

Lecture 11 – Probability

DSC 10, Fall 2025

Disc today

Quiz 2 Weds

## Agenda


We'll cover the basics of probability theory. This is a math lesson; take written notes 📝.

## Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5.**
- **Theory Meets Data, Chapters 1 and 2.**
- **Khan Academy's unit on Probability.**

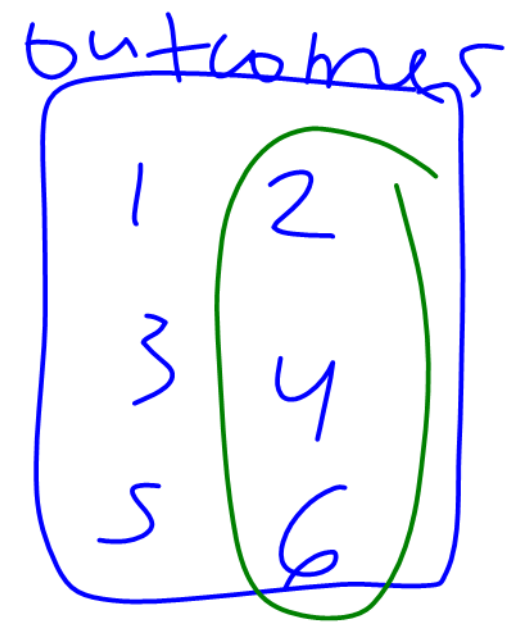
## Probability theory

- Some things in life *seem* random.
  - e.g., flipping a coin or rolling a die .
- The **probability** of seeing "heads" when flipping a fair coin is  $\frac{1}{2}$ .
- One interpretation of probability says that if we flipped a coin infinitely many times, then  $\frac{1}{2}$  of the outcomes would be heads.

# Prob (even #)

## Terminology

- **Experiment:** A process or action whose result is random.
  - e.g., rolling a die.
  - e.g., flipping a coin twice.
- **Outcome:** The result of an experiment.
  - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
  - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event:** A set of outcomes.
  - e.g., the event that the die lands on a even number is the set of outcomes {2, 4, 6}.
  - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
  - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.



## Terminology

- **Probability:** A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
  - 0: The event never happens.
  - 1: The event always happens.
- Notation: If  $A$  is an event,  $P(A)$  is the probability of that event.

$$P(\text{even } \#) = \frac{3}{6} = \frac{1}{2} \text{ outcomes}$$

1	2
3	4
5	6

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of  $A$  is

$$P(A) = \frac{\# \text{ of outcomes satisfying } A}{\text{total } \# \text{ of outcomes}}$$

- Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

outcomes

HHH	THH
HTH	TTH
HHT	THT
HTT	TTT

$$2 \times 2 \times 2 = 2^3$$



$$\frac{3}{8}$$

not ok  
outcomes

0H	1H
2H	3H

$$\frac{1}{4}$$

$$P(A \text{ and } B) = P(A) * P(B \text{ given } A)$$

multiply

Concept Check ☒ – Answer at [cc.dsc10.com](http://cc.dsc10.com)

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A)  $\frac{1}{9}$
- B)  $\frac{1}{6}$
- C)  $\frac{1}{3}$
- D)  $\frac{2}{3}$
- E) None of the above.

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{6}$$

green     red

outcome

$$P(A) * P(B \text{ given } A)$$

RB	BR	GR
RG	BG	GB

>  $\frac{1}{6}$   
direct



## Conditional probabilities

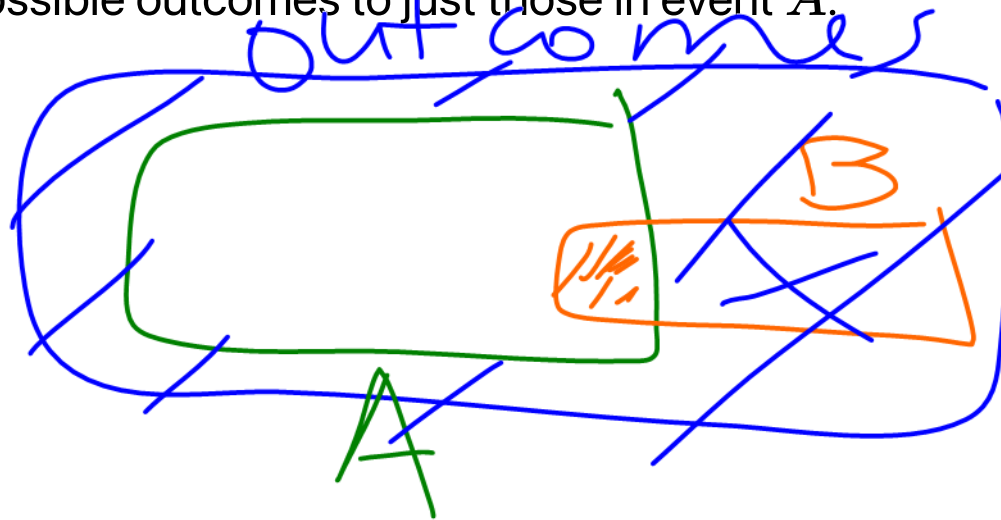
- Two events  $A$  and  $B$  can both happen. Suppose that we know  $A$  has happened, but we don't know if  $B$  has.
- If all outcomes are equally likely, then the conditional probability of  $B$  given  $A$  is:

$$P(B \text{ given } A) = \frac{\text{\# of outcomes satisfying both } A \text{ and } B}{\text{\# of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of  $B$ :

$$P(B) = \frac{\text{\# of outcomes satisfying } B}{\text{total \# of outcomes}}$$

if you restrict the set of possible outcomes to just those in event  $A$ .



A = Know it happened

B = Wondering

Concept Check ☒ – Answer at [cc.dsc10.com](https://cc.dsc10.com)

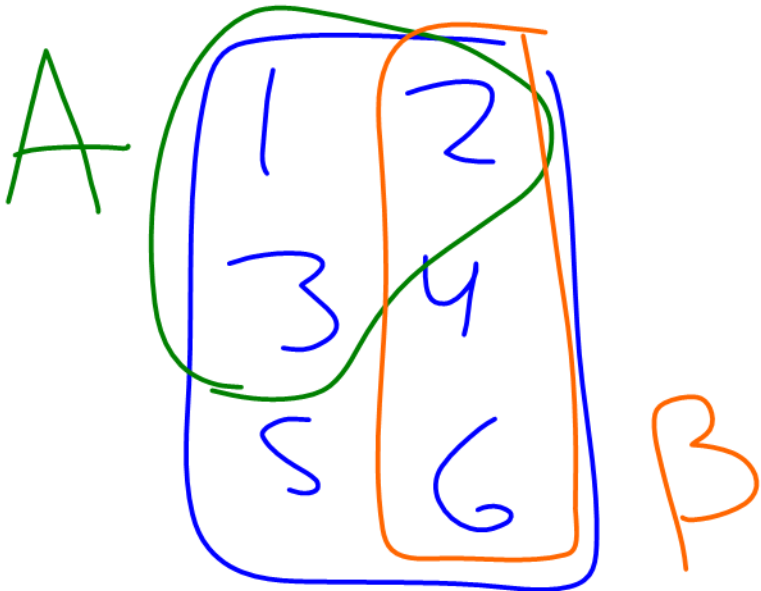
$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

if it happened

A I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A)  $\frac{1}{2}$
- B)  $\frac{1}{3}$
- C)  $\frac{1}{4}$
- D) None of the above.

B



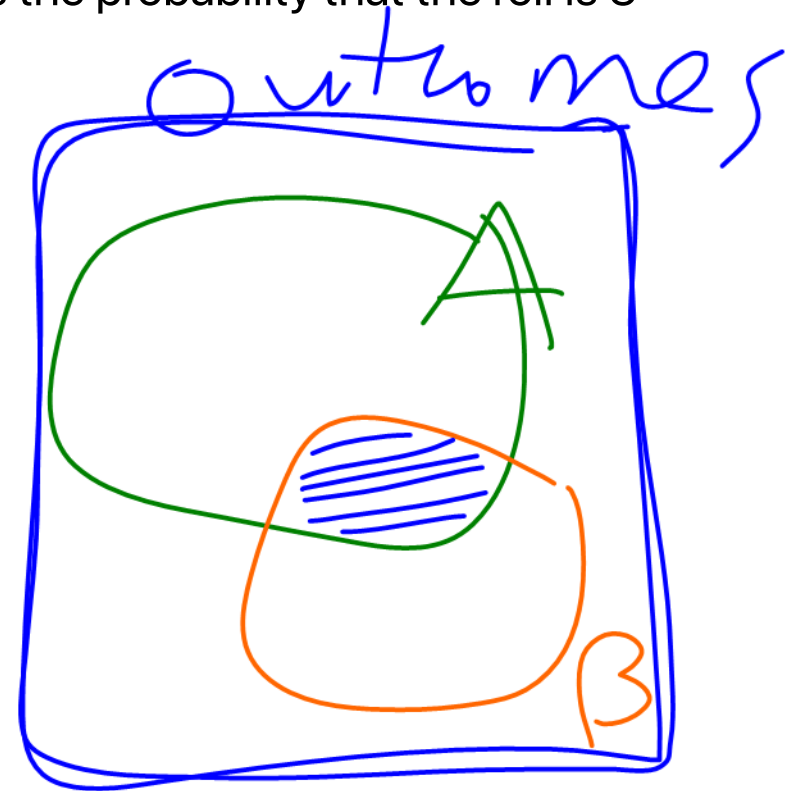
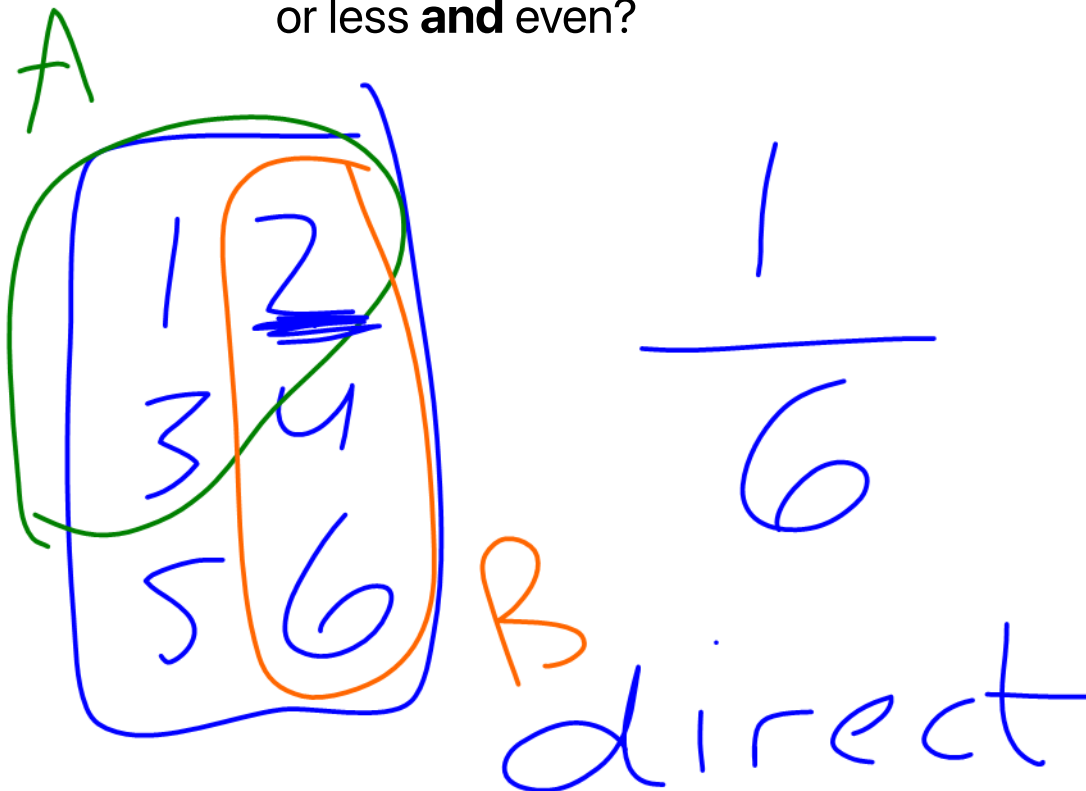
$$\frac{1}{3}$$

Probability that two events both happen

- Suppose again that  $A$  and  $B$  are two events, and that all outcomes are equally likely. Then, the probability that both  $A$  and  $B$  occur is

$$P(A \text{ and } B) = \frac{\text{\# of outcomes satisfying both } A \text{ and } B}{\text{total \# of outcomes}}$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?



## The multiplication rule

- The multiplication rule specifies how to compute the probability of both  $A$  and  $B$  happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?



$$\frac{3}{6} \times \frac{1}{3} = \frac{1}{6}$$

What if  $A$  isn't affected by  $B$ ? 🤔

- The multiplication rule states that, for any two events  $A$  and  $B$ ,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that  $A$  happens doesn't tell you anything about the likelihood of  $B$  happening?
  - Suppose we flip a fair coin three times.
  - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is  $P(A \text{ and } B)$ ?

Special  
Case:  
"given A"  
is irrelevant

Independent events

definition

- Two events  $A$  and  $B$  are independent if  $P(\underline{B \text{ given } A}) = P(B)$ , or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

$P(\text{H on } 1^{\text{st}} \text{ AND H on } 2^{\text{nd}} \text{ AND } \dots)$

$0.7 \times 0.7 \times \dots$

$0.7^5$

Probability that an event *doesn't* happen

- The probability that  $A$  **doesn't** happen is  $1 - P(A)$ .
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

$$1 - P(\text{never talk to gma})$$

Concept Check  – Answer at [cc.dsc10.com](https://cc.dsc10.com)

Every time I call my grandma 📞, the probability that she answers her phone is  $\frac{1}{3}$ , independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A)  $\frac{1}{3}$
- B)  $\frac{2}{3}$
- C)  $\frac{1}{2}$
- D) 1
- E) None of the above.

$$\begin{aligned} & 1 - P(NNN) \\ &= 1 - \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \\ &= 1 - \frac{8}{27} = \boxed{\frac{19}{27}} \end{aligned}$$

Y = yes  
N = no

8 outcomes

N	N	N
N	N	Y
N	Y	N
N	Y	Y
Y	N	N
Y	N	Y
Y	Y	N
Y	Y	Y

$$\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$$

not equally likely



not ok!

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

even

Probability of either of two events happening

- Suppose again that  $A$  and  $B$  are two events, and that all outcomes are equally likely. Then, the probability that either  $A$  or  $B$  occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or at least 5?



$$\frac{4}{6}$$



The addition rule

- Suppose that if  $A$  happens, then  $B$  doesn't, and if  $B$  happens, then  $A$  doesn't.

- Such events are called **mutually exclusive** – they have **no overlap**.

- If  $A$  and  $B$  are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5:** Suppose I have two biased coins, a red coin and a blue coin.

The red coin flips heads with probability 0.6, and the blue coin flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

$$P(\text{one H, one T}) = P(\text{H on red and T on blue OR T on red and H on blue})$$

$$0.6 \times 0.7 + 0.4 \times 0.3$$
$$\boxed{0.54}$$



special case

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If  $A$  and  $B$  are events consisting of equally likely outcomes, and furthermore  $A$  and  $B$  are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned} P(A \text{ or } B) &= \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A) + (\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= P(A) + P(B) \end{aligned}$$

## Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
  - The **multiplication rule**, which states that for any two events,  $P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A)$  .
  - The **addition rule**, which states that for any two **mutually exclusive** events,  $P(A \text{ or } B) = P(A) + P(B)$ .
- **Next time:** Simulations.