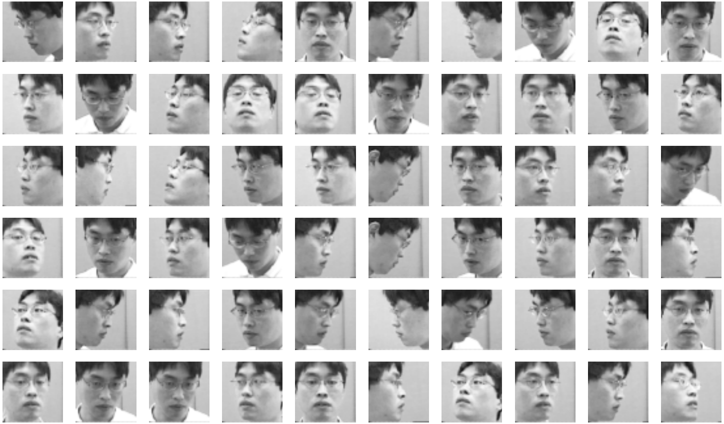
DSC 140B
Representation Learning

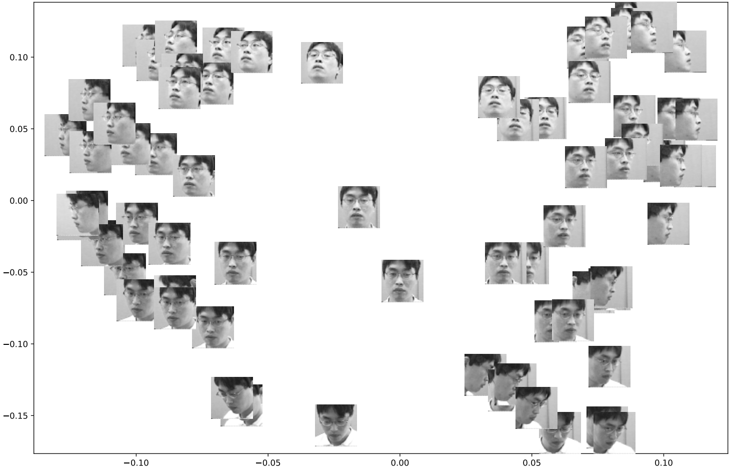
Lecture 02 | Part 1

Why Linear Algebra?

Last Time



Last Time



Dimensionality Reduction

- This is an example of **dimensionality reduction**:
  - Input: vectors in $\mathbb{R}^{10{,}000}$.
  - Output: vectors in $\mathbb{R}^2$.
- The method which produced this result is called **Laplacian Eigenmaps**.
- How does it work?

A Preview of Laplacian Eigenmaps

To reduce dimensionality from d to d':

1. Create an undirected **similarity graph** G
   - Each vector in $\mathbb{R}^d$ becomes a node in the graph.
   - Make edge (u, v) if u and v are "close"
2. Form the **graph Laplacian matrix**, L:
   - Let A be the adjacency matrix, D be the degree matrix.
   - Define the graph Laplacian matrix, $L = D - A$.
3. Compute d' **eigenvectors** of L.
   - Each eigenvector gives one new feature.

Why eigenvectors?

- We will cover Laplacian Eigenmaps in much greater detail.

- For now: why do eigenvectors appear here?
  - What are eigenvectors?
  - How are they useful?
  - Why is linear algebra important in ML?

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Representation Learning

Lecture 02 | Part 2

Coordinate Vectors

Coordinate Vectors

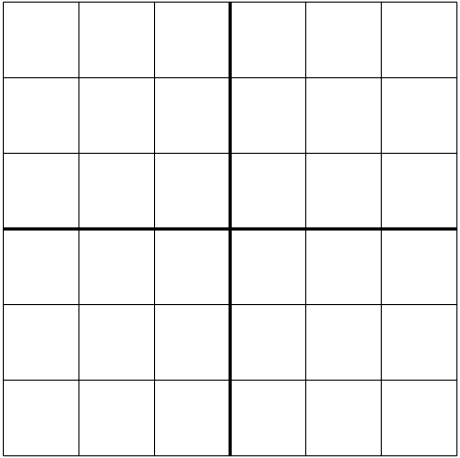
- We can write a vector $\vec{x} \in \mathbb{R}^d$ as a **coordinate vector**:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

Example

$$\vec{x} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

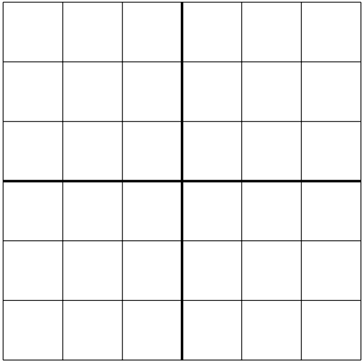
$$\vec{y} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



Standard Basis

- Writing a vector in coordinate form requires choosing a **basis**.

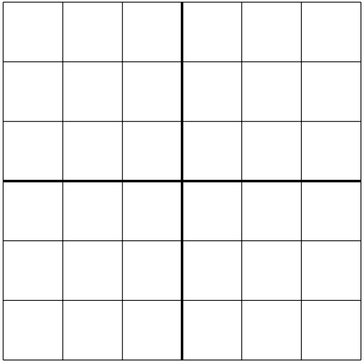
- The “default” is the **standard basis**: $\hat{e}^{(1)}, \ldots, \hat{e}^{(d)}$.



Standard Basis

- When we write $\vec{x} = (x_1, \ldots, x_d)^T$, we mean that $\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)} + \ldots x_d \hat{e}^{(d)}$.

Example: $\vec{x} = (3, -2)^T$



Standard Basis Coordinates

- In coordinate form:

$$\hat{e}^{(i)} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

where the 1 appears in the ith place.

Exercise

Let $\vec{x} = (3, 7, 2, -5)^T$. What is $\vec{x} \cdot \hat{e}^{(4)}$?

Recall: the Dot Product

- The **dot product** of $\vec{u}$ and $\vec{v}$ is defined as:

  $$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

  where θ is the angle between $\vec{u}$ and $\vec{v}$.

- $\vec{u} \cdot \vec{v} = 0$ if and only if $\vec{u}$ and $\vec{v}$ are orthogonal

Dot Product (Coordinate Form)

- In terms of coordinate vectors:

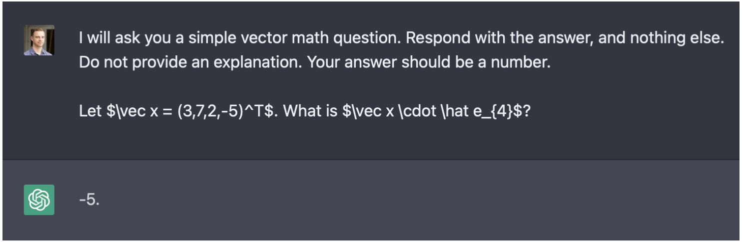
$$\begin{aligned}\vec{u} \cdot \vec{v} &= \vec{u}^T \vec{v} \\ &= \begin{pmatrix} u_1 & u_2 & \cdots & u_d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_d \end{pmatrix} \\ &= \end{aligned}$$

- This definition assumes the standard basis.

Example

$$\begin{pmatrix} 3 \\ 7 \\ 2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

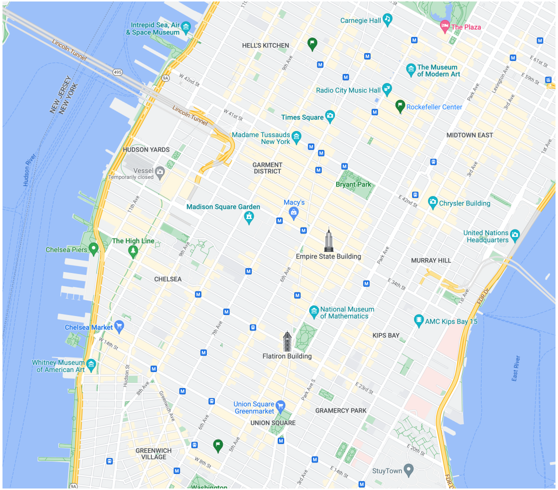
What does ChatGPT say?



Other Bases

- The standard basis is not the **only** basis.
- Sometimes more convenient to use another.

Example



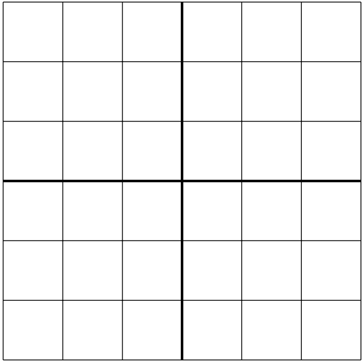
Carnegie Hall
The Plaza
Intrepid Sea, Air & Space Museum
HELL'S KITCHEN
The Museum of Modern Art
Radio City Music Hall
Rockefeller Center
Lincoln Tunnel
NEW JERSEY
NEW YORK
W 42nd St
W 41st St
Times Square
Madame Tussauds New York
MIDTOWN EAST
HUDSON YARDS
GARMENT DISTRICT
Vessel
Temporarily closed
Bryant Park
Hudson River
Madison Square Garden
Macy's
Chrysler Building
E 42nd St
United Nations Headquarters
The High Line
Chelsea Piers
Empire State Building
MURRAY HILL
CHELSEA
E 34th St
National Museum of Mathematics
AMC Kips Bay 15
Chelsea Market
KIPS BAY
W 14th St
Flatiron Building
Whitney Museum of American Art
East River
E 23rd St
Union Square Greenmarket
GRAMERCY PARK
UNION SQUARE
GREENWICH VILLAGE
W 8th St
E 14th St
E 20th St
StuyTown
FDR Dr
Washington

Orthonormal Bases

- **Orthonormal bases** are particularly nice.
- A set of vectors $\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}$ forms an **orthonomal basis** $\mathcal{U}$ for R^d if:
  - They are mutually orthogonal: $\hat{u}^{(i)} \cdot \hat{u}^{(j)} = 0$.
  - They are all unit vectors: $\|\hat{u}^{(i)}\| = 1$.

Example

$$\hat{u}^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \hat{u}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$



Coordinate Vectors

- A vector's coordinates depend on the basis used.
- If we are using the basis $\mathcal{U} = \{\hat{u}^{(1)}, \hat{u}^{(2)}\}$, then $\vec{x} = (x_1, x_2)^T$ means $\vec{x} = x_1 \hat{u}^{(1)} + x_2 \hat{u}^{(2)}$.
- We will write $[\vec{x}]_{\mathcal{U}} = (x_1, \dots, x_d)^T$ to denote that the coordinates are with respect to the basis $\mathcal{U}$.

Exercise

Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1,1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1,1)^T$. Suppose $[\vec{x}]_{\mathcal{U}} = (3,-4)^T$. What is $\vec{x} \cdot \hat{u}^{(1)}$?

What did ChatGPT say?

Let $\bu{1} = \frac{1}{\sqrt2}(1,1)^T$ and $\bu{2} = \frac{1}{\sqrt2}(-1,1)^T$. Suppose $[\vec x]_\mathcal{U} = (3, -4)^T$. What is $\vec x \cdot \bu{1}$?

-0.5. (Note: This answer assumes that $[\vec x]_\mathcal{U}$ represents the coordinate vector of $\vec x$ with respect to the basis $\mathcal{U}$, and that $\bu{1}$ and $\bu{2}$ are orthonormal basis vectors.) The dot product between $\vec x$ and $\bu{1}$ represents the projection of $\vec x$ onto the direction of $\bu{1}$. In this case, $\bu{1}$ is a unit vector, so the dot product gives the signed length of the projection of $\vec x$ onto the direction of $\bu{1}$. The result of -0.5 indicates that the projection of $\vec x$ onto the direction of $\bu{1}$ is in the opposite direction of $\bu{1}$, and the magnitude of the projection is 0.5 times the length of $\bu{1}$. Note that the negative sign indicates that the projection is in the opposite direction of $\bu{1}$. The dot product can be used to determine the alignment or misalignment of vectors, and is a useful tool in various applications, such as in physics, engineering, and computer graphics. It is important to understand how to compute and interpret dot products in the context of vector projections and geometric relationships between vectors. Therefore, the result of -0.5 indicates that $\vec x$ is projected onto the opposite direction of $\bu{1}$ with a magnitude of 0.5 times the length of $\bu{1}$. Keep practicing vector math problems to further strengthen your understanding of dot products and their applications!

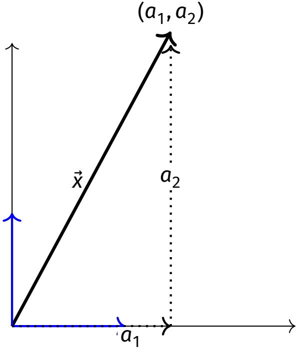
Exercise

Consider $\vec{x} = (2, 2)^T$ and let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$. What is $[\vec{x}]_{\mathcal{U}}$?

Change of Basis

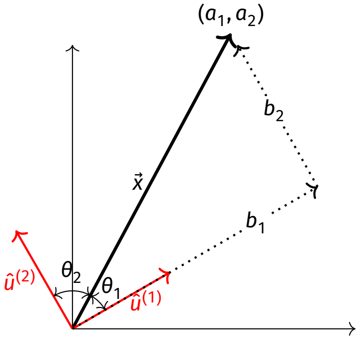
- How do we compute the coordinates of a vector in a new basis, $\mathcal{U}$?
- Some trigonometry is involved.
- **Key Fact**: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

Change of Basis



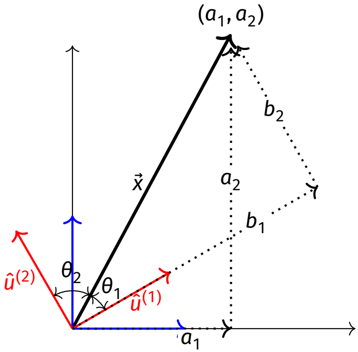
- Suppose we know $\vec{x} = (a_1, a_2)^T$ w.r.t. standard basis.
- Then $\vec{x} = a_1 \hat{e}^{(1)} + a_2 \hat{e}^{(2)}$

Change of Basis



- Want to write:
  $\vec{x} = b_1 \hat{u}^{(1)} + b_2 \hat{u}^{(2)}$
- Need to find b_1 and b_2.

Change of Basis



- **Exercise:** Solve for b_1, writing the answer as a dot product.
- Hint: $\cos\theta$ = adjacent/hypotenuse

Change of Basis

- Let $\mathcal{U} = \{\hat{u}^{(1)}, \ldots, \hat{u}^{(d)}\}$ be an orthonormal basis.
- The coordinates of $\vec{x}$ w.r.t. $\mathcal{U}$ are:

$$[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} \vec{x} \cdot \hat{u}^{(1)} \\ \vec{x} \cdot \hat{u}^{(2)} \\ \vdots \\ \vec{x} \cdot \hat{u}^{(d)} \end{pmatrix}$$

Exercise

Suppose $\vec{x} = (2, 1)^T$ and let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$. What is $[\vec{x}]_{\mathcal{U}}$?

Exercise

Let $\vec{x} = (-1, 4)^T$ and suppose:

$$\hat{u}^{(1)} \cdot \hat{e}^{(1)} = 3 \qquad \hat{u}^{(2)} \cdot \hat{e}^{(1)} = -1$$

$$\hat{u}^{(1)} \cdot \hat{e}^{(2)} = -2 \qquad \hat{u}^{(2)} \cdot \hat{e}^{(2)} = 5$$

What is $[\vec{x}]_{\mathcal{U}}$?

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Lecture 02 | Part 3

Functions of a Vector

Functions of a Vector

- In ML, we often work with functions of a vector: $f : \mathbb{R}^d \to \mathbb{R}^{d'}$.
- Example: a prediction function, $H(\vec{x})$.
- Functions of a vector can return:
  - a number: $f : \mathbb{R}^d \to \mathbb{R}^1$
  - a vector $\vec{f} : \mathbb{R}^d \to \mathbb{R}^{d'}$
  - something else?

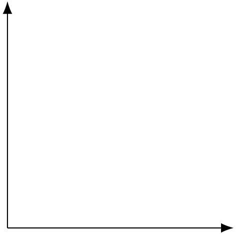
Transformations

- A **transformation** $\vec{f}$ is a function that takes in a vector, and returns a vector *of the same dimensionality*.

- That is, $\vec{f} : \mathbb{R}^d \to \mathbb{R}^d$.

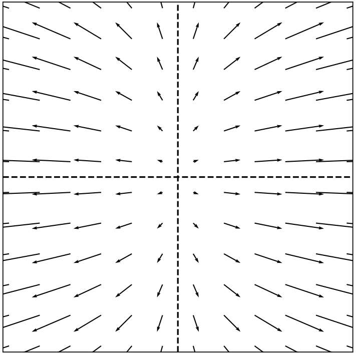
Visualizing Transformations

- A transformation is a **vector field**.
  - Assigns a vector to each point in space.
  - Example: $\vec{f}(\vec{x}) = (3x_1, x_2)^T$



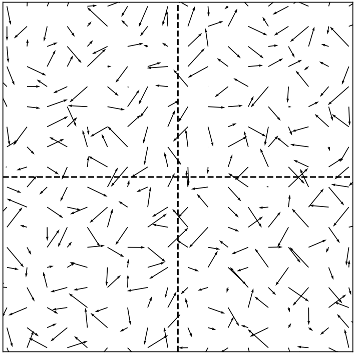
Example

- $\vec{f}(\vec{x}) = (3x_1, x_2)^T$



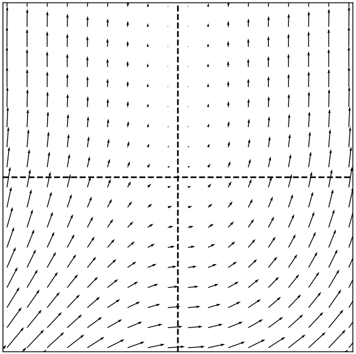
Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Arbitrary Transformations

- Arbitrary transformations can be quite complex.



Linear Transformations

- Luckily, we often[1] work with simpler, **linear transformations**.

- A transformation f is linear if:

$$\vec{f}(\alpha\vec{x} + \beta\vec{y}) = \alpha\vec{f}(\vec{x}) + \beta\vec{f}(\vec{y})$$

[1] Sometimes, just to make the math tractable!

Checking Linearity

- To check if a transformation is linear, use the definition.

- **Example:** $\vec{f}(\vec{x}) = (x_2, -x_1)^T$

Exercise

Let $\vec{f}(\vec{x}) = (x_1 + 3, x_2)$. Is $\vec{f}$ a linear transformation?

Implications of Linearity

- Suppose $\vec{f}$ is a linear transformation. Then:

$$\begin{aligned}\vec{f}(\vec{x}) &= \vec{f}(x_1\hat{e}^{(1)} + x_2\hat{e}^{(2)}) \\ &= x_1\vec{f}(\hat{e}^{(1)}) + x_2\vec{f}(\hat{e}^{(2)})\end{aligned}$$

- I.e., $\vec{f}$ is **totally determined** by what it does to the basis vectors.

The **Complexity** of Arbitrary Transformations

- Suppose f is an **arbitrary** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

The **Simplicity** of Linear Transformations

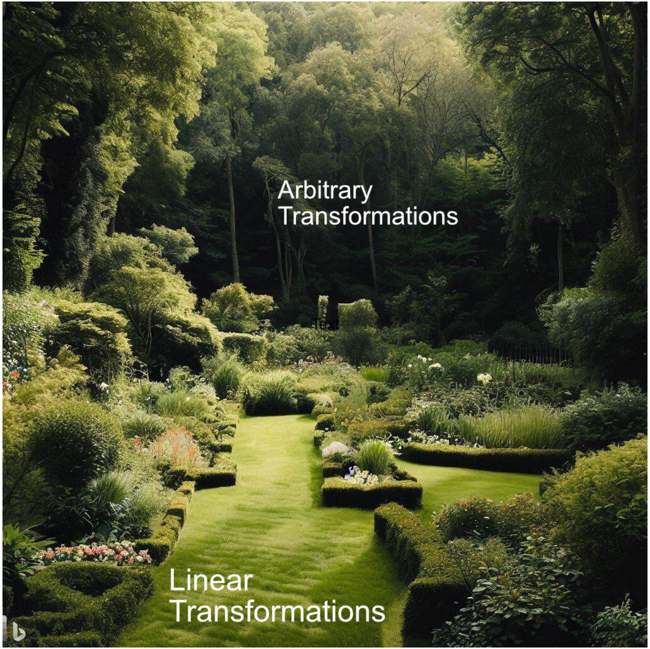
- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (x_1, x_2)^T$.
- What is $\vec{f}(\vec{x})$?

Exercise

- Suppose f is a **linear** transformation.
- I tell you $\vec{f}(\hat{e}^{(1)}) = (2, 1)^T$ and $\vec{f}(\hat{e}^{(2)}) = (-3, 0)^T$.
- I tell you $\vec{x} = (3, -4)^T$.
- What is $\vec{f}(\vec{x})$?

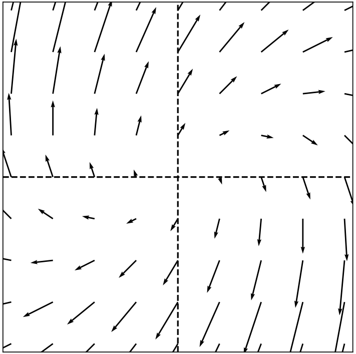
Key Fact

- Linear functions are determined **entirely** by what they do on the basis vectors.
- I.e., to tell you what f does, I only need to tell you $\vec{f}(\hat{e}^{(1)})$ and $\vec{f}(\hat{e}^{(2)})$.
- This makes the math easy!


Arbitrary
Transformations
Linear
Transformations

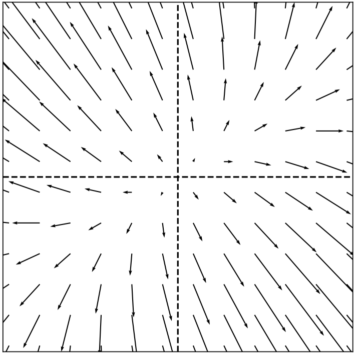
Example Linear Transformation

- $\vec{f}(\vec{x}) = (x_1 + 3x_2, -3x_1 + 5x_2)^T$



Another Example Linear Transformation

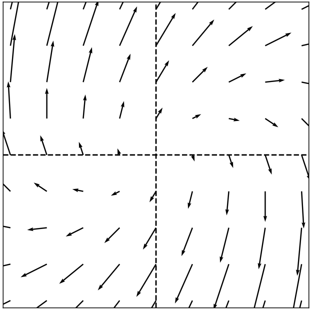
- $\vec{f}(\vec{x}) = (2x_1 - x_2, -x_1 + 3x_2)^T$

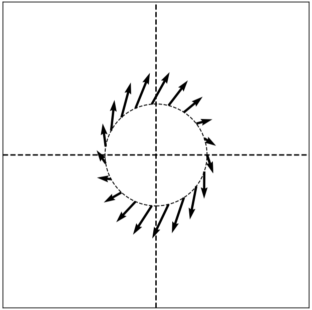


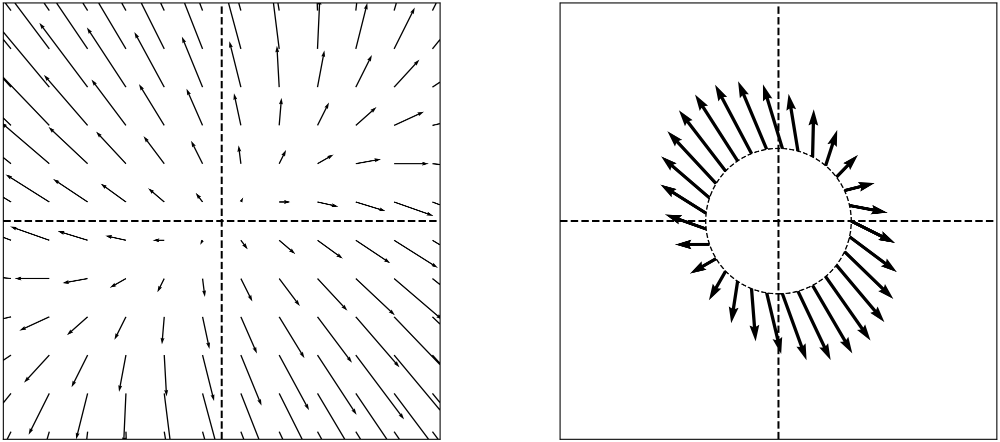
Note

- Because of linearity, along any given direction $\vec{f}$ changes only in scale.

$$\vec{f}(\lambda\hat{x}) = \lambda\vec{f}(\hat{x})$$







Linear Transformations and Bases

- We have been writing transformations in coordinate form. For example:

$$\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$$

- To do so, we assumed the **standard basis**.
- If we use a different basis, the formula for $\vec{f}$ changes.

Example

- Suppose that in the standard basis, $\vec{f}(\vec{x}) = (x_1 + x_2, x_1 - x_2)^T$.
- Let $\hat{u}^{(1)} = \frac{1}{\sqrt{2}}(1, 1)^T$ and $\hat{u}^{(2)} = \frac{1}{\sqrt{2}}(-1, 1)^T$.
- Write $[\vec{x}]_{\mathcal{U}} = (z_1, z_2)^T$.
- What is $[\vec{f}(\vec{x})]_{\mathcal{U}}$ in terms of z_1 and z_2?

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Lecture 02 | Part 4

Matrices

Matrices?

- I thought this was supposed to be about linear algebra... Where are the matrices?

Matrices?

- I thought this was supposed to be about linear algebra… Where are the matrices?

- What is a matrix, anyways?

What is a matrix?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Recall: Linear Transformations

- A **transformation** $\vec{f}(\vec{x})$ is a function which takes a vector as input and returns a vector of the same dimensionality.

- A transformation f is **linear** if

$$\vec{f}(\alpha\vec{u} + \beta\vec{v}) = \alpha\vec{f}(\vec{u}) + \beta\vec{f}(\vec{v})$$

Recall: Linear Transformations

- A **key** property: to compute $\vec{f}(\vec{x})$, we only need to know what f does to basis vectors.

- Example:

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$

Matrices

- f defined by what it does to basis vectors
- Place $\vec{f}(\hat{e}^{(1)}), \vec{f}(\hat{e}^{(2)}), \ldots$ into a table as columns
- This is the **matrix** representing[2] f

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}$$

[2]with respect to the standard basis $\hat{e}^{(1)}, \hat{e}^{(2)}$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = (1, 4, 7)^T$$

$$\vec{f}(\hat{e}^{(2)}) = (2, 5, 7)^T$$

$$\vec{f}(\hat{e}^{(3)}) = (3, 6, 9)^T$$

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$.

What is matrix multiplication?

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

A low-level definition

$$(A\vec{x})_i = \sum_{j=1}^{n} A_{ij} x_j$$

A low-level interpretation

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

In general...

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{a}^{(1)} & \vec{a}^{(2)} & \vec{a}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \vec{a}^{(1)} + x_2 \vec{a}^{(2)} + x_3 \vec{a}^{(3)}$$

Matrix Multiplication

$$\vec{x} = x_1 \hat{e}^{(1)} + x_2 \hat{e}^{(2)} + x_3 \hat{e}^{(3)} = (x_1, x_2, x_3)^T$$

$$\vec{f}(\vec{x}) = x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)})$$

$$A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$\begin{aligned} A\vec{x} &= \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{f}(\hat{e}^{(1)}) & \vec{f}(\hat{e}^{(2)}) & \vec{f}(\hat{e}^{(3)}) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= x_1 \vec{f}(\hat{e}^{(1)}) + x_2 \vec{f}(\hat{e}^{(2)}) + x_3 \vec{f}(\hat{e}^{(3)}) \end{aligned}$$

Matrix Multiplication

- Matrix A represents a linear transformation $\vec{f}$
  - With respect to the standard basis
  - If we use a different basis, the matrix changes!

- Matrix multiplication $A\vec{x}$ **evaluates** $\vec{f}(\vec{x})$

What are they, *really*?

- Matrices are sometimes just tables of numbers.
- But they often have a deeper meaning.

Main Idea

A square $(n \times n)$ matrix can be interpreted as a compact representation of a linear transformation $\vec{f} : \mathbb{R}^n \to \mathbb{R}^n$.

What's more, if A represents $\vec{f}$, then $A\vec{x} = \vec{f}(\vec{x})$; that is, multiplying by A is the same as evaluating $\vec{f}$.

Example

$$\vec{x} = 3\hat{e}^{(1)} - 4\hat{e}^{(2)} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

$$\vec{f}(\hat{e}^{(1)}) = -\hat{e}^{(1)} + 3\hat{e}^{(2)}$$

$$\vec{f}(\hat{e}^{(2)}) = 2\hat{e}^{(1)}$$

$$\vec{f}(\vec{x}) =$$

$$A =$$

$$A\vec{x} =$$

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.

Note

- All of this works because we assumed $\vec{f}$ is **linear**.
- If it isn't, evaluating $\vec{f}$ isn't so simple.
- Linear algebra = simple!