Lecture 19 – Combinatorics

DSC 40A, Fall 2022 @ UC San Diego
Dr. Truong Son Hy, with help from **many others**

Agenda

- Sequences, permutations, and combinations.
- Practice problems.

Sequences, permutations, and combinations

Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.

$$52^4$$

- **Example:** A UCSD PID starts with “A” then has 8 digits. How many UCSD PIDs are possible?

$$10^8 = 100,000,000$$

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is n^k.

(Note: We mentioned this fact in the first lecture on clustering!)

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.
- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?
  President: 8 choices
  Vice: 7 choices
  Secretary: 6 choices
  Total: $8 \cdot 7 \cdot 6 = 336$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n,k) = (n)(n-1)\ldots(n-k+1)$$

- To simplify: recall that the definition of $n!$ is

$$n! = (n)(n-1)\ldots(2)(1)$$

- Given this, we can write

$$P(n,k) = \frac{n!}{(n-k)!}$$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

A) 21
B) 210
C) 343
D) 2187
E) None of the above

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

A) 21
B) 210
C) 343
D) 2187
E) None of the above

Answer: B) $7 \cdot 6 \cdot 5 = 210$

Special case of permutations

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$$

- This is consistent with the formula

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Combinations

- A **combination** is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**.
- **Example:** There are 24 ice cream flavors. How many ways can you pick two flavors?

$$\frac{24 \cdot 23}{2} = 276$$

From permutations to combinations

- There is a close connection between:
  - the number of permutations of k elements selected from a group of n, and
  - the number of combinations of k elements selected from a group of n

$$\#\text{ combinations} = \frac{\#\text{ permutations}}{\#\text{ orderings of } k \text{ items}}$$

- Since # permutations = $\frac{n!}{(n-k)!}$ and # orderings of k items = $k!$, we have

$$C(n,k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

Example: committees

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$8 \cdot 7 \cdot 6 = 336$$

- How many ways are there to select a committee of 3 people from a group of 8 people?

$$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{6} = 56$$

- If you're ever confused about the difference between permutations and combinations, **come back to this example**.

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in the set of dominoes?

A) $\binom{7}{2}$

B) $\binom{7}{1} + \binom{7}{2}$

C) $P(7, 2)$

D) $\frac{P(7,2)}{P(7,1)} 7!$

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.
How many dominoes are in the set of dominoes?

A) $\binom{7}{2}$
B) $\binom{7}{1} + \binom{7}{2}$
C) $P(7,2)$
D) $\frac{P(7,2)}{P(7,1)} 7!$

Answer: A

More examples

Counting and probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Selecting students — overview

We're going to start by answering the same question using several different techniques.

Question 1: There are 20 students in a class. Billy is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Billy is among the 5 selected students?

Selecting students (Method 1: using permutations)

The total number of permutations is: $n! = 20!$
If Billy ranks 1st then, there are exactly $(n-1)! = 19!$ permutations so.
If Billy ranks 2nd then, there are exactly $(n-1)! = 19!$ permutations so.

...

If Billy ranks 5th then, there are exactly $(n-1)! = 19!$ permutations so.

$$|A| = 19! + 19! + 19! + 19! + 19! = 5 \cdot 19!$$

We have $|S| = 20!$. The probability is:

$$\frac{|A|}{|S|} = \frac{5 \cdot 19!}{20!} = \frac{1}{4} = 25\%$$

Selecting students (Method 2: using permutations and the complement)

If Billy ranks k-th then, there are exactly $(n - 1)! = 19!$ permutations so.
Probability that Billy is not in the top 5 is:

$$\frac{\sum_{k=6}^{20} 19!}{20!} = \frac{15 \cdot 19!}{20!} = \frac{15}{20} = \frac{3}{4}$$

Probability that Billy is in the top 5 is: $1 - \frac{3}{4} = \frac{1}{4} = 25\%$.

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

$$|S| = \binom{20}{5}$$

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Billy?

Suppose Billy is already in top 5, we need to select 4 more people out of 19. Thus:

$$|A| = \binom{19}{4}$$

Selecting students (Method 3: using combinations)

The probability that Billy is in top 5 is:

$$\frac{|A|}{|S|} = \frac{\binom{19}{4}}{\binom{20}{5}} = \frac{\frac{19!}{15!4!}}{\frac{20!}{15!5!}} = \frac{5}{20} = \frac{1}{4} = 25\%$$

Selecting students (Method 4: “the easy way”)

We leave out Billy for now. There are 19 other people and 19! ways to rank them. For each way, we try to add Billy in with the top 4, and there are 5 ways to do so. By the **multiplication rule**, we have:

$$|A| = 5 \cdot 19!$$

Therefore, the final result is still: 25%.

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Billy (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

A) Equal to
B) Greater than
C) Less than

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Billy (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

A) Equal to
B) Greater than
C) Less than

Answer: C) Less than. Let's justify it mathematially!

Here, we will use the **complement rule**. We have: $|S| = 20^5$.
The number of ways to sample 5 people without ever picking up Billy is: 19^5.
The result **with replacement** becomes:

$$1 - \left(\frac{19}{20}\right)^5 \approx 22.62\% < 25\%$$

Another example

Question 2, Part 1: We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets?

$$\binom{12}{4} = 495$$

Another example

Question 2, Part 2: We have 12 pets, 5 dogs and 7 cats. In how many ways can we select 4 pets such that we have...

1. 2 dogs and 2 cats?

$$\binom{5}{2} \cdot \binom{7}{2} = 10 \cdot 21 = 210$$

2. 3 dogs and 1 cat?

$$\binom{5}{3} \cdot \binom{7}{1} = 10 \cdot 7 = 70$$

3. At least 2 dogs?

$$\binom{5}{2} \cdot \binom{7}{2} + \binom{5}{3} \cdot \binom{7}{1} + \binom{5}{4} \cdot \binom{7}{0} = 210 + 70 + 5 = 285$$

Another example

Question 2, Part 3: We have 12 pets, 5 dogs and 7 cats. We randomly select 4 pets. What's the probability that we selected at least 2 dogs?

$$\frac{285}{495} \approx 57.57\%$$

Yet another example

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
$$\frac{1}{2^{10}}$$
2. What is the probability that we see an equal number of heads and tails?
The number of ways to choose 5 out of 10 places is: $\binom{10}{5}$.
For the chosen places, we set them 1 and the rest to be 0.
The final result is:
$$\frac{\binom{10}{5}}{2^{10}}$$

One step further

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?

$$\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^5 = \frac{2^5}{3^{10}}$$

2. What is the probability that we see an equal number of heads and tails?

$$\binom{10}{5} \cdot \frac{2^5}{3^{10}}$$

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.