Lecture 12 – Probability

DSC 10, Summer 2022

Note: No code in today's lecture. I recommend pen-and-paper instead.

Announcements

- Lab 4 is due **tomorrow at 11:59pm**.
- The midterm is **Fri 7/29 at 11:00am**.
- Homework 4 is due on **Sat 7/30 at 11:59pm**.
- The Midterm Project is **released**.
    - Open it to see how long it is and plan accordingly!
- Grade report on Gradescope gives summary of grades so far.

Agenda

- Motivation for probability.
- Probability theory.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- Note 18 in the course notes (https://notes.dsc10.com/04-probability_and_simulation/probability_and_simulation.html).
- Computational and Inferential Thinking, Chapter 9.5 (https://inferentialthinking.com/chapters/09/5/Finding_Probabilities.html).
- Theory Meets Data, Chapters 1 and 2 (http://stat88.org/textbook/notebooks/Chapter_01/00_The_Basics.html).
- Khan Academy's unit on Probability (https://www.khanacademy.org/math/probability/xa88397b6:probability).

Motivation for probability

Swain vs. Alabama, 1965

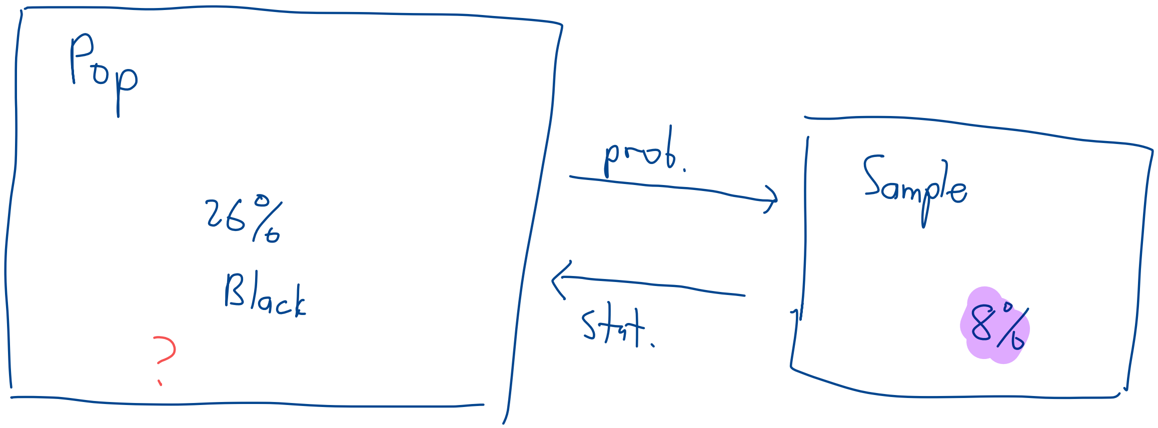
- Robert Swain was a black man convicted of crime in Talladega County, Alabama.
- He appealed the jury's decision all the way to the Supreme Court, on the grounds that Talladega County systematically excluded Black people from juries.
- At the time, only men 21 years or older were allowed to serve on juries. 26% of this population was Black.
- But of the 100 men on Robert Swain's jury panel, only 8 were Black (8%).
    - Suspiciously low!

The Supreme Court's ruling ⚖️

- About disparities between the percentages in the eligible population and the jury panel, the Supreme Court wrote:

"... the overall percentage disparity has been small..."

Probability and Statistics



Probability theory

Probability theory

- Some things in life *seem* random.
    - e.g. flipping a coin or rolling a die 🎲.
- The **probability** of seeing "heads" when flipping a fair coin is 50%.
- One interpretation of probability says that if we flipped a coin infinitely many times, then 50% of outcomes would be heads.

Terminology

- **Experiment**: A process or action whose result is random.
- **Outcome**: The result of an experiment.
- **Event**: A set of outcomes.

Exp.	Outcome	Event
rolling die once	$1 \to 6$	$\{2, 4\}$
flipping coin twice	HH, HT, TH, TT	$\{TT\}$
jury of 100 ppl	$0 \to 100$ Black	$\leq 10\%$ Black

Terminology

- **Probability**: A number between 0 and 1 (equivalently, between 0% and 100%) which describes the likelihood of an event.
    - 0: the event never happens.
    - 1: the event always happens.
- Notation: if A is an event, $P(A)$ is the probability of that event.

$P(A) = 0.5$

A: 1 head

Equally-likely outcomes → all events equally likely

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

1. Outcomes?
2. Events?

$$P(A) = \frac{\#\text{ outcomes in } A}{\text{total }\#\text{ outcomes}}$$

Outcomes

HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

$$\boxed{\frac{3}{8}}$$

Example 1 solved

You Try

I have three cards: red, blue, and green. What is the chance that I choose a card at random, and it is green, then – **without putting it back** – I choose another card at random and it is red?

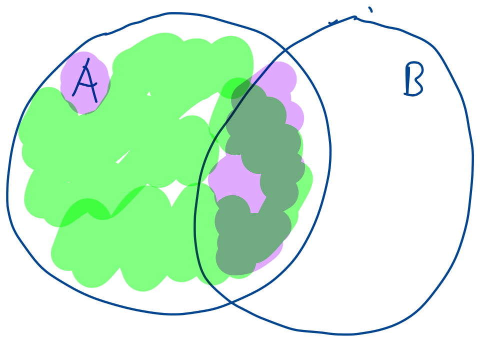
E : drawing 2 cards

$$\frac{O}{\begin{array}{l} RB \\ RG \\ BR \\ BG \\ GR \\ GB \end{array}}$$

$$P(A) = \frac{1}{6}$$

Solution

Conditional probabilities



A and B

A happened, B?

$$P(B \text{ given } A) = \frac{\#\text{ outcomes in both } A \text{ and } B}{\#\text{ outcomes in } A}$$

Conditional probabilities

- **Example 2:** Suppose I roll a fair six-sided die, and suppose A is the event "roll is 3 or less" and B is the event "roll is even". What is $P(B \text{ given } A)$?

A: 1, 2, 3

A and B: 2

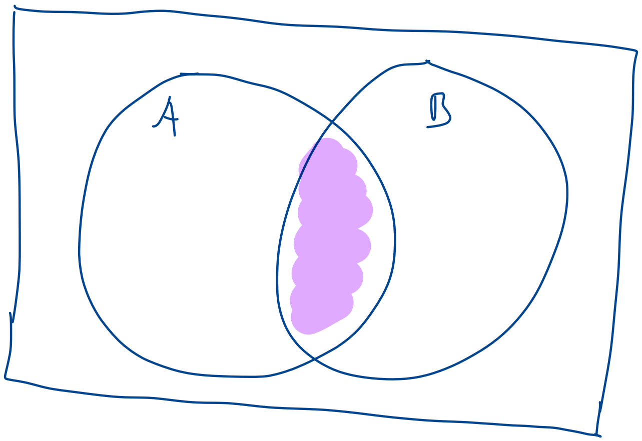
$P(B \text{ given } A) = \frac{1}{3}$

$P(B \mid A)$

Example 2 solved

Probability that two events both happen

$$P(A \text{ and } B) = \frac{\#\text{ total outcomes in } A \text{ and } B}{\#\text{ total outcomes}}$$



Probability that two events both happen

- **Example 3:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

O : 2

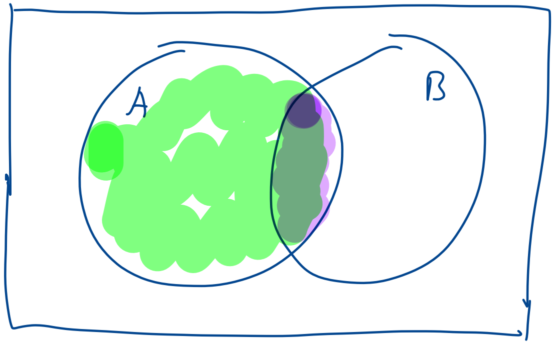
$P(A \text{ and } B) = \frac{1}{6}$

all outcomes : $1 \rightarrow 6$

Example 3 solved

The multiplication rule

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$



The multiplication rule

- **Example 3, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

A B

$= P(B) \cdot P(A \text{ given } B)$

$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$

$P(A) = \frac{1}{2}$

$P(B \text{ given } A) = \frac{1}{3}$

$P(A \text{ and } B) = \frac{1}{6}$

Example 3 solved, again

What if A isn't affected by B? 🤔

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

A and B are independent, then

$$P(B \text{ given } A) = P(B)$$

$$P(A \text{ and } B) = P(A)\,P(B)$$

Independent events

- **Example 4:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$$P(HHHHH) = P(H)P(H)P(H)P(H)P(H)$$

$$= 0.7^5$$

Outcomes

HHHHH

HHHHT

HHHTH

Independent events

- **Example 4:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

Example 4 solved

Probability that an event *doesn't* happen

$$P(A \text{ doesn't happen}) = 1 - P(A)$$

$$P(\text{tomorrow sunny}) = 0.85$$

$$P(\text{not sunny}) = 1 - 0.85 = 0.15$$

Discussion Question

Every time I call my grandma 👵, the probability that she answers her phone is $\frac{1}{3}$. If I call my grandma three times today, what is the chance that I will talk to her at least once?

$$P(\text{at least once}) = P(1 \text{ time } \underline{\text{or}} \ 2 \text{ times } \underline{\text{or}} \ 3 \text{ times})$$

$$= 1 - P(0 \text{ times})$$

$$P(0 \text{ times}) = \underline{P(\text{no})\,P(\text{no})\,P(\text{no})}$$

$$= \left(\frac{2}{3}\right)^3$$

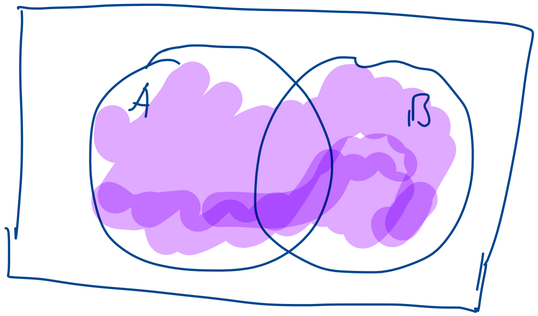
$$= 1 - \left(\frac{2}{3}\right)^3$$

$$= \frac{19}{27}$$

Discussion Question solved

Probability of either of two events happening

$$P(A \text{ or } B) = \frac{\# \text{ outcomes in } A \text{ or } B}{\# \text{ total}}$$

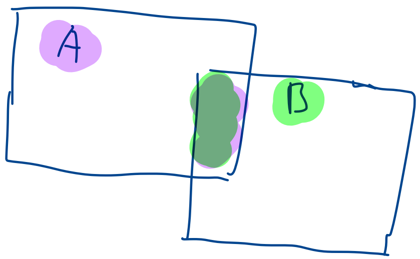


Probability of either of two events happening

- **Example 5:** I roll a fair six-sided die. What is the probability that the roll is even or more than 5?

Outcomes
1
2
3
4
5
6

$$P(A \text{ or } B) = \frac{4}{6} = \frac{2}{3}$$



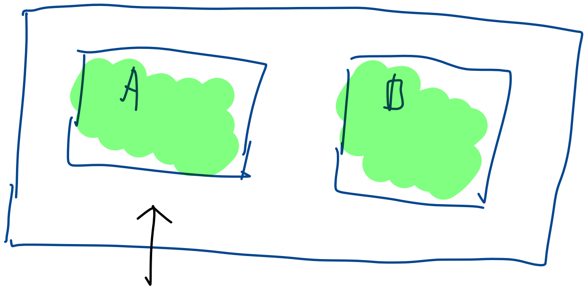
highlighted twice, but not counted twice

Example 5 solved

The addition rule

If A and B disjoint:

$$P(A \text{ or } B) = P(A) + P(B)$$



Disjoint means A and B are dependent.

The addition rule

- **Example 6:** Suppose I have two biased coins, coin A and coin B. Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

Outcomes

H H

H T

T H

T H

not equally likely

$\frac{2}{4}$

$$P(HT \text{ or } TH)$$
$$= P(HT) + P(TH)$$
$$= (0.6)(0.7) + (0.4)(0.3)$$
$$= 0.54$$

Usually:

$P(A \text{ and } B)$? Multiplication

$P(A \text{ or } B)$? Addition

Example 6 solved

Aside: proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned}
P(A \text{ or } B) &= \frac{\text{\# of outcomes satisfying either } A \text{ or } B}{\text{total \# of outcomes}} \\
&= \frac{(\text{\# of outcomes satisfying } A) + (\text{\# of outcomes satisfying } B)}{\text{total \# of outcomes}} \\
&= \frac{(\text{\# of outcomes satisfying } A)}{\text{total \# of outcomes}} + \frac{(\text{\# of outcomes satisfying } B)}{\text{total \# of outcomes}} \\
&= P(A) + P(B)
\end{aligned}$$

Summary

Summary

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
  - The **multiplication rule**, which states that for any two events, $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$.
  - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** simulations.

negation