Linear Algebra Review

Matrices

An $m \times n$ **matrix** is a table of numbers with m rows, n columns:

- Example: 2×3 matrix:

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{pmatrix}$$

- Example: 3×3 "square" matrix:

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Matrix Notation

- We use upper-case letters for matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

- Sometimes use subscripts to denote particular elements: $A_{13} = 3$, $A_{21} = 4$

- A^T denotes the transpose of A:

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Matrix Addition and Scalar Multiplication

- We can add two matrices only if they are the same size.

- Addition occurs elementwise:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 3 & 3 & 3 \end{pmatrix}$$

- Scalar multiplication occurs elementwise, too:

$$2 \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

Matrix-Matrix Multiplication

- We can multiply two matrices A and B only if # cols in A is equal to # rows in B

- If $A = m \times n$ and $B = n \times p$, the result is $m \times p$.
  - This is **very useful**. Remember it!

- The low-level definition. the ij entry of the product is:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Matrix-Matrix Multiplication Example

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 6 \\ 1 & 3 \\ 4 & 8 \end{pmatrix}$$

- What is the size of AB?
- What is $(AB)_{12}$?

Matrix-Matrix Multiplication Properties

- Distributive: $A(B + C) = AB + AC$
- Associative: $(AB)C = A(BC)$
- **Not commutative in general**: $AB \neq BA$

Identity Matrices

- The $n \times n$ **identity matrix** I has ones along the diagonal:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- If A is $n \times m$, then $IA = A$.

- If B is $m \times n$, then $BI = B$.

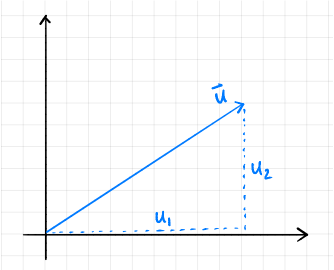
Vectors

- An d-**vector** is an $d \times 1$ matrix.
- Often use arrow, lower-case letters to denote: $\vec{x}$.
- Often write $\vec{x} \in \mathbb{R}^d$ to say $\vec{x}$ is a d vector.
- Example. A 4-vector:

$$\begin{pmatrix} 2 \\ 1 \\ 5 \\ -3 \end{pmatrix}$$

Geometric Meaning of Vectors

- A vector $\vec{u} = (u_1, \ldots, u_d)^T$ is an arrow to the point $(u_1, \ldots, u_d)$:



- The length, or **norm**, of $\vec{u}$ is $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_d^2}$.

Dot Products

- The **dot product** of two d-vectors $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

- Using low-level matrix multiplication definition:

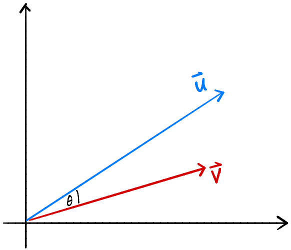
$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$
$$= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Dot Product Example

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \vec{u} \cdot \vec{v} =$$

Geometric Interpretation of Dot Product

- $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$



Exercise

Which of these is another expression for the norm of $\vec{u}$?

a) $\vec{u} \cdot \vec{u}$
b) $\sqrt{\vec{u}^2}$
c) $\sqrt{\vec{u} \cdot \vec{u}}$
d) $\vec{u}^2$

Properties of the Dot Product

- Commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Distributive: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- Linear: $\vec{u} \cdot (\alpha\vec{v} + \beta\vec{w}) = \alpha\vec{u} \cdot v + \beta\vec{u} \cdot \vec{w}$

Matrix-Vector Multiplication

- Special case of matrix-matrix multiplication.
- Result is always a vector with same number of rows as the matrix.
- One view: a “mixture” of the columns.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 6 & 3 \\ 2 & 7 & 1 \end{pmatrix}$ and $Ax = \begin{pmatrix} 1 \\ 9 \\ 4 \end{pmatrix}$.

Which of theses is a possible x?

a) $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ b) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ c) $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ d) $\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

Exercise

Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 2 \\ 2 & 7 & 1 \end{pmatrix}$.

For what x would $xA = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$?

a) $\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} .5 & .5 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 2 & 0 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 0 & .5 & 0 \end{pmatrix}$

Matrices and Functions

- Matrix-vector multiplication takes in a vector, outputs a vector.
- An $m \times n$ matrix is an encoding of a function mapping $\mathbb{R}^n$ to $\mathbb{R}^m$.
- Matrix multiplication evaluates that function.

Exercise

Suppose $f(x) = \begin{pmatrix} x_3 & x_2 \end{pmatrix}$ where $x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \in \mathbb{R}^3$. What could A such that $f(x) = Ax$ be?

a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$