Integration

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INTEGRATION [**80 Marks**]

1 Integrate the following:

(a) $\int \tan^{-1}(\frac{\pi}{3}x)\,\mathrm{d}x$ [4]

Integrating by Parts:

$$\begin{aligned}\int \tan^{-1}(\frac{\pi}{3}x)\,\mathrm{d}x &= x\tan^{-1}(\frac{\pi}{3}x) - \int \frac{\pi/3\cdot x}{1+\pi^2/9\cdot x^2}\,\mathrm{d}x \\ &= x\tan^{-1}(\frac{\pi}{3}x) - \int \frac{3\pi x}{\pi^2x^2+9}\,\mathrm{d}x \\ &= x\tan^{-1}(\frac{\pi}{3}x) - \frac{3}{2\pi}\ln|\pi^2x^2+9| + C\end{aligned}$$

(b) $\int x^3\mathrm{e}^{3x}\,\mathrm{d}x$ [4]

Integrating by Parts Repeatedly:

$$\begin{aligned}\int x^3\mathrm{e}^{3x}\,\mathrm{d}x &= \frac{1}{3}x^3\mathrm{e}^{3x} - \int x^2\mathrm{e}^{3x}\,\mathrm{d}x \\ &= \frac{1}{3}x^3\mathrm{e}^{3x} - \frac{1}{3}x^2\mathrm{e}^{3x} + \frac{2}{3}\int x\mathrm{e}^{3x}\,\mathrm{d}x \\ &= \frac{1}{3}x^3\mathrm{e}^{3x} - \frac{1}{3}x^2\mathrm{e}^{3x} - \frac{2}{9}x\mathrm{e}^{3x} - \frac{2}{27}\mathrm{e}^{3x} + C\end{aligned}$$

(c) $\int_{-\frac{1}{2}}^{1} \frac{1}{\sqrt{(1-x)(x+2)}}\,\mathrm{d}x$ [4]

Completing the Square and then Integrating:

$$\begin{aligned}
\int_{-\frac{1}{2}}^{1} \frac{1}{\sqrt{(1-x)(x+2)}}\,\mathrm{d}x &= \int_{-\frac{1}{2}}^{1} \frac{1}{\sqrt{-x^2-x+2}}\,\mathrm{d}x \\
&= \int_{-\frac{1}{2}}^{1} \frac{1}{\sqrt{(\frac{3}{2})^2-(x+\frac{1}{2})^2}}\,\mathrm{d}x \\
&= [\sin^{-1}\frac{x+\frac{1}{2}}{\frac{3}{2}}]_{-\frac{1}{2}}^{1} \\
&= \sin^{-1}1-\sin^{-1}0 = \frac{\pi}{2}
\end{aligned}$$

(d) $\int_0^{\ln 2} \frac{1}{e^x+e^{2x}}\,\mathrm{d}x$, using the substitution $x=\ln u$. [5]

Substituting $x=\ln u$ and $\mathrm{d}x = \frac{1}{u}\,du$:

$$\int_0^{\ln 2} \frac{1}{e^x+e^{2x}}\,\mathrm{d}x = \int_1^2 \frac{1}{u^2(u+1)}\,du$$

Using the Partial Fraction Decomposition:

$$\frac{1}{u^2(u+1)} = \frac{A}{u}+\frac{B}{u^2}+\frac{C}{u+1} \quad \text{where} \quad A,B,C\in\mathbb{Q}$$

We can use the Cover-Up Method to get B and C:

$$B=\frac{1}{0+1}=1, \quad C=\frac{1}{(-1)^2}=1$$

Multiplying both sides by $u^2(u+1)$:

$$1 = Au(u+1)+(u+1)+u^2$$

By Comparing the x^2 coefficient, it is evident that $A=-1$ Thus, our integration simplifies to:

$$\int_0^{\ln 2} \frac{1}{e^x + e^{2x}}\,\mathrm{d}x = \int_1^2 \frac{1}{u^2(u+1)}\,\mathrm{d}u = \int_1^2 -\frac{1}{u} + \frac{1}{u^2} + \frac{1}{u+1}\,\mathrm{d}u$$
$$= \left[-\ln|u| - \frac{1}{u} + \ln|u+1|\right]_1^2 = \frac{1}{2} - 2\ln 2 + \ln 3$$

2 Show that $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$.

Using the Fraction Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{\sin x}\right) = \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

Hence or otherwise, find $\int \csc^3 x\,\mathrm{d}x$, given that $0 < x < \pi$. [6]

Integrating by Parts, then using the Pythagorean Identity $\cot^2 x + 1 = \csc^2 x$:

$$\int \csc^3 x\,\mathrm{d}x = -\csc x \cot x - \int \csc x \cot^2 x\,\mathrm{d}x$$
$$= -\csc x \cot x - \int \csc x(\csc^2 x - 1)\,\mathrm{d}x$$
$$= -\csc x \cot x + \int \csc x\,\mathrm{d}x - \int \csc^3 x\,\mathrm{d}x$$
$$= -\csc x \cot x - \ln|\csc x + \cot x| - \int \csc^3 x\,\mathrm{d}x$$

Grouping $\int \csc^3 x\,\mathrm{d}x$ on the LHS,

$$2\int \csc^3 x\,\mathrm{d}x = -\csc x \cot x - \ln|\csc x + \cot x| + C$$
$$\int \csc^3 x\,\mathrm{d}x = -\frac{1}{2}(\csc x \cot x + \ln|\csc x + \cot x|) + C$$

3 Show that $\cos a\theta \cos 3a\theta = \dfrac{\cos 2a\theta + \cos 4a\theta}{2}$.

Using the identity $\cos p + \cos q = 2\cos\frac{p+q}{2}\cos\frac{p-q}{2}$:

$$\begin{aligned}\cos a\theta\cos 3a\theta &= \frac{\cos 2a\theta + \cos 4a\theta}{2}\\ &= \frac{1}{2}\times(2\cos\frac{4a\theta-2a\theta}{2}\cos\frac{4a\theta+2a\theta}{2})\\ &= \frac{\cos 2a\theta+\cos 4a\theta}{2}\end{aligned}$$

Hence by Integration by Parts, find $\int_0^{\pi/4} 2\theta\cos\theta\cos 3\theta\, d\theta$. [6]

Using the above result:

$$\begin{aligned}\int_0^{\pi/4} 2\theta\cos\theta\cos 3\theta\,\mathrm{d}\theta &= \int_0^{\pi/4}\theta(\cos 2\theta+\cos 4\theta)\,\mathrm{d}\theta\\ &= [\theta(\frac{1}{2}\sin 2\theta+\frac{1}{4}\sin 4\theta)]_0^{\pi/4} - \int_0^{\pi/4}\frac{1}{2}\sin 2\theta+\frac{1}{4}\sin 4\theta\,\mathrm{d}\theta\\ &= [\theta(\frac{1}{2}\sin 2\theta+\frac{1}{4}\sin 4\theta)+\frac{1}{4}\cos 2\theta+\frac{1}{16}\cos 4\theta]_0^{\pi/4}\\ &= [\frac{\pi}{4}(\frac{1}{2}\sin\frac{\pi}{2})+\frac{1}{16}\cos\pi]-(\frac{1}{4}+\frac{1}{16})\\ &= \frac{\pi-3}{8}\end{aligned}$$

4 A geometric progression has first term a and common ration x, where $a, x \in \mathbb{R}$. The sum of the first n terms of the geometric progression is denoted by S_n.

(a) State in terms of a and x, the nth term of the geometric progression. [1]

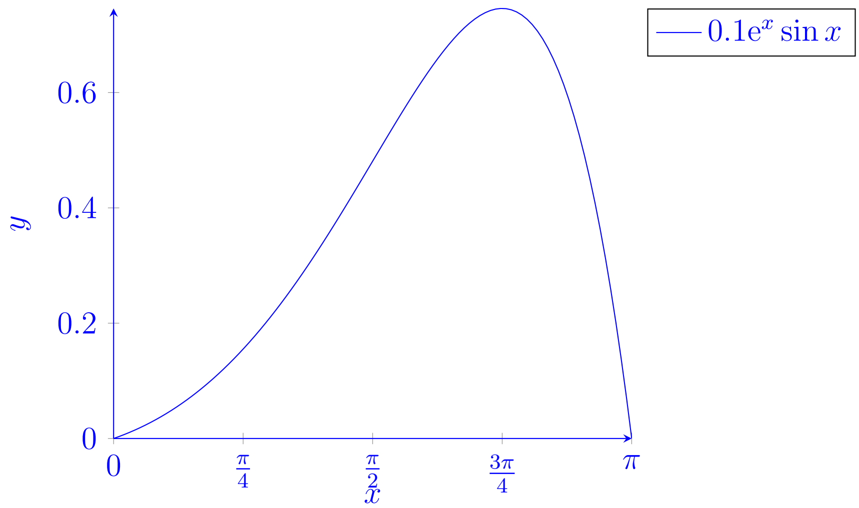
By the definition of a Geometric Progression, the nth term is ax^{n-1}

(b) Given that the sum of the first n terms of the Harmonic Series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = H_n$, show that $\int_0^1 S_n\,\mathrm{d}x = aH_n$ [3]

Using the fact that $S_n = a\frac{x^n - 1}{x - 1}$:

$$\int_0^1 S_n \,\mathrm{d}x = \int_0^1 a\frac{x^n - 1}{x - 1}\,\mathrm{d}x = a\int_0^1 x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x^2 + x + 1 \,\mathrm{d}x$$
$$= a[\frac{x^n}{n} + \frac{x^{n-1}}{n-1} + \frac{x^{n-2}}{n-2} + \cdots + \frac{x^3}{3} + \frac{x^2}{2} + x]_0^1$$
$$= a[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{3} + \frac{1}{2} + 1]$$
$$= aH_n$$

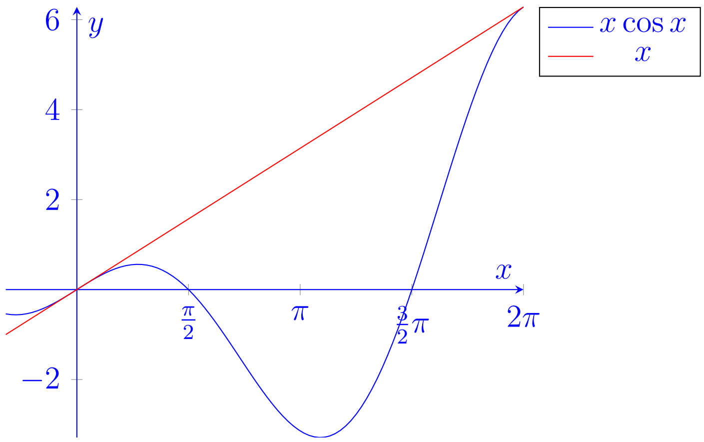
5 Sketch the graph of $f(x) = 0.1\mathrm{e}^x$ for $0 \leq x \leq \pi, x \in \mathbb{R}$. Using a Graphing Calculator, find the region enclosed by $y = f(x)$ and the x-axis. [5]



From G.C:

$$\int_0^{\pi} 0.1\mathrm{e}^x \sin x \,\mathrm{d}x = 1.21 \text{ units}^2$$

6 Curves C_1 and C_2 are given by the equations $y = x\cos x$ and $y = x$ respectively. For $0 \leq x \leq 2\pi$, find the exact area enclosed by C_1 and C_2. [4]



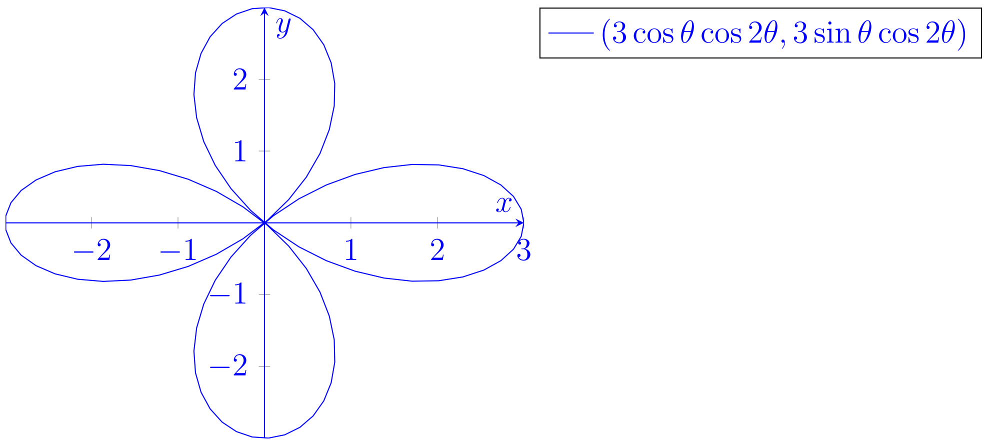
The curves intersect at (0,0) and $(2\pi,2\pi)$ with $C_2 \geq C_1$ for the given range

$$\therefore \text{Area} = \int_0^{2\pi} C_2 - C_1 \, \mathrm{d}x = \int_0^{2\pi} x - x\cos x \, \mathrm{d}x = [\frac{1}{2}x^2 - x\sin x]_0^{2\pi} + \int_0^{2\pi} \sin x \, \mathrm{d}x$$

$$= [\frac{1}{2}x^2 - x\sin x - \cos x]_0^{2\pi}$$

$$= 2\pi^2 \text{ units}^2$$

7 An engineer from FlowerFans.com models a new fan with the parametric equations $x = 3\cos\theta\cos 2\theta, y = 3\sin\theta\cos 2\theta$ for $0 \leq x \leq 2\pi$. Using a graphing calculator, find the total area of the blades of the fan in units^2. [4]



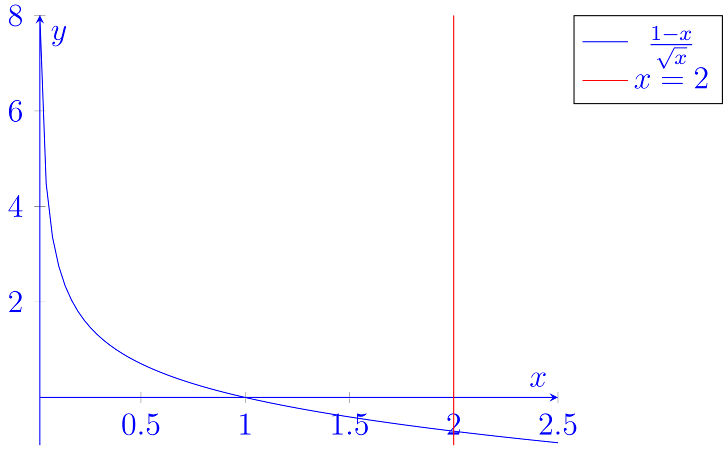
We can find the area of the fan by dividing it into 8 even segments. To find the area of one segment we can find the area bounded by the curve and the y-axis where $x, y \geq 0$

$$y = 3\sin\theta\cos 2\theta \implies \mathrm{d}y = (-6\cos\theta\sin 2\theta - 3\cos\theta\sin\theta)\,\mathrm{d}\theta$$

$$\text{Area} = 8\int_{\pi/4}^{0} (3\cos\theta\cos 2\theta)(-6\cos\theta\sin 2\theta - 3\cos\theta\sin\theta)\,\mathrm{d}\theta$$

$$= 14.1\text{units}^2 \quad \text{(From G.C.)}$$

8 Find the exact volume of region R rotated through 2π radians about the x-axis, where R is the region enclosed by $y = \dfrac{1-x}{\sqrt{x}}$, the line $x = 2$ and the x-axis. [4]



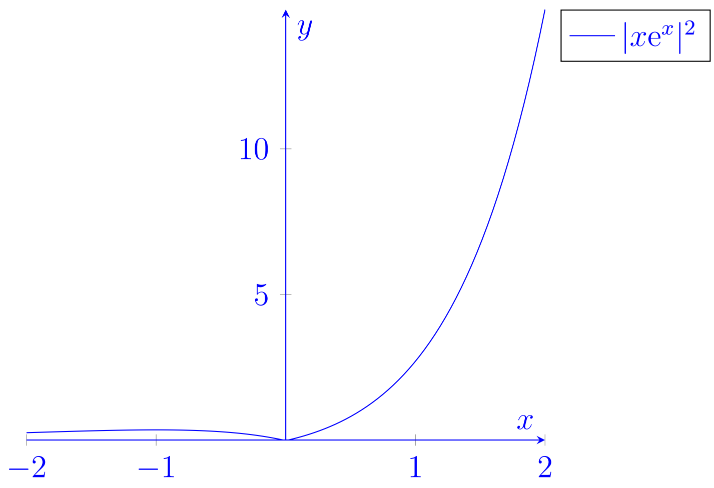
$$\text{Volume} = \int_1^2 \pi\left(\frac{1-x}{\sqrt{x}}\right)^2 \mathrm{d}x = \pi\int_1^2 \frac{1-2x+x^2}{x}\,\mathrm{d}x$$

$$= \pi\int_1^2 \frac{1}{x} - 2 + x\,\mathrm{d}x$$

$$= \pi[\ln|x| - 2 + x]_1^2 = \pi(\ln 2 - \frac{1}{2})\text{ units}^3$$

9 The curve C has the equation $y = |xe^x|$.

(a) Sketch the curve C. [2]



(b) Find the volume of region S rotated through 2π radians about the x-axis, where S is the region enclosed by C, the lines $x = -1$ and $x = 1$, and the x-axis. [4]

Integrating Repeatedly by Parts:

$$\begin{aligned}\text{Volume} = \pi\int_{-1}^{1} |xe^x|^2 \, dx &= \pi\int_{-1}^{1} x^2e^{2x} \, dx \\ &= \pi[\frac{1}{2}x^2e^{2x}]_{-1}^{1} - \pi\int_{-1}^{1} xe^{2x} \, dx \\ &= \pi[\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x}]_{-1}^{1} + \pi\int_{-1}^{1} \frac{1}{2}e^{2x} \, dx \\ &= \pi[\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x}]_{-1}^{1} \\ &= \pi(\frac{e^2}{4} - \frac{5e^{-2}}{4}) \text{ units}^3\end{aligned}$$

10 The curve C has the equation $e^{\frac{1}{2}x-1}$.

(a) Find the equation of L, where L is the tangent to the curve C at $x = 4$. [2]

First we differentiate C to get the gradient of the tangent.

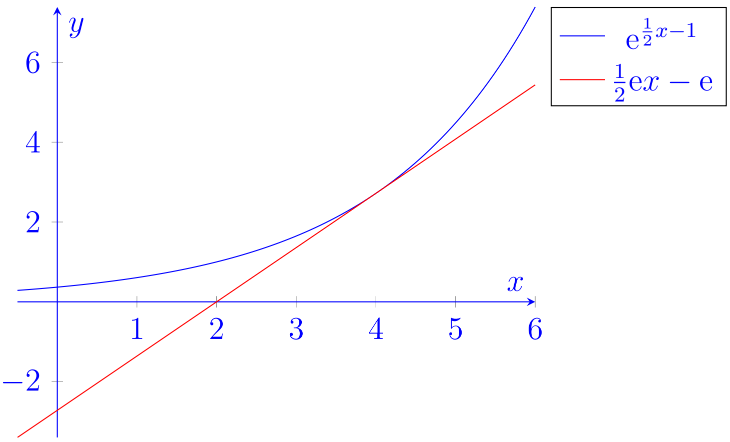
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\frac{1}{2}x-1}) = \frac{\mathrm{e}^{\frac{x}{2}-1}}{2}$$

$$\text{At} \quad x = 4, y = \mathrm{e} \quad \text{and the gradient,} \quad m = \frac{1}{2}\mathrm{e}$$

$$L : y - \mathrm{e} = \frac{1}{2}\mathrm{e}(x-4)$$

$$\implies y = \frac{1}{2}\mathrm{e}x - \mathrm{e}$$

(b) Find the region enclosed by C, L, and the axes. [3]



To find the area of the region, we find the area bounded by C and the x-axis from $(0, 4)$ and then subtract the area of the triangle bounded by L and the x-axis from $(2, 4)$

$$\text{Area} = \int_0^4 \mathrm{e}^{\frac{1}{2}x-1}\,\mathrm{d}x - \frac{1}{2} \times 2 \times \mathrm{e} = [2\mathrm{e}^{\frac{1}{2}x-1}]_0^4 - \mathrm{e} = [\mathrm{e} - \frac{2}{\mathrm{e}}] \text{ units}^2$$

(c) Find the volume of the above-mentioned region when it is rotated by 2π radians about the y-axis. [3]

To find the Volume of the region, we will take the volume of the of the region enclosed by L and the y-axis from $(0, \mathrm{e})$ and subtract from it the volume of the region enclosed by C and the y-axis from $(\frac{1}{e}, \mathrm{e})$

$$\text{Volume} = \pi[\int_0^e (\frac{2y + 2e}{e})^2\,\mathrm{d}y - \int_0^e (2\ln y + 2)^2\,\mathrm{d}y] = 20.6 \text{ units}^3$$

11 A ball partially submerged in a beaker of water can be expressed as the following graph from the side view, where the ball is a circle and the water surface is the x-axis. Given

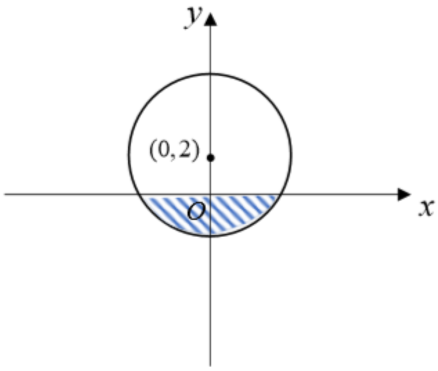
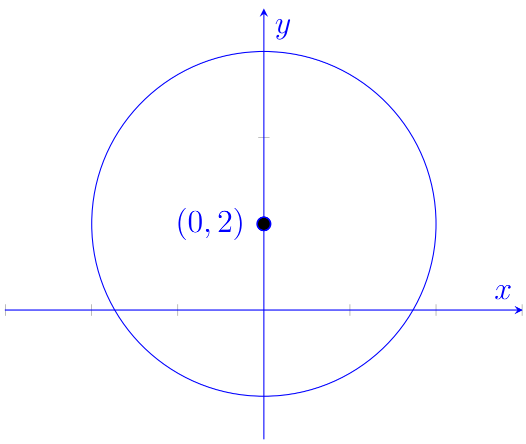


Figure 1: Partially Submerged Ball

that the centre of the ball is at $(0, 2)$ and that the radius of the ball is 4 units, show that the volume of the ball submerged in water is equal to $\frac{40}{3}\pi$ units^3. [7]



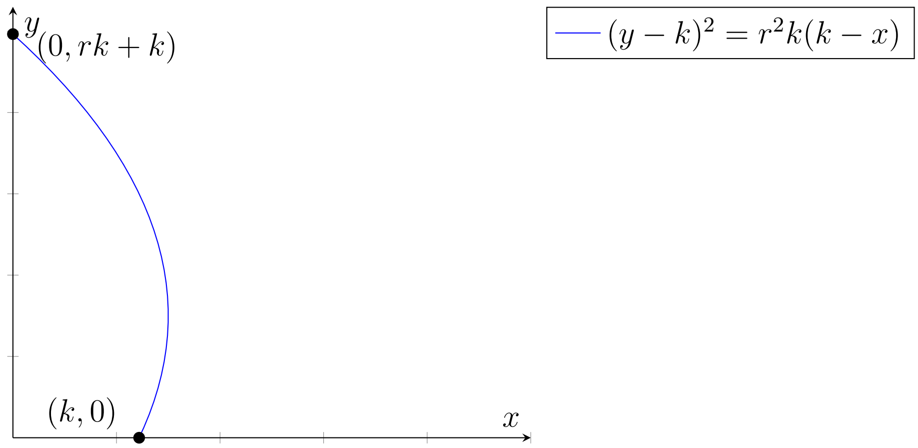
The circle can be modelled by the equation $x^2 + (y-2)^2 = 4^2$. Thus, the axial intercepts are $(0, -2)$, $(\pm 2\sqrt{3}, 0)$. Let us only consider the right half of the circle. It has the equation $x = \sqrt{16 - (y-2)^2}$ where $x \geq 0$. We will rotate the region bounded by this new curve and the y-axis for the range $(0, -2)$ by 2π radians to get the volume of the submerged region.

$$\text{Volume} = \int_{-2}^{0} \pi \left[16 - (y-2)^2\right] \, \mathrm{d}y = [16\pi y - \frac{\pi\,(y-2)^3}{3}]_{-2}^{0} = \frac{40}{3}\pi \text{ units}^3$$

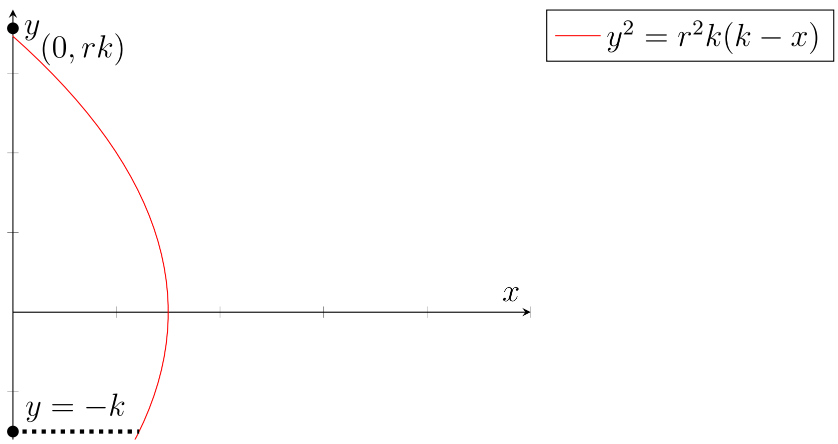
12 An architecture student is attempting to model the famous Burj Al Arab in wood (sculpting wood comes at \$1000.00/units^3). He approximates the shape of the building to the curve C with equation $(y-k)^2 = r^2k(k-x)$ for $x, y \geq 0$ where k and r are positive constants used for scaling purposes.

Figure 2: Burj Al Arab, taken from Wikipedia

(a) Given that the wooden model is formed by rotating the region enclosed by C and the axes $\frac{\pi}{4}$ radians around the y-axis, calculate the volume of the model in terms of k and r. [6]



First, we transpose C down by k-units, effectively replacing $y-k$ with y.



Following which, we find the volume of the region bounded by C and the y-axis by integrating with respect to y from $(-k, rk)$. Since we are rotating the region by only $\frac{\pi}{4}$ radians, we multiply the integral by only $\frac{\pi}{8}$ instead of π

$$\begin{aligned}
\text{Volume} &= \frac{\pi}{8}\int_{-k}^{rk}\left(k - \frac{y^2}{r^2k}\right)^2 \mathrm{d}y \\
&= \frac{\pi}{8}\int_{-k}^{rk}\left(\frac{y^4}{k^2r^4} - \frac{2y^2}{r^2} + k^2\right) \mathrm{d}y \\
&= \frac{\pi}{8}[\frac{y^5}{5k^2r^4} - \frac{2y^3}{3r^2} + k^2y]_{-k}^{rk} \\
&= \frac{\pi k^3\left(8r^5 + 15r^4 - 10r^2 + 3\right)}{120r^4}
\end{aligned}$$

(b) Calculate the amount of money spent on wood by the architecture student to construct the model if $k = 1, r = \frac{3}{2}$. [3]

$$\begin{aligned}
\text{Cost} &= \frac{\pi \cdot 1^3\left[8(1.5)^5 + 15(1.5)^4 - 10(1.5)^2 + 3\right]}{120(1.5)^4} \times \$1000 \\
&= \frac{125\pi}{648} \times \$1000 \\
&= \$606.02 \quad \text{(Nearest cent)}
\end{aligned}$$