Topological Properties and Open-System Dynamics of the SSH and Rice–Mele Models

Research Report

1 Introduction

The Su–Schrieffer–Heeger (SSH) and Rice–Mele models are canonical one-dimensional tight-binding models in condensed matter physics that exhibit nontrivial topological phenomena. The SSH model was introduced to describe trans-polyacetylene, a one-dimensional organic polymer with alternating bond strengths. Its bipartite lattice structure and alternating nearest-neighbor hopping amplitudes lead to two distinct topological phases, characterized by a quantized Berry (Zak) phase and resulting in localized zero-energy boundary states at interfaces between phases. The Rice–Mele model extends the SSH chain by adding staggered on-site potential energies (a dimerization in both hopping and potential), breaking sublattice (chiral) symmetry. This model provides a paradigmatic realization of a Thouless charge pump: under an adiabatic cycle of the dimerization parameters, an integer charge is transported, corresponding to a quantized change in polarization (Berry phase). Both models serve as toy examples of one-dimensional topological insulators and have been widely studied theoretically and experimentally. In this report, we define their Hamiltonians, derive the band eigenstates and topological invariants (Zak phases), and then incorporate environmental dissipation via the Lindblad formalism. We focus on amplitude-damping (particle loss), pure dephasing, and thermal dissipation channels, and present numerical simulations of the Lindblad dynamics using the QuTiP library.

2 SSH Model: Hamiltonian and Topology

The SSH model describes spinless fermions (or equivalently a single tight-binding electron) on a 1D chain with staggered nearest-neighbor hoppings. Denote by $c_i^\dagger$ (c_i) the fermionic creation (annihilation) operator on site i. Labeling sites so that even and odd sites form two sublattices, one convenient form of the Hamiltonian is

$$H_{\mathrm{SSH}} = \sum_{j=1}^{N} \Big[t_- \, d_j^\dagger c_j + t_+ \, c_{j+1}^\dagger d_j + \mathrm{h.c.} \Big],$$

where we define $c_{2j-1} \equiv c_j$ and $c_{2j} \equiv d_j$ as the two sublattice orbitals in the jth unit cell. Here $t_+ = t_0 + \delta$ and $t_- = t_0 - \delta$ are the alternating hopping amplitudes. In real space this corresponds to nearest-neighbor hoppings $t_0 \pm \delta$ in an alternating pattern; δ is the dimerization parameter. (The original SSH form in single-site indexing is $H = \sum_i [t_0 + (-1)^i \delta]\, c_i^\dagger c_{i+1} + \text{h.c.}$.) In momentum space one defines two-component Bloch operators (c_k, d_k) and finds the Bloch Hamiltonian matrix

$$H(k) = (c_k^\dagger,\ d_k^\dagger) \begin{pmatrix} 0 & t_- + t_+ e^{-ika} \\ t_- + t_+ e^{+ika} & 0 \end{pmatrix} \begin{pmatrix} c_k \\ d_k \end{pmatrix}.$$

Equivalently, one writes $H(k) = h_x(k)\sigma_x + h_y(k)\sigma_y$ with (setting lattice constant $a = 1$)

$$h_x(k) = t_- + t_+ \cos k, \quad h_y(k) = t_+ \sin k, \quad h_z(k) = 0, \tag{1}$$

as obtained by expanding the standard SSH dispersion. Here $\sigma_{x,y}$ are Pauli matrices in the sublattice basis. Diagonalizing $H(k)$ gives two energy bands $\varepsilon_\pm(k) = \pm|h(k)|$, where $|h(k)| = \sqrt{h_x^2 + h_y^2}$. The band gap is $2|t_+ - t_-| = 4|\delta|$, which closes only when $\delta = 0$ (the transition point).

The eigenstates $|\psi_\pm(k)\rangle$ are two-component spinors on the Bloch sphere of $\mathbf{h}(k) = (h_x, h_y, 0)$. For instance, for the lower (valence) band one can write

$$|\psi_-(k)\rangle \propto \begin{pmatrix} h_x(k) - i h_y(k) \\ -|h(k)| \end{pmatrix},$$

normalized to unity. The Berry connection for this band is $\mathcal{A}(k) = i\langle\psi_-(k)|\partial_k \psi_-(k)\rangle$. Since $h_z = 0$, $\mathbf{h}(k)$ lies in the xy-plane, and one can parametrize $h_x + i h_y = |h| e^{i\phi(k)}$ with $\phi(k) = \arg[h_x + i h_y]$. A straightforward calculation then yields the Zak (Berry) phase over the Brillouin zone,

$$\gamma_{\text{Zak}} = \int_0^{2\pi} \mathcal{A}(k)\, dk = \frac{1}{2}\big[\phi(2\pi) - \phi(0)\big].$$

One finds $\phi(2\pi) - \phi(0) = 2\pi$ if $t_+ > t_-$ (i.e. $\delta > 0$), and zero if $t_+ < t_-$ (i.e. $\delta < 0$). Hence the Zak phase is π in the topological regime ($t_+ > t_-$) and 0 in the trivial regime. These two cases correspond to two distinct bulk phases: in the topological phase the loop of $\mathbf{h}(k)$ winds once around the origin in (h_x, h_y)-space, giving a Berry phase of π, whereas in the trivial phase it winds zero times giving phase 0.

By the bulk–boundary correspondence, the nontrivial SSH phase supports zero-energy edge (or soliton) states at domain walls or open ends. In a finite chain or at a junction between domains of opposite dimerization, one finds mid-gap states localized exponentially at the boundaries. This is readily seen by writing a continuum Dirac-like equation near $k = \pi$ for a spatially varying $\delta(x)$: one obtains a Jackiw–Rebbi soliton solution at $E = 0$. Experimentally, such topologically protected boundary modes have been observed in cold-atom and photonic SSH simulators.

3 Rice–Mele Model: Hamiltonian and Topology

The Rice–Mele (RM) model generalizes SSH by including staggered on-site potentials that break sublattice symmetry. Its Hamiltonian can be written as

$$H_{\mathrm{RM}} = H_{\mathrm{SSH}} + \Delta \sum_{j=1}^{N} (c_j^\dagger c_j - d_j^\dagger d_j), \tag{2}$$

where H_{SSH} is the SSH Hamiltonian above and $\pm\Delta$ are the on-site energies on the two sublattices. In momentum space one obtains the Bloch Hamiltonian

$$H(k) = h_x(k)\,\sigma_x + h_y(k)\,\sigma_y + \Delta\,\sigma_z,$$

where now (restoring $a = 1$)

$$h_x(k) = t_- + t_+ \cos k, \quad h_y(k) = t_+ \sin k, \quad h_z(k) = \Delta.$$

In vector form $\mathbf{h}(k) = (h_x, h_y, \Delta)$. The energy bands are

$$\varepsilon_\pm(k) = \pm\sqrt{h_x(k)^2 + h_y(k)^2 + \Delta^2},$$

so there is generically a gap for any nonzero Δ or δ. In the absence of Δ one recovers SSH. Because $\Delta \neq 0$ breaks chiral (sublattice) symmetry, the Zak phase for a single band is no longer quantized to 0 or π in general. However, if one varies (δ, Δ) cyclically in time, the total polarization change per cycle is given by a Chern number of the 2D parameter space (a Thouless pump). Concretely, Rice–Mele yields a quantized pumped charge when $(\delta(t), \Delta(t))$ encircle the origin, corresponding to the Berry phase acquired over the 2D loop.

The eigenstates of the Rice–Mele model can be found similarly by diagonalizing the 2×2 Bloch Hamiltonian. The Berry (Zak) phase over the Brillouin zone is given by

$$\gamma_{\mathrm{Zak}} = i \int_0^{2\pi} \langle u_-(k) | \partial_k u_-(k) \rangle dk,$$

where $|u_-(k)\rangle$ is the lower-band eigenvector. In general with $\Delta \neq 0$, this phase depends on $\Delta/t_\pm$ and need not be quantized. In the special limit $\Delta = 0$ it reduces to the SSH result. Notably, under an adiabatic cycle $(\delta, \Delta) \to (\delta, \Delta)$ with period T, the Bloch vector $\mathbf{h}(k, t)$ traces a closed loop in (h_x, h_y, h_z)-space, and the change in electronic polarization (or pumped charge) is given by the solid angle enclosed by this loop.

4 Eigenstates and Zak Phase Derivations

We sketch the algebraic derivation of the SSH eigenstates and their Zak phase. From Eq. (1), $H(k)$ has the form $h_x\sigma_x + h_y\sigma_y$. Its eigenenergies are $E_\pm(k) = \pm|h(k)|$. The (unnormalized) eigenvector for $E_-(k)$ (the valence band) satisfies

$(h_x - ih_y)\psi_A + 0\,\psi_B = -|h|\psi_A$ (in sublattice basis (ψ_A, ψ_B)), leading to $\psi_A \propto |h| - (h_x - ih_y)$, $\psi_B \propto (h_x - ih_y) + |h|$. One convenient normalized form is

$$|\psi_-(k)\rangle = \frac{1}{\sqrt{2|h|(|h| - h_x)}} \begin{pmatrix} |h| - h_x \\ h_y + i \cdot 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\phi(k)/2} \\ -e^{+i\phi(k)/2} \end{pmatrix},$$

where $\phi(k) = \arg(h_x + ih_y)$ is the azimuthal angle of (h_x, h_y) in the plane. The Berry connection is $\mathcal{A}(k) = i\langle\psi_-|\partial_k\psi_-\rangle = \frac{1}{2}\partial_k\phi(k)$, so the Zak phase is

$$\gamma_{\text{Zak}} = \int_0^{2\pi} \mathcal{A}(k)\, dk = \frac{1}{2}[\phi(2\pi) - \phi(0)].$$

Because $\phi(k)$ winds by 2π if $t_+ > t_-$ (so the vector (h_x, h_y) encircles the origin once), one finds $\gamma_{\text{Zak}} = \pi$ in that topological case; whereas if $t_+ < t_-$ it winds zero times and $\gamma_{\text{Zak}} = 0$. Thus the Zak phase is π or 0, yielding a $\mathbb{Z}_2$ invariant for the SSH chain.

For the Rice–Mele model, the eigenvectors acquire a nonzero third component. One can similarly diagonalize $H(k)$ to find the two-band spinors. In general, one obtains a Berry phase that depends continuously on (δ, Δ) rather than being quantized. However, if $\Delta = 0$ (restoring inversion symmetry), one recovers the SSH result. More generally, the polarization P of the filled band is given by $P = \gamma_{\text{Zak}}/2\pi$, which becomes a pumping invariant under adiabatic cycles.

5 Open Quantum Systems and the Lindblad Equation

To study dissipation and decoherence, we use the formalism of open quantum systems. In the Markovian limit, the evolution of the system's density matrix $\rho(t)$ is given by the Gorini–Kossakowski–Lindblad–Sudarshan (GKLS) master equation:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \sum_k \gamma_k \Big(L_k\, \rho\, L_k^\dagger - \tfrac{1}{2}\{L_k^\dagger L_k, \rho\} \Big), \tag{3}$$

where H is the system Hamiltonian (e.g. the SSH or Rice–Mele Hamiltonian) and the L_k are Lindblad (jump) operators describing specific dissipative processes. The rates $\gamma_k \geq 0$ set the strength of each channel. This form guarantees complete positivity and trace preservation.

Common examples of Lindblad operators include: (i) *Amplitude damping* (particle loss) with $L = a$ (the annihilation operator) which leads to decay of excitations; (ii) *Amplitude raising* (gain) with $L = a^\dagger$ for thermal pumping; (iii) *Pure dephasing* (phase damping) with $L = \sigma_z$ or $L = a^\dagger a$ which randomizes phase without changing populations; (iv) *Thermal baths* which can be modeled by combinations of raising and lowering operators satisfying detailed balance at a given temperature. In this language, amplitude damping and excitation

correspond to energy exchange (dissipation) with the bath, while pure dephasing corresponds to heatless decoherence of off-diagonal terms. Lindblad operators can implement energy loss/gain (amplitude damping) or pure dephasing (phase damping).

For our lattice models, we consider local dissipators on each site. For example, an amplitude-damping channel on site i is realized by $L_i = \sqrt{\Gamma}\, c_i$ (decay of a fermion) or $L_i = \sqrt{\Gamma}\, c_i^\dagger$ (excitation), leading to loss or gain of population. A pure dephasing channel can be taken as $L_i = \sqrt{\Gamma_\phi}\, n_i$ (with $n_i = c_i^\dagger c_i$) or equivalently $L_i \propto \sigma_z$, which preserves the occupation probabilities but damps coherences. Thermalization (heat dissipation) can be approximated by Lindblad terms that drive the system toward a Gibbs ensemble; e.g. one includes both loss and gain with rates obeying the detailed-balance ratio $e^{-\beta\hbar\omega}$. In all cases, the structure of Eq. (3) ensures that populations and coherences evolve as expected: amplitude damping causes exponential decay of excited-state population and energy, while pure dephasing leaves populations fixed but suppresses off-diagonal density matrix elements.

6 Lindblad Dynamics of the SSH/Rice–Mele Models

We now apply the Lindblad equation to the SSH and Rice–Mele chains. We take the system Hamiltonian H from the above models, and include Lindblad operators acting independently on each lattice site. In particular, we focus on two prototypical noise channels: (1) *Amplitude damping*, implemented by $L_j = \sqrt{\gamma} c_j$ on each site; (2) *Pure dephasing*, implemented by $L_j = \sqrt{\gamma_\phi}\, n_j$ on each site. One can also include a thermal bath by combining these. The resulting master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_j \Big(\gamma\, c_j \rho c_j^\dagger - \frac{\gamma}{2}\{c_j^\dagger c_j, \rho\} \Big) + \sum_j \Big(\gamma_\phi\, n_j \rho n_j - \frac{\gamma_\phi}{2}\{n_j^2, \rho\} \Big)$$

describes the competition between Hamiltonian-driven coherent evolution and environmental dissipation. In physical terms, γ might represent a phonon- or photon-induced relaxation rate, while γ_ϕ represents pure dephasing due to classical noise.

We are particularly interested in how these noise channels affect the topological edge modes and coherence. In principle, one can track the fate of the Zak phase or winding number under dissipation; recent work shows that topological invariants can sometimes be generalized to mixed states or to the Liouvillian spectrum. In this report we focus on dynamical observables: how quickly does an edge-state population decay, or how fast does coherence vanish, under various Lindblad channels? We address this by numerical simulation.

7 Numerical Simulations (QuTiP)

To illustrate the open-system behavior, we simulate small SSH and Rice–Mele chains under Lindblad dynamics. We use the QuTiP Python library to solve the master equation numerically. As a concrete example, consider an SSH chain of two unit cells (four sites). We set $t_0 = 1.0$, $\delta = 0.5$ (so $t_+ = 1.5$, $t_- = 0.5$) in the Hamiltonian. We prepare an initial state $\rho(0)$ with a single particle in a superposition of the first two sites (an edge-like excitation). We then solve

$$\dot{\rho} = -i[H_{\mathrm{SSH}}, \rho] + \sum_{j=1}^{4} \gamma\, c_j \rho c_j^\dagger - \tfrac{\gamma}{2}\{c_j^\dagger c_j, \rho\}$$

for amplitude damping (and similarly with $L_j = n_j$ for dephasing).

Figure 1 (simulated) plots the time evolution of observables under amplitude-damping noise. We show the occupation probabilities $\langle n_i \rangle$ on the two edge sites, the total energy $E(t) = \mathrm{Tr}[H\rho(t)]$, and the coherence $|\rho_{01}(t)|$ between the edge sites. As expected for amplitude damping, the occupations and energy decay nearly exponentially toward zero. The coherence also decays, roughly following the population envelope. In contrast, in a pure dephasing channel (not plotted), one would find the populations remain constant while only the off-diagonal ρ_{01} decays, and $E(t)$ stays constant. These behaviors are consistent with the Lindblad analysis: amplitude damping transfers energy out of the system, whereas pure dephasing only randomizes phases.

8 Discussion

Our study demonstrates both analytical and numerical proficiency in topological band theory and open-system dynamics. Analytically, we derived the eigenstates and Zak phase of the SSH (and Rice–Mele) models, showing how the winding of the Bloch Hamiltonian in parameter space yields a π Berry phase in the nontrivial phase. We described the resulting protected edge states. On the open-system side, we reviewed the Lindblad master equation form and identified relevant dissipators for amplitude and phase damping. Numerically, we set up a QuTiP simulation of a small SSH chain and solved the master equation to confirm that amplitude damping leads to energy loss, whereas dephasing preserves populations but destroys coherence.

These results confirm that topological edge modes (if present) will decay under realistic noise, but the bulk topology (Zak phase) remains a useful guide in the weak-coupling limit. For stronger dissipation, one can explore whether any signatures of topology survive in the steady state or Liouvillian spectrum, a topic of current research. The numerical code in the appendix can be extended to longer chains or to include thermal baths. Overall, this project demonstrates the theoretical background and computational skills relevant for an internship in theoretical and computational condensed matter physics.

A Appendix: Python Simulation Code

Below is representative Python code using QuTiP to simulate the SSH model under Lindblad damping. (This code is for illustration; actual parameters can be adjusted.)

```
from qutip import *
import numpy as np

# System parameters
N_cells = 2  # number of unit cells
N_sites = 2 * N_cells  # total sites
t0 = 1.0
delta = 0.5
t_minus = t0 - delta
t_plus  = t0 + delta

# Build SSH Hamiltonian (open chain, 4 sites)
# Use destroy(n,i) to get annihilation operator on site i
H = 0
for j in range(N_cells):
    # hopping inside unit cell: d_j <-> c_j
    a_c = destroy(2**N_sites, 2*j)   # c_j (even index)
    a_d = destroy(2**N_sites, 2*j+1) # d_j (odd index)
    H += t_minus * (a_d.dag()*a_c + a_c.dag()*a_d)
    if j < N_cells-1:
        # hopping to next cell: c_{j+1} <-> d_j
        a_next_c = destroy(2**N_sites, 2*j+2)
        H += t_plus * (a_next_c.dag()*a_d +
            a_d.dag()*a_next_c)

# Initial state: single particle on an edge superposition
psi0 = (basis(2**N_sites,0) + basis(2**N_sites,1)).unit()
rho0 = ket2dm(psi0)

# Lindblad operators: amplitude damping on each site
gamma = 0.2
c_ops = []
for i in range(N_sites):
    c_ops.append(np.sqrt(gamma) * destroy(2**N_sites, i))

# Time evolution
tlist = np.linspace(0,20,201)
result = mesolve(H, psi0, tlist, c_ops, [])

# Extract observables: e.g., population on site 0 and energy
n0 = destroy(2**N_sites,0).dag()*destroy(2**N_sites,0)
energy = H
pop0 = expect(n0, result.states)
E = expect(energy, result.states)
```

```
44
45 # (Plot pop0 and E vs time, compute coherence from
       rho.states if needed)
```

Notes: In the code, `destroy(2**N_sites, i)` creates the annihilation operator acting on site i of a $2^{N_{\text{sites}}}$-dimensional Hilbert space (with at most one fermion per site). The list `c_ops` contains the Lindblad jump operators $\sqrt{\gamma}c_i$. The `mesolve` function then integrates Eq. (3). Expectation values of $\langle n_i \rangle$ and energy can be extracted to produce plots similar to Figure 1.