DSBA Calculus HW2

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1. Prove by definition that:

   (a) $\lim_{n\to\infty} \frac{(-1)^n}{5n-2} = 0$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n > N \left|\frac{(-1)^n}{5n-2}\right| < \epsilon$$

$$\left|\frac{(-1)^n}{5n-2}\right| = \frac{1}{5n-2} < \epsilon$$

$$n > \frac{\frac{1}{\epsilon}+2}{5}$$

$$N = \left[\frac{1}{5\epsilon} + \frac{2}{5}\right] + 1$$

   (b) $\lim_{n\to\infty} \frac{2n-1}{3n+5} = \frac{2}{3}$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n > N \left|\frac{2n-1}{3n+5} - \frac{2}{3}\right| < \epsilon$$

$$\left|\frac{2n-1}{3n+5} - \frac{2}{3}\right| = \left|\frac{6n-3-6n-10}{3(3n+5)}\right| = \frac{13}{3(3n+5)} < \epsilon$$

$$N = \left[\left|\frac{\frac{13}{3\epsilon} - 5}{3}\right|\right] + 1$$

2. Prove that the sequence $x_n = (-1)^n n - 7$ is divergent:

   Let's rewrite a sequence as a sum: $x_n = y_n + z_n$, where $y_n = (-1)^n n$ and $z_n = -7$. z_n is a constant therefore it converges. Then in order to x_n be divergent y_n must diverge by arithmetic properties of limits.

   With respect to theorem that if some sequence converges that it is bounded, let's show that y_n is unbounded, hence it diverges.

$$\forall M > 0, \exists n : \left|(-1)^n n\right| \geq M$$

$$\left|(-1)^n n\right| = n \geq M$$

$$N = [M] + 1$$

Therefore y_n diverges $\Rightarrow x_n$ is a divergent sequence.

3. Suppose the sequences a_n and b_n are both convergent. Is it true that if $a_n > b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n > \lim_{n\to\infty} b_n$?

   Let's show a counter example that it is not true with the case of equal limits.

   For instance, if a_n is a monotone decreasing sequence, b_n is a monotone increasing sequence, $a_n > b_n$, $\forall n \in \mathbb{N}$ and they both have the same limit.

   $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n} \Rightarrow lim_{n\to\infty} \frac{1}{n} = lim_{n\to\infty}(-\frac{1}{n}) = 0$

   We need to make inequalities non-strict in order to make them work properly.

   So, if $a_n \geq b_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n$. Let's prove this statement.

   Let $c_n = a_n - b_n \geq 0$. Then basically we need to prove that $\lim_{n\to\infty} c_n = c \geq 0$, because $\lim_{n\to\infty} c_n = \lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$

   By definition of limit:

   $$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n > N, \left|c_n - c\right| < \epsilon$$

   Proof by contradiction. Suppose $c < 0$. If it is valid for all ϵ, that it should be valid for $\epsilon = -c$.

   $$\left|c_n - c\right| < \epsilon$$

   $$\left|c_n - c\right| < -c$$

   $$c < c_n - c < -c$$

   $$2c < c_n < 0$$

   But we assumed that $c_n \geq 0$. We've found a contradiction, therefore $c \geq 0$

4. Suppose x_n and y_n are both divergent. Do the sequences $x_n + y_n$ and $x_n y_n$ **must** be divergent?

   Let $z_n = x_n + y_n$. Let's find a counter example that shows that z_n can be convergent sequence. For instance, if we take $z_n = 1$ which is obviously convergent and $x_n = (-1)^n$ that is divergent, then $y_n = 1 - (-1)^n$ is also divergent because this sequence jumps between 2 and 0.

   Let use this example to show that the statement above is not always true for $z_n = x_n \cdot y_n$.

   Let $z_n = 1, x_n = (-1)^n \Rightarrow y_n = \frac{1}{(-1)^n}$ which is obviously divergent.

   So, it is not always the case that $x_n + y_n$ or $x_n \cdot y_n$ are divergent.