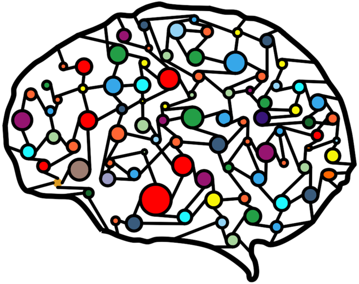
Lecture 19 – More Probabability and Combinatorics Examples



DSC 40A, Spring 2023

Announcements

- Discussion is tonight at 7pm or 8pm in FAH 1101.
  - Come to work on Groupwork 6, which is due **tonight at 11:59pm**.
- Homework 6 is released, due **Tuesday at 11:59pm**.
- Don’t forget to read through the solutions to past assignments before doing the next assignment. This is especially useful for probability and combinatorics to learn new ways of solving problems.
  - See the pinned post on Campuswire.

Agenda

- Lots of examples.

Last time

Last time we answered the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

With vs. without replacement

ex) 20 select 20
- without replacement
P(Avi) = 1

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

- with replacement
P(Avi) < 1

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

$$\binom{12}{4} = C(12,4)$$

order of selection doesn't matter

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?

$$\underbrace{C(5,2)}_{\text{of 5 markers, pick any 2}} * \underbrace{C(7,2)}_{\text{of 7 crayons, pick any 2}}$$

$$C(5,3) * \underbrace{C(7,1)}_{7}$$

good to know! $C(5,3) = \frac{5!}{3!2!}$

$= C(5,2)$

choose which 3 I want

choose which 2 I don't want

= 6 outfits

$C(n,K) = C(n,n-K)$

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

S = all sets of 4 art supplies

$|S| = C(12,4)$

all equally likely

# markers	
0	$C(5,0)\cdot C(7,4)$
1	$C(5,1)\cdot C(7,3)$
2	$C(5,2)\cdot C(7,2)$
3	$C(5,3)\cdot C(7,1)$
4	$C(5,4)\cdot C(7,0)$

$$p(\text{at least 2 markers}) = \frac{\#\text{ ways to choose 4 art supplies such that at least 2 are markers}}{C(12,4)} = \frac{C(12,4) - \left(C(5,0)\cdot C(7,4) + C(5,1)\cdot C(7,3)\right)}{C(12,4)}$$

Fair coin

T H T . . .

$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

Question 3: Suppose we flip a **fair coin** 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$\boxed{\left(\frac{1}{2}\right)^{10}} = \frac{1}{2^{10}}$

$\frac{\text{\# seq with 5 H, 5 T}}{\text{\# seq. of 10 H's, T's}} = \frac{C(10,5)}{2^{10}}$

C(10,5) = # ways to choose 5 positions for the H's

T T H T H T H H T H

Why C, not P?
order doesn't matter for positions of H's

Unfair coin

Question 4: Suppose we flip an **unfair coin** 10 times. The coin is biased such that for each flip, $\boxed{P(\text{heads}) = \frac{1}{3}}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$$\cancel{\frac{\#\text{ good seq}}{\text{total \# seq}} = \frac{1}{2^{10}}}$$

because not all outcomes equally likely

T H T T H . . .

$$\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdots$$

how many T's?
five

$$\left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$$

$P(H) = 1/3$, flip 10 times, prob of 5H + 5T

THTTHTHHTH prob: $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$

TTTTTHHHHH prob: $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$

TTTTTTTTTT prob: $\left(\frac{2}{3}\right)^{10}$

HHHHHHHHHH prob: $\left(\frac{1}{3}\right)^{10}$

S = all sequences of H's + T's

event I care about includes all outcomes with 5H and 5T

prob of event E

$$= \sum_{s \in E} \text{prob}(s)$$

$$= \sum_{s \in E} \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$$

$$= \left(\#\text{ outcomes in } E\right) * \left(\frac{2}{3}\right)^5 * \left(\frac{1}{3}\right)^5$$

$$= \boxed{C(10,5) * \left(\frac{2}{3}\right)^5 * \left(\frac{1}{3}\right)^5}$$

Deck of cards

- There are 52 cards in a standard deck (4 suits, 13 values).

4 suits

13 values

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Sets of cards

Deck of cards

1. How many 5 card hands are there in poker?

$C(52,5)$

2. How many 5 card hands are there where all cards are of the same suit (a **flush**)?

ex.) $\underset{52}{3\heartsuit}, \underset{12}{Q\heartsuit}, \underset{11}{A\heartsuit}, \underset{10}{5\heartsuit}, \underset{9}{10\heartsuit}$

$\boxed{\dfrac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}}$

1) what suit? 4 options

2) which 5 cards? $C(13,5)$

$\boxed{4 * C(13,5)}$

1) what card to start? 52 options

2) then which card? 12 options

⋮

3. How many 5 card hands are there that include a **four-of-a-kind** (four cards of the same value)?

$13 \cdot 12 \cdot 4$

ex.) 5♢, 5♤, 5♧, 5♡, 8♢

1) which value to repeat? 13 options

2) what other card? $48 = 12 \cdot 4$

4. How many 5 card hands are there that have a **straight** (all card values consecutive)?

9 options for which 5 values to use

(start 2, 3, 4, 5, 6, 7, 8, 9, 10) ex.) 4, 5, 6, 7, 8

4^5 options for assigning suits

$9 \cdot 4^5$

5. How many 5 card hands are there that are a **straight flush** (all card values consecutive and of the same suit)?

6. How many 5 card hands are there that include exactly **one pair** (values aabcd)?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.