Simulation of Three-Body Problem Using Euler-Cromer Method

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1 Introduction

The N-body problem is often considered to be *the* fundamental problem in celestial mechanics. For $N = 2$, an analytical solution can be found. However, for $N >= 3$, the system is chaotic and requires a numerical approach. Here, we make use of the Euler-Cromer method to simulate the system for $N = 3$, where the three bodies are Earth, Jupiter, and the Sun. This program is based on Exercise 4.16 in *Computational Physics.*[1]

2 Method

Note: We are using astronomical units (AU) for units of length, years (yr) for units of time, and kilograms (kg) for units of mass.

2.1 Two-Body Problem: Earth and Sun

Consider Newton's Law of Universal Gravitation,

$$F_G = \frac{GM_\odot M_\oplus}{r^2}, \tag{1}$$

where F_G is force due to gravity, G is the gravitational constant[1], $M_\odot = 2.0 * 10^{30}$ kg is the mass of the Sun, $M_\oplus = 6.0 * 10^{24}$ kg is the Mass of earth, and r is the distance between the Earth and the Sun. Because $M_\odot >> M_\oplus$, we assume the Sun to be the center of mass of the system while Earth orbits.

We now consider the x and y components of F_G.

$$F_{G,x} = -\frac{GM_\odot M_\oplus}{r^2}\cos\theta = -\frac{GM_\odot M_\oplus x}{r^3}, \tag{2}$$

$$F_{G,y} = -\frac{GM_\odot M_\oplus}{r^2}\sin\theta = -\frac{GM_\odot M_\oplus y}{r^3}. \tag{3}$$

[1]Found from $GM_\odot = 4\pi^2$ AU^3/yr^2. See **Critique** for explanation.

It follows from Newton's Second Law that

$$\frac{dv_x}{dt} = -\frac{GM_\odot x}{r^3}, \tag{4}$$

$$\frac{dx}{dt} = v_x, \tag{5}$$

$$\frac{dv_y}{dt} = -\frac{GM_\odot x}{r^3}, \tag{6}$$

$$\frac{dy}{dt} = v_y, \tag{7}$$

where v_x and v_y are the Earth's velocity components.
We now consider the Euler-Cromer step for time difference Δt.

$$v_{x,i+1} = v_{x,i} - \frac{GM_\odot x_i}{r_i^3}\Delta t, \tag{8}$$

$$x_{i+1} = x_i + v_{x,i+1}\Delta t, \tag{9}$$

$$v_{y,i+1} = v_{y,i} - \frac{GM_\odot y_i}{r_i^3}\Delta t, \tag{10}$$

$$y_{i+1} = y_i + v_{y,i+1}\Delta t. \tag{11}$$

Using this, we can write our loop for the simulation.

2.2 Three-Body Problem: Adding Jupiter to the System

Now, we consider the three-body problem with Earth, Jupiter, and the Sun. In this system, it is sufficient to assume $M_\odot >> M_{♃} > M_\oplus$, where $M_{♃} = 1.9 * 10^{27}$ kg is the mass of Jupiter. Hence, the Sun is still assumed to be the center of mass of the system. However, we must consider the additional relationship between the Earth and Jupiter when calculating the accelerations for each body. We find for the force on Earth due to Jupiter,

$$F_{\oplus ♃,x} = -\frac{GM_{♃} M_\oplus}{r_{\oplus ♃}^2}\cos\theta_{\oplus ♃} = -\frac{GM_{♃} M_\oplus (x_\oplus - x_{♃})}{r_{\oplus ♃}^3}, \tag{12}$$

$$F_{\oplus ♃,y} = -\frac{GM_{♃} M_\oplus}{r_{\oplus ♃}^2}\sin\theta_{\oplus ♃} = -\frac{GM_{♃} M_\oplus (y_\oplus - y_{♃})}{r_{\oplus ♃}^3}, \tag{13}$$

and vice versa,

$$F_{♃ \oplus,x} = -\frac{GM_{♃} M_\oplus (x_{♃} - x_\oplus)}{r_{\oplus ♃}^3}, \tag{14}$$

$$F_{\jupiter\oplus,y} = -\frac{GM_{\jupiter}M_{\oplus}(y_{\jupiter} - y_{\oplus})}{r^3_{\oplus\jupiter}}. \tag{15}$$

It follows from Newton's Second Law that

$$\frac{dv_{x,\oplus}}{dt} = -\frac{GM_{\odot}x_{\oplus}}{r^3_{\odot\oplus}} - \frac{GM_{\jupiter}(x_{\oplus} - x_{\jupiter})}{r^3_{\oplus\jupiter}}, \tag{16}$$

$$\frac{dv_{y,\oplus}}{dt} = -\frac{GM_{\odot}y_{\oplus}}{r^3_{\odot\oplus}} - \frac{GM_{\jupiter}(y_{\oplus} - y_{\jupiter})}{r^3_{\oplus\jupiter}}, \tag{17}$$

$$\frac{dv_{x,\jupiter}}{dt} = -\frac{GM_{\odot}x_{\jupiter}}{r^3_{\odot\jupiter}} - \frac{GM_{\oplus}(x_{\jupiter} - x_{\oplus})}{r^3_{\oplus\jupiter}}, \tag{18}$$

$$\frac{dv_{y,\jupiter}}{dt} = -\frac{GM_{\odot}y_{\jupiter}}{r^3_{\odot\jupiter}} - \frac{GM_{\oplus}(y_{\jupiter} - y_{\oplus})}{r^3_{\oplus\jupiter}}. \tag{19}$$

With these equations, we are now ready to rewrite the previous Euler-Cromer step to create our three-body simulation with the Sun as the assumed center of mass.

2.3 Three-Body Problem: Accounting for the True Center of Mass

We note that in our previous three body simulation, should we multiply the mass of any of the bodies such that we can no longer assume $M_{\odot} >> M_{\jupiter} > M_{\oplus}$, the simulation becomes inaccurate, for the sun can no longer be assumed as the center of mass. Hence, we must now find additional relationships between the Sun and the other two bodies, Earth and Jupiter. This gives us a total of six relationships for three bodies.

Luckily, we can write a general formula for acceleration in an N-body system. Let k represent a body. It follows that

$$\frac{d\mathbf{v}_k}{dt} = \sum_{j \neq k}^{N} GM_j \frac{\mathbf{r}_{kj}}{r^3_{kj}}, \tag{20}$$

where $\mathbf{v}_k$ is the general velocity of a body k and $\mathbf{r}_{kj}$ is the general displacement between two bodies k and j.[2]

Our Euler-Cromer step for $N = 3$ is as follows:

$$\mathbf{v}_{k,i+1} = \mathbf{v}_{k,i} - \sum_{j \neq k}^{3} GM_j \frac{\mathbf{r}_{kj}}{r^3_{kj}} \Delta t, \tag{21}$$

$$\mathbf{r}_{k,i+1} = \mathbf{r}_{k,i} + \mathbf{v}_{k,i+1}. \tag{22}$$

3 Verification of program

We consider the initial conditions for the system in a simulation where the masses for Earth, Jupiter, and the Sun are set to their approximately true values. We start the simulation with each of the three bodies aligned on the x-axis and initial velocities only along the y-axis. Earth is 1 AU away from the center of mass while Jupiter is 5.2 AU away. Since the eccentricity of the orbits of Earth and Jupiter are very low, it suffices to give them initial velocities assuming circular orbits. Hence, the initial velocity of Earth given a 1 yr orbit is 2π AU/yr while the initial velocity of Jupiter given a 12 yr orbit is $2\pi * (12/5.2)$ AU/yr. In this case, we expect (and find in **Figures 1-3**) that the simulation will have outputs where the Sun is at the origin, Earth forms a circular 1-yr-orbit with a radius of 1 AU, and Jupiter forms a circular 12-yr-orbit with a radius of 5.2 AU.

In **Figure 3** and in **Figure 5**, we consider the system where the center of mass is calculated rather than assumed as the sun. We set the initial conditions such that the origin is the center of mass. The origin stays as the center of the mass, since the initial velocities are set such that there is a net-zero momentum in the system.

4 Data

Note: Only the mass-multiplier variables were adjusted in these simulations.

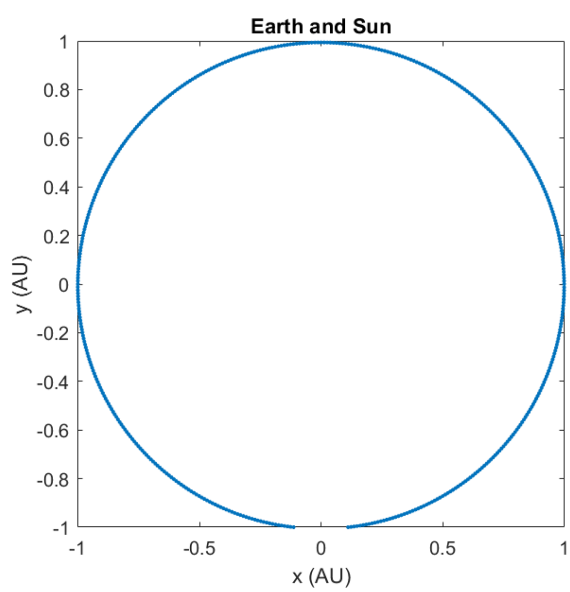

Figure 1: Earth and sun system with real masses.

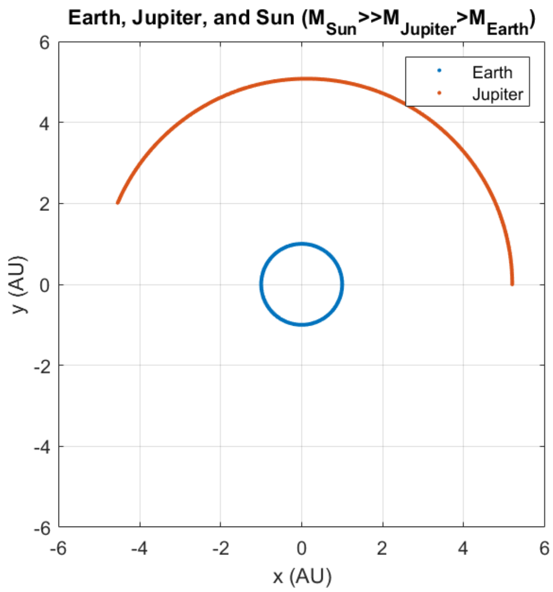

Figure 2: Earth, Jupiter, and Sun with real masses (assuming sun is the center of mass).

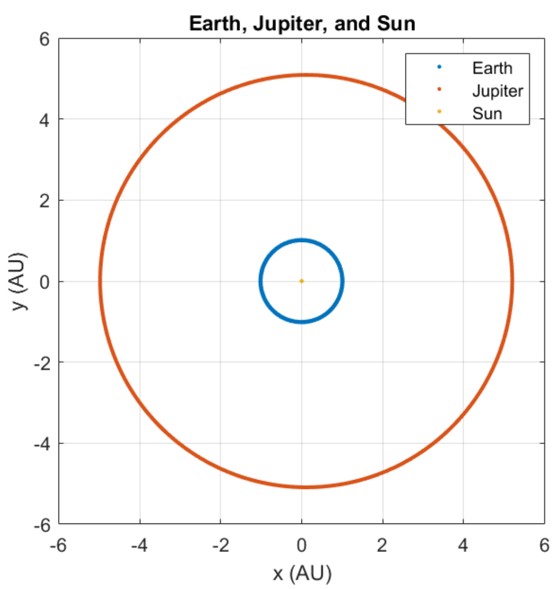

Figure 3: Earth, Jupiter, and Sun with real masses. The center of mass is the origin.

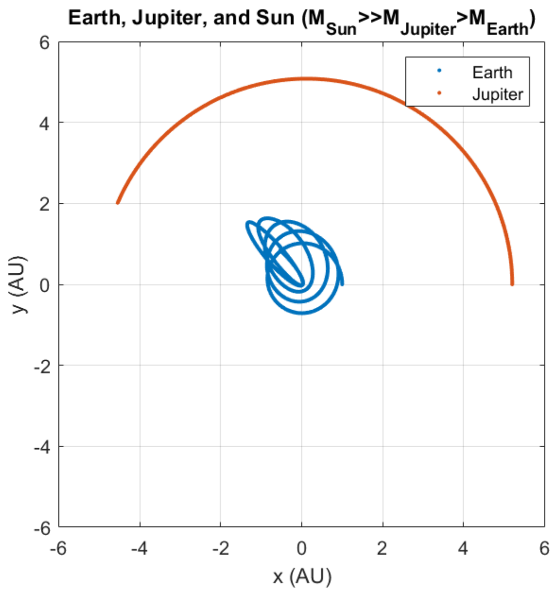

Figure 4: Earth, Jupiter, and Sun with 1000× Jupiter mass $M_{♃}$. The assumption that the sun is the center of mass makes this a faulty simulation.

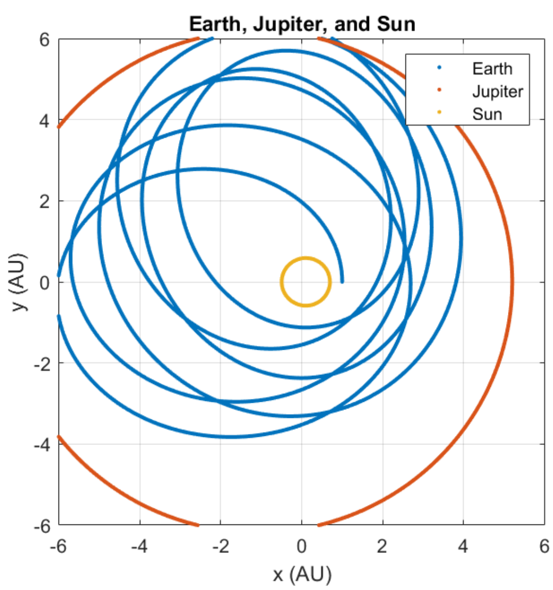

Figure 5: Earth, Jupiter, and Sun with 100× Jupiter mass $M_{♃}$. The center of mass is the origin.

5 Analysis

As expected, the real masses of the system produce a stable circular orbit in **Figures 1-3**. Given the very low eccentricities of Earth and Jupiter, this is a good approximation.

As for the the chaos observed in **Figures 4-5**, it is clear that Jupiter's effect would be far greater on the system, including the Sun, if it mass were significantly multiplied. At 100× Jupiter mass $M_{♃}$ in **Figure 5**, chaos is observed in the orbit of Earth; we also note that the Sun is very obviously no longer the center of mass, since it now forms a circular orbit around the origin. From this, it becomes quite obvious that **Figure 4** is quite inaccurate! The Sun is certainly should no longer the center of mass here; Jupiter's orbit seems to be unaltered, and Earth's orbit, though quite altered, is significantly less altered than in **Figure 5** with a smaller Jupiter mass.

6 Critique

The Euler-Cromer method is not the best algorithm that could have been used here, but it was very easy to implement. Furthermore, we note that this is a classical system and does not account for relativistic corrections to Newtonian gravity.

We also note that in the program, we calculate the gravitational constant G from $GM_{\odot} = 4\pi^2$ AU^3/yr^2. We do this because it turns out that this is a quite accurate approximation, as the eccentricity of Earth is very low ($e_{\oplus} = 0.017$), making Earth's orbit approximately circular. Hence, knowing that the radius of this circular orbit is 1 AU (by definition of the astronomical unit AU) and that the orbit takes 1 yr, the linear velocity of Earth is found to be $v = 2\pi$ AU/yr. Through the comparison

$$\frac{M_{\oplus} v^2}{r} = \frac{G M_{\odot} M_{\oplus}}{r^2}, \tag{23}$$

where r is the distance between Earth and the Sun, we find our approximation.

Of course, this is not something that should be done in a future implementation of this program should we want more significant figures for even greater accuracy. Nonetheless, this is still quite accurate for several significant figures.

Finally, users of this program should be warned that in its current state, it is quite memory intensive! Future implementations should discard the data as it is plotted rather than storing it onto vectors. Another recommendation for future implementations of this program is to animate plots.

References

[1] Nicholas J. Giordano and Hisao Nakanishi. *Computational Physics*. Pearson, 2 edition, 2006.

[2] Wilhelm Kley and Christopher Schäfer. Chaos in planetary systems. 2021.