Functions

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FUNCTIONS [60 Marks]

1 The function f is defined by

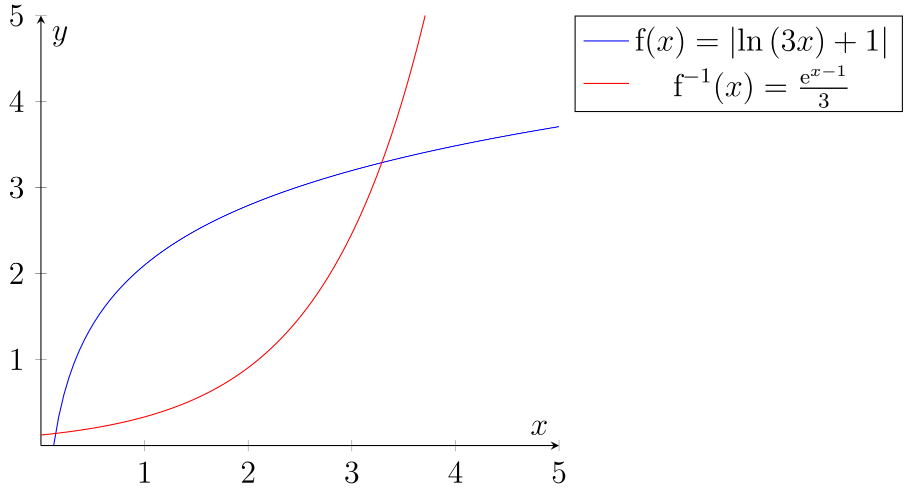
$$\mathrm{f} : x \mapsto |\ln(3x) + 1|\,, x > a.$$

(a) Find the smallest value of a such that f^{-1} exists. [1]

Solving $\mathrm{f}(x) = 0$, we get $x = \dfrac{1}{3\mathrm{e}}$. Thus, the smallest value of a, $a_{\text{min}} = \dfrac{1}{3\mathrm{e}}$

(b) Using the domain of f found in **(a)**, sketch the graph of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram. [4]

To get $\mathrm{f}^{-1}(x)$, we make y the subject of the equation $x = |\ln(3y) + 1|$ so we get $\mathrm{f}^{-1}(x) = \dfrac{\mathrm{e}^{x-1}}{3}$



2 The function f is defined by

$$\mathrm{f} : x \mapsto \sqrt{a^2 - x^2} + k,\ 0 < x < a.$$

(a) Given that f is *self-inverse*, meaning that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, state the conditions that must be fulfilled by a and k. [2]

Let us make x the subject of the equation $y = \sqrt{a^2 - x^2} + k$

$$\begin{aligned} y &= \sqrt{a^2 - x^2} + k \\ (y-k)^2 &= a^2 - x^2 \\ x &= \sqrt{a^2 - (y-k)^2} \end{aligned}$$

It is evident that the equation is *self-inverse* for $a \in \mathbb{R}^+, k = 0$

(b) Hence, find the exact value of x such that $\mathrm{f}^{-1}\mathrm{f}^{-1}(x) = \mathrm{fff}(x)$. [3]

Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\mathrm{f}^{-1}\mathrm{f}^{-1}(x) = \mathrm{fff}(x)$ can be found by solving $x = \mathrm{f}x$

$$\begin{aligned} x &= \sqrt{a^2 - x^2} \\ \implies x &= \sqrt{\frac{a^2}{2}} \end{aligned}$$

3 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{1}{2}\sqrt{4 - x^2},\ 0 \le x \le 2.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state the domain and range of f^{-1}. [2]

To find $\mathrm{f}^{-1}(x)$, let us make x the subject of $y = \frac{1}{2}\sqrt{4 - x^2}$

$$
\begin{aligned}
y &= \frac{1}{2}\sqrt{4 - x^2} \\
\implies 4y^2 &= 4 - x^2 \\
\implies x &= 2\sqrt{1 - y^2}
\end{aligned}
$$

Thus, $\mathrm{f}^{-1}(x) = 2\sqrt{1 - x^2}$. Further, we know that $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = [0, 1], \mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_{\mathrm{f}} = [0, 2]$

(b) State the domains and ranges of ff^{-1} and $\mathrm{f}^{-1}\mathrm{f}$. [2]

$\mathrm{D}_{\mathrm{ff}^{-1}} = \mathrm{D}_{\mathrm{f}^{-1}} = [0, 1]$, $\mathrm{R}_{\mathrm{ff}^{-1}} = [0, 1]$

$\mathrm{D}_{\mathrm{f}^{-1}\mathrm{f}} = \mathrm{D}_{\mathrm{f}} = [0, 2]$, $\mathrm{R}_{\mathrm{f}^{-1}\mathrm{f}} = [0, 2]$

(c) State the set of values of x for which $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [1]

The set of values is $[0, 1] \cap [0, 2] = [0, 1]$

(d) Find the exact solution of $\mathrm{ff}^{-1}\mathrm{f}(x) = \frac{1}{2}$. [2]

Since $\mathrm{f}^{-1}\mathrm{f}(x) = x$, we can solve for x in $\mathrm{f}(x) = \frac{1}{2}$ which is equivalent to $\mathrm{f}^{-1}(\frac{1}{2}) = \sqrt{3}$

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{1 - x}, \ x \in \mathbb{R}$$

(a) Show that $\mathrm{fff}(x) = x$. [2]

$$
\begin{aligned}
\mathrm{fff}(x) &= \mathrm{ff}\left(\frac{1}{1-x}\right) \\
&= \mathrm{f}\left(\frac{1}{1-\frac{1}{1-x}}\right) \\
&= \mathrm{f}\left(\frac{1-x}{1-x-1}\right) \\
&= \mathrm{f}\left(-\frac{1}{x}+1\right) \\
&= -\frac{1}{\left(\frac{1}{1-x}\right)}+1 \\
&= x \text{ (shown)}
\end{aligned}
$$

(b) Show that $\mathrm{f}^{-1}\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

To find $\mathrm{f}^{-1}(x)$, let us make x the subject of $y = \frac{1}{1-x}$

$$
\begin{aligned}
y &= \frac{1}{1-x} \\
1-x &= \frac{1}{y} \\
x &= 1-\frac{1}{y}
\end{aligned}
$$

Therefore, $\mathrm{f}^{-1}(x) = 1 - \frac{1}{x}$.

We now simplify the composite function $\mathrm{f}^{-1}\mathrm{f}^{-1}(x)$.

$$\begin{aligned}
\mathrm{f}^{-1}\mathrm{f}^{-1}(x) &= \mathrm{f}^{-1}(1-\frac{1}{x}) \\
&= 1-\frac{1}{1-\frac{1}{x}} \\
&= 1-\frac{x}{x-1} \\
&= \frac{x-1-x}{x-1} \\
&= \frac{1}{1-x} \\
&= \mathrm{f}(x) \text{ (shown)}
\end{aligned}$$

(c) Hence, find the exact value(s) of a such that $\mathrm{f}(\mathrm{fff}(a)+\mathrm{ff}(a)+\mathrm{f}(a)) = \mathrm{f}^{-1}(a)$. [4]

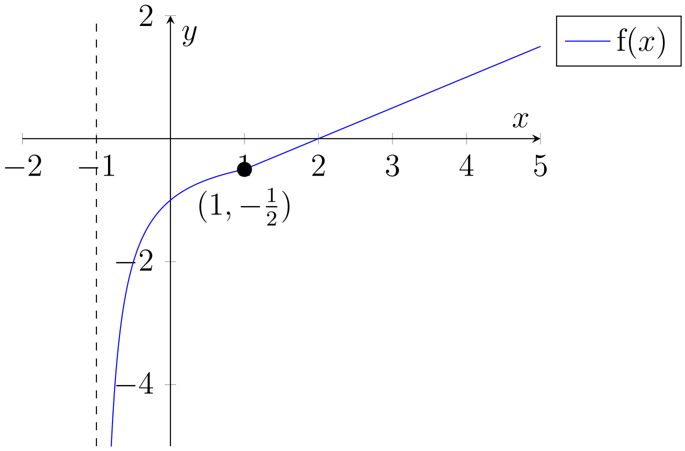
We will apply f^{-1} on both sides so the RHS of the equation becomes $\mathrm{f}^{-1}\mathrm{f}^{-1}(a)$ to utilise our result in **(b)**. We then simplify the composite functions fff and ff using our working in **(a)**.

$$\begin{aligned}
\mathrm{f}(\mathrm{fff}(a)+\mathrm{ff}(a)+\mathrm{f}(a)) &= \mathrm{f}^{-1}(a) \\
\mathrm{fff}(a)+\mathrm{ff}(a)+\mathrm{f}(a) &= \mathrm{f}^{-1}\mathrm{f}^{-1}(a) \\
a-\frac{1}{a}+\mathrm{f}(a) &= \mathrm{f}(a) \\
a^2-1 &= 0 \\
\therefore a &= \pm 1
\end{aligned}$$

5 The function f is defined by

$$\mathrm{f}(x) = \begin{cases} -\dfrac{1}{1+x} & \text{for } -1<x<1, \\ \frac{1}{2}x-1 & \text{for } x \geq 1. \end{cases}$$

(a) Sketch the graph of f. Hence, show that f is one-one. [3]



The graph of $y = a$ cuts $\mathrm{f}(x)$ at one and only one point $\forall a \in \mathbb{R} \implies$ the function is one-one.

(b) Find f^{-1}. [2]

Making x the subject for both $y = -\dfrac{1}{1+x}$ and $y = \dfrac{1}{2}x - 1$:

$$\implies x = -\frac{1}{y} - 1 \quad \text{and} \quad x = 2(y+1)$$

$$\therefore \mathrm{f}^{-1}(x) = \begin{cases} -\frac{1}{x} - 1 & \text{for } x < -\frac{1}{2}, \\ 2(x+1) & \text{for } x \geq -\frac{1}{2}. \end{cases}$$

6 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2+1}{x-1}, \; x > a.$$

(a) Given that f is one-one, state the exact value of a. [2]

By plotting the result on a G.C., we can see that a must be the x-coordinate of the minimum point of the curve. To obtain stationary points, we have to differentiate f(x) with respect to x.

$$\begin{aligned}
\mathrm{f}(x) &= x + 1 + \frac{2}{x-1} \\
\mathrm{f}'(x) &= 1 - \frac{2}{(x-1)^2} = 0 \text{ for stationary points} \\
&\implies (x-1)^2 = 2 \\
x &= 1 \pm \sqrt{2}
\end{aligned}$$

From the G.C., we can observe that the x-coordinate of the minimum point is greater than that of the maximum point. Therefore, $a = 1 + \sqrt{2}$.

(b) Find $\mathrm{f}^{-1}(x)$ and state the domain and range of f^{-1}. [5]

To find $\mathrm{f}^{-1}(x)$, let us make x the subject of $y = \frac{x^2+1}{x-1}$.

$$\begin{aligned}
y &= \frac{x^2+1}{x-1} \\
y(x-1) &= x^2 + 1 \\
x^2 - yx + y + 1 &= 0 \\
x &= \frac{y \pm \sqrt{y^2 - 4y - 4}}{2}
\end{aligned}$$

To eliminate the $\pm$, we must find the range of values of y, or $\mathrm{R_f}$. From the G.C. we know that the y-coordinate of the minimum point is the lower limit of $\mathrm{R_f}$.

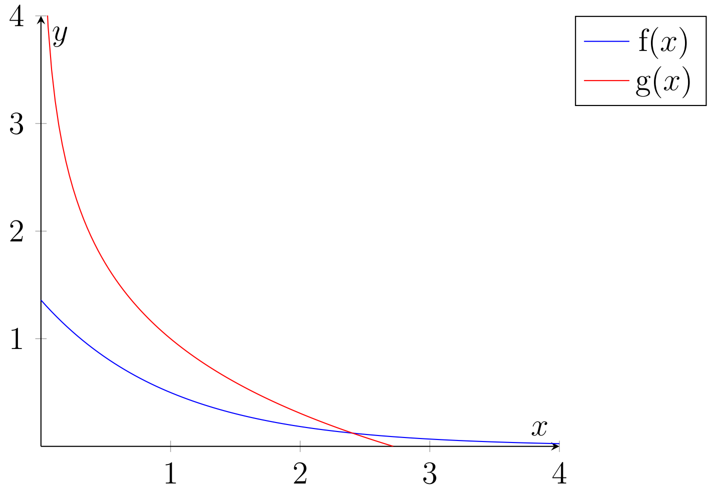
$$\begin{aligned}
\text{Substituting } x = 1 + \sqrt{2},\ y &= \frac{(1+\sqrt{2})^2 + 1}{1 + \sqrt{2} - 1} \\
&= \frac{(4 + 2\sqrt{2})}{\sqrt{2}} \\
&= 2\sqrt{2} + 2
\end{aligned}$$

Since $y > 2\sqrt{2} + 2$, $\sqrt{y^2 - 4y - 4} > \sqrt{(2\sqrt{2}+2)^2 - 4(2\sqrt{2}+2) - 4} = 0$.
It follows that $\frac{y - \sqrt{y^2-4y-4}}{2} < 1 + \sqrt{2}$.
However, $x \in (1 + \sqrt{2}, \infty)$, so we can only take $\frac{y + \sqrt{y^2-4y-4}}{2}$.
$\therefore\ \mathrm{f}^{-1}(x) = \frac{x + \sqrt{x^2-4x-4}}{2}$.
As found earlier, $\mathrm{D_{f^{-1}}} = \mathrm{R_f} = (2 + 2\sqrt{2}, \infty)$ and $\mathrm{R_{f^{-1}}} = \mathrm{D_f} = (1 + \sqrt{2}, \infty)$

7 Functions f and g are defined as follows:

$$f(x) = \frac{1}{2}e^{1-x}, \ x \geq 0,$$
$$g(x) = 1 - \ln(x), \ 0 < x \leq e.$$

(a) Show that both fg and gf exist. [3]



$R_g = \mathbb{R}_0^+ = D_f \implies$ fg exists

$R_f = (0, \frac{1}{2}e] \subset (0, e] = D_g \implies$ gf exists

(b) By finding expressions for fg and gf, find the exact solution of $fg(2) = gf(\ln(x))$. [3]

$fg(x) = \frac{1}{2}e^{1-(1-\ln x)} = \frac{1}{2}x$

$gf(x) = 1 - \ln\left(\frac{1}{2}e^{1-x}\right) = 1 - \ln\left(\frac{1}{2} \cdot e \cdot e^{-x}\right) = 1 - (-\ln 2 + 1 - x) = x + \ln 2$

$$gf(\ln x) = \ln 2x$$
$$= fg(2) = 1$$
$$\implies x = \frac{e}{2}$$

8 The functions f and g are defined by:

$$\mathrm{f}(x) = \begin{cases} 2\sqrt{x-2} & \text{for } 2 \le x < 6, \\ 6 - \sqrt{\frac{2x}{3}} & \text{for } x \ge 6. \end{cases} \quad \text{and} \quad \mathrm{g}(x) = x^2,\ x \in \mathbb{R}.$$

(a) Show that gf exists. [2]

Notice that the maximum value of (x) is equal to 4 when $x = 6$.

$\mathrm{R_f} = (-\infty, 4] \subset \mathbb{R} = \mathrm{D_g} \implies$ gf exists.

(b) Find the exact value of gf(4). [2]

Since $4 \in [2, 6)$ for which $\mathrm{f}(x) = 2\sqrt{x-2}$,

$$\begin{aligned} \mathrm{gf}(x) &= \mathrm{g}(2\sqrt{x-2}) \\ &= (2\sqrt{x-2})^2 \\ &= 4(x-2) \end{aligned}$$

$\therefore \mathrm{gf}(4) = 4(2) = 8$

(c) Find the exact value of x such that $\mathrm{gf}(x) = 5$. [2]

We have already found $\mathrm{gf}(x)$ in terms of x in the previous answer. All we need to do is to equate this expression to 5.

$$\begin{aligned} \mathrm{gf}(x) = 4(x-2) &= 5 \\ x &= \frac{13}{4} \end{aligned}$$

9 The function h is defined by:

$$\mathrm{h}(n) = \begin{cases} n(\mathrm{h}(n-1)) & \text{for } n > 1, \\ 1 & \text{for } n = 1. \end{cases}$$

where $n \in \mathbb{Z}^+$.

(a) State the values of h(2), h(3) and h(4). [1]

$h(2) = 2(h(1)) = 2, h(3) = 3(h(2)) = 6, h(4) = 4(h(3)) = 24$

(b) Deduce the use of function h. [1]

$h(n) = n!$

(c) Evaluate $\dfrac{\mathrm{h}(40)}{\mathrm{h}(10)\mathrm{h}(30)}$. [2]

Notice that this is just $\binom{40}{30}$.

$$\frac{\mathrm{h}(40)}{\mathrm{h}(10)\mathrm{h}(30)} = \frac{40!}{10!30!} = 847660528$$

10 The function f has an inverse and is such that

$$\mathrm{f} : x^2 + 3 \mapsto x,\ x > 0.$$

(a) Find $\mathrm{f}(x)$, and write down its domain and range. [3]

Making x the subject of $y = x^2 + 3$, we get $x = \pm\sqrt{y-3}$. Since $\mathrm{D}_{\mathrm{f}^{-1}} = \mathbb{R}^+$, $f(x) = \sqrt{x-3}$ and $\mathrm{R}_\mathrm{f} = \mathbb{R}^+$ and $\mathrm{D}_\mathrm{f} = (3, \infty)$

(b) The function g is such that $\mathrm{g}(3x + 2) = \mathrm{f}(x)$.

Find $\mathrm{g}(x)$. State its domain and range. [4]

Given that $\mathrm{g}(3x + 2)$ has exactly the same graph as $\mathrm{f}(x)$, we can manipulate the terms within the brackets.

Subtracting 2 units on both sides, $\mathrm{g}(3x) = \mathrm{f}(x - 2)$.

Dividing by 3 on both sides,

$$\begin{aligned} g(x) &= f\left(\frac{x-2}{3}\right) \\ &= \sqrt{\frac{x-2}{3} - 3} \\ &= \sqrt{\frac{x-11}{3}} \end{aligned}$$

To find the domain of g, we perform the same transformations.

Since $x > 3$, $3x + 2 > 11$. $D_g = (11, \infty)$

The range of g is clearly $R_g = \mathbb{R}^+$.

Alternatively, one can deduce this using the fact that $g(x)$ is a square root function, which means it cannot take negative values.