DSBA Calculus HW5

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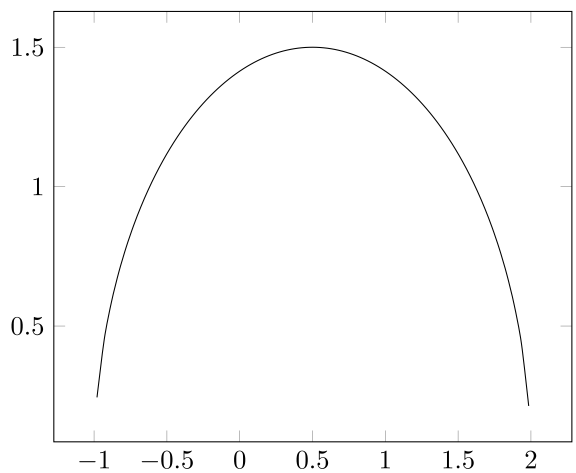
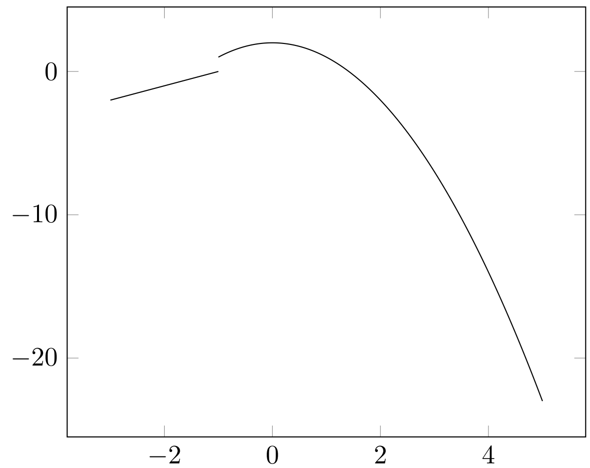
1. Find the domain and the range for the following functions:

$$f(x) = \sqrt{2 + x - x^2}$$
$$D(f) = \{x \in \mathbb{R} \mid 2 + x - x^2 \geq 0\}$$
$$x_1, x_2 = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot (-1)}}{-2} = \frac{-1 \pm 3}{-2} = 2; -1$$
$$-(x+1)(x-2) \geq 0$$
$$(x+1)(x-2) \leq 0$$
$$x \in [-1; 2]$$



In $y = 2 + x - x^2$ branches are directed downwards. It reaches maximum at $x_0 = \frac{1}{2} \Rightarrow y_0 = \frac{9}{4} \Rightarrow f(x) \leq \sqrt{\frac{9}{4}} = \frac{3}{2}$
Also square root must be greater or equal than zero, hence range
$y \in [0; \frac{3}{2}]$

$$f(x) = \begin{cases} x+1, & -3 \le x < -1 \\ 2-x^2, & x \ge -1 \end{cases}$$



There is no limitations on functions defined on these intervals except that $y = x + 1$ starts from -3, so domain of f(x) is $[-3; +\infty)$. Linear function is less than 0 on $-3 \le x < -1$ and monotone increases. Parabola has branches directed downwards, so the maximum will be at $f(0) = 2$. Despite the gap between these functions at $x = -1$, the right branch of parabola tends to minus infinity, so the range is $(-\infty, 2]$

2. Use the $\epsilon - \delta$ definition of limit to prove that $\lim_{x \to 9} \sqrt{x} = 3$. Find such δ such that $|\sqrt{x} - 3| < 0.01$ whenever $0 < |x - 9| < \delta$

$$\begin{aligned} \forall \epsilon > 0, \exists \delta > 0 : \forall x \quad 0 < |x-9| < \delta \quad &\Rightarrow \quad |\sqrt{x} - 3| < \epsilon \\ |\sqrt{x} - 3| = \frac{|x-9|}{\sqrt{x}+3} < \frac{\delta}{\sqrt{x}+3} &< \epsilon \\ \forall x, \sqrt{x} + 3 \ge 3 \Rightarrow \delta < (\sqrt{x}+3) &\cdot \epsilon \end{aligned}$$

So we can take $\delta = \epsilon$

$$\begin{aligned} &|\sqrt{x} - 3| < 0.01 \\ &3 - 0.01 < \sqrt{x} < 3 + 0.01 \\ &9 - 6 \cdot 0.01 + 0.01^2 < x < 9 + 6 \cdot 0.01 + 0.01^2 \end{aligned}$$

If we take $\delta = 0.01^2$, then $9 - 0.01^2 < x < 9 + 0.01^2$, which satisfies the upper inequality.

3. Find the following limits:

$$\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to 0}\frac{(x-1)(x+1)}{(x-1)(2x+1)}=\lim_{x\to 0}\frac{x+1}{2x+1}=1$$

$$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(2x+1)}=\lim_{x\to 1}\frac{x+1}{2x+1}=\frac{2}{3}$$

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2-x-1}=\lim_{x\to\infty}\frac{(x-1)(x+1)}{(x-1)(2x+1)}=\lim_{x\to\infty}\frac{x+1}{2x+1}=\lim_{x\to\infty}\frac{1+\frac{1}{x}}{2+\frac{1}{x}}=\frac{1}{2}$$

$$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x} \quad S_3=\frac{1+x+1+3x}{2}3=3(1+2x)$$
$$\lim_{x\to 0}\frac{(1+x)(1+2x)(1+3x)-1}{x}=\lim_{x\to 0}\frac{3(1+2x)-1}{x}=\lim_{x\to 0}\frac{2+6x}{x}=6$$

4. Find the following limits:

$$lim_{x\to\infty}\left(\sqrt{x^2+3x-1}-\sqrt{x^2+7}\right)=$$
$$lim_{x\to\infty}\frac{3x-8}{\sqrt{x^2+3x-1}+\sqrt{x^2+7}}=$$
$$lim_{x\to\infty}\frac{3-\frac{8}{x}}{\sqrt{1+\frac{3}{x}-\frac{1}{x^2}}+\sqrt{1+\frac{7}{x^2}}}=\frac{3}{2}$$

$$\lim_{x\to 3}\frac{\sqrt{6+x}-x}{\sqrt{28-x}-5}=$$
$$\lim_{x\to 3}\frac{(6+x-x^2)(\sqrt{28-x}+5)}{(28-x-25)(\sqrt{6+x}+x)}=$$
$$\lim_{x\to 3}\frac{-(x-3)(x+2)(\sqrt{28-x}+5)}{(3-x)(\sqrt{6+x}+x)}=$$
$$\lim_{x\to 3}\frac{(x+2)(\sqrt{28-x}+5)}{(\sqrt{6+x}+x)}=\frac{5\cdot 10}{3+3}=\frac{25}{3}$$

5. Find the following limits:

$$lim_{x\to+\infty}\frac{3\cdot 2^x-7\cdot 3^x}{2^{x+2}+5\cdot 3^x}=$$
$$lim_{x\to+\infty}\frac{3\cdot(\frac{2}{3})^x-7}{4\cdot(\frac{2}{3})^x+5}=-\frac{7}{5}$$

$$lim_{x\to-\infty}\frac{3\cdot 2^x-7\cdot 3^x}{2^{x+2}+5\cdot 3^x}=$$
$$lim_{x\to-\infty}\frac{3-7\cdot(\frac{3}{2})^x}{4+5\cdot(\frac{3}{2})^x}=\frac{3}{4}$$