DSBA Calculus HW4

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2th of October, 2020

1. Find the following limits:

   (a) $\lim_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-1})\cdot\sin(n^2+3)$
   Because $-1 \le sin(n^2+3) \le 1$

   $$x_n = -(\sqrt{n^2+1}-\sqrt{n^2-1}) \le (\sqrt{n^2+1}-\sqrt{n^2-1})\cdot\sin(n^2+3)$$
   $$(\sqrt{n^2+1}-\sqrt{n^2-1})\cdot\sin(n^2+3) \le (\sqrt{n^2+1}-\sqrt{n^2-1}) = z_n$$
   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} -(\sqrt{n^2+1}-\sqrt{n^2-1}) = -\lim_{n\to\infty}\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}} = 0$$
   $$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-1}) = \lim_{n\to\infty}\frac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}} = 0$$

   Then by sandwich theorem:

   $$\lim_{n\to\infty}(\sqrt{n^2+1}-\sqrt{n^2-1})\cdot\sin(n^2+3) = 0$$

   (b) $\lim_{n\to\infty}\frac{(4\cos n-3n)^2(2n^5-n^3+1)}{(6n^3+5n\sin n)(n+2)^4}$
   If we multiply inf. small by bounded sequence, we will get inf. small again.

   $$\lim_{n\to\infty}\frac{(4\cos n-3n)^2(2n^5-n^3+1)}{(6n^3+5n\sin n)(n+2)^4} = \lim_{n\to\infty}\frac{n^2(\frac{4\cos n}{n}-3)^2n^5(2-\frac{1}{n^2}+\frac{1}{n^5})}{n^3(6+\frac{5\sin n}{n^2})n^4(1+\frac{2}{n})^4} = \lim_{n\to\infty}\frac{3^2\cdot 2}{6\cdot 1} = 3$$

2. Prove that the following sequences are the Cauchy sequences:

   (a) $x_n = 0.77...7$ (n digits)

   $$\forall\epsilon>0, \exists N\in\mathbb{N}: \forall n\ge N, \forall p>0 \ |\underbrace{0.77..77}_{\text{n + p 7's}} - \underbrace{0.77..77}_{\text{n 7's}}| < \epsilon$$
   $$|0.77..77 - 0.77..7| = |0.\underbrace{000...}_{\text{n digits}}\underbrace{777...}_{\text{p digits}}|$$

For $\epsilon \geq 1$ inequality is obvious, suppose $0 < \epsilon < 1$, then we can represent ϵ as:

$$\epsilon = 0.a_1 a_2 ... a_m$$

where $a_i \in \{0, ..., 9\}, \forall i \in [m]$
Then starting from the left we can find first non-zero digit a_j.
Afterwards we choose $N = j + 1$ so that:

$$|0.\underbrace{000}_{\text{j + 1 digits}} 777..| < 0.\underbrace{000}_{\text{j - 1 digits}} a_j....a_m$$

Therefore $\{x_n\}$ is a Cauchy sequence.

3. Prove that the sequence converges:

   (a) $\sum_{k=1}^{n} \frac{\cos ka}{2^k}$
   It is equivalent to prove that sequence is a Cauchy sequence. By using the definition written above:

$$\left|\frac{\cos 1a}{2^1} + ... + \frac{\cos(n+p)}{2^{n+p}} - \frac{\cos 1a}{2^1} - ... - \frac{\cos n}{2^n}\right| = \left|\frac{\cos(n+1)}{2^{n+1}} + ... + \frac{\cos(n+p)}{2^{n+p}}\right| \leq$$
$$\left|\frac{\cos(n+1)}{2^{n+1}}\right| + ... + \left|\frac{\cos(n+p)}{2^{n+p}}\right| \leq \frac{1}{2^{n+1}} + ... + \frac{1}{2^{n+p}} = \frac{1}{2^n}\left(\frac{1}{2} + ... + \frac{1}{2^p}\right) =$$
$$\frac{1}{2^n} \cdot \frac{1}{2} \cdot \frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \frac{1}{2^n} \cdot (1 - (\frac{1}{2})^n) < \frac{1}{2^n} < \epsilon$$
$$2^n > \frac{1}{\epsilon}$$
$$n > \log_2 \frac{1}{\epsilon}$$
$$N = \left[\left|log_2 \frac{1}{\epsilon}\right|\right] + 1$$