Lecture 19 - More Probability and Combinatorics Examples

DSC 40A, Spring 2023

Announcements

- ▶ Discussion is tonight at 7pm or 8pm in FAH 1101.
  - ▶ Come to work on Groupwork 6, which is due **tonight at 11:59pm**.

- ▶ Homework 6 is released, due **Tuesday at 11:59pm**.

- ▶ Don't forget to read through the solutions to past assignments before doing the next assignment. This is especially useful for probability and combinatorics to learn new ways of solving problems.
  - ▶ See the pinned post on Campuswire.

Agenda

- ▶ Lots of examples.

Last time

Last time we answered the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$1/4$

With vs. without replacement

always!
get 5 people

may not get 5 unique people, may just get same person repeated

get ≤ 5 people

another way!

— without replacement

$$P(\text{Avi on 1}^{st}\text{ pick}) = \frac{1}{20}$$

$$P(\text{Avi on 2}^{nd}\text{ pick} \mid \text{didn't get Avi on 1}^{st}\text{ pick}) = \frac{1}{19}$$

— with replacement

$$P(\text{Avi on 1}^{st}\text{ pick}) = \frac{1}{20}$$

$$P(\text{Avi on 2}^{nd}\text{ pick} \mid \text{~~didn't get Avi on 1}^{st}\text{ pick}~~) = \frac{1}{20}$$

← irrelevant (independent)

extreme: randomly sample 20 people from 20 —

without replacement

$$P(\text{Avi}) = 1$$

with replacement

$$P(\text{Avi}) < 1$$

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

$$C(12,4) = \binom{12}{4}$$

context: order doesn't matter

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?

$$C(5,2) * C(7,2)$$

$$3 * 2 = 6$$
outfits

$$C(5,3) * C(7,1)$$

7

choose which 2 to take

$$\frac{5!}{2! \, 3!} =$$

choose which 3 to not take

Another thing to know!

$$C(5,2) = C(5,3)$$
$$C(n,k) = C(n, n-k)$$

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

$S =$ all sets of 4 art supplies

$|S| = C(12,4)$

are all elements of S equally likely? yes

$P\begin{pmatrix} \text{at least} \\ 2 \text{ markers} \end{pmatrix} = \dfrac{\text{\# sets of 4 art supplies that include at least 2 markers}}{\text{\# sets of 4 art supplies}}$

0 markers $\begin{pmatrix} 5 \\ 0 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

1 marker $\begin{pmatrix} 5 \\ 1 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix}$

2 markers $\begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix}$

3 markers $\begin{pmatrix} 5 \\ 3 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}$

4 markers $\begin{pmatrix} 5 \\ 4 \end{pmatrix}\begin{pmatrix} 7 \\ 0 \end{pmatrix}$

total is $|S|$

$= \dfrac{\begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 3 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}\begin{pmatrix} 7 \\ 0 \end{pmatrix}}{\begin{pmatrix} 12 \\ 4 \end{pmatrix}} = \dfrac{\begin{pmatrix} 12 \\ 4 \end{pmatrix} - \left(\begin{pmatrix} 5 \\ 0 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} \right)}{\begin{pmatrix} 12 \\ 4 \end{pmatrix}}$

Fair coin

Question 3: Suppose we flip a **fair coin** 10 times.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$$

$$\frac{\text{\# seq. with } 5H, 5T}{\text{\# seq}}$$

$$= \frac{C(10,5)}{2^{10}} \leftarrow \text{of 10 positions}$$

HHTHHTTHTT
select 5 of them for tails

Unfair coin

Question 4: Suppose we flip an **unfair coin** 10 times. The coin is biased such that for each flip, P(heads) = $\frac{1}{3}$.

1. What is the probability that we see the specific sequence THTTHTHHTH?

2. What is the probability that we see an equal number of heads and tails?

"good" seq = $\frac{1}{2^{10}}$
total # seq

not all outcomes are equally likely

S
Prob prob

↳ add up probs for each outcome in event

T H T T H T H H T H
$\frac{2}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{2}{3}$ $\frac{1}{3}$ ... $= \left(\frac{2}{3}\right)^5 \cdot \left(\frac{1}{3}\right)^5$

$P(H) = 1/3$ flip 10 times

Prob of equal # of H and T

δ = all sequences of H's + T's

THTTHTHHTH   prob: $(\frac{2}{3})^5 (\frac{1}{3})^5$

prob: $(\frac{2}{3})^5 (\frac{1}{3})^5$

HHHHHHHHHH   prob: $(\frac{1}{3})^{10}$

event I care about is equal # of H and T (five of each)

total prob of event E is

$$\sum_{s \in E} \text{prob}(s)$$
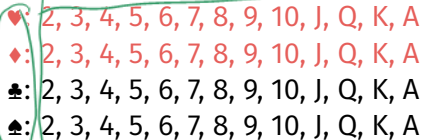
$$\sum_{s \in E} (\frac{2}{3})^5 \cdot (\frac{1}{3})^5$$

$$= \binom{\text{# elements}}{\text{in } E} * (\frac{2}{3})^5 * (\frac{1}{3})^5$$

$$= C(10, 5) * (\frac{2}{3})^5 * (\frac{1}{3})^5$$

Deck of cards

▶ There are 52 cards in a standard deck (4 suits, 13 values).

4 suits   13 values

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

▶ In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Set

Deck of cards

1. How many 5 card hands are there in poker?

$$C(52,5)$$

2. How many 5 card hands are there where all cards are of the same suit (a **flush**)?

$$\rightarrow \frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

$$4 \cdot \binom{13}{5}$$

ex.) $3\diamond, 10\diamond, A\diamond, 4\diamond, 2\diamond$

$\underset{52}{}\quad \underset{12}{}\quad \underset{11}{}\quad \underset{10}{}\quad \underset{9}{}$

what suit? 4 options

what 5 values? $C(13,5)$

52 options for 1st card
*12 options for 2nd card ...

3. How many 5 card hands are there that include a $13 \cdot 48$ **four-of-a-kind** (four cards of the same value)?

$12 \cdot 4$

ex.) $5\heartsuit, 5\diamondsuit, 5\spadesuit, 5\clubsuit, 10\heartsuit$

what # to be repeated ? 13 values

what other card ? 52-4 = 48 other cards

4. How many 5 card hands are there that have a **straight** (all card values consecutive)?

5. How many 5 card hands are there that are a **straight flush** (all card values consecutive and of the same suit)?

6. How many 5 card hands are there that include exactly **one pair** (values aabcd)?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.

- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n, k) = \frac{n!}{(n-k)!}$.

- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.