

Lecture 16 - Conditional Probability, Sequences and Permutations



DSC 40A, Spring 2023

Announcements

- ▶ Discussion is tonight at 7pm or 8pm in FAH 1101. Groupwork 6 is due **tonight at 11:59pm**.
- ▶ Homework 5 is released, due **Tuesday at 11:59pm**.
- ▶ **Important:** We've posted **many** probability resources on the [resources tab of the course website](#). These will no doubt come in handy.
 - ▶ No more DSC 40A-specific readings, though the Probability Roadmap was written specifically for students of this course.

Agenda

- ▶ Conditional probability.
- ▶ Simpson's Paradox.
- ▶ Sequences and permutations.

Conditional probability

Last time

- ▶ \bar{A} is the complement of event A. $P(\bar{A}) = 1 - P(A)$.

① Complement rule
(NOT)

- ▶ For any two events A and B

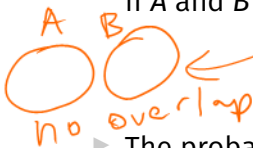
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

② addition rule
(OR)

If A and B are mutually exclusive, this simplifies to

$$P(A \cup B) = P(A) + P(B).$$

may be zero



- ▶ The probability that events A and B both happen is

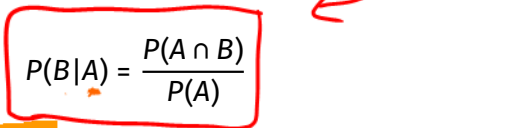
$$P(A \cap B) = P(A)P(B|A).$$

③ multiplication rule
(AND)

- ▶ $P(B|A)$ is the conditional probability of B occurring, given that A occurs. If $P(B|A) = P(B)$, then events A and B are independent.

Conditional probability

- ▶ The probability of an event may **change** if we have additional information about outcomes.
- ▶ Starting with the multiplication rule, $P(A \cap B) = P(A)P(B|A)$, we have that


$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

assuming that $P(A) > 0$.

Example: pets

Suppose a family has two pets. Assume that it is equally likely that each pet is a dog or a cat. Consider the following two probabilities:

1. The probability that both pets are dogs given that **the oldest is a dog**.
2. The probability that both pets are dogs given that **at least one of them is a dog**.

Discussion Question

Are these two probabilities equal?

a) Yes, they're equal

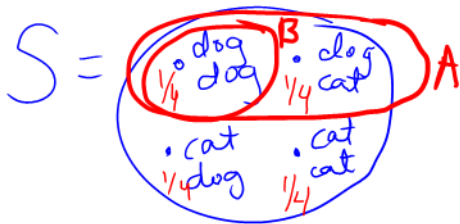
☒ b) No, they're not equal

Example: pets

$$P(B|A)$$

B

Let's compute the probability that both pets are dogs given that the oldest is a dog. A



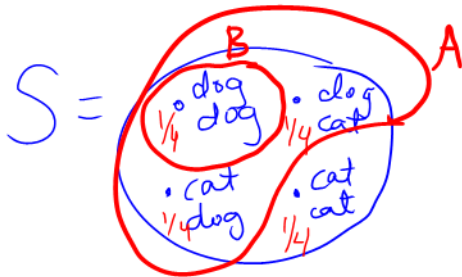
$$\begin{aligned} P(B|A) &= \frac{P(A \text{ and } B)}{P(A)} \\ &= \frac{1/4}{1/2} \\ &= 1/2 \end{aligned}$$

Example: pets

$$P(B|A)$$

B

Let's now compute the probability that both pets are dogs given that at least one of them is a dog.

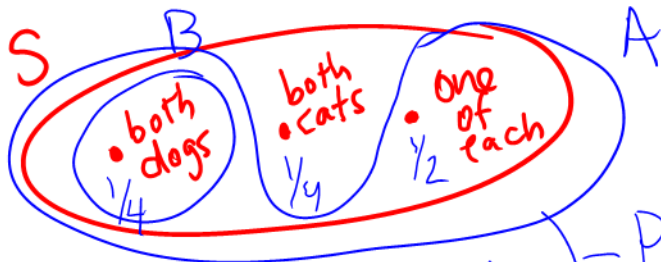


$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$= \frac{1/4}{3/4}$$

$$= 1/3$$

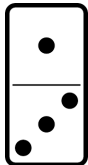
How do you know that order matters?



$$P(\underbrace{\text{both dogs}}_B \mid \underbrace{\text{at least one dog}}_A) = \frac{P(\text{both dogs and at least one dog})}{P(\text{at least one dog})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example: dominoes (source: 538)

In a set of dominoes, each tile has two sides with a number of dots on each side: zero, one, two, three, four, five, or six. There are 28 total tiles, with each number of dots appearing alongside each other number (including itself) on a single tile.



	0	1	2	3	4	5	6	
0	X	X	X	X	X	X	X	→ 7
1	•	X	X	X	X	X	X	→ 6
2	•	•	X	X	X	X	X	→ 5
3	•	•	•	X	X	X	X	→ 4
4	•	•	•	•	X	X	X	→ 3
5	•	•	•	•	•	X	X	→ 2
6	•	•	•	•	•	•	X	→ 1

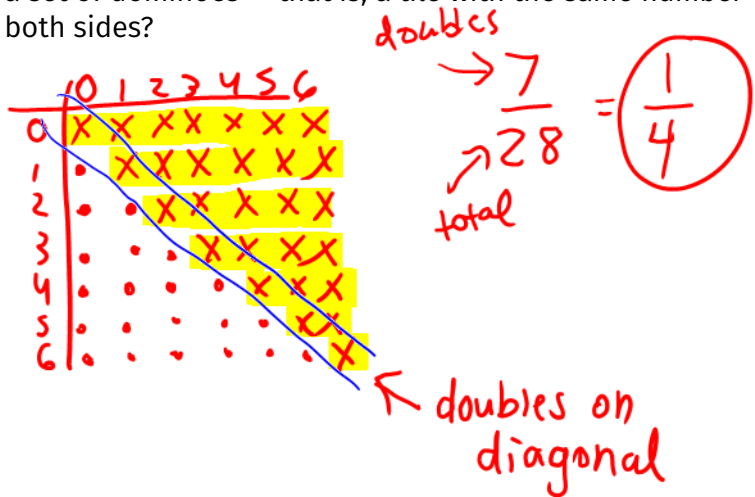
$$49 - 7 = 42$$

$$\frac{42}{2} = 21$$

$$21 + 7 = 28$$

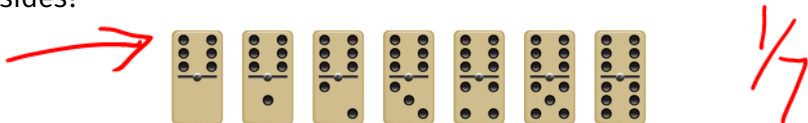
Example: dominoes (source: 538)

Question 1: What is the probability of drawing a “double” from a set of dominoes — that is, a tile with the same number on both sides?



Example: dominoes (source: 538)

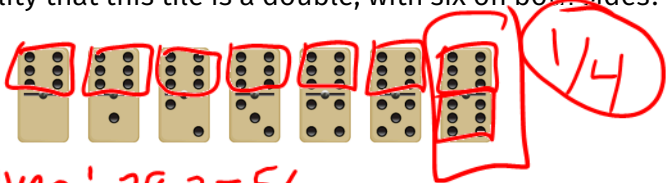
Question 2: Now your friend picks a random tile from the set and tells you that at least one of the sides is a 6. What is the probability that your friend's tile is a double, with 6 on both sides?



	0	1	2	3	4	5	6
0	X	X	X	X	X	X	X
1	•	X	X	X	X	X	X
2	•	•	X	X	X	X	X
3	•	•	•	X	X	X	X
4	•	•	•	•	X	X	X
5	•	•	•	•	•	X	X
6	•	•	•	•	•	•	X

Example: dominoes (source: 538)

Question 3: Now you pick a random tile from the set and uncover only one side, revealing that it has six dots. What is the probability that this tile is a double, with six on both sides?



domino halves: $28 \cdot 2 = 56$

8 domino halves are 6's

how many are across from
another 6? 2 \Rightarrow $\frac{2}{8} = \frac{1}{4}$

See 538's explanation here.

← direct interpretation

$$P(GG/\text{revealed } G) = \frac{P(GG \text{ and } \cancel{\text{revealed } G})}{P(\text{revealed } G)}$$

$$= \frac{1/28}{8/56}$$

$$= \frac{2/\cancel{56}}{8/\cancel{56}} = \boxed{\frac{1}{4}}$$

from cond.
prob. formula

$$P(B|A) = P(A \text{ and } B) / P(A)$$

Simpson's Paradox

Simpson's Paradox (source: [nih.gov](https://www.nih.gov))

	Treatment A	Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Discussion Question

Which treatment is better?

- a) Treatment A for all cases.
- b) Treatment B for all cases.
- c) Treatment A for small stones and B for large stones.
- d) Treatment A for large stones and B for small stones.

Simpson's Paradox (source: [nih.gov](https://www.nih.gov))

depts

	<i>male</i> Treatment A	<i>female</i> Treatment B
Small kidney stones	81 successes / 87 (93%)	234 successes / 270 (87%)
Large kidney stones	192 successes / 263 (73%)	55 successes / 80 (69%)
Combined	273 successes / 350 (78%)	289 successes / 350 (83%)

Simpson's Paradox occurs when an association between two variables exists when the data is divided into subgroups, but reverses or disappears when the groups are combined.

- See more in DSC 80.

Sequences and permutations

Motivation

- ▶ Many problems in probability involve counting.
 - ▶ Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
 - ▶ Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- ▶ In order to solve such problems, we first need to learn how to count. *→ # objects that satisfy*
- ▶ The area of math that deals with counting is called *certain* **combinatorics**. *properties*

Selecting elements (i.e. sampling)

- ▶ Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a population.
 - ▶ If drawing cards from a deck, the population is the deck of all cards.
 - ▶ If selecting people from DSC 40A, the population is everyone in DSC 40A.
- ▶ Two decisions:
 - ▶ Do we select elements with or without replacement?
 - ▶ Does the order in which things are selected matter?

Sequences

- ▶ A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.
- ▶ **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.
- ▶ **Example:** A UCSD PID starts with “A” then has 8 digits. How many UCSD PIDs are possible?

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is n^k .

(Note: We mentioned this fact in the lecture on clustering!)

Permutations

- ▶ A **permutation** is obtained by selecting k elements from a group of n possible elements **without replacement**, such that **order matters**.
- ▶ **Example:** Draw 4 cards (without replacement) from a standard 52-card deck.
- ▶ **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

Permutations

- ▶ In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n, k) = (n)(n - 1) \dots (n - k + 1)$$

- ▶ To simplify: recall that the definition of $n!$ is

$$n! = (n)(n - 1) \dots (2)(1)$$

- ▶ Given this, we can write

$$P(n, k) = \frac{n!}{(n - k)!}$$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

- a) 21
- b) 210
- c) 343
- d) 2187
- e) None of the above.

Special case of permutations

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

- This is consistent with the formula

$$P(n, n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Summary, next time

Summary

- ▶ The **conditional probability** of B given A is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

- ▶ A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
 - ▶ Number of sequences: n^k .
- ▶ A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
 - ▶ Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.
- ▶ **Next time: combinations**, where order doesn't matter.