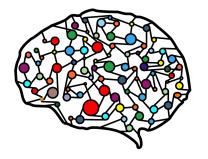# Lecture 13 – Combinatorics

DSC 40A, Fall 2021 @ UC San Diego
Suraj Rampure, with help from **many others**

Announcements

- ▶ **Please submit Survey 5!**

- ▶ Groupwork 6 due **Friday 11/12 at 11:59pm**.

- ▶ Homework 6 due **Tuesday 11/16 at 11:59pm**.

- ▶ Homework 4 grades are out.

- ▶ **Note:** No lecture or OH on Thursday.
  - ▶ Use it to take a break! :)

Agenda

- ▶ Sequences, permutations, and combinations.

- ▶ Practice problems.

Sequences, permutations, and combinations

Motivation

- ▶ Many problems in probability involve counting.
    - ▶ Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?

    - ▶ Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?

- ▶ In order to solve such problems, we first need to learn how to count.

- ▶ The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

▶ Many experiments involve choosing *k* elements randomly from a group of *n* possible elements. This group is called a **population**.

  ▶ If drawing cards from a deck, the population is the deck of all cards.
  ▶ If selecting people from DSC 40A, the population is everyone in DSC 40A.

▶ Two decisions:

  ▶ Do we select elements with or without **replacement**?
  ▶ Does the **order** in which things are selected matter?

Sequences

▶ A **sequence** of length *k* is obtained by selecting *k* elements from a group of *n* possible elements **with replacement**, such that **order matters**.

▶ **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.

▶ **Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is n^k.

(Note: We mentioned this fact in the first lecture on clustering!)

Permutations

▶ A **permutation** is obtained by selecting *k* elements from a group of *n* possible elements **without replacement**, such that **order matters**.

▶ **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

Permutations

▶ In general, the number of ways to select *k* elements from a group of *n* possible elements such that **repetition is not allowed** and **order matters** is

$$P(n, k) = (n)(n - 1)...(n - k + 1)$$

▶ To simplify: recall that the definition of *n*! is

$$n! = (n)(n - 1)...(2)(1)$$

▶ Given this, we can write

$$P(n, k) = \frac{n!}{(n - k)!}$$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?
- A) 21
- B) 210
- C) 343
- D) 2187
- E) None of the above

To answer, go to `menti.com` **and enter 3779 0977.**

Special case of permutations

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

- This is consistent with the formula

$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Combinations

- ▶ A **combination** is a set of *k* items selected from a group of *n* possible elements **without replacement**, such that **order does not matter**.

- ▶ **Example:** There are 24 ice cream flavors. How many ways can you pick two flavors?

From permutations to combinations

▶ There is a close connection between:
  ▶ the number of permutations of k elements selected from a group of n, and

  ▶ the number of combinations of k elements selected from a group of n

$$\text{\# combinations} = \frac{\text{\# permutations}}{\text{\# orderings of } k \text{ items}}$$

▶ Since # permutations = $\frac{n!}{(n-k)!}$ and
  # orderings of k items = $k!$, we have

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Combinations

In general, the number of ways to select *k* elements from a group of *n* elements such that **repetition is not allowed** and **order does not matter** is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The symbol $\binom{n}{k}$ is pronounced "*n* choose *k*", and is also known as the **binomial coefficient**.

Example: committees

- ▶ How many ways are there to select a president, vice president, and secretary from a group of 8 people?

- ▶ How many ways are there to select a committee of 3 people from a group of 8 people?

- ▶ If you're ever confused about the difference between permutations and combinations, **come back to this example**.

Probability examples

Counting and probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.

- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.

- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Selecting students — overview

We're going to start by answering the same question using several different techniques.

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Billy?

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 4: "the easy way")

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Billy (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

   A) Equal to
   B) Greater than
   C) Less than

To answer, go to `menti.com` **and enter 3779 0977.**

Another example

Question 2: Suppose we flip a fair coin 10 times. What is the probability that we see the exact sequence HTTHTTHHHT?

Yet another example

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

Question to think about

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. What is the probability that we see an equal number of heads and tails?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
    - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
    - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
    - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.