DSBA Calculus HW9

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1. Find vertical and oblique asymptotes of the following functions:

   (a) $f(x) = \frac{x+1}{x^2+3x-4}$

   $$f(x) = \frac{x+1}{x^2+3x-4} = \frac{x+1}{(x+4)(x-1)}$$

   Firstly, let's find vertical asymptotes by checking several potential points $x = -4$ and $x = 1$.

   $$\lim_{x\to-4+} \frac{x+1}{(x+4)(x-1)} = +\infty$$
   $$\lim_{x\to-4-} \frac{x+1}{(x+4)(x-1)} = -\infty$$
   $$\lim_{x\to1+} \frac{x+1}{(x+4)(x-1)} = +\infty$$
   $$\lim_{x\to1-} \frac{x+1}{(x+4)(x-1)} = -\infty$$

   So, these are vertical asymptotes $x = -4$ and $x = 1$. Then consider oblique asymptotes.

   $$k_{+-} = \lim_{x\to\infty} \frac{x+1}{(x+4)(x-1)x} = 0$$
   $$b_{+-} = \lim_{x\to\infty} \frac{x+1}{(x+4)(x-1)} = 0$$

   Therefore $y = 0$ is an oblique asymptote.

   (b) $f(x) = \sqrt{\frac{x^3}{x-2}}$

   This function is defined while $x \in (-\infty, 0] \cup [2, +\infty)$. So let's check the margin points $x = 0$ and $x = 2$ by considering one-sided limits.

$$\lim_{x\to 0-}\sqrt{\frac{x^3}{x-2}} = 0$$

$$\lim_{x\to 2+}\sqrt{\frac{x^3}{x-2}} = +\infty$$

So, $x = 2$ is a vertical asymptote. What about oblique asymptotes?

$$k_+ = \lim_{x\to+\infty}\frac{\sqrt{\frac{x^3}{x-2}}}{x} = \lim_{x\to+\infty}\sqrt{\frac{x}{x-2}} = 1$$

$$b_+ = \lim_{x\to+\infty}\sqrt{\frac{x^3}{x-2}} - x = \lim_{x\to+\infty}\frac{\frac{x^3}{x-2} - x^2}{\sqrt{\frac{x^3}{x-2}} + x} = \lim_{x\to+\infty}\frac{2x^2}{(x-2)(\sqrt{\frac{x^3}{x-2}} + x)} =$$

$$= \lim_{x\to+\infty}\frac{2x^2}{(x^2-2x)(\sqrt{\frac{x}{x-2}} + 1)} = 1$$

$$k_- = \lim_{x\to-\infty}\frac{\sqrt{\frac{x^3}{x-2}}}{x} = \lim_{x\to-\infty} -\sqrt{\frac{x}{x-2}} = -1$$

$$b_- = \lim_{x\to-\infty}\sqrt{\frac{x^3}{x-2}} + x = \lim_{x\to-\infty}\frac{\frac{x^3}{x-2} + x^2}{\sqrt{\frac{x^3}{x-2}} + x} = \lim_{x\to-\infty}\frac{-2x^2}{(x-2)(\sqrt{\frac{x^3}{x-2}} + x)} =$$

$$= \lim_{x\to-\infty}\frac{-2x^2}{(x^2-2x)(\sqrt{\frac{x}{x-2}} + 1)} = -1$$

$y = x + 1$ and $y = -x - 1$ are oblique asymptotes.

(c) $f(x) = \sqrt{x^2-1} - x$
Function is defined while $x \in (-\infty, -1] \cup [1, +\infty)$

$$\lim_{x\to -1-}\sqrt{x^2-1} - x = 1$$

$$\lim_{x\to 1+}\sqrt{x^2-1} - x = -1$$

These limits are finite, therefore there are no vertical asymptotes.

$$k_{+-} = \lim_{x\to\infty} \frac{\sqrt{x^2-1}-x}{x} = \lim_{x\to\infty} \frac{|x|\sqrt{1-\frac{1}{x^2}}-x}{x} =$$
$$= \lim_{x\to\infty} sgn(x)\sqrt{1-\frac{1}{x^2}}-1 = sgn(x)-1 = 0; -2$$
$$b_+ = \lim_{x\to+\infty} \sqrt{x^2-1}-x = \lim_{x\to+\infty} \frac{-1}{\sqrt{x^2-1}+x} = 0$$
$$b_- = \lim_{x\to-\infty} \sqrt{x^2-1}-x+2x = \lim_{x\to-\infty} \sqrt{x^2-1}+x = \lim_{x\to-\infty} \frac{-1}{\sqrt{x^2-1}-x} = 0$$

Oblique asymptotes are $y=-2x$ and $y=0$.

(d) $f(x) = \frac{\sqrt{4x^4+1}}{|x|}$

The only suspicious point is $x=0$.

$$\lim_{x\to 0-} \frac{\sqrt{4x^4+1}}{|x|} = +\infty$$
$$\lim_{x\to 0+} \frac{\sqrt{4x^4+1}}{|x|} = +\infty$$

$x=0$ is a vertical asymptote.

$$k_+ = \lim_{x\to+\infty} \frac{\sqrt{4x^4+1}}{x^2} = \lim_{x\to+\infty} \frac{x^2\sqrt{4+\frac{1}{x^4}}}{x^2} = 2$$
$$k_- = \lim_{x\to-\infty} \frac{\sqrt{4x^4+1}}{-x^2} = \lim_{x\to-\infty} \frac{x^2\sqrt{4+\frac{1}{x^4}}}{-x^2} = -2$$
$$b_+ = \lim_{x\to+\infty} \frac{\sqrt{4x^4+1}}{x} - 2x = \lim_{x\to+\infty} \frac{\sqrt{4x^4+1}-2x^2}{x} = \lim_{x\to+\infty} \frac{1}{x(\sqrt{4x^4+1}+2x^2)} = 0$$
$$b_- = \lim_{x\to-\infty} \frac{\sqrt{4x^4+1}}{x} + 2x = \lim_{x\to-\infty} \frac{\sqrt{4x^4+1}+2x^2}{x} = \lim_{x\to-\infty} \frac{1}{x(\sqrt{4x^4+1}-2x^2)} = 0$$

$y=2x$ and $y=-2x$ are oblique asymptotes.

(e) $f(x) = x\arctan\frac{x}{2}$

There are no vertical asymptotes, function is defined everywhere.

$$k_+ = \lim_{x\to+\infty} \frac{x\arctan\frac{x}{2}}{x} = \arctan\frac{x}{2} = \frac{\pi}{2}$$
$$k_- = \lim_{x\to-\infty} \frac{x\arctan\frac{x}{2}}{x} = \arctan\frac{x}{2} = -\frac{\pi}{2}$$

I'll use the following identity $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$, $x > 0$ to find the next limit. Brief proof of it, for example, if we've got a right-triangle with sides 1 and x, then $\tan\alpha = x$ and $\tan\beta = \frac{1}{x}$, we know that the sum of angles are equal to $\frac{\pi}{2}$, so then $\alpha + \beta = \arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$. For negative x we know that $\arctan x$ is an odd function, so $\arctan(-x) = -\arctan x$, therefore $\arctan(-x) + \arctan(-\frac{1}{x}) = -(\arctan x + \arctan \frac{1}{x}) = -\frac{\pi}{2}$.
Also, I'll use the equivalence $\arctan x \sim x$, $x \to 0$.

$$b_+ = \lim_{x\to+\infty} x\arctan\frac{x}{2} - \frac{\pi}{2}x = \left|\left|t = \frac{x}{2}\right|\right| = \lim_{t\to+\infty} 2t(\arctan t - \frac{\pi}{2}) = \lim_{t\to+\infty} -2t\arctan\frac{1}{t} =$$
$$= \lim_{t\to+\infty} -\frac{2t}{t} = -2$$
$$b_- = \lim_{x\to-\infty} x\arctan\frac{x}{2} + \frac{\pi}{2}x = \left|\left|t = \frac{x}{2}\right|\right| = \lim_{t\to-\infty} 2t(\arctan t + \frac{\pi}{2}) = \lim_{t\to-\infty} -2t\arctan\frac{1}{t} =$$
$$= \lim_{t\to-\infty} -\frac{2t}{t} = -2$$

Therefore, we got $y = \frac{\pi}{2}x - 2$ and $y = -\frac{\pi}{2}x - 2$.

2. Use the definition to find the derivatives of the following functions:

(a) $f(x) = 8x^2 - x + 2$

$$f'(x) = \lim_{h\to0}\frac{8(x+h)^2 - (x+h) + 2 - (8x^2 - x + 2)}{h} =$$
$$= \lim_{h\to0}\frac{8x^2 + 16xh + 8h^2 - x - h + 2 - 8x^2 + x - 2}{h} =$$
$$= \lim_{h\to0}\frac{16xh + 8h^2 - h}{h} = \lim_{h\to0}(16x + 8h - 1) = 16x - 1$$

(b) $f(x) = 2\sqrt{x+4} \quad x_0 = 5$

$$f'(x_0) = \lim_{h\to0}\frac{2\sqrt{x_0+h+4} - 2\sqrt{x_0+4}}{h} = \lim_{h\to0}\frac{2\sqrt{h+9} - 6}{h} =$$
$$= \lim_{h\to0}\frac{2(h+9-9)}{h(\sqrt{h+9}+3)} = \frac{2}{6} = \frac{1}{3}$$

(c) $f(x) = \cos x$

$$f'(x) = \lim_{h\to0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to0}\underbrace{\frac{-2\sin\frac{h}{2}\sin\frac{2x+h}{2}}{h}}_{\text{apply first remarkable limit}} = \lim_{h\to0} -\sin\frac{2x+h}{2} = -\sin x$$

3. Compute the derivatives of the following functions using basic differentiation rules:

   (a) $f(x) = 5^{\cos x} \cdot \ln x + \frac{x^2+\sin x}{\sqrt{5x^2+3x-7}}$

$$f'(x) = 5^{\cos x} \cdot \ln 5 \cdot (-\sin x) \cdot \ln x + \frac{5^{\cos x}}{x} +$$
$$+ \frac{(2x+\cos x)\sqrt{5x^2+3x-7} - (x^2+\sin x)\frac{10x+3}{2\sqrt{5x^2+3x-7}}}{5x^2+3x-7}$$

   (b) $f(x) = (x^2+3) \cdot \tan\sqrt{x} + \frac{5^x}{7x-\ln x}$

$$f'(x) = 2x \cdot \tan\sqrt{x} + \frac{x^2+3}{2\sqrt{x}\cos^2\sqrt{x}} + \frac{5^x \cdot \ln 5 \cdot (7x-\ln x) - 5^x(7-\frac{1}{x})}{(7x-\ln x)^2}$$

4. Compute the derivatives of the following functions:

   (a) $f(x) = (\arctan x)^{\cos^2 x}$

$$f'(x) = (e^{\ln(\arctan x)^{\cos^2 x}})' = (e^{\cos^2(x)\ln(\arctan x)})' = (\arctan x)^{\cos^2 x}(-2\cos x \sin x \ln(\arctan x) +$$
$$+ \frac{\cos^2(x)}{\arctan x \cdot (1+x^2)})$$

   (b) $f(x) = \frac{e^{\arccos x}(x+7)^9}{(1+x^2)^4} = e^{\arccos x}\frac{(x+7)^9}{(1+x^2)^4}$

$$f'(x) = -\frac{e^{\arccos x}\frac{(x+7)^9}{(1+x^2)^4}}{\sqrt{1-x^2}} + e^{\arccos x}\frac{9(x+7)^8(1+x^2)^4 - 4(x+7)^9(1+x^2)^3 \cdot 2x}{(1+x^2)^8}$$