1) $\mathcal{A} = \begin{bmatrix} a & c^t \\ c & D \end{bmatrix}$

$$\mathcal{A} = \begin{bmatrix} \sqrt{a} & 0 \\ \frac{c}{\sqrt{a}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D - \frac{1}{a}cc^t \end{bmatrix} \begin{bmatrix} \sqrt{a} & \frac{c^t}{\sqrt{a}} \\ 0 & I \end{bmatrix} = L_0 A_1 L_0^t$$

(a) $x^t \mathcal{A} x > 0 \quad \forall x \neq 0$

$x = \begin{pmatrix} x_1 \\ x_{n-1} \end{pmatrix} \in \mathbb{R}^n$

$$\mathcal{A}x = \begin{bmatrix} \sqrt{a} & 0 \\ \frac{c}{\sqrt{a}} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & D - \frac{1}{a}cc^t \end{bmatrix} \begin{bmatrix} \sqrt{a} & \frac{c^t}{\sqrt{a}} \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_{n-1} \end{bmatrix} =$$

$$= L_0 A_1 \begin{bmatrix} x_1\sqrt{a} + \frac{c^t x_{n-1}}{\sqrt{a}} \\ x_{n-1} \end{bmatrix} = L_0 \begin{bmatrix} x_1\sqrt{a} + \frac{c^t x_{n-1}}{\sqrt{a}} \\ (D - \frac{1}{a}cc^t)x_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} x_1 a + c^t x_{n-1} \\ c x_1 + \frac{c(c^t x_{n-1})}{a} + (D - \frac{1}{a}cc^t)x_{n-1} \end{bmatrix}$$

$$x^t \mathcal{A} x = (x_1\ x_{n-1}^t) \begin{pmatrix} x_1 a + c^t x_{n-1} \\ c x_1 + \frac{c(c^t x_{n-1})}{a} + (D - \frac{1}{a}cc^t)x_{n-1} \end{pmatrix} =$$

$$= x_1^2 a + (c^t x_{n-1})x_1 + (x_{n-1}^t c)x_1 + \frac{(x_{n-1}^t c)(c^t x_{n-1})}{a} + x_{n-1}^t (D - \frac{1}{a}cc^t)x_{n-1} > 0$$

$$\Leftrightarrow a x_1^2 + 2x_1(x_{n-1}, c) + \frac{1}{a}(x_{n-1}, c)^2 + x_{n-1}^t (D - \frac{1}{a}cc^t)x_{n-1} > 0$$

$$\Leftrightarrow \left(\sqrt{a}\,x_1 + \frac{1}{\sqrt{a}}(x_{n-1}, c)\right)^2 + x_{n-1}^t (D - \frac{1}{a}cc^t)x_{n-1} > 0$$

(1) Это нер-во для всех $x \neq 0$, в частности для $x = \begin{pmatrix} 0 \\ b \end{pmatrix}$ $\quad b \in \langle c \rangle^\perp$, $b \in \mathbb{R}^{n-1}$

Тогда $(\ldots)^2 = 0$ и $x_{n-1}^t (D - \frac{1}{a}cc^t)x_{n-1} > 0 \quad x_{n-1} \in \langle c \rangle^\perp$, $x_{n-1} \neq 0$

(2) если $x_{n-1} = \lambda c$ $\quad$ Возьмем $x_1 = -\frac{\lambda \|c\|^2}{a}$ и получим $(\ldots)^2 = 0$

$\sqrt{a}x_1 + \frac{1}{\sqrt{a}}\lambda\|c\|^2$

Тогда тоже $x_{n-1}^t (D - \frac{1}{a}cc^t)x_{n-1} > 0 \quad x_{n-1} \in \langle c \rangle$, $x_{n-1} \neq 0$

$\Rightarrow$ для $\forall x_{n-1} \in \mathbb{R}^{n-1}$, $x_{n-1} \neq 0$ $\quad x_{n-1}^t (D - \frac{1}{a}cc^t)x_{n-1} > 0$

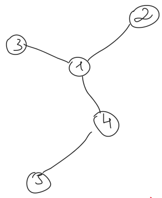
$\Rightarrow$ она пол. опр.

(b)

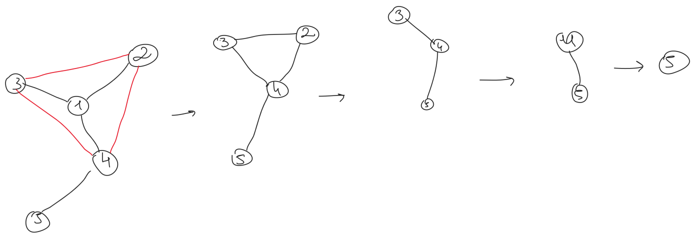
$$\mathcal{A} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & \\ \end{array}$$

$$\begin{array}{c} \\ \mathcal{A} = \end{array} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \end{array}$$

$$\mathcal{A} = \begin{pmatrix} 3 & 1 & 9 & 7 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 9 & 0 & 5 & 0 & 0 \\ 7 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 & 4 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$



(c)



(e) $val = [3, 1, 9, 7, 1, 2, 9, 5, 7, 1, 3, 3, 4]$

$row = [0, 4, 6, 8, 11, 13]$

$col = [0, 1, 2, 3, 0, 1, 0, 2, 0, 3, 4, 3, 4]$

Задача мин. $\|\ldots\|^2$ эквив. мин $\|\ldots\|$

2) $\mathcal{A} \in \mathbb{R}^{n \times n}$ $\quad \det A \neq 0$

$x_{k+1} = x_k + \tau_k r_k$, $\quad r_k = b - \mathcal{A}x_k$

$$\tilde{J}(x) = \|b - \mathcal{A}x\|_2^2 = (b - \mathcal{A}x)^t(b - \mathcal{A}x) =$$

$$= b^t b - b^t \mathcal{A}x - \underbrace{x^t \mathcal{A}^t b}_{(x^t A^t b)^t = b^t A x} + x^t A^t A x = b^t b - 2b^t A x + x^t A^t A x$$

$$\tilde{J}(x+h) = b^t b - 2b^t A(x+h) + (x+h)^t A^t A(x+h) = \tilde{J}(x) - 2b^t A h + h^t \mathcal{A}^t \mathcal{A} h + h^t \mathcal{A}^t A x + x^t A^t A h =$$

$$= \tilde{J}(x) - 2b^t A h + 2x^t A^t A h + h^t A^t \mathcal{A} h = \tilde{J}(x) + (2A^t A x - 2A^t b)^t h + h^t A^t A h$$

по опр. $D\tilde{J}(x) = 2\mathcal{A}^t \mathcal{A}x - 2\mathcal{A}^t b = 2\mathcal{A}^t(\mathcal{A}x - b)$

Для минимизации нужно идти в направлении антиградиента

$d_k = -(2\mathcal{A}^t\mathcal{A}x_k - 2\mathcal{A}^t b) = -2\mathcal{A}^t(\mathcal{A}x_k - b) = 2\mathcal{A}^t r_k$

Найдем τ_k: $\quad \tau_k = \operatorname{argmin}_{\tau \in \mathbb{R}} \tilde{J}(x_k + \tau d_k)$

$$\tilde{J}(x_k + \tau d_k) = \tilde{J}(x_k) + (2\tau\mathcal{A}^t(-r_k))^t 2\mathcal{A}^t r_k + \tau^2 2r_k^t A A^t A 2\mathcal{A}^t r_k$$

$$\frac{\partial \tilde{J}(x_k + \tau d_k)}{\partial \tau} = -4r_k^t \mathcal{A}\mathcal{A}^t r_k + 8\tau r_k^t \mathcal{A}\mathcal{A}^t \mathcal{A}\mathcal{A}^t r_k = 0$$

$$r_k^t A A^t r_k = 2\tau \cdot r_k^t \mathcal{A}\mathcal{A}^t \mathcal{A}\mathcal{A}^t r_k$$

$$\tau_* = \frac{r_k^t \mathcal{A}\mathcal{A}^t r_k}{2r_k^t \mathcal{A}\mathcal{A}^t \mathcal{A}\mathcal{A}^t r_k}$$

$$\frac{\partial^2 \tilde{J}(x_k + \tau d_k)}{\partial \tau^2} = 8r_k^t(\mathcal{A}\mathcal{A}^t)^2 r_k = 8(\mathcal{A}\mathcal{A}^t r_k)^t \mathcal{A}\mathcal{A}^t r_k = 8\|\mathcal{A}\mathcal{A}^t r_k\|^2 \geq 0$$

$\Rightarrow \tau_*$ — точка min $\quad$ и $\quad \tau_k = \frac{r_k^t \mathcal{A}\mathcal{A}^t r_k}{2r_k^t \mathcal{A}\mathcal{A}^t \mathcal{A}\mathcal{A}^t r_k} = \frac{\|\mathcal{A}r_k\|^2}{2\|\mathcal{A}\mathcal{A}^t r_k\|^2}$

(b) $\mathcal{A} = \mathcal{A}^t > 0$

$$\|r_{k+1}\|_2 \overset{?}{\leq} \frac{\operatorname{cond}_2(A) - 1}{\operatorname{cond}_2(A) + 1}\|r_k\|_2 \qquad \tau_k = \operatorname{argmin}_\tau J(x_k + \tau r_k)$$

$$\forall t \quad \leq \|\underbrace{b - \mathcal{A}x_k}_{r_k} - t\mathcal{A}r_k\|_2 =$$

$$\|r_{k+1}\|_2 = \|b - \mathcal{A}x_{k+1}\|_2 = \min_\tau \|b - \mathcal{A}(x_k + \tau r_k)\|_2$$

$$= \|r_k(I - t\mathcal{A})\|_2 = \|r_k\|_2 \cdot \|I - t\mathcal{A}\|_2 = \|r_k\|_2 \frac{\operatorname{cond}_2(A) - 1}{\operatorname{cond}_2(A) + 1}$$

$t = \frac{2}{\lambda_1 + \lambda_n}$ мин. $\|I - t\mathcal{A}\|_2$ с лекции

(3) $\frac{\|e_k\|_2}{\|e_0\|_2} \leq \varepsilon \qquad 0 < \varepsilon < 1$

τ — оптим.

$$N(\varepsilon) = 1 + \frac{\operatorname{cond}_2 A}{2}\ln \varepsilon^{-1}$$

не более $N(\varepsilon)$ итераций

$$\|e_k\|_2 = \|I - \tau\mathcal{A}\|_2 \|e_{k-1}\|_2 \leq \ldots \leq \|I - \tau\mathcal{A}\|_2^k \|e_0\|_2$$

$$\tau_{opt} = \frac{2}{\lambda_1 + \lambda_n} \qquad \frac{\|e_k\|_2}{\|e_0\|_2} \leq \frac{\|I - \tau\mathcal{A}\|_2^k \|e_0\|_2}{\|e_0\|_2} = \|I - \tau A\|_2^k < \varepsilon$$

$$\left(\frac{\operatorname{cond}\mathcal{A} - 1}{\operatorname{cond}\mathcal{A} + 1}\right)^k < \varepsilon$$

$$k \ln\left(\frac{\operatorname{cond} A - 1}{\operatorname{cond} A + 1}\right) < \ln \varepsilon$$

$$\ln\left(\frac{\operatorname{cond} A - 1}{\operatorname{cond} A + 1}\right) = \ln\left(1 - \frac{2}{\operatorname{cond} A + 1}\right) \approx -\frac{2}{\operatorname{cond} A + 1} + o\left(\frac{1}{\operatorname{cond} A}\right)$$

если $\operatorname{cond} A \to \infty$

$$k\left(-\frac{2}{\operatorname{cond} A + 1} + o\left(\frac{1}{\operatorname{cond} A}\right)\right) < \ln \varepsilon$$

экспериментальн $\quad k < \frac{\ln \varepsilon}{-\frac{2}{\operatorname{cond} A + 1} + o\left(\frac{1}{\operatorname{cond} A}\right)} \leq \ln \varepsilon\left(\frac{\operatorname{cond} A + 1}{-2}\right) = \ln \varepsilon^{-1}\left(\frac{\operatorname{cond} A + 1}{2}\right) =$

Подберем k:

$$\left(1 + \frac{\operatorname{cond} A}{2}\ln \varepsilon^{-1}\right)\left(-\frac{2}{\operatorname{cond} A + 1} + o\left(\frac{1}{\operatorname{cond} A}\right)\right) \overset{?}{<} \ln \varepsilon$$

$$\underbrace{\frac{\operatorname{cond} A}{\operatorname{cond} A + 1}}_{< 1}\ln \varepsilon + \underbrace{\frac{\ln \varepsilon^{-1}}{2} \cdot o(1)}_{\to 0} - \underbrace{\frac{2}{\operatorname{cond} A + 1}}_{\to 0} + \underbrace{o\left(\frac{1}{\operatorname{cond} A}\right)}_{\to 0} < \ln \varepsilon$$

Доказали.