Vectors

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VECTORS [150 Marks]

1 Solve the following:

a) $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin. Given that lines OA and OB are parallel, $|\mathbf{a}| = 2$ and $\mathbf{a} \cdot \mathbf{b} = -2$, express $\mathbf{b}$ in terms of $\mathbf{a}$. [2]

$$\text{Let } \mathbf{b} = k\mathbf{a} \text{ for some } k \in \mathbb{R}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{a}) = k|\mathbf{a}|^2$$

$$-2 = k(2)^2 \Rightarrow k = -\frac{1}{2}$$

$$\therefore \mathbf{b} = -\frac{1}{2}\mathbf{a}$$

b) A vector $\mathbf{a}$ is such that $\mathbf{a} = (\sqrt{2}\cos\alpha)\mathbf{i} - (\cos\alpha)\mathbf{j} + (\sqrt{2}\sin\alpha)\mathbf{k}$, where $0 \leq \alpha \leq 2\pi$ and $|\mathbf{a}| = \sqrt{2}$. Find the value(s) of α. [2]

$$|\mathbf{a}| = \sqrt{2\cos^2\alpha + \cos^2\alpha + 2\sin^2\alpha}$$

$$\sqrt{2} = \sqrt{2 - \cos^2\alpha}$$

$$2 = 2 - \cos^2\alpha$$

$$\cos\alpha = 0$$

$$\alpha = 0 \text{ or } 2\pi$$

c) The points A, B and C with respect to the origin are represented by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $|\mathbf{b}| = 2$, $\mathbf{a} \cdot \mathbf{b} = k$ and $\mathbf{b} \cdot \mathbf{c} = 2$. Given further that point C divides the line AB such that $AC : CB = 2 : 1$, find k. [3]

$$\mathbf{c} = \frac{\mathbf{a} + 2\mathbf{b}}{3}$$
$$\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \left(\frac{\mathbf{a} + 2\mathbf{b}}{3}\right)$$
$$2 = \frac{1}{3}(\mathbf{b} \cdot \mathbf{a} + 2|\mathbf{b}|^2)$$
$$2 = \frac{1}{3}(k + 8)$$
$$\therefore k = -2$$

d) Four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ exist such that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = \mathbf{0}$. Show that $\mathbf{b} \times (\mathbf{a}+\mathbf{c}) = \mathbf{d} \times \mathbf{b}$. [2]

$$\begin{aligned} \mathbf{b} \times (\mathbf{a} + \mathbf{c}) &= \mathbf{b} \times (-\mathbf{b} - \mathbf{d}) \\ &= \mathbf{b} \times (-\mathbf{b}) - \mathbf{b} \times \mathbf{d} \\ &= \mathbf{d} \times \mathbf{b} \end{aligned}$$

e) Point A referred from the origin has vector $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. The line OA makes an angle of α with the y-axis and β with the z-axis, where $\alpha, \beta < \pi$. Show that

$\alpha + \beta = \pi$. [3]

$$\cos\alpha = \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{(1)\sqrt{1^2+2^2+2^2}} = \frac{2}{3}$$

$$\cos\beta = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}{(1)(3)} = -\frac{2}{3}$$

$$\text{Since } \cos\alpha = -\cos\beta, \text{ and } \alpha, \beta < \pi,$$

$$\cos\alpha = \cos(\pi - \beta)$$

$$\alpha = \pi - \beta$$

$$\alpha + \beta = \pi$$

2 Referred to the origin O, points A and B have position vectors given by $\mathbf{a}$ and $\mathbf{b}$ respectively. C_0 is the foot of perpendicular from A to OB with position vector $\mathbf{c}_0$. The angle between lines OA and OB is α , where $0 < \alpha < \frac{\pi}{2}$.

a) By considering $\cos\alpha$, show that $|\mathbf{c}_0| = \mathbf{a} \cdot \hat{\mathbf{b}}$. [2]

$$\cos\alpha = \frac{|\mathbf{c}_0|}{|\mathbf{a}|}$$

$$\text{Also, } \cos\alpha = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

$$\text{Hence, } \frac{|\mathbf{c}_0|}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$

$$|\mathbf{c}_0| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \cdot \hat{\mathbf{b}}$$

b) The foot of perpendicular from C_0 to OA is C_1. Show that $|\mathbf{c_1}| = \mathbf{a} \cdot \hat{\mathbf{b}}(\cos\alpha)$. [1]

$$\cos\alpha = \frac{|\mathbf{c}_1|}{|\mathbf{c}_0|}$$

$$|\mathbf{c}_1| = |\mathbf{c}_0|\cos\alpha$$

$$= \mathbf{a}\cdot\hat{\mathbf{b}}\cos\alpha$$

c) C_n is the nth foot of perpendicular. State $|\mathbf{c}_n|$ in terms of a, b, n and α. [1]

$$|\mathbf{c}_n| = \mathbf{a}\cdot\hat{\mathbf{b}}\cos^n\alpha$$

d) State the sum to infinity of scalar projections $|\mathbf{c}_0| + |\mathbf{c}_1| + ... + |\mathbf{c}_n| + ...$ [1]

$$\text{Sum to infinity} = \mathbf{a}\cdot\hat{\mathbf{b}} + \mathbf{a}\cdot\hat{\mathbf{b}}\cos\alpha + ... + \mathbf{a}\cdot\hat{\mathbf{b}}\cos^n\alpha + ...$$

$$= \frac{\mathbf{a}\cdot\hat{\mathbf{b}}}{1-\cos\alpha}$$

3 Referred to the origin O, points A and B have position vectors given by: $\mathbf{a} = \mathbf{i} - p^2\mathbf{k}$ and $\mathbf{b} = \frac{2}{p}\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively, where p is to be found. Given that $|\mathbf{a}\times\mathbf{b}|^2 = 4p^2 + 2$, find the value(s) that p can take. [4]

$$|\mathbf{a}\times\mathbf{b}|^2 = \left|\begin{pmatrix}1\\0\\-p^2\end{pmatrix}\times\begin{pmatrix}2p^{-1}\\-1\\1\end{pmatrix}\right|^2$$

$$= \left|\begin{pmatrix}-p^2\\-1-2p\\-1\end{pmatrix}\right|^2$$

$$= \left(p^2\right)^2 + (1+2p)^2 + 1$$

$$= p^4 + 4p^2 + 4p + 2$$

$$= 4p^2 + 2$$

$$p^4 + 4p = 0$$

$$p(p^3+4) = 0$$

$$p = -\sqrt[3]{4} \text{ since } p \neq 0$$

4 The vector equation of l is given by $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$. Point F is the foot of perpendicular from origin O to the line l. If $|\mathbf{b}| = 1$ and $\mathbf{a}\cdot\mathbf{b} = 1$, express the position vector $\overrightarrow{OF}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

$$\text{Let } \overrightarrow{OF} = \mathbf{a} + k\mathbf{b}\text{for some } k \in \mathbb{R}$$

$$\text{Since} OF \perp l, (\mathbf{a} + k\mathbf{b})\cdot\mathbf{b} = 0$$

$$\mathbf{a}\cdot\mathbf{b} + k|\mathbf{b}|^2 = 0$$

$$1 + k(1)^2 = 0$$

$$k = -1$$

$$\therefore \overrightarrow{OF} = \mathbf{a} - \mathbf{b}$$

5 The equations of l and m are given by $l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ and $m : \mathbf{r} = \mathbf{b} + \mu\mathbf{a},\ \mu \in \mathbb{R}$, where $\mathbf{a}$ and $\mathbf{b}$ are co-planar vectors. State the conditions such that lines l and m are skew lines. [2]

Equating l and m and since skew lines are parallel and do not intersect,

$$\mathbf{a} + \lambda\mathbf{b} = \mu\mathbf{a} + \mathbf{b}$$

$$\lambda \neq 1, \mu \neq 1 \text{ and } \mathbf{a} \neq k\mathbf{b} \text{ for all } k \in \mathbb{R}$$

6 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the origin O. It is given that $(\mathbf{a} - 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) = 11$. Find the perpendicular distance from point A to line OB if $|\mathbf{b}| = 11$. [4]

$$\begin{aligned}
(\mathbf{a} - \mathbf{3b}) \times (\mathbf{5a} + \mathbf{7b}) &= \mathbf{a} \times \mathbf{a} + 7\mathbf{a} \times \mathbf{b} - 15\mathbf{b} \times \mathbf{a} - 21\mathbf{b} \times \mathbf{b} \\
11 &= 22\mathbf{a} \times \mathbf{b} \\
\mathbf{a} \times \mathbf{b} &= \frac{1}{2} \\
\text{Perpendicular distance from A to OB } &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \\
&= \frac{\left(\frac{1}{2}\right)}{11} \\
&= \frac{1}{22}\text{units}
\end{aligned}$$

7 Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to the origin O. It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 2$.

a) State the vector equation of line AB. [1]

$$l_{AB} : \mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}),\ \lambda \in \mathbb{R}$$

b) Find $|\mathbf{b} - \mathbf{a}|$. [2]

$$\begin{aligned}
|\mathbf{b} - \mathbf{a}| &= \sqrt{(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})} \\
&= \sqrt{|\mathbf{b}|^2 - 2\mathbf{b} \cdot \mathbf{a} + |\mathbf{a}|^2} \\
&= \sqrt{1 - 2(2) + 3^2} \\
&= \sqrt{6}
\end{aligned}$$

c) Find the position vector of F, the foot of perpendicular from O to AB, in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

$$\text{Let}\overrightarrow{OF} = \mathbf{a} + k(\mathbf{b} - \mathbf{a})\text{for some } k \in \mathbb{R}$$

$$\text{Since } OF \perp AB \text{ , } (\mathbf{a} + k(\mathbf{b} - \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$\mathbf{a} \cdot \mathbf{b} - |\mathbf{a}|^2 + k|\mathbf{b} - \mathbf{a}|^2 = 0$$

$$2 - 3^2 + k(\sqrt{6})^2 = 0$$

$$\therefore k = \frac{7}{6}$$

$$\text{Substituting } k = \frac{7}{6} \text{ back into } \overrightarrow{OF},$$

$$\overrightarrow{OF} = \mathbf{a} + \frac{7}{6}(\mathbf{b} - \mathbf{a})$$

$$\overrightarrow{OF} = \frac{1}{6}(7\mathbf{b} - \mathbf{a})$$

d) Find $|7\mathbf{b} - \mathbf{a}|$. Hence, find the exact area of triangle OAB. [3]

$$|7\mathbf{b} - \mathbf{a}| = \sqrt{(7\mathbf{b} - \mathbf{a}) \cdot (7\mathbf{b} - \mathbf{a})}$$

$$= \sqrt{49|\mathbf{b}|^2 - 14\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2}$$

$$= \sqrt{49 - 14(2) + 3^2}$$

$$= \sqrt{30}$$

$$\text{Area of triangle OAB} = \frac{1}{2} \times \left|\overrightarrow{AB}\right| \times \left|\overrightarrow{OF}\right|$$

$$= \frac{1}{2}|\mathbf{b} - \mathbf{a}| \left|\frac{1}{6}(7\mathbf{b} - \mathbf{a})\right|$$

$$= \frac{1}{2}\left(\sqrt{6}\right)\left(\frac{1}{6}\right)(\sqrt{30})$$

$$= \frac{\sqrt{5}}{2}\text{units}^2$$

8 Referred to the origin O, points A and B have the position vectors $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{k}$ and $\overrightarrow{OB} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ respectively.

a) Verify that $P(3, -2, -6)$ lies on line AB. [2]

$$\text{Line AB} \parallel \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$$

$$\text{Equation of line AB is } \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

$$\text{Substitute } \mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}:$$

$$\begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

There is a solution $\lambda = -2$

Hence, P lies on AB.

b) Find the position vector of F, the foot of perpendicular from P to AB. [3]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{PF} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + k\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$$
$$= \begin{pmatrix} -2-k \\ -2+k \\ 4+2k \end{pmatrix}$$

c) Hence, find the equation of line PF. [3]

Since $PF \perp AB$,

$$\begin{pmatrix} -2-k \\ -2+k \\ 4+2k \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 0$$

$\therefore k = -\dfrac{4}{3}$

Substituting $k = -\dfrac{4}{3}$back into $\overrightarrow{PF}$,

$$\overrightarrow{PF} = \begin{pmatrix} -2+\frac{8}{3} \\ -2-\frac{4}{3} \\ 4-\frac{8}{3} \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$$

Equation of line PF : $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix}$, $\mu \in \mathbb{R}$

9 The equations of lines l_1 and l_2 are given by:

$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and $l_2 : \dfrac{x+1}{9} = \dfrac{y}{7} = \dfrac{4-z}{3}$ respectively.

Point A has coordinates $(2, -1, 1)$ while the foot of perpendicular from A to l_2 is F.

a) Find the position vector of P, the point of intersection between l_1 and l_2. [2]

$$\text{Vector equation of line } l_2 : \mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\text{Equating both lines, } \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}$$

$$\text{Using G.C., we obtain } \lambda = \frac{3}{2},\ \mu = \frac{1}{2}.$$

$$\overrightarrow{OP} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix}$$

b) Find vector $\overrightarrow{AF}$. [3]

$$\text{Let } \overrightarrow{OF} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix}, \text{ for some } k \in \mathbb{R}$$

$$\overrightarrow{AF} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 + 9k \\ 1 + 7k \\ 3 - 3k \end{pmatrix}$$

$$\text{Since } AF \perp l_2, \begin{pmatrix} -3+9k \\ 1+7k \\ 3-3k \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} = 0$$

$$\therefore k = \frac{29}{139}$$

Substituting $k = \dfrac{29}{139}$ back into $\overrightarrow{AF}$, $\overrightarrow{AF} = \begin{pmatrix} -3+9\left(\frac{29}{139}\right) \\ 1+7\left(\frac{29}{139}\right) \\ 3-3\left(\frac{29}{139}\right) \end{pmatrix} = \dfrac{2}{139}\begin{pmatrix} -78 \\ 171 \\ 165 \end{pmatrix}$

c) Hence, find the vector equation of l_3, the reflection of l_1 in l_2. [3]

Let A' on l_3 be the reflection of A in l_2.

$$\begin{aligned}
\overrightarrow{AF} &= \overrightarrow{FA'} \\
&= \overrightarrow{OA'} - \overrightarrow{OF} \\
\overrightarrow{OA'} &= \overrightarrow{AF} + \overrightarrow{OF} \\
&= \frac{2}{139}\begin{pmatrix} -78 \\ 171 \\ 165 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \frac{29}{139}\begin{pmatrix} 9 \\ 7 \\ -3 \end{pmatrix} \\
&= \frac{1}{139}\begin{pmatrix} -34 \\ 545 \\ 973 \end{pmatrix}
\end{aligned}$$

$$l_3 // \frac{1}{139}\begin{pmatrix}-34\\545\\973\end{pmatrix} - \frac{1}{2}\begin{pmatrix}7\\7\\5\end{pmatrix} = \begin{pmatrix}-\frac{1041}{278}\\ \frac{117}{278}\\ \frac{9}{2}\end{pmatrix}$$

$$\therefore l_3 : \mathbf{r} = \frac{1}{2}\begin{pmatrix}7\\7\\5\end{pmatrix} + \alpha\begin{pmatrix}-1041\\117\\1251\end{pmatrix},\ \alpha \in \mathbb{R}$$

10 Points A and B with position vectors $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} + \mathbf{k}$ respectively both lie on l_1.

The line l_2 has Cartesian equation $l_2 : x = 7,\ y - 3 = z$.

a) Show that l_1 and l_2 are skew lines. [2]

$$l_1 // \begin{pmatrix}3\\0\\1\end{pmatrix} - \begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}4\\-2\\2\end{pmatrix}$$

$$l_1 : \mathbf{r} = \begin{pmatrix}3\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix},\ \lambda \in \mathbb{R}$$

$$l_2 : \mathbf{r} = \begin{pmatrix}7\\3\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix},\ \mu \in \mathbb{R}$$

$$\text{Equating both lines, } \begin{pmatrix}3\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}7\\3\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\text{We have the following equations } \begin{cases} 2\lambda = 4 \\ -\lambda - \mu = 3 \\ \lambda - \mu = -1 \end{cases}$$

Using a G.C., there is no solution found.

Hence, l_1 and l_2 are skew lines.

b) Find a vector that is perpendicular to both l_1 and l_2. [1]

A vector that is perpendicular to both l_1 and l_2 // $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$

Let the vector be $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

c) Hence, find the shortest distance between l_1 and l_2. [3]

Let point C on l_2 be $C(7, 3, 0)$.

$$\text{Shortest distance} = \text{Projection of } \overrightarrow{BC} \text{ onto } \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

$$= \frac{\left(\begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\left|\left(\begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}\right)\right| \cdot \left|\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right|}$$

$$= \frac{\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}\right| \cdot \left|\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right|}$$

$$= \frac{8}{\sqrt{26}\sqrt{3}}$$

$$= \frac{4\sqrt{78}}{39} \text{units}$$

11 Referred to an origin O, points A and B have coordinates $(-1, 2, 2)$ and $(0, 1, 2)$ respectively. The point P on OA is such that $OP : PA = \lambda : 1$ and the point Q on OB is such that $OQ : QB = \lambda : 1 - \lambda$, where λ is a real constant to be determined.

a) Find the area of ΔOAB. [2]

$$\begin{aligned}\text{Area of } \Delta OAB &= \tfrac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right| \\ &= \tfrac{1}{2}\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\2\end{pmatrix}\right| \\ &= \tfrac{1}{2}\left|\begin{pmatrix}2\\2\\-1\end{pmatrix}\right| \\ &= \tfrac{1}{2}\sqrt{4+4+1} \\ &= \tfrac{3}{2}\text{units}^2\end{aligned}$$

b) Express the ratio $\frac{\text{Area of } \Delta OAB}{\text{Area of } \Delta OPQ}$ in terms of λ. [3]

$$\begin{aligned}\text{Area of } \Delta OPQ &= \tfrac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right| \\ &= \tfrac{1}{2}\left|\left(\tfrac{\lambda}{\lambda+1}\right)\mathbf{a}\times\left(\tfrac{\lambda}{\lambda+1-\lambda}\right)\mathbf{b}\right| \\ &= \tfrac{1}{2}\left|\tfrac{\lambda^2}{\lambda+1}\mathbf{a}\times\mathbf{b}\right| \\ &= \left(\tfrac{\lambda^2}{\lambda+1}\right)\left(\tfrac{1}{2}\left|\mathbf{a}\times\mathbf{b}\right|\right) \text{ , since } 0<\lambda<1 \text{ so } \tfrac{1}{\lambda+1}>0 \\ &= \left(\tfrac{\lambda^2}{\lambda+1}\right)(\text{Area of } \Delta OAB)\end{aligned}$$

$$\therefore \frac{\text{Area of } \Delta OAB}{\text{Area of } \Delta OPQ} = \frac{\lambda+1}{\lambda^2}$$

c) Deduce if PQ is ever parallel to AB for some value of λ. [3]

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \lambda\mathbf{b} - \left(\tfrac{\lambda}{\lambda+1}\right)\mathbf{a}\end{aligned}$$

Assuming $PQ \,//\, AB$,

$\lambda\mathbf{b} - \left(\frac{\lambda}{\lambda+1}\right)\mathbf{a} = k(\mathbf{b}-\mathbf{a})$ for some $k \in \mathbb{R}$

Equating scalar multiples of $\mathbf{b}$ and $\mathbf{a}$,

$\lambda = k$ and $\frac{\lambda}{\lambda+1} = k$

$\lambda = \frac{\lambda}{\lambda+1}$

$\lambda^2 = 0$

However, clearly $0 < \lambda < 1$, so no value of $k \in \mathbb{R}$ exists for $PQ \,//\, AB$.

12 Line l has the equation $-x = \frac{y-3}{2} = \frac{z+4}{2}$. Line m, which is parallel to $\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}$ where c is some real constant, is obtained by rotating line l 45° about the point $A(0, 3, -4)$. Find the possible vector equations of line m. [5]

$$\text{Equation of line } l: \ \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

$$\text{Equation of line } m: \ \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} = \frac{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\right| \left|\begin{pmatrix} c \\ 0 \\ 1 \end{pmatrix}\right|} = \frac{2-c}{3\sqrt{c^2+1}}$$

$$\frac{1}{2} = \frac{(2-c)^2}{9(c^2+1)}$$

$$9c^2 + 9 = 2c^2 - 8c + 8$$

$$7c^2 + 8c + 1 = 0$$

$$c = -\frac{1}{7} \text{ or } -1$$

The two equations of line m are:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 7 \end{pmatrix}, \mu \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

13 Three points A, B and C referred from the origin O have position vectors given by: $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \frac{3}{2}\mathbf{i} + \frac{5}{2}\mathbf{j} - 3\mathbf{k}$.

a) Find the vector equations of lines AB and AC. [2]

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1.5 \\ 2.5 \\ -3 \end{pmatrix} = \tfrac{1}{2} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

Equations of lines AB and AC are:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}, \ \lambda \in \mathbb{R} \ \text{ and } \ \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \ \mu \in \mathbb{R} \ \text{ respectively.}$$

b) Find two vector equations of l, where l is the line representing the all the midpoints of lines AB and AC. [4]

$$\text{Unit vector of } AB,\ \mathbf{u}_1 = \frac{1}{\sqrt{4^2+1+3^2}}\begin{pmatrix}4\\-1\\-3\end{pmatrix} = \frac{1}{\sqrt{26}}\begin{pmatrix}4\\-1\\-3\end{pmatrix}$$

$$\text{Unit vector of } AC,\ \mathbf{u}_2 = \frac{1}{\sqrt{1+3^2+4^2}}\begin{pmatrix}1\\3\\4\end{pmatrix} = \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}$$

Two possible midpoints have position vectors $\frac{1}{2}(\mathbf{u}_1+\mathbf{u}_2)$ and $\frac{1}{2}(\mathbf{u}_1-\mathbf{u}_2)$.

$$\frac{1}{2}(\mathbf{u}_1+\mathbf{u}_2) = \frac{1}{2}\left(\frac{1}{\sqrt{26}}\begin{pmatrix}4\\-1\\-3\end{pmatrix} + \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}\right) = \frac{1}{2\sqrt{26}}\begin{pmatrix}5\\2\\1\end{pmatrix}$$

$$\frac{1}{2}(\mathbf{u}_1-\mathbf{u}_2) = \frac{1}{2}\left(\frac{1}{\sqrt{26}}\begin{pmatrix}4\\-1\\-3\end{pmatrix} - \frac{1}{\sqrt{26}}\begin{pmatrix}1\\3\\4\end{pmatrix}\right) = \frac{1}{2\sqrt{26}}\begin{pmatrix}3\\-4\\-7\end{pmatrix}$$

Hence, two possible equations of l are:

$$l_1 : \mathbf{r} = \begin{pmatrix}2\\4\\-1\end{pmatrix} + s\begin{pmatrix}5\\2\\1\end{pmatrix},\ s \in \mathbb{R} \text{ and}$$

$$l_2 : \mathbf{r} = \begin{pmatrix}2\\4\\-1\end{pmatrix} + t\begin{pmatrix}-3\\4\\7\end{pmatrix},\ t \in \mathbb{R}$$

14 Point A with position vector $\mathbf{a}$ lies on plane π with normal parallel to vector $\mathbf{n}$. Given that $|\mathbf{a}-\mathbf{n}|^2 = 3$ and $|\mathbf{n}|^2 = 4 - |\mathbf{a}|^2$, find the value of d if the equation of plane π is $\mathbf{r}\cdot\mathbf{n} = d$. [4]

The equation of plane π is $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$.

To find $\mathbf{a}\cdot\mathbf{n}$,

$$\begin{aligned}
|\mathbf{a}-\mathbf{n}|^2 &= 3\\
(\mathbf{a}-\mathbf{n})\cdot(\mathbf{a}-\mathbf{n}) &= 3\\
|\mathbf{a}|^2+|\mathbf{n}|^2-2\mathbf{a}\cdot\mathbf{n} &= \mathbf{3}\\
4-2\mathbf{a}\cdot\mathbf{n} &= \mathbf{3}\\
\mathbf{a}\cdot\mathbf{n} &= \frac{1}{2}
\end{aligned}$$

$\therefore$ The equation of π is $\mathbf{r}\cdot\mathbf{n}=\frac{1}{2}$.

15 The equations of parallel planes p and q are given by $p:\mathbf{r}\cdot\mathbf{n}=d$ and $q:\mathbf{r}\cdot\mathbf{n}=kd$. Line l given by equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b},\ \lambda\in\mathbb{R}$ intersects planes p and q at points A and B respectively.

a) Show that $\overrightarrow{AB}=\mathbf{b}\left(\dfrac{d(k-1)}{\mathbf{b}\cdot\mathbf{n}}\right)$. [4]

Substitute equation of l into p and q to get $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

$$\begin{aligned}
\text{For } A, (\mathbf{a}+\lambda\mathbf{b})\cdot\mathbf{n} &= d\\
\lambda &= \frac{d-\mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\\
\overrightarrow{OA} &= \mathbf{a}+\left(\frac{d-\mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b}\\
\text{For } B, (\mathbf{a}+\lambda\mathbf{b})\cdot\mathbf{n} &= kd\\
\lambda &= \frac{kd-\mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}
\end{aligned}$$

$$
\begin{aligned}
\overrightarrow{OB} &= \mathbf{a} + \left(\frac{kd - \mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b} \\
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= \mathbf{a} + \left(\frac{kd - \mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b} - \left(\mathbf{a} + \left(\frac{d - \mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b}\right) \\
&= \left(\frac{kd - \mathbf{a}\cdot\mathbf{n} - d + \mathbf{a}\cdot\mathbf{n}}{\mathbf{b}\cdot\mathbf{n}}\right)\mathbf{b} \\
&= \mathbf{b}\left(\frac{d(k-1)}{\mathbf{b}\cdot\mathbf{n}}\right)
\end{aligned}
$$

b) Hence, or otherwise, show that the perpendicular distance between planes p and q is equal to $\dfrac{d(k-1)}{|\mathbf{n}|}$ units. [2]

$$
\begin{aligned}
\text{Perpendicular distance between planes } p \text{ and } q &= \text{Projection of } \overrightarrow{AB} \text{ onto } \mathbf{n} \\
&= \frac{\left(\mathbf{b}\left(\frac{d(k-1)}{\mathbf{b}\cdot\mathbf{n}}\right)\right)\cdot\mathbf{n}}{|\mathbf{n}|} \\
&= \frac{\left(\frac{d(k-1)}{\mathbf{b}\cdot\mathbf{n}}\right)(\mathbf{b}\cdot\mathbf{n})}{|\mathbf{n}|} \\
&= \frac{d(k-1)}{|\mathbf{n}|}
\end{aligned}
$$

16 The equations of plane π and l are given by:

$$\pi : \mathbf{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = -1 \text{ and } l : \mathbf{r} = \begin{pmatrix}-1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\k\end{pmatrix},\ \lambda, k \in \mathbb{R} \text{ respectively.}$$

a) Show that, for π and l to intersect, $k \neq -\frac{7}{2}$. [1]

If π and l do not intersect, l is perpendicular to the normal of π.

$$\text{i.e.} \quad \begin{pmatrix} 3 \\ 2 \\ k \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 2k + 7 = 0$$

$$k = -\frac{7}{2}$$

Hence, for intersection, $k \neq -\frac{7}{2}$.

For the rest of the question, assume $k = 1$.

b) Find the coordinates of point P, the point of intersection of π and l. [2]

Equating line and plane,

$$\left(\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -1$$

$$9s + 3 = -1$$

$$s = -\frac{4}{9}$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -\frac{1}{9} \begin{pmatrix} 21 \\ 8 \\ -14 \end{pmatrix}$$

c) Find the shortest distance from $A(-1, 0, 2)$ to π. [3]

$$\overrightarrow{AP} = -\tfrac{1}{9} \begin{pmatrix} 21 \\ 8 \\ -14 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -\tfrac{4}{9} \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix}$$

$$\begin{aligned}\text{Shortest distance } &= \text{Projection of } \overrightarrow{AP} \text{ onto normal of } \pi \\ &= \frac{\left| -\frac{4}{9}\begin{pmatrix}3\\2\\8\end{pmatrix} \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} \right|}{\left|\begin{pmatrix}1\\2\\2\end{pmatrix}\right|} \\ &= \frac{4}{9}\left(\frac{23}{3}\right) \\ &= \frac{92}{27}\text{units}\end{aligned}$$

d) Find the acute angle between π and l. [2]

Let acute angle be α.

$$\sin\alpha = \frac{\left(\frac{92}{27}\right)}{\left|\overrightarrow{AP}\right|} = \frac{\left(\frac{92}{27}\right)}{\frac{4}{9}\left|\begin{pmatrix}3\\2\\8\end{pmatrix}\right|} = \frac{\left(\frac{92}{27}\right)}{\frac{4}{9}\sqrt{77}}$$

$\therefore \alpha = 60.9°$ (1 d.p.)

17 Plane π has a normal parallel to $\begin{pmatrix}-1\\1\\3\end{pmatrix}$ and has the equation:

$$\pi : \mathbf{r} = \begin{pmatrix}7\\4\\3\end{pmatrix} + t\begin{pmatrix}3\\0\\1\end{pmatrix} + s\begin{pmatrix}a\\2\\1\end{pmatrix}, \quad t, s \in \mathbb{R},$$

where a is some real constant to be determined.

a) Find the value of a. [2]

$$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} // \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ a-3 \\ 6 \end{pmatrix}$$

$$\text{Clearly, } \begin{pmatrix} -2 \\ a-3 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore a = 5$$

b) Find the scalar product equation of plane π. [1]

$$\text{Equation is } \mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix} = 6$$

c) Line l passes through π, the origin O and $A(3, 2, 5)$. Find the position vector of P, the point of intersection between line l and plane π. [2]

$$\text{Equation of } l \text{ is } \mathbf{r} = \lambda \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

$$\text{Substituting into equation of } \pi,\ k \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = 6 \text{ for some } k \in \mathbb{R}$$

$$14k = 6$$

$$k = \frac{4}{7}$$

$$\therefore \overrightarrow{OP} = \tfrac{4}{7}\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$$

d) Find $\left|\overrightarrow{PF}\right|$, where F is the foot of perpendicular from A to plane π. [3]

$$\begin{aligned}
\left|\overrightarrow{PF}\right| &= \frac{\left|\overrightarrow{PA} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}\right|}{\left|\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}\right|} \\
&= \frac{\left|\frac{3}{7}\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}\right|}{\sqrt{11}} \\
&= \frac{\frac{3}{7}\left|\begin{pmatrix} 1 \\ -14 \\ 5 \end{pmatrix}\right|}{\sqrt{11}} \\
&= 37\sqrt{1+14^2+5^2}\sqrt{11} \\
&= 37\sqrt{\frac{222}{11}} \text{ units}
\end{aligned}$$

e) Hence find $\left|\overrightarrow{PG}\right|$, where G is the foot of perpendicular from O to plane π. [2]

$\dfrac{\left|\overrightarrow{PG}\right|}{\left|\overrightarrow{PF}\right|} = \dfrac{\left|\overrightarrow{OP}\right|}{\left|\overrightarrow{PA}\right|} = \dfrac{4}{3}$ by similar triangles, so we have:

$$\begin{aligned}\left|\overrightarrow{PG}\right| &= \frac{4}{3}\left|\overrightarrow{PF}\right| \\ &= \frac{4}{3}\left(\frac{3}{7}\sqrt{\frac{222}{11}}\right) \\ &= \frac{4}{7}\sqrt{\frac{222}{11}} \text{ units}^2\end{aligned}$$

18 The equations of planes π_1, π_2 and π_3 are such that:

$$\pi_1 : 2x + 3y + 4z = -1, \quad \pi_2 : -2x + y - z = 5 \quad \text{and} \quad \pi_3 : \mathbf{r} \cdot \begin{pmatrix} a \\ -5 \\ -a \end{pmatrix} = k.$$

a) Find the vector equation of l, the line of intersection between π_1 and π_2. [3]

We have the two equations, $\begin{cases} 2x + 3y + 4z = -1 \\ -2x + y - z = 5 \end{cases}$

Using a G.C., we obtain $x = -2 - \frac{7}{8}z$, $y = 1 - \frac{3}{4}z$, $z \in \mathbb{R}$.

Equation of l is $\mathbf{r} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

b) Given that $a = 2$, find the value of k such that π_3 contains l. [2]

Assuming that π_3 contains l, substituting equation of l into equation of π,

$$\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 7 \\ 6 \\ -8 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix} = k \text{ for all } \lambda \in \mathbb{R}$$

$-4 + 14\lambda - 5 - 30\lambda + 16\lambda = k$

$k = -9$

c) Given that $a = 1,\ k = 3$, find the point of intersection of π_1, π_2 and π_3. [2]

We have the three equations, $\begin{cases} 2x + 3y + 4z = -1 \\ -2x + y - z = 5 \\ x - 5y - z = 3 \end{cases}$

Using a G.C., we obtain $x = -\dfrac{20}{3}$, $y = -3$ and $z = \dfrac{16}{3}$.

19 An incident beam of light was reflected perfectly ($\theta_1 = \theta_2$) on a round mirror.

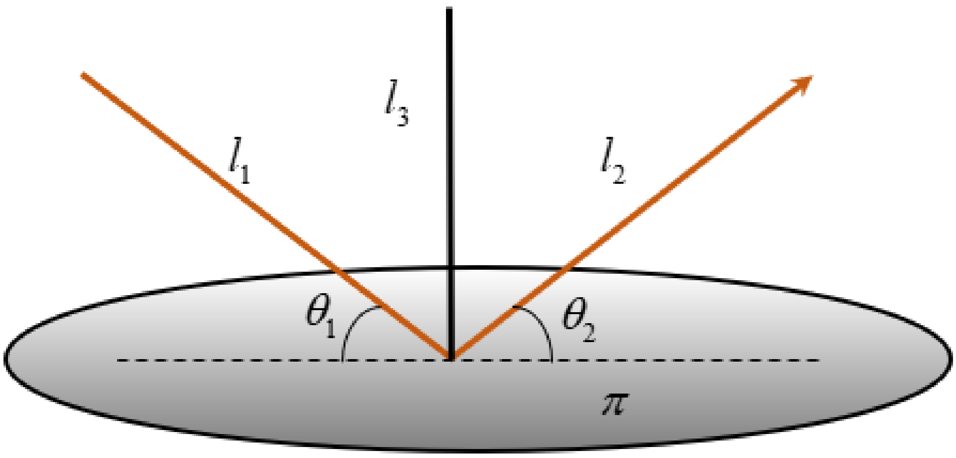

Figure 1: Mirror

A student modelled the scenario such that the incident beam is l_1, the reflected beam is l_2 and the mirror is π, where π contains the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} - \mathbf{j}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

a) Find the equation of the plane π in the form $\mathbf{r}\cdot\mathbf{n} = d$. [3]

$$\text{Normal of plane } \pi \ // \ \left(\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\0\end{pmatrix}\right) \times \left(\begin{pmatrix}2\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\\0\end{pmatrix}\right)$$

$$= \begin{pmatrix}0\\2\\1\end{pmatrix} \times \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-4\\1\\-2\end{pmatrix}$$

$$d = \begin{pmatrix}1\\-1\\0\end{pmatrix} \cdot \begin{pmatrix}4\\-1\\2\end{pmatrix} = 5$$

$$\therefore \text{Equation of } \pi \text{ is } \mathbf{r}\cdot\begin{pmatrix}4\\-1\\2\end{pmatrix} = 5$$

b) Show that $P(1,3,2)$, the point of intersection between l_1 and l_2 lies on π. [1]

$$\text{Substituting } \mathbf{r} = \begin{pmatrix}1\\3\\2\end{pmatrix} \text{ into equation of } \pi,$$

$$\text{LHS} = \begin{pmatrix}1\\3\\2\end{pmatrix} \cdot \begin{pmatrix}4\\-1\\2\end{pmatrix} = 5$$

$$= \text{RHS}$$

c) State the vector equation of l_3, the axis of reflection between l_1 and l_2. [1]

$$l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

d) Given that $A(t, 1, 1)$ lies on l_1, where $t > 0$, find t such that $\theta_1 = \theta_2 = \frac{\pi}{4}$. [4]

$$\overrightarrow{PA} = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix}$$

Let α be the angle between l_1 and l_3.

$$\cos\alpha = \frac{\begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}}{\left|\begin{pmatrix} t-1 \\ -2 \\ -1 \end{pmatrix}\right| \cdot \left|\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}\right|}$$
$$= \frac{4t-4}{\sqrt{t^2-2t+6\sqrt{21}}}$$

Since $\theta_1 = \frac{\pi}{4}$, $\alpha = \frac{3\pi}{4}$ or $\frac{\pi}{4}$.

Regardless, $\cos^2\alpha = \frac{1}{2}$.

$$\text{Squaring both sides, } \frac{1}{2} = \frac{16t^2 - 32t + 16}{21t^2 - 42t + 126}$$
$$11t^2 - 22t - 94 = 0$$

Using G.C., $t = 2.50$ or -3.41 (3s.f.)

Since $t > 0$, $t = 2.50$

For the rest of the question, take $t = 4$.

e) Find the shortest distance from A to π. [2]

$$\text{Shortest distance} = \frac{\left| \overrightarrow{AP} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}$$

$$= \frac{\left| \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right|}{\sqrt{21}}$$

$$= \frac{\left| \begin{pmatrix} 5 \\ 10 \\ -5 \end{pmatrix} \right|}{\sqrt{21}}$$

$$= \frac{5}{\sqrt{21}}\sqrt{1+2^2+1}$$

$$= \frac{5\sqrt{14}}{7}\text{units}$$

f) Find the coordinates of F, the foot of perpendicular from A to l_3. Hence, or otherwise, find the equation of l_2. [6]

Let $\overrightarrow{OF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$, for some $k \in \mathbb{R}$

$\overrightarrow{AF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3+4k \\ 2-k \\ 1+2k \end{pmatrix}$

Since $\overrightarrow{AF} \perp l_3$,

$\begin{pmatrix} -3+4k \\ 2-k \\ 1+2k \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = 0$

$-12 + 21k = 0$

$k = \frac{4}{7}$

$$\overrightarrow{OF} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 23 \\ 17 \\ 22 \end{pmatrix}$$
$$\therefore F \left(\frac{23}{7}, \frac{17}{7}, \frac{22}{7} \right)$$

Let A' be the reflection of point A in l_3.

$$
\begin{aligned}
\overrightarrow{OA'} &= \overrightarrow{OA} + 2\overrightarrow{AF} \\
&= \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + 2\left(\frac{1}{7}\begin{pmatrix} 23 \\ 17 \\ 22 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}\right) \\
&= \frac{1}{7}\begin{pmatrix} 18 \\ 27 \\ 37 \end{pmatrix} \\
\overrightarrow{A'P} &= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - \frac{1}{7}\begin{pmatrix} 18 \\ 27 \\ 37 \end{pmatrix} \\
&= \frac{1}{7}\begin{pmatrix} -11 \\ -6 \\ -23 \end{pmatrix}
\end{aligned}
$$

Since l_2 is parallel to $\overrightarrow{A'P}$ and contains P, equation of l_2 is

$$
\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 11 \\ 6 \\ 23 \end{pmatrix},\ \mu \in \mathbb{R}.
$$

20 A professional card stacker stacks two cards P and Q as follows:

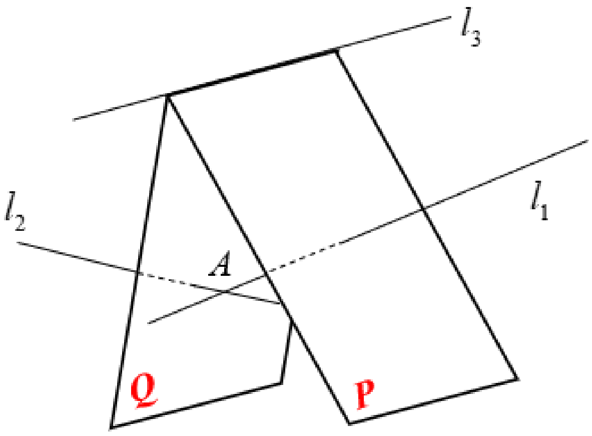

Figure 2: Card Stack

Mr Poh models the scenario such that the two cards are planes P and Q, where the equation of plane P is $P : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1$, where line l_1 is a normal to plane P that contains $A(-3, 3, 2)$. Line l_2, a normal to plane Q, also contains point A and is parallel to the vector $\begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}$ where $a < 0$. l_3 is the line of intersection between planes P and Q.

a) Find the coordinates of B, the point of intersection between l_1 and plane P. [2]

$$\text{Equation of } l_1 \text{ is } \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$$

Substitute equation of l_1 into equation of P :

$$\left(\begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + k \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 1 \text{ for some } k \in \mathbb{R}$$

$-10 + 11k = 1$

$k = 1$

$$\overrightarrow{OB} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$\therefore B(0,\ 2,\ 3)$

b) Given that l_2 is obtaining by rotating l_1 $\cos^{-1}\dfrac{9}{11}$ about point A, find a; Hence, find the vector equation of l_2. [4]

$$\pm \cos\left(\cos^{-1}\frac{9}{11}\right) = \frac{\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ -1 \\ a \end{pmatrix}\right|}$$

$\pm\frac{9}{11} = \frac{10+a}{\sqrt{11}\sqrt{10+a^2}}$

$\frac{81}{121} = \frac{100+a^2+20a}{110+11a^2}$

$8910 + 891a^2 = 12100 + 121a^2 + 2420a$

$770a^2 - 2420a - 3190 = 0$

Using a G.C., we obtain $a=-1$ or $\dfrac{29}{7}$. Since $a<0$, $a=-1$.

$$\therefore l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

c) The point of intersection between l_2 and plane Q is point B'. Given that $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AB'}\right|$, find the position vector of B' given that the x-coordinate of $B' < 0$. [3]

$$\begin{aligned} \left|\overrightarrow{AB'}\right| &= \left|\overrightarrow{AB}\right| \\ &= \left| \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{11} \end{aligned}$$

$$\text{Unit vector along } AB' = \frac{\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}}{\left| \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \right|} = \frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{OB'} = \overrightarrow{OA} \pm \frac{\left|\overrightarrow{AB'}\right|}{\sqrt{11}} \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \pm \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix}$$

Since the x - coordinate of $B' < 0, \overrightarrow{OB'} = \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix}$

d) Find the equation of the plane Q in the form $\mathbf{r} \cdot \mathbf{n} = d$. [2]

Since plane Q contains B', equation of plane Q is

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 25$$

e) Find the vector equation of l_3. [2]

We have the two equations, $\begin{cases} 3x - y + z = 1 \\ 3x - y - z = -25 \end{cases}$

Using a G.C., we obtain $x = -4 + \frac{1}{3}y,\ y \in \mathbb{R},\ z = 13.$

Equation of l_3 is $\mathbf{r} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + \tau \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix},\ \tau \in \mathbb{R}.$

The equation of plane R is such that it reflects the image of plane P to form plane

Q.

f) Find the vector $\overrightarrow{AF}$, where F is the foot of perpendicular from A to l_3.

Hence, find the equation of plane R in the form $\mathbf{r}\cdot\mathbf{n} = d$. [5]

Let $\overrightarrow{OF} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + k\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ for some $k \in \mathbb{R}$

$\overrightarrow{AF} = \begin{pmatrix} -4 \\ 0 \\ 13 \end{pmatrix} + k\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1+k \\ -3+3k \\ 11 \end{pmatrix}$

Since $\overrightarrow{AF} \perp l_3$,

$\begin{pmatrix} -1+k \\ -3+3k \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 0$

$-10 + 10k = 0$

$k = 1$

$\therefore \overrightarrow{AF} = \begin{pmatrix} -1+1 \\ -3+3 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 11 \end{pmatrix}$

Since plane R is parallel to both $\overrightarrow{AF}$ and l_3,

Normal to plane R // $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$

Equation of plane R is $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} = 12$