

# *DSC 140B*

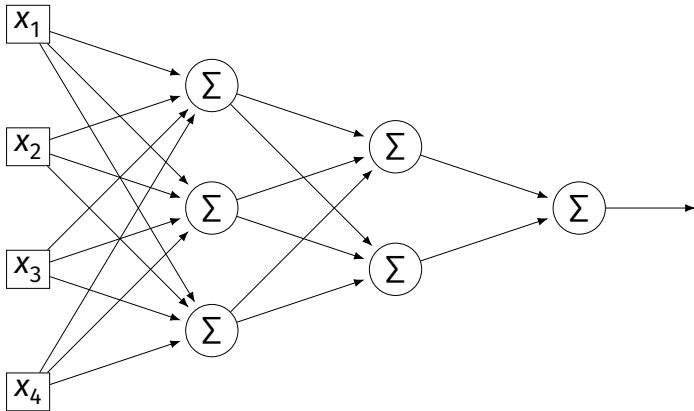
## *Representation Learning*

Lecture 14 | Part 1

**Training Neural Networks**

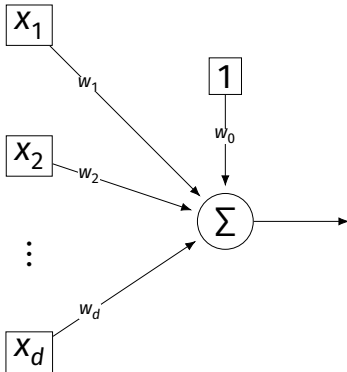
# Training

- How do we learn the weights of a (deep) neural network?



# Remember...

- How did we learn the weights in linear least squares regression?



# Empirical Risk Minimization

0. Collect a training set,  $\{(\vec{x}^{(i)}, y_i)\}$
1. Pick the form of the prediction function,  $H$ .
2. Pick a loss function.
3. Minimize the empirical risk w.r.t. that loss.

# Remember: Linear Least Squares

0. Pick the form of the prediction function,  $H$ .
  - ▶ E.g., linear:  $H(\vec{x}; \vec{w}) = w_0 + w_1 x_1 + \dots + w_d x_d = \text{Aug}(\vec{x}) \cdot \vec{w}$
1. Pick a loss function.
  - ▶ E.g., the square loss.
2. Minimize the empirical risk w.r.t. that loss:

$$R_{\text{sq}}(\vec{w}) = \frac{1}{n} \sum_{i=1}^n (H(\vec{x}^{(i)}) - y_i)^2 = \frac{1}{n} \sum_{i=1}^n (\text{Aug}(\vec{x}^{(i)}) \cdot \vec{w} - y_i)^2$$

# Minimizing Risk

- ▶ To minimize risk, we often use **vector calculus**.
  - ▶ Either set  $\nabla_{\vec{w}} R(\vec{w}) = 0$  and solve...
  - ▶ Or use gradient descent: walk in opposite direction of  $\nabla_{\vec{w}} R(\vec{w})$ .
- ▶ Recall,  $\nabla_{\vec{w}} R(\vec{w}) = (\partial R / \partial w_0, \partial R / \partial w_1, \dots, \partial R / \partial w_d)^T$

# In General

- ▶ Let  $\ell$  be the loss function, let  $H(\vec{x}; \vec{w})$  be the prediction function.
- ▶ The empirical risk:

$$R(\vec{w}) = \frac{1}{n} \sum_{i=1}^n \ell(H(\vec{x}^{(i)}; \vec{w}), y_i)$$

- ▶ Using the chain rule:

$$\nabla_{\vec{w}} R(\vec{w}) = \frac{1}{n} \sum_{i=1}^n \frac{\partial \ell}{\partial H} \nabla_{\vec{w}} H(\vec{x}^{(i)}; \vec{w})$$

# Gradient of $H$

- ▶ To minimize risk, we want to compute  $\nabla_{\vec{w}} R$ .
- ▶ To compute  $\nabla_{\vec{w}} R$ , we want to compute  $\nabla_{\vec{w}} H$ .
- ▶ This will depend on the form of  $H$ .



## Example: Linear Model

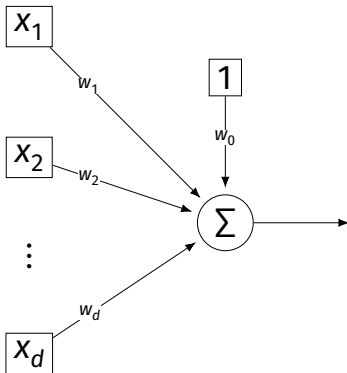
- ▶ Suppose  $H$  is a linear prediction function:

$$H(\vec{x}; \vec{w}) = w_0 + w_1 x_1 + \dots + w_d x_d$$

- ▶ What is  $\nabla_{\vec{w}} H$  with respect to  $\vec{w}$ ?

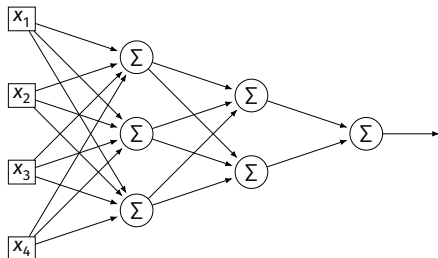
# Example: Linear Model

- Consider  $\partial H / \partial w_1$ :



# Example: Neural Networks

- ▶ Suppose  $H$  is a neural network (with nonlinear activations).
- ▶ What is  $\nabla H$ ?
  - ▶ It's more complicated...



# Parameter Vectors

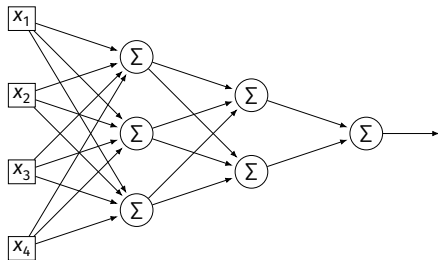
- ▶ It is often useful to pack all of the network's weights into a **parameter vector**,  $\vec{w}$ .
- ▶ Order is arbitrary:

$$\vec{w} = (W_{11}^{(1)}, W_{12}^{(1)}, \dots, b_1^{(1)}, b_2^{(1)}, W_{11}^{(2)}, W_{12}^{(2)}, \dots, b_1^{(2)}, b_2^{(2)}, \dots)^T$$

- ▶ The network is a function  $H(\vec{x}; \vec{w})$ .
- ▶ Goal of learning: find the “best”  $\vec{w}$ .

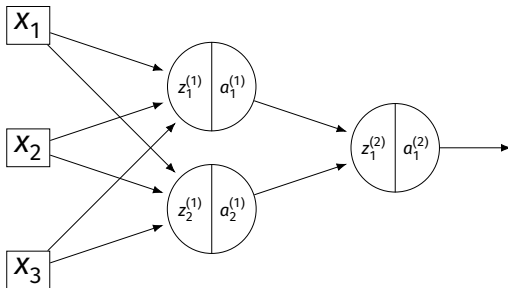
# Gradient of Neural Network

- ▶  $\nabla_{\vec{w}} H$  is a vector-valued function.
- ▶ Plugging a data point,  $\vec{x}$ , and a parameter vector,  $\vec{w}$ , into  $\nabla_{\vec{w}} H$  “evaluates the gradient”, results in a vector, same size as  $\vec{w}$ .



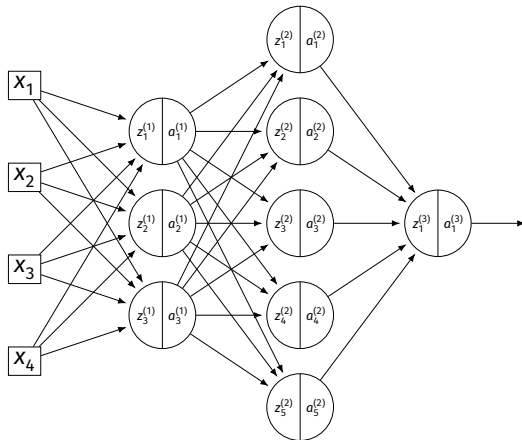
## Exercise

Suppose  $W_{11}^{(1)} = -2, W_{21}^{(1)} = -5, W_{31}^{(1)} = 2$  and  $\vec{x} = (3, 2, -2)^T$  and all biases are 0. ReLU activations are used. What is  $\partial H / \partial W_{11}^{(1)}(\vec{x}, \vec{w})$ ?



# Example

- Consider  $\partial H / \partial W_{11}^{(1)}$ :



# A Better Way

- ▶ Computing the gradient is straightforward...
- ▶ But can involve a lot of repeated work.
- ▶ **Backpropagation** is an algorithm for efficiently computing the gradient of a neural network.



# DSC 140B

## Representation Learning

Lecture 14 | Part 2

**Backpropagation**

# Gradient of a Network

- ▶ We want to compute the gradient  $\nabla_{\vec{w}} H$ .
  - ▶ That is,  $\partial H / \partial W_{ij}^{(\ell)}$  and  $\partial H / \partial b_i^{(\ell)}$  for all valid  $i, j, \ell$ .
- ▶ A network is a composition of functions.
- ▶ We'll make good use of the **chain rule**.

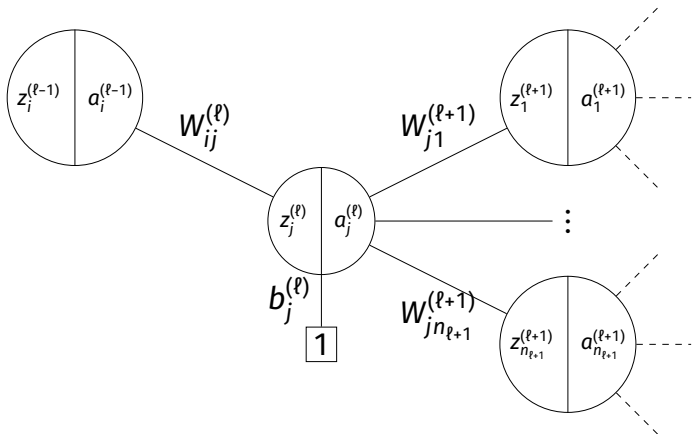
## Recall: The Chain Rule

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \frac{df}{dg} \frac{dg}{dx} \\ &= f'(g(x)) g'(x)\end{aligned}$$

## Some Notation

- ▶ We'll consider an arbitrary node in layer  $\ell$  of a neural network.
- ▶ Let  $g$  be the activation function.
- ▶  $n_\ell$  denotes the number of nodes in layer  $\ell$ .

# Arbitrary Node

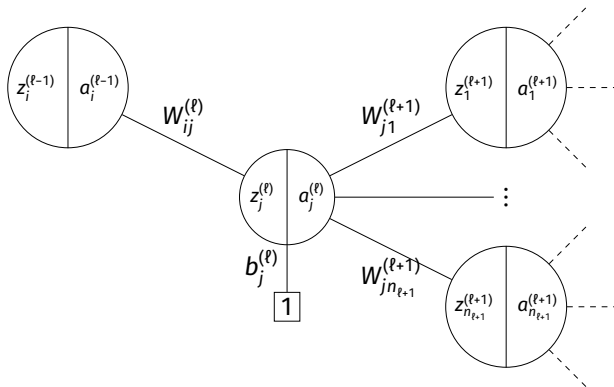


►  $\frac{\partial H}{\partial W_{ij}^{(\ell)}}?$

►  $\frac{\partial H}{\partial b_j^{(\ell)}}?$

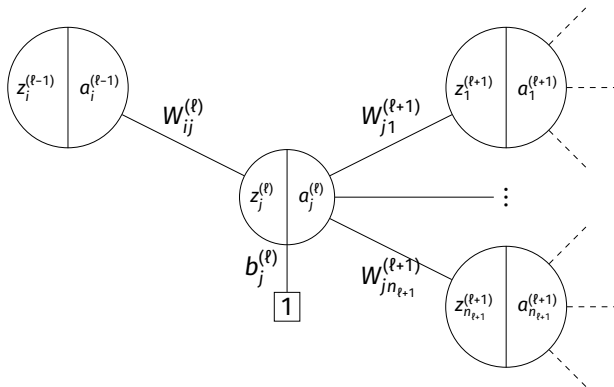
# Claim #1

$$\frac{\partial H}{\partial W_{ij}^{(\ell)}} = \frac{\partial H}{\partial z_j^{(\ell)}} a_i^{(\ell-1)}$$



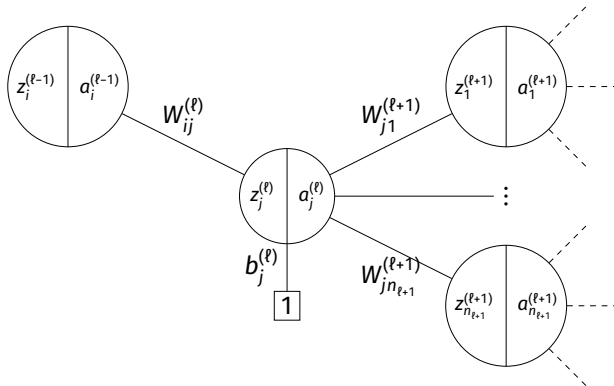
# Claim #2

$$\frac{\partial H}{\partial z_j^{(\ell)}} = \frac{\partial H}{\partial a_j^{(\ell)}} g'(z_j^\ell)$$



# Claim #3

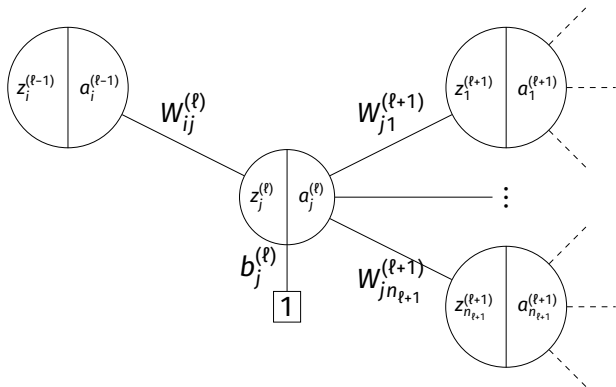
$$\frac{\partial H}{\partial a_j^{(\ell)}} = \sum_{k=1}^{n_{\ell+1}} \frac{\partial H}{\partial z_k^{(\ell+1)}} W_{jk}^{(\ell+1)}$$





## Exercise

What is  $\partial H / \partial b_j^{(\ell)}$ ?



# General Formulas

- ▶ For any node in any neural network<sup>1</sup>, we have the following recursive formulas:

- ▶  $\frac{\partial H}{\partial a_j^{(\ell)}} = \sum_{k=1}^{n_{\ell+1}} \frac{\partial H}{\partial z_k^{(\ell+1)}} W_{jk}^{(\ell+1)}$

- ▶  $\frac{\partial H}{\partial z_j^{(\ell)}} = \frac{\partial H}{\partial a_j^{(\ell)}} g'(z_j^{(\ell)})$

- ▶  $\frac{\partial H}{\partial W_{ij}^{(\ell)}} = \frac{\partial H}{\partial z_j^{(\ell)}} a_i^{(\ell-1)}$

- ▶  $\frac{\partial H}{\partial b_j^{(\ell)}} = \frac{\partial H}{\partial z_j^{(\ell)}}$

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<sup>1</sup>Fully-connected, feedforward network

## Main Idea

The derivatives in layer  $\ell$  depend on derivatives in layer  $\ell + 1$ .

# Backpropagation

- ▶ **Idea:** compute the derivatives in last layers, first.
- ▶ That is:
  - ▶ Compute derivatives in last layer,  $\ell$ ; store them.
  - ▶ Use to compute derivatives in layer  $\ell - 1$ .
  - ▶ Use to compute derivatives in layer  $\ell - 2$ .
  - ▶ ...

# Backpropagation

Given an input  $\vec{x}$  and a current parameter vector  $\vec{w}$ :

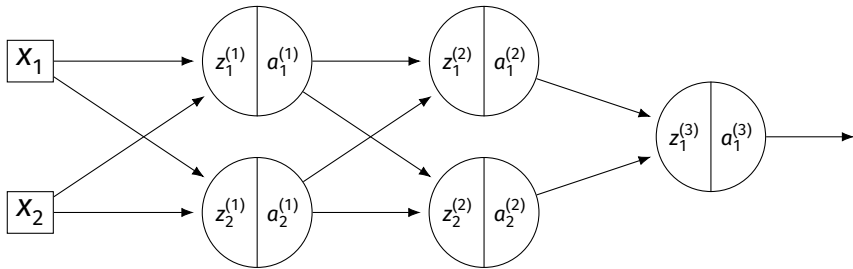
1. Evaluate the network to compute  $z_i^{(\ell)}$  and  $a_i^{(\ell)}$  for all nodes.
2. For each layer  $\ell$  from last to first:

- ▶ Compute  $\frac{\partial H}{\partial a_j^{(\ell)}} = \sum_{k=1}^{n_{\ell+1}} \frac{\partial H}{\partial z_k^{(\ell+1)}} W_{jk}^{(\ell+1)}$
- ▶ Compute  $\frac{\partial H}{\partial z_j^{(\ell)}} = \frac{\partial H}{\partial a_j^{(\ell)}} g'(z_j^{(\ell)})$
- ▶ Compute  $\frac{\partial H}{\partial W_{ij}^{(\ell)}} = \frac{\partial H}{\partial z_j^{(\ell)}} a_i^{(\ell-1)}$
- ▶ Compute  $\frac{\partial H}{\partial b_j^{(\ell)}} = \frac{\partial H}{\partial z_j^{(\ell)}}$

# Example

Compute the entries of the gradient given:

$$W^{(1)} = \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix} \quad W^{(2)} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad W^{(3)} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \quad \vec{x} = (2, 1)^T \quad g(z) = \text{ReLU}$$

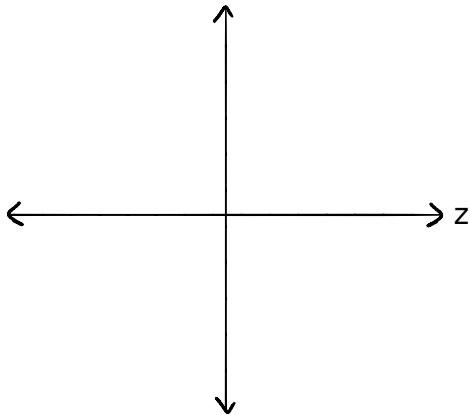


$$\frac{\partial H}{\partial a_j^{(\ell)}} = \sum_{k=1}^{n_{\ell+1}} \frac{\partial H}{\partial z_k^{(\ell+1)}} W_{jk}^{(\ell+1)} \quad \frac{\partial H}{\partial z_j^{(\ell)}} = \frac{\partial H}{\partial a_j^{(\ell)}} g'(z_j^{(\ell)}) \quad \frac{\partial H}{\partial W_{ij}^{(\ell)}} = \frac{\partial H}{\partial z_j^{(\ell)}} a_i^{(\ell-1)}$$

## Aside: Derivative of ReLU

$$g(z) = \max\{0, z\}$$

$$g'(z) = \begin{cases} 0, & z < 0 \\ 1, & z > 0 \end{cases}$$



## Summary: Backprop

- ▶ **Backprop** is an algorithm for efficiently computing the gradient of a neural network
- ▶ It is not an algorithm **you** need to carry out by hand: your NN library can do it for you.