DSC 190 - Homework 01

Due: Wednesday, April 6

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Unless otherwise noted by the problem's instructions, show your work or provide some justification for your answer. Homeworks are due via Gradescope at 11:59 PM.

Problem 1.

Consider a linear prediction function H used for binary classification, and assume that when the output of H is positive we predict for class +1, and when it's negative we predict for class -1. This means that the **decision boundary** is where $H(\vec{x}) = 0$.

In lecture, we saw that a linear prediction function has the form:

$$\begin{aligned} H(\vec{x}) &= w_0 + w_1 x_1 + \ldots + w_d x_d \\ &= \vec{w} \cdot \text{Aug}(\vec{x}) \end{aligned}$$

where $\vec{w} = (w_0, w_1, \ldots, w_d)^T$. In this problem, it will also be useful to define the vector $\vec{w}' = (w_1, \ldots, w_d)^T$, which is the same as $\vec{w}$ except that it does not include w_0. Note that $\vec{x} \cdot \vec{w}' = w_1 x_1 + \ldots w_d x_d$. With this definition, we can write $H(\vec{x})$ in a slightly different way:

$$H(\vec{x}) = w_0 + \vec{w}' \cdot \vec{x}$$

Remember this formula, as it will be useful several times below!

Over the course of this problem, we'll answer the question: how is the magnitude of $H(\vec{x})$ related to the distance between $\vec{x}$ and the decision boundary?

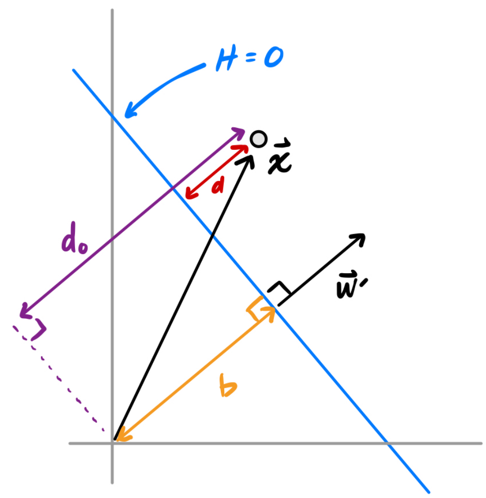
Note: for this problem it may be useful to review the properties of the dot product and vector algebra. In particular, remember that $\|u\|$ denotes the norm (length) of a vector, and that $\vec{u} \cdot \vec{u} = \|u\|^2$. By dividing a vector by its norm, as in $\vec{u}/\|\vec{u}\|$, we obtain a *unit vector* with unit length – a unit vector is useful for specifying a direction. To find the component of a vector $\vec{u}$ that points in the same direction as another vector $\vec{v}$, we write $\vec{u} \cdot \vec{v}/\|\vec{v}\|$. Also remember that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, and that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{w}$.

a) Suppose that $\vec{z}$ is a point on the decision boundary, which implies that $H(\vec{z}) = 0$. Show that $\vec{w}' \cdot \vec{z} = -w_0$.

b) Argue that $\vec{w}'$ is orthogonal to the decision boundary.

Hint: take two arbitrary points $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ that are assumed to be on the decision boundary. Then, since we know that the boundary is linear (it is a line, plane, etc.), the difference of these vectors, $\vec{\delta} = \vec{x}^{(1)} - \vec{x}^{(2)}$ is parallel to the boundary. To show that $\vec{w}'$ is orthogonal to the boundary, it suffices to show that $\vec{w}' \cdot \vec{\delta} = 0$. Make sure you somehow use the fact that $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are on the decision boundary.

c) Now that we know that $\vec{w}'$ is orthogonal to the decision boundary, we can draw a better picture of the situation:



The blue line is the decision boundary – it is where $H = 0$. We have drawn an arbitary point $\vec{x}$, along with several distances:

- d: the (signed) distance from the decision boundary to $\vec{x}$
- b: the distance from the the origin to the decision boundary
- d_0: the length of the component of $\vec{x}$ that is orthogonal to the decision boundary.

We're most interested in knowing d. First, though, we need to find b.

Consider the vector $b\vec{w}'/\|\vec{w}'\|$; this is a vector from the origin to the decision boundary that is orthogonal to the boundary and with length b.

Since this vector is on the decision boundary, $H(b\vec{w}'/\|\vec{w}'\|) = 0$. Using this fact, show that $b = -\dfrac{w_0}{\|\vec{w}'\|}$.

Hint: first show that $H(b\vec{w}'/\|\vec{w}'\|) = b\|\vec{w}'\| + w_0$, then set this to zero and solve for b.

d) Recall that d_0 is the component of $\vec{x}$ that is orthogonal to the decision boundary; this is simply $\vec{x} \cdot \vec{w}'/\|\vec{w}'\|$. From the picture, $d_0 = d + b$. We know d_0 and b, and can therefore solve for d.

Use this to show that $|d| = |H(\vec{x})|/\|\vec{w}'\|$.

We have shown that the distance between $\vec{x}$ and the decision boundary is proportional to the output of the prediction function, $H(\vec{x})$. This gives us a very useful interpretation of $|H(\vec{x})|$! For example, this means if $H(\vec{x}^{(1)}) > H(\vec{x}^{(2)}) > 0$, then $\vec{x}^{(1)}$ is further from the decision boundary than $\vec{x}^{(2)}$.