

Lecture 6 – Gradient Descent in Action



DSC 40A, Spring 2023

Announcements

- ▶ Homework 2 is due **Tuesday at 11:59pm.**
 - ▶ Come to office hours before then for help!
 - ▶ See dsc40a.com/calendar for the office hours schedule.
- ▶ Solutions to Groupwork 2 and Homework 1 are available on Campuswire.
 - ▶ Reviewing them will help you write better solutions in future assignments.
 - ▶ You should also make sure you know how to do all groupwork and homework questions for exams.

Agenda

- ▶ Brief recap of Lecture 5.
- ▶ Gradient descent demo.
- ▶ When is gradient descent guaranteed to work?

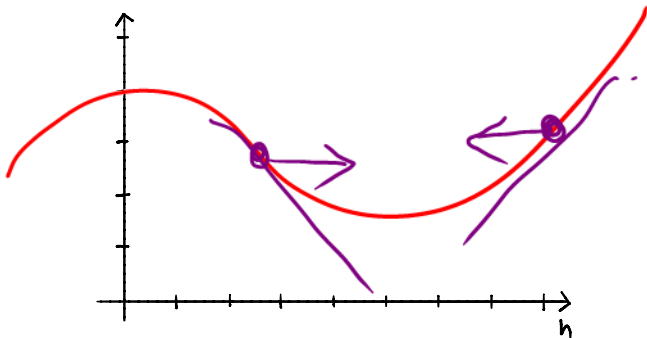
Gradient descent fundamentals

The general problem

- ▶ **Given:** a differentiable function $R(h)$.
- ▶ **Goal:** find the input h^* that minimizes $R(h)$.

Key idea behind **gradient descent**

- ▶ If the slope of R at h is **positive** then we'll **decrease** h .
- ▶ If the slope of R at h is **negative** then we'll **increase h .**



Gradient descent

- ▶ Pick a positive constant, α , for the **learning rate**.
- ▶ Pick a starting prediction, h_0 .
- ▶ Repeatedly apply the gradient descent update rule.

$$h_i = h_{i-1} - \alpha \cdot \frac{dR}{dh}(h_{i-1})$$

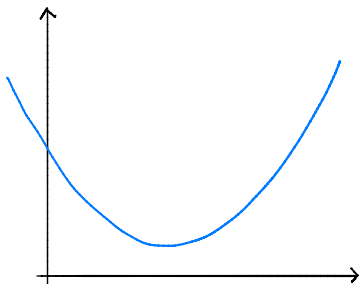
- old pred* \rightarrow *minus* \leftarrow *a multiple of slope*
- ▶ Repeat until convergence (when h doesn't change much).

Gradient descent demo

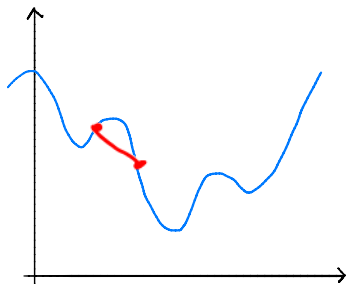
Let's see gradient descent in action. [Follow along here.](#)

When is gradient descent guaranteed to work?

Convex functions



Convex



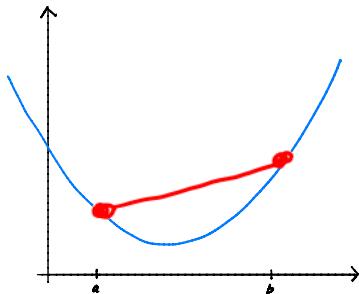
Non-convex

Convexity: Definition

- f is **convex** if for **every** a, b in the domain of f , the line segment between

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

does not go below the plot of f .

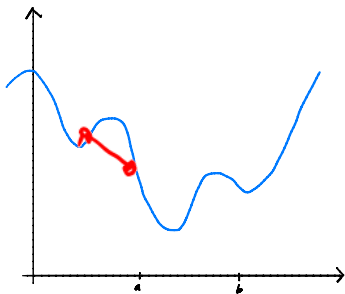


Convexity: Definition

- f is **convex** if for **every** a, b in the domain of f , the line segment between

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

does not go below the plot of f .



Convexity: Formal definition

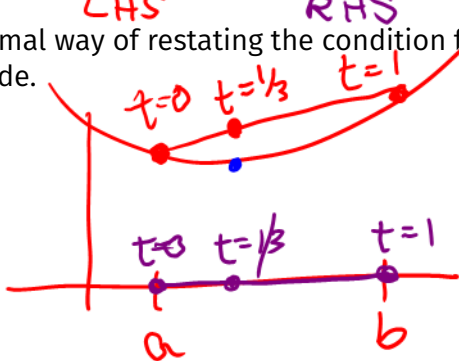
- ▶ A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **convex** if for every choice of a, b and $t \in [0, 1]$:

line above function

$$\underbrace{(1-t)f(a) + tf(b)}_{\text{LHS}} \geq \underbrace{f((1-t)a + tb)}_{\text{RHS}}$$

- ▶ This is a formal way of restating the condition from the previous slide.

parameterized line



Discussion Question

Which of these functions is not convex?

a) $f(x) = |x|$

b) $f(x) = e^x$

c) $f(x) = \sqrt{x-1}$

d) $f(x) = (x-3)^{24}$



Why does convexity matter?

- ▶ Convex functions are (relatively) easy to minimize with gradient descent.
- ▶ **Theorem:** if $R(h)$ is convex and differentiable then gradient descent converges to a **global minimum** of R *provided* that the step size is small enough.
- ▶ **Why?**
 - ▶ If a function is convex and has a local minimum, that local minimum must be a global minimum.
 - ▶ In other words, gradient descent won't get stuck/terminate in local minimums that aren't global minimums.

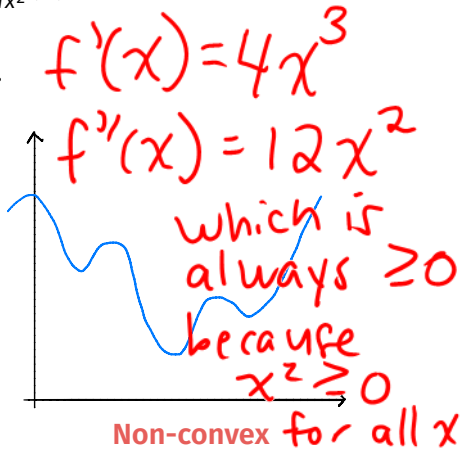
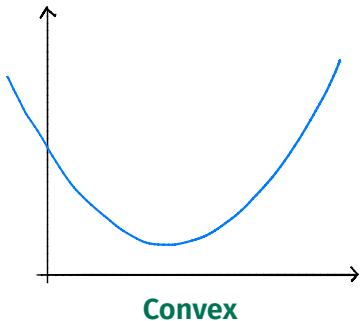
Nonconvexity and gradient descent

- ▶ We say a function is **nonconvex** if it does not meet the criteria for convexity.
- ▶ Nonconvex functions are (relatively) hard to minimize.
- ▶ Gradient descent can still be useful, but it's not guaranteed to converge to a global minimum.
 - ▶ We saw this when trying to minimize $R_{ucsd}(h)$ with a smaller σ .



Second derivative test for convexity

- ▶ If $f(x)$ is a function of a single variable and is twice differentiable, then:
- ▶ $f(x)$ is convex if and only if $\frac{d^2f}{dx^2}(x) \geq 0$ for all x .
- ▶ Example: $f(x) = x^4$ is convex.



$$f'(x) = 4x^3$$

$$f''(x) = 12x^2$$

which is
always ≥ 0
because
 $x^2 \geq 0$

Convexity of empirical risk

- ▶ If $L(h, y)$ is a convex function (when y is fixed) then

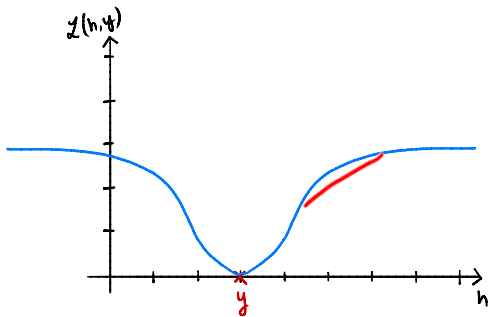
$$R(h) = \frac{1}{n} \sum_{i=1}^n L(h, y_i)$$

is convex.

- ▶ More generally, sums of convex functions are convex.
- ▶ What does this mean?
 - ▶ If a loss function is convex, then the corresponding empirical risk will also be convex.

Convexity of loss functions

- ▶ Is $L_{sq}(h, y) = (y - h)^2$ convex? **Yes** or No.
- ▶ Is $L_{abs}(h, y) = |y - h|$ convex? **Yes** or No.
- ▶ Is $L_{ucsd}(h, y)$ convex? **Yes** or **No**.



Convexity of R_{ucsd}

- ▶ A function can be convex in a region.
- ▶ If σ is large, $R_{ucsd}(h)$ is convex in a big region around data.
 - ▶ A large σ led to a very smooth, parabolic-looking empirical risk function with a single local minimum (which was a global minimum).
- ▶ If σ is small, $R_{ucsd}(h)$ is convex in only small regions.
 - ▶ A small σ led to a very bumpy empirical risk function with many local minimums.

Discussion Question

Recall the empirical risk for absolute loss,

$$R_{abs}(h) = \frac{1}{n} \sum_{i=1}^n |y_i - h|$$

Is $R_{abs}(h)$ **convex**? Is gradient descent **guaranteed** to find a global minimum, given an appropriate step size?

- a) **YES** convex, **YES** guaranteed
- b) **YES** convex, **NOT** guaranteed
- c) **NOT** convex, **YES** guaranteed
- c) **NOT** convex, **NOT** guaranteed

Summary

Summary

- ▶ Gradient descent is a general tool used to minimize differentiable functions.
- ▶ Convex functions are (relatively) **easy** to optimize with gradient descent.
- ▶ We like **convex loss functions**, such as the squared loss and absolute loss, because the corresponding empirical risk functions are also convex.

What's next?

- ▶ So far, we've been predicting future values (salary, for instance) without using any information about the individual.
 - ▶ GPA.
 - ▶ Years of experience.
 - ▶ Number of LinkedIn connections.
 - ▶ Major.
- ▶ How do we incorporate this information into our prediction-making process?