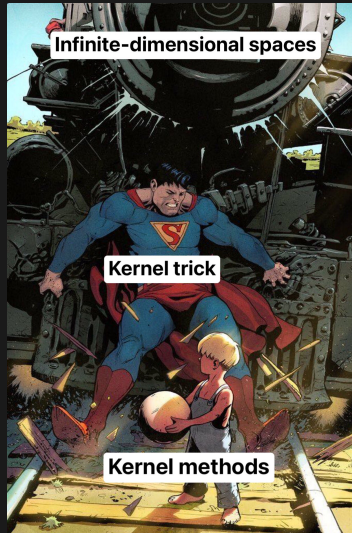


Nonparametrics and kernels

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Lecture 04.1 (v2.0.1)



<https://x.com/miniapaur/status/1682949029297025025>

Signposting

- ▶ We'll cover the following topics:
 - ▶ Nonparametric statistics - a birds eye view
 - ▶ Transforms - how to make good features
 - ▶ Density estimation
 - ▶ The Kernel Trick

Questions

- ▶ What is non-parametric statistics?
- ▶ Could we use a Fourier Transform in data science?
- ▶ How do we estimate density?
- ▶ Can we computationally compare in “infinite” dimensions?

Non-parametric statistics - overview

- ▶ Non-parametric statistics come in several flavours:
 1. Parameter-free hypothesis tests
 2. **Zero-parameter** representations which can be thought of as a **data transformation**.
 - ▶ examples include: Time-Frequency transforms, Kernel methods
 3. **Infinite-parameter** representations which can be thought of as generalisations of parametric models.
 - ▶ examples include: Hierarchical Dirichlet Process, the Stochastic Block Model for graphs
- ▶ We covered 1 in testing. We touch on 3 later. This lecture is about 2.
- ▶ Most methods are **parametric nonparametrics**: it is rare that a data transformation method isn't naturally thought of with a parameter!

Transforming data

- ▶ In previous practical problems we've used simple transforms to make the data easier to model:
 - ▶ log-transform
 - ▶ square-root/power transform
- ▶ Some data simplify greatly when transformed appropriately:
 - ▶ periodic data are simpler after taking a frequency transform
- ▶ Transformed data can be seen as feature augmentation, latent embedding, etc, depending on use.
- ▶ Generally, the goal is to make the noise **additive** so that it **averages out**.

The Basis Expansion

- ▶ Many transforms are designed to exactly reproduce the data.
 - ▶ These are **basis expansions** and are typically invertible.
- ▶ They make good feature sets if they result in a **dimensionality reduction**;
 - ▶ that is, they lead to a useful approximation using only a few features.
- ▶ PCA is one example of this.
- ▶ There are many others. . .

Fourier transform

- ▶ The Fourier transform is written:

$$\hat{f}(\eta) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \eta} dx$$

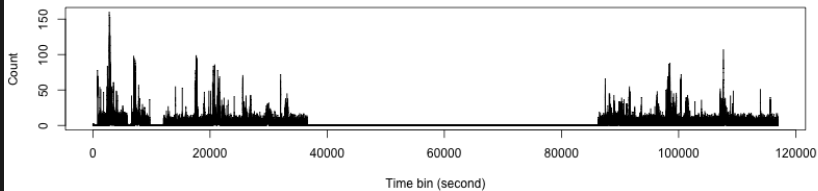
- ▶ The **Discrete Fourier Transform (DFT)** is used in practice as datasets typically have a minimum sampling rate δ .
- ▶ It is usually computed using the **Fast Fourier Transform (FFT)**.
- ▶ Consider using it for periodic data, or to look for periodicity.
- ▶ The **power** in any frequency i is proportional to $|\hat{f}(\eta_i)|^2$.
 - ▶ High power means this frequency is present in your data.
 - ▶ There are formal tests for “significance” of high power.

Fourier transform example

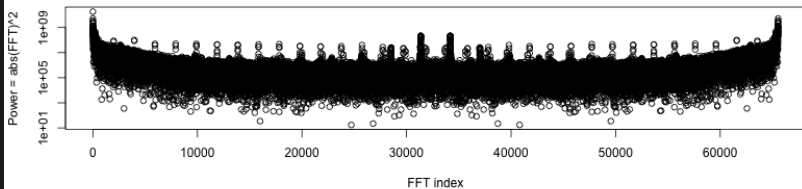
```
conndata_ts=data.frame(t=seq(min(conndata$ts),  
                             max(conndata$ts),by=1),x=0)  
for(i in 1:dim(conndata)[1]){  
  conndata_ts[ceiling(conndata[i,"ts"]-  
                      conndata_ts[1,"t"]), "x"] =  
    conndata_ts[ceiling(conndata[i,"ts"]-  
                      conndata_ts[1,"t"]), "x"] + 1  
}  
# Not fast unless length(x)=2^k  
myx=1:(2^16) # Largest valid choice  
conndata_fft=fft(conndata_ts[myx,"x"])
```

Fourier transform example

a) Time domain



b) Frequency domain



Walsh-Hadamard transform

- ▶ The Walsh-Hadamard transform is a version of the Fourier Transform that is useful for **Binary data**.
- ▶ It is defined recursively via the **Hadamard Matrix**:

$$H_0 = 1,$$

$$H_m = \frac{1}{\sqrt{2}} \begin{pmatrix} H_{m-1} & H_{m-1} \\ H_{m-1} & -H_{m-1} \end{pmatrix}$$

- ▶ For N total bits, the whole matrix is of size $2^m \times 2^m = N \times N$.
- ▶ The transform is $\mathbf{w} = \mathbf{H}\mathbf{x}$.
- ▶ \mathbf{w} can be computed efficiently with the **fast** Walsh-Hadamard transform in complexity $O(N \log(N))$.
- ▶ It was developed in encryption & signals processing but is useful to generate features in many contexts.

Walsh-Hadamard matrices

$$H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Walsh-Hadamard matrices

$$H_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

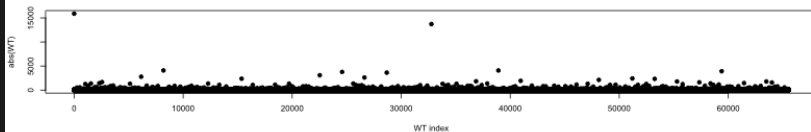
$$H_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Walsh-Hadamard transform examples

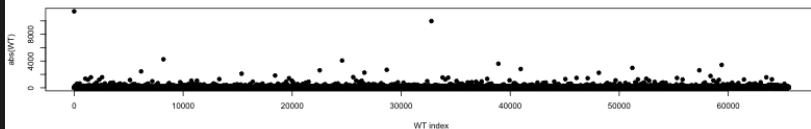
- ▶ Examples:
 - ▶ 00000... \rightarrow 00000...
 - ▶ 11111... \rightarrow +0000...
 - ▶ 01010... \rightarrow +-000...
 - ▶ 10101... \rightarrow ++000...
 - ▶ 00010001... \rightarrow ++++000....
- ▶ i.e. the i -th bit is **activated by periodicity** of length i
- ▶ The details are sensitive to the “phase”, i.e. exactly where in the sequence the periodicity lies.

Walsh-Hadamard transform example

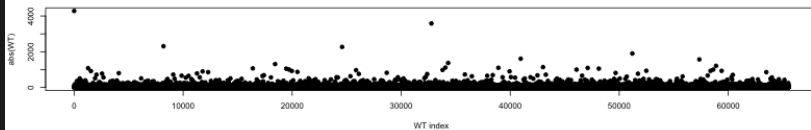
a) Walsh-Hadamard transform of $x > 0$



b) Walsh-Hadamard transform of $x > 1$



c) Walsh-Hadamard transform of $x > 10$



Other transforms

- ▶ Other transforms can be useful. For example:
 - ▶ Wavelets (time and space decomposition)
 - ▶ Laplace transform
 - ▶ Sine/ Cosine transforms
 - ▶ Hankel transform (radial basis function)
 - ▶ Polynomials
 - ▶ ... etc
- ▶ All you need is a **basis function** and you have a **transform**.

Density estimation overview

- ▶ Density estimation is an extremely hard problem because it maps **discrete data** into a **continuous estimator**.
- ▶ This is only conceptually well founded if we are willing to make (strong) **assumptions** regarding smoothness.
- ▶ There is no way to perform an entirely data-driven analysis!
- ▶ Many popular methods are very bad if, for example, the smoothness varies by location.

Kernel density estimation (KDE)

- ▶ Let $\{\vec{x}_i\}_{i=1}^N$ be a dataset on some space (for simplicity taken as \mathbb{R}^d).
- ▶ Then the Kernel K provides the density estimate for **any point** \vec{y} as:

$$f_{\mathbf{H}}(\vec{y}) = \frac{1}{N} \sum_{i=1}^N K_{\mathbf{H}}(\vec{y} - \vec{x}_i),$$

where \mathbf{H} is a matrix of bandwidths.

- ▶ In other words, its a **sum of independent contributions** from each datapoint.
- ▶ It can be written:

$$K_{\mathbf{H}}(\vec{y} - \vec{x}_i) = \frac{1}{\det(\mathbf{H})} K\left(\mathbf{H}^{-1}(\vec{y} - \vec{x}_i)\right)$$

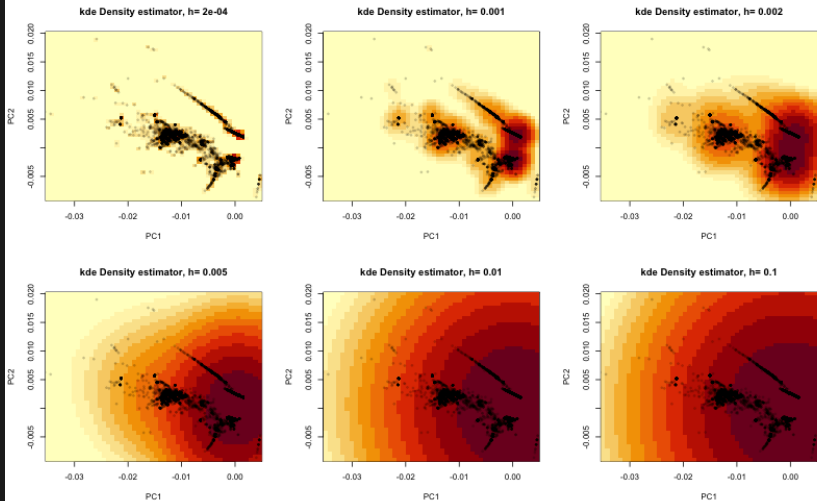
KDE in 1d

- ▶ In 1D:

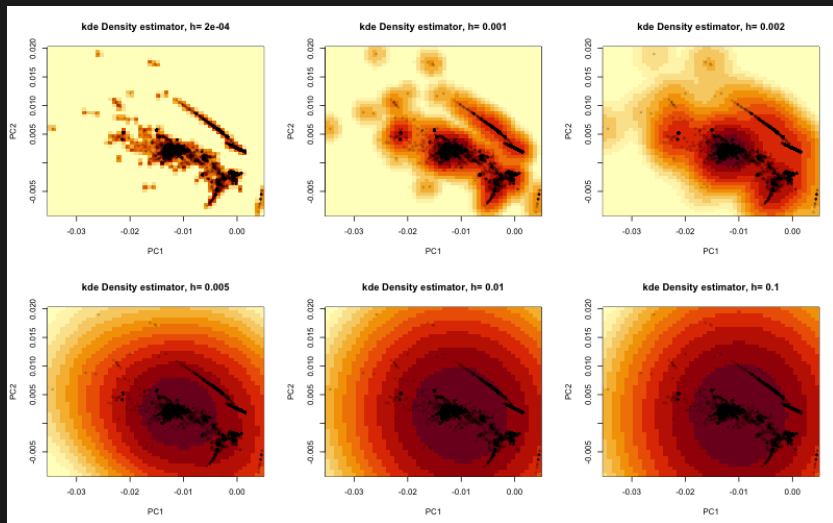
$$f_h(\vec{y}) = \frac{1}{N} \sum_{i=1}^N K\left(\frac{\vec{y} - \vec{x}_i}{h}\right)$$

- ▶ Its common to use a Normal kernel
 $K(x) = \text{Normal}(x; \mu = 0, \sigma = 1)$.
- ▶ h can be chosen by minimising the “Mean Integrated Square Error”...
- ▶ which theoretically suggests a functional form $h \propto N^{-1/5}$.
- ▶ Most density tools in packages use a reasonable default (which also depends on dimension).
 - ▶ This is appropriate for statistical inference of the density estimate at an unspecified point x .
- ▶ In practice the **“right” bandwidth** is a function of the **question**, so defaults might work poorly.
 - ▶ For **EDA**, we often want a **smaller bandwidth** to reveal potential data features

KDE Example



KDE with unique points



KDE kernels

- ▶ Some **important multivariate kernels**:
 - ▶ Spheroid Gaussian (\mathbf{H} and Σ are diagonal)
 - ▶ Rectangular (\mathbf{H} is diagonal, Uniform kernel)
 - ▶ Product Gaussian (\mathbf{H} off-diagonals are products, Σ is diagonal)
- ▶ \mathbf{H} is a parameter. It can be estimated by Cross-Validation but it is high dimensional so this is hard.

Applications of KDE

- ▶ Kernel density estimates are considered important in many applications, including:
 - ▶ Smoothing
 - ▶ Clustering
 - ▶ Topological Data Analysis
 - ▶ Level set estimation
 - ▶ Feature Extraction
 - ▶ ... etc!

K-Nearest neighbours

- ▶ Measuring neighbourhoods is a very important component of many applications.
- ▶ A fast way to do this is by computing k-Nearest neighbours (k-NN) for each point.
- ▶ Note the requirement for a **distance measure** (metric or otherwise).
- ▶ Algorithms to do this are called **nearest neighbour search**:
 - ▶ **Linear algorithms**: Check all distances for all points. $O(N^2)$ to compute the structure.
 - ▶ **Space partitioning**: KD-trees etc partition the space. $O(N \log(N))$ but are less good in high dimensions...
 - ▶ **Approximate methods**: there are many great methods for this problem, which are often nearly perfect and much faster. **Locality Sensitive Hashing** is popular.

k-NN density estimation

- ▶ A **Density estimate** using **k-NN**:

$$\hat{p}_{kNN}(x) = \frac{k}{N} \cdot \frac{1}{V_d R_k^d(x)}$$

- ▶ where:

- ▶ d is the dimension of the space,
- ▶ k is the number of neighbours,
- ▶ N is the sample size,
- ▶ $R_k^d(x)$ is the “radius”, i.e. the distance to the k -th closest neighbour of x , and
- ▶ V_d is the volume of a unit ball:

$$V_d = \frac{\pi^{d/2}}{\Gamma(d/2 + 1)}$$

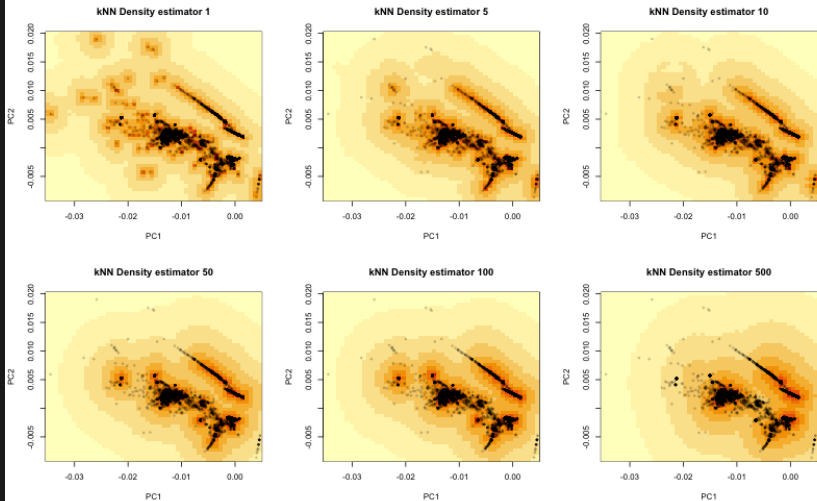
- ▶ so $V_1 = 2$, $V_2 = \pi$, $V_3 = \frac{4}{3}\pi^{3/2}$.
- ▶ NB Like the distance function, k is a **parameter**!

k-NN density estimation

```
library("TDA")
Xseq <- seq(-0.035, 0.0046, length.out=50)
Yseq <- seq(-0.009, 0.02, length.out=50)
Grid <- expand.grid(Xseq, Yseq)

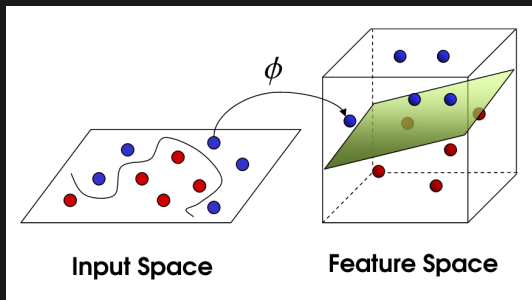
klist=c(1,2,5,10,20,50)
knnlist=lapply(klist,function(k){
  KNN <- knnDE(testdata_all.svd$u[,1:2], Grid, k)
  KNNm=matrix(KNN,nrow=length(Xseq),ncol=length(Yseq))
})
```

k-NN density estimation



The Kernel trick - a Motivation

- ▶ What if there is a nonlinearity in the data?
- ▶ **Solution:** map the data into a higher dimensional space in which the relationship is (approximately) linear



The Kernel Trick

- ▶ **Problem:** High dimensional spaces are hard to work with and computationally costly
- ▶ **Solution:** Make the space **implicit**: all computation is done using a **Kernel** that uses a map $\phi : X \rightarrow \mathbb{R}^n$ for data in the original space $x, y \in X$:

$$K(x, y) = \langle \phi(x), \phi(y) \rangle$$

- ▶ Kernels are any function that can be expressed as an **inner product**..

Kernel example

- **Input space** $X \subseteq \mathbb{R}^2$ with the **map**:

$$\phi : X = (x_1, x_2) \rightarrow (x_1^2, x_2^2, \sqrt{2}x_1x_2) \in \mathbb{R}^3$$

- i.e. the **second moments**. Then:

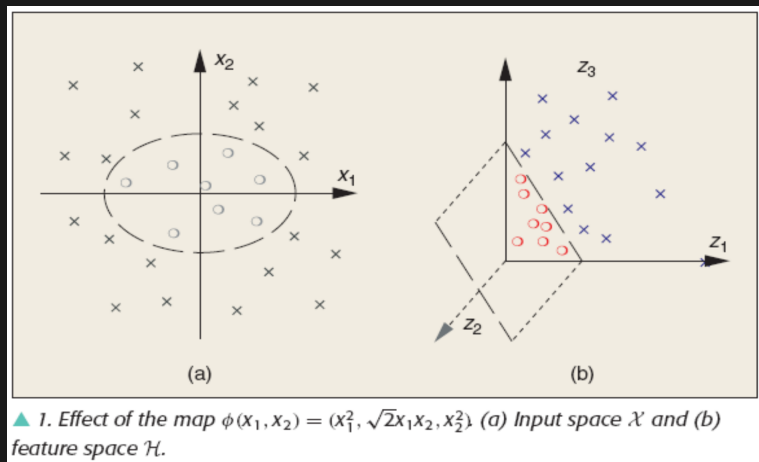
$$\langle \phi(x), \phi(y) \rangle = \langle (x_1^2, x_2^2, \sqrt{2}x_1x_2), (y_1^2, y_2^2, \sqrt{2}y_1y_2) \rangle \quad (1)$$

$$= (x_1^2y_1^2 + x_2^2y_2^2 + 2x_1y_1x_2y_2) \quad (2)$$

$$= (x_1y_1 + x_2y_2)^2 = \langle \mathbf{x}, \mathbf{y} \rangle, \quad (3)$$

- i.e. the (squared) **dot product**.

Kernel examples¹



Kernel properties

- ▶ Kernel spaces are **closed** under many operations.
- ▶ Being **closed** under f means that if x is in the space, $f(x)$ is also in the space.
- ▶ The operations are:
 1. Addition: $K(x, y) = K_1(x, y) + K_2(x, y)$
 2. Multiplication of a scalar: $K(x, y) = \alpha K_1(x, y)$
 3. Kernel Product: $K(x, y) = K_1(x, y)K_2(x, y)$
 4. Functional Product: $K(x, y) = f(x)f(y)$
 5. Kernel of a Kernel: $K(x, y) = K_3(\phi(x), \phi(y))$
 6. Matrix operation: $K(x, y) = x^T B y$
- ▶ It is therefore possible to make **modular kernels**.

Gram Matrix

- ▶ The **Gram matrix** is used by many methods exploiting the Kernel Trick:

$$\mathbf{K} \equiv (k(x_i, x_j))_{ij}, \quad \forall i, j$$

- ▶ This is a pre-computation: we compute the kernel between all pairs once, at the beginning, from which all subsequent computations follow.
- ▶ Gram matrices should be positive semi-definite. You can do the theory, or just check. . .
- ▶ The resulting space is called a **Reproducing Kernel Hilbert Space** (RKHS).
- ▶ It provides several important properties² and underpins many applications. . .

²Hofmann, Schoelkopf, & Smola (2008) “**Kernel Methods in Machine Learning**” (Ann. Stat.)

Important applications (later)

- ▶ Support Vector Machines
- ▶ Kernel Regression
- ▶ Kernel models on graphs (random walk, etc)
- ▶ Causal inference (Markov graphs)
- ▶ Kernel PCA

Kernel PCA

- ▶ For illustration we'll consider **kernel PCA**. Map $x_i \in \mathbb{R}^d$ to an arbitrary feature space $\phi(x_i) \in \mathbb{R}^n$ using the **Gram Matrix**:

$$K(x, y) = \phi(x)^T \phi(y)$$

- ▶ For which we'll consider the **eigenvector equation** for $v \in \mathbb{R}^n$:

$$Cv = \lambda v$$

- ▶ with the usual properties for the mean $\mu = \frac{1}{n} \sum_{i=1}^n \phi(x_i) = 0$ and covariance $C = \frac{1}{n} \sum_{i=1}^n \phi(x_i) \phi(x_i)^T$.

Kernel PCA continued

- ▶ Eigenvectors are **linear combinations** of the features:
 $v = \sum_{i=1}^n \alpha_i \phi(x_i).$
- ▶ It turns out that kernel PCA requires only solving the **regular eigenvector problem** for the eigenvalues α_i of a Kernel matrix \tilde{K} :

$$\tilde{K}\alpha_i = \lambda_i\alpha_i$$

Because the feature space may not be mean centred, $\tilde{K} \neq K$ in general but is simply related:

$$\tilde{K} = K - 2\mathbf{1}_{1/n}K + \mathbf{1}_{1/n}K\mathbf{1}_{1/n}$$

- ▶ where $\mathbf{1}_{1/n}$ is a vector of length n with elements $1/n$.

Kernel PCA example

► See ³.

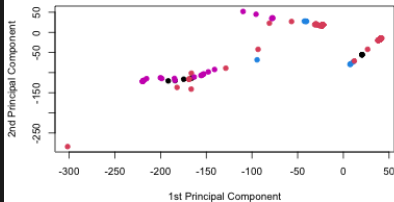
```
library("kernlab")
kpcvanilla=kpca(~.,data=testdata_sample,
  kernel="vanilladot",kpar=list(),features=4)
kpc=kpca(~.,data=testdata_sample,
  kernel="rbfdot",kpar=list(sigma=0.02),features=4)
kpclaplace=kpca(~.,data=testdata_sample,
  kernel="laplacedot",kpar=list(),features=10)
kpcpoly=kpca(~.,data=testdata_sample,
  kernel="polydot",kpar=list(),features=10)

plot(kpc@eig) # Plot eigenvalues
```

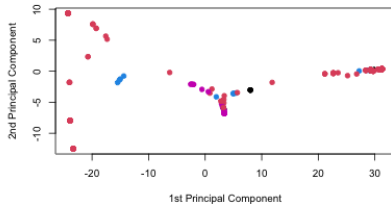
³Hofmann, Schoelkopf, & Smola (2008) “Kernel Methods in Machine Learning” (Ann. Stat.)

Kernel PCA example

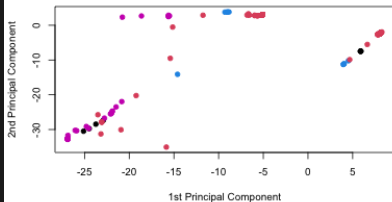
Vanilla



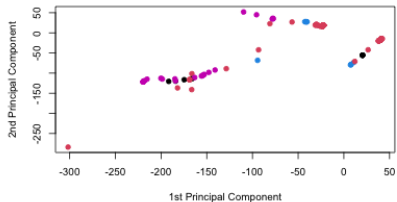
Laplace



Radial



Polynomial



Example Kernels:

- ▶ Linear Kernel: $k(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \mathbf{y} + c$
 - ▶ The regular dot product.
- ▶ Gaussian Kernel: $k(\mathbf{x}, \mathbf{y}) = \exp\left(\frac{-|\mathbf{x}-\mathbf{y}|^2}{2\sigma^2}\right) + c$
 - ▶ Very susceptible to outliers due to the “narrow tails”
- ▶ Exponential Kernel: $k(\mathbf{x}, \mathbf{y}) = \exp\left(\frac{-|\mathbf{x}-\mathbf{y}|}{2\sigma^2}\right) + c$
 - ▶ Also called the radial kernel
 - ▶ Related to the Laplacian kernel
- ▶ Power Kernel: $k(\mathbf{x}, \mathbf{y}) = -|\mathbf{x} - \mathbf{y}|^p$
 - ▶ conditionally positive definite, so needs extra care
- ▶ Log Kernel: $k(\mathbf{x}, \mathbf{y}) = -\log(|\mathbf{x} - \mathbf{y}| + 1)$
 - ▶ conditionally positive definite, so needs extra care
- ▶ Histogram Intersection Kernel
- ▶ ... and so on!

Thoughts on kernels

- ▶ The choice of Kernel is a **parameter**
- ▶ Which may itself **contain additional parameters**, e.g. bandwidths
- ▶ How to estimate? Evaluating performance requires calculating the whole N^2 matrix so it will be slow to iterate!
- ▶ Machine Learning thrives on usage cases where these decisions are either **relatively unimportant** or **determined by the method**.
- ▶ As we've seen, **adaptive** kernels such as nearest neighbour density estimation may be more robust than **parametric** kernels. Similar guidance holds here.

Reflection

- ▶ What role could transforms play in classification?
 - ▶ How do you know if they are working?
- ▶ How do these transforms generalise? What parameters does this introduce?
- ▶ What is the benefit of the Kernel Trick? What is the cost?
 - ▶ How would you apply it in practice?
- ▶ When should you estimate density by KDE vs KNN?
 - ▶ What does the density estimate at a point mean?
 - ▶ How could it be used in classification?
 - ▶ What are its other uses?

Signposting

- ▶ Transforms are clearly linked to PCA from Block 03
- ▶ Further reading for nonparametric statistics:
 - ▶ [Nonparametric Statistics by Eduardo García Portugués](#)
 - ▶ Basis Expansions: Chapter 5 of [The Elements of Statistical Learning: Data Mining, Inference, and Prediction](#) (Friedman, Hastie and Tibshirani).
- ▶ Further reading for Kernel Density Estimation:
 - ▶ Kernel Smoothing: Chapter 6 of [The Elements of Statistical Learning: Data Mining, Inference, and Prediction](#) (Friedman, Hastie and Tibshirani).
 - ▶ For kNN [Yen-Chi Chen's notes on kNN and the Basis](#)

Signposting (2)

- ▶ For the Kernel Trick [Dave Krebs' Intro to Kernels](#)
- ▶ For the Kernel PCA: [Rita Osadchi's Kernel PCA notes](#)
- ▶ Hofmann, Schoelkopf, & Smola (2008) "[Kernel Methods in Machine Learning](#)" (Ann. Stat.)
- ▶ Schoelkopf B., A. Smola, K.-R. Mueller (1998) "[Nonlinear component analysis as a kernel eigenvalue problem](#)".