Functions

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FUNCTIONS [60 Marks]

1 The function f is defined by

$$\mathrm{f} : x \mapsto |\ln(3x) + 1|, x > a.$$

(a) Find the smallest value of a such that f^{-1} exists. [1]

(b) Using the domain of f found in **(a)**, sketch the graph of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram. [4]

2 The function f is defined by

$$\mathrm{f} : x \mapsto \sqrt{a^2 - x^2} + k,\ 0 < x < a.$$

(a) Given that f is *self-inverse*, meaning that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, state the conditions that must be fulfilled by a and k. [2]

(b) Hence, find the exact value of x such that $\mathrm{f}^{-1}\mathrm{f}^{-1}(x) = \mathrm{fff}(x)$. [3]

3 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{1}{2}\sqrt{4 - x^2},\ 0 \le x \le 2.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state the domain and range of f^{-1}. [2]

(b) State the domains and ranges of ff^{-1} and $\mathrm{f}^{-1}\mathrm{f}$. [2]

(c) State the set of values of x for which $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [1]

(d) Find the exact solution of $\mathrm{ff}^{-1}\mathrm{f}(x) = \frac{1}{2}$. [2]

4 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{1-x}, \; x \in \mathbb{R}$$

(a) Show that $\mathrm{fff}(x) = x$. [2]

(b) Show that $\mathrm{f}^{-1}\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(c) Hence, find the exact value(s) of a such that $\mathrm{f}(\mathrm{fff}(a) + \mathrm{ff}(a) + \mathrm{f}(a)) = \mathrm{f}^{-1}(a)$. [4]

5 The function f is defined by

$$\mathrm{f}(x) = \begin{cases} -\dfrac{1}{1+x} & \text{for } -1 < x < 1, \\ \frac{1}{2}x - 1 & \text{for } x \geq 1. \end{cases}$$

(a) Sketch the graph of f. Hence, show that f is one-one. [3]

(b) Find f^{-1}. [2]

6 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2+1}{x-1}, \; x > a.$$

(a) Given that f is one-one, state the exact value of a. [2]

(b) Find $\mathrm{f}^{-1}(x)$ and state the domain and range of f^{-1}. [5]

7 Functions f and g are defined as follows:

$$\mathrm{f}(x) = \frac{1}{2}\mathrm{e}^{1-x}, \; x \geq 0,$$

$$\mathrm{g}(x) = 1 - \ln(x), \; 0 < x \leq \mathrm{e}.$$

(a) Show that both fg and gf exist. [3]

(b) By finding expressions for fg and gf, find the exact solution of $\mathrm{fg}(2) = \mathrm{gf}(\ln(x))$. [3]

8 The functions f and g are defined by:

$$\mathrm{f}(x) = \begin{cases} 2\sqrt{x-2} & \text{for } 2 \leq x < 6, \\ 6 - \sqrt{\frac{2x}{3}} & \text{for } x \geq 6. \end{cases} \quad \text{and} \quad \mathrm{g}(x) = x^2,\ x \in \mathbb{R}.$$

(a) Show that gf exists. [2]

(b) Find the exact value of gf(4). [2]

(c) Find the exact value of x such that $\mathrm{gf}(x) = 5$. [2]

9 The function h is defined by:

$$\mathrm{h}(n) = \begin{cases} n(\mathrm{h}(n-1)) & \text{for } n > 1, \\ 1 & \text{for } n = 1. \end{cases}$$

where $n \in \mathbb{Z}^+$.

(a) State the values of h(2), h(3) and h(4). [1]

(b) Deduce the use of function h. [1]

(c) Evaluate $\dfrac{\mathrm{h}(40)}{\mathrm{h}(10)\mathrm{h}(30)}$. [2]

10 The function f has an inverse and is such that

$$\mathrm{f} : x^2 + 3 \mapsto x,\ x > 0.$$

(a) Find $\mathrm{f}(x)$, and write down its domain and range. [3]

(b) The function g is such that $\mathrm{g}(3x+2) = \mathrm{f}(x)$.

Find $\mathrm{g}(x)$. State its domain and range. [4]