Vectors

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VECTORS [150 Marks]

1 Solve the following:

a) $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin. Given that lines OA and OB are parallel, $|\mathbf{a}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = -2$, express $\mathbf{b}$ in terms of $\mathbf{a}$. [2]

b) A vector $\mathbf{a}$ is such that $\mathbf{a} = (\sqrt{2}\cos\alpha)\mathbf{i} - (\cos\alpha)\mathbf{j} + (\sqrt{2}\sin\alpha)\mathbf{k}$, where $0 \leq \alpha \leq 2\pi$ and $|\mathbf{a}| = \sqrt{2}$. Find the value(s) of α. [2]

c) The points A, B and C with respect to the origin are represented by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $|\mathbf{b}| = 2$, $\mathbf{a} \cdot \mathbf{b} = k$ and $\mathbf{b} \cdot \mathbf{c} = 2$. Given further that point C divides the line AB such that $AC : CB = 2 : 1$, find k. [3]

d) Four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ exist such that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = \mathbf{0}$. Show that $\mathbf{b} \times (\mathbf{a}+\mathbf{c}) = \mathbf{d} \times \mathbf{b}$. [2]

e) Point A referred from the origin has vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. The line OA makes an angle of α with the y-axis and β with the z-axis, where $\alpha, \beta < \pi$. Show that $\alpha + \beta = \pi$. [3]

2 Referred to the origin O, points A and B have position vectors given by $\mathbf{a}$ and $\mathbf{b}$ respectively. C_0 is the foot of perpendicular from A to OB with position vector $\mathbf{c}_0$. The angle between lines OA and OB is α , where $0 < \alpha < \frac{\pi}{2}$.

a) By considering $\cos\alpha$, show that $|\mathbf{c}_0| = \mathbf{a} \cdot \hat{\mathbf{b}}$. [2]

b) The foot of perpendicular from C_0 to OA is C_1. Show that $|\mathbf{c_1}| = \mathbf{a} \cdot \hat{\mathbf{b}}(\cos\alpha)$. [1]

c) C_n is the nth foot of perpendicular. State $|\mathbf{c}_n|$ in terms of a, b, n and α. [1]

d) State the sum to infinity of scalar projections $|\mathbf{c}_0| + |\mathbf{c}_1| + ... + |\mathbf{c}_n| + ...$ [1]

3 Referred to the origin O, points A and B have position vectors given by: $\mathbf{a} = \mathbf{i} - p^2\mathbf{k}$ and $\mathbf{b} = \frac{2}{p}\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively, where p is to be found. Given that $|\mathbf{a} \times \mathbf{b}|^2 = 4p^2 + 2$, find the value(s) that p can take. [4]

4 The vector equation of l is given by $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$. Point F is the foot of perpendicular from origin O to the line l. If $|\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 1$, express the position vector $\overrightarrow{OF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

5 The equations of l and m are given by $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $m : \mathbf{r} = \mathbf{b} + \mu\mathbf{a},\ \mu \in \mathbb{R}$, where $\mathbf{a}$ and $\mathbf{b}$ are co-planar vectors. State the conditions such that lines l and m are skew lines. [2]

6 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the origin O. It is given that $(\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) = 11$. Find the perpendicular distance from point A to line OB if $|\mathbf{b}| = 11$. [4]

7 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the origin O. It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 2$.

a) State the vector equation of line AB. [1]

b) Find $|\mathbf{b} - \mathbf{a}|$. [2]

c) Find the position vector of F, the foot of perpendicular from O to AB, in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

d) Find $|7\mathbf{b} - \mathbf{a}|$. Hence, find the exact area of triangle OAB. [3]

8 Referred to the origin O, points A and B have the position vectors $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}$ and $\overrightarrow{OB} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively.

a) Verify that $P(3, -2, -6)$ lies on line AB. [2]

b) Find the position vector of F, the foot of perpendicular from P to AB. [3]

c) Hence, find the equation of line PF. [3]

9 The equations of lines l_1 and l_2 are given by:

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R} \text{ and } l_2 : \frac{x+1}{9} = \frac{y}{7} = \frac{4-z}{3} \text{ respectively.}$$

Point A has coordinates $(2, -1, 1)$ while the foot of perpendicular from A to l_2 is F.

a) Find the position vector of P, the point of intersection between l_1 and l_2. [2]

b) Find vector $\overrightarrow{AF}$. [3]

c) Hence, find the vector equation of l_3, the reflection of l_1 in l_2. [3]

10 Points A and B with position vectors $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + \mathbf{k}$ respectively both lie on l_1. The line l_2 has Cartesian equation $l_2 : x = 7, \ y - 3 = z$.

a) Show that l_1 and l_2 are skew lines. [2]

b) Find a vector that is perpendicular to both l_1 and l_2. [1]

c) Hence, find the shortest distance between l_1 and l_2. [3]

11 Referred to an origin O, points A and B have coordinates $(-1, 2, 2)$ and $(0, 1, 2)$ respectively. The point P on OA is such that $OP : PA = \lambda : 1$ and the point Q on OB is such that $OQ : QB = \lambda : 1 - \lambda$, where λ is a real constant to be determined.

a) Find the area of ΔOAB. [2]

b) Express the ratio $\frac{\text{Area of } \Delta OAB}{\text{Area of } \Delta OPQ}$ in terms of λ. [3]

c) Deduce if PQ is ever parallel to AB for some value of λ. [3]

12 Line l has the equation $-x = \frac{y-3}{2} = \frac{z+4}{2}$. Line m, which is parallel to $\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$ where c is some real constant, is obtained by rotating line l 45° about the point $A(0, 3, -4)$. Find the possible vector equations of line m. [5]

13 Three points A, B and C referred from the origin O have position vectors given by: $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \frac{3}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} - 3\mathbf{k}$.

a) Find the vector equations of lines AB and AC. [2]

b) Find two vector equations of l, where l is the line representing the all the midpoints of lines AB and AC. [4]

14 Point A with position vector $\mathbf{a}$ lies on plane π with normal parallel to vector $\mathbf{n}$. Given that $|\mathbf{a} - \mathbf{n}|^2 = 3$ and $|\mathbf{n}|^2 = 4 - |\mathbf{a}|^2$, find the value of d if the equation of plane π is $\mathbf{r} \cdot \mathbf{n} = d$.[4]

15 The equations of parallel planes p and q are given by $p : \mathbf{r} \cdot \mathbf{n} = d$ and $q : \mathbf{r} \cdot \mathbf{n} = kd$. Line l given by equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ intersects planes p and q at points A and B respectively.

a) Show that $\overrightarrow{AB} = \mathbf{b}\left(\dfrac{d(k-1)}{\mathbf{b} \cdot \mathbf{n}}\right)$. [4]

b) Hence, or otherwise, show that the perpendicular distance between planes p and q is equal to $\dfrac{d(k-1)}{|\mathbf{n}|}$ units. [2]

16 The equations of plane π and l are given by:

$$\pi : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -1 \text{ and } l : \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix},\ \lambda, k \in \mathbb{R} \text{ respectively.}$$

a) Show that, for π and l to intersect, $k \neq -\frac{7}{2}$. [1]

For the rest of the question, assume $k = 1$.

b) Find the coordinates of point P, the point of intersection of π and l. [2]

c) Find the shortest distance from $A(-1, 0, 2)$ to π. [3]

d) Find the acute angle between π and l. [2]

17 Plane π has a normal parallel to $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and has the equation:

$$\pi : \mathbf{r} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}, \quad t, s \in \mathbb{R},$$

where a is some real constant to be determined.

a) Find the value of a. [2]

b) Find the scalar product equation of plane π. [1]

c) Line l passes through π, the origin O and $A(3, 2, 5)$. Find the position vector of P, the point of intersection between line l and plane π. [2]

d) Find $\left|\overrightarrow{PF}\right|$, where F is the foot of perpendicular from A to plane π. [3]

e) Hence find $\left|\overrightarrow{PG}\right|$, where G is the foot of perpendicular from O to plane π. [2]

18 The equations of planes π_1, π_2 and π_3 are such that:

$$\pi_1 : 2x + 3y + 4z = -1, \quad \pi_2 : -2x + y - z = 5 \quad \text{and} \quad \pi_3 : \mathbf{r} \cdot \begin{pmatrix} a \\ -5 \\ -a \end{pmatrix} = k.$$

a) Find the vector equation of l, the line of intersection between π_1 and π_2. [3]

b) Given that $a = 2$, find the value of k such that π_3 contains l. [2]

c) Given that $a = 1,\ k = 3$, find the point of intersection of π_1, π_2 and π_3. [2]

19 An incident beam of light was reflected perfectly ($\theta_1 = \theta_2$) on a round mirror.

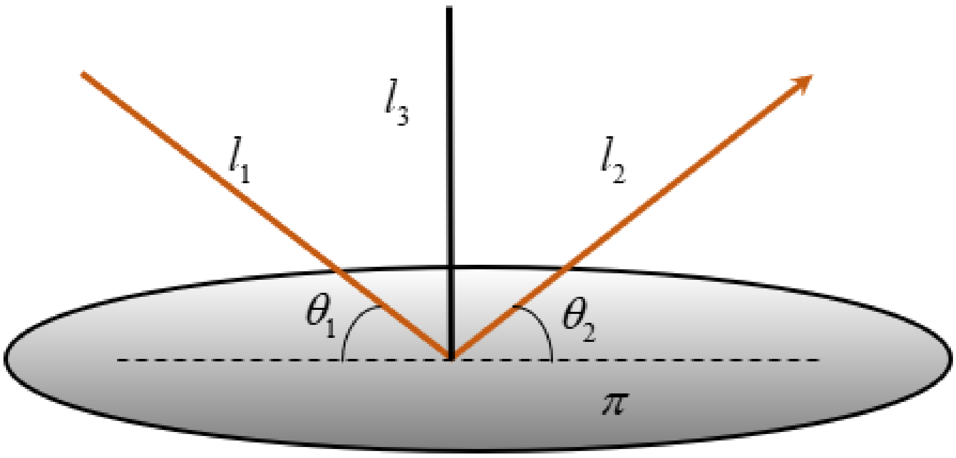

Figure 1: Mirror

A student modelled the scenario such that the incident beam is l_1, the reflected beam is l_2 and the mirror is π, where π contains the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

a) Find the equation of the plane π in the form $\mathbf{r} \cdot \mathbf{n} = d$. [3]

b) Show that $P(1, 3, 2)$, the point of intersection between l_1 and l_2 lies on π. [1]

c) State the vector equation of l_3, the axis of reflection between l_1 and l_2. [1]

d) Given that $A(t, 1, 1)$ lies on l_1, where $t > 0$, find t such that $\theta_1 = \theta_2 = \frac{\pi}{4}$. [4]

For the rest of the question, take $t = 4$.

e) Find the shortest distance from A to π. [2]

f) Find the coordinates of F, the foot of perpendicular from A to l_3. Hence, or otherwise, find the equation of l_2. [6]

20 A professional card stacker stacks two cards P and Q as follows:

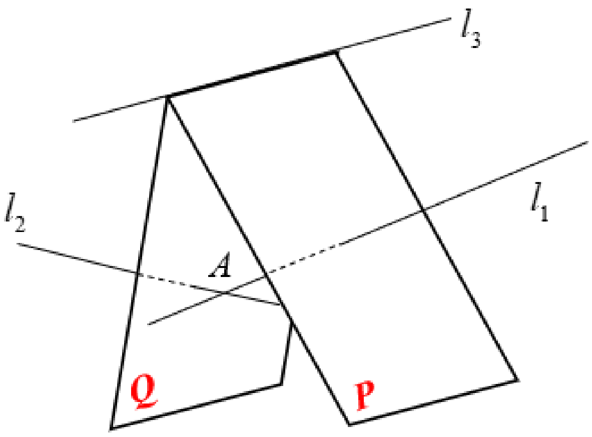

Figure 2: Card Stack

Mr Poh models the scenario such that the two cards are planes P and Q, where the equation of plane P is $P : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1$, where line l_1 is a normal to plane P that contains $A(-3, 3, 2)$. Line l_2, a normal to plane Q, also contains point A and is parallel to the vector $\begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}$ where $a < 0$. l_3 is the line of intersection between planes P and Q.

a) Find the coordinates of B, the point of intersection between l_1 and plane P. [2]

b) Given that l_2 is obtaining by rotating l_1 $\cos^{-1}\dfrac{9}{11}$ about point A, find a; Hence, find the vector equation of l_2. [4]

c) The point of intersection between l_2 and plane Q is point B'. Given that $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AB'}\right|$, find the position vector of B' given that the x-coordinate of $B' < 0$. [3]

d) Find the equation of the plane Q in the form $\mathbf{r} \cdot \mathbf{n} = d$. [2]

e) Find the vector equation of l_3. [2]

The equation of plane R is such that it reflects the image of plane P to form plane Q.

f) Find the vector $\overrightarrow{AF}$, where F is the foot of perpendicular from A to l_3.

Hence, find the equation of plane R in the form $\mathbf{r} \cdot \mathbf{n} = d$. [5]