DSBA Linear Algebra HW7

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1. Solve the following system using Cramer's rule:

$$\begin{cases} \lambda x + 3y = \lambda - 2 \\ 3x + \lambda y = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} \lambda & 3 \\ 3 & \lambda \end{vmatrix} = \lambda^2 - 9 \quad \Delta_1 = \begin{vmatrix} \lambda - 2 & 3 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 \quad \Delta_2 = \begin{vmatrix} \lambda & \lambda - 2 \\ 3 & 1 \end{vmatrix} = -2\lambda + 6$$

$$x = \frac{\Delta_1}{\Delta} = \frac{\lambda^2 - 2\lambda - 3}{\lambda^2 - 9} = \frac{(\lambda - 3)(\lambda + 1)}{(\lambda - 3)(\lambda + 3)}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-2\lambda + 6}{\lambda^2 - 9} = \frac{-2(\lambda - 3)}{(\lambda - 3)(\lambda + 3)}$$

   (a) $\lambda = -3$
   There are no solutions, because in this case $\Delta = 0$ and $\Delta_i \neq 0, \forall i$. Precisely, we have a contradiction $\Delta \cdot x_i = 0 \cdot x_i \neq \Delta_i$.

   (b) $\lambda = 3$
   In this case there are infinitely many solutions, because there is one equation with two variables.

$$\begin{cases} 3x + 3y = 1 \\ 3x + 3y = 1 \end{cases} \iff 3x + 3y = 1 \iff x = \frac{1}{3} - y, \quad y \in \mathbb{R}$$

   (c) In other cases there is a unique solution.

$$x = \frac{\lambda + 1}{\lambda + 3}$$
$$y = -\frac{2}{\lambda + 3}$$

2. Solve the following system using Cramer's rule:

$$\begin{cases} x + y + 2z = -1 \\ 2x - y + 2z = 3 \\ 4x + y + 4z = -3 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 4 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & -2 \\ 0 & -3 & -4 \end{vmatrix} = 6$$

$$\Delta_1 = \begin{vmatrix} -1 & 1 & 2 \\ 3 & -1 & 2 \\ -3 & 1 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 8 \\ 0 & -2 & -2 \end{vmatrix} = -12$$

$$\Delta_2 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 2 \\ 4 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 5 & -2 \\ 0 & 1 & -4 \end{vmatrix} = -18$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -3 & 5 \\ 0 & -3 & 1 \end{vmatrix} = 12$$

$$\begin{cases} x = \frac{\Delta_1}{\Delta} = -\frac{12}{6} = -2 \\ y = \frac{\Delta_2}{\Delta} = -\frac{18}{6} = -3 \\ z = \frac{\Delta_3}{\Delta} = \frac{12}{6} = 2 \end{cases}$$

3. Let a, b and c be real numbers. Then, without resorting to a direct calculation (Sarrus' Rule is a direct calculation!), find the value of the following determinant of matrix X.

$$|X| = \begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix}$$

Let A = $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$. Then evidently $A^T \cdot A = X$.

By **theorem 7.2** $|X| = |A^T| \cdot |A|$. Determinant of the Vandermonde matrix A equals to $(b-a)(c-a)(c-b)$. But $|A^T| = |A|$, hence

$$|X| = (b-a)^2(c-a)^2(c-b)^2$$