Discrete Mathematics HW2

Kirill Korolev, DSBA 203-1

20th of September 2020

1. Translate the following statements into/from logical symbolism and if they are true:

   (a) There is a street with at least two different Bank offices.

   $$\exists a_1, a_2, s\ (B(a_1, s) \wedge B(a_2, s) \to (a_1 \neq a_2))$$

   True, for instance, for fourth street.

   (b) Only Restaurants and Supermarkets can share a crossing with a Hotel.

   $$\forall a, s\ (H(a, s) \to \neg B(a, s))$$

   True.

   (c) Every avenue with a Supermarket has a Restaurant as well.

   $$\forall a\ (\exists s\ (S(a, s)) \to \exists s\ (R(a, s)))$$

   False for first avenue.

   (d) If you mistake streets for avenues and vice versa, the map is still accurate.

   $$\forall a, s((H(a, s) \equiv H(s, a)) \wedge (R(a, s) \equiv R(s, a)) \wedge (S(a, s) \equiv S(s, a)) \wedge (B(a, s) \equiv B(s, a)))$$

   False.

   (e) $\exists m \forall n\, S(n, m) \to \forall n \exists m\, R(n, m)$
   If there exists street with Supermarket on each avenue then there is a Restaurant on each avenue.
   False, because $s = 3$ satisfies first condition, but there exists such avenue $a = 1$ where there is no Restaurant.

   (f) $\exists i, j, n, m(B(i, j) \wedge S(i, m) \wedge S(n, j) \wedge B(n, m))$
   There exists a Bank with a Supermarket on the same street and also with a Supermarket on the same avenue, and also there is a Bank on a avenue of a first Supermarket and on a street of the second Supermarket.
   True, for $i = n = 2, j = m = 3$

2. Put the following arguments in symbols and check their validity:

(a) Only birds have feathers. No mammal is a bird. Therefore each mammal is featherless.
B(x) = "x is a bird", F(x) = "x has feathers", M(x) = "x is a mammal", $\forall x \in A$ - set of animals.

$$(\forall x(B(x) \equiv F(x)) \wedge \forall x(M(x) \to \neg B(x))) \to \forall x(M(x) \to \neg F(x))$$

Proof. Suppose the formula is false.

false	true
$\forall x(M(x) \to \neg F(x))$ **(1)**	$\forall x(B(x) \equiv F(x)) \wedge \forall x(M(x) \to \neg B(x))$ **(2)**
	$\forall x(B(x) \equiv F(x))$ **(2.1)**
	$\forall x(M(x) \to \neg B(x))$ **(2.2)**
	$\exists x'(M(x') \wedge F(x'))$ **(1)**
	$M(x'), F(x')$ **(1.1)** **(1.2)**
$B(x')$ **(2.2)**	$\neg B(x')$ **(2.2)**

$\neg(\forall x(M(x) \to \neg F(x))) = \exists x' \neg(\neg M(x') \vee \neg F(x')) = \exists x'(M(x') \wedge F(x'))$ **(1)**
If formula works for any x, then it should work for x'. Therefore, we can substitute x by x' in **(2)**. Eventually, **(2.1)** is true, but it contradicts with **(1.2)** and **(2.2)**. Hence, formula is valid.

(b) Everyone loves himself. Therefore someone is loved by somebody.
L(x, y) = "x loves y", $\forall x, y \in P$ - set of people.

$$(\forall x(L(x,x)) \to \exists x, y(L(x,y))$$

Proof. Suppose the formula is false.

false	true
$\exists x, y(L(x,y))$ **(1)**	$\forall x(L(x,x))$ **(2)**
	$\forall x, y \neg(L(x,y))$ **(1)**

But (**1**) contradicts with (**2**) because at least we can find people who love themselves. Therefore, formula is valid.

(c) Any mathematician can solve this problem if anyone can. Paul is a mathematician and cannot solve the problem. Therefore, the problem cannot be solved.
S(x) = "x can solve this problem", M(x) = "x is mathematician", $\forall x \in P$ - set of people. Paul is $p_0 \in P$.

$$(\forall x(S(x)) \to \forall x(M(x) \to S(x))) \wedge (M(p_0) \wedge \neg S(p_0)) \to \forall x(\neg S(x))$$

Proof. Suppose the formula is false.

false	true
$\forall x(\neg S(x))$ (**1**)	$(\forall x(S(x)) \to \forall x(M(x) \to S(x))) \wedge (M(p_0) \wedge \neg S(p_0))$ (**2**)
	$(\forall x(S(x)) \to \forall x(M(x) \to S(x)))$ (**2.1**)
	$M(p_0) \wedge \neg S(p_0)$ (**2.2**)
	$M(p_0), \neg S(p_0)$ (**2.2.1**) (**2.2.2**)
	$\exists x'(S(x'))$ (**1**)

If we put (**2.2.1**) and (**2.2.2**) into (**2.1**), we will get zero in the right implication. Hence in order to (**2.1**) be equal to true, $(\forall x(S(x))$ must be false, in other words $\exists x' \neg S(x')$. So, there is no contradiction, therefore argument is invalid. We can consider domain, where some people can solve this problem, but not any person, and Paul is mathematician and cannot solve the problem, which leads this statement to be false.

(d) Anyone who can solve this problem is a mathematician. Paul cannot solve this problem. Therefore, Paul is not a mathematician.

$$(\forall x(S(x) \to M(x)) \wedge \neg S(p_0)) \to \neg M(p_0)$$

Proof. Suppose the formula is false.

false	true
$\neg M(p_0)$ (**1**)	$\forall x(S(x) \to M(x)) \wedge \neg S(p_0)$ (**2**)
	$\forall x(S(x) \to M(x))$ (**2.1**)
	$\neg S(p_0)$ (**2.2**)
	$M(p_0)$ (**1**)

If we plug in $M(p_0)$ and $S(p_0)$ to (**2.1**) it will work. So, there is no contradiction and this argument is invalid. Basically, if you cannot prove Fermat's Last Theorem, this doesn't mean that you're not mathematician :) The domain for counter example would be that Paul is mathematician and the problem is unsolvable.

(e) Anyone who can solve this problem is a mathematician. No mathematician can solve this problem. Therefore, the problem cannot be solved.

$$(\forall x(S(x) \to M(x)) \wedge \forall x(M(x) \to \neg S(x))) \to \forall x(\neg S(x))$$

Proof. Suppose the formula is false.

false	true
$\forall x(\neg S(x))$ (**1**)	$\forall x(S(x) \to M(x)) \wedge \forall x(M(x) \to \neg S(x))$ (**2**)
	$\forall x(S(x) \to M(x))$ (**2.1**)
	$\forall x(M(x) \to \neg S(x))$ (**2.2**)
	$\exists x' S(x')$ (**1**)
	$M(x')$ (**2.1.1**)
	$\neg S(x')$ (**2.2.1**)

(**1**) and (**2.2.1**) contradict with each other, so, the formula is valid.