DSC 40A - Group Work Session 1

due January 5, 2022 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **tag all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive credit, you must work in a group of two to four students for at least 50 minutes, at one of the scheduled groupwork sessions. You may not do the groupwork alone or meet outside of the scheduled sessions.

1 Summation Notation

You can often verify for yourself if something is true about summation notation by "expanding" the summation symbol and seeing if the property holds. For instance, suppose we want to see if it is true that

$$\sum_{i=1}^{n} c \cdot x_i = c \sum_{i=1}^{n} x_i$$

We start by "expanding" $\displaystyle\sum_{i=1}^{n} c \cdot x_i$:

$$\sum_{i=1}^{n} c \cdot x_i = cx_1 + cx_2 + cx_3 + \ldots + cx_n$$

Now we see that the c can be factored out:

$$\begin{aligned} &= c(x_1 + x_2 + x_3 + \ldots + x_n) \\ &= c \sum_{i=1}^{n} x_i. \end{aligned}$$

This is a simple proof that the property is true. On the other hand, we can prove that a property doesn't hold in the same way: by expanding both sides and showing that they are not equal.

Problem 1.

Show that $\displaystyle\sum_{i=1}^{n}(x_i + y_i) = \left(\sum_{i=1}^{n} x_i\right) + \left(\sum_{i=1}^{n} y_i\right)$.

Solution:

$$\begin{aligned}\sum_{i=1}^{n}(x_i + y_i) &= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n)\\ &= (x_1 + x_2 + \ldots x_n) + (y_1 + y_2 + \ldots y_n)\\ &= \left(\sum_{i=1}^{n} x_i\right) + \left(\sum_{i=1}^{n} y_i\right)\end{aligned}$$

Problem 2.

Find a simple expression for $\sum_{i=1}^{n} c$ not involving summation notation. Show that your expression is correct.

Solution: The simple expression is $c * n$, as shown by expanding the sum:

$$\begin{aligned}\sum_{i=1}^{n} c &= c + c + \cdots + c\\ &= c * n.\end{aligned}$$

2 Inequalities

Inequalities are a fundamental part of mathematical proofs. We will go over the basic properties to brush up on things.

- **Law of Trichotomy:** For all $x, y \in \mathbb{R}$, either $x < y$, $x = y$ or $x > y$..
- **Transitive property:** For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
- **Addition property:** For all $x, y, c \in \mathbb{R}$, if $x \leq y$, then $x + c \leq y + c$.
- **Multiplication property:** For all $x, y \in \mathbb{R}$, if $x \leq y$, then
  - $cx \leq cy$ for any $c \geq 0 \in \mathbb{R}$, and
  - $cx \geq cy$ for any $c \leq 0 \in \mathbb{R}$.

Problem 3.

Suppose $a \leq b$ and $c \leq d$. Which of the statements below are always true? Remember to justify your answers.

1. $a + c \leq b + d$
2. $a - c \leq b + d$
3. $a \leq bc$
4. $ac \leq bd$
5. $|ac| \leq |bd|$
6. $a^2 \leq b^2$
7. $\min(a, c) \leq \min(b, d)$
8. $\min(a, c) \leq \max(b, d)$
9. $\min(a, \max(b, d)) \leq \min(c, \max(b, d))$
10. $\min(a, \max(b, d)) \leq \max(b, d)$

Solution:

1. **True.** We know $a \le b$. First, we add d to both sides, $a + d \le b + d$. We also know that $c \le d$. Adding a to both sides gives us $a + c \le a + d$. Then, we use transitivity to say that $a + c \le b + d$.
2. **Not always true.** Pick $a = 3, b = 3, c = -2, d = -1$. Then, $a - c = 5 > b + d = 2$.
3. **Not always true.** Pick $a = 3, b = 3, c = -2, d = -1$. Then, $a = 3 > bc = -6$.
4. **Not always true.** Pick $a = 3, b = 10, c = -2, d = -1$. Then, $ac = -6 > bd = -10$.
5. **Not always true**. Pick $a = -100, b = 10, c = 1, d = 2$. Then, $|ac| = 100 > |bd| = 20$.
6. $a^2 \le b^2$. **Not always true**. Pick $a = -2, b = 0$. Then, $a^2 = 4 > b^2 = 0$.
7. $\min(a, c) \le \min(b, d)$. **True**. $\min(a, c) \le a$ and $\min(a, c) \le c$. Also, $a \le b$ and $c \le d$. By transitivity, $\min(a, c) \le b$ and $\min(a, c) \le d$. Since $\min(b, d)$ is either b or d and $\min(a, c)$ is smaller than or equal to both, $\min(a, c) \le \min(b, d)$.
8. $\min(a, c) \le \max(b, d)$. **True**. Using the same argument above, $\min(a, c) \le b$ and $\min(a, c) \le d$. Hence, $2 * \min(a, c) \le b + d \le 2 * \max(b, d)$. Then, $\min(a, c) \le \max(b, d)$
9. $\min(a, \max(b, d)) \le \min(c, \max(b, d))$. **Not always true**. Pick $a = 10, b = 20, c = -2, d = 100$. Then, $\min(10, \max(20, 100)) = 10 > \min(-2, \max(20, 100)) = -2$.
10. $\min(a, \max(b, d)) \le \max(b, d)$. **True**. Let $\max(b, d) = e$. Then the equation becomes $\min(a, e) \le e$. But we know that $\min(a, e) \le a$ and $\min(a, e) \le e$!

3 Chaining Inequalities

Suppose we have collected a bunch of numbers, $y_1, \ldots, y_n$. Let's assume, too, that these numbers are in sorted order, so that $y_1 \le y_2 \le \ldots \le y_n$.

The *midpoint* of $y_1, \ldots, y_n$ is the average of the smallest and largest number:

$$\text{midpoint} = \frac{y_1 + y_n}{2}.$$

Intuitively, the midpoint is at most y_n and is at least y_1; it lies somewhere in the middle of these two numbers. We can easily prove this with a *chain* of inequalities.

First, we show that the midpoint is at most y_n. We start with the definition:

$$\text{midpoint} = \frac{y_1 + y_n}{2}$$

We can do anything to the right hand side that makes it bigger, keeping in mind that we're trying to get it to look like y_n. Right now there is y_1 hanging out; can we simply change it to a y_n? Yes! Remember that $y_n \ge y_1$, so this would make the right hand side bigger. Therefore, we have to write $\le$:

$$\le \frac{y_n + y_n}{2}$$

We can simplify this:

$$= \frac{2y_n}{2}$$

Notice that we wrote $=$ on the last line, not $\le$. This is because the line is indeed equal to the one before it.

$$= y_n$$

We have made a chain of inequalities and equalities; this one looks like $=, \leq, =, =$. Since $\leq$ is the "weakest link" in the chain, the strongest statement we can make is that the midpoint is $\leq y_n$, but this is what we wanted to say.

Problem 4.

Prove that the midpoint is $\geq y_1$.

Solution:

$$\begin{aligned} y_1 &\leq y_n \\ y_1 + y_1 &\leq y_n + y_1 \\ \frac{2y_1}{2} &\leq \frac{y_n + y_1}{2} \\ y_1 &\leq \mathit{midpoint} \end{aligned}$$

Problem 5.

Suppose $y_1, \ldots, y_n$ are all positive numbers. The *geometric mean* of $y_1, \ldots, y_n$ is defined to be:

$$(y_1 \cdot y_2 \cdots y_n)^{1/n}.$$

Prove that the geometric mean is less than or equal to y_n and greater than or equal to y_1 using a chain of inequalities.

Solution: Assuming the numbers are ordered, let's first show that the geometric mean $\geq y_1$. We know that the below inequalities hold by definition .

$$\begin{aligned} y_1 &\leq y_1 \\ y_1 &\leq y_2 \\ y_1 &\leq y_3 \\ y_1 &\leq y_4 \\ &\ldots \\ y_1 &\leq y_n \end{aligned}$$

Since $y_i > 0 \ \forall i$, we can multiply the n inequalities to get

$$y_1 y_1 \ldots y_1 \leq y_1 y_2 \ldots y_n$$

So,

$$\begin{aligned} y_1^n &\leq y_1 y_2 \ldots y_n \\ (y_1^n)^{1/n} &\leq (y_1 y_2 \ldots y_n)^{1/n} \\ y_1 &\leq \mathit{geometric\ mean} \end{aligned}$$

You can similarly show that *geometric mean* $\leq y_n$ by using the fact that $y_i \leq y_n$ for $i = 1, 2, \ldots n$.