Vectorisation, Mapping and Reducing

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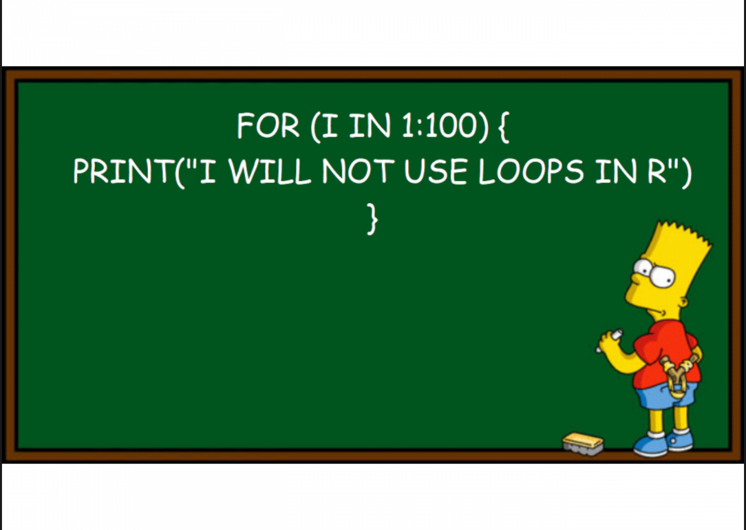
Lecture 10.2 (v2.0.0)

How to write your lines faster



Vectorisation

- Vectorised code is parallelised code.
  - Each operation for vectorised code is computable independently
  - The **same** operation is applied to each element (with different data)
  - **CPU optimisation** is possible and may be straightforward
  - GPU acceleration is possible
  - Vectorisations are always **one dimensional** representations
- A set of standardized elementwise computations is possible:
  - addition, subtraction, multiplication, division
  - other operations are possible, this becomes architecture dependent

Vectorisation of K-dimensional objects

- Matrices can be represented by standardized vectorisation procedures
  - $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$
  - **Row major** order: $\mathrm{vec}(A) = (a, b, c, d, e, f)$
  - **Column major** order: $\mathrm{vec}(A) = (a, d, b, e, c, f)$
- Matrix multiplication:
  - Is just sums of the correct components of the vectorised matrices
  - Choice of row vs column major order affects efficiency!
- Parallelization:
  - On a **shared memory** machine, the computations are distributed
  - Otherwise a **memory distribution** problem
- Efficient implementations for many common computations

Vectorisation and time complexity

- Assuming no parallelization:
- A **for loop** with N iterations is $O(N)$
- A **vectorisation** with N elements is $O(N)$
- But the vectorised code may still be orders of magnitude faster:
  - It often can be pushed into low-level code (C backend)
  - It can exploit CPU memory architecture: **caching** the correct content to avoid overhead
  - It can exploit CPU compute architecture: **multiple registers** in parallel
- Vectorisation also leads directly into parallel implementations:
  - It emphasises dependencies,
  - It encourages reordering of loops which can reduce time complexity.

Distributed computation

- On a shared memory system, parallel computation is trivial:

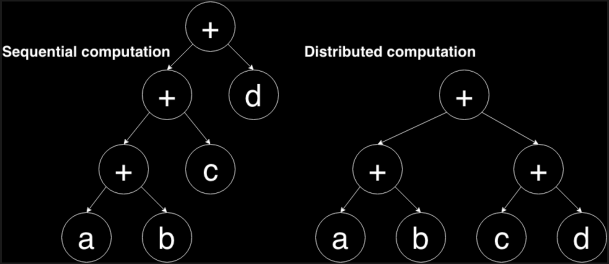
1. **Initialise** a parallelisable step, i.e.
   - enumerate the computations to be performed.
2. **Assign** them to *worker threads*:
   - either evenly if compute resource is guaranteed and tasks take equal time, e.g. on a GPU,
   - or as a **queue**.
3. **Action** the computation,
4. **Block**, i.e. wait for all computations to complete.

- On completion, the results are in the same place in memory as if the computation was performed in series.

Accumulate/Reduce

- Suppose that you wanted to compute the cumulative sum. Then the elements become **dependent** and you cannot use a purely independent vectorization.
- How can we combine results from N parallel computations?
  - **accumulate** is a vectorisation of any (binary, i.e. pairwise) (associative and commutative) function returning a single value
  - It may or may not provide access to intermediate function evaluations
  - It is often called a **Reduce** operation
  - It is a natively parallelisable way to view combining

Accumulate/Reduce computation graph



- Computational graph properties:
  - Nodes n internal to binary tree: $n(d) = \sum_{i=0}^{d} 2^i = 2^{d+1} - 1$
  - Depth d: $d(n) = \Theta(\log(n))$
- Algorithm properties:
  - Maximum compute could use $2^{d-1} = 2^{\log_2(n)} = n/2$ cores,
  - Parallel maximum speedup: $\Theta(\log(n))$ due to depth,
  - Simple blocking queue would reserve $n \log(n)/2$ processes,
  - Parallel efficiency cost: $E = \Theta(n/(n \log(n)))$ if all memory operations are in place.

Map/Reduce parallel framework

- For general purpose computation, the concepts of **mapping** and **reducing** enable efficient parallel code.
  - This uses the concept of a **key-value** tuple.
- The data are **mapped**: each **value** is assigned one or more **keys**
- Data associated with each **key** is passed to a **reducer**
- The **reducer** completes the computation
- More precisely,
  - **Map**: $M(k_0, v_0) \to ((k_1, v_1), \cdots, (k_K, v_K))$ is a function taking an input key/value pair to a list of output key/value pairs
  - **Reduce**: $R(k, (v_1, \cdots, v_R)) \to (k, v)$ is a function taking an input key and list of values, to a single (list-valued) value.

Map/Reduce vector averaging example

- Let X be a vector of length N.
- **Map**: $(k_0, v) \rightarrow (k, \{w = 1, v = v\})$
  - Assign each element a **key** $k \in [1, \cdots, K]$,
  - Assign a weight in the value,
  - The key acts as a **fold** of data.
  - Here, we are using the key as an arbitrary index, but this can be exploited.
- **Reduce**: $(k, \{v\}) \rightarrow (k, v)$
  - Count within each fold:
  - Return $(k, v) = (k, \{w = \sum_{k=1}^{K} v_w, v = \sum_{k=1}^{K} v_v\})$
- **Postprocess**: Return $\mathrm{mean} = \frac{\sum_{k=1}^{K} v_{k,v}}{\sum_{k=1}^{K} v_{k,w}}$

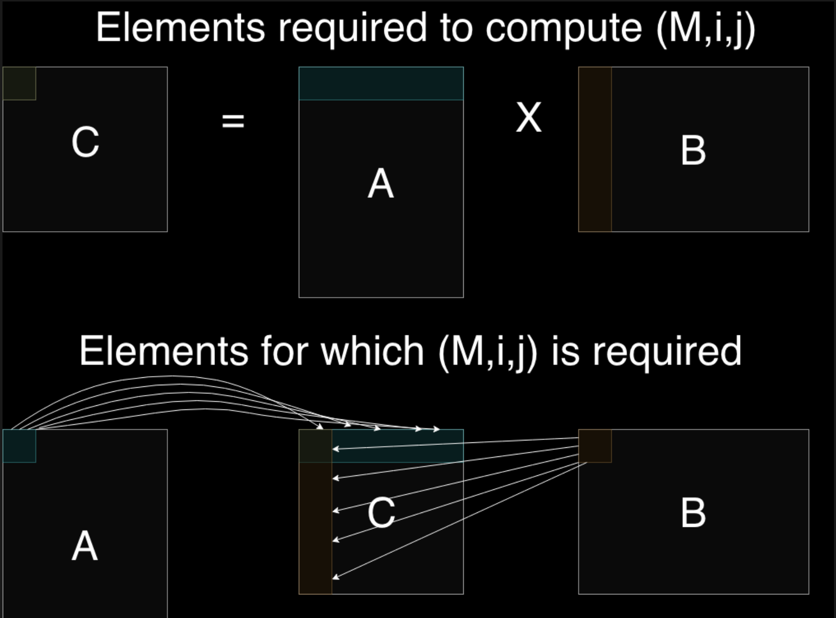
Map/Reduce analysis

- Assume **within-memory** implementation
- Use $p \leq K$ parallel threads (assume an integer multiple for simplicity...)
- The **map** stage is entirely parallel for cost $\Theta(\lceil n/p \rceil)$
- There is a **sort** stage which would be handled by a set of K lists
  - Independently parallelised construction of the K lists for cost $\Theta(\lceil n/p \rceil)$
  - In memory concatenation cost is negligible
- The **reduce** stage is parallel across $\Theta(\lceil K \lceil n/K \rceil / p \rceil) \approx \Theta(\lceil n/p \rceil)$ processes
- The **postprocess** stage is naively sequential with compute cost K
  - Total parallel time: $T_p = \Theta(\lceil n/p \rceil + \lceil n/p \rceil + \lceil n/p \rceil + K)$
  - Total sequential time: $T_s = \Theta(n)$
  - Total efficiency loss: $T_p / T_s \sim \Theta(1 + Kp)$

Map/Reduce reducer parallelisation

- **Practical concerns**:
  - Reducers don't automatically provide parallelism: we have to ask for it
  - This is because the reducer is not assumed to be commutative
  - But if the keys explicitly specify the desired folds, the reduce can be parallelised
  - In Hadoop/Spark Map/Reduce, reduction is parallelised **across keys**
  - In python/local Map/Reduce, reduction parallelisation is manual
- We can also **map** the **postprocess** k-fold reduction sum. Using p_2 processes:
  - Reduce the postprocess time from K to $T'_p = \Theta(\lceil K/p_2 \rceil + \lceil K/p_2 \rceil + p_2)$
  - Minimized at $p_2 = \sqrt{K}$
  - So we should use $K = p^2$ keys, keeping $p_2 = p$.
  - Total parallel time: $T_p = \Theta(\lceil n/p \rceil + \lceil n/p \rceil + \lceil n/p \rceil + \sqrt{p})$

Map/Reduce Matrix Example



Map/Reduce Matrix Example

$$C = \begin{bmatrix} k & l \\ m & n \end{bmatrix} = AB = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

- C has dimension $L \times L$, A has dimension $L \times K$
- where $k = au + bw + cy$, etc
- For a **one-stage** implementation, each of the **four** computations requires access to **three** elements from each array
- Represent the matrices in index form: $(key, value)$ where $key = (M, i, j)$ is the position (row and column) index and records the matrix type $M \in [A, B, C]$.
- Computing (C, i, j) requires all elements of A from row i and all elements of B from row j
- There will be $K = 3$ such elements
- Required to compute L^2 entries of C

Map/Reduce Matrix Multiplication Algorithm

- **Map**: each element is mapped independently to a list of K elements:
- $\mathrm{Map}((M,i,j),v):$
  - $((A,i,j),v) \to ((i,k),(A,j,v)) \quad \forall k = 1,\ldots,K$
  - $((B,i,j),v) \to ((k,j),(B,i,v)) \quad \forall k = 1,\ldots,K$
  - Cost: *2K* for each of L^2 independent entries
- **Reduce**: each key (i,j) is received $2K$ times, K from A and K from B.
- $\mathrm{Reduce}((i,j),(M,k,v)):$
  - $v_{i,j} = \sum_{k=1}^{K} v_{(A,k,v)} v_{(B,k,v)}$
  - Return $((i,j),v_{i,j})$
  - Cost: *K* for each of L^2 independent entries
- Cost:
  - Parallel time $T_p = \Theta(\lceil L^2/p \rceil K)$
  - Sequential time $T_s = \Theta(L^2 K)$
  - Efficiency 1
  - Despite inefficient duplication of data, which **fast** algorithms avoid!

Map/Reduce paradigm

- Map/Reduce is an essential tool in low-effort parallelism.
- The main computational advantage is that it is **scalable**: it can be parallelised across machines.
- So far we've described Map/Reduce as an **in memory** algorithm.
- In this case it naturally leads to fast analogues for a single computer:
  - We can imagine each **reducer key** being a memory location and the mappers are providing data fed to that location;
  - This is essentially how vectorised matrix computations are implemented efficiently.

Summary

- Vectorised code is efficiently computed
- Vectorised code is parallelisable with little effort
- Embarrassingly parallel algorithms are common
- Map/Reduce is a powerful paradigm for non-trivial parallelism and is the heart of massively parallel data processing
- Map/Reduce comes at an efficiency cost

References

- Chapter 27 of Cormen et al 2010 Introduction to Algorithms covers some of these concepts.
- Numpy vectorisation
- MapReduce algorithm for matrix multiplication
- Chrys Woods Parallel Python
- See the non-taught **Block 12** content on Spark and HDFS if you want to learn about how this works on **Distributed data**.