Discrete Mathematics HW1

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1. **Check whether the following statements are tautologies:**

   Tautology is such a statement that is true for any combinations of truth values of its variables. So we can check whether the statement is tautology by assuming that it is not one and trying to find a contradiction or by finding a combination which corresponds to the false value of a whole expression.

   a) $((((A \to B) \to A) \to A) \to A) \to A$

   Let's simplify the expression step by step starting from the left. We'll use several known helpful facts[1]:

   - Equivalence of the implication and such disjunction
   $$A \to B \equiv \overline{A} \lor B$$
   - De Morgan's law
   $$\overline{A \lor B} \equiv \overline{A} \land \overline{B}$$
   - Double negation
   $$\overline{\overline{A}} \equiv A$$
   - Elimination law
   $$A \land B \lor A \equiv A$$

$$\begin{aligned}((((A \to B) \to A) \to A) \to A) \to A &= \\ ((((\overline{A} \lor B) \to A) \to A) \to A) \to A &= \\ (((\overline{\overline{A} \lor B} \lor A) \to A) \to A) \to A &= \\ (((A \land \overline{B} \lor A) \to A) \to A) \to A &= \\ (((A \to A) \to A) \to A\end{aligned}$$

   Finally, implication is false only if the premise is true and conclusion is false. Here that's not the case, hence obviously for any value of **A** the statement is valid, so it is a tautology.

[1] These facts were proved during the seminar and were allowed to use by prof.

b) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Let's prove by contradiction that the following statement is a tautology. So assume that it is not and the whole expression equals to zero.

The outer implication is zero if the left hand side is one and the right hand side is zero.

$$A \to (B \to C) = 1 \tag{1}$$
$$(A \to B) \to (A \to C) = 0 \tag{2}$$

Use the same logic with the 2nd equation.

$$A \to B = 1 \tag{3}$$
$$A \to C = 0 \tag{4}$$

So in order to satisfy the 3rd and 4th equations in this case:

$$A = 1, C = 0 \Rightarrow B = 1$$

Let's plug in these values to the first equation and we'll get a contradiction because the expression will be equal to zero. Therefore it's been proved that the statement is a tautology.

c) $(A \to (C \wedge D)) \to (((A \to B) \wedge (E \to \overline{D})) \to ((C \to B) \vee (D \wedge B \wedge \overline{E})))$

Once again let's assume that the statement is not valid for some combination of atoms values.

Then we have

$$A \to (C \wedge D) = 1 \tag{5}$$
$$((A \to B) \wedge (E \to \overline{D})) \to ((C \to B) \vee (D \wedge B \wedge \overline{E})) = 0 \tag{6}$$

By observing the consequent of the 6th equation we'll notice that two operands of the disjunction must be false at the same time.

$$C \to B = 0 \tag{7}$$
$$D \wedge B \wedge \overline{E} = 0 \tag{8}$$

Hence $B = 0, C = 1$ and for the 8th equation values of D and E don't matter because B is false. So we proceed further and take a look on the premise of 6th equation. It must be equal to one therefore if B is zero then A also must be zero because two operands in the conjunction have to be true.

So now we can already see that for $A = 0, B = 0, C = 1, D = 0, E = 0$, for instance, the statement is false. Eventually, the counter example has been found and it has been proved that it is not a tautology.

2. **Prove the following logical equivalences:**

a) $(A \land B) \lor C \equiv (A \lor C) \land (B \lor C)$

For proving the equivalence of these statements let's use the truth table method. There are 3 variables so $2^3 = 8$ combinations of values.

Truth tables for disjunction and conjunction of A and B are well-known:

A	B	$A \lor B$
0	0	0
0	1	1
1	0	1
1	1	1

A	B	$A \land B$
0	0	0
0	1	0
1	0	0
1	1	1

A	B	C	$(A \land B) \lor C$	$(A \lor C) \land (B \lor C)$
0	0	0	0	0
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

We can see that truth tables for these expressions are identical.

b) $(A \land B) \lor A \equiv A \equiv A \land (A \lor B)$

This equivalence is also known as an elimination law which we've used in the **1a** example.

We can see that corner statements are fully depend on **A**. For the left one, if A is zero it doesn't matter what value B has because the conjunction with zero gives us false and overall we get false in the disjunction. For the right one, if A is zero then immediately the whole statement is false. If A is one then the disjunction is one therefore the whole statement is true.

c) $\overline{A} \to B \equiv \overline{B} \to A$

By using equivalence for implication and negation law:

$$\overline{A} \to B = A \vee B$$
$$\overline{B} \to A = B \vee A$$

Disjunction is commutative so these statements are equivalent.

3. **Alice has chosen a natural number x. She makes the following statements, of which exactly one is true. Which statement is true?**

   (a) 12 is divisible by x
   (b) x = 4 or x = 10
   (c) if x is even then x = 6
   (d) $4 <= x <= 6$
   (e) 22 is divisible by x but $x < 22$
   (f) x = 7 or x = 12

   Let's proceed in a brute-force manner :) For each statement suppose it is true and others are false. Then consider the domain of x's for that valid statement. For each x check if the other statements rather true or false. If some of them is also true then the former statement is not unique.

   (a) All valid $x \in \{1, 2, 3, 4, 6, 12\}$, but 1 and 2 are also valid for **e**, 3 and 6 for **c**, 4 for **b**, 12 for **f**
   (b) $x \in \{4, 11\}$, but 4 is also valid for **d** and 11 is valid for **e**
   (c) $x \in \{n \in N \mid n \mod 2 \equiv 1 \vee n = 6\}$, notice that for some odd values of x only this statement holds, for example, if x = 9. Actually, there are many more such numbers like 13, 15, 17, ...
   (d) $x \in \{4, 5, 6\}$, 4 is also valid for **b**, 5 and 6 are valid for **c**
   (e) $x \in \{1, 2, 11\}$, 1 and 2 are valid for **a**, 11 is valid for **c**
   (f) $x \in \{7, 12\}$, 7 is valid for **c**, 12 is valid for **a**

   So, only **c** is true for some particular values listed above.