DSBA Calculus HW1

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18th of September, 2020

1. Verify whether the following set is bounded or not:

$$X = \left\{ \frac{n + (-1)^n}{3n - 1} : n \in \mathbb{N} \right\}$$

By definition set X is bounded if $\exists M > 0 : \forall x \in \mathbb{X}, |x| \leq M$

$$\left| \frac{n + (-1)^n}{3n - 1} \right| \leq \left| \frac{n}{3n-1} \right| + \left| \frac{(-1)^n}{3n-1} \right| = \frac{n}{3n-1} + \frac{1}{3n-1} =$$
$$\frac{1}{3} + \frac{1}{3(3n-1)} + \frac{1}{3n-1} \leq \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1$$

$\Rightarrow X$ is bounded.

2. Define an unbounded set. Prove that the following set is unbounded:

$$X = \left\{ n^2 : n \in \mathbb{N} \right\}$$

X is unbounded if $\forall M > 0, \exists x \in X : |x| > M$

$$n^2 > M$$

For instance, we can take $n = [\sqrt{M}] + 1$

3. Prove that the following sequence is bounded and find $\inf X, \sup X$:

   (a) $x_n = \frac{(-1)^n n}{n+1}$

$$\exists M > 0 : \forall n \in \mathbb{N}, \left| \frac{(-1)^n n}{n+1} \right| \leq M$$
$$\left| \frac{(-1)^n n}{n+1} \right| = \frac{n}{n+1} = 1 - \frac{1}{n+1} < 1$$

$\Rightarrow x_n$ is bounded.
By the construction of proof that x_n is bounded, assume that $\inf x_n = -1, \sup x_n = 1$ First point of definition is obvious, because we've proved that $|x_n| < 1 \Rightarrow -1 < x_n < 1, \forall n \in \mathbb{N}$. Let's check that there are no greater or lesser boundaries respectively.

Infinum

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : x_N < \inf x_n + \epsilon$$

Suppose $n = 2k + 1$:

$$\begin{aligned} -\frac{n}{n+1} &< -1 + \epsilon \\ 1 - \epsilon &< 1 - \frac{1}{n+1} \\ \frac{1}{n+1} &< \epsilon \\ n &> \frac{1}{\epsilon} - 1 \end{aligned}$$

We've assumed that n is odd, so the number in the following form will satisfy the condition:

$$N = 2\left[\frac{1}{\epsilon}\right] + 1$$

Supremum

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : x_N > \sup x_n - \epsilon$$

Suppose $n = 2k$:

$$\begin{aligned} \frac{n}{n+1} &> 1 - \epsilon \\ 1 - \frac{1}{n+1} &> 1 - \epsilon \\ \frac{1}{n+1} &< \epsilon \\ n &> \frac{1}{\epsilon} - 1 \end{aligned}$$

Also, given that n is even:

$$N = 2\left[\frac{1}{\epsilon}\right] + 2$$

So, we've proved that these are the greatest and the least boundaries respectively. Therefore, $\inf x_n = -1, \sup x_n = 1$

4. Prove that the following sequence is monotone, starting from some term:

$$x_n = \frac{n+2}{5n+1}$$

Suppose that this sequence is decreasing $\Rightarrow x_n > x_{n+1}$

$$\frac{n+2}{5n+1} > \frac{n+3}{5n+6}$$
$$\frac{(n+2)(5n+6)-(n+3)(5n+1)}{(5n+1)(5n+6)} > 0$$
$$\frac{5n^2+6n+10n+12-5n^2-n-15n-3}{(5n+1)(5n+6)} > 0$$
$$\frac{9}{(5n+1)(5n+6)} > 0$$

This holds $\forall n \in \mathbb{N} \Rightarrow x_n$ is a monotone decreasing sequence.