Lecture 18 - Probababability and Combinatorics Examples

DSC 40A, Spring 2023

Announcements

- Homework 5 is due **tomorrow at 11:59pm**.
- RSVP for the DSC Undergrad Town Hall tomorrow from 1:30-3:30pm in the SDSC Auditorium.
  - A chance to talk about what's going on in the department, raise concerns, talk to professors, etc.

Agenda

- Review of combinatorics.
- Lots of examples.

Review of combinatorics

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** You roll a die 10 times. How many different sequences of results are possible?

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is

$$n^k.$$

Permutations

- A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$\begin{aligned} P(n,k) &= (n)(n-1)\ldots(n-k+1) \\ &= \frac{n!}{(n-k)!} \end{aligned}$$

Combinations

- A combination is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**.

- **Example:** How many ways are there to select a committee of 3 people from a group of 8 people?

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

$$\begin{aligned} C(n,k) &= \binom{n}{k} \\ &= \frac{P(n,k)}{k!} \\ &= \frac{n!}{(n-k)!k!} \end{aligned}$$

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

Lots of examples

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.

How many dominoes are in the set of dominoes?

a) $\binom{7}{2}$

b) $\binom{7}{1} + \binom{7}{2}$

c) $P(7, 2)$

d) $\frac{P(7,2)}{P(7,1)} 7!$

Selecting students — overview

We're going answer the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 4: “the easy way”)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?
3. At least 2 markers?

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

Fair coin

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Unfair coin

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Summary

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- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.