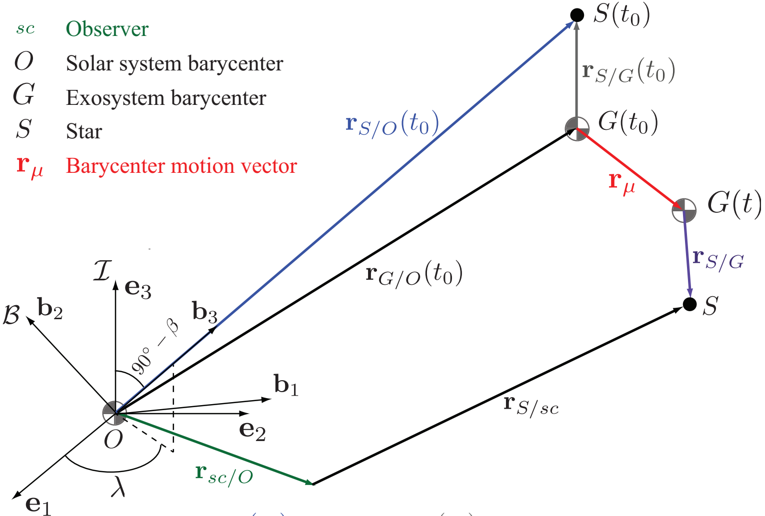
D. Savransky, E. Cady, N. J. Kasdin, *Parameter distributions of Keplerian orbits.* ApJ 2011.

Outline

Locating Stars in the Sky

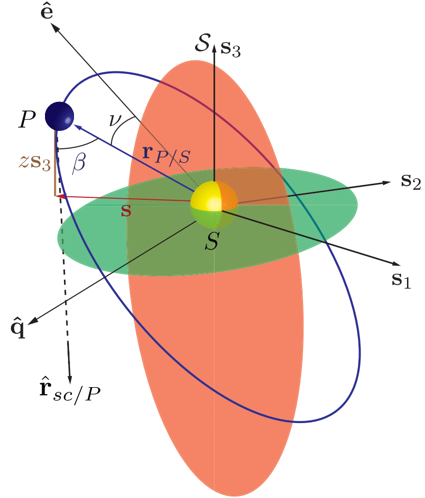
[Savransky and Kasdin, 2010]

sc	Observer
O	Solar system barycenter
G	Exosystem barycenter
S	Star
$\mathbf{r}_\mu$	Barycenter motion vector



$$\mathbf{r}_{S/sc} = \mathbf{r}_{S/O}(t_0) + \mathbf{r}_\mu - \mathbf{r}_{S/G}(t_0) + \mathbf{r}_{S/G} - \mathbf{r}_{sc/O}$$

System Orientation and Dynamics



$$\mathbf{r}_{P/S} = r\left(\cos\nu\hat{\mathbf{e}} + \sin\nu\hat{\mathbf{q}}\right)$$

$$r \triangleq \|\mathbf{r}_{P/S}\| = \frac{a(1-e^2)}{e\cos(\nu)+1}$$

$$= \text{Orbital radius}$$

$$^{\mathcal{I}}\mathbf{v}_{P/S} = \sqrt{\frac{\mu_P + \mu_S}{\ell}} \times$$

$$\left(-\sin\nu\hat{\mathbf{e}} + (e + \cos\nu)\hat{\mathbf{q}}\right)$$

$$^{\mathcal{I}}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}_{P/S} + (\mu_S + \mu_P)\frac{\mathbf{r}_{P/S}}{\|\mathbf{r}_{P/S}\|^3} = 0$$

$$\beta \approx \cos^{-1}\left(\frac{z}{r}\right) = \text{Phase angle}$$

a	Semi-major axis	ν	True anomaly
e	Eccentricity	$\mathbf{s}$	Projected separation

Light From Planets

[Brown, 2005, Barman et al., 2001]

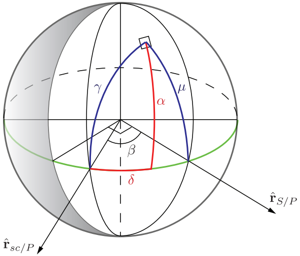


Figure : Reflecting spherical body [Sobolev, 1975]

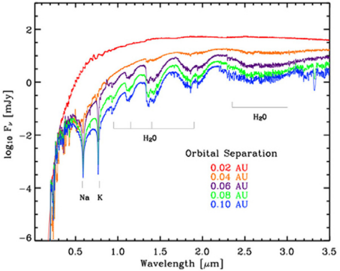


Figure : Irradiated planet model spectra [Hauschildt et al., 2008]

- Flux Ratio:

$$\frac{F_P^{\text{ref}}}{F_S} = p\left(\frac{R}{r}\right)^2 \Phi(\beta)$$

$$\Phi_l = \frac{\sin(\beta) + (\pi - \beta)\cos(\beta)}{\pi}$$

- Day side equilibrium tempearture:

$$\sigma T_{\text{eq}}^4 = \sigma T_{\text{int}}^4 + (1 - A)\left(\frac{R}{r}\right)^2 F_S$$

where $L_P = 4\pi R^2 \sigma T_{\text{int}}^4$

The Parameter Set

- Dynamics of exosystem with n planets can be encoded with the state:

$$X_D = \begin{bmatrix} \mathbf{r}_{P_1/G} & \dot{\mathbf{r}}_{P_1/G} & m_1 & \ldots & \mathbf{r}_{P_n/G} & \dot{\mathbf{r}}_{P_n/G} & m_n & \mathbf{r}_{S/G} & \dot{\mathbf{r}}_{S/G} & m_S \end{bmatrix}^T$$

  where $\dot{\mathbf{r}} \equiv {}^{\mathcal{I}}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}$.
- Augment state with astrometric and physical constants:

$$X_C = \begin{bmatrix} \dot{\mathbf{r}}_\mu & \varpi & \{R_i\}_1^n & \{p_i\}_1^n & R_S & \left\{T^{\mathrm{eff}_i}\right\}_1^n \end{bmatrix}^T$$

$$X = \begin{bmatrix} X_D \\ X_C \end{bmatrix}$$

- Treat observer position $\mathbf{r}_{sc}$ as known
- May be possible to simplify (or constrain) parameter set by modeling dependencies between mass, radius, age and temperature [Fortney et al., 2007, Sudarsky et al., 2005]

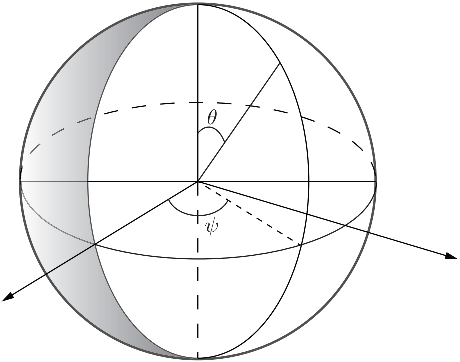
Exosystem Generation

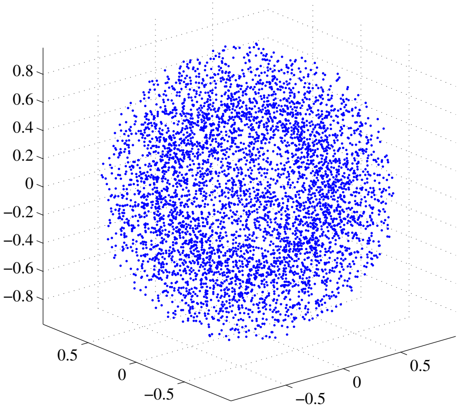
How do you simulate populations of exosystems?

1. Planetary population of interest (i.e., 'Earth-like' planets) - set values or ranges for all orbital and physical planetary parameters
    - Good for studying missions/surveys with specific goals, or doing mission comparisons/trade studies
2. Solar system analogue - exosystems composed of subset of solar system bodies in the solar system
    - Fast way of evaluating effects of multi-planet systems
3. Model actual distribution of physical and orbital parameters of known planets based on all available data
    - If trying to closely predict actual results of a survey
4. Generate systems starting with simulated nebulae
    - Only needed for testing specific formation theories

Exosystem Orientation

There are no known biases on exosystem orientation





$$\psi \sim U([0, 2\pi]) \qquad \theta \sim \cos^{-1}\left(U([-1, 1])\right)$$

Exosystem Propagation

Once systems are generated, need to be able to propagate planets forward in time

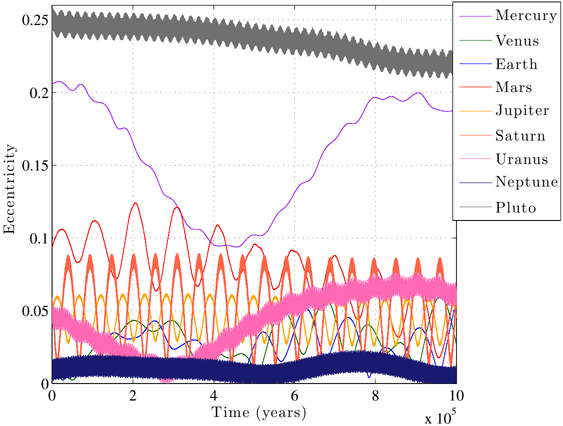


Figure : Variation in orbital eccentricities of solar system bodies over one million years.

- Store Keplerian orbital elements, update anomalies (fast but misses n-body effects)
- Store positions and velocities and integrate (slow but nearly exact)
- Hybrid scheme [Chambers, 1999]

Multi-Planet Stability

Its highly unlikely that we'll observe unstable systems, so simulated systems should have long-term stability

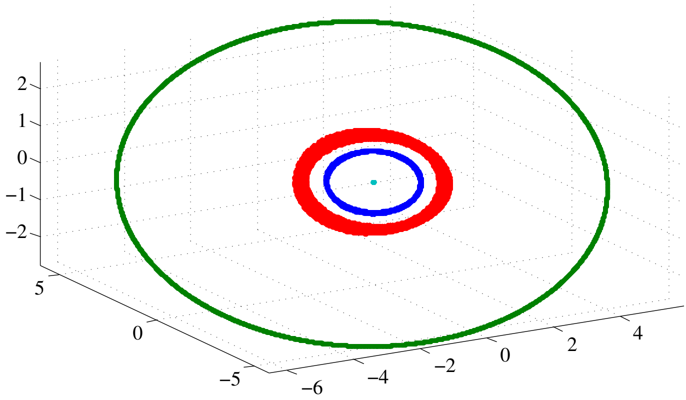


Figure : Orbits over 1 million years for exosystem composed of analogues of Earth (blue), Mars (red) and Jupiter (green), with a 1.5 $L_\odot$ star.

Mapping Instruments to the Parameter Set

Each observation technique can be described in terms of elements of this parameter set

- Imaging - planet position and physical parameters
- Transit photometry - planet and star positions and radii
- Doppler spectroscopy - star position and velocity
- Interferometric astrometry - planet and star positions and astrometric parameters

We can now describe exosystems and the data produced by observing them with various instruments using one unified parameter set.

Outline

Direct Detection Observable Distribution

[Brown, 2005]

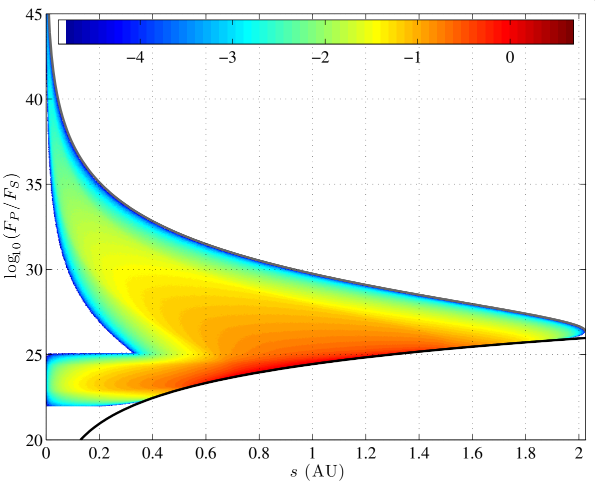


Figure : Joint probability density function of $(\bar{s} = s, \bar{F}_P/F_S = F_P/F_S)$ for Earth-twin planets ($a \in [0.7, 1.5]$, $e \in [0, 0.35]$, $p = p_\oplus$, $R = R_\oplus$) sampled via 1 billion Monte Carlo trials.

Direct Detection Observable Distribution

[Brown, 2005]

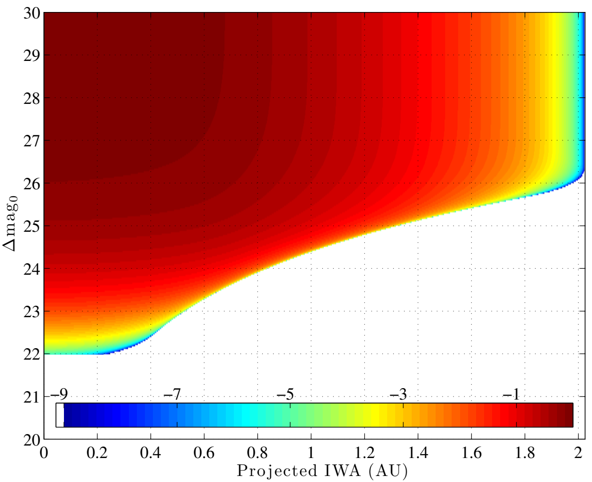


Figure : Joint cumulative distribution function of $(\bar{s} = s, \bar{F}_P/F_S = F_P/F_S)$ for Earth-twin planets ($a \in [0.7, 1.5]$, $e \in [0, 0.35]$, $p = p_\oplus$, $R = R_\oplus$) sampled via 1 billion Monte Carlo trials.

Monte Carlo is Inefficient

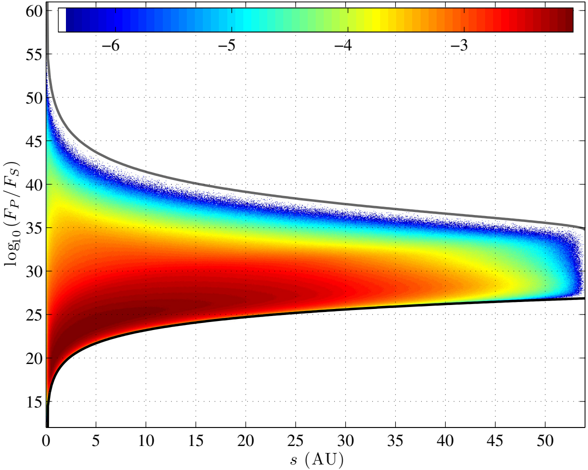


Figure : Joint probability density function of $(\bar{s} = s, \bar{F}_P/F_S = F_P/F_S)$ for a randomized planetary population ($a \in [0.4, 30]$, $e \in [0, 0.8]$, $p \in [0.1, 0.5]$, $R \in [0.7, 11.2]R_\oplus$) sampled via 1 billion Monte Carlo trials.

Monte Carlo is Inefficient

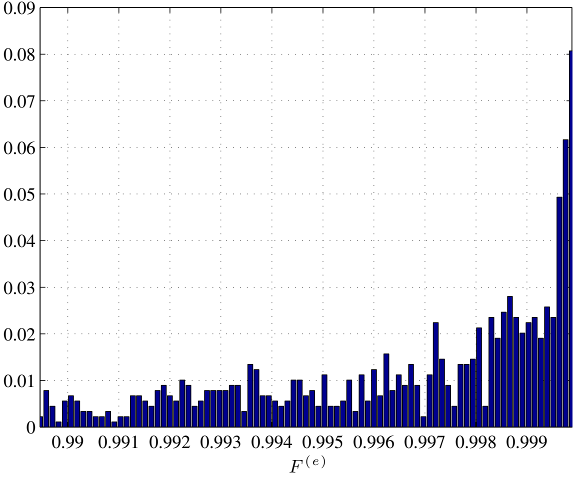


Figure : Probability density function for transit flux ratio ($\bar{F}^{(e)} = F^{(e)}$).

Distributions of Keplerian Orbital Elements

[Savransky et al., 2011]

If you know the distribution functions of the observables, you can directly sample the completeness function. Assume:

- Exosystem orientations are uniform wrt the observer
- Distributions for semi-major axis and eccentricity are known

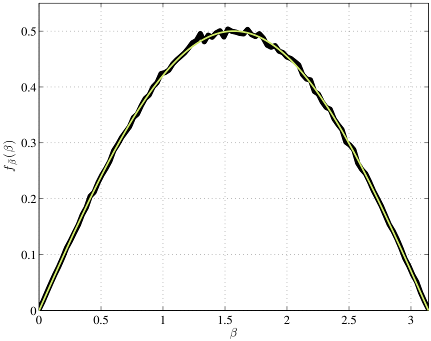


Figure : Phase angle PDF: Monte Carlo (black) and algebraic solution (green)

β is always sinusoidally distributed!

Direct Sampling vs. Monte Carlo

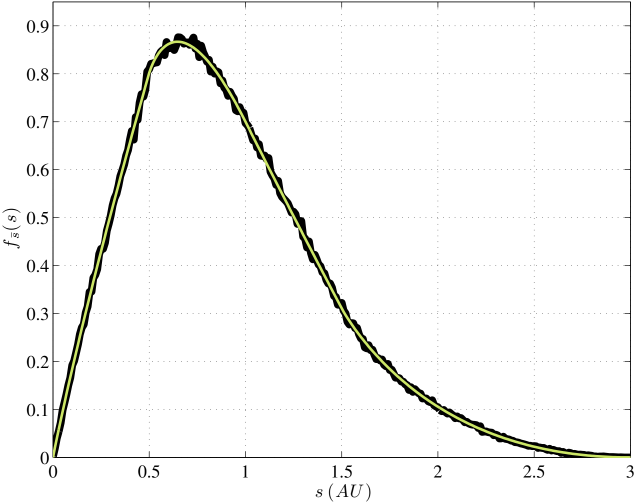


Figure : Probability density function of apparent separation using Monte Carlo (black) and algebraic solution (green) for uniform a and e.

Direct Sampling vs. Monte Carlo

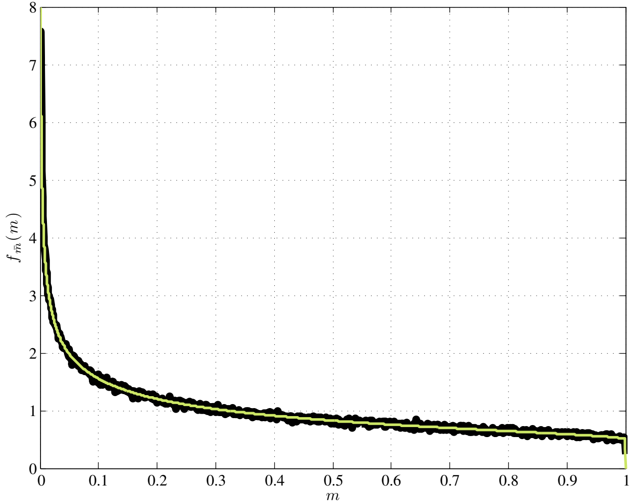


Figure : Probability density function of Lambert phase function $m = \Phi(\beta)$ using Monte Carlo (black) and algebraic solution (green).

Period Estimation

$$P_{orb} = 2\pi\sqrt{a^3/(\mu_S + \mu_P)}$$

- Assume $\mu_S >> \mu_P$ and get μ_S from mass-luminosity relationship [Henry and McCarthy Jr., 1993]
- Need to estimate the semi-major axis

$$f_{\bar{s}|\bar{a}}\left(s|a\right) = \frac{1}{\pi}\int_0^1\int_0^1 \frac{s}{a\sqrt{(1-l^2)\left[(ael)^2-(al-s)^2\right]}} f_{\bar{e}}(e)\,\mathrm{d}e\,\mathrm{d}l$$

$$\hat{a} = \arg\max_{a\in\bar{a}} \frac{1}{\pi}\int_0^1\int_0^1 \frac{s_0}{a\sqrt{(1-l^2)\left[(ael)^2-(al-s_0)^2\right]}} f_{\bar{e}}(e)\,\mathrm{d}e\,\mathrm{d}l$$

$$\hat{a} = s_0$$

Semi-major Axis Estimation

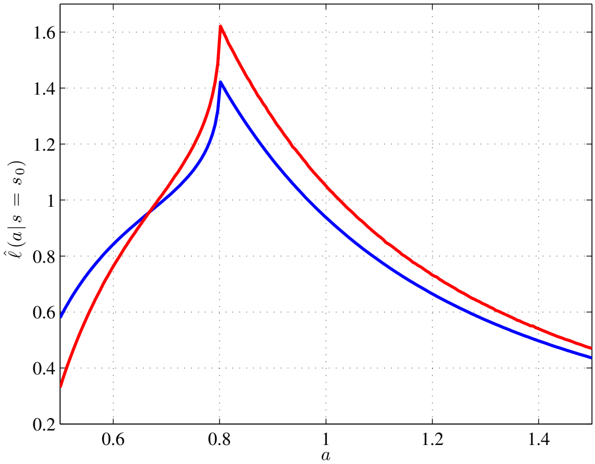


Figure : Likelihood function for semi-major axis given one observation of apparent separation $s_0 = 0.8$ for uniform distribution of a (in AU) and uniformly distributed (red) or step-distributed e (blue).

Mission Simulation

[Savransky et al., 2010]

- Create descriptions of instruments, planetary orbits/properties and observations
- Generate ensembles of full mission simulations (timelines of observations)
- Extract distributions of science yield/performance metrics

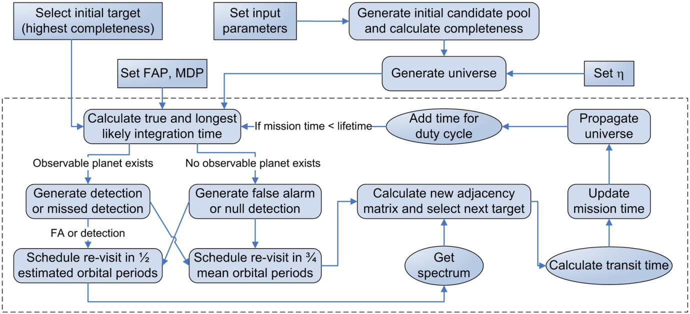


Figure : Flowchart of simulation framework

Visits as a Graph

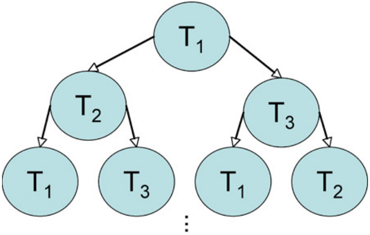


Figure : Visit graph for 3 target pool.

- Each set of possible transitions on the visit graph can be represented as a weighted adjacency matrix.
- The weights of the matrix entries represent the 'cost' of choosing the next star.

The cost of transitioning from target i to target j is:

$$A_{ij} = \left[a_1 \frac{\cos^{-1}(u_i \cdot u_j)}{2\pi} B_{inst} + a_2 \text{comp}_j - a_3 e^{t_c - t_f} B_{unvis} + a_4 B_{vis}(1 - B_{revis}) \right.$$
$$\left. - a_5 B_{revis} \left(\frac{N_j}{N_{req}} \right) (N_j < N_{req}) - a_6 \frac{\tau_j}{\text{vis}_j} \right] / (1 - B_{keepout})$$

Local Optimality of Decision Modeling

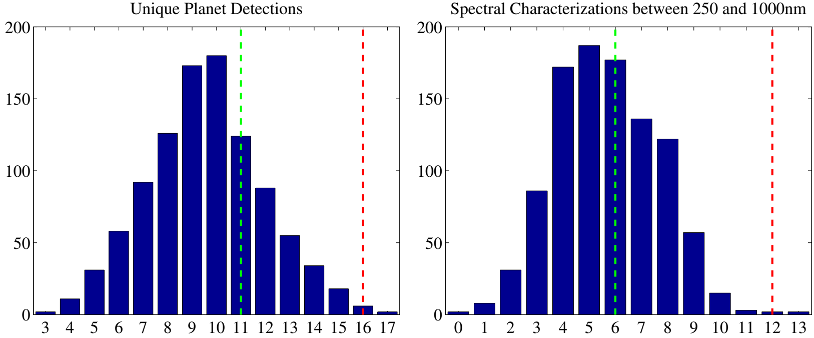
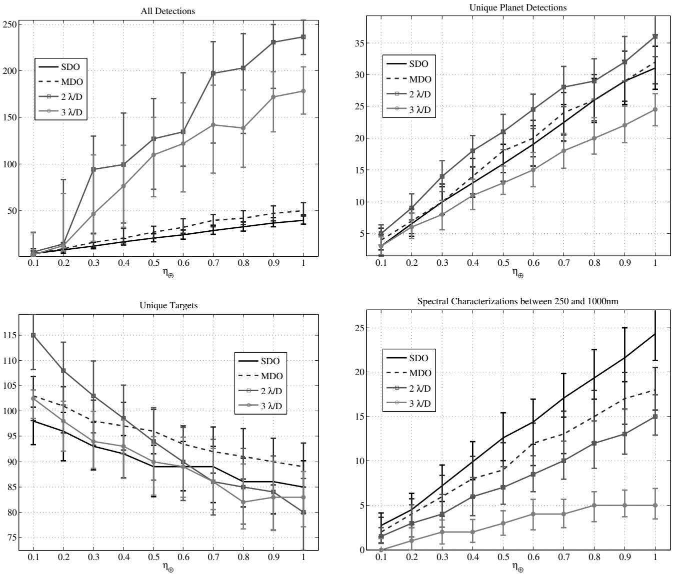


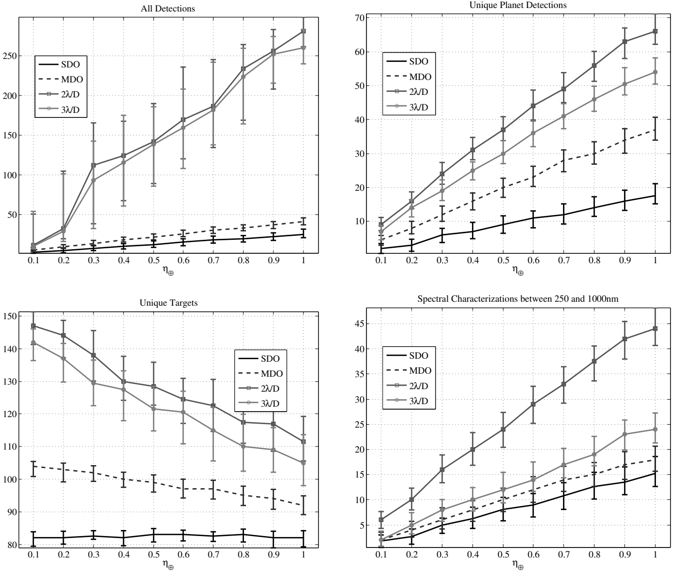
Figure : Comparison of scientific yield from automated visit order selection and randomized visit order. The blue bars are histograms of results from 1000 mission simulations using randomized visit order. Red dashed lines are results from the automated visit order. Green dashed lines are results obtained by always going to the next highest completeness target.

Mission Analysis



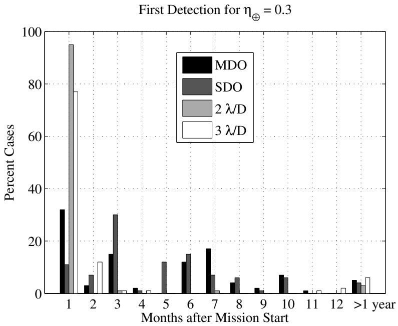
4 m Telescope

Mission Analysis

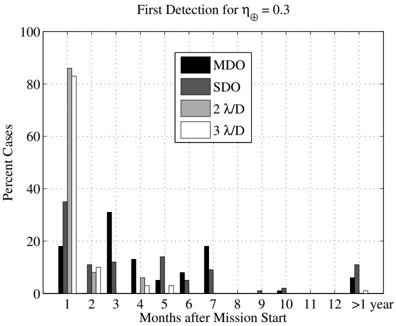


8 m Telescope

Mission Analysis



4 m Telescope



8 m Telescope

Mission Analysis

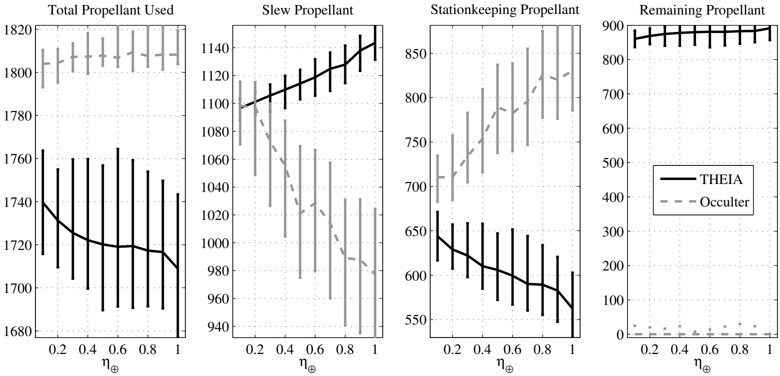
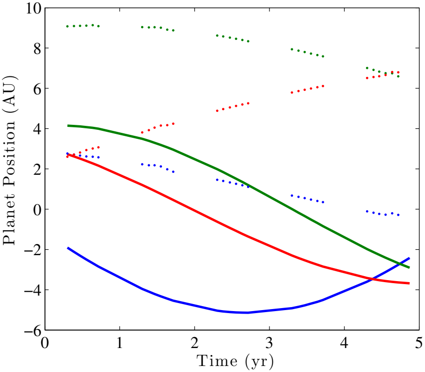


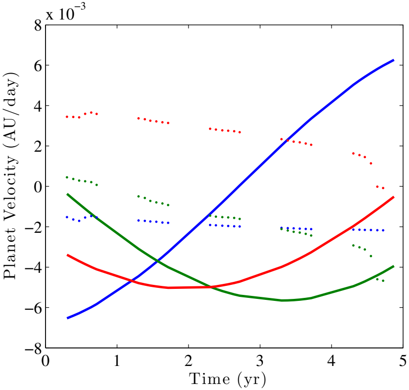
Figure : Spacecraft propellant use (in kg).

State Estimation

- Any exosystem measurement is a partial observation of an underlying dynamical system
$$\mathbf{z}_k = \boldsymbol{h}_k(\mathbf{x}_k, k) + \boldsymbol{n}$$
- The system evolves via known physical laws
$$\mathbf{x}_{k+1} = \boldsymbol{f}(\mathbf{x}_k, k)$$
- State estimation seeks to reconstruct the full state
$$\hat{\mathbf{x}}_k = \arg\min E\{\|\mathbf{x}_k - \hat{\mathbf{x}}_k\|^2\}$$
- Can be formulated as a recursive two-step process:
$$p(\mathbf{x}_k|\mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_k|\mathbf{x}_{k-1})p(\mathbf{x}_{k-1}|\mathbf{z}_{1:k-1})d\mathbf{x}_{k-1}$$
$$p(\mathbf{x}_k|\mathbf{z}_{1:k}) = \frac{p(\mathbf{z}_k|\mathbf{x}_k)p(\mathbf{x}_k|\mathbf{z}_{1:k-1})}{p(\mathbf{z}_k|\mathbf{z}_{1:k-1})}$$

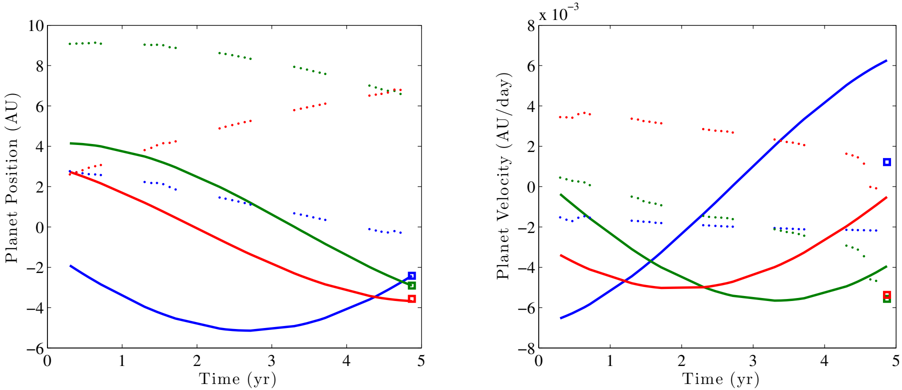
Combining Data Sets





True state values (solid) and filter estimates (dotted) for x (blue) y (green) and z (red) components of position and velocity.

Combining Data Sets



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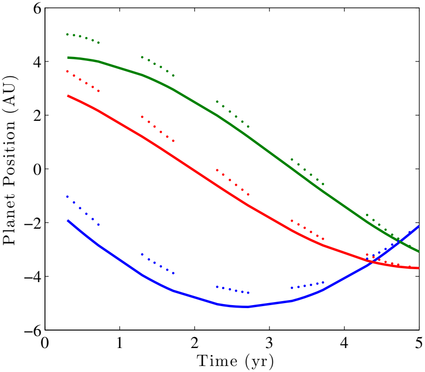

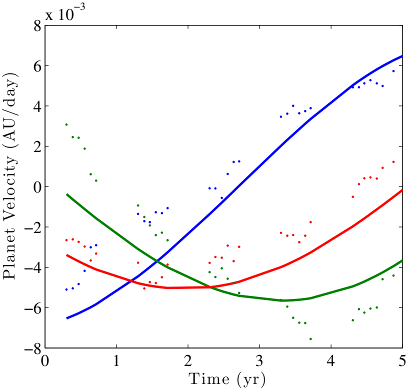

True state values (solid) and filter estimates (dotted) for x (blue) y (green) and z (red) components of position and velocity.

Conclusions

- Detailed exosystem and instrument modeling is very important to all aspects of planet-finding
  - Mission design
  - Survey design
  - Data analysis
- Statistical analyses can provide powerful tools for planning and data processing
- Optimal estimation techniques are highly applicable to this area

Thanks!

Many, many thanks to:

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Ed Turner	**Stephanie Goldfarb**	**Jessica O'Leary**
Michael Littman	**Don Lindler**	**Jill Ray**

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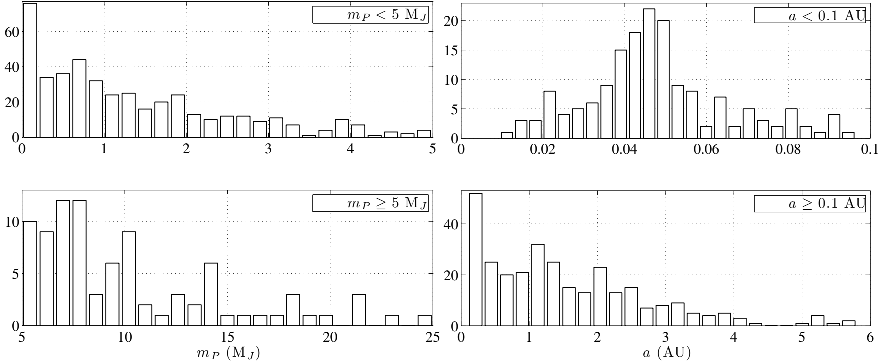
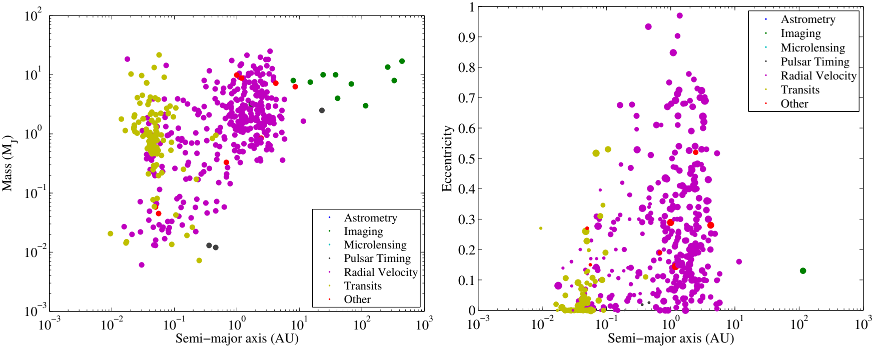
Current Exoplanet Statistics



Data from `http://nsted.ipac.caltech.edu/`. Retrieved 06/01/2011.

Current Exoplanet Statistics



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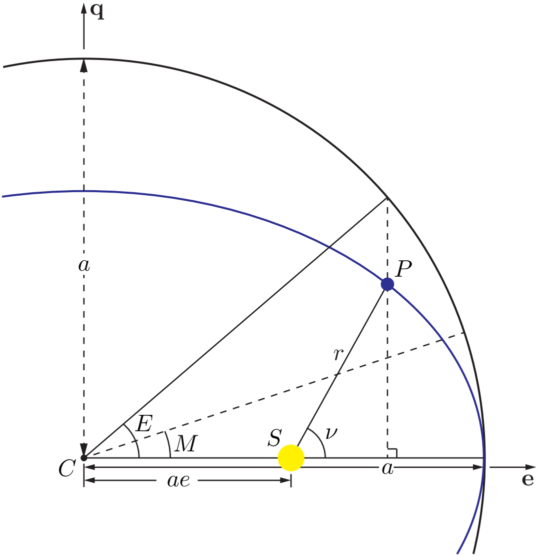
Keplerian Orbits



Starlight Suppression

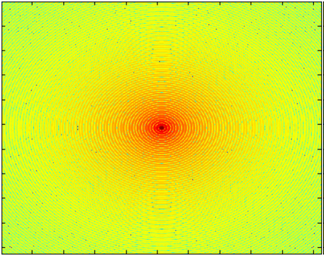


Figure : Open aperture PSF.

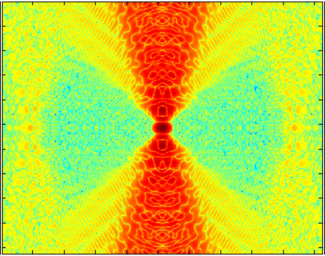


Figure : Pupil mask PSF.

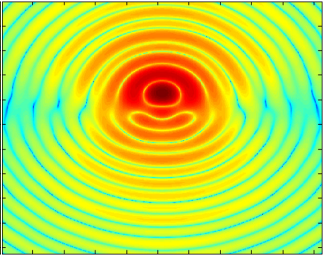
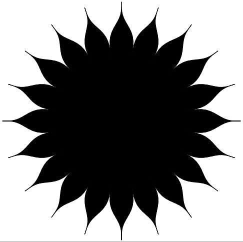
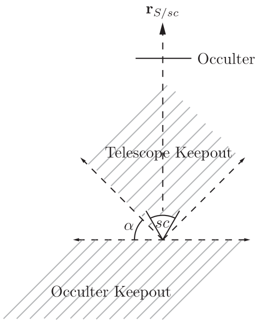


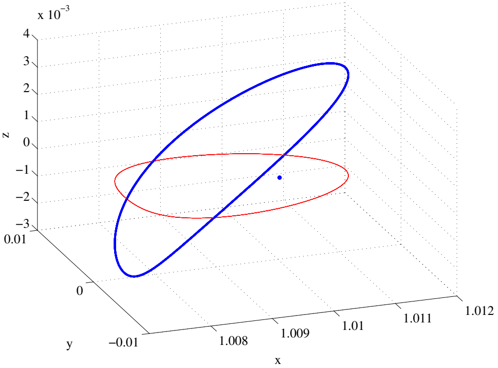
Figure : Apodized Pupil PSF.

Two basic options: internal or external suppression system (with lots of variations).



Imaging Constraints

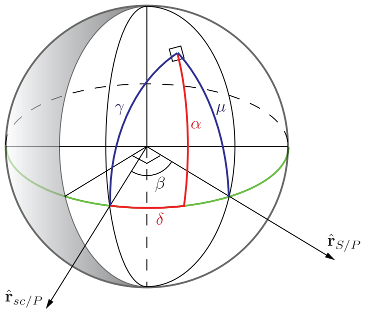




Star observability is determined by spacecraft orbit and time of year.

Derivation of Lambert Phase Function

[Sobolev, 1975]



Let (α, δ) be the planetary latitude and longitude, μ the angle of incident radiation, γ the angle of emergent radiation at a surface point, and ξ the azimuthal angle between them:

$$\begin{aligned}
\cos\mu &= \cos\alpha\cos(\beta-\delta) \triangleq C_\mu \\
\cos\gamma &= \cos\alpha\cos\delta \triangleq C_\gamma \\
\cos\beta &= \cos\mu\cos\gamma - \sin\mu\sin\gamma\cos\xi
\end{aligned}$$

The emergent intensity is given by $I = C_\gamma F\rho(C_\mu, C_\gamma, \xi)$ where $\pi C_\gamma F$ is the flux on a patch of the surface and ρ is the reflection coefficient.

The energy crossing a surface patch into unit solid angle is $C_\gamma C_\mu F\rho dA$ where $dA = R_p^2\cos\alpha d\alpha d\delta$.

Derivation of Lambert Phase Function (contd.)

[Sobolev, 1975]

So, the energy per second per unit area per unit solid angle received by an observer is:

$$E(\beta) = \frac{FR_p^2}{d^2}\int_{\beta-\pi/2}^{\pi/2}\cos(\beta-\delta)\cos\delta d\delta\int_{-\pi/2}^{\pi/2}\rho(C_\mu, C_\gamma, \xi)\cos^3\alpha d\alpha$$

The phase function is defined as the ratio of $E(\beta)/E(0)$. For isotropic scattering, ρ is constant so:

$$\phi_L(\beta) = \frac{\int_{\beta-\pi/2}^{\pi/2}\cos(\beta-\delta)\cos\delta d\delta}{\int_{-\pi/2}^{\pi/2}\cos(-\delta)\cos\delta d\delta} = \frac{\sin(\beta)+(\pi-\beta)\cos(\beta)}{\pi}$$

Imaging

[Kasdin and Braems, 2006, Savransky et al., 2010]

- Model observation as:

$$\mathbf{z}(x,y) = C_p\bar{P}(x-\xi, y-\eta) + \boldsymbol{n}$$

where C_p is the mean photon count at planet location - pixel (ξ,η), $\bar{P}$ is the normalized PSF, and $\boldsymbol{n}$ is the noise.

- Pixel location maps to on-sky separation as:

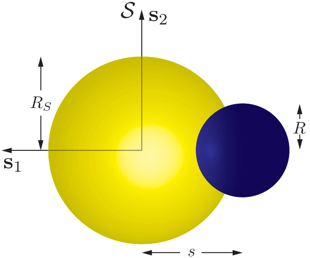
$$\mathbf{s} = {}^{\mathcal{I}}C^{\mathcal{S}} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \frac{\sqrt{\Delta\alpha}}{f\varpi}$$

where $\Delta\alpha$ is the physical pixel size, f is the focal length, and ${}^{\mathcal{I}}C^{\mathcal{S}}$ rotates the sky frame into the inertial frame.

- $C_p \propto F_P$, with constant depending on camera characteristics.

Transit Photometry

[Mandel and Agol, 2002]



$$\frac{F^{(e)}}{F} = \begin{cases} 1 & R_S + R < s \\ 1 - \frac{1}{\pi}\left[\frac{R^2}{R_S^2}\kappa_0 + \kappa_1 - \sqrt{\frac{s^2}{R_S^2} - \frac{\left(R_S^2+s^2-R^2\right)^2}{4R_S^4}}\right] & |R_S - R| < s \leq R_S + R \\ 1 - \left(\frac{R}{R_S}\right)^2 & s \leq R_S - R \\ 0 & s \leq R - R_S \end{cases}$$

$$\kappa_0 = \cos^{-1}\left(\frac{R^2 + s^2 - R_S^2}{2Rs}\right) \qquad \kappa_1 = \cos^{-1}\left(\frac{R_S^2 - R^2 + s^2}{2R_S s}\right)$$

Doppler Spectroscopy

[Marcy and Butler, 1992, Butler et al., 1996]

- The observed spectrum as a function of wavelength:

$$I_{obs}(\lambda) = \kappa \left[I_S(\lambda + \Delta\lambda_S) T_C(\lambda + \Delta\lambda_C) \right] \otimes \mathrm{PSF}$$

  where I_S is the stellar spectrum and T_C is the transmission function of the absorption cell.

- After deconvolution and fitting, relate to parameter set via relativistic Doppler equation:

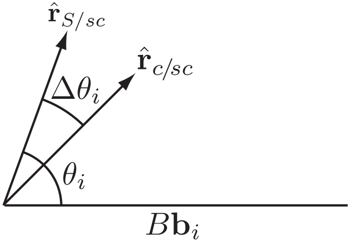
$$\frac{\Delta\lambda_S - \Delta\lambda_C}{\lambda} = \frac{\left(1 + \left(\frac{v}{c}\right)^2 \cos\theta\right)(1 + \rho_g)}{n\sqrt{1 - \left(\frac{v}{c}\right)^2}} - 1$$

  where $v = \|\dot{\mathbf{r}}_{S/sc}\|$ and $\cos\theta = \frac{\mathbf{r}_{S/sc}}{\|\mathbf{r}_{S/sc}\|} \cdot \frac{\dot{\mathbf{r}}_{S/sc}}{\|\dot{\mathbf{r}}_{S/sc}\|}$

Interferometric Astrometry

[Konacki et al., 2002, Savransky and Kasdin, 2010]

Narrow-angle astrometric measurement with interferometer baseline B:



$$\begin{aligned} d_i &= B\left(\cos\theta_i - \cos(\theta_i - \Delta\theta_i)\right) \\ &= B\left(\cos\theta_i(1-\cos\Delta\theta_i) - \sin\theta_i\sin\Delta\theta_i\right) \end{aligned}$$

$$\mathbf{d} = B\begin{bmatrix} \mathbf{b}_1 \cdot \left(\hat{\mathbf{r}}_{S/sc} - \hat{\mathbf{r}}_{c/sc}\right) \\ \mathbf{b}_2 \cdot \left(\hat{\mathbf{r}}_{S/sc} - \hat{\mathbf{r}}_{c/sc}\right) \end{bmatrix} + \mathbf{n}$$

$$\begin{aligned} \hat{\mathbf{r}}_{S/sc} = \frac{\mathbf{r}_{S/sc}}{\|\mathbf{r}_{S/sc}\|} = &\left(\mathbf{r}_{S/O}(t_0) + \mathbf{r}_\mu + \Delta\mathbf{r}_{S/G} - \mathbf{r}_{sc/O}\right)\times \\ &\left\{\mathbf{r}_{S/O}(t_0)\cdot\mathbf{r}_{S/O}(t_0) + \mathbf{r}_\mu\cdot\mathbf{r}_\mu + \Delta\mathbf{r}_{S/G}\cdot\Delta\mathbf{r}_{S/G} + \mathbf{r}_{sc/O}\cdot\mathbf{r}_{sc} \right.\\ &+ 2\mathbf{r}_{S/O}(t_0)\cdot\mathbf{r}_\mu + 2\mathbf{r}_{S/O}(t_0)\cdot\Delta\mathbf{r}_{S/G} - 2\mathbf{r}_{S/O}(t_0)\cdot\mathbf{r}_{sc/O} \\ &\left. + 2\mathbf{r}_\mu\cdot\Delta\mathbf{r}_{S/G} - 2\mathbf{r}_\mu\cdot\mathbf{r}_{sc/O} - 2\Delta\mathbf{r}_{S/G}\cdot\mathbf{r}_{sc/O}\right\}^{-\frac{1}{2}} \end{aligned}$$

Derivation of Δmag Extrema

[Brown, 2004]

The relative magnitude assuming a Lambert phase function is:

$$\Delta\text{mag} = -2.5\log\left(p\left(\frac{R}{r}\right)^2\Phi(\beta)\right), \Phi = \frac{\sin(\beta)+(\pi-\beta)\cos(\beta)}{\pi},$$

where the phase angle is given by $\beta = \sin^{-1}\left(\frac{s}{r}\right)$. So, for constant p, R and s, the extrema of Δmag occurs when:

$$\frac{\sin(\beta)}{\pi}\left[2\sin(\beta)\cos(\beta)+(\pi-\beta)\left(2\cos^2(\beta)-\sin^2(\beta)\right)\right] = 0$$

The maximum occurs when $\beta = 1.10473$, so the minimum Δmag as a function of s is given by

$$\Delta\text{mag}_{min} = -2.5\log\left(0.459455\frac{p_{max}R_{max}^2}{s^2}\right)$$

Derivation of Δmag Extrema

The maximum value of Δmag for a fixed s occurs when the planet is in front of the star, such that

$$\beta = \pi - \sin^{-1}\left(\frac{s}{r_{max}}\right)$$

where $r_{max} = a_{max}(1 + e_{max})$.
Thus

$$\Delta\text{mag}_{max} = -2.5\log\left(p_{min}R_{min}^2 \frac{s - \sqrt{r_{max}^2 - s^2}\sin^{-1}\left(\frac{s}{r_{max}}\right)}{\pi r_{max}^3}\right)$$

The Distribution of ν

- Let $\nu = g(M, e)$ and $M = h(\nu, e)$ with $f_{\bar{e}}$ the probability density function of $\bar{e}$. The cumulative distribution function of $\bar{\nu}$ is:

$$F_{\bar{\nu}}(\nu) = P(g(\bar{M}, \bar{e}) \leq \nu) = \int_{-\infty}^{\infty} P(g(\bar{M}, e) \leq \nu | \bar{e} = e) f_{\bar{e}}(e) \, \mathrm{d}e$$

- $\bar{M}$ and $\bar{e}$ are independent and $e \in [0, 1]$ so:

$$F_{\bar{\nu}}(\nu) = \int_0^1 P(\bar{M} \leq h(\nu, e)) f_{\bar{e}}(e) \, \mathrm{d}e$$

Probability Density Function of ν

$$f_{\bar{\nu}}(\nu) = \frac{\mathrm{d}}{\mathrm{d}\nu} F_{\bar{\nu}}(\nu) = \frac{1}{2\pi} \int_0^1 \frac{\partial h}{\partial \nu} f_{\bar{e}}(e) \, \mathrm{d}e = \frac{1}{2\pi} \int_0^1 \frac{\left(1 - e^2\right)^{\frac{3}{2}}}{\left(1 + e \cos \nu\right)^2} f_{\bar{e}}(e) \, \mathrm{d}e$$

The Distribution of Apparent Separation

- $s \approx r \sin \beta$
- Let $\bar{l} = \sin \bar{\beta}$

$$f_{\bar{l}}(l) = f_{\bar{\beta}}(\sin^{-1}(l)) \left| \frac{\mathrm{d}}{\mathrm{d}l} \sin^{-1}(l) \right| = \begin{cases} \frac{l}{\sqrt{1-l^2}} & l \in [0,1) \\ 0 & \text{else} \end{cases}$$

- Since $\bar{s} = \bar{r}\bar{l}$

$$f_{\bar{s}}(s) = \int_{-\infty}^{\infty} \frac{1}{l} f_{\bar{r}}\left(\frac{s}{l}\right) f_{\bar{l}}(l) \, \mathrm{d}l$$

Probability Density Function of s

$$f_{\bar{s}}(s) = \frac{1}{\pi} \int_0^1 \int_0^\infty \int_0^1 \frac{s}{a\sqrt{(1-l^2)\left[(ael)^2 - (al-s)^2\right]}} f_{\bar{e}}(e) \, \mathrm{d}e \, f_{\bar{a}}(a) \, \mathrm{d}a \, \mathrm{d}l$$

Distributions of Keplerian Orbital Elements

[Savransky et al., 2011]

$$f_{\bar{\nu}}(\nu) = \frac{1}{2\pi}\int_0^1 \frac{\left(1-e^2\right)^{\frac{3}{2}}}{\left(1+e\cos\nu\right)^2} f_{\bar{e}}(e)\,\mathrm{d}e$$

$$f_{\bar{r}}(r) = \frac{1}{\pi}\int_0^\infty\int_0^1 \frac{r}{a\sqrt{(ae)^2-(a-r)^2}} f_{\bar{e}}(e)\,\mathrm{d}e\, f_{\bar{a}}(a)\,\mathrm{d}a$$

$$f_{\bar{s}}(s) = \frac{1}{\pi}\int_0^1\int_0^\infty\int_0^1 \frac{s}{a\sqrt{(1-l^2)\left[(ael)^2-(al-s)^2\right]}} f_{\bar{e}}(e)\,\mathrm{d}e\, f_{\bar{a}}(a)\,\mathrm{d}a\,\mathrm{d}l$$

$$f_{\bar{F}_R}(F_R) = \int\limits_{-\infty}^{\infty} \frac{f_{\bar{n}}(n)}{npR^2}\cos\left(\sum_{k=1}^{\infty} b_k\left(\frac{F_R}{npR^2}-\frac{1}{\pi}\right)^k\right)\left|\sum_{k=1}^{\infty} k b_k\left(\frac{F_R}{npR^2}-\frac{1}{\pi}\right)^{k-1}\right|\mathrm{d}n$$

$$b_k = \frac{1}{k\alpha_1^k}\sum_{x\in X}(-1)^{|x|}\prod_{r=1}^{|x|}(k-1+r)\prod_{i=1}^{k-1}\frac{\left(\alpha_{i+1}/\alpha_1\right)^{x_i}}{x_i!}$$

Transit Probability

- Transits occur when
$$s < R_S + R$$
so probability of transit is
$$P(s < R_S + R) = \int_0^{R_S+R} f_{\bar{s}}(s)\,\mathrm{d}s$$
- Assuming a specific observing cadence, occurrence of transits is modeled as a Poisson process. In each time interval Δt, the probability of transits is
$$P[N_{\text{transits}}(\Delta t) > 0] = 1 - e^{-\lambda \Delta t}$$
- To capture all transits of orbits with semi-major axis a_0 in one orbital period
$$\Delta t \leq \frac{1}{\pi}\frac{R_s + R}{a_0}$$

Transit Detection

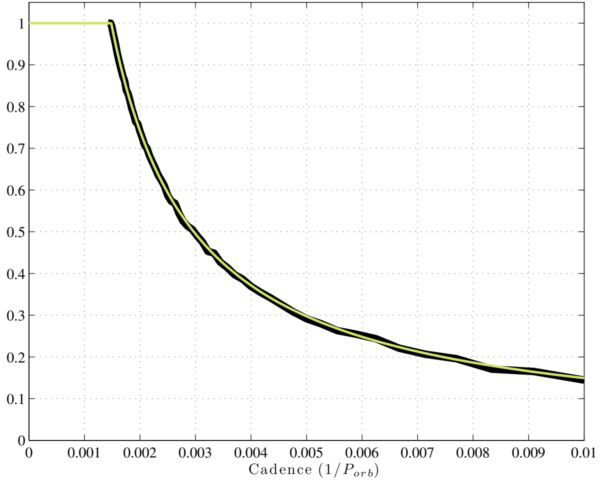


Figure : Portion of observable transits detected over one orbital period for $a_0 = 1, R_S = R_\odot$ and $R = 0$ as a function of observation cadence using Monte Carlo (black) and algebraic solution (green).

Dynamic Filtering (The Optimal Estimation Way)

- Measurements ($\mathbf{z}$) at time k are a function of a state vector ($\mathbf{x}$) describing the positions of all orbiting planets, and time, with added noise $\boldsymbol{n}$ of covariance R:

$$\mathbf{z}_k = \boldsymbol{f}(\mathbf{x}_k, k) + \boldsymbol{n}$$

- The solution to this problem is a minimization with respect to $\mathbf{x}$ for N observations of the cost function:

$$J = \sum_{k=1}^{N} \left[\mathbf{z}_k - \boldsymbol{f}(\mathbf{x}_k, k)\right]^T R^{-1} \left[\mathbf{z}_k - \boldsymbol{f}(\mathbf{x}_k, k)\right]$$

  subject to the constraints of the physical system (i.e. Newtonian dynamics) and any inherent constraints in the formulation of the state (i.e., quaternion definition, eccentricity bounds, etc.).

- Can be re-formulated as a recursive filter

Dynamic Filtering (The Bayesian Way)

- Assume a Markov process with state $\mathbf{x}$ and observation $\mathbf{z}$. Then

$$p(\mathbf{x}_0, \mathbf{x}_1 \cdots \mathbf{x}_n, \mathbf{z}_1, \mathbf{z}_2 \cdots \mathbf{z}_n) = p(\mathbf{x}_0) \prod_{j=1}^{n} p(\mathbf{z}_j | \mathbf{x}_j) p(\mathbf{x}_j | \mathbf{x}_{j-1})$$

- Predict the next state given the observed history

$$p(\mathbf{x}_j | \mathbf{z}_{1:k-1}) = \int p(\mathbf{x}_j | \mathbf{x}_{j-1}) p(\mathbf{x}_{j-1} | \mathbf{z}_{1:j-1}) d\mathbf{x}_{j-1}$$

- Update the state estimate given the current observation

$$p(\mathbf{x}_j | \mathbf{z}_{1:j}) = \frac{p(\mathbf{z}_j | \mathbf{x}_j) p(\mathbf{x}_j | \mathbf{z}_{1:j-1})}{p(\mathbf{z}_j | \mathbf{z}_{1:j-1})}$$

Extended Kalman Filter

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), t) \qquad \dot{\mathbf{P}}(t) = \mathbf{F}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(t) + \mathbf{Q}$$
$$\hat{\mathbf{x}}_0 = E[\mathbf{x}(0)] \qquad \mathbf{P}_0 = E[(\mathbf{x}(0) - \hat{\mathbf{x}}_0)(\mathbf{x}(0) - \hat{\mathbf{x}}_0)^T]$$
$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t)} \qquad \mathbf{Q}(t) = E[\mathbf{w}(t)\mathbf{w}^T(\tau)]$$

$$\hat{\mathbf{x}}^+_{k_i} = \hat{\mathbf{x}}^-_k + \mathbf{K}_{k_i}\left(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}^+_{k_{i-1}}) - \mathbf{H}_{k_i}(\hat{\mathbf{x}}^-_k - \hat{\mathbf{x}}^+_{k_{i-1}})\right) \qquad \hat{\mathbf{x}}^+_{k_0} = \hat{\mathbf{x}}^-_k$$
$$\mathbf{K}_{k_i} = \mathbf{P}^-_k\mathbf{H}^T_{k_i}\left(\mathbf{H}_{k_i}\mathbf{P}^-_k\mathbf{H}^T_{k_i} + \mathbf{R}_k\right)^{-1} \qquad \mathbf{H}_{k_i} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}^+_{k_i}}$$
$$\mathbf{P}^+_{k_i} = (\mathbf{I} - \mathbf{K}_{k_i}\mathbf{H}_{k_i})\,\mathbf{P}^-_k \qquad \mathbf{R}(t) = E[\mathbf{v}(t)\mathbf{v}^T(\tau)]$$

Various enhancements include:

- Simultaneous integration of state and covariance
- Iteration on nonlinear observer functions
- Introduction of inequality state constraints

EKF Modifications

- Position and Velocity state makes it easy to describe open orbits
  - Introduce inequality constraints of the form $D\bar{X} \leq d$ to constrain orbital specific energy
  - At each time step, solve quadratic programming problem of the form $\min_{\tilde{x}} \left(\tilde{x}^T W \tilde{x} - 2\bar{X}^T W \tilde{x}\right)$ s.t. $D\tilde{x} \leq d$ [Simon and Simon, 2006]
- Nonlinearities in state propagation make filter very sensitive to initial conditions
  - Attempt to constraint initial conditions via periodograms and other coarse analysis of data
  - Introduce random restarts when state or covariance diverges
- Covariance estimate extrapolation is potential source of problems for nonlinear state update
  - Evaluated particle filter-like approach
  - Generate set of random states distributed according to current covariance estimate with mean of current state
  - Propagate random states and find covariance

Priming Initial State with Update Step

- We can use our initial measurement to improve the initialization step
- For example, use formalism of the unscented transform to define the initial state:

$$\hat{\mathbf{x}}_0 = XWY^T\left(YWY^T + R\right)^{-1}(\mathbf{z}_0 - Yw)$$

where

$$X = c[\mathbf{0} \quad A \quad -A]$$

$$Y = \boldsymbol{h}(\mathbf{x}, 0)$$

and c, w and W are matrices of weights determined by the expected noise, R is the covariance of $\boldsymbol{n}$ and A is the Cholesky decomposition of the expected state covariance.