

Intro/Refresher on MDPs and Reinforcement Learning



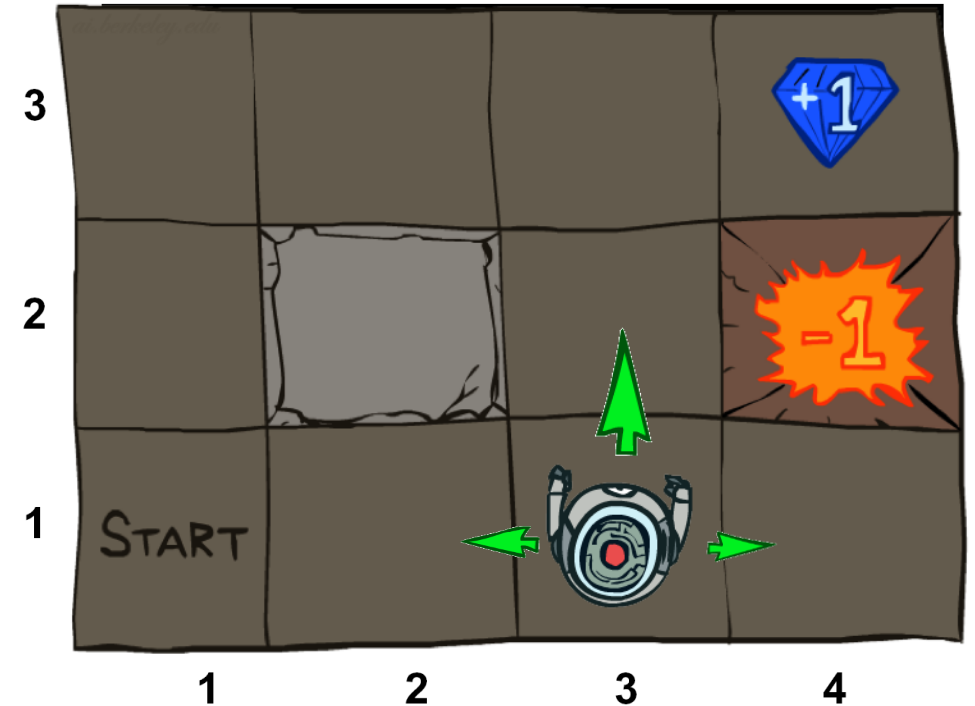
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University of Utah

[Based on slides created by Dan Klein and Pieter Abbeel for CS188 Intro to AI at UC Berkeley. <http://ai.berkeley.edu>.]

Markov Decision Processes

- An MDP is defined by:
 - A **set of states** $s \in S$
 - A **set of actions** $a \in A$
 - A **transition function** $T(s, a, s')$
 - Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - Also called the model or the dynamics
 - A **reward function** $R(s, a, s')$
 - Sometimes just $R(s)$, $R(s,a)$, or $R(s')$
 - A **start state**
 - Maybe a **terminal state**
- MDPs are non-deterministic search problems
 - One way to solve them is with expectimax search
 - We'll have a new tool soon



Other examples of MDPs

- Checkers Boardgame
- Medication treatment

Other examples of MDPs

- Self-driving car
- Language Generation (ChatGPT)

What is Markov about MDPs?

- “Markov” generally means that given the present state, the future and the past are independent. **Conditional Independence!**
- For Markov decision processes, “Markov” means action outcomes depend only on the current state

$$\begin{aligned} &P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots, S_0 = s_0) \\ &= \\ &P(S_{t+1} = s' | S_t = s_t, A_t = a_t) \end{aligned}$$

- This is just like search, where the successor function could only depend on the current state (not the history)



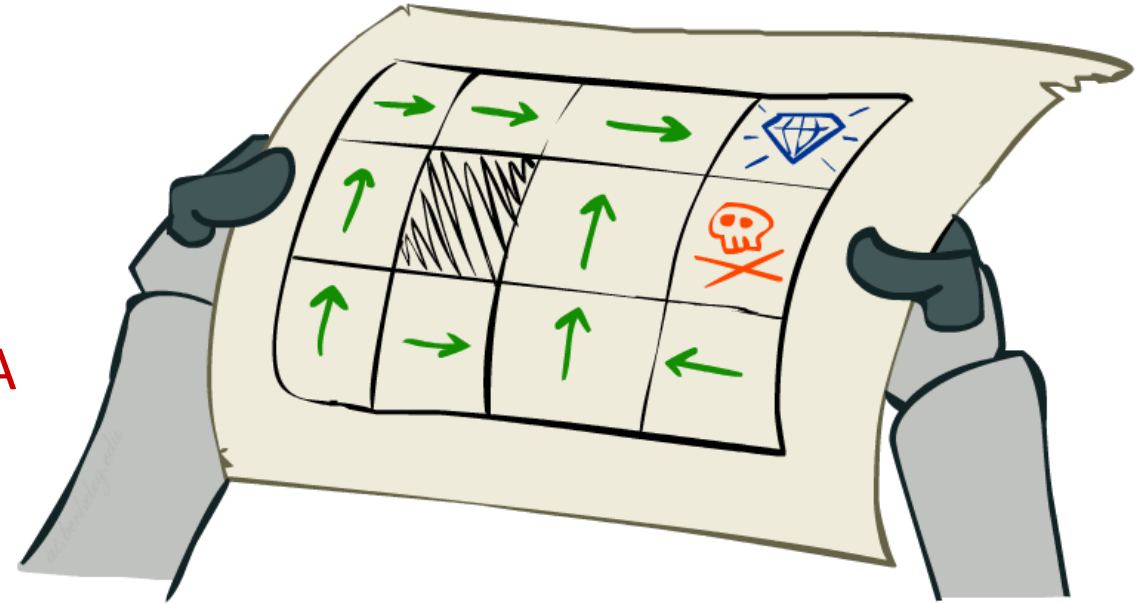
Andrey Markov
(1856-1922)

Types of Markov Models

	System state is fully observable	System state is partially observable
System is autonomous	Markov chain	Hidden Markov model (HMM)
System is controlled	Markov decision process (MDP)	Partially observable Markov decision process (POMDP)

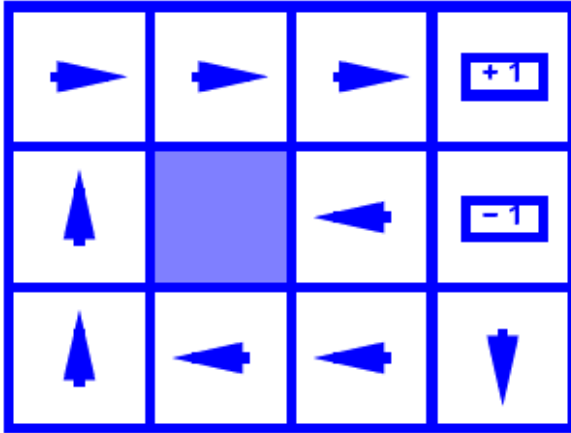
Policies

- In deterministic single-agent search problems, we wanted an optimal **plan**, or sequence of actions, from start to a goal
- For MDPs, we want an optimal **policy** $\pi^*: S \rightarrow A$
 - A policy π gives an action for each state
 - An optimal policy is one that maximizes expected utility if followed
 - An explicit policy defines a reflex agent

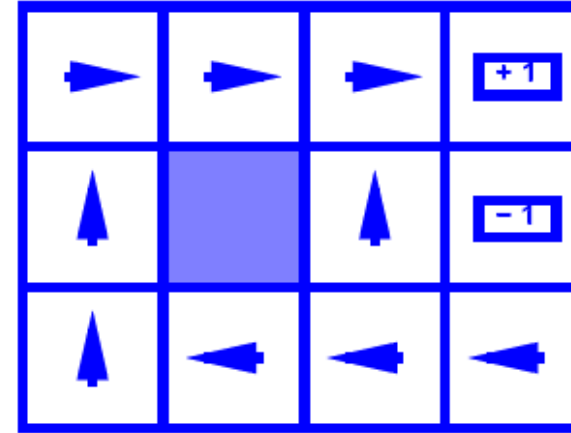


Optimal policy when $R(s, a, s') = -0.03$
for all non-terminals s

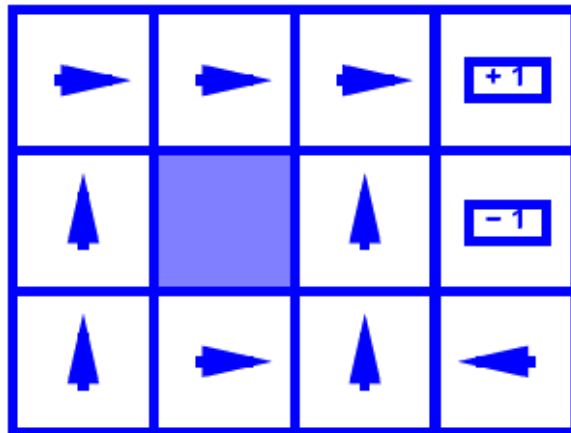
Optimal Policies



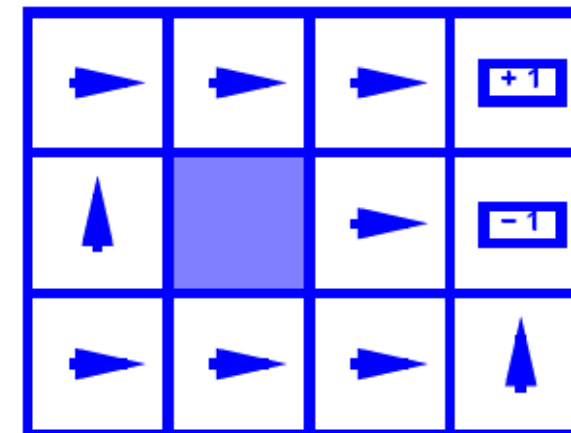
$R(s) = -0.01$



$R(s) = -0.03$



$R(s) = -0.4$



$R(s) = -2.0$

Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially



1

Worth Now



γ

Worth Next Step



γ^2

Worth In Two Steps

Discounting

- How to discount?

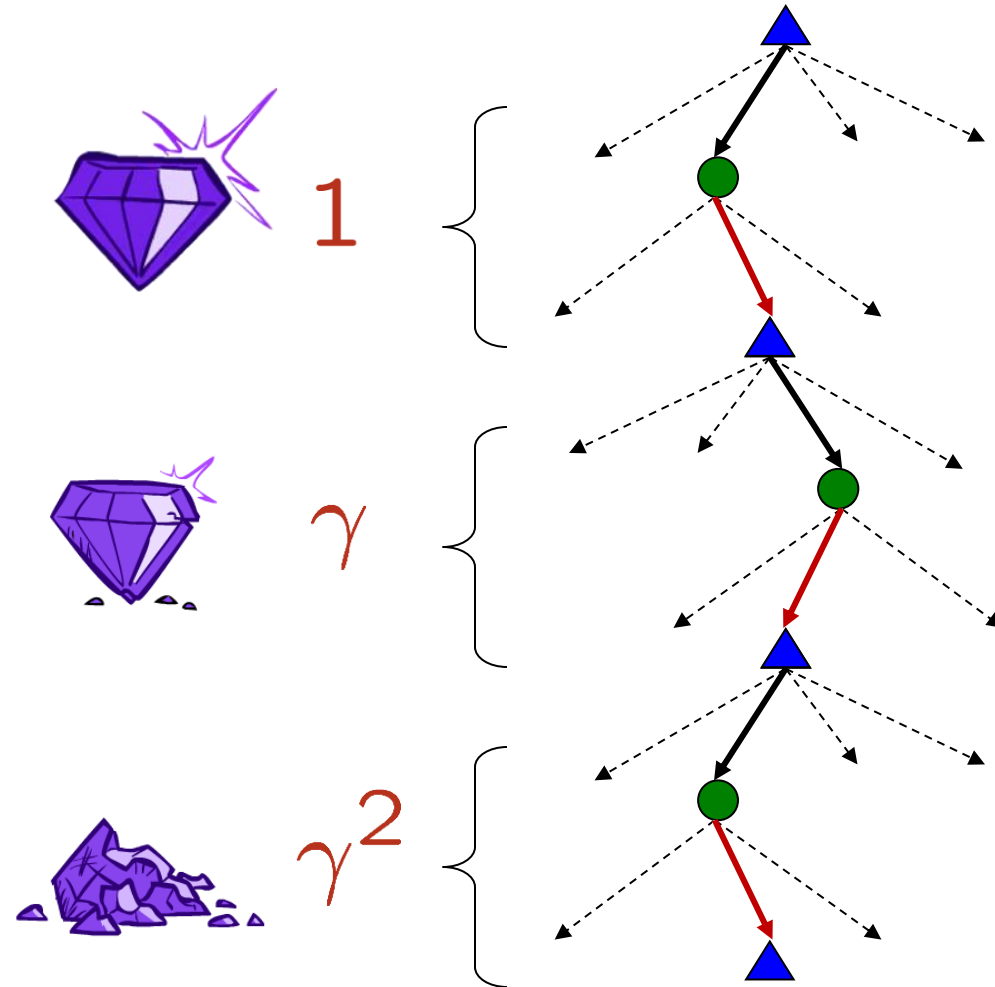
- Each time we descend a level, we multiply in the discount once

- Why discount?

- Sooner rewards probably do have higher utility than later rewards
- Also helps our algorithms converge

- Example: discount of 0.5

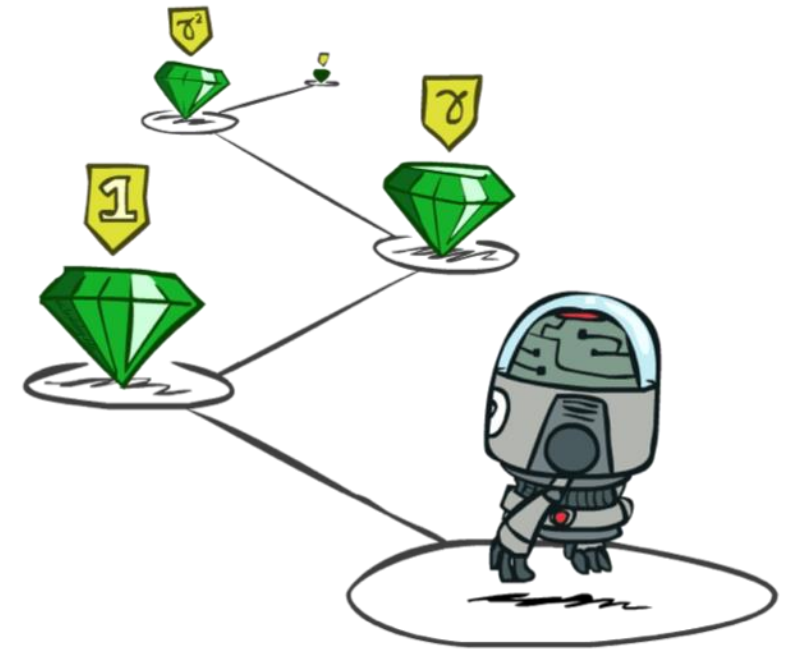
- $U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3$
- $U([1,2,3]) < U([3,2,1])$



Stationary Preferences

- Theorem: if we assume **stationary preferences**:

$$\begin{aligned} [a_1, a_2, \dots] &\succ [b_1, b_2, \dots] \\ &\Updownarrow \\ [r, a_1, a_2, \dots] &\succ [r, b_1, b_2, \dots] \end{aligned}$$



- Then: there are only two ways to define utilities

- Additive utility: $U([r_0, r_1, r_2, \dots]) = r_0 + r_1 + r_2 + \dots$
- Discounted utility: $U([r_0, r_1, r_2, \dots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \dots$

Quiz: Discounting

- Given: reward

10				1
a	b	c	d	e

- Actions: East, West, and Exit (only available in exit states a, e)
- Transitions: deterministic

- Quiz 1: For $\gamma = 1$, what is the optimal policy?

10				1
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- Quiz 2: For $\gamma = 0.1$, what is the optimal policy?

10				1
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- Quiz 3: For which γ are West and East equally good when in state d?



Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?

- Solutions:

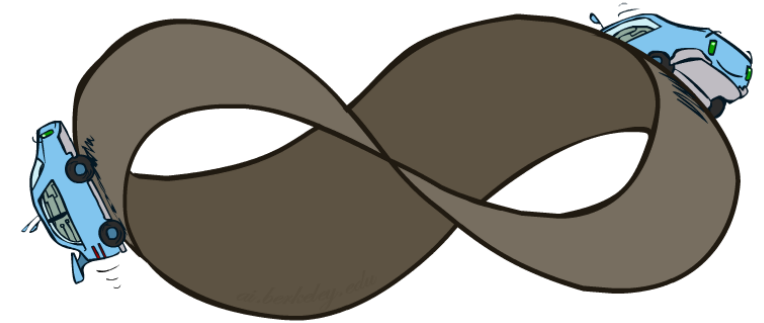
- Finite horizon: (similar to depth-limited search)
 - Terminate episodes after a fixed T steps (e.g. life)
 - Gives nonstationary policies (π depends on time left)

- Discounting: use $0 < \gamma < 1$

Why?

$$U([r_0, \dots, r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max} / (1 - \gamma)$$

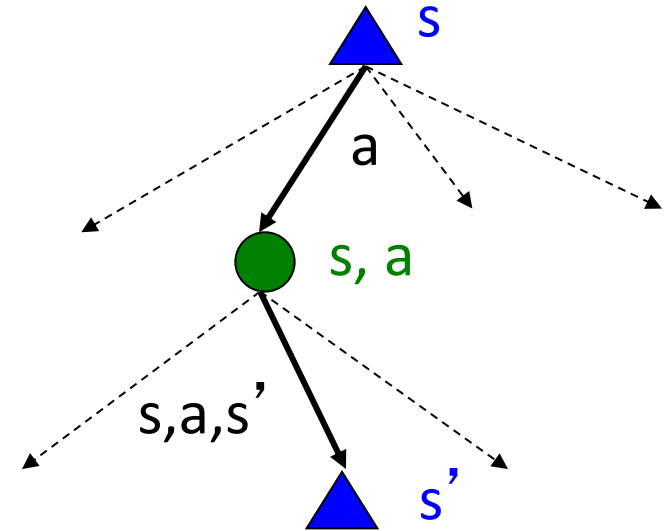
- Smaller γ means smaller “horizon” – shorter term focus
- Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “overheated” for racing)



MDP Notation

- Markov decision processes:

- Set of states S
- Start state s_0
- Set of actions A
- Transitions $P(s' | s, a)$ (or $T(s, a, s')$)
- Rewards $R(s, a, s')$ (and discount γ)

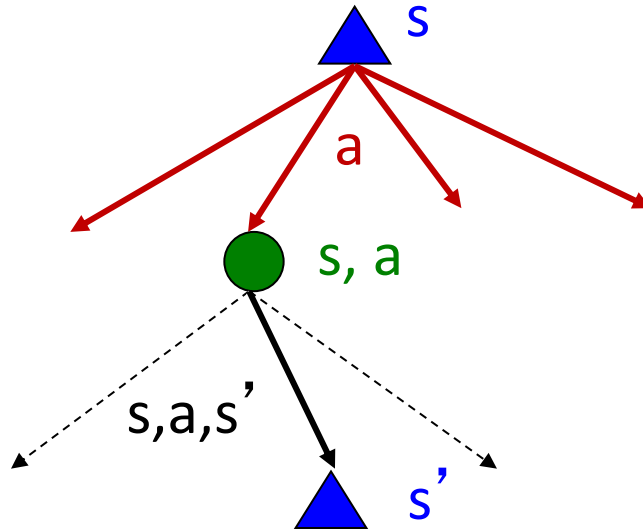


- Important MDP quantities:

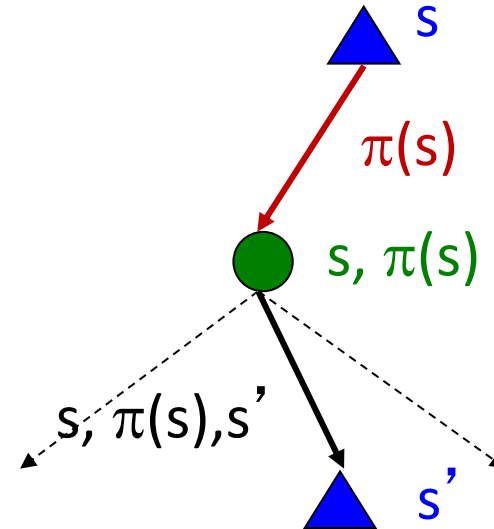
- Policy = Choice of action for each state
- Utility = expected sum of (discounted) rewards = “expected return”

Fixed Policies

Choosing actions



Do what π says to do

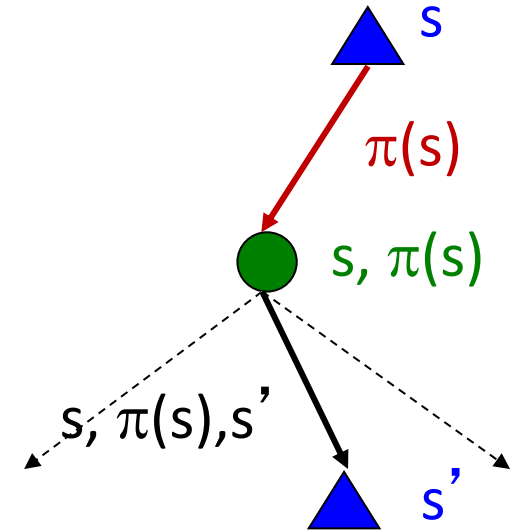


- If we fixed some policy $\pi(s)$, then the computation is simpler – only one action per state
 - ... though the performance now depend on which policy we fixed

Performance of a Fixed Policy

- Goal: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s , under a fixed policy π :
 $V^\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')]$$



Example: Policy Evaluation

Always Go Right



Always Go Forward



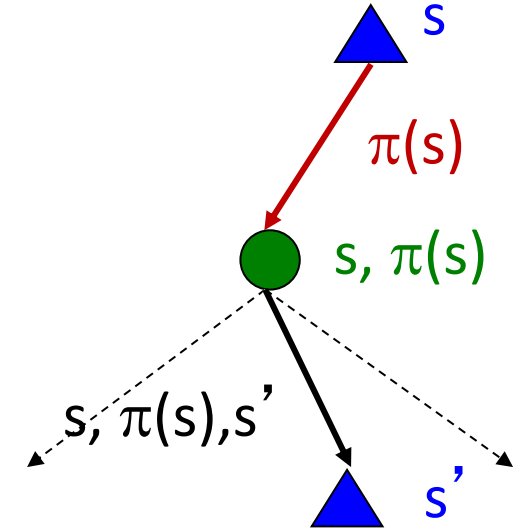
Policy Evaluation

- How do we calculate the V 's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

- Efficiency: $O(S^2)$ per iteration
- Idea 2: Just a linear system
 - Solve with Numpy or Matlab (or your favorite linear system solver)



Policy Evaluation

- Idea 2: Without the maxes, the Bellman equations are just a linear system
 - Solve with Numpy or Matlab (or your favorite linear system solver)

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')]$$

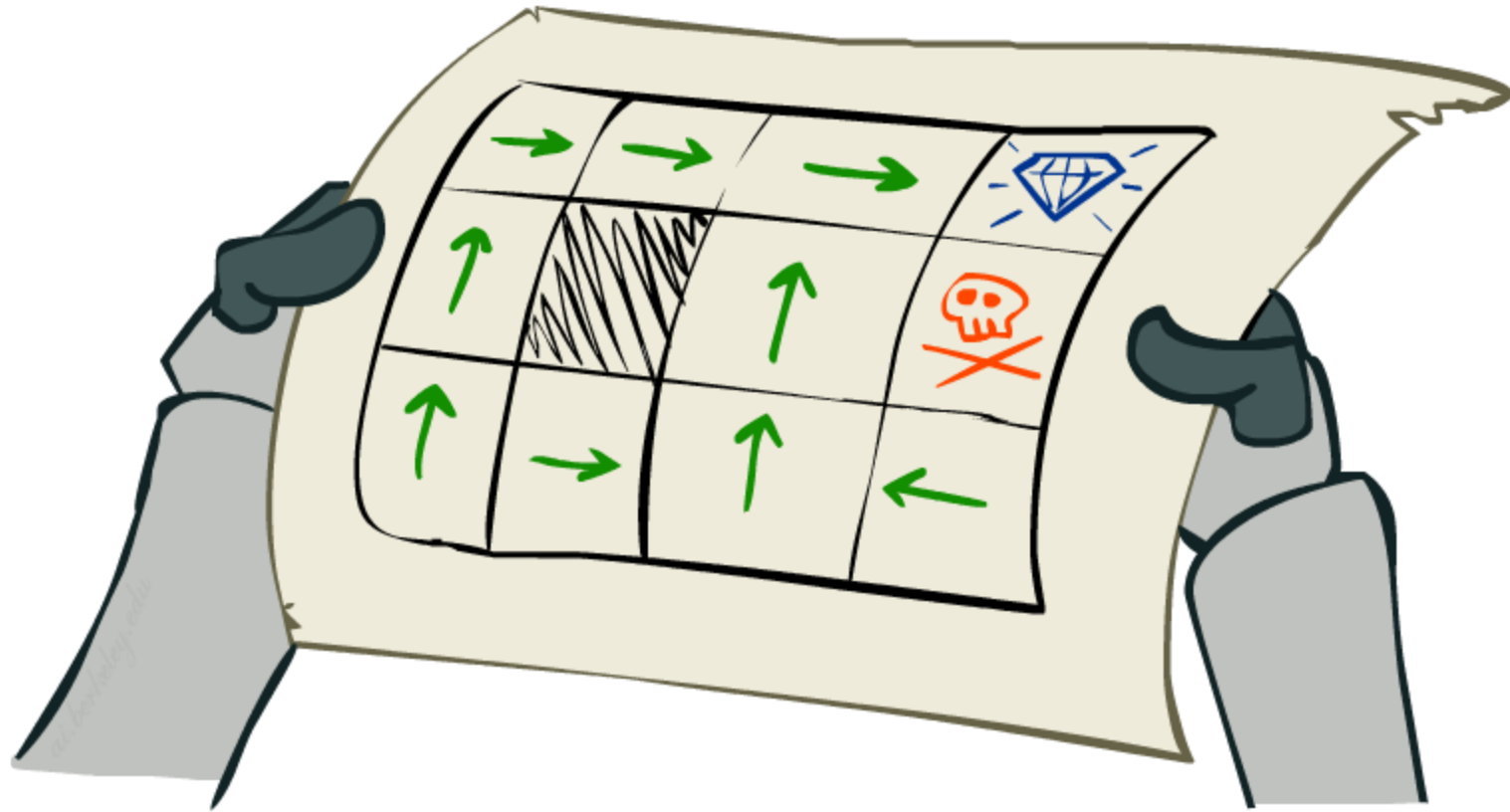
$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') R(s, \pi(s), s') + \gamma \sum_{s'} T(s, \pi(s), s') V^\pi(s')$$

$$V^\pi(s) = \bar{R}(s) + \gamma \sum_{s'} T(s, \pi(s), s') V^\pi(s')$$

$$V^\pi = \bar{R} + \gamma T^\pi V^\pi$$

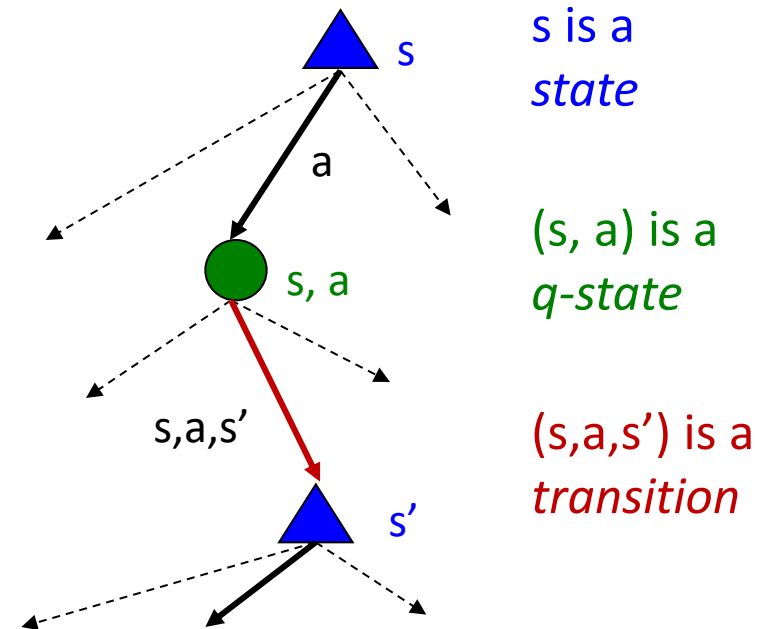
$$(I - \gamma T^\pi) V^\pi = \bar{R} \quad \Rightarrow \quad V^\pi = (I - \gamma T^\pi)^{-1} \bar{R}$$

Solving MDPs



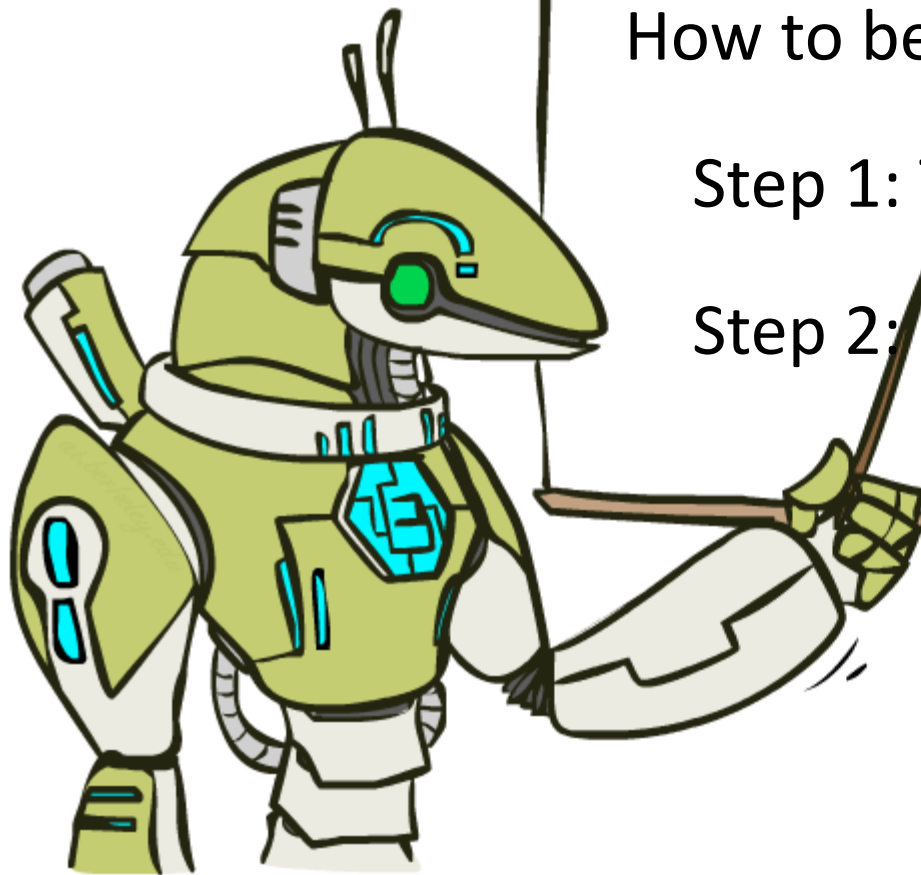
Optimal Quantities

- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s
 $\pi^*(s) = \arg \max_a Q^*(s,a)$



Can we write the optimal policy in terms of Q^* ?

The Bellman Equations



How to be optimal:

Step 1: Take correct first action

Step 2: Keep being optimal

Bellman Equations

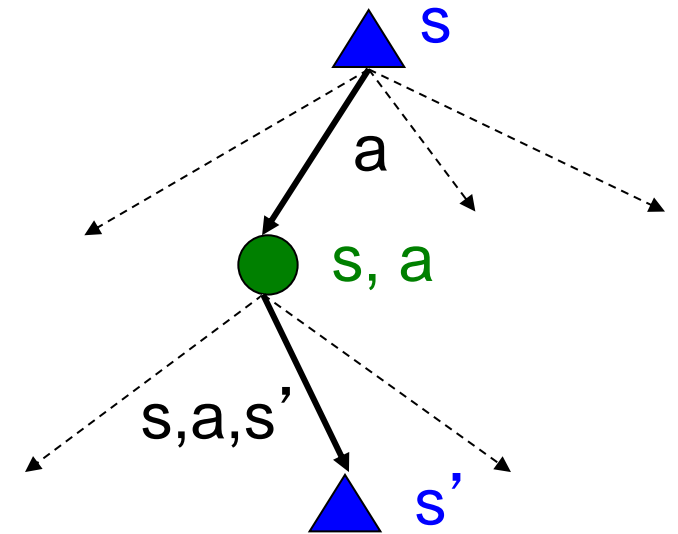
- Fundamental operation: compute the (expectimax) value of a state
 - Expected utility under optimal action
 - Average sum of (discounted) rewards
 - This is just what expectimax computed!

- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



Aside: Different ways to write Bellman Eqns

- What if R only depends on state and action? e.g. $R(s,a,s') = R(s,a)$

$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) =$$

Aside: Different ways to write Bellman Eqns

- What if R only depends on state? e.g. $R(s,a,s') = R(s)$

$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) =$$

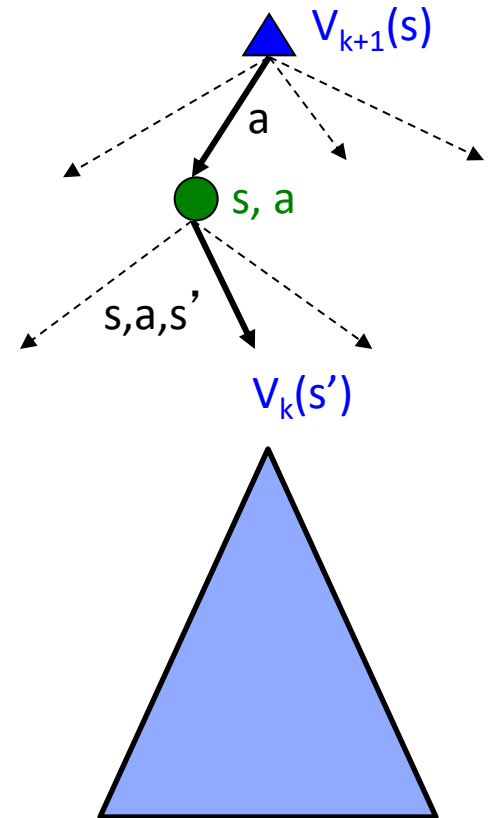
Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

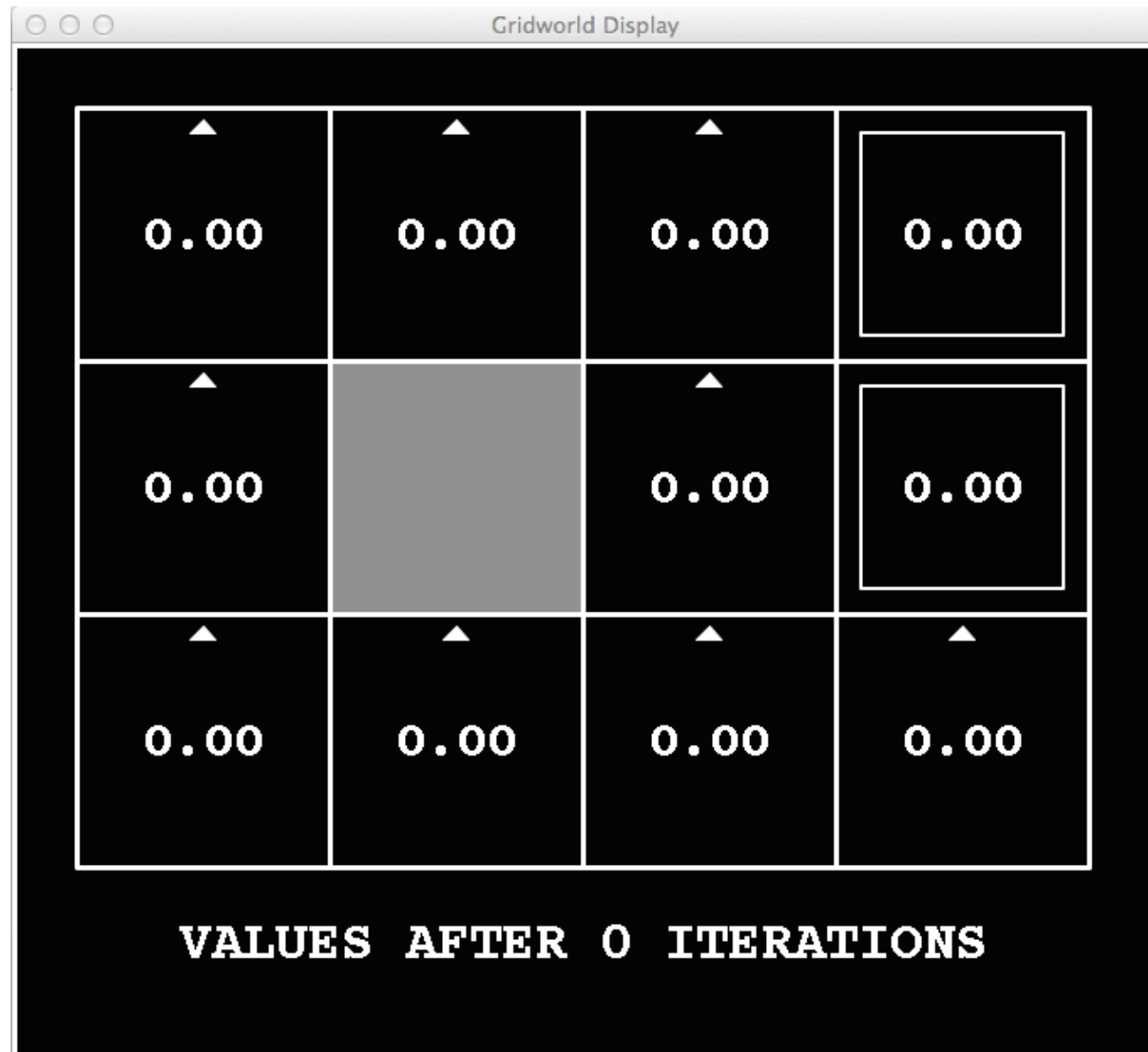
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

Bellman Update Equation

- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
 - Basic idea: approximations get refined towards optimal values
 - Policy may converge long before values do



$k=0$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=1



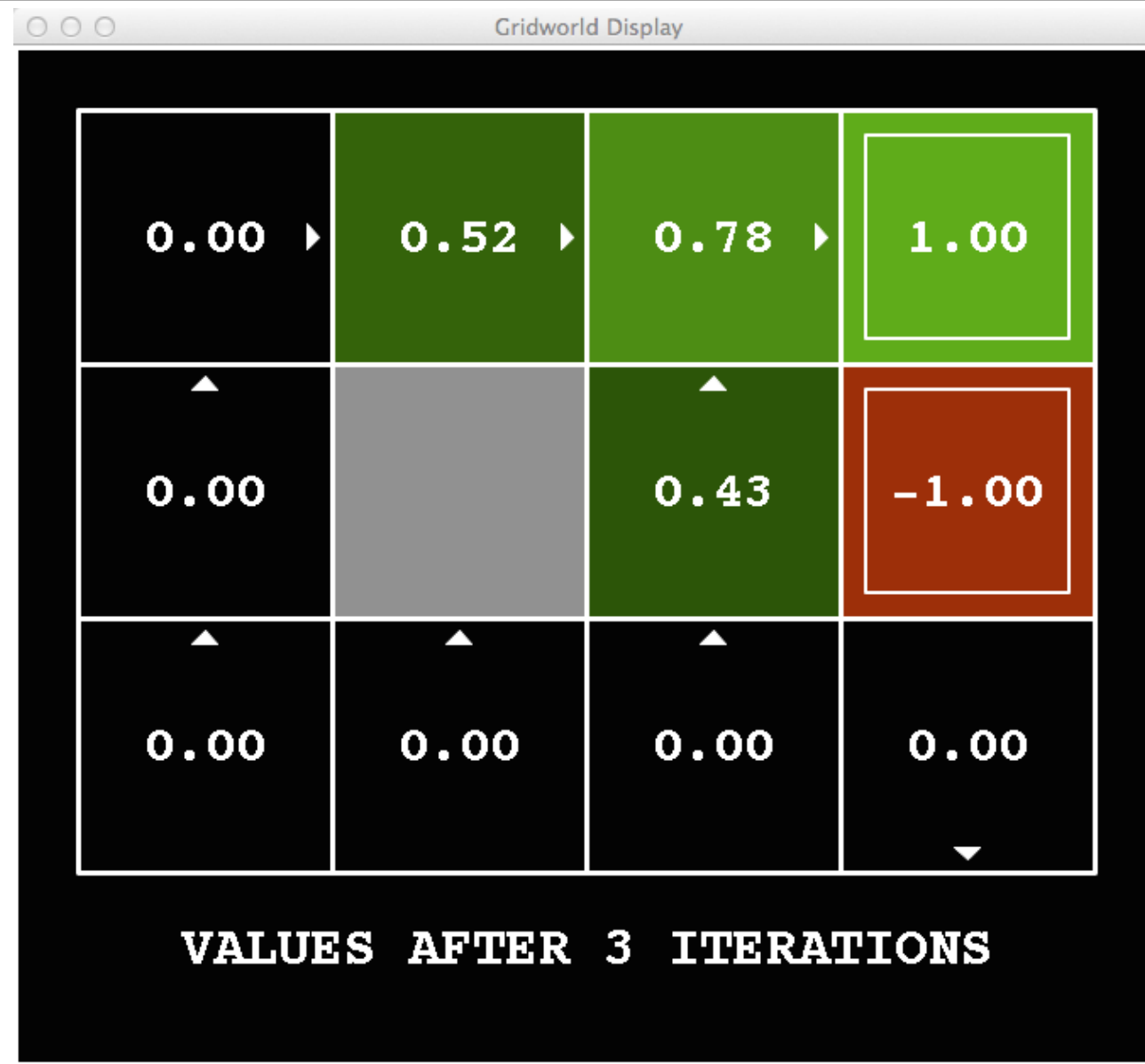
Noise = 0.2
Discount = 0.9
Living reward = 0

k=2



Noise = 0.2
Discount = 0.9
Living reward = 0

k=3



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=4$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

k=7



Noise = 0.2
Discount = 0.9
Living reward = 0

k=8



Noise = 0.2
Discount = 0.9
Living reward = 0

k=9



Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Value Iteration

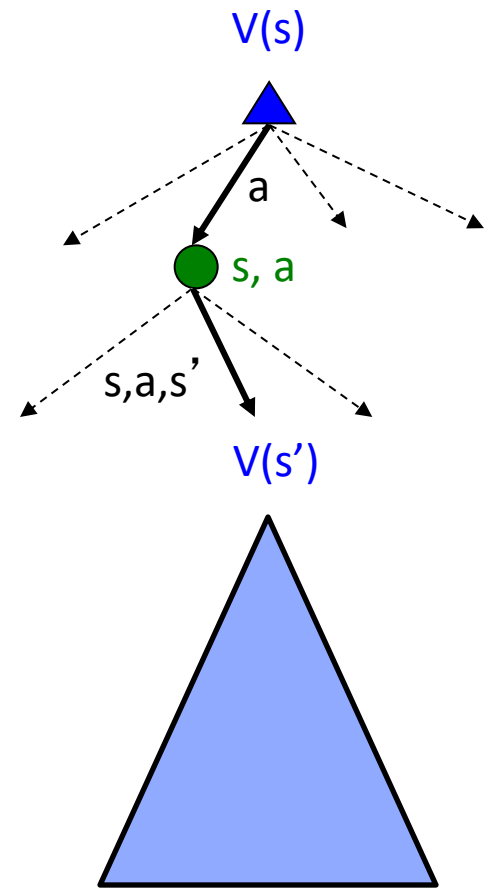
- Bellman equations **characterize** the optimal values:

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

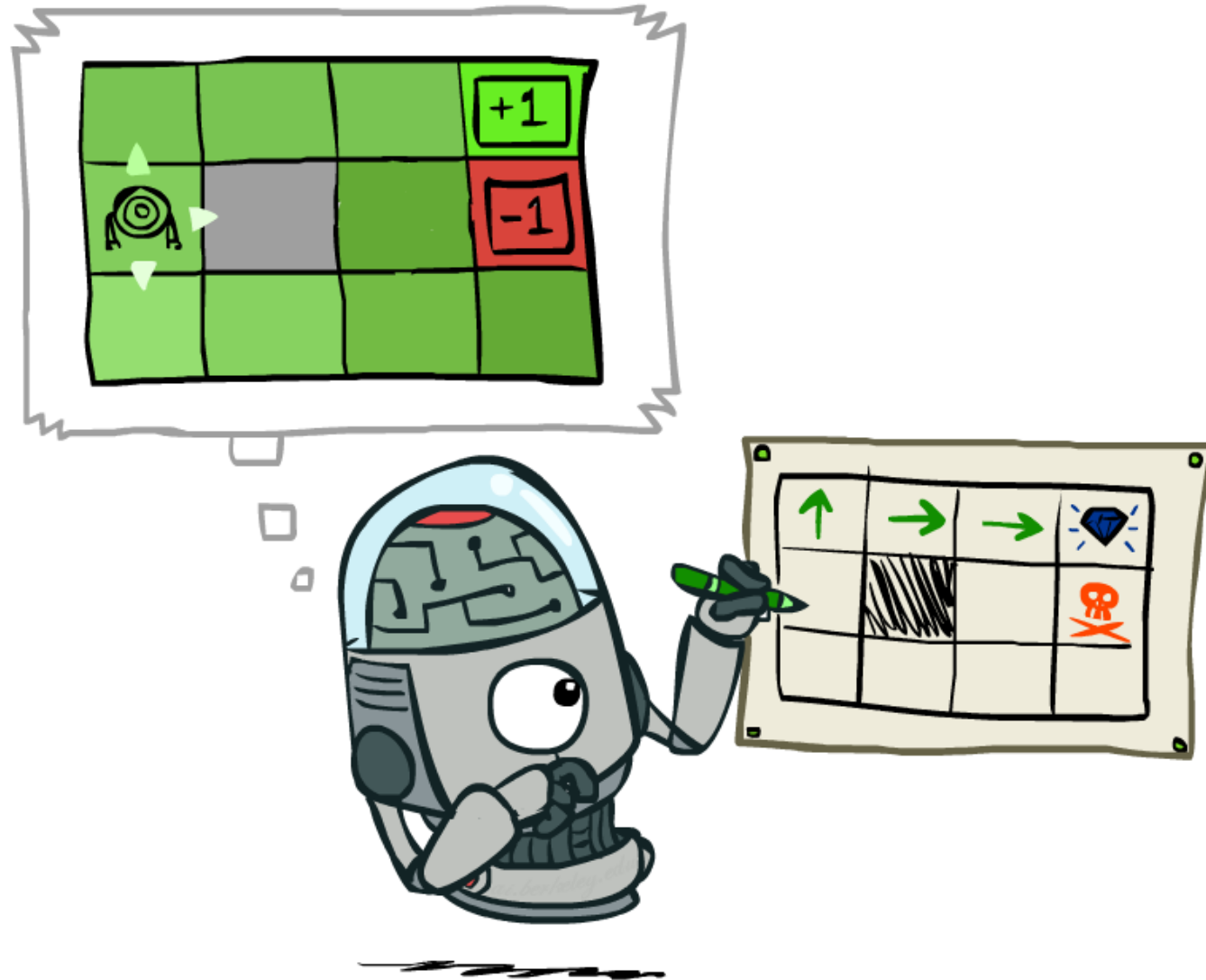
- Value iteration **computes** them:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Value iteration is just a fixed point solution method
 - ... though the V_k vectors are also interpretable as time-limited values



Policy Extraction



Computing Actions from Values

- Let's imagine we have the optimal values $V^*(s)$
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)



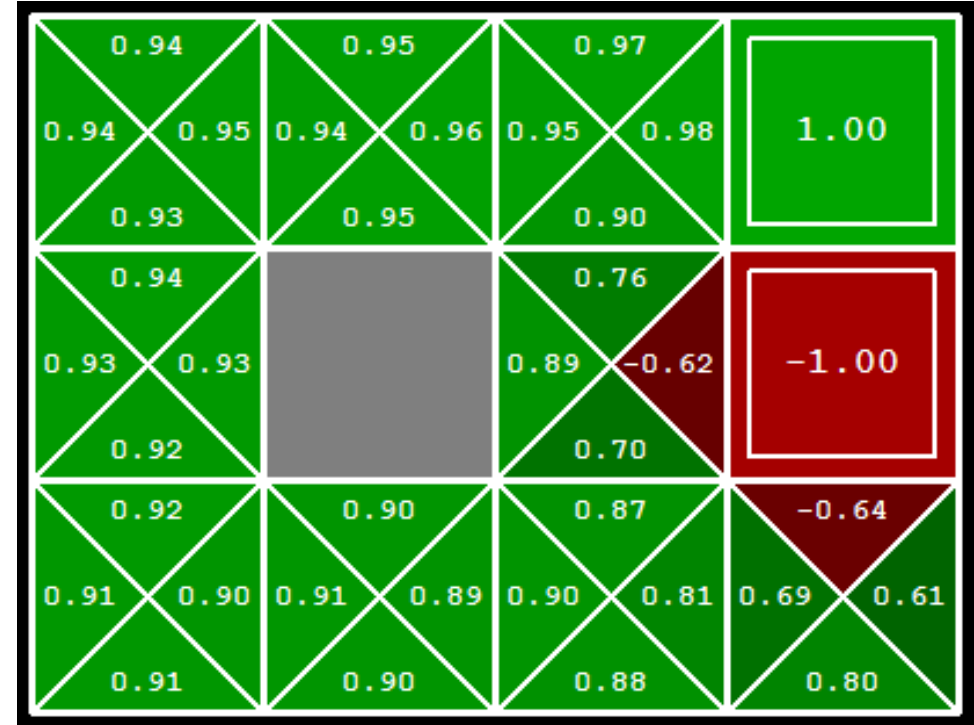
$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- This is called **policy extraction**, since it gets the policy implied by the values

Computing Actions from Q-Values

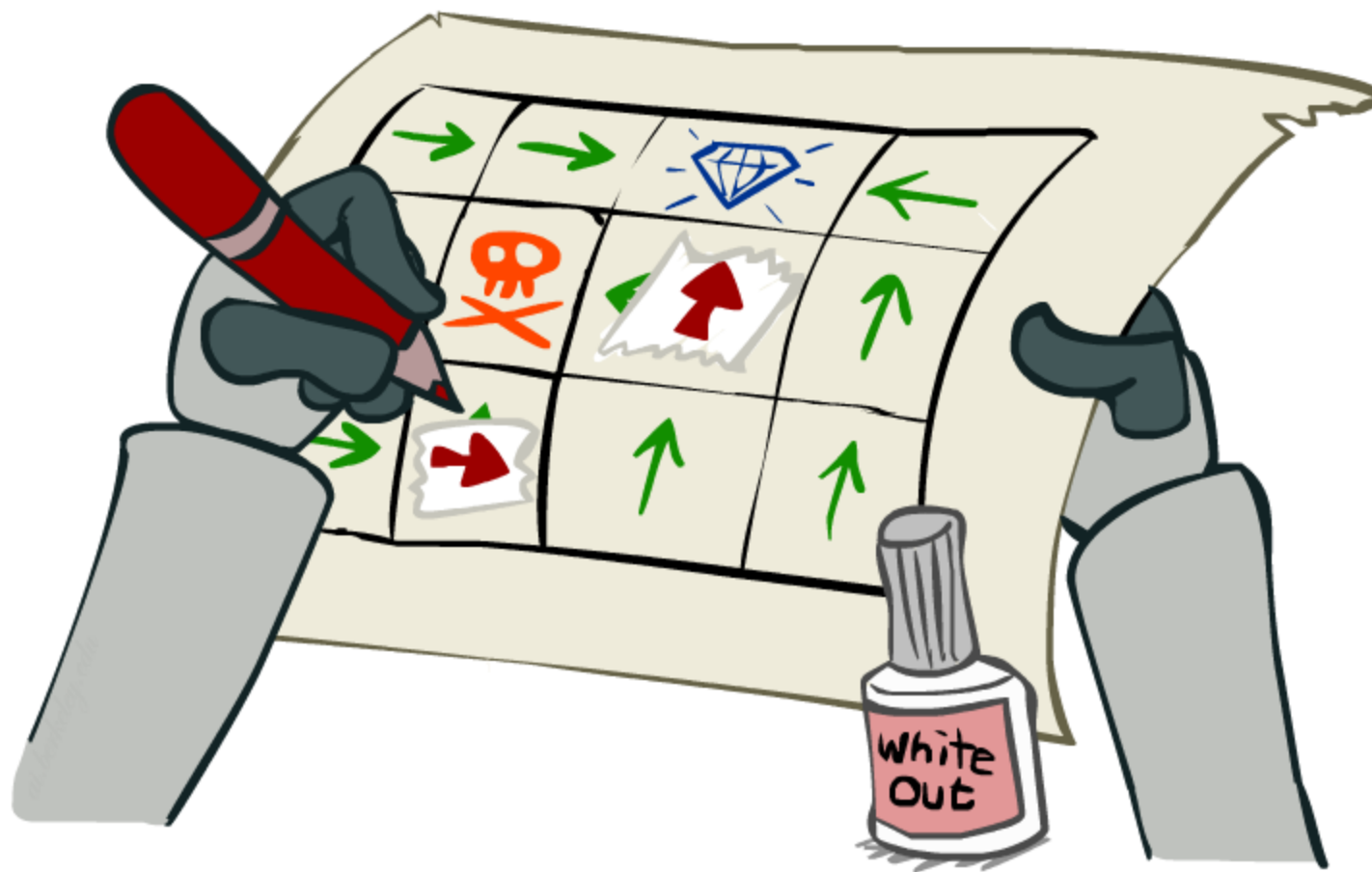
- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$



- Important lesson: actions are easier to select from q-values than values!

Policy Iteration

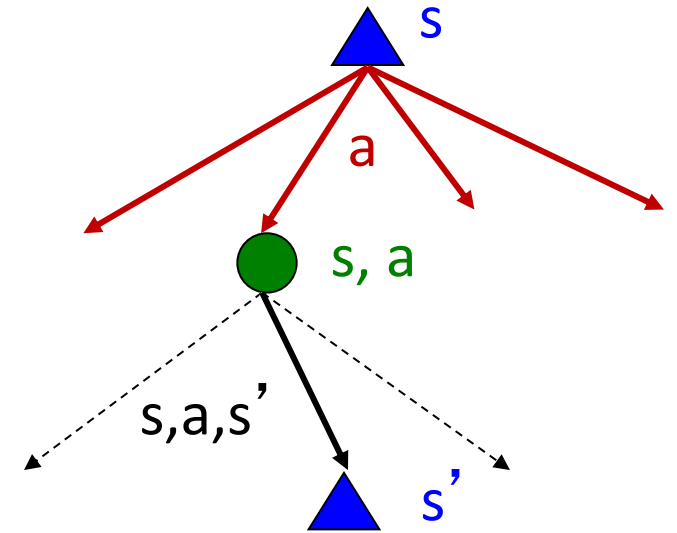


Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values



k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Iteration

- Alternative approach for optimal values:
 - **Step 1: Policy evaluation:** calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - **Step 2: Policy improvement:** update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges
- This is **policy iteration**
 - It's still optimal!
 - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') [R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:

$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

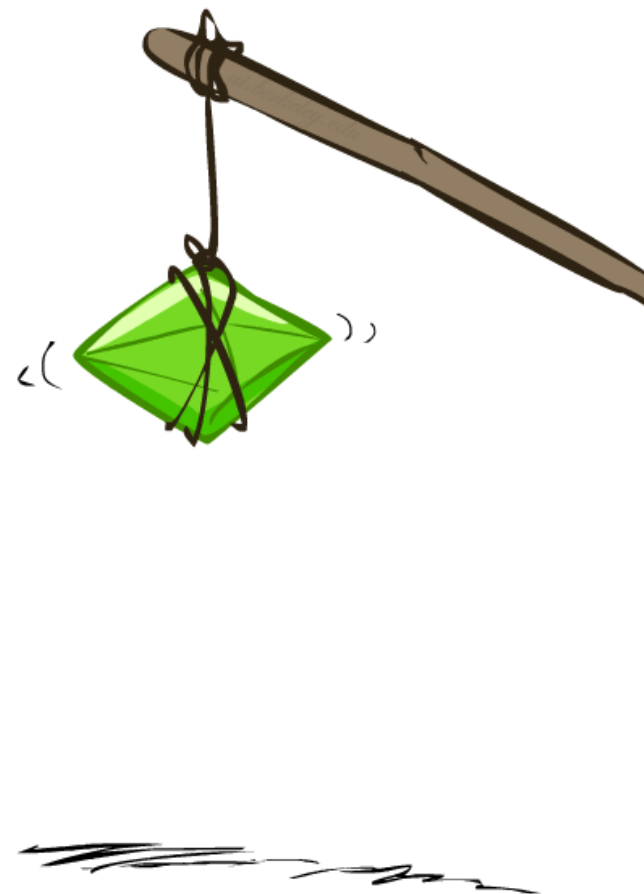
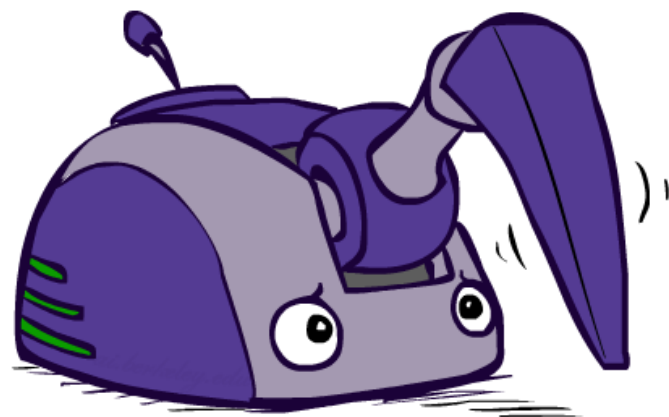
Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

- So you want to....
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
 - They basically are – they are all variations of Bellman updates
 - They all use one-step lookahead computations

Reinforcement Learning



What changes?

- Rather than planning, we now need to learn!
 - No access to underlying MDP, can't solve it with just computation
 - You needed to actually act to figure it out
 - Extension and generalization of Multi-Armed Bandits
- Important ideas in reinforcement learning that came up
 - Exploration: you have to try unknown actions to get information
 - Exploitation: eventually, you have to use what you know
 - Regret: even if you learn intelligently, you make mistakes
 - Sampling: because of chance, you have to try things repeatedly
 - Difficulty: learning can be much harder than solving a known MDP



Example: Learning to Walk



Initial



A Learning Trial



After Learning [1K Trials]

Example: Learning to Walk



Initial

Example: Learning to Walk



Training

Example: Learning to Walk



Finished

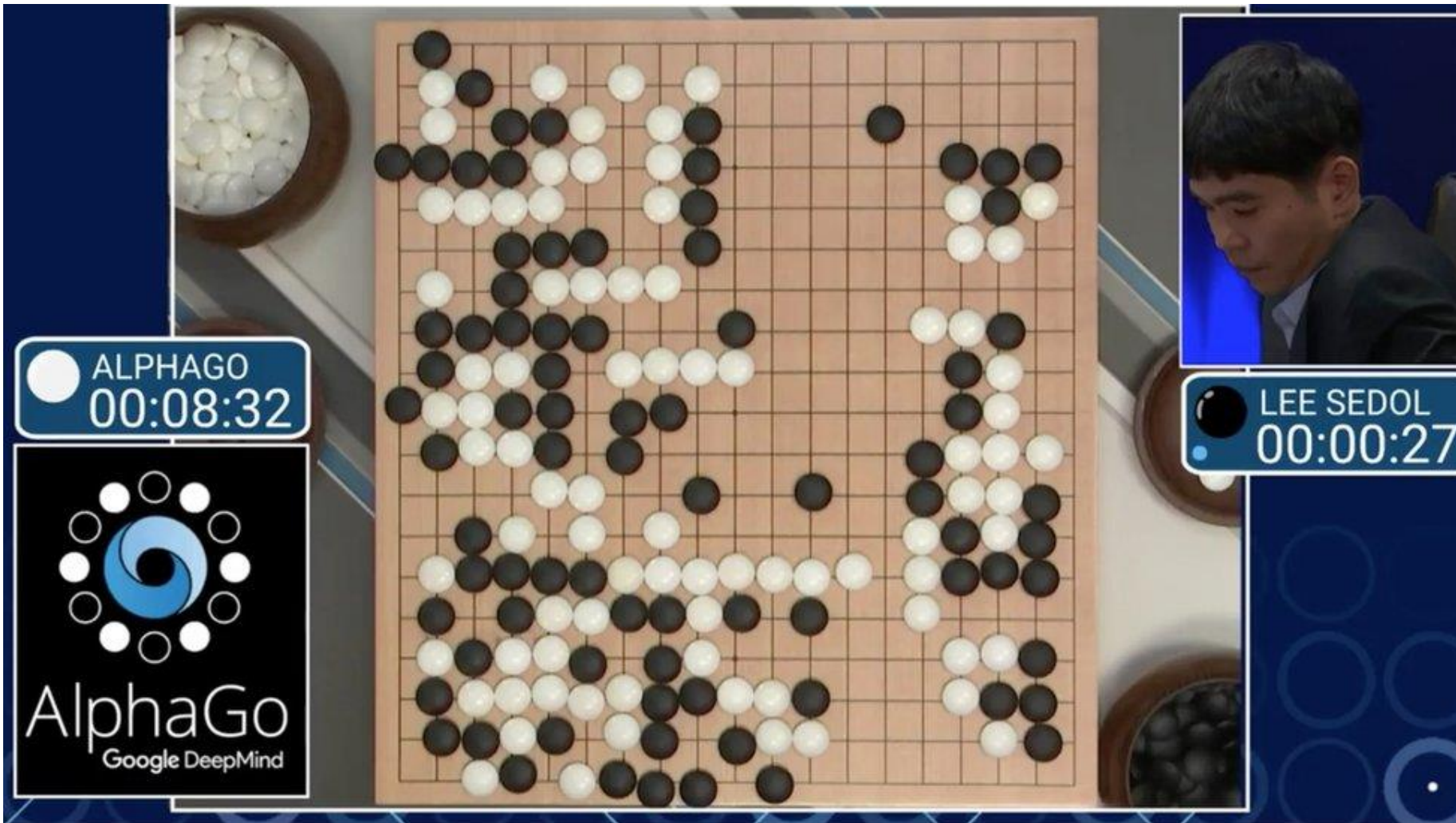
<https://vision-locomotion.github.io/>





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● ALPHAGO
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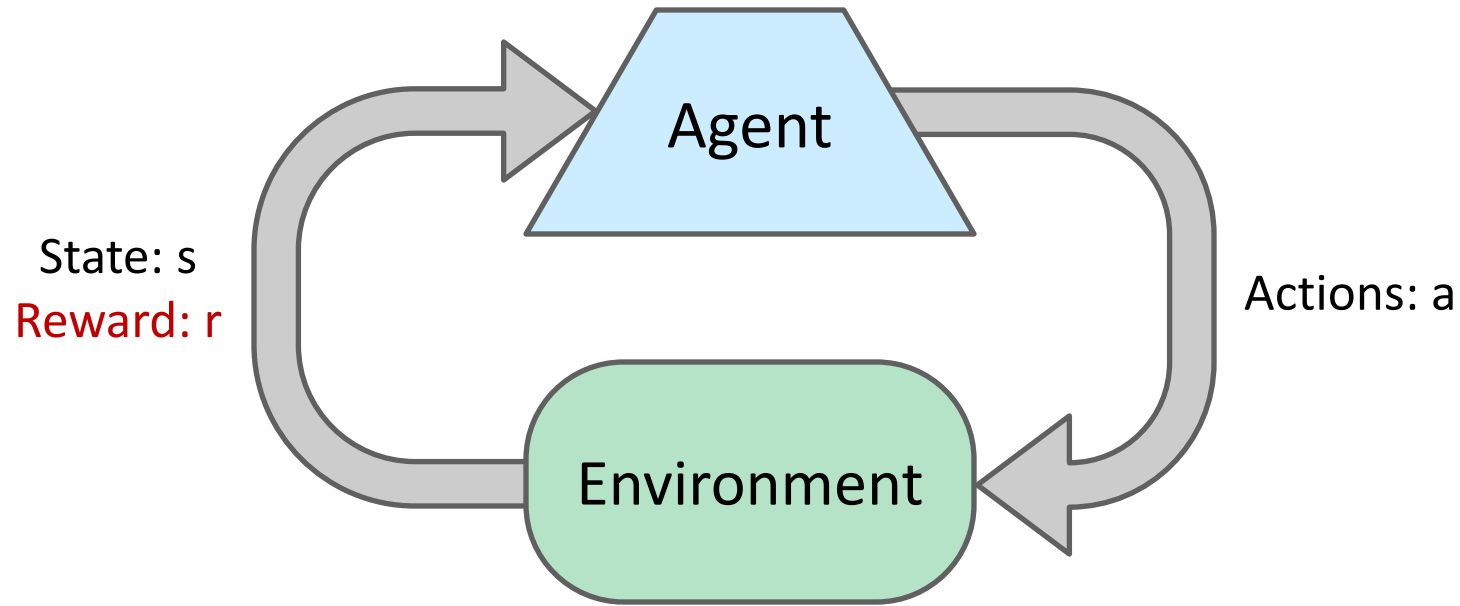


● LEE SEDOL
00:00:27

ChatGPT



Reinforcement Learning



- **Basic idea:**

- Receive feedback in the form of **rewards**
- Agent's utility is defined by the reward function
- Must (learn to) act so as to **maximize expected rewards**
- All learning is based on observed samples of outcomes!

Why Reinforcement Learning?

- Takes inspiration from nature
- Often easier to encode a task as a sparse reward (e.g. recognize if goal is achieved) but hard to hand-code how to act so reward is maximized (e.g. Go)
- General purpose AI framework

Reinforcement Learning

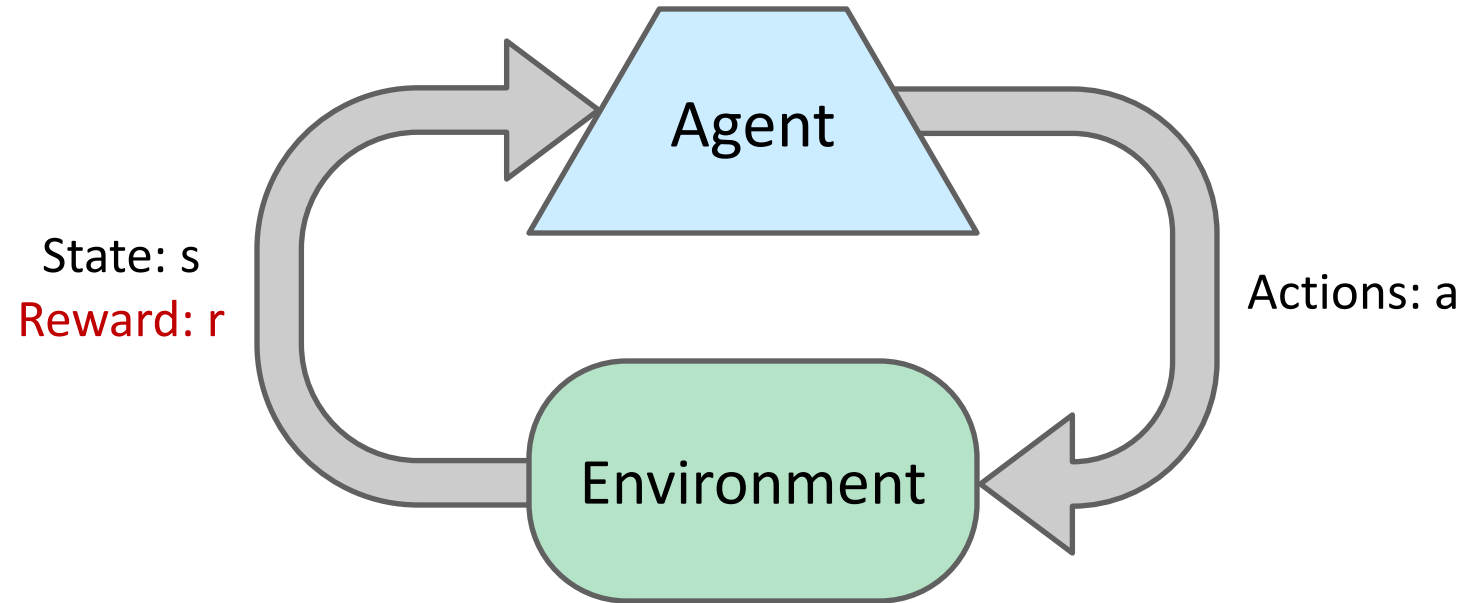
- Still assume a Markov decision process (MDP):

- A set of states $s \in S$
- A set of actions (per state) A
- A model $T(s,a,s')$
- A reward function $R(s,a,s')$

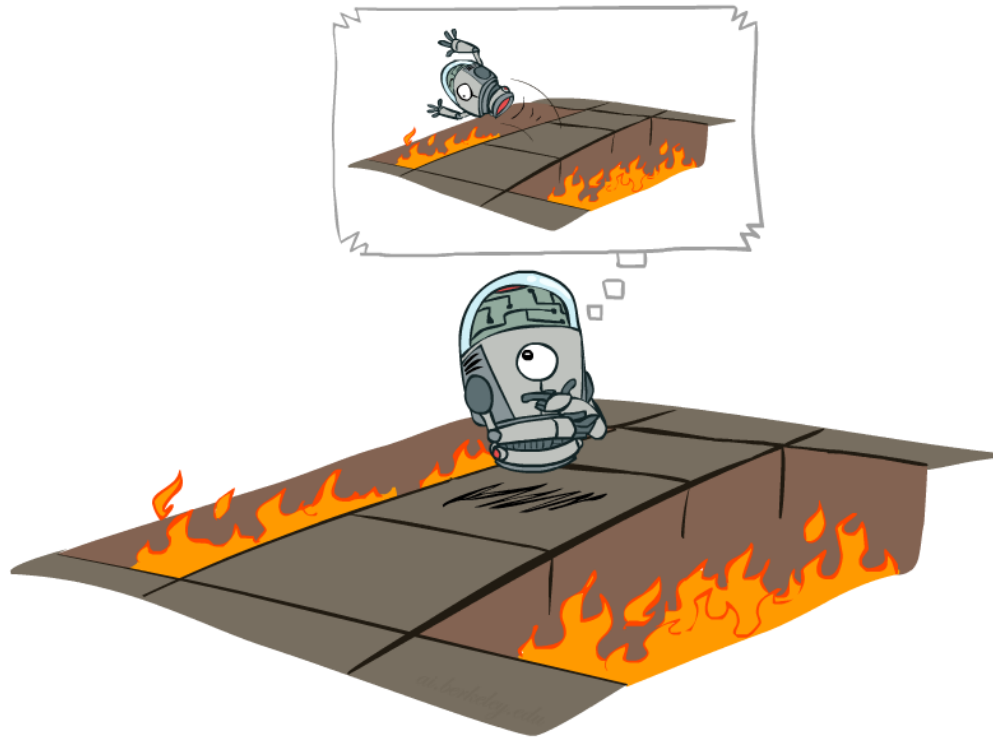
- Still looking for a policy $\pi(s)$

- New twist: don't know T or R

- I.e. we don't know which states are good or what the actions do
- Must actually try actions and states out to learn



Offline (MDPs) vs. Online (RL)

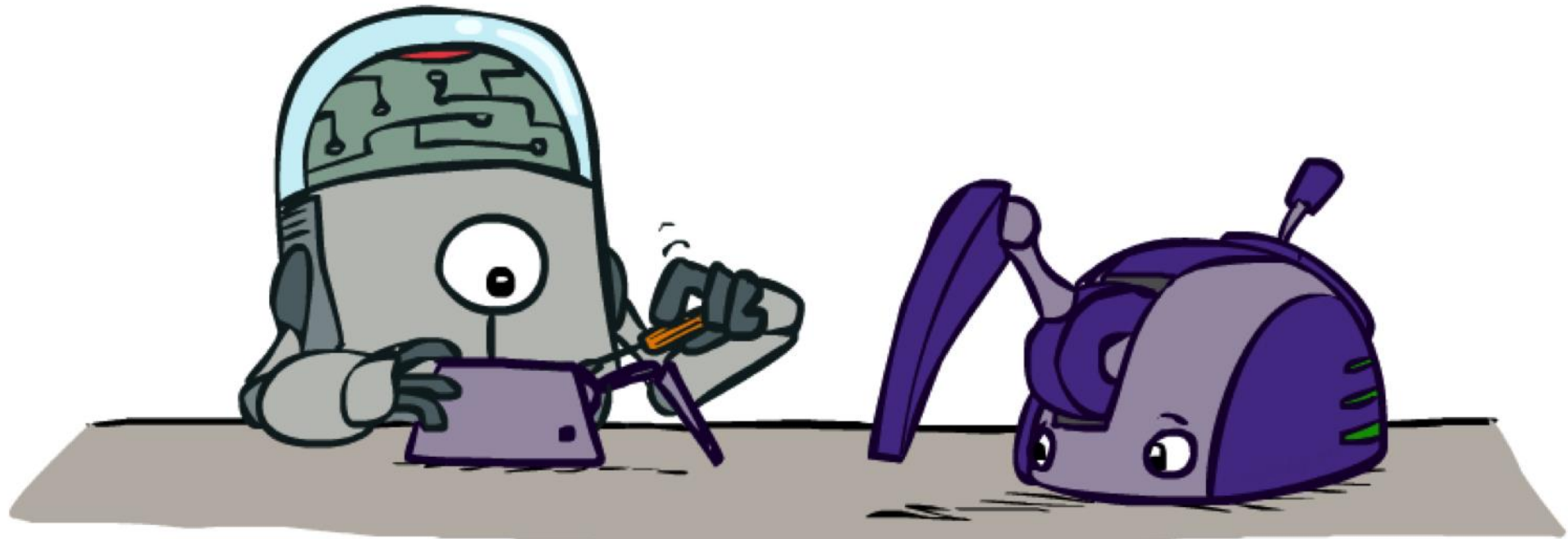


Offline Solution



Online Learning

Model-Based Learning

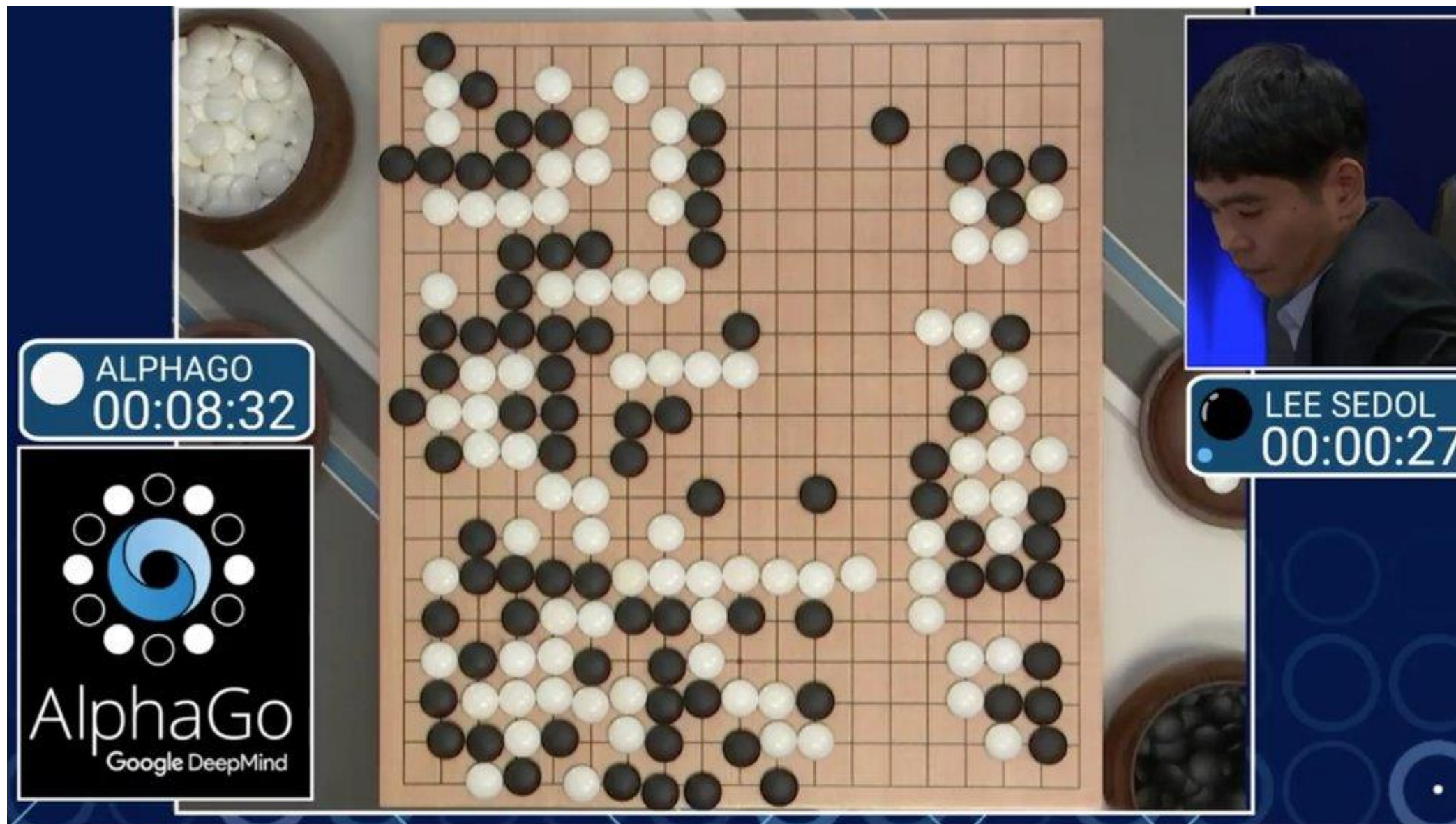


Simple View of Model-Based RL

- **Model-Based Idea:**
 - Learn an approximate model based on experiences
 - Solve for values as if the learned model were correct
- **Step 1: Learn empirical MDP model**
 - Count outcomes s' for each s, a
 - Normalize to give an estimate of $\hat{T}(s, a, s')$
 - Discover each $\hat{R}(s, a, s')$ when we experience (s, a, s')
- **Step 2: Solve the learned MDP**
 - For example, use value iteration, as before



Sometimes Model of World is Known



Deep RL Makes a Big Splash!

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Google Acquires Artificial Intelligence Startup DeepMind For More Than \$500M

Catherine Shu @catherineshu / 6:20 PM MST • January 26, 2014

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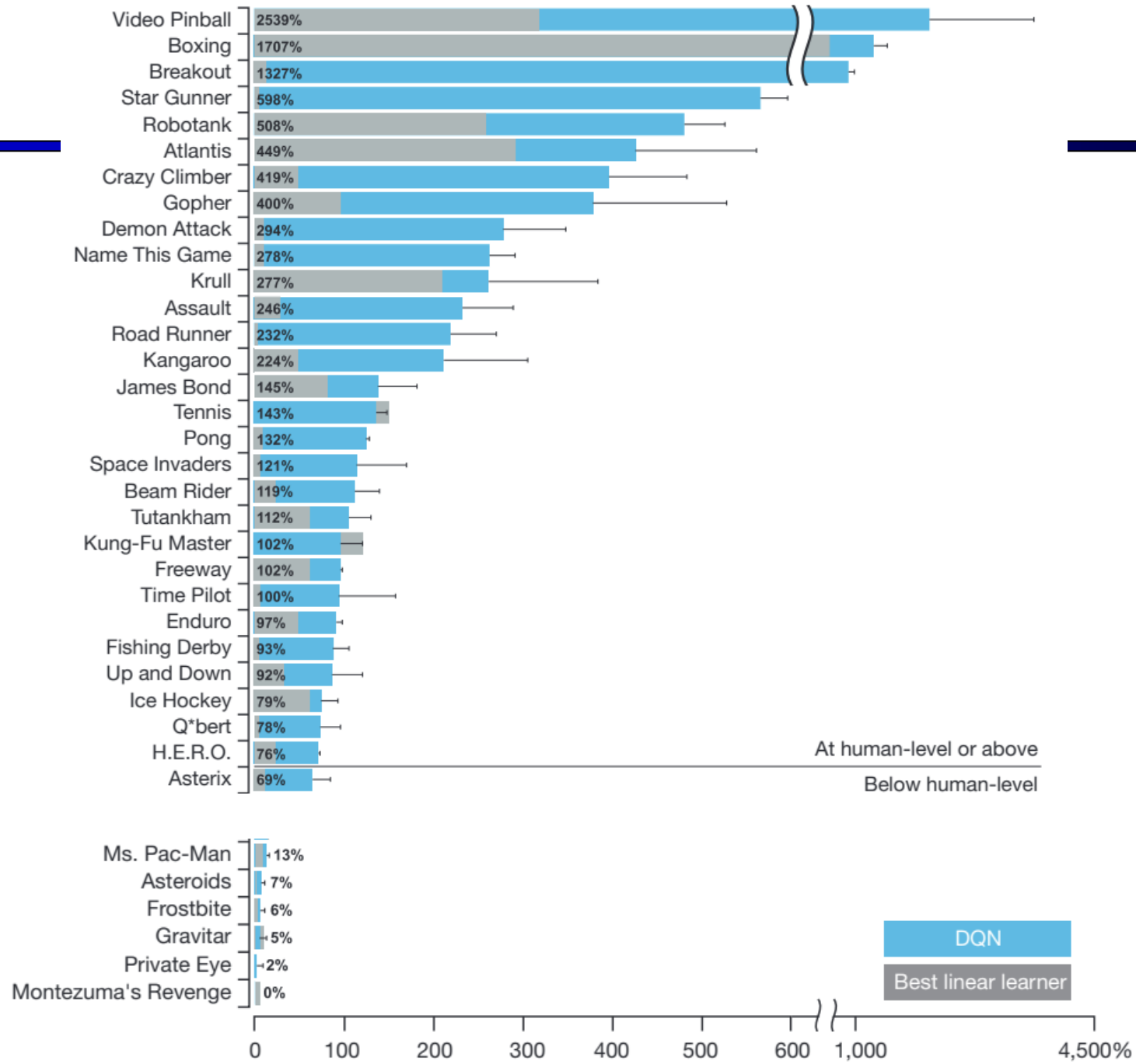
TechCrunch
Early Stage

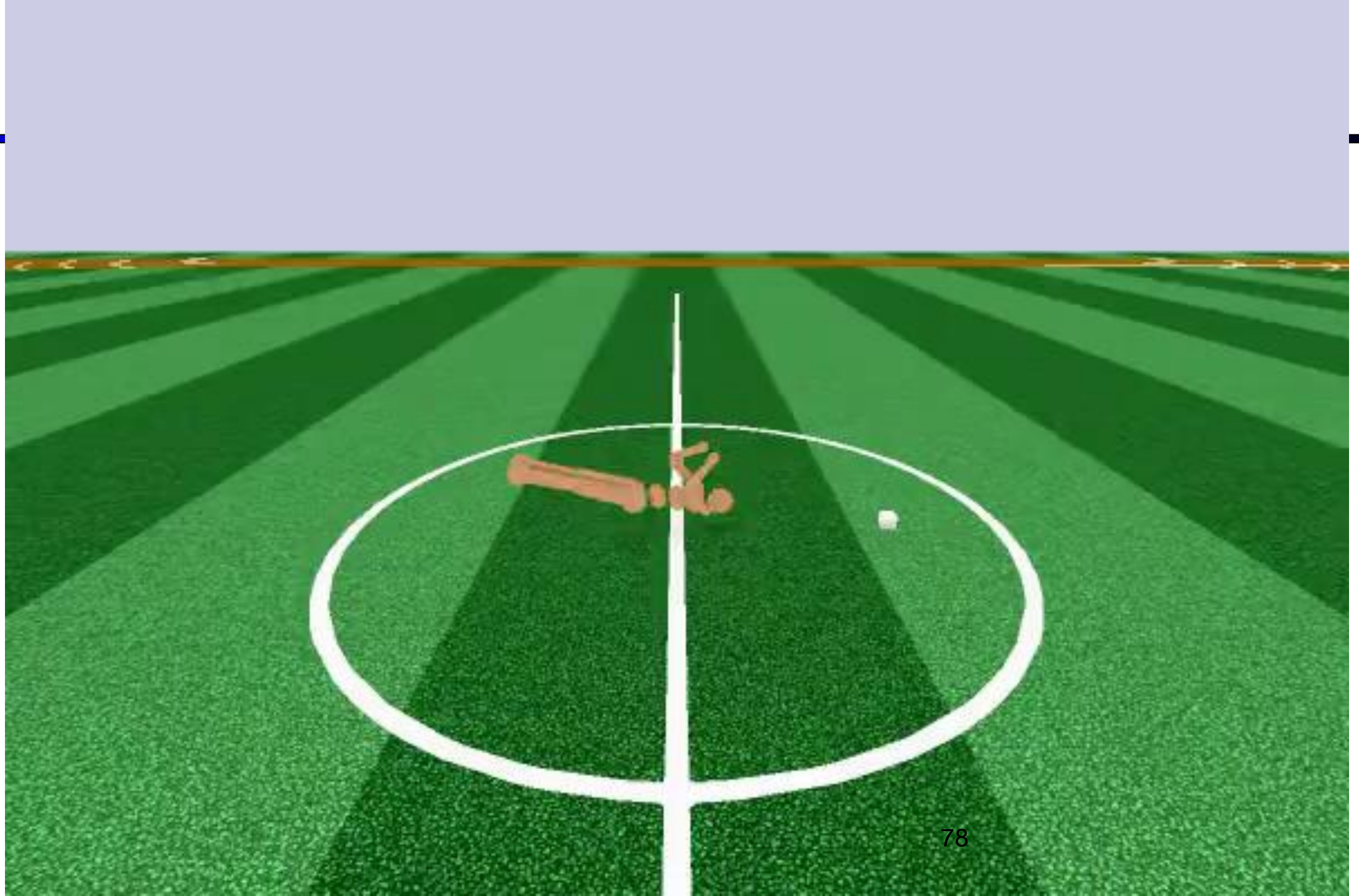
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When might RL be a good tool for your problem?

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- Is your problem a sequential decision making problem?
- Are there “actions” that effect the next “state”?
- Do you know the rules of these effects?
- Can you write down a clear objective/score/reward/cost?
- Do you have a simulator?
- Lots of examples of sequences of decisions and their long-term consequences?
- Is it unclear what to do in each state? Exploration required?
- Are you looking for unique/creative/super-human solutions?

When might RL not be a good tool?

When might RL not be a good tool?

- Single step or static problem
- No clear reward signal.
- Reward signal is unavailable or very hard to write down.
- Well-known model of the environment.
- Deterministic environment
- Low-tolerance for exploration and trial and error
- No need for adaptive or novel solutions. The goal is to perform the task in a very predictable way.