DSBA Linear Algebra HW5

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17th of October, 2020

1. Find the number of inversions in the following ordered line:

$$(9, 1, 3, 7, 2, 8, 5, 4, 6)$$

Inversion is such pair $(f(i), f(j))$ of permutation f for which valid $i < j$ and $f(i) > f(j)$. Basically, for one-line notation number of inversions means number of non-sorted pairs in ascending order.

Let's list these pairs for each number:

9: 1, 3, 7, 2, 8, 5, 4, 6
1: none
3: 2
7: 2, 5, 4, 6
2: none
8: 5, 4, 6
5: 4
4: none
6: none

Overall there are $8 + 1 + 4 + 3 + 1 = 17$ inversions.

2. Let $f = \begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 4 & 1 & 2 & 3 & 5 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 5 & 3 & 4 & 2 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

(a) cycle notation for f and g

Each permutation can be represented as product of independent cycles whereas a cycle is a list of $(a_1, a_2, ..., a_k)$ with mapping $a_1 \to a_2, a_2 \to a_3, ..., a_k \to a_1$.

$f = (245)(13)$ and $g = (15423)$

(b) $f \circ g$ (using two-line notation)

$$\left(\begin{array}{ccccc} 1 & 5 & 3 & 4 & 2 \\ 5 & 4 & 1 & 2 & 3 \\ \hline 5 & 4 & 1 & 2 & 3 \\ 2 & 5 & 3 & 4 & 1 \end{array}\right) \Rightarrow f \circ g = \begin{pmatrix} 1 & 5 & 3 & 4 & 2 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}$$

(c) $g \circ f$ (using cycle notation)

$$g \circ f = (15423)(245)(13) = (2)(4)(53)(1) \Rightarrow$$
$$g \circ f = \begin{pmatrix} 2 & 3 & 5 & 1 & 4 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$$

(d) $f^{-1} \circ g^{-1}$
Obtain inverse permutations by swapping rows in two-line notation:

$$f^{-1} = \begin{pmatrix} 4 & 1 & 2 & 3 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix} \quad g^{-1} = \begin{pmatrix} 5 & 4 & 1 & 2 & 3 \\ 1 & 5 & 3 & 4 & 2 \end{pmatrix}$$
$$\left(\begin{array}{ccccc} 5 & 4 & 1 & 2 & 3 \\ 1 & 5 & 3 & 4 & 2 \\ \hline 1 & 5 & 3 & 4 & 2 \\ 3 & 4 & 1 & 2 & 5 \end{array}\right) \Rightarrow f^{-1} \circ g^{-1} = \begin{pmatrix} 5 & 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$$

(e) $sgn(f)$
Instead of calculating $sgn(f)$ by definition we can use the fact, which was proved on seminar, that $sgn(f) = (-1)^{n-d}$ where n is the number of elements in permutation and d is the number of independent cycles, therefore $sgn(f) = (-1)^{5-2} = -1$

(f) $sgn(g)$
$sgn(g) = (-1)^{5-1} = 1$

(g) Verify that $sgn(f) \circ sgn(g) = sgn(f \circ g)$
Remember that $f \circ g = \begin{pmatrix} 1 & 5 & 3 & 4 & 2 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}$ and we can factorize it on independent cycles like this: $f \circ g = (12)(5)(3)(4)$. Therefore $sgn(f \circ g) = (-1)^{5-4} = -1 = -1 \cdot 1 = sgn(f) \cdot sgn(g)$

3. Let
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 7 & 1 & 3 & 2 & 6 \end{pmatrix}$$
$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 8 & 11 & 7 & 12 & 9 & 1 & 6 & 3 & 2 & 13 & 5 & 4 & 10 \end{pmatrix}$$

(a) Find f^{2020}
Split f on cycles $\Rightarrow f = (14)(25376)$, then:

$$f^{2020} = (14)^{2020}(25376)^{2020}$$

Let's use the fact that the order of a cycle of length n, in other words the minimum power to raise a permutation to obtain an identity permutation, is also n. It is easy to see that we need to scroll a cycle n times to get the same permutation. Therefore, $(14)^{2k} = id$ and $(25376)^{5k} = id$. 2020 is divisible by 2 and 5, hence after putting everything together:

$$f^{2020} = (14)^{2020}(25376)^{2020} = id \cdot id = id$$

(b) Find g^{2077}

Split g on cycles $\Rightarrow g = (18376)(2, 11, 5, 9)(10, 13)(4, 12)$, then:

$g^{2077} = (18376)^{2077}(2, 11, 5, 9)^{2077}(10, 13)^{2077}(4, 12)^{2077}$

$2077 \equiv 2(5) \quad 2077 \equiv 1(4) \quad 2077 \equiv 1(2) \Rightarrow$

$(18376)^{2077}(2, 11, 5, 9)^{2077}(10, 13)^{2077}(4, 12)^{2077} = (18376)^2(2, 11, 5, 9)(10, 13)(4, 12) =$

$(13687)(2, 11, 5, 9)(10, 13)(4, 12)$

$g^{2077} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 3 & 11 & 6 & 12 & 9 & 8 & 1 & 7 & 2 & 13 & 5 & 4 & 10 \end{pmatrix}$

4. Find all $f \in S_{2020}$ such that

$$f^2 = (1, 2, 3, ..., 2019, 2020)$$

where $(1, 2, 3, ..., 2019, 2020)$ is a long cycle of length 2020.

Let's denote this cycle as σ_{2020} and use lemma about parity of composition:

$$sgn(f \circ g) = sgn(f) \circ sgn(g)$$

We can see that there is no way to find such permutation f, because $sgn(\sigma_{2020}) = (-1)^{2020-1} = -1 \neq sgn^2(f)$