

Lecture 14

More Combinatorics Examples

DSC 40A, Summer 2024

Announcements

- Homework 6 is due on **Friday at 11:59PM.**
 - Combinatorics is hard! Come to office hours.
- How are office hours working for everyone?
 - Balance of in-person/virtual? Days of the week? Morning/afternoon?

Agenda

- Example: Selecting students.
- Example: An unfair coin.
- Example: Poker.
- Example: Chess.

Remember, we've posted **many** probability resources on the [resources tab of the course website](#). These will come in handy! Specific resources you should look at:

- The [DSC 40A probability roadmap](#), written by Janine Tiefenbruck.
- The textbook [Theory Meets Data](#), which explains many of the same ideas and contains more practice problems.

For combinatorics specifically, there are two supplementary videos that you should watch. Both are linked in [this playlist](#), which is also linked at dsc40a.com.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at [q.dsc40a.com!](http://q.dsc40a.com)

If the direct link doesn't work, click the " Lecture Questions" link in the top right corner of dsc40a.com.

Example: Selecting students

We'll start by working through a few examples we didn't get to finish on Thursday.

Summary

Combinatorics = "counting"
|
sequences permutations combinations

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	No, order doesn't matter
With replacement Repetition allowed	n^k possible sequences	more complicated: watch this video , or see the domino example
Without replacement Repetition not allowed	$\frac{n!}{(n - k)!}$ possible permutations	$\binom{n}{k}$ combinations

Using the binomial coefficient, $\binom{n}{k}$

As an example, let's evaluate:

$$\begin{aligned}
 &= \frac{q \cdot 8 \cdot 7!}{7! 2!} + \frac{q \cdot 8 \cdot 7 \cdot 6!}{6! 3!} = \frac{\cancel{q \cdot 8 \cdot 7!}}{\cancel{7! 2!}} + \frac{\cancel{3^3 q \cdot 8 \cdot 7}}{\cancel{3 \cdot 2}} \\
 &= 36 + 84 = 120
 \end{aligned}$$

$$\binom{n}{k} = \frac{n!}{(n-k)! k!}$$

$P(n, k)$

$$= \binom{n}{n-k}$$

$$\text{e.g. } \binom{7}{2} = \binom{7}{5}$$

↑

choosing
2 to take

↑

choosing
5
to not
take

Aside: Combinatorial proof

We showed that $\binom{9}{2} + \binom{9}{3} = 120$.

It turns out that $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = 120$.

Can we argue why $\binom{9}{2} + \binom{9}{3} = \binom{10}{3}$?

Example : Coin flipping. How many ways can we see 3 heads in 10 flips of a coin? $\Rightarrow \binom{10}{3}$

Equivalent: $\underbrace{\text{HHH TTT TTTT}}_{\text{number of permutations}}$ $= \frac{10!}{3! 7!} = \binom{10}{3}$

\uparrow
ways to rearrange
3 heads

$$\text{Proof of } \binom{9}{2} + \binom{9}{3} = \binom{10}{3}.$$

Claim: Both sides count # of ways to see 3 heads in 10 flips.

Left side: consider 2 cases

① First flip heads



② First flip tails



Key idea: both $\binom{9}{2} + \binom{9}{3}$ and $\binom{10}{3}$ count the same thing,
so they must be equal! "Combinatorial proof"

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $\mathbb{P}(A) = \frac{|A|}{|S|}$.
$$S = \left\{ \text{set of all possible outcomes} \right\}$$
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Overview: Selecting students

We're going answer the same question using several different techniques.

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random without replacement. What is the probability that Avi is among the 5 selected students?

- Avocado

can't think about sequences

all students are equally likely to be chosen

weren't told whether order matters:
both permutations or combinations
give the same results!

Selecting students

Method 1: Using permutations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$S = \{ \text{all permutations of 5 students selected from 20} \}$

To find $|S|$: $\frac{20}{\text{1st}} \cdot \frac{19}{\text{2nd}} \cdot \frac{18}{\text{3rd}} \cdot \frac{17}{\text{4th}} \cdot \frac{16}{\text{5th}} = 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

To find # with Avi: 5 cases

$\boxed{A} \underline{19} \underline{18} \underline{17} \underline{16} \oplus \underline{19} \boxed{A} \underline{18} \underline{17} \underline{16} \oplus \underline{19} \underline{18} \boxed{A} \underline{17} \underline{16} \oplus \underline{19} \underline{18} \underline{17} \boxed{A} \underline{16} \oplus \underline{19} \underline{18} \underline{17} \underline{16} \boxed{A}$

\Rightarrow permutations with Avi = $5 \cdot 19 \cdot 18 \cdot 17 \cdot 16$

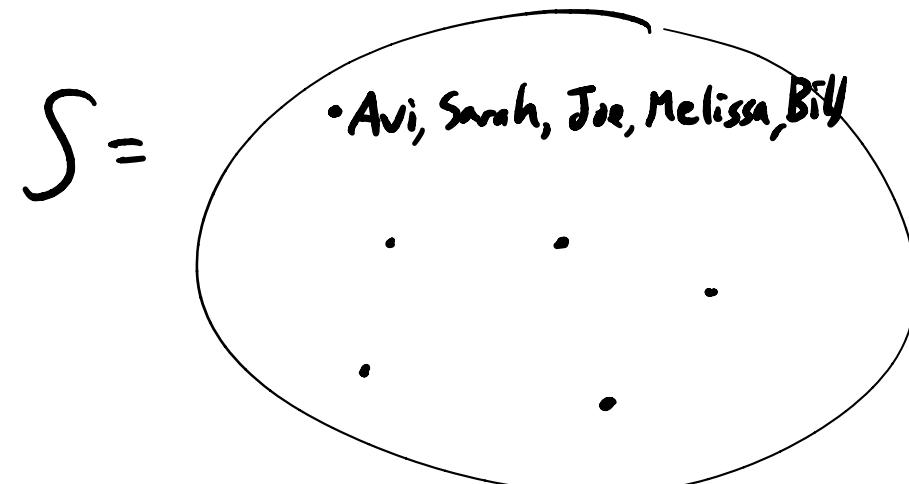
$$P(\text{Avi selected}) = \frac{\# \text{ permutations including Avi}}{\# \text{ total permutations}}$$

$$= \frac{5 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}$$

$$= \frac{5}{20}$$

$$= \boxed{\frac{1}{4}}$$

Remember, $S = \{ \text{all permutations} \}$



permutations: $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16$
of them

Selecting students

Method 2: Using permutations and the complement

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

We want $P(\text{Avi selected})$

→ Idea: Start with $P(\text{Avi not selected})$, then find $P(\text{Avi selected}) = 1 - P(\text{Avi not selected})$.

$$P(\text{Avi not selected}) = \frac{\# \text{permutations WITHOUT Avi}}{\# \text{total permutations}} = \frac{\cancel{19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}}{\cancel{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}} = \frac{15}{20} = \frac{3}{4}$$

$\frac{19}{1^{\text{st}}} \quad \frac{18}{2^{\text{nd}}} \quad \frac{17}{3^{\text{rd}}} \quad \frac{16}{4^{\text{th}}} \quad \frac{15}{5^{\text{th}}}$

$$\Rightarrow P(\text{Avi selected}) = 1 - \frac{3}{4} = \frac{1}{4}$$

benefit: didn't need 5 cases! easier :)

Selecting students

Method 3: Using combinations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

$$S = \{ \text{all combinations of 5 students selected from 20} \}$$
$$|S| = \binom{20}{5} \leftarrow \begin{array}{l} \text{the number of ways to choose 5 students} \\ \text{from 20 such that order doesn't matter} \end{array}$$

$$A = \{ \text{all combinations of 5 including Avi} \}$$

$$|A| = \binom{19}{4} \leftarrow \begin{array}{l} \text{Avi is already included:} \\ \text{choose 4 of the remaining 19} \end{array}$$



Benefit of combinations:
didn't need
5 cases!

$$P(\text{Avi selected}) = \frac{\# \text{ combinations with Avi}}{\# \text{ total combinations}}$$

$$= \frac{\binom{19}{4}}{\binom{20}{5}} = \frac{19!}{4! 15!} \cdot \frac{5! 15!}{20!}$$

↑
reciprocal of $\binom{20}{5}$

$$= \frac{19! 5!}{4! 20!} = \frac{19! 5 \cdot 4!}{4! 20 \cdot 19!}$$

$$= \frac{5}{20} = \frac{1}{4}$$

We could
use either
permutations or
combinations
because we weren't
told if order matters!

$$\text{Notice : } P(\text{Avi selected}) = \frac{5}{20} = \frac{1}{4}$$

Selecting students

Method 4: The "easy" way

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Thought experiment: shuffle all 20 students in a line,
select the first 5  5 "good" positions

$$S = \{ \text{all possible positions Avi could be in} \} \Rightarrow |S| = 20$$
$$P(\text{Avi selected}) = \frac{\# \text{ "good" positions}}{\# \text{ total positions}} = \frac{5}{20} = \boxed{\frac{1}{4}}$$

Question 🤔

Answer at q.dsc40a.com

With vs. without replacement

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

- A. Equal to.
- B. Greater than.
- C. Less than.

without rep: $P(Avi \text{ selected}) = \frac{1}{4} = 25\%$.

with rep: $P(Avi \text{ selected}) = P(\text{Avi selected at least once})$
 $= 1 - P(\text{Avi never selected})$
 $= 1 - \left(\frac{19}{20}\right)^5 \approx 22\%$.

without replacement: pool gets smaller after every selection, increasing the chance Avi is picked

$$\frac{1}{20} \rightarrow \frac{1}{19} \rightarrow \frac{1}{18} \rightarrow \frac{1}{17} \rightarrow \frac{1}{16}$$

getting bigger

vs. with replacement: all $\frac{1}{20}$

⇒ with replacement, could have duplicates; without, all 5 are unique.

Example: An unfair coin

Followup: An unfair coin

In the video you were asked to watch, we flipped an unfair coin 10 times, where the coin was biased such that for each flip, $P(\text{heads}) = \frac{1}{3}$. \rightarrow assume each flip independent

1. What is the probability that we see the specific sequence HHHHTTTTTT?

$$P(\text{HHHHHTTTTT}) = \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^6$$

2. What is the probability that we see exactly 4 heads?

Many such sequences: HHHHHHTTTTTT, HTHTHTHTHT, TTHHHHHHTTTT, ...

ways to arrange 4H, 6T: $\binom{10}{4}$

$$P(4 \text{ H, 6 T}) = \frac{\# \text{ ways of seeing } 4 \text{ H, 6 T}}{\# \text{ sequences}} \times P(\text{one such sequence}) = \binom{10}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^6$$

Followup: An unfair coin

3. What is the probability that we see exactly k heads, where $0 \leq k \leq 10$?

$$P(k \text{ heads}, 10-k \text{ tails}) = \left| \binom{10}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{10-k} \right|$$

\uparrow
ways to arrange k heads, $10-k$ tails

4. What is the probability that we see at least k heads, where $0 \leq k \leq 10$?

$$\begin{aligned} P(\text{at least } k \text{ heads}) &= \overline{P(k \text{ heads}) + P(k+1 \text{ heads}) + \dots + P(10 \text{ heads})} \\ &= \left| \sum_{i=k}^{10} \binom{10}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{10-i} \right| \end{aligned}$$

Example: Poker

Deck of cards

- There are **52** cards in a standard deck.
- Each card has a suit (4 possibilities) and a value (13 possibilities).
 “*face values*”

hearts ♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

diamonds ♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

clubs ♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

spades ♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Deck of cards

1. How many 5 card hands are there in poker?

52 possible, choose 5 \Rightarrow order doesn't matter!



$$\frac{52}{\underline{51}} \frac{50}{\underline{49}} \frac{48}{\underline{47}} \Rightarrow \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}$$

this assumes order matters,
but it doesn't!

2. How many 5 card hands are there where all cards are of the same suit (i.e. a "flush")?

① all hearts

$$\binom{13}{5}$$

② all diamonds

$$\binom{13}{5}$$

③ all clubs

$$\binom{13}{5}$$

④ all spades

$$\binom{13}{5}$$

$$\# = 4 \cdot \binom{13}{5}$$

$\binom{4}{1}$: "4 suits,
choose 1"

13 values,
choose 5

Deck of cards

3. How many 5 card hands are there that include four cards of the same value (i.e. a "four-of-a-kind")?

aaaaab ← 52 - 4

① four 2s : $2, 2, 2, 2, \underline{48}$

② four 3s : $3, 3, 3, 3, \underline{48}$
⋮

⑬ four As : $A, A, A, A, \underline{48}$

$$\# = \boxed{13 \cdot 48}$$

face values

Deck of cards

4. How many 5 card hands are there that have all card values consecutive (i.e. a "straight")? Assume A is the highest

9 cases / starting points

2 - 6

3 - 7

4 - 8

5 - 9

6 - 10

7 - J

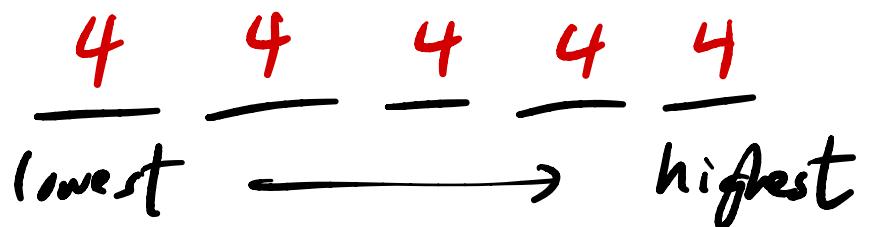
8 - Q

9 - K

10 - A

i.e. 4, 5, 6, 7, 8

after picking a starting point,
for each card, 4 possible suits



$$\# = \boxed{9 \cdot 4^5}$$

starting points

ways of selecting
suits

Deck of cards

5. How many 5 card hands are there that have all card values consecutive and of the same suit (i.e. a "straight flush")?

Same 9 starting points as on the last slide

BUT

all 5 cards need the same suit!

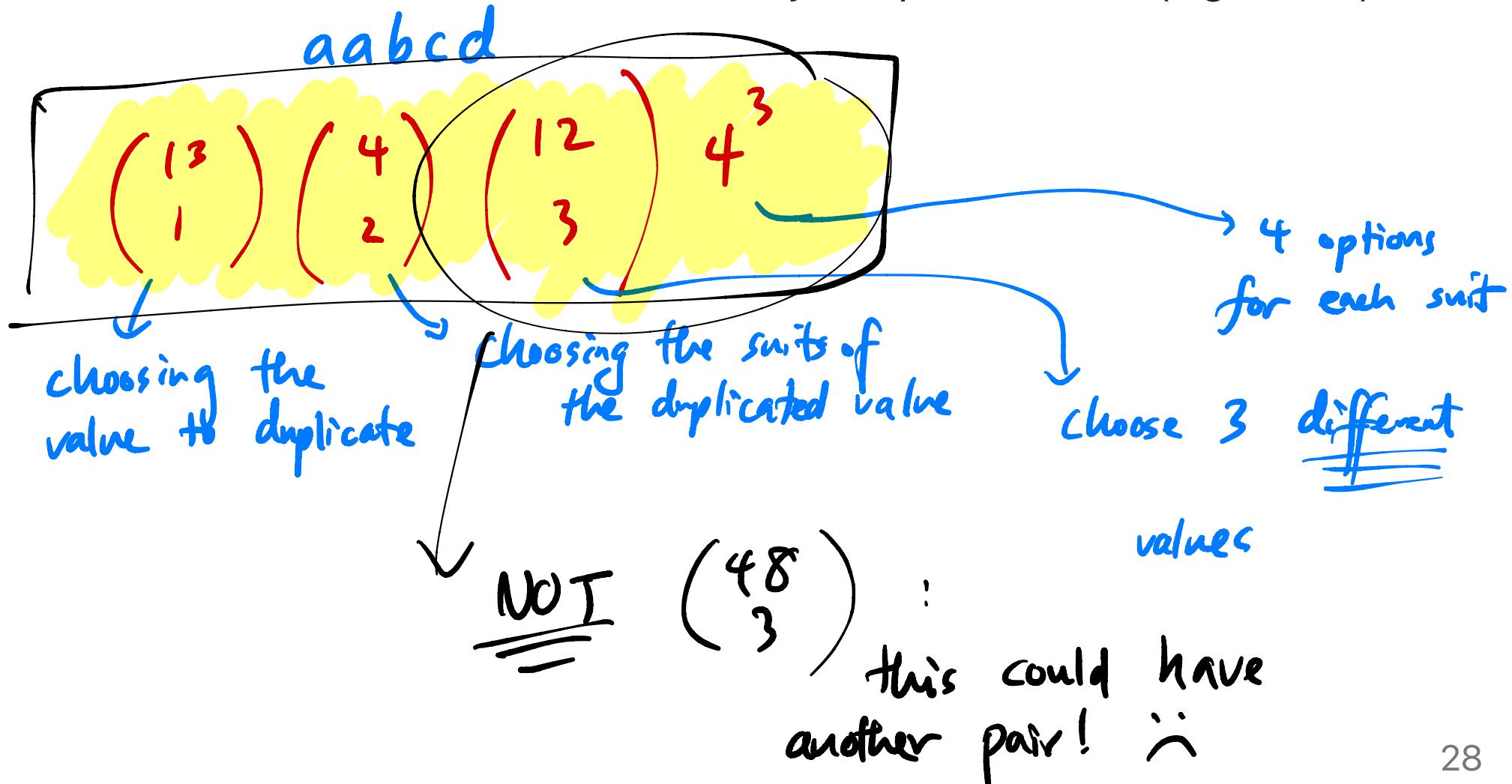
4 possible suits:

$$\# = \boxed{9 \cdot 4 = 36}$$

starting points # suits

Deck of cards

6. How many 5 card hands are there that include exactly one pair of values (e.g. aabcd)?



see practice site for additional explanations

Example: Chess

Chess pieces

(Source: [Spring 2023 Midterm 2, Problem 3](#))

A set of chess pieces has 32 pieces. 16 of these are black and 16 of these are white. In each color, the 16 pieces are:

8 pawns, 2 bishops, 2 knights, 2 rooks, 1 queen, and 1 king

When there are multiple pieces of a given color and type (for example, 8 white pawns), we will assume they are **indistinguishable** from one another.

In this problem, a **lineup** is a way of arranging items in a straight line.

on practice site!

Chess pieces

1. A chess player lines up all 16 white pieces from the set of chess pieces. How many different-looking lineups can be created? Remember, some pieces look the same.

①

8 pawns
2 bishops
2 Knights
2 rooks
1 king
1 queen

$$\frac{16!}{8! \cdot 2! \cdot 2! \cdot 2!}$$

repeat
pawns

repeat
bishops

equal!

②

16 positions, choose 8 to
fill with pawns

$$\binom{16}{8} \binom{8}{2} \binom{6}{2} \binom{4}{2} \binom{2}{1} \binom{1}{1}$$

of the 8 remaining positions, fill 2 with
bishops

of the remaining
6 positions,
fill 2 with
knights:

Chess pieces

2. A chess player lines up all 16 pawns from the set of chess pieces. How many lineups have white pawns on both ends?

fill with 14 pieces:
6 W, 8 B

$$\binom{14}{6} = \binom{14}{8}$$

14 positions, choose 6 to fill with white

Chess pieces

3. A chess player lines up all 16 pawns from the set of chess pieces. Assuming that each different-looking lineup is equally likely, what is the probability that the lineup has two of the same-colored pawns on both ends (both black or both white)?

$$W \text{ --- } W : \binom{14}{6}$$

$$B \text{ --- } B : \binom{14}{6}$$

$$P(\text{same colored pawns}) = \frac{2 \binom{14}{6}}{\binom{16}{8}} = \frac{2 \cdot 14!}{6! 8!} \cdot \frac{8! 8!}{16!}$$

$$= \frac{2 \cdot 14! \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 16 \cdot 15 \cdot 14!} = \boxed{\frac{7}{15}}$$

16 positions, choose 8 to be white

Summary

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	No, order doesn't matter
With replacement Repetition allowed	n^k possible sequences	more complicated: watch this video , or see the domino example
Without replacement Repetition not allowed	$\frac{n!}{(n - k)!}$ possible permutations	$\binom{n}{k}$ combinations