DSC 190 - Homework 01

Due: Wednesday, April 12

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

Suppose that in a group of 1000 people, 600 currently live in California and 400 currently live in Texas. In any given year, 5% of the people living in California move to Texas, and 3% of the people living in Texas move to California. You may assume that the people do not move to any other states.

We can represent the current number of people living in California and Texas with a *population vector*:

$$\vec{p} = (\#\text{ in California}, \#\text{ in Texas})^T.$$

The initial situation described above is represented by the population vector $(600, 400)^T$.

a) After one year, how many people will be living in Calfornia and Texas? What about after two years?

Do this problem by hand, showing your calculations. State your answers in the form of a population vector. It is OK for your results to be decimals (*don't* round them to the nearest integer).

b) Let $\vec{f}(\vec{p})$ be the transformation which takes in a current population vector, $\vec{p} = (c, t)^T$, and returns the population vector after one year has passed.

Write the formula of the transformation in coordinate form with respect to the standard basis. That is, fill in:

$$\vec{f}(\vec{p}) = (\ldots, \quad \ldots)^T.$$

Example: consider the transformation $\vec{g}$ which doubles the population of California each year, and triples population of Texas. Written in coordinate form, that transformation has the formula $\vec{g}(\vec{p}) = (2c, 3t)^T$.

Hint: your answer should have the form:

$$\vec{f}(\vec{p}) = (\alpha_1 c + \alpha_2 t, \quad \alpha_3 c + \alpha_4 t)^T,$$

where $\alpha_1, \ldots, \alpha_4$ are real constants that you should provide.

c) Prove that the transformation $\vec{f}(\vec{p})$ you derived above is a *linear* transformation by showing that it satisfies the definition. That is, show that for any vector $\vec{u} = (c_1, t_1)^T$ and $\vec{v} = (c_2, t_2)^T$, and scalars α, β:

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

d) In 50 years, how many people will live in California, and how many will live in Texas? That is, what is the population vector $\vec{p}$ after $\vec{f}$ is applied 50 times? Your answer can include decimal numbers.

You (probably) do not want to carry out these calculations by hand. Instead, implement it in code (attaching a screenshot to show your work).

e) You might wonder: if this process is allowed to continue, will it ever *converge*? That is, will there ever be a time where the populations in California and Texas do not change from year to year?

To answer this, repeat the previous question, but increase the number of iterations until the population vector does not change. Report as your answer this population vector.

Hint: your answer should contain nice, whole numbers which add to 1000.

f) Let $\vec{u}^{(1)}$ be the vector you found in the previous part. What you saw above is that $\vec{f}(\vec{u}^{(1)}) = \vec{u}^{(1)}$. In the language of linear algebra, $\vec{u}^{(1)}$ is an *eigenvector* of $\vec{f}$ with eigenvalue 1, since $\vec{f}(\vec{u}^{(1)}) = 1 \cdot \vec{u}^{(1)}$.

It can be shown that another eigenvector of $\vec{f}$ is $\vec{u}^{(2)} = (1, -1)^T$, and that $\vec{f}(\vec{u}^{(2)}) = 0.92\vec{u}^{(2)}$. In the language of linear algebra, the eigenvalue associated with $\vec{u}^{(2)}$ is 0.92.

It is often useful to use the eigenvectors of a linear transformation as basis vectors. In some situations, the eigenvectors are guaranteed to be orthogonal, although that is not the case here. However, we can still use the above eigenvectors as a basis, but it will not be an *orthonormal basis*.

Write the vector $\vec{p} = (600, 400)^T$ as a coordinate vector in the basis $\mathcal{U} = \{\vec{u}^{(1)}, \vec{u}^{(2)}\}$. That is, find $[\vec{p}]_{\mathcal{U}}$.

Hint: you cannot use the approach described in class, where we found the coordinates by computing $\vec{p} \cdot \vec{u}^{(1)}$ and $\vec{p} \cdot \vec{u}^{(2)}$. That approach only works in the case of an *orthonormal* basis. Instead, recognize that we want to find α and β so that $\vec{p} = \alpha\vec{u}^{(1)} + \beta\vec{u}^{(2)}$. This amounts to solving the system of two equations:

$$\underbrace{\begin{pmatrix} 600 \\ 400 \end{pmatrix}}_{\vec{p}} = \alpha \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{\vec{u}^{(1)}} + \beta \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\vec{u}^{(2)}},$$

where a and b are the coordinates of $\vec{u}^{(1)}$ that you found in the last problem (i.e., they are whole numbers which add to 1000).

g) Let $[\vec{x}]_{\mathcal{U}} = (x_1, x_2)^T$ be a population vector with respect to the basis $\mathcal{U}$. Write the formula for $\vec{f}(\vec{x})$ with respect to the basis $\mathcal{U}$. That is, what is $[\vec{f}(\vec{x})]_{\mathcal{U}}$?

Hint: $\vec{f}$ is linear, so $\vec{f}(\alpha\vec{u}^{(1)} + \beta\vec{u}^{(2)}) = \alpha\vec{f}(\vec{u}^{(1)}) + \beta\vec{f}(\vec{u}^{(2)})$. We already know what $\vec{f}(\vec{u}^{(1)})$ and $\vec{f}(\vec{u}^{(2)})$ are from above.

h) Suppose $[\vec{x}]_{\mathcal{U}} = (x_1, x_2)^T$ is current population vector, expressed with respect to the basis $\mathcal{U}$. Write a formula for the population vector after k years have passed, also expressed in the basis $\mathcal{U}$.

Hint: your formula should involve x_1, x_2, k and some constants.

Your formula should be pretty simple – this was enabled by using the eigenvectors as our basis. The formula written in the standard basis is not nearly as simple!

i) Suppose the current population vector, expressed in the basis $\mathcal{U}$, is $(1, 225)^T$. What will the population vector be in 50 years, expressed in terms of the basis $\mathcal{U}$?

j) Express the vector your found in the last part as a coordinate vector in the standard basis.

Hint: your result should be familiar as an answer to a previous part. However, it might not be *exactly* the same due to some roundoff error – you presumably calculated the other answer on a computer with finite numerical precision.