Integration

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INTEGRATION [80 Marks]

1 Integrate the following:

(a) $\int \tan^{-1}(\frac{\pi}{3}x)\,\mathrm{d}x$ [4]

(b) $\int x^3 \mathrm{e}^{3x}\,\mathrm{d}x$ [4]

(c) $\int_{-\frac{1}{2}}^{1} \frac{1}{\sqrt{(1-x)(x+2)}}\,\mathrm{d}x$ [4]

(d) $\int_0^{\ln 2} \frac{1}{e^x + e^{2x}}\,\mathrm{d}x$, using the substitution $x = \ln u$. [5]

2 Show that $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$.

Hence or otherwise, find $\int \csc^3 x\,\mathrm{d}x$, given that $0 < x < \pi$. [6]

3 Show that $\cos a\theta \cos 3a\theta = \dfrac{\cos 2a\theta + \cos 4a\theta}{2}$.

Hence by Integration by Parts, find $\int_0^{\pi/4} 2\theta \cos\theta \cos 3\theta\,d\theta$. [6]

4 A geometric progression has first term a and common ration x, where $a, x \in \mathbb{R}$. The sum of the first n terms of the geometric progression is denoted by S_n.

(a) State in terms of a and x, the nth term of the geometric progression. [1]

(b) Given that the sum of the first n terms of the Harmonic Series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = H_n$, show that $\int_0^1 S_n\,\mathrm{d}x = aH_n$ [3]

5 Sketch the graph of $f(x) = 0.1\mathrm{e}^x$ for $0 \leq x \leq \pi, x \in \mathbb{R}$. Using a Graphing Calculator, find the region enclosed by $y = f(x)$ and the x-axis. [5]

6 Curves C_1 and C_2 are given by the equations $y = x\cos x$ and $y = x$ respectively. For $0 \leq x \leq 2\pi$, find the exact area enclosed by C_1 and C_2. [4]

7 An engineer from FlowerFans.com models a new fan with the parametric equations $x = 3\cos\theta\cos 2\theta, y = 3\sin\theta\cos 2\theta$ for $0 \leq x \leq 2\pi$. Using a graphing calculator, find the total area of the blades of the fan in units^2. [4]

8 Find the exact volume of region R rotated through 2π radians about the x-axis, where R is the region enclosed by $y = \dfrac{1-x}{\sqrt{x}}$, the line $x = 2$ and the x-axis. [4]

9 The curve C has the equation $y = |xe^x|$.

(a) Sketch the curve C. [2]

(b) Find the volume of region S rotated through 2π radians about the x-axis, where S is the region enclosed by C, the lines $x = -1$ and $x = 1$, and the x-axis. [4]

10 The curve C has the equation $e^{\frac{1}{2}x-1}$.

(a) Find the equation of L, where L is the tangent to the curve C at $x = 4$. [2]

(b) Find the region enclosed by C, L, and the axes. [3]

(c) Find the volume of the above-mentioned region when it is rotated by 2π radians about the y-axis. [3]

11 A ball partially submerged in a beaker of water can be expressed as the following graph from the side view, where the ball is a circle and the water surface is the x-axis. Given

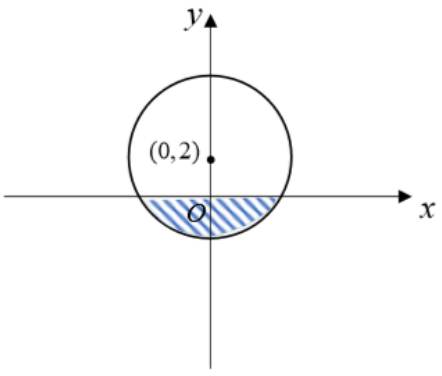

Figure 1: Partially Submerged Ball

that the centre of the ball is at $(0, 2)$ and that the radius of the ball is 4 units, show that the volume of the ball submerged in water is equal to $\frac{40}{3}\pi$ units^3. [7]

12 An architecture student is attempting to model the famous Burj Al Arab in wood (sculpting wood comes at \$1000.00/units^3). He approximates the shape of the building to the curve C with equation $(y-k)^2 = r^2k(k-x)$ for $x, y \geq 0$ where k and r are positive constants used for scaling purposes.

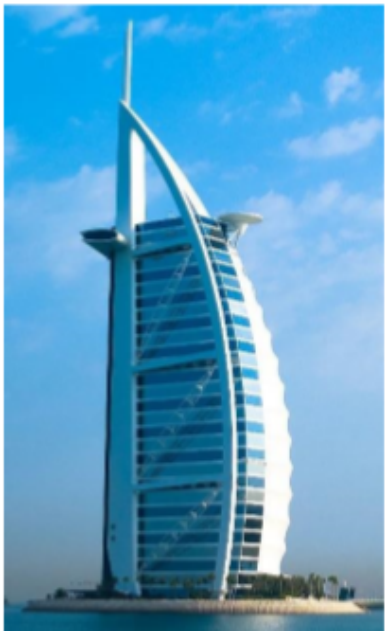

Figure 2: Burj Al Arab, taken from Wikipedia

(a) Given that the wooden model is formed by rotating the region enclosed by C and the axes $\frac{\pi}{4}$ radians around the y-axis, calculate the volume of the model in terms of k and r. [6]

(b) Calculate the amount of money spent on wood by the architecture student to construct the model if $k = 1, r = \frac{3}{2}$. [3]