

Lecture 11 – Probability

DSC 10, Winter 2025

Agenda


We'll cover the basics of probability theory. This is a math lesson; take written notes 🖋️.

Probability resources

Probability is a tricky subject. If it doesn't click during lecture or on the assignments, take a look at the following resources:

- **Computational and Inferential Thinking, Chapter 9.5.**
- **Theory Meets Data, Chapters 1 and 2.**
- **Khan Academy's unit on Probability.**

Probability theory

- Some things in life *seem* random.
 - e.g., flipping a coin or rolling a die .
- The **probability** of seeing "heads" when flipping a fair coin is $\frac{1}{2}$.
- One interpretation of probability says that if we flipped a coin infinitely many times, then $\frac{1}{2}$ of the outcomes would be heads.

Terminology

- **Experiment:** A process or action whose result is random.
 - e.g., rolling a die.
 - e.g., flipping a coin twice.
- **Outcome:** The result of an experiment.
 - e.g., the possible outcomes of rolling a six-sided die are 1, 2, 3, 4, 5, and 6.
 - e.g., the possible outcomes of flipping a coin twice are HH, HT, TH, and TT.
- **Event:** A set of outcomes.
 - e.g., the event that the die lands on an even number is the set of outcomes {2, 4, 6}.
 - e.g., the event that the die lands on a 5 is the set of outcomes {5}.
 - e.g., the event that there is at least 1 head in 2 flips is the set of outcomes {HH, HT, TH}.

Terminology

- **Probability:** A number between 0 and 1 (equivalently, between 0% and 100%) that describes the likelihood of an event.
 - 0: The event never happens.
 - 1: The event always happens.
- Notation: If A is an event, $P(A)$ is the probability of that event.

Equally-likely outcomes

- If all of the possible outcomes are equally likely, then the probability of A is

$$P(A) = \frac{\# \text{ of outcomes satisfying } A}{\text{total } \# \text{ of outcomes}}$$

- **Example 1:** Suppose we flip a fair coin 3 times. What is the probability we see exactly 2 heads?

Concept Check  – Answer at cc.dsc10.com

I have three cards: red, blue, and green. What is the chance that I choose a card at random and it is green, then – **without putting it back** – I choose another card at random and it is red?

- A) $\frac{1}{9}$
- B) $\frac{1}{6}$
- C) $\frac{1}{3}$
- D) $\frac{2}{3}$
- E) None of the above.

Conditional probabilities

- Two events A and B can both happen. Suppose that we know A has happened, but we don't know if B has.
- If all outcomes are equally likely, then the conditional probability of B given A is:

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

- Intuitively, this is similar to the definition of the regular probability of B ,
 $P(B) = \frac{\# \text{ of outcomes satisfying } B}{\# \text{ of total outcomes}}$

$\{ \# \text{ of total outcomes} \}$

, if you restrict these to possible outcomes to be just those in event A .

Concept Check  – Answer at cc.dsc10.com

$$P(B \text{ given } A) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\# \text{ of outcomes satisfying } A}$$

I roll a six-sided die and don't tell you what the result is, but I tell you that it is 3 or less. What is the probability that the result is even?

- A) $\frac{1}{2}$
- B) $\frac{1}{3}$
- C) $\frac{1}{4}$
- D) None of the above.

Probability that two events both happen

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that both A and B occur is

$$P(A \text{ and } B) = \frac{\# \text{ of outcomes satisfying both } A \text{ and } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 2:** I roll a fair six-sided die. What is the probability that the roll is 3 or less **and** even?

The multiplication rule

- The multiplication rule specifies how to compute the probability of both A and B happening, even if all outcomes are not equally likely.

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- **Example 2, again:** I roll a fair six-sided die. What is the probability that the roll is 3 or less and even?

What if A isn't affected by B ? 🤔

- The multiplication rule states that, for any two events A and B ,

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

- What if knowing that A happens doesn't tell you anything about the likelihood of B happening?
 - Suppose we flip a fair coin three times.
 - The probability that the second flip is heads doesn't depend on the result of the first flip.
- Then, what is $P(A \text{ and } B)$?

Independent events

- Two events A and B are independent if $P(B \text{ given } A) = P(B)$, or equivalently if

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

- **Example 3:** Suppose we have a coin that is **biased**, and flips heads with probability 0.7. Each flip is independent of all other flips. We flip it 5 times. What's the probability we see 5 heads in a row?

Probability that an event *doesn't* happen

- The probability that A **doesn't** happen is $1 - P(A)$.
- For example, if the probability it is sunny tomorrow is 0.85, then the probability it is not sunny tomorrow is 0.15.

Concept Check  – Answer at cc.dsc10.com

Every time I call my grandma 🧓, the probability that she answers her phone is $\frac{1}{3}$, independently for each call. If I call my grandma three times today, what is the chance that I will talk to her at least once?

- A) $\frac{1}{3}$
- B) $\frac{2}{3}$
- C) $\frac{1}{2}$
- D) 1
- E) None of the above.

Probability of either of two events happening

- Suppose again that A and B are two events, and that all outcomes are equally likely. Then, the probability that either A or B occur is

$$P(A \text{ or } B) = \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}}$$

- **Example 4:** I roll a fair six-sided die. What is the probability that the roll is even or at least 5?

The addition rule

- Suppose that if A happens, then B doesn't, and if B happens, then A doesn't.
 - Such events are called **mutually exclusive** – they have **no overlap**.
- If A and B are any two mutually exclusive events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

- **Example 5:** Suppose I have two biased coins, coin A and coin B . Coin A flips heads with probability 0.6, and coin B flips heads with probability 0.3. I flip both coins once. What's the probability I see two different faces?

Aside: Proof of the addition rule for equally-likely events

You are not required to know how to "prove" anything in this course; you may just find this interesting.

If A and B are events consisting of equally likely outcomes, and furthermore A and B are mutually exclusive (meaning they have no overlap), then

$$\begin{aligned} P(A \text{ or } B) &= \frac{\# \text{ of outcomes satisfying either } A \text{ or } B}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A) + (\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= \frac{(\# \text{ of outcomes satisfying } A)}{\text{total } \# \text{ of outcomes}} + \frac{(\# \text{ of outcomes satisfying } B)}{\text{total } \# \text{ of outcomes}} \\ &= P(A) + P(B) \end{aligned}$$

Summary, next time

- Probability describes the likelihood of an event occurring.
- There are several rules for computing probabilities. We looked at many special cases that involved equally-likely events.
- There are two general rules to be aware of:
 - The **multiplication rule**, which states that for any two events,
$$P(A \text{ and } B) = P(B \text{ given } A) \cdot P(A) .$$
 - The **addition rule**, which states that for any two **mutually exclusive** events, $P(A \text{ or } B) = P(A) + P(B)$.
- **Next time:** Simulations.