Lecture 12 – Probability

DSC 10, Spring 2022

Announcements

- Lab 4 is due **tomorrow at 11:59pm**.
- Homework 4 is due on **Tuesday 4/26 at 11:59pm**.
- The Midterm Project is **released**.
    - Start right away!
- Grade report on Gradescope gives summary of grades so far.
    - Check it out before the drop deadline tonight.

Agenda

- Motivation for probability.
- Probability theory.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Note 18 in the course notes**.
- **Computational and Inferential Thinking, Chapter 9.5**.
- **Theory Meets Data, Chapters 1 and 2**.
- **Khan Academy's unit on Probability**.

Motivation for probability

Swain vs. Alabama, 1965

- Robert Swain was a black man convicted of crime in Talladega County, Alabama.
- He appealed the jury's decision all the way to the Supreme Court, on the grounds that Talladega County systematically excluded Black people from juries.
- At the time, only men 21 years or older were allowed to serve on juries. 26% of this population was Black.
- But of the 100 men on Robert Swain's jury panel, only 8 were Black.

The Supreme Court's ruling ⚖️

- About disparities between the percentages in the eligible population and the jury panel, the Supreme Court wrote:

> "... *the overall percentage disparity has been small...*"

- The Supreme Court denied Robert Swain's appeal and he was sentenced.
- The fact that the jury panel had far fewer Black men proportionally than Talladega County is an example of racial bias.
- Over the next few weeks, we will give you tools to quantitatively highlight this bias.
    - We will try to answer the question, "what are the chances that this disparity was due to random chance?"

- If this chance is small, we know something is wrong.
- But first: we need to formalize what **probability** is.

Probability theory

Probability theory

- Some things in life *seem* random.
    - e.g. flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is 50%.
- One interpretation of probability says that if we flipped a coin infinitely many times, then 50% of outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome**: The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event**: A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.

- e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
    - 0: the event never happens.
    - 1: the event always happens.
- Notation: if A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all outcomes in event A are equally likely, then the probability of A is

$$\boxed{P(A) = \frac{\text{\# of outcomes satisfying } A}{\text{total \# of outcomes}}}$$

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

possible outcomes ← 8

HHH THH
HTH TTH
HHT THT
HTT TTT

$= \frac{3}{8}$

can't do this

Outcomes
0 H
1 H
2 H
3 H

$= \frac{1}{4}$

because all outcomes not equally likely

Example 1 solved

- When we flip a fair coin 3 times, there are 8 possible outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, and TTT.
- These outcomes are all equally likely.
- 3 of these outcomes have exactly 2 heads: HHT, HTH, and THH.
- So, the probability of seeing exactly 2 heads in 3 flips of a fair coin is $\frac{3}{8}$.

Discussion Question

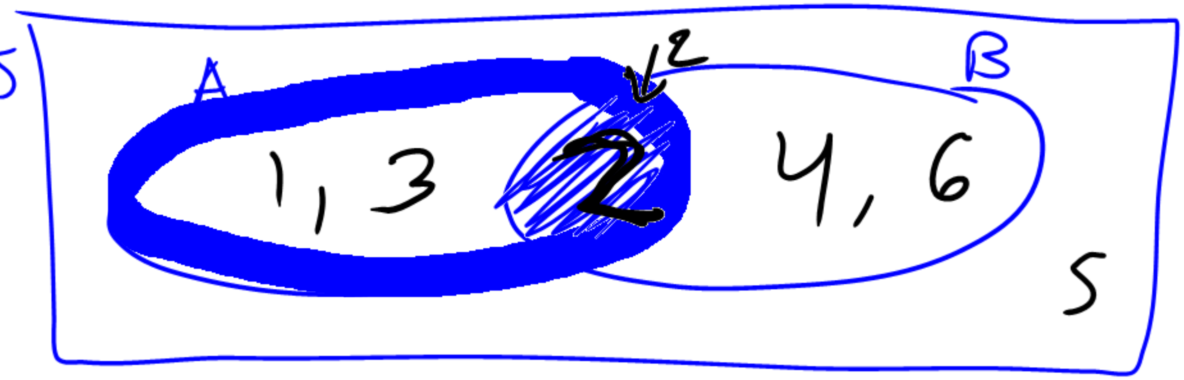
I have three cards: red, blue, and green. What is the chance that I choose a card at random, and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

To answer, go to **menti.com** and enter the code 7703 6292 or **click here**.

Discussion Question solved

- There are 6 possible outcomes: RG, RB, GR, GB, BR, and BG.
- These outcomes are equally likely.
- There is only 1 outcome which makes the event happen: GR.
- Hence the probability is $\frac{1}{6}$.



Conditional probabilities

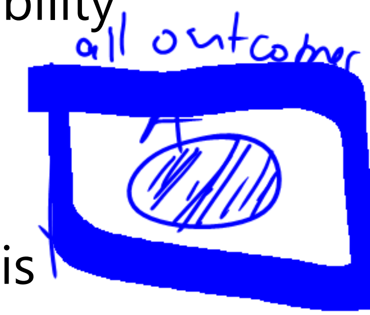
- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A} = \frac{1}{3}$$

1 → just "2"

3 → "1", "2", "3"

- Intuitively, this is similar to the definition of the regular probability of B, $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\text{total } \# \text{ of outcomes}}$, if you restrict the set of possible outcomes to be just those in event A.



- **Example 2:** Suppose I roll a fair six-sided die, and suppose A is the event $\underbrace{\text{"roll is 3 or less"}}_{A}$ and B is the event $\underbrace{\text{"roll is even"}}_{B}$. What is $P(B \text{ given } A)$?

1, 2, 3

2, 4, 6

Example 2 solved

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

Another way of phrasing the problem: I roll a fair six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

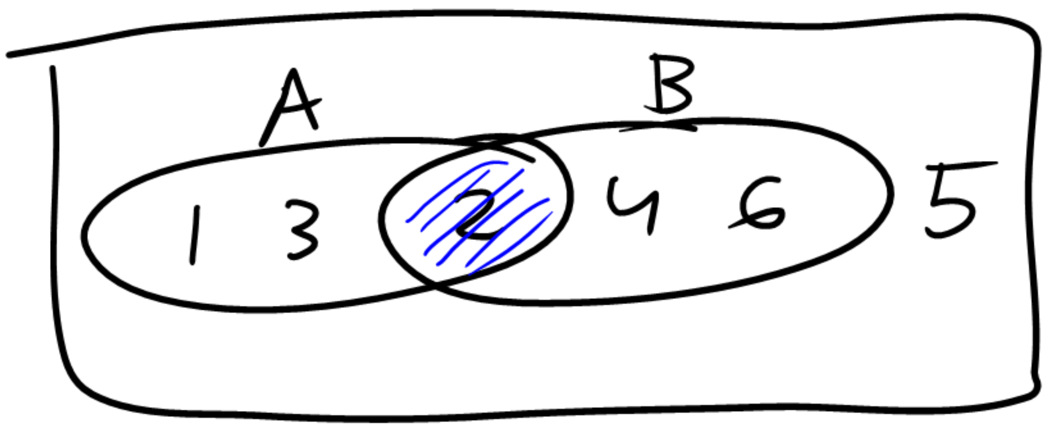
- There are three outcomes where the roll is 3 or less: 1, 2, and 3.
- There is only one outcome where the roll is 3 or less and even: 2.
- So the probability that the roll is even given that it is 3 or less is $P(B \text{ given } A) = \frac{1}{3}$.

Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\#\text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \#\text{ of outcomes}}$$

- **Example 3:** I roll a fair six-sided die. What is the probability that the roll is $\underbrace{\text{3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?



$$= \frac{1}{6}$$

Example 3 solved

I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

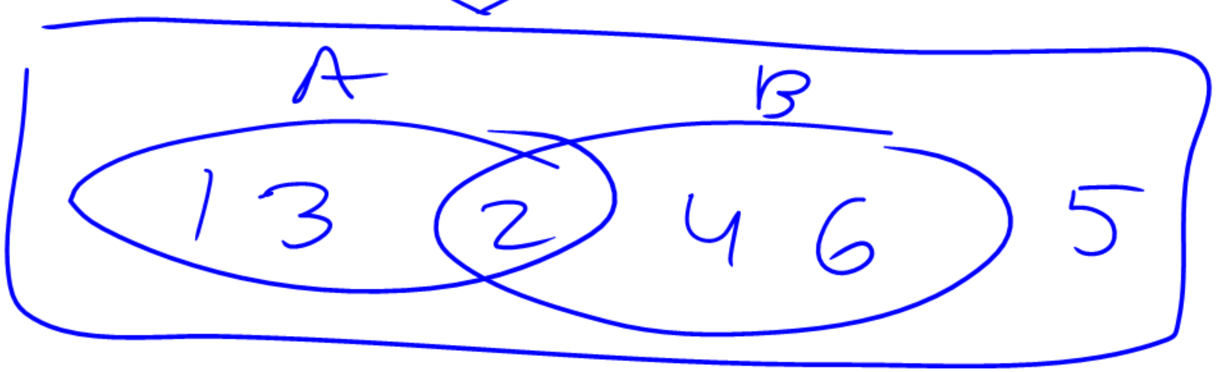
- Only one outcome is both 3 or less and even: 2.
- There are 6 total outcomes.
- Thus, $P(A \text{ and } B) = \frac{1}{6}$.

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$\underline{P(A \text{ and } B)} = \underline{P(A)} \cdot \underline{P(B \text{ given } A)}$$

- **Example 3, again:** I roll a fair six-sided die. What is the probability that the $\underbrace{\text{roll is 3 or less}}_{A}$ and $\underbrace{\text{even}}_{B}$?



$$P(A \text{ and } B) = P(A) * P(B \text{ given } A)$$
$$= 3/6 * 1/3 = 1/6$$

Example 3 solved, again

I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

- The probability that the roll is 3 or less is $P(A) = \frac{1}{2}$.
- From before, the probability that the roll is even given that the roll is 3 or less is $P(B \text{ given } A) = \frac{1}{3}$.
- Thus, the probability the roll is both 3 or less and even is $P(A \text{ and } B) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.
- Note that an equivalent formula is $P(A \text{ and } B) = P(B) \cdot P(A \text{ given } B)$.

Generally, situations involving an ”and” involve multiplication.

What if A isn't affected by B? 🤔

- The multiplication rule states that, for any two events A and B,

true all the time

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

when A, B are not influenced by one another, this term is P(B)

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

$$P(A \text{ and}) = P(A) * P(B)$$

only when A, B ind.

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 4:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$$P(HHHHH) = P(H \text{ on } 1\text{st} \text{ AND } H \text{ on } 2\text{nd} \text{ AND} \ldots)$$

$$= P(H \text{ on } 1\text{st}) * P(H \text{ on } 2\text{nd}) * \ldots$$

$$= 0.7 * 0.7 * \ldots$$

$$= (0.7)^5$$

$$\neq \frac{1}{32} \rightarrow \text{possible coin toss sequences for 5 coins}$$

Example 4 solved

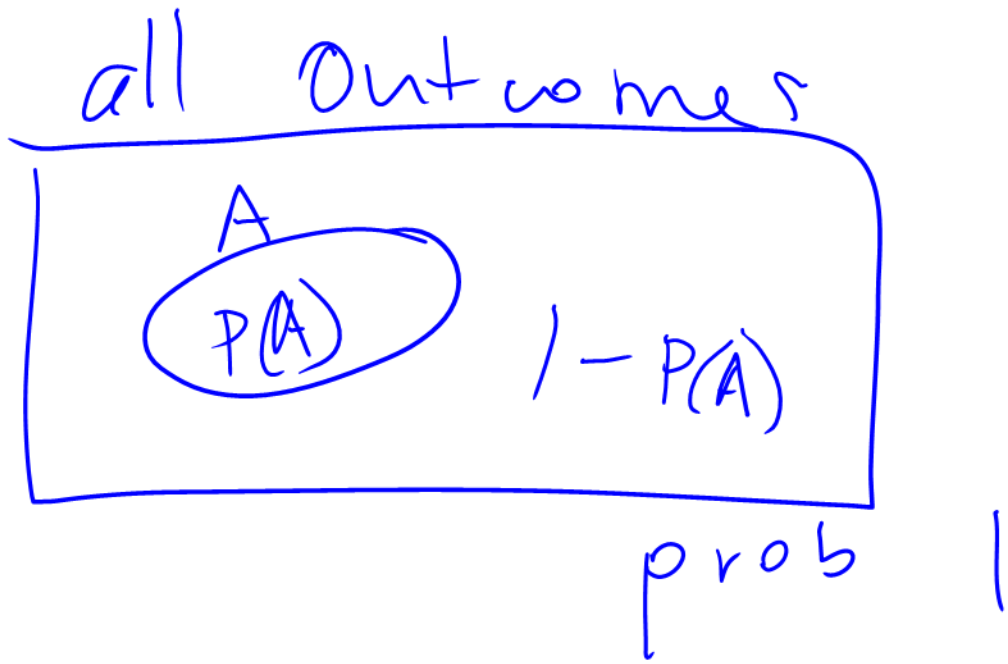
Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

- The probability of seeing heads on a single flip is 0.7.
- Each flip is independent.
- So, the probability of seeing 5 heads in a row is

$$0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 \cdot 0.7 = 0.7^5$$

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.



Discussion Question

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- D) 1
- E) None of the above.

at least one Y

outcomes

Y/N

$\left\{ \begin{matrix} NNY \\ YYY \\ YNY \end{matrix} \right\}$

all equally likely?

NO

To answer, go to **menti.com** and enter the code 7703 6292 or **click here**.

opposite of "at least once" is never

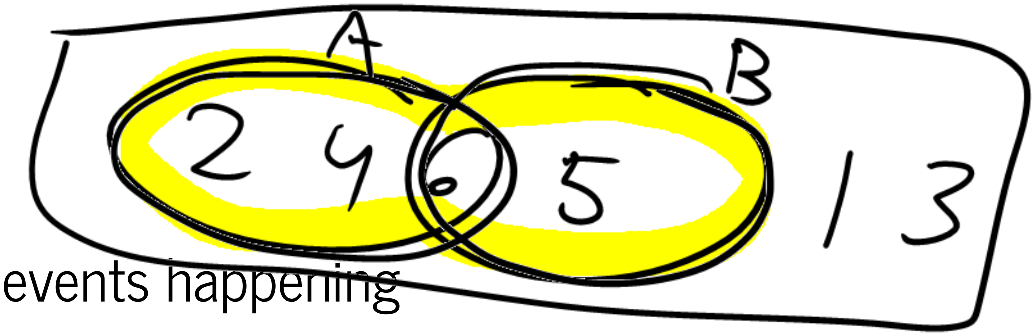
$$1 - P(NNN) = 1 - P(\text{No on 1st call AND} \ldots)$$

$$1 - \left(\frac{2}{3}\right) * \left(\frac{2}{3}\right) * \left(\frac{2}{3}\right)$$

Discussion Question solved

- Let's first calculate the probability that she **doesn't** answer her phone in three tries.
  - The probability she doesn't answer her phone on any one attempt is $\frac{2}{3}$.
  - So the probability she doesn't answer her phone in three tries is $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$.
- But we want the probability of her answering **at least** once. So we subtract the above result from 1.
  - $1 - \frac{8}{27} = \boxed{\frac{19}{27}}$; none of the above!

$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$ wrong

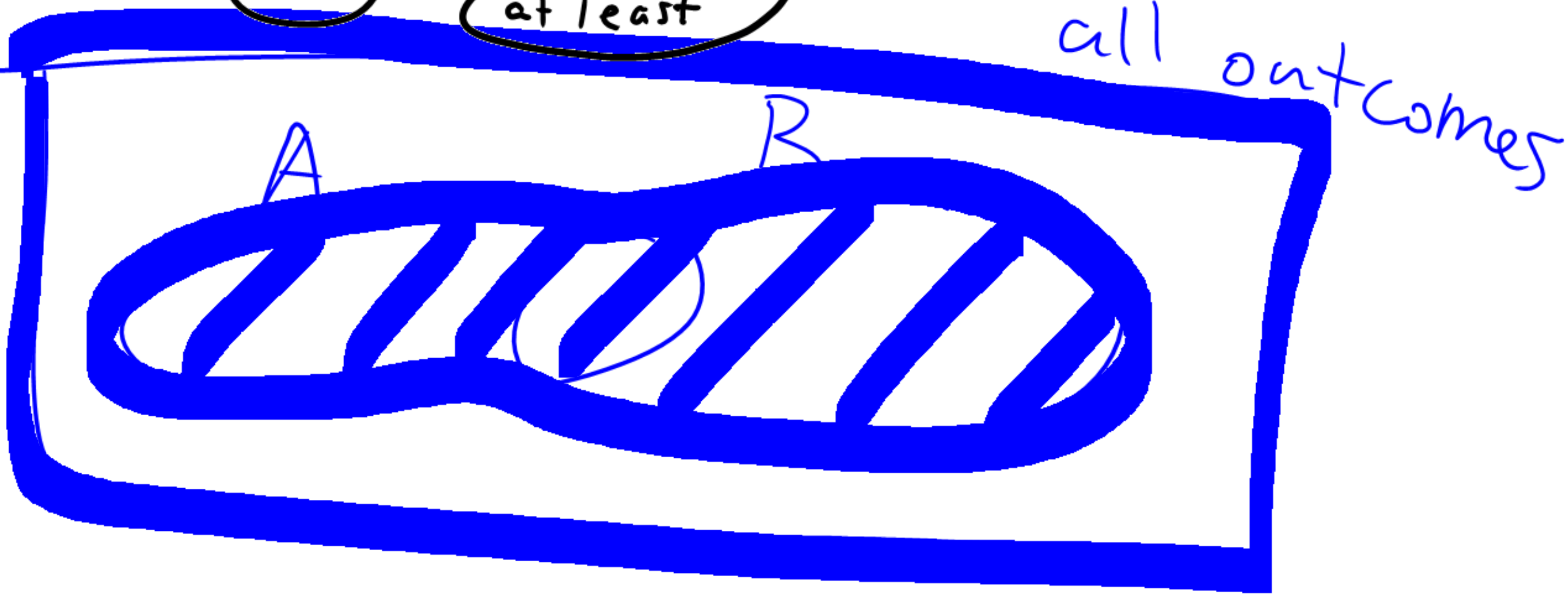


Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$\boxed{P(A \text{ or } B) = \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}}}$$

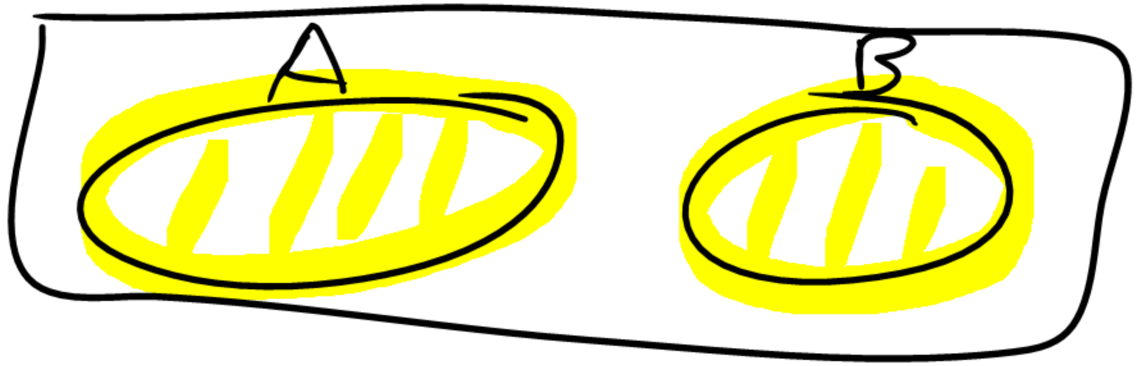
- **Example 5:** I roll a fair six-sided die. What is the probability that the roll is even or ~~more than~~ at least 5? $= \frac{4}{6}$



Example 5 solved

I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

- There are three outcomes that are even: 2, 4, 6.
- There are two outcomes that are at least 5: 5, 6.
- There are four total outcomes that satisfy at least one of the two conditions: 2, 4, 5, 6.
- Thus, the probability that the roll is even or at least 5 is $\boxed{\frac{4}{6}} = \frac{2}{3}$.
  - Note that this is not $P(A) + P(B)$, which would be $\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$, because there is **overlap** between events A and B.



The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
  - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$\boxed{P(A \text{ or } B) = P(A) + P(B)}$$

← "or" becomes addition when no overlap

- **Example 6:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P\left(\frac{A}{H}\Big|\frac{B}{T} \text{ or } \frac{A}{T}\Big|\frac{B}{H}\right)$$

$$= P\left(\frac{A}{H}\Big|\frac{B}{T}\right) + P\left(\frac{A}{T}\Big|\frac{B}{H}\right)$$

$$= 0.6 * 0.7 + 0.4 * 0.3$$

OR

can't be in both at same time

Example 6 solved

Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. The two coins are independent of one another. I flip both coins once. What's the probability I see two different faces?

- The event we see two different faces corresponds to either seeing a head then a tail, **or** a tail then a head (i.e. not both heads and not both tails).
- The probability of seeing a head then a tail is $0.6 \cdot (1 - 0.3)$, because the two coins are independent of one another.
- The probability of seeing a tail then a head is $(1 - 0.6) \cdot 0.3$.
- So, the probability of seeing two different faces is

$$\boxed{0.6 \cdot (1 - 0.3) + (1 - 0.6) \cdot 0.3 = 0.54}$$

Generally, situations involving an "or" involve addition. ← only when no overlap

Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned}
P(A \text{ or } B) &= \frac{\#\text{ of outcomes satisfying either } A \text{ or } B}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A) + (\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= \frac{(\#\text{ of outcomes satisfying } A)}{\text{total }\#\text{ of outcomes}} + \frac{(\#\text{ of outcomes satisfying } B)}{\text{total }\#\text{ of outcomes}} \\
&= P(A) + P(B)
\end{aligned}$$

Summary

Summary

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
  - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
  - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** simulations.