DSBA Linear Algebra HW6

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1. (a) What is the coefficient (in the formula for the determinant of order 7) before $a_{33}a_{16}a_{72}a_{27}a_{55}a_{61}a_{44}$?

   Recall the definition of the determinant of order 7:

   $$|A| = \sum_{\sigma \in S_7} sgn(\sigma) \prod_{i \in [7]} a_{i,\sigma(i)}$$

   So, basically we need to find a parity of the corresponding permutation:

   $$\sigma = \begin{pmatrix} 3 & 1 & 7 & 2 & 5 & 6 & 4 \\ 3 & 6 & 2 & 7 & 5 & 1 & 4 \end{pmatrix}$$

   Rewrite it in a cycle form as $\sigma = (16)(27)$, therefore:

   $$sgn(\sigma) = (-1)^{7-5} = 1$$

   By formula $sgn(\sigma) = (-1)^{n-d}$ where n is a length of permutation and d is the number of cycles.

   (b) Find all possible i, j, k such that $a_{47}a_{63}a_{i2}a_{55}a_{7k}a_{j4}a_{31}$ occurs in the determinant of order 7 with positive sign.

   Once again in order to solve this problem we need to find a parity of a permutation:

   $$\sigma = \begin{pmatrix} 4 & 6 & i & 5 & 7 & j & 3 \\ 7 & 3 & 2 & 5 & k & 4 & 1 \end{pmatrix}$$

   The only possible value for k is 6 and i, j can be either $1, 2$ or $2, 1$, so we have the following permutations respectively:

   $$\sigma_1 = (4763124) \quad \sigma_2 = (476314)$$

   Then $sgn(\sigma_1) = (-1)^{7-2} = -1$ and $sgn(\sigma_2) = (-1)^{7-3} = 1$ by the same formula.

   Therefore, only $i = 2, j = 1, k = 6$ will give a positive sign.

2. Transforming the matrix into Row Echelon Form, evaluate the following determinant[1]:

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -2 & 0 & -5 & -1 \\ 4 & -1 & 13 & 6 \\ 6 & -3 & 14 & 12 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 4 \\ -2 & 0 & -5 & -1 \\ 0 & 1 & 7 & -2 \\ 6 & -3 & 14 & 12 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 4 \\ -2 & 0 & -5 & -1 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 5 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & -1 & -2 & 3 \\ 0 & 1 & 7 & -2 \\ 0 & 0 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 5 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 4 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 2(-1)5(-1) = 10$$

3. Using the Laplace Expansion theorem, evaluate the following determinant:

$$\begin{vmatrix} 5 & 1 & 2 & 7 \\ 3 & 0 & 0 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 1 & 1 & -4 \end{vmatrix} = 3(-1)^{2+1} \begin{vmatrix} 1 & 2 & 7 \\ 3 & 4 & 5 \\ 1 & 1 & -4 \end{vmatrix} + 2(-1)^{2+4} \begin{vmatrix} 5 & 1 & 2 \\ 1 & 3 & 4 \\ 2 & 1 & 1 \end{vmatrix} =$$

$$= -3 \begin{vmatrix} 1 & 2 & 7 \\ 0 & -2 & -16 \\ 0 & -1 & -11 \end{vmatrix} + 2 \begin{vmatrix} 0 & -14 & -18 \\ 1 & 3 & 4 \\ 0 & -5 & -7 \end{vmatrix} = -3 \begin{vmatrix} -2 & -16 \\ -1 & -11 \end{vmatrix} - 2 \begin{vmatrix} -14 & -18 \\ -5 & -7 \end{vmatrix} =$$

$$= -3(22 - 16) - 2(98 - 90) = -34$$

[1] I hope that elementary transformations are obvious without explanations

4. Evaluate $det A$, where

$$A = \begin{bmatrix} k+1 & k+2 & k+3 & \dots & k+n \\ k+n+1 & k+n+2 & k+n+3 & \dots & k+2n \\ \dots & \dots & \dots & \dots & \dots \\ k+(n-1)n+1 & k+(n-1)n+2 & k+(n-1)n+3 & \dots & k+n^2 \end{bmatrix}$$

(a) $n \geq 3$

Subtract 2nd row from the 3rd and the 1st from the 2nd one. By obtaining two same rows we can conclude that the determinant is equal to 0.

$$|A| = \begin{vmatrix} k+1 & k+2 & k+3 & \dots & k+n \\ k+n+1 & k+n+2 & k+n+3 & \dots & k+2n \\ k+2n+1 & k+2n+2 & k+2n+3 & \dots & k+3n \\ \dots & \dots & \dots & \dots & \dots \\ k+(n-1)n+1 & k+(n-1)n+2 & k+(n-1)n+3 & \dots & k+n^2 \end{vmatrix} =$$

$$= \begin{vmatrix} k+1 & k+2 & k+3 & \dots & k+n \\ k+n+1 & k+n+2 & k+n+3 & \dots & k+2n \\ n & n & n & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ k+(n-1)n+1 & k+(n-1)n+2 & k+(n-1)n+3 & \dots & k+n^2 \end{vmatrix} =$$

$$= \begin{vmatrix} k+1 & k+2 & k+3 & \dots & k+n \\ n & n & n & \dots & n \\ n & n & n & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ k+(n-1)n+1 & k+(n-1)n+2 & k+(n-1)n+3 & \dots & k+n^2 \end{vmatrix} = 0$$

(b) $n = 1, 2$

Previous transformations won't work because number of rows less than 3.

$$\begin{vmatrix} k+1 \end{vmatrix} = k+1$$

$$\begin{vmatrix} k+1 & k+2 \\ k+3 & k+4 \end{vmatrix} = \begin{vmatrix} k+1 & 1 \\ k+3 & 1 \end{vmatrix} = k+1-k-3 = -2$$