DSBA Discrete Mathematics HW4

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1. Prove that for each natural $n > 1$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by induction.

$$IB. \quad 1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$$

$$IS. \quad \sum_{k=1}^{n+1} k^2 = \underbrace{\frac{n(n+1)(2n+1)}{6}}_{ind.hyp.} + (n+1)^2 =$$

$$= \frac{(n+1)\Big(n(2n+1) + 6(n+1)\Big)}{6} = \frac{(n+1)(2n^2 + n + 6n + 6)}{6}$$

$$n_1, n_2 = \frac{-7 \pm \sqrt{49 - 4 \cdot 6 \cdot 2}}{4} = \frac{-7 \pm 1}{4} = -\frac{3}{2}; -2$$

$$\frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

2. In some country, there are finitely many cities, each two of which are connected by a one-way road. Prove that there is a city from which any other is accessible.

Let C_n will be a set of n cities. Denote $P(x, y)$ as path from x to y and $R(x, y)$ as one-way road from x to y, where $x, y \in C_n$.

$P(x, y) = \exists\{z_i\} : R(x, z_1) \wedge R(z_1, z_2) \wedge ... \wedge R(z_m, y) \quad z_i \in C_n,\ m \in \mathbb{N}$.

Then we need to prove $A(n) = \exists x \in C_n : \forall y \in C_n\ P(x, y)$

Also, by condition holds $\forall x, y \in C_n \quad R(x, y) \vee R(y, x)$

Proof by induction on $A(n)$**.** $A(2)$ is obvious by condition. Suppose that statement is valid for $A(n)$. So there is a city x_0 with path to other $n - 1$ cities:

$$\exists x_0 \in C_n : \forall y \in C_n \quad P(x_0, y)$$

Let's add another city y_0. Then by condition y_0 is also connected with other n cities. There are two cases.

Case 1 $\exists z \in C_n\ (P(x_0, z) \wedge R(z, y_0)) \rightarrow P(x_0, y_0) \Rightarrow x_0$ is the source city.

Case 2 $\forall z \in C_n\ R(y_0, z) \Rightarrow y_0$ is the source city.

Therefore, the statement has been proved.

3. Prove that for each natural n, there exists some k s.t. $1 + \frac{1}{2} + ... + \frac{1}{k} \geq n$

Proof by induction.

$$IB. \quad 1 \geq 1$$

$$\Delta_{k,m} = \frac{1}{k+1} + ... + \frac{1}{k+m} \quad k, m \in \mathbb{N}$$

$$IS. \quad 1 + \frac{1}{2} + ... + \frac{1}{k} + \Delta_{k,m} \geq n + \Delta_{k,m}$$

To prove given statement we need to show that $n + \Delta_{k,m} \geq n + 1 \Rightarrow \Delta_{k,m} \geq 1$. Let's prove last inequality by induction on k.

$$IB. (k = 1) \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 \quad m = 4$$

$$IS. \quad \frac{1}{k+1} + \frac{1}{k+2} + ... + \frac{1}{k+m} + \frac{1}{k+m+1} \geq 1 + \frac{1}{k+m+1} \geq 1$$

$$\Rightarrow \Delta_{k,m} \geq 1$$

$$\Rightarrow 1 + \frac{1}{2} + ... + \frac{1}{k} + \Delta_{k,m} \geq n + 1$$