Complex Numbers

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COMPLEX NUMBERS [90 Marks]

1 Solve the following:

(a) Given that $z = a + ib$ and $w = c + id$, where $a, b, c, d \in \mathbb{R}$,

(i) Show that $(zw)^* = z^*w^*$. [2]

(ii) Hence, show that $(vzw)^* = v^*z^*w^*$ where $v \in \mathbb{C}$. [1]

(b) The locus of a parabola is given by $z = at^2 + i(2at)$ for $a, t \in \mathbb{R}$.

Show that $|z - a| = |\text{Re}(z) + a|$. [3]

(c) It is given that $\text{Im}(\dfrac{a + bi}{a - bi}) = 0$ for some $a, b \in \mathbb{R}$.

Find the possible values of $\dfrac{a + bi}{a - bi}$. [3]

(d) For some $a, b \in \mathbb{R}$, $z^2 + az - b = 0$ has no real roots and $aw^2 + bw + a = 0$ has two real distinct roots. Given that $a > 1$, find an inequality satisfied by b. [3]

(e) It is given that $k = \dfrac{a - z^*}{z} + \dfrac{a - z}{z^*}$, where $z = a + ib$ for some $a, b \in \mathbb{R}$.

Show that $k = \dfrac{2b^2}{a^2 + b^2}$. [3]

(f) Given that $z = a + bi$ and $w = b - ai$, express v in terms of a and b if $v^* = \dfrac{z + w}{zw}$.

[3]

(g) Find $w \in \mathbb{C}$ in the equation $z^2 + wz + 2 = 0$ if one of the roots is i. [3]

(h) For $a, b \in \mathbb{R}$ and $z \in \mathbb{C}$,

(i) Express $|(a - bi)^n|$ in terms of a, b and n. [2]

(ii) Hence, if $(\sqrt{3} - \sqrt{2}i)^n = z$ where $|z| = 625$, find the value of n. [2]

2 Solve the following:

(a) Prove that $\mathrm{Re}(\dfrac{1}{\mathrm{e}^{i\theta}+\mathrm{e}^{-3i\theta}}) \equiv \dfrac{\cos\theta}{2\cos 2\theta}$. [4]

(b) Show that $\dfrac{\csc\theta(\cot\theta+i)}{2\cos\theta(\cot\theta-i)} = \cot 2\theta + i$. [4]

3 Solve the simultaneous equations $zw = \dfrac{5}{2}(1+i)$ and $(1-i)w = \dfrac{iz+3}{2}$. [5]

4 It is given that $\arg(z^6(w^*)^5) = \dfrac{3\pi}{4}$. Given also that $|z^6(w^*)^5| = |z|$ and $z = \sqrt{3}+3i$, find w. [4]

5 Given $k \in \mathbb{R}$ such that $\sqrt{11+ki} = a+bi$, where a and b are positive real numbers,

(a) Express k in terms of a. [2]

(b) Find the values of a and b if $k = 60$. [2]

6 It is given that $z = 1 + \mathrm{e}^{-i\frac{\pi}{4}}$.

(a) Show that $\mathrm{e}^{-i\dfrac{3\pi}{4}} = -\mathrm{e}^{i\dfrac{\pi}{4}}$. [1]

(b) Hence, find $(z-1)^3 + (z-1)^2 + z + i$. [3]

7 Given that $1+i$ is a root of the equation

$$4z^4 - 8z^3 + 17z^2 - 18z + 18 = 0,$$

find all the other roots in the form $a+bi$. [4]

8 The equation $3z^3 + az^2 + bz - 5 = 0$ has a root $z = \dfrac{1}{3}$, where a and b are real, non-zero constants.

Given that the sum of roots is $\dfrac{7}{3}$,

(a) Find the values of a and b. [3]

(b) Find the remaining roots without using a calculator. [2]

9 On an Argand diagram, referred from the origin O, points Z and W represent the complex numbers $z = 1+i$ and $w = -1+\sqrt{3}i$ respectively. Points A and B represent $\mathrm{Re}(z)$ and $\mathrm{Re}(w)$ respectively.

(a) Express z and w in the form $re^{i\theta}$. [1]

(b) Find the area of trapezium $BAZW$. Hence or otherwise, find the area of ΔOZW. [2]

(c) Hence, prove that $\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$. [3]

10 Given that $e^{i\theta} = \cos\theta + i\sin\theta$,

(a) Show that $e^{i(3\theta)} = \cos 3\theta + i\sin 3\theta$. [1]

(b) Find $\text{Im}((\cos\theta + i\sin\theta)^3)$. [2]

(c) Using your answers in **(a)** and **(b)**, express $\sin 3\theta$ in terms of $\sin\theta$. [2]

11 It is given that $z = 1 + ki$, where $k > 0$.

(a) Express z in the form $re^{i\theta}$. [2]

(b) Given that z^5 is a positive real number, find the value of k. [2]

12 The complex numbers z and w are given by $z = e^{i\frac{\pi}{6}}$ and $w = -1 - \sqrt{3}i$. If $\frac{z^2 p^*}{w^3}$ is a positive real number and $\left|\frac{p^2 w^2}{z^3}\right| = \frac{4}{9}$, find p in the form $a + bi$. [4]

13 In an Argand diagram, $ABCDEF$ is a regular hexagon centred at the origin O where point A represents complex number a, B represents b, and so on. Given that a and d are purely real, and $|a| = |d| = 4$, find the area of:

(a) Rectangle $BCDF$; [2]

(b) Regular Hexagon $ABCDEF$. [2]

14 The complex number z has modulus 2 and argument $-\frac{2\pi}{3}$.

(a) Sketch an Argand diagram showing the points P, Q and R representing z, z^2 and z^3. [2]

(b) Using your diagram, find the area of ΔPQR. [3]

15 In an Argand diagram, the points A, B, C and D represent complex numbers $a = 5 + 3i$, b, $c = 1 - i$ and d respectively such that $ABCD$ is a circle described in a clockwise sense with AC as its diameter.

(a) Calculate the area of the circle $ABCD$. [2]

(b) Given that $AB = 2BC$ and $AB = CD$, find b and d. [5]