Lecture 18 - Probabability and Combinatorics Examples

DSC 40A, Spring 2023

Announcements

- Homework 5 is due **tomorrow at 11:59pm**.
- RSVP for the DSC Undergrad Town Hall tomorrow from 1:30-3:30pm in the SDSC Auditorium.
  - A chance to talk about what's going on in the department, raise concerns, talk to professors, etc.

Agenda

- Review of combinatorics.
- Lots of examples.

Review of combinatorics

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** You roll a die 10 times. How many different sequences of results are possible?

$$\underset{1^{st}}{\underline{6}} \cdot \underset{2^{nd}}{\underline{6}} \cdot \underset{3^{rd}}{\underline{6}} \cdots \underset{10^{th}}{\underline{\quad}}$$

$$\boxed{6^{10}}$$

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is

$$\boxed{n^k.}$$

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$$\underset{\text{pres}}{\underline{8}} \cdot \underset{\text{vp}}{\underline{7}} \cdot \underset{\text{sec}}{\underline{6}}$$

$$P(8,3) = \frac{8!}{5!} = 8 \cdot 7 \cdot 6$$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$\boxed{\begin{aligned} P(n,k) &= (n)(n-1)\ldots(n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}}$$

Combinations

– which elements are included

- A **combination** is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**.

- **Example:** How many ways are there to select a committee of 3 people from a group of 8 people?

A B C
A C B
B A C
B C A
C B A
C A B

$$\frac{P(8,3)}{3!} = \underbrace{C(8,3)}_{\text{choose a set of 3 people from 8}}$$

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

$$\boxed{\begin{aligned} C(n,k) &= \binom{n}{k} \leftarrow \text{notation} \\ &= \frac{P(n,k)}{k!} \leftarrow \\ &= \frac{n!}{(n-k)!k!} \leftarrow \end{aligned}}$$

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

order matters?

	Yes	NO
with replacement	sequence	dominoes
without replacement	permutation	combinations

Lots of examples

Sum of combinations

options for each half are 0, 1, 2, 3, 4, 5, 6

also need doubles

0 1 2 3 4 5 6

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.
How many dominoes are in the set of dominoes?

a) ~~$\binom{7}{2}$~~ $= C(7,2)$ gives dominoes with two different halves

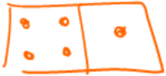
b) $\binom{7}{1} + \binom{7}{2}$ $= \underbrace{C(7,1)}_{7} + C(7,2)$

c) ~~$P(7,2)$~~

d) ~~$\frac{P(7,2)}{P(7,1)} 7!$~~

Counts

$\neq$



with/without replacement? with

order matters? no

Selecting students — overview

We're going answer the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

all equally likely to be chosen

5 different people
no repeats

give "names" to our students:

$A, B, C, D, \ldots, T = 20$ students

A = Avi

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = a permutation (ordered selection) of 5 students chosen from A, B, . . ., T

ex.) LFGAT → are all perms equally likely? yes!

$$\text{prob A included} = \frac{\#\text{perms including A}}{\text{total \#perms}}$$

denom.

$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

$= P(20, 5)$

numerator: # perms including A

ex.) ACJTE

how many perms include A?

A $\underset{19}{\_}$ $\underset{18}{\_}$ $\underset{17}{\_}$ $\underset{16}{\_}$ → $19\cdot 18\cdot 17\cdot 16$

just counts perms that start with A, not perms that include A because A can be elsewhere

5 cases:

1) A _ _ _ _ $19\cdot 18\cdot 17\cdot 16$

2) _ A _ _ _ $19\cdot 18\cdot 17\cdot 16$

⋮

total # perms is $\boxed{5\cdot 19\cdot 18\cdot 17\cdot 16}$

$= 5\cdot P(19,4)$

$$\frac{\text{num}}{\text{denom}} = \frac{5 \cdot P(19,4)}{P(20,5)} = \frac{5 \cdot \frac{19!}{15!}}{\left(\frac{20!}{15!}\right)}$$

$$= 5 \cdot \frac{19!}{\cancel{15!}} * \frac{\cancel{15!}}{20!}$$

$$= 5 \cdot \frac{1}{20}$$

$$= 5/20 = \boxed{1/4}$$

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{\#\text{ perms including A}}{\text{total }\#\text{ perms}} = \frac{\text{total }\#\text{ perms} - \#\text{ perms not including A}}{\text{total }\#\text{ perms}}$$

$\#$ perms not including A $= 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 = P(19,5)$

ex.) CDJTL

$$\frac{P(20,5) - P(19,5)}{P(20,5)} = \frac{1}{4}$$

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

S = sets of 5 students, chosen from A, B, ..., T

ex.) {B, D, G, H, M}

$$\frac{\text{\# sets of 5 students including A}}{\text{\# sets of 5 students}}$$

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

$$C(20,5) = \binom{20}{5} = \frac{20!}{15!\,5!}$$

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

$C(19, 4)$

from among the other 19 students B, C, ..., T

choose 4 to go with Avi

to try: use complement and combinations. should still give 1/4

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$\frac{C(19,4)}{C(20,5)} = \frac{\frac{19!}{15!\,4!}}{\left(\frac{20!}{15!\,5!}\right)} = \frac{19!}{\cancel{15!}\,4!} \times \frac{\cancel{15!}\,5!}{20!}$$

$$= \frac{5}{20}$$

$$= 1/4$$

Selecting students (Method 4: “the easy way”)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

think of selecting 5 students as follows:

randomize all 20 students, line them up in random order, take first five

S = positions where A may end up

$$\frac{\#\text{"good" positions}}{\#\text{ positions}} = \frac{5}{20} = \frac{1}{4}$$

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\boxed{\frac{1}{4}.}$

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?
3. At least 2 markers?

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

Fair coin

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Unfair coin

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.