Lecture 4

Comparing Loss Functions

DSC 40A, Fall 2024

Announcements

- Homework 1 will be relased by tomorrow and will be due on **Friday, October 11th**.
  - Before working on it, watch the Walkthrough Videos on problem solving and using Overleaf.
  - Using the Overleaf template is required for Homework 2 (and only Homework 2).
- Remember that in, general, groupwork worksheets are released on Sunday and due Monday.
- Look at the office hours schedule here and plan to start regularly attending!
- Remember to take a look at the supplementary readings linked on the course website.

Agenda

- Recap: Empirical risk minimization.
- Choosing a loss function.
    - The role of outliers.
- Center and spread.
- Towards linear regression.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

Recap: Empirical risk minimization

Goal

We had one goal in Lectures 2 and 3: given a dataset of values from the past, **find the best constant prediction** to make.

$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 92$$

Key idea: Different definitions of "best" give us different "best predictions."

The modeling recipe

In Lectures 2 and 3, we made two full passes through our "modeling recipe."

1. Choose a model.

$$H(x) = h$$

2. Choose a loss function.

$$L_{\text{sq}}(y_i, h) = (y_i - h)^2 \qquad\qquad L_{\text{abs}}(y_i, h) = |y_i - h|^2$$

3. Minimize average loss to find optimal model parameters.

$$h* = \text{mean}(y_1, \ldots, y_n) \qquad\qquad h* = \text{median}(y_1, \ldots, y_n)$$

Empirical risk minimization

- The formal name for the process of minimizing average loss is **empirical risk minimization**.
- Another name for "average loss" is **empirical risk**.
- When we use the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$, the corresponding empirical risk is mean squared error:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$$

- When we use the absolute loss function, $L_{\text{abs}}(y_i, h) = |y_i - h|$, the corresponding empirical risk is mean absolute error:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

Empirical risk minimization, in general

Key idea: If $L(y_i, h)$ is **any** loss function, the corresponding empirical risk is:

$$R(h) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, h)$$

Question 🤔

Answer at q.dsc40a.com

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$$

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

Is the following statement true, for any dataset $y_1, y_2, \ldots, y_n$ and prediction h?

$$\left(R_{\text{abs}}(h)\right)^2 = R_{\text{sq}}(h)$$

- A. It's true for any h and any dataset.
- B. It's true for at least one h for any dataset, but not in general.
- C. It's never true.

Choosing a loss function

Now what?

- We know that, for the constant model $H(x) = h$, the **mean** minimizes mean **squared** error.
- We also know that, for the constant model $H(x) = h$, the **median** minimizes mean **absolute** error.
- **How does our choice of loss function impact the resulting optimal prediction?**

Comparing the mean and median

- Consider our example dataset of 5 commute times.

$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 92$$

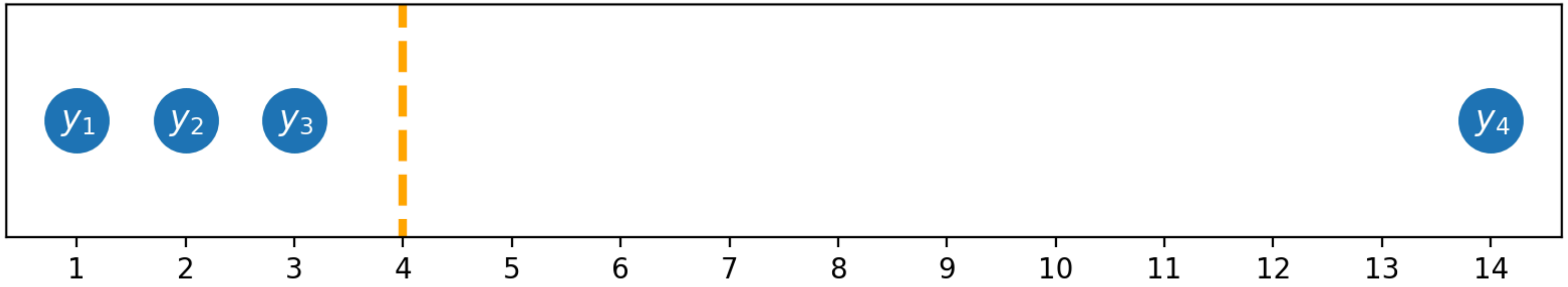
- As of now, the **median is 85** and the **mean is 80**.
- What if we add 200 to the largest commute time, 92?

$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 292$$

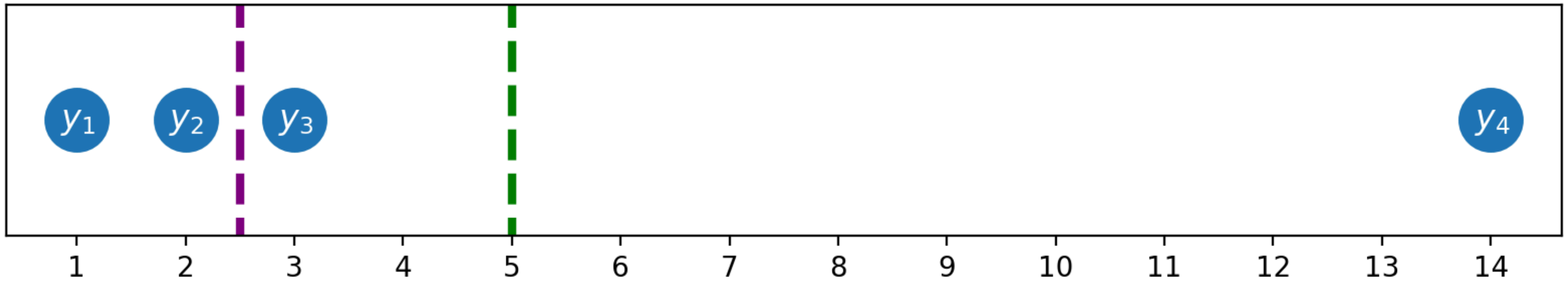
- Now, the median is but the mean is !
- **Key idea**: The mean is quite **sensitive** to outliers.

Outliers

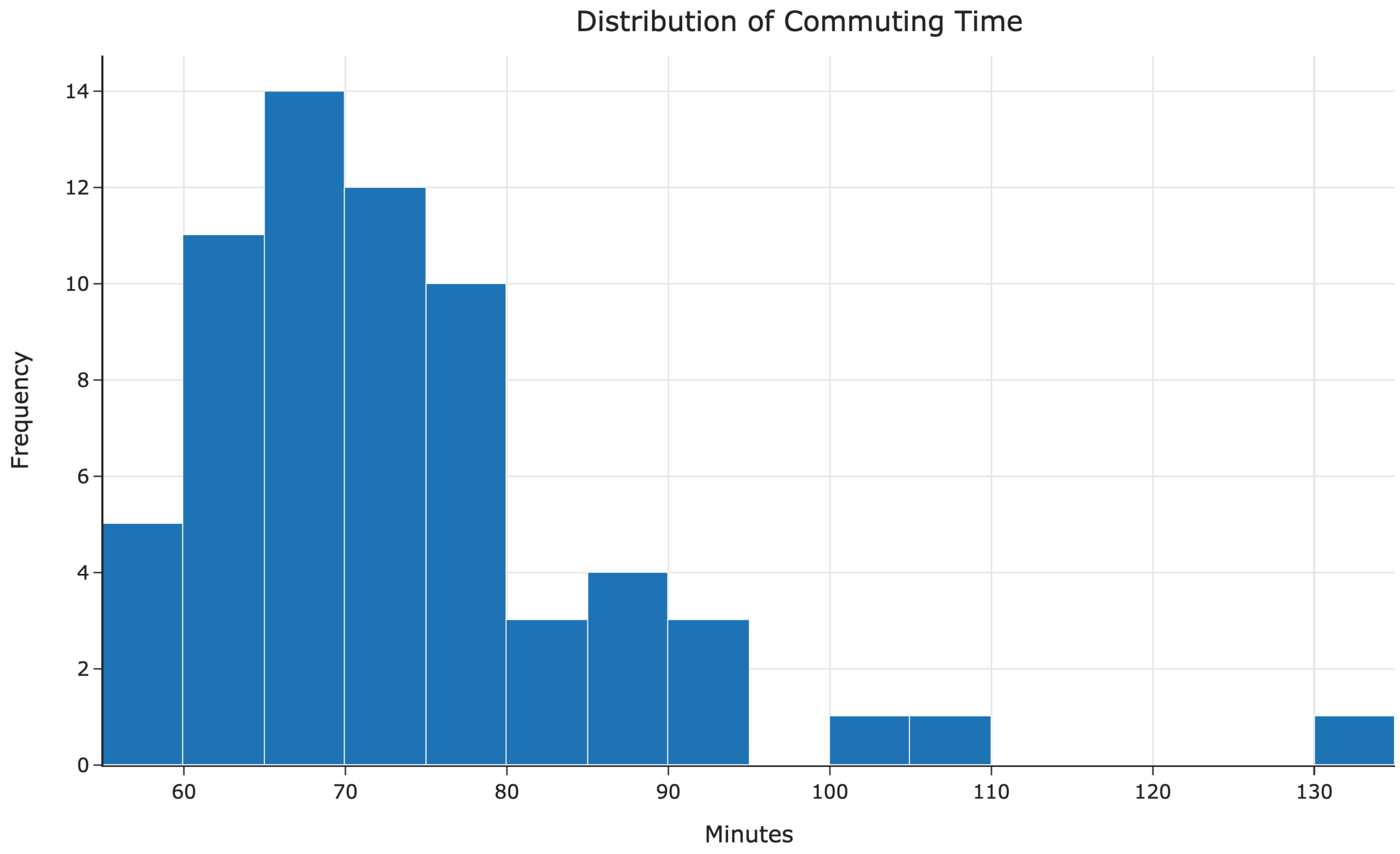
Below, $|y_4 - h|$ is 10 times as big as $|y_3 - h|$, but $(y_4 - h)^2$ is 100 times $(y_3 - h)^2$.



The result is that the **mean** is "pulled" in the direction of outliers, relative to the **median**.

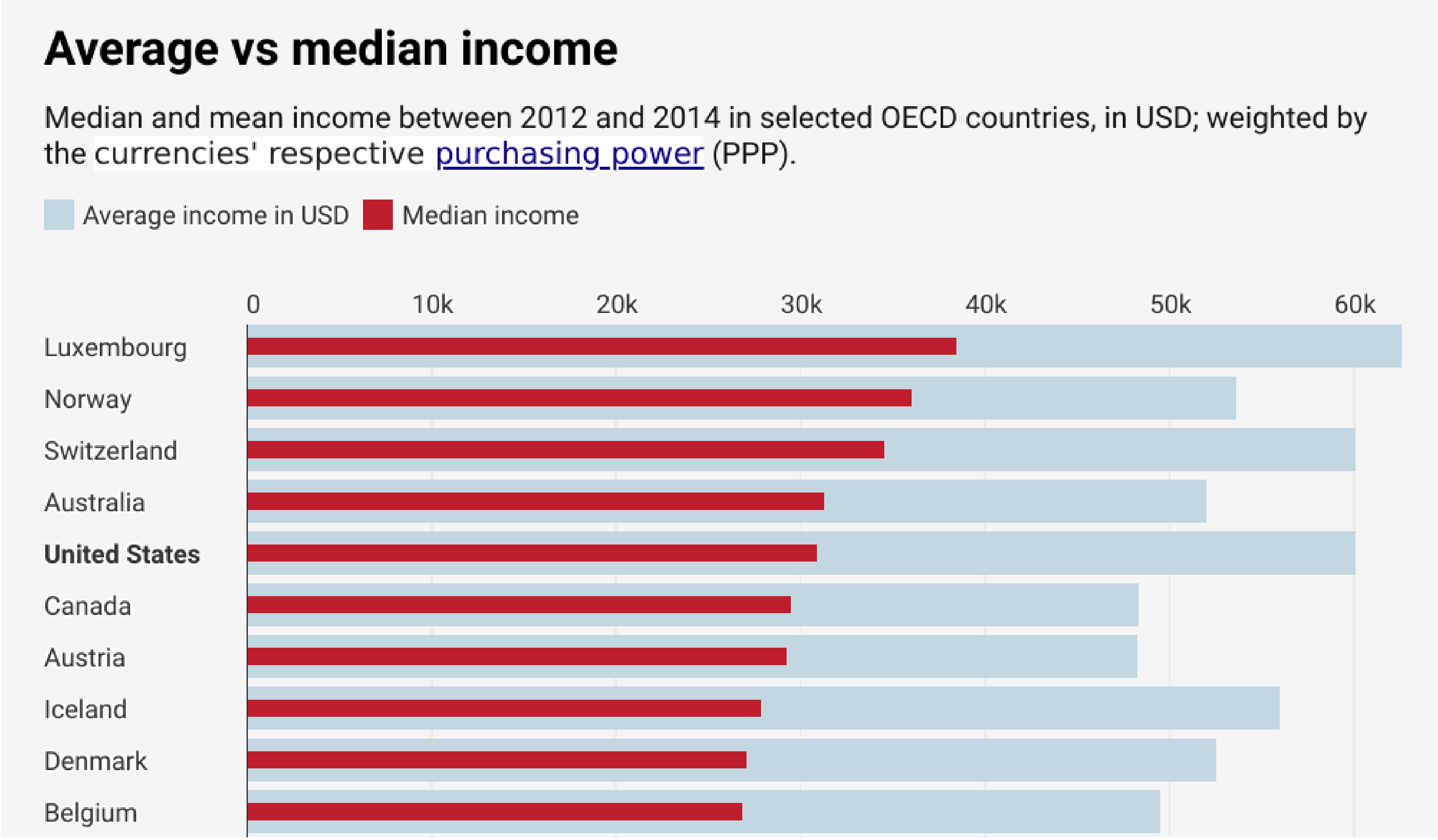


As a result, we say the **median** is **robust** to outliers. But the **mean** was easier to solve for.



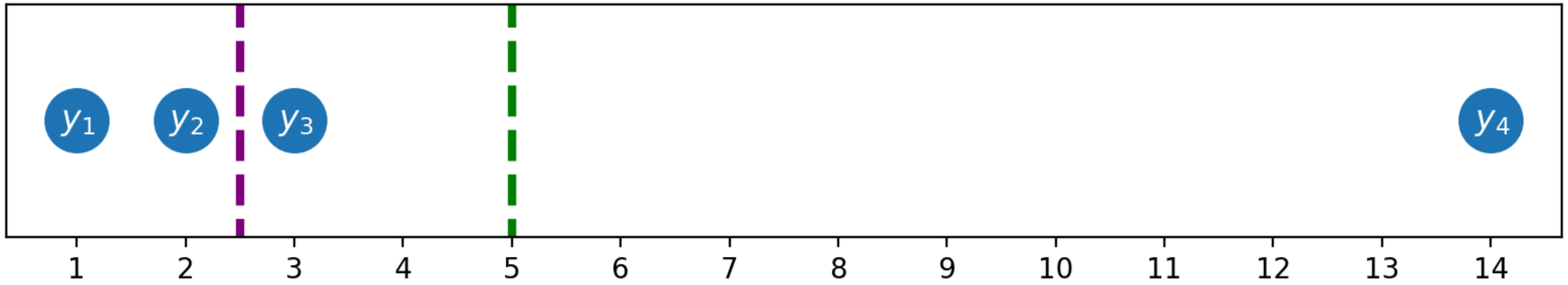
Distribution of Commuting Time
Frequency
0
2
4
6
8
10
12
14
60
70
80
90
100
110
120
130
Minutes

Example: Income inequality



Balance points

Both the **mean** and **median** are "balance points" in the distribution.



- The **mean** is the point where $\sum_{i=1}^{n}(y_i - h) = 0$.

- The **median** is the point where $\#(y_i < h) = \#(y_i > h)$.

Why stop at squared loss?

Empirical Risk, $R(h)$	Derivative of Empirical Risk, $\frac{d}{dh} R(h)$	Minimizer
$\frac{1}{n} \sum_{i=1}^n \lvert y_i - h \rvert$	$\frac{1}{n} \left(\sum_{y_i < h} 1 - \sum_{y_i > h} 1 \right)$	median
$\frac{1}{n} \sum_{i=1}^n (y_i - h)^2$	$\frac{-2}{n} \sum_{i=1}^n (y_i - h)$	mean
$\frac{1}{n} \sum_{i=1}^n \lvert y_i - h \rvert^3$		???
$\frac{1}{n} \sum_{i=1}^n (y_i - h)^4$		???
$\frac{1}{n} \sum_{i=1}^n (y_i - h)^{100}$		???
...	...	...

Generalized L_p loss

For any $p \geq 1$, define the L_p loss as follows:

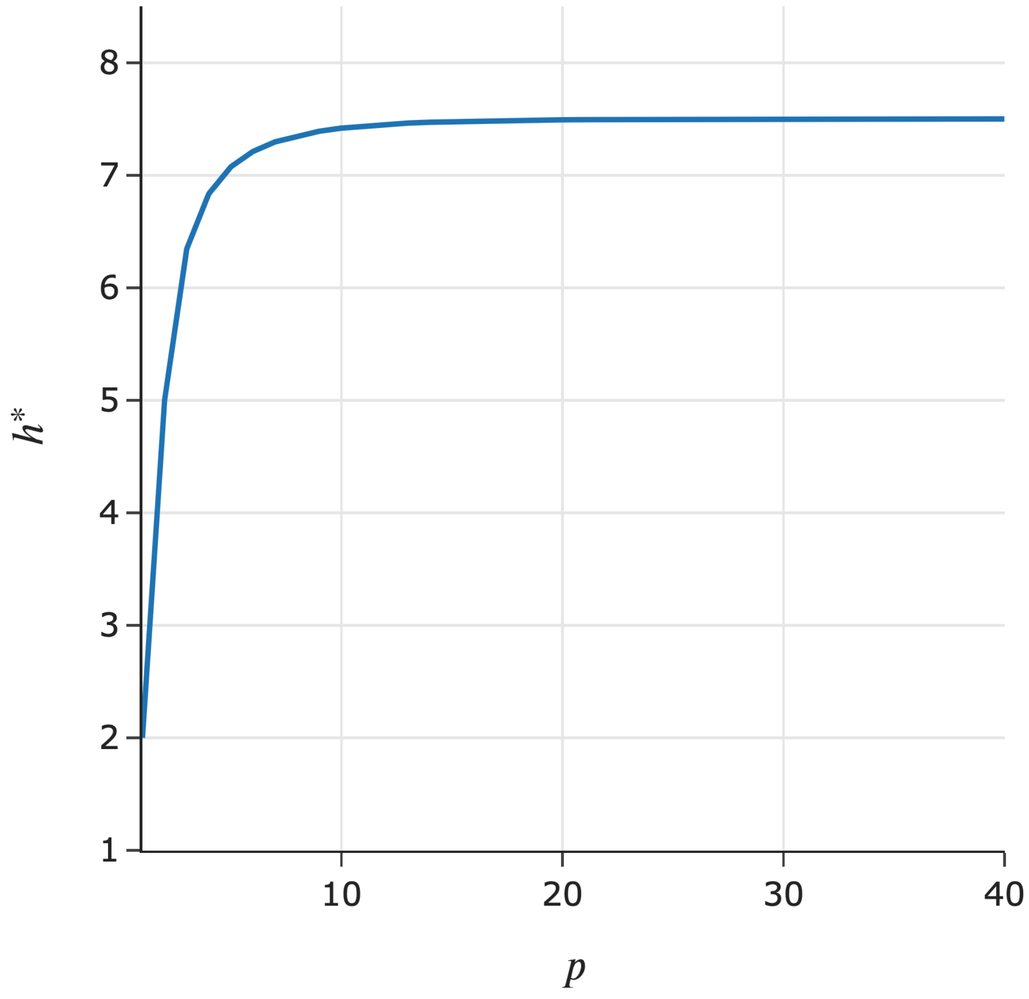
$$L_p(y_i, h) = |y_i - h|^p$$

The corresponding empirical risk is:

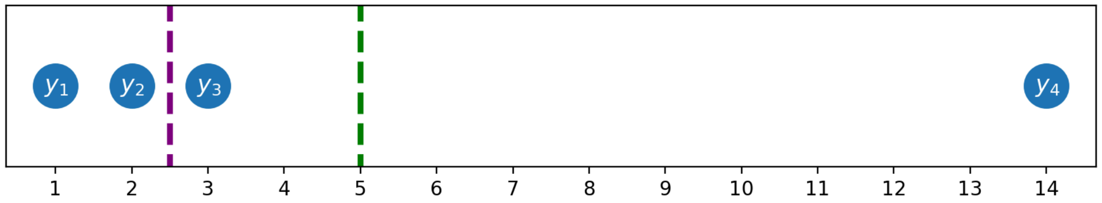
$$R_p(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|^p$$

- When $p = 1$, $h^* = \text{Median}(y_1, y_2, \dots, y_n)$.
- When $p = 2$, $h^* = \text{Mean}(y_1, y_2, \dots, y_n)$.
- What about when $p = 3$?
- What about when $p \to \infty$?

What value does h^* approach, as $p \to \infty$?



Consider the dataset $1, 2, 3, 14$:



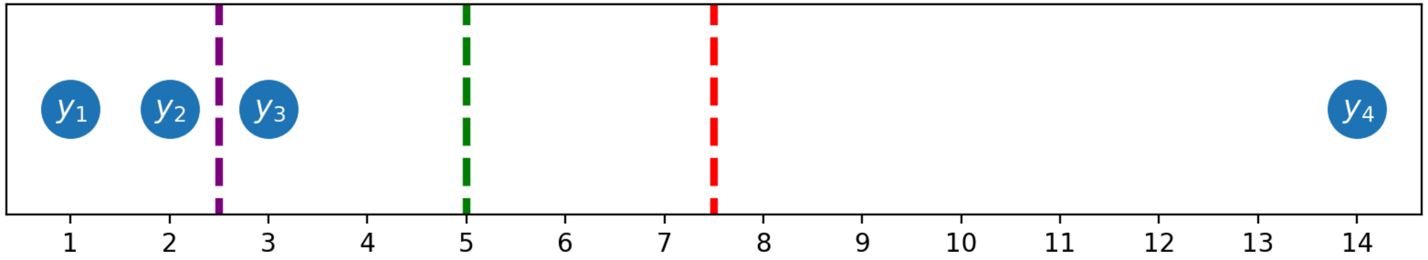
On the left:

- The x-axis is p.
- The y-axis is h^*, the optimal constant prediction for L_p loss:

$$h^* = \underset{h}{\text{argmin}} \frac{1}{n} \sum_{i=1}^{n} |y_i - h|^p$$

The *midrange* minimizes average L_∞ loss!

On the previous slide, we saw that as $p \to \infty$, the minimizer of mean L_p loss approached **the midpoint of the minimum and maximum values in the dataset**, or the midrange.



- As $p \to \infty$, $R_p(h) = \frac{1}{n} \sum_{i=1}^n |y_i - h|^p$ minimizes **the "worst case" distance from any data point"**. (Read more here).
- If your measure of "good" is "not far from any one data point", then the midrange is the best prediction.

Another example: 0-1 loss

Consider, for example, the **0-1 loss**:

$$L_{0,1}(y_i, h) = \begin{cases} 0 & y_i = h \\ 1 & y_i \neq h \end{cases}$$

The corresponding empirical risk is:

$$R_{0,1}(h) = \frac{1}{n} \sum_{i=1}^{n} L_{0,1}(y_i, h)$$

Question 🤔

Answer at q.dsc40a.com

$$R_{0,1}(h) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 0 & y_i = h \\ 1 & y_i \neq h \end{cases}$$

Suppose $y_1, y_2, \ldots, y_n$ are all unique. What is $R_{0,1}(y_1)$?

- A. 0.
- B. $\frac{1}{n}$.
- C. $\frac{n-1}{n}$.
- D. 1.

Minimizing empirical risk for 0-1 loss

$$R_{0,1}(h) = \frac{1}{n}\sum_{i=1}^{n}\begin{cases}0 & y_i = h\\1 & y_i \neq h\end{cases}$$

Summary: Choosing a loss function

Key idea: Different loss functions lead to different best predictions, h^*!

Loss	Minimizer	Always Unique?	Robust to Outliers?	Differentiable?
L_{sq}	mean	yes ✅	no ❌	yes ✅
L_{abs}	median	no ❌	yes ✅	no ❌
L_{∞}	midrange	yes ✅	no ❌	no ❌
$L_{0,1}$	mode	no ❌	yes ✅	no ❌

The optimal predictions, h^*, are all **summary statistics** that measure the **center** of the dataset in different ways.

Center and spread

What does it mean?

- The general form of empirical risk, for any loss function $L(y_i, h)$, is:

$$R(h) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, h)$$

- As we just saw, the input h^* that minimizes $R(h)$ is some measure of the **center** of the dataset.
    - Examples include the mean (L_{sq}), median (L_{abs}), and mode ($L_{0,1}$).
- The minimum output, $R(h^*)$, represents some measure of the **spread**, or variation, in the dataset.

Squared loss

- The empirical risk for squared loss, i.e. mean squared error, is:

$$R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$$

- $R_{\text{sq}}(h)$ is minimized when $h^* = \text{Mean}(y_1, y_2, \dots, y_n)$.
- Therefore, the minimum value of $R_{\text{sq}}(h)$ is:

$$\begin{aligned} R_{\text{sq}}(h^*) &= R_{\text{sq}}\left(\text{Mean}(y_1, y_2, \dots, y_n)\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \text{Mean}(y_1, y_2, \dots, y_n)\right)^2 \end{aligned}$$

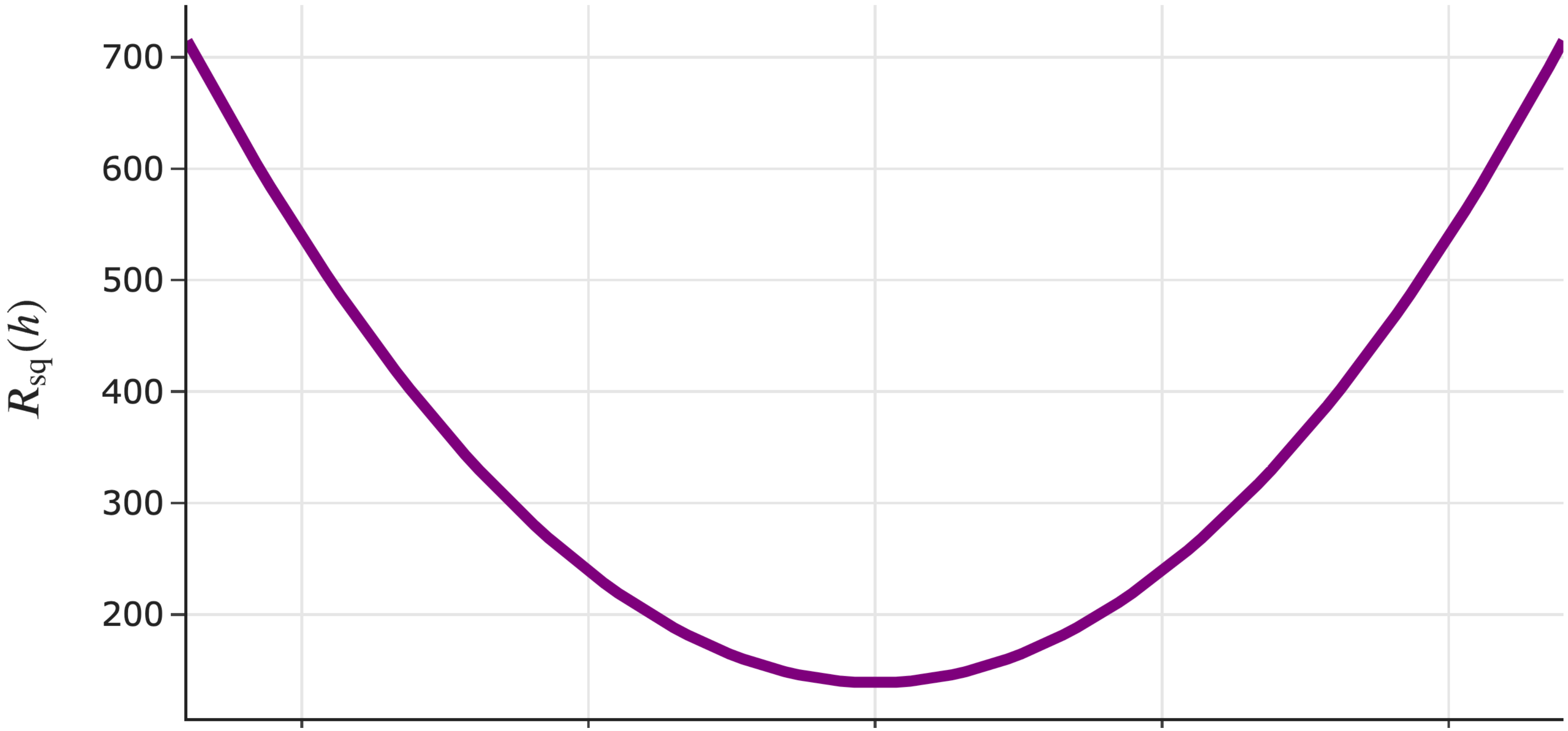
Variance

- The minimum value of $R_{\text{sq}}(h)$ is the mean squared deviation from the mean, more commonly known as the **variance**.

$$\text{Variance}(y_1, y_2, \ldots, y_n) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - \text{Mean}(y_1, y_2, \ldots, y_n)\right)^2$$

- It measures the squared distance of each data point from the mean, on average.
- Its square root is called the **standard deviation**.

$$R_{\text{sq}}(h) = \frac{1}{5}\left((72-h)^2 + (90-h)^2 + (61-h)^2 + (85-h)^2 + (92-h)^2\right)$$



Absolute loss

- The empirical risk for absolute loss, i.e. mean absolute error, is:

$$R_{\text{abs}}(h) = \frac{1}{n}\sum_{i=1}^{n}|y_i - h|$$

- $R_{\text{abs}}(h)$ is minimized when $h^* = \text{Median}(y_1, y_2, \dots, y_n)$.
- Therefore, the minimum value of $R_{\text{abs}}(h)$ is:

$$\begin{aligned} R_{\text{abs}}(h^*) &= \frac{1}{n}\sum_{i=1}^{n}|y_i - h| \\ &= R_{\text{abs}}(h) = \frac{1}{n}\sum_{i=1}^{n}|y_i - \text{Median}(y_1, y_2, \dots, y_n)| \end{aligned}$$

Mean absolute deviation from the median

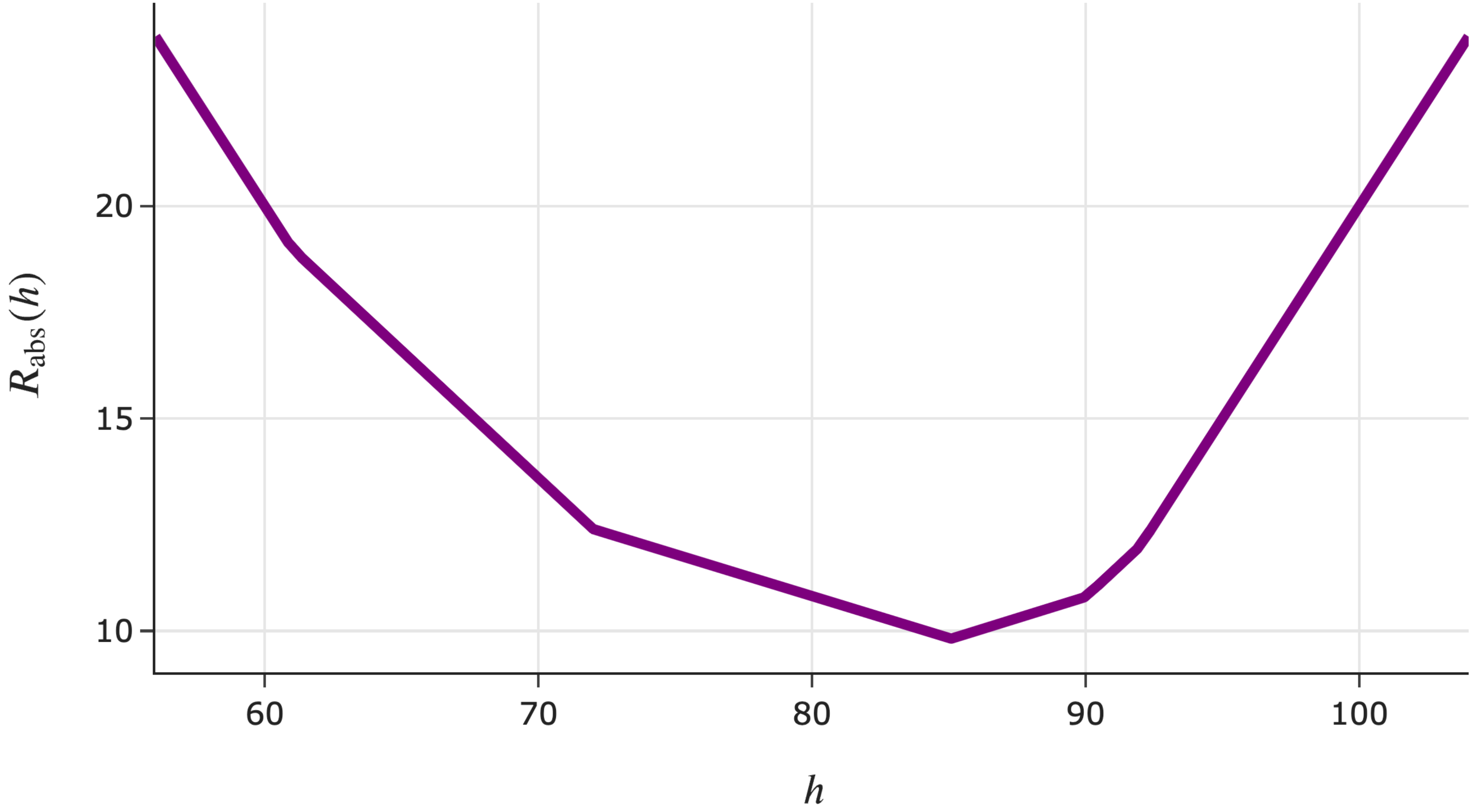
- The minimum value of $R_{\text{abs}}(h)$ is the **mean absolute deviation from the median**.

$$\text{MAD from the median}(y_1, y_2, \ldots, y_n) = \frac{1}{n} \sum_{i=1}^{n} |y_i - \text{Median}(y_1, y_2, \ldots, y_n)|$$

- It measures how far each data point is from the median, on average.
- **Example**: What's the MAD from the median in the dataset $2, 3, 3, 4, 5$?

Mean absolute deviation from the median

$$R_{\text{abs}}(h) = \frac{1}{5}(|72 - h| + |90 - h| + |61 - h| + |85 - h| + |92 - h|)$$



0-1 loss

- The empirical risk for the 0-1 loss is:

$$R_{0,1}(h) = \frac{1}{n}\sum_{i=1}^{n}\begin{cases} 0 & y_i = h \\ 1 & y_i \neq h \end{cases}$$

- This is the proportion (between 0 and 1) of data points not equal to h.
- $R_{0,1}(h)$ is minimized when $h^* = \text{Mode}(y_1, y_2, \ldots, y_n)$.
- Therefore, $R_{0,1}(h^*)$ is the proportion of data points not equal to the mode.
- **Example**: What's the proportion of values not equal to the mode in the dataset $2, 3, 3, 4, 5$?

A poor way to measure spread

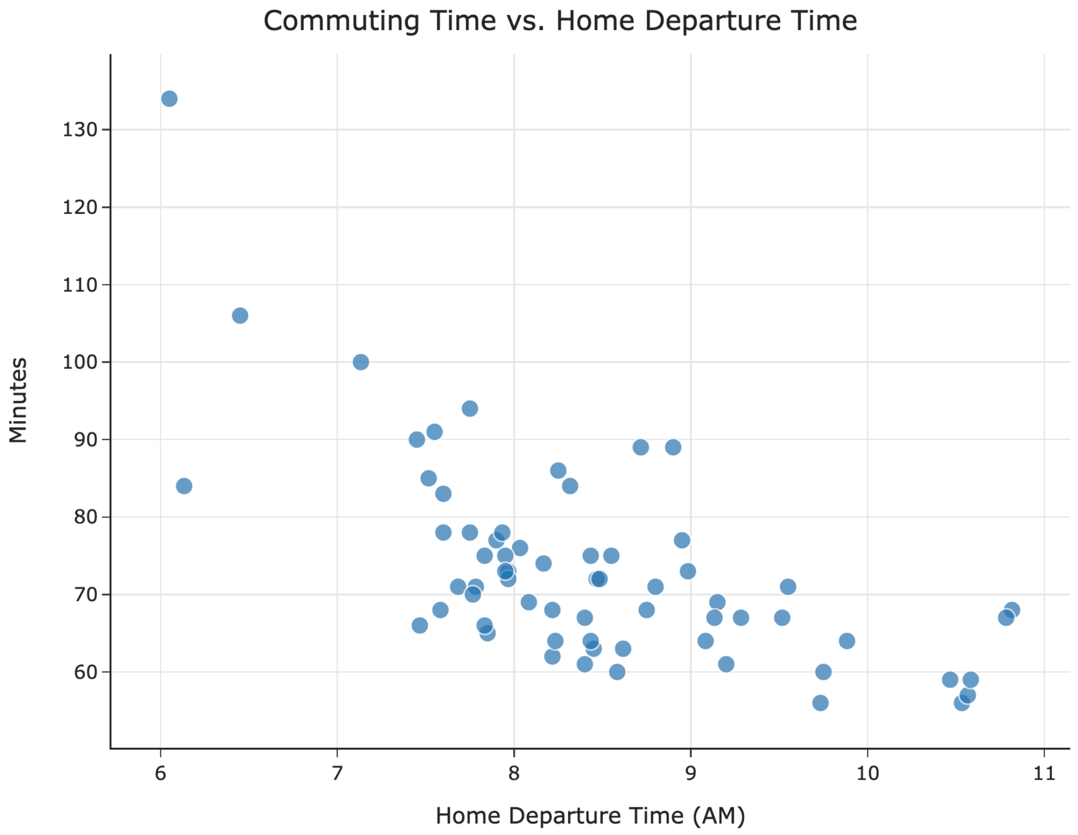
- The minimum value of $R_{0,1}(h)$ is the proportion of data points not equal to the mode.
- A higher value means less of the data is clustered at the mode.
- Just as the mode is a very basic way of measuring the center of the data, $R_{0,1}(h^*)$ is a very basic and uninformative way of measuring spread.

Summary of center and spread

- Different loss functions $L(y_i, h)$ lead to different empirical risk functions $R(h)$, which are minimized at various measures of **center**.
- The minimum values of empirical risk, $R(h^*)$, are various measures of **spread**.
- There are many different ways to measure both center and spread; these are sometimes called **descriptive statistics**.

What's next?

Towards simple linear regression



- In Lecture 1, we introduced the idea of a hypothesis function, $H(x)$.
- We've focused on finding the best **constant model**, $H(x) = h$.
- Now that we understand the modeling recipe, we can apply it to find the best **simple linear regression model**, $H(x) = w_0 + w_1 x$.
- This will allow us to make predictions that aren't all the same for every data point.

The modeling recipe

1. Choose a model.

2. Choose a loss function.

3. Minimize average loss to find optimal model parameters.