DSBA Linear Algebra HW4

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1. Solve the following system:

$$\left[\begin{array}{ccccc|c} 6 & 4 & 5 & 2 & 3 & 1 \\ 9 & 6 & 1 & 3 & 2 & 2 \\ 3 & 2 & -2 & 1 & 0 & -7 \\ 3 & 2 & 4 & 1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 6 & 4 & 5 & 2 & 3 & 1 \\ 9 & 6 & 1 & 3 & 2 & 2 \\ 3 & 2 & -2 & 1 & 0 & -7 \end{array}\right] \to \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & -3 & 0 & -1 & -5 \\ 0 & 0 & -11 & 0 & -4 & -7 \\ 0 & 0 & -6 & 0 & -2 & -10 \end{array}\right] \to$$

$$\left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & -3 & 0 & -1 & -5 \\ 0 & 0 & -11 & 0 & -4 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 3 & 2 & 4 & 1 & 2 & 3 \\ 0 & 0 & -3 & 0 & -1 & -5 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{34}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_5 = -34 \\ x_3 = \frac{5}{3} + \frac{34}{3} = 13 \\ x_1 = 1 + 34 \cdot \frac{2}{3} - \frac{x_4}{3} - 13 \cdot \frac{4}{3} - \frac{2x_2}{3} = \frac{19}{3} - \frac{2x_2}{3} - \frac{x_4}{3} \end{cases}$$

$$X = \begin{bmatrix} \frac{19}{3} - \frac{2c_1}{3} - \frac{c_2}{3} \\ c_1 \\ 13 \\ c_2 \\ -34 \end{bmatrix} \quad \forall c_1, c_2 \in \mathbb{R}$$

2. Solve the following system with a parameter:

$$\left[\begin{array}{cccc|c} \lambda & 1 & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & \lambda & 1 & 1 & 1 \\ \lambda & 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & \lambda-1 & 0 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 & 1-\lambda \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & \lambda-1 & 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda & 2-\lambda-\lambda^2 & 1-\lambda \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1-\lambda & 2-\lambda-\lambda^2 & 1-\lambda \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & \lambda & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3-2\lambda-\lambda^2 & 1-\lambda \end{array}\right]$$

If $(\lambda+3)(\lambda-1) \neq 0$:

$$\begin{cases} x_1 + x_2 + x_3 + \lambda x_4 = 1 \\ x_2 - x_4 = 0 \\ x_3 - x_4 = 0 \\ -(\lambda+3)(\lambda-1)x_4 = 1-\lambda \end{cases} \iff \begin{cases} x_1 = 1 - \frac{2+\lambda}{\lambda+3} = \frac{1}{\lambda+3} \\ x_2 = x_3 = x_4 \\ x_4 = \frac{1}{\lambda+3} \end{cases}$$

$$X = \begin{bmatrix} \frac{1}{\lambda+3} \\ \frac{1}{\lambda+3} \\ \frac{1}{\lambda+3} \\ \frac{1}{\lambda+3} \end{bmatrix}$$

If $\lambda = 1$:

$$\begin{cases} x_1 = 1 - 3c \\ x_2 = x_3 = x_4 = c \quad c \in \mathbb{R} \end{cases}$$

$$X = \begin{bmatrix} 1-3c \\ c \\ c \\ c \end{bmatrix} \quad \forall c \in \mathbb{R}$$

If $\lambda = -3$ there are no solutions.

3. Find a polynomial $f(x) = ax^3 + bx^2 + cx + d$ such that

$$f(-2) = -1, f(-1) = 2, f(1) = 14, f(2) = 35$$

We can rewrite it as system of equations with unknowns a, b, c, d:

$$\left[\begin{array}{cccc|c} -8 & 4 & -2 & 1 & -1 \\ -1 & 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 14 \\ 8 & 4 & 2 & 1 & 35 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ -8 & 4 & -2 & 1 & -1 \\ -1 & 1 & -1 & 1 & 2 \\ 8 & 4 & 2 & 1 & 35 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 8 & 0 & 2 & 34 \\ 0 & 2 & 0 & 2 & 16 \\ 8 & 4 & 2 & 1 & 35 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 4 & 0 & 1 & 17 \\ 0 & -4 & -6 & -7 & -77 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 4 & 0 & 1 & 17 \\ 0 & 0 & -6 & -6 & -60 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & 0 & -3 & -15 \\ 0 & 0 & -6 & -6 & -60 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 1 & 8 \\ 0 & 0 & 1 & 1 & 10 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\begin{aligned} d &= 5 \\ c + 5 = 10 &\Rightarrow c = 5 \\ b + 5 = 8 &\Rightarrow b = 3 \\ a + 3 + 5 + 5 = 14 &\Rightarrow a = 1 \end{aligned}$$

Hence the polynomial will be in a form of:

$$f(x) = x^3 + 3x^2 + 5x + 5$$

4. Find numbers $a, b, c \in \mathbb{R}$ such that the following equality holds true:

$$\frac{x(5+x)}{(1-x)(2+x^2)} = \frac{a}{1-x} + \frac{b+cx}{2+x^2}$$
$$x(5+x) = a(2+x^2) + (b+cx)(1-x)$$
$$x^2 + 5x = (a-c)x^2 + (c-b)x + 2a + b$$

Coefficients of members with same powers should be equal, therefore:

$$\begin{cases} a - c = 1 \\ c - b = 5 \\ 2a + b = 0 \end{cases} \iff \begin{cases} -\frac{b}{2} - b - 5 = 1 \\ c = b + 5 \\ a = -\frac{b}{2} \end{cases} \iff \begin{cases} b = -4 \\ c = 1 \\ a = 2 \end{cases}$$

Indeed $2(2+x^2)+(x-4)(1-x) = 4+2x^2+x-x^2-4+4x = x^2+5x = x(5+x)$