DSBA Discrete Mathematics HW6

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1. Suppose that each of the digits 0, 1, and 2 has exactly 100 occurrences in the decimal notation of a certain integer x. No other digit occurs there. Prove there is no such integer y that $x = y^2$.

   $x = \overline{x_{300}x_{299}\dots x_2x_1} \quad x_i \in \{0, 1, 2\}$

   Then if we sum up all the digits of x we get $100 \cdot 0 + 100 \cdot 1 + 100 \cdot 2 = 300 \equiv 0(3) \Rightarrow 3 \mid x$.

   Suppose $y = \overline{y_m y_{m-1} \dots y_1 y_0} = 10^m y_m + 10^{m-1} y_{m-1} + \dots + 10y_1 + y_0 \Rightarrow$

   $$\begin{aligned} y^2 &\equiv (10^m y_m + 10^{m-1} y_{m-1} + \dots + 10y_1 + y_0)^2 \equiv \\ &\equiv (y_m + y_{m-1} + \ldots + y_1 + y_0)^2 \not\equiv 0(3) \end{aligned}$$

   Because either $y_i = 0 \Rightarrow y = 0$ or $(y_m + y_{m-1} + \ldots + y_1 + y_0)^2 = 3k$, which is obviously not the case because there is no such integer which gives 3 being squared.

2. Prove that there are infinitely many primes of the form $6k + 5$.

   Suppose there are finite number of prime numbers of the form $6k + 5$. Let $P = (p_1, p_2, \dots, p_m)$ would be the set of such primes. Notice that $6k, 6k+2, 6k+4$ cannot be primes because these numbers would be even. So, only $6k + 1$ and $6k + 3$ are left except $6k + 5$.

   Let's choose $N = p_1p_2 \dots p_m - 1 = 6(p_1p_2 \dots p_m - 1) + 5 = 6k + 5$. If N is prime then it has to be in P because $N = 6k + 5$, but it is not, therefore N is composite.

   If the divisors are only of the form $6k + 1$, then the product would be also in this form:

   $$(6k + 1)(6k + 1) = 6(6k^2 + 2k) + 1 = 6k' + 1$$

   The same holds for divisors $6k + 3$:

   $$(6k + 3)(6k + 3) = 6(6k^2 + 6k) + 9 = 6(6k^2 + 6k + 1) + 3 = 6k' + 3$$

   And for any combination of those:

   $$(6k + 3)(6k + 1) = 6(6k^2 + 4k) + 3$$

   Assuming that there must be at least one divisor $6k + 5 = p_i \in P$ of N. This divisor must be prime because $p_i < N$. Because of the construction of N $p_i \mid p_1p_2 \dots p_m$. But then $p_i \mid 1 \Rightarrow p_i = 1$ which leads to contradiction.