H2 MATHEMATICS

Derek, Vignesh

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COMPLEX NUMBERS [90 Marks]

1 Solve the following:

(a) Given that $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$,

(i) Show that $(zw)^* = z^*w^*$. [2]

We will perform algebraic manipulation of the given complex numbers using a, b, c, d

$$(zw)^* = [(a + ib)(c + id)]^* = [ac - bd + (bc + ad)i]^* = ac - bd - (bc + ad)i$$

$$z^*w^* = (a - ib)(c - id) = ac - bd - (bc + ad)i \quad \text{(Shown)}$$

(ii) Hence, show that $(vzw)^* = v^*z^*w^*$ where $v \in \mathbb{C}$. [1]

Using the identity from above:

$$(vzw)^* = v^*(zw)^*$$

$$\therefore (vzw)^* = v^*z^*w^* \quad \text{(Shown)}$$

OR Let $v = x + iy$ where $x, y \in \mathbb{R}$

$$\begin{aligned}
(vzw)^* &= [(x+iy)(ac-bd+(bc+ad)i)]^* \\
&= [acx - bdx - (bc+ad)y + (bc+ad)xi]^* \\
&= acx - bdx - (bc+ad)y - (bc+ad)xi \\
v^*z^*w^* &= (x-iy)[ac-bd-(bc+ad)i] \\
&= acx - bdx - (bc+ad)y - (bc+ad)xi \quad \text{(Shown)}
\end{aligned}$$

(b) The locus of a parabola is given by $z = at^2 + i(2at)$ for $a, t \in \mathbb{R}$.

Show that $|z - a| = |\mathrm{Re}(z) + a|$. [3]

We will just perform basic algebraic manipulation to prove this. Recall that the magnitude of a positive real number is the number itself and the magnitude of a complex number can be thought of as the distance from the origin of the complex number and is given by the equation $|z| = \sqrt{[\mathrm{Re}(z)]^2 + [\mathrm{Im}(z)]^2}$

$|z - a| = |at^2 - a + i(2at)| = \sqrt{(at^2-a)^2 + (2at)^2} = \sqrt{a^2t^4 + a^2 + 2(at)^2} = at^2 + a$

$|\mathrm{Re}(z) + a| = at^2 + a \quad$ (Shown)

(c) It is given that $\mathrm{Im}(\frac{a+bi}{a-bi}) = 0$ for some $a, b \in \mathbb{R}$.

Find the possible values of $\frac{a+bi}{a-bi}$. [3]

When $\mathrm{Im}(\frac{a+bi}{a-bi}) = 0$, it means that the coefficient of i after performing algebraic manipulation of $\frac{a+bi}{a-bi}$ is 0. We first multiply the top and bottom of the given fraction with the conjugate of the denominator.

$$\frac{a+bi}{a-bi} = \frac{(a+bi)(a+bi)}{(a-bi)(a+bi)} = \frac{a^2-b^2+2abi}{a^2+b^2}$$

$$\implies 2ab = 0 \implies a = 0 \quad \text{OR} \quad b = 0$$

$$\therefore \frac{a+bi}{a-bi} = \pm 1$$

(d) For some $a, b \in \mathbb{R}$, $z^2 + az - b = 0$ has no real roots and $aw^2 + bw + a = 0$ has two real distinct roots. Given that $a > 1$, find an inequality satisfied by b. [3]

Recall that for any quadratic equation with real coefficients, the discriminant's value,Δ, defines the nature of its roots. Further recall that $\Delta = b^2 - 4ac$

$$\Delta(z^2 + az - b) = a^2 - 4(1)(-b) = a^2 + 4b < 0 \quad \text{(no real roots)}$$

$$\Delta(aw^2 + bw + a = 0) = b^2 - 4(a)(a) = b^2 - 4a^2 > 0 \quad \text{(2 distinct real roots)}$$

From the above equations we can glean that $b < \frac{-a^2}{4}$ and $b^2 > 4a^2$ which implies that $b < -2a$ since $a > 1 \implies a^2 > 1$. Equating $\frac{a^2}{4} = 2a$:

$$\frac{a^2}{4} = 2a$$

$$a(a-8) = 0$$

For $1 < a \leq 8$, $2a \geq \frac{a^2}{4}$ and for $a > 8$, $2a \leq \frac{a^2}{4}$. This gives us the inequality below:

$$\left.\begin{array}{l} b < -2a, 1 < a \leq 8 \\ b < \frac{-a^2}{4}, a > 8 \end{array}\right\}$$

(e) It is given that $k = \dfrac{a - z^*}{z} + \dfrac{a - z}{z^*}$, where $z = a + ib$ for some $a, b \in \mathbb{R}$.

Show that $k = \dfrac{2b^2}{a^2+b^2}$. [3]

We will just use the equation $z = a + ib$ to solve this equation after making the denominator of the fractions to be $zz^* = |z|^2$

$$
\begin{aligned}
k = \frac{a - z^*}{z} + \frac{a - z}{z^*} &= \frac{(a - z^*)z^*}{zz^*} + \frac{(a - z)z}{zz^*} \\
&= \frac{a(z^* + z) - (z^*)^2 - z^2}{|z|^2} \\
&= \frac{2a^2 - (a^2 - b^2 - 2abi) - (a^2 - b^2 + 2abi)}{a^2 + b^2} \\
&= \frac{2b^2}{a^2 + b^2} \text{ (Shown)}
\end{aligned}
$$

(f) Given that $z = a + bi$ and $w = b - ai$, express v in terms of a and b if $v^* = \dfrac{z + w}{zw}$. [3]

Recall that $\dfrac{1}{z} = \dfrac{z^*}{|z|^2}$. Let us apply this fact to find v^* and subsequently v.

$$
\begin{aligned}
v^* = \frac{z + w}{zw} &= \frac{1}{z} + \frac{1}{w} \\
&= \frac{z^*}{|z|^2} + \frac{w^*}{|w|^2} \\
&= \frac{a - bi}{a^2 + b^2} + \frac{b + ai}{a^2 + b^2} \\
&= \frac{a + b + (a - b)i}{a^2 + b^2} \\
\implies v &= \frac{a + b - (a - b)i}{a^2 + b^2}
\end{aligned}
$$

(g) Find $w \in \mathbb{C}$ in the equation $z^2 + wz + 2 = 0$ if one of the roots is i. [3]

We can just substitute $z = i$ into this equation.

$$i^2 + w(i) + 2 = -1 + wi + 2 = 0$$

$$\implies w = -1/i = i$$

(h) For $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$,

(i) Express $|(a-bi)^n|$ in terms of a, b and n. [2]

For any complex number $z = a + bi$, we can express it in polar form as $(\sqrt{a^2+b^2})e^{i\theta}$ where $\theta = \arg(z)$. Thus, $z^n = (a^2+b^2)^{\frac{n}{2}} e^{ni\theta}$ and it is evident that $|z^n| = (a^2+b^2)^{\frac{n}{2}}$. $\therefore |(a-bi)^n| = (a^2+b^2)^{\frac{n}{2}}$

(ii) Hence, if $(\sqrt{3}-\sqrt{2}i)^n = z$ where $|z| = 625$, find the value of n. [2]

Given $(\sqrt{3}-\sqrt{2}i)^n = z$, we know that $|z| = (3+2)^{\frac{n}{2}} = 625 \implies \dfrac{n}{2} = 4 \implies n = 8$

2 Solve the following:

(a) Prove that $\mathrm{Re}(\dfrac{1}{\mathrm{e}^{i\theta}+\mathrm{e}^{-3i\theta}}) \equiv \dfrac{\cos\theta}{2\cos 2\theta}$. [4]

First, let us multiply the numerator and denominator of this fraction with $\mathrm{e}^{i\theta}$. Then, we will use the fact that $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$ to prove this identity.

$$\begin{aligned}
\frac{1}{\mathrm{e}^{i\theta}+e^{-3i\theta}} &= \frac{\mathrm{e}^{i\theta}}{\mathrm{e}^{2i\theta}+\mathrm{e}^{-2i\theta}} \\
&= \frac{\cos\theta + i\sin\theta}{2\cos 2\theta} \\
\implies \mathrm{Re}(\frac{1}{\mathrm{e}^{i\theta}+\mathrm{e}^{-3i\theta}}) &= \frac{\cos\theta}{2\cos 2\theta} \quad \text{(Shown)}
\end{aligned}$$

(b) Show that $\dfrac{\csc\theta(\cot\theta + i)}{2\cos\theta(\cot\theta - i)} = \cot 2\theta + i$. [4]

$$\begin{aligned}
\frac{\csc\theta(\cot\theta+i)}{2\cos\theta(\cot\theta-i)} &= \frac{(\cot\theta+i)}{2\sin\theta\cos\theta(\cot\theta-i)} \\
&= (\csc 2\theta)\frac{(\cot\theta+i)(\sin\theta)}{(\cot\theta-i)(\sin\theta)} \\
&= (\csc 2\theta)\frac{(\cos\theta+i\sin\theta)}{(\cos\theta-i\sin\theta)} \\
&= (\csc 2\theta)\frac{\mathrm{e}^{i\theta}}{\mathrm{e}^{-i\theta}} \\
&= (\csc 2\theta)\mathrm{e}^{2i\theta} \\
&= (\csc 2\theta)(\cos 2\theta + i\sin 2\theta) \\
&= \cot 2\theta + i
\end{aligned}$$

3 Solve the simultaneous equations $zw = \dfrac{5}{2}(1+i)$ and $(1-i)w = \dfrac{iz+3}{2}$. [5]

Notice that we can remove w by taking $\dfrac{zw}{(1-i)w}$

$$\begin{aligned}
\frac{zw}{(1-i)w} &= \frac{5}{2}(1+i) \div \frac{iz+3}{2} \\
\frac{z}{1-i} &= \frac{5(1+i)}{iz+3} \\
iz^2 + 3z &= 10
\end{aligned}$$

Now we can collect z and apply the quadratic formula

$$\begin{aligned}
iz^2 + 3z - 10 = 0& \\
z &= \frac{-3 \pm \sqrt{3^2 - 4(i)(10)}}{2i} \\
&= \frac{-3 \pm \sqrt{9-40i}}{2i} \\
&= \frac{-3 \pm (5-4i)}{2i} \\
&= \frac{-8+4i}{2i} \quad \text{OR} \quad \frac{2-4i}{2i} \\
&= 2+4i \quad \text{OR} \quad -2-i
\end{aligned}$$

Let us substitute this value back into the first equation, $zw = \frac{5}{2}(1+i)$ to get w.

$$w_1 = \frac{5(1+i)}{2(2+4i)} \qquad\qquad w_2 = \frac{5(1+i)}{2(-2-2i)}$$
$$= \frac{5(1+i)(2-4i)}{2(2+4i)(2-4i)} \qquad\qquad = \frac{5(1+i)(-2+2i)}{2(-2-2i)(-2+2i)}$$
$$= \frac{30-10i}{40} \qquad\qquad = \frac{-20}{16}$$
$$= \frac{3-i}{4} \qquad\qquad = -\frac{5}{4}$$

When $z + 2 + 4i, w = \frac{3-i}{4}$ and when $z = -2-i, w = -\frac{5}{4}$

4 It is given that $\arg\left(z^6(w^*)^5\right) = \frac{3\pi}{4}$. Given also that $|z^6(w^*)^5| = |z|$ and $z = \sqrt{3}+3i$, find w. [4]

Since this question consists of purely multiplying complex numbers, let us work in coordinate form:

$$z = \sqrt{3} + 3i = \sqrt{3^2+3}\,\mathrm{e}^{i\tan^{-1}\sqrt{3}} = 2\sqrt{3}\,\mathrm{e}^{i\frac{\pi}{3}}$$

We will use this to compute $\arg(w)$ and $|z|$

$$\begin{aligned}\arg\left(z^6(w^*)^5\right) &= 6\arg(z) - 5\arg(w)\\ &= 2\pi - 5\arg(w)\\ &= \frac{3\pi}{4}\\ \implies \arg(w) &= \frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}|w| &= \left(\frac{|z|}{|z|^6}\right)^{\frac{1}{5}}\\ &= |z|^{-1}\\ &= \frac{1}{2\sqrt{3}}\\ &= \frac{\sqrt{3}}{6}\end{aligned}$$

Thus, $w = \frac{\sqrt{3}}{6}\mathrm{e}^{i\dfrac{\pi}{4}}$

5 Given $k \in \mathbb{R}$ such that $\sqrt{11+ki} = a+bi$, where a and b are positive real numbers,

(a) Express k in terms of a. [2]

$$\sqrt{11+ki} = a+bi$$
$$11+ki = a^2-b^2+2abi$$

Equating real and imaginary parts, we have

$$11 = a^2-b^2 \implies b = \sqrt{a^2-11} \text{ and } k = 2ab$$

Making k the subject, we have $k = 2a\sqrt{a^2-11}$

(b) Find the values of a and b if $k = 60$. [2]

Substituting $k = 60$ into the equation found in **(a)**,

$$60 = 2a\sqrt{a^2-11}$$
$$900 = (a^2)(a^2-11)$$
$$a^4 - 11a^2 - 3600 = 0$$
$$a = 6 \text{ (Using a G.C.)}$$

Using equation $b = \sqrt{a^2-11}, b = 5$

$$\therefore a = 6, b = 5$$

6 It is given that $z = 1 + \mathrm{e}^{-i\frac{\pi}{4}}$.

(a) Show that $\mathrm{e}^{-i\dfrac{3\pi}{4}} = -\mathrm{e}^{i\dfrac{\pi}{4}}$. [1]

Recall that $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$

$$\mathrm{e}^{-i\frac{3\pi}{4}} = \cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$
$$-\mathrm{e}^{i\frac{\pi}{4}} = -(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = -(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \quad \text{(Shown)}$$

(b) Hence, find $(z-1)^3 + (z-1)^2 + z + i$. [3]

We will substitute the value of z into the expression and simplify the result using the answer from **(a)**.

$$\begin{aligned}
(z-1)^3 + (z-1)^2 + z + i &= (1 + \mathrm{e}^{-i\frac{\pi}{4}} - 1)^3 + (1 + \mathrm{e}^{-i\frac{\pi}{4}} - 1)^2 + (1 + \mathrm{e}^{-i\frac{\pi}{4}}) + i \\
&= (\mathrm{e}^{-i\frac{\pi}{4}})^3 + (\mathrm{e}^{-i\frac{\pi}{4}})^2 + (1 + \mathrm{e}^{-i\frac{\pi}{4}}) + i \\
&= \mathrm{e}^{-i\frac{3\pi}{4}} + \mathrm{e}^{-i\frac{\pi}{2}} + 1 + \mathrm{e}^{-i\frac{\pi}{4}} + i \\
&= -\mathrm{e}^{i\frac{\pi}{4}} + (-i) + 1 + \mathrm{e}^{-i\frac{\pi}{4}} + i \\
&= 1 - 2i(\sin\frac{\pi}{4}) \\
&= 1 - \sqrt{2}\, i
\end{aligned}$$

7 Given that $1 + i$ is a root of the equation

$$4z^4 - 8z^3 + 17z^2 - 18z + 18 = 0,$$

find all the other roots in the form $a + bi$. [4]

Since all coefficients of the quartic equation are real, $1 - i$ is also a root of the given equation. Thus, the equation simplifies to:

$$\begin{aligned}
4z^4 - 8z^3 + 17z^2 - 18z + 18 &= [z - (1+i)][z - (1-i)](az^2 + bz + c) \quad \text{where} \quad a, b, c \in \mathbb{R} \\
&= (z^2 - 2z + 2)(az^2 + bz + c) \\
&= (z^2 - 2z + 2)(4z^2 + 9) = 0
\end{aligned}$$

Thus, the roots are $1 \pm i$ and $\pm\dfrac{3}{2}i$

8 The equation $3z^3 + az^2 + bz - 5 = 0$ has a root $z = \dfrac{1}{3}$, where a and b are real, non-zero constants.

Given that the sum of roots is $\dfrac{7}{3}$,

(a) Find the values of a and b. [3]

Substitute $z = \frac{1}{3}$ into the equation to get:

$$3\left(\frac{1}{3}\right)^3 + a\left(\frac{1}{3}\right)^2 + b\left(\frac{1}{3}\right) - 5 = 0$$
$$\implies a + 3b = 44 \;---\; (1)$$

We can use the formula $\frac{a_2}{a_3} = -(\alpha+\beta+\gamma)$ where α, β, γ are roots of cubic polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$.

$$\text{Sum of Roots} = -\frac{a}{3} = \frac{7}{3}$$
$$\therefore a = -7$$

Solving for b using (1) and $a = -7$, we get $b = 17$

(b) Find the remaining roots without using a calculator. [2]

We now factorise out $3z - 1$ from $3z^3 - 7z^2 + 16z - 5$. The remaining roots should form a conjugate pair.

$$\begin{aligned} 3z^3 - 7z^2 + 16z - 5 &= (3z-1)(z^2 - 2z + 5) = 0 \\ z &= \frac{2 \pm \sqrt{4 - 4(5)}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \end{aligned}$$

The remaining roots are $1 \pm 2i$.

9 On an Argand diagram, referred from the origin O, points Z and W represent the complex numbers $z = 1 + i$ and $w = -1 + \sqrt{3}i$ respectively. Points A and B represent $\text{Re}(z)$ and $\text{Re}(w)$ respectively.

(a) Express z and w in the form $re^{i\theta}$. [1]

$z = \sqrt{2}e^{i\frac{\pi}{4}}, w = 2e^{i\frac{2\pi}{3}}$

(b) Find the area of trapezium $BAZW$. Hence or otherwise, find the area of ΔOZW.

[2]

$$\begin{aligned}\text{Area of trapezium } BAZW &= \frac{1}{2}(1+\sqrt{3})(2)\\ &= (1+\sqrt{3}) \text{ units}^2\end{aligned}$$

$$\begin{aligned}\text{Area of } \Delta OZW &= \text{Area of trapezium } BAZW - \text{Area of } \Delta OAZ - \text{Area of } \Delta OBW\\ &= (1+\sqrt{3}) - \frac{1}{2}(1)(1) - \frac{1}{2}(1)(\sqrt{3})\\ &= (\frac{1+\sqrt{3}}{2}) \text{ units}^2\end{aligned}$$

(c) Hence, prove that $\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$. [3]

Recall the formula for the area of Triangle $= \frac{1}{2}ab\sin C$. In our case, the two sides a and b are $|z|$ and $|w|$. The angle $C =$ angle OZW which is equal to $\arg(w) - \arg(z) = \frac{2\pi}{3} - \frac{\pi}{4} = \frac{5\pi}{12}$ on an Argand diagram.

$$\begin{aligned}\text{Area of } \Delta OZW &= \frac{1}{2}|z||w|\sin\frac{5\pi}{12}\\ &= \frac{1}{2}(\sqrt{2})(2)\sin\frac{5\pi}{12}\\ &= \sqrt{2}\sin\frac{5\pi}{12}\\ &= \frac{1+\sqrt{3}}{2}\end{aligned}$$

$$\therefore \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

10 Given that $e^{i\theta} = \cos\theta + i\sin\theta$,

(a) Show that $e^{i(3\theta)} = \cos 3\theta + i\sin 3\theta$. [1]

Replacing θ with 3θ, $e^{i(3\theta)} = \cos 3\theta + i\sin 3\theta$

(b) Find $\mathrm{Im}((\cos\theta + i\sin\theta)^3)$. [2]

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \binom{3}{0}\cos^3\theta + \binom{3}{1}i\cos^2\theta\sin\theta - \binom{3}{2}\cos\theta\sin^2\theta - \binom{3}{3}i\sin^3\theta \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta\end{aligned}$$

$$\therefore \operatorname{Im}((\cos\theta + i\sin\theta)^3) = 3\cos^2\theta\sin\theta - \sin^3\theta$$

(c) Using your answers in **(a)** and **(b)**, express $\sin 3\theta$ in terms of $\sin\theta$. [2]

$\sin 3\theta$ can be found in the imaginary part of the answer in **(a)**. We now equate this with the answer found in **(b)** and simplify the result.

$$\begin{aligned}\sin 3\theta &= \operatorname{Im}((\cos\theta + i\sin\theta)^3) \\ &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)(\sin\theta) - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

11 It is given that $z = 1 + ki$, where $k > 0$.

(a) Express z in the form $re^{i\theta}$. [2]

Since $k > 0$, $\arg(z)$ is in the first quadrant and thus it is simply $\tan^{-1} k$.

$$z = (\sqrt{1+k^2})\mathrm{e}^{i\tan^{-1}k}$$

(b) Given that z^5 is a positive real number, find the value of k. [2]

$$
\begin{aligned}
\arg(z^5) &= 5\arg(z) \\
&= 5\tan^{-1} k \\
&= n(2\pi), \text{where } n \text{ is a non-negative integer}
\end{aligned}
$$

$$\text{Since } 0 < \tan^{-1} k < \frac{\pi}{2} \implies 0 < 5\tan^{-1} k < \frac{5\pi}{2}, n = 1$$

$$\therefore 5\tan^{-1} k = 2\pi \implies k = \frac{2\pi}{5}$$

12 The complex numbers z and w are given by $z = \mathrm{e}^{i\frac{\pi}{6}}$ and $w = -1 - \sqrt{3}i$. If $\frac{z^2 p^*}{w^3}$ is a positive real number and $\left|\frac{p^2 w^2}{z^3}\right| = \frac{4}{9}$, find p in the form $a + bi$. [4]

We first convert w into polar form, which allows us to manipulate powers of complex numbers more easily. We then simplify the argument of $\frac{z^2 p^*}{w^3}$ and modulus of $\frac{p^2 w^2}{z^3}$ to find the argument and modulus of p.

With $z = e^{i\frac{\pi}{6}}$ and $w = 2e^{-i\frac{2\pi}{3}}$,

$$
\begin{aligned}
\arg\left(\frac{z^2 p^*}{w^3}\right) &= 2\arg(z) - \arg(p) - 3\arg(w) \\
&= \frac{\pi}{3} - \arg(p) - (-2\pi) \\
&= 2\pi
\end{aligned}
$$

$$\therefore \arg(p) = \frac{\pi}{3}$$

$$
\begin{aligned}
\left|\frac{p^2 w^2}{z^3}\right| &= \frac{|p|^2 |w|^2}{|z|^3} \\
&= \frac{|p|^2 (4)}{(1)} \\
&= \frac{4}{9}
\end{aligned}
$$

$$\therefore |p| = \frac{1}{3}$$

$$\therefore p = \frac{1}{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{6}(1 + i\sqrt{3})$$

13 In an Argand diagram, $ABCDEF$ is a regular hexagon centred at the origin O where point A represents complex number a, B represents b, and so on. Given that a and d are purely real, and $|a| = |d| = 4$, find the area of:

(a) Rectangle $BCDF$; [2]

$$\begin{aligned}\text{Area of rectangle } BCDF &= (4)(2)(4\sin\frac{\pi}{3})\\ &= 16\sqrt{3}\text{ units}^2\end{aligned}$$

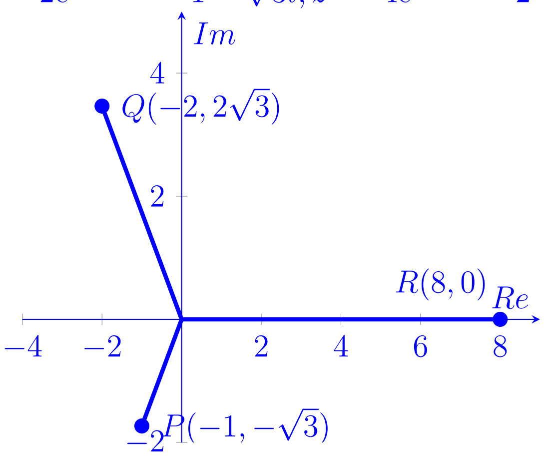
(b) Regular Hexagon $ABCDEF$. [2]

$$\begin{aligned}6\times\text{ Area of triangle } OAB &= (6)(\frac{1}{2})(4)(4)(\sin\frac{\pi}{3})\\ &= 24\sqrt{3}\text{ units}^2\end{aligned}$$

14 The complex number z has modulus 2 and argument $-\frac{2\pi}{3}$.

(a) Sketch an Argand diagram showing the points P, Q and R representing z, z^2 and z^3. [2]

$z = 2e^{-i\frac{2\pi}{3}} = -1-\sqrt{3}i$, $z^2 = 4e^{i\frac{2\pi}{3}} = -2+2\sqrt{3}i$, $z^3 = 8e^{-2\pi i} = 8$



(b) Using your diagram, find the area of ΔPQR. [3]

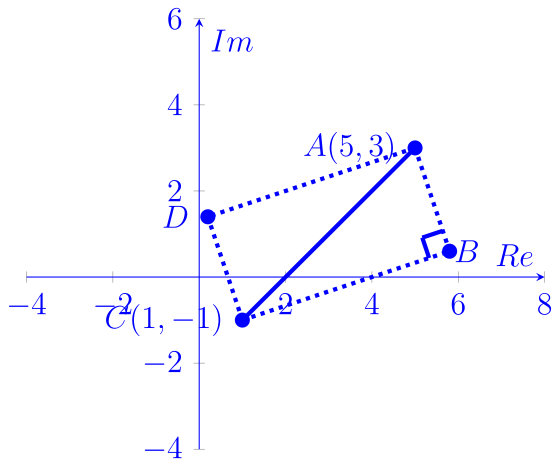
We will use the formula, Area of Triangle $= \frac{1}{2}ab\sin C$, for each of the triangles ΔOPQ, ΔOQR and ΔORP and sum up the three areas.

$$\begin{aligned}
\text{Area of } \Delta PQR &= \text{Area of } \Delta OPQ + \text{Area of } \Delta OQR + \text{Area of } \Delta ORP \\
&= \frac{1}{2}|z||z^2|\sin\frac{2\pi}{3} + \frac{1}{2}|z^2||z^3|\sin\frac{2\pi}{3} + \frac{1}{2}|z^3||z|\sin\frac{2\pi}{3} \\
&= \frac{1}{2}(\frac{\sqrt{3}}{2})(|z^3| + |z^4| + |z^5|) \\
&= \frac{\sqrt{3}}{4}|z^3|(1 + |z| + |z^2|) \\
&= \frac{\sqrt{3}}{4}(8)(1 + 2 + 4) \\
&= 14\sqrt{3}\text{units}^2
\end{aligned}$$

15 In an Argand diagram, the points A, B, C and D represent complex numbers $a = 5 + 3i$, b, $c = 1 - i$ and d respectively such that $ABCD$ is a circle described in a clockwise sense with AC as its diameter.

(a) Calculate the area of the circle $ABCD$. [2]

To visualise the problem, we will sketch an Argand diagram and plot the points A and C.



$$\begin{aligned}\text{Area of circle } ABCD = \frac{1}{2}\pi r^2 &= \frac{1}{2}\pi(\frac{1}{2}|a-c|)^2 \\ &= \frac{1}{8}\pi(|5+3i-(1-i)|)^2 \\ &= \frac{1}{8}\pi(\sqrt{4^2+4^2})^2 \\ &= 4\pi \text{ units}^2\end{aligned}$$

(b) Given that $AB = 2BC$ and $AB = CD$, find b and d. [5]

The vectors $\overrightarrow{BC}$ and $\overrightarrow{BA}$ are represented by complex numbers $c - b$ and $a - b$ respectively. Recall that multiplying a complex number by i gives it a 90-degree anti-clockwise rotation. Given then BC is 2 times of BA and is rotated 90 degrees anti-clockwise from BA, we let $b = x + iy$ and we have:

$$\overrightarrow{BC} = (2)(i)\overrightarrow{BA}$$

$$c - b = (2i)(a - b)$$

$$1 - i - x - iy = 2i(5 + 3i - x - iy)$$

$$(1 - x) + i(-1 - y) = (2y - 6) + i(-2x + 10)$$

Equating real and imaginary parts, $1 - x = 2y - 6$ and $-1 - y = 10 - 2x$.

Using a G.C., we obtain $x = 5.8, y = 0.6$.

$\therefore b = 5.8 + 0.6i$

Since $\overrightarrow{AB} = \overrightarrow{DC}$ we have:

$$\begin{aligned} b - a &= c - d \\ d &= c - b + a \\ &= (1 - i) - (5.8 + 0.6i) + (5 + 3i) \\ &= 0.2 + 1.4i \end{aligned}$$