Lecture 16

Independence and Conditional Independence

DSC 40A, Summer 2024

Announcements

- There's a lecture note associated with today's lecture.
- Homework 7 is due **tonight**.
    - Fixed a small typo in Problem 5 (missing percentages).
    - One conditional independence question we'll cover today.
- Homework 8 is due **Thursday**, but it's short: only 2 questions.
- By Friday 8AM, please fill out SETs and the Final Survey.
    - If 90% of the class fills out both, everyone gets 2% extra credit.

The Final Exam is this Friday!

- The Final Exam is on **Friday, September 6th from 11:30AM-2:30PM** in WLH 2113.
- 180 minutes, on paper, no calculators or electronics.
  - **You are allowed to bring two double-sided index cards (4 inches by 6 inches) of notes that you write by hand (no iPad).**
- Content: All lectures (including this week), homeworks (including HW 8), and groupworks. No clustering
- Prepare by practicing with old exam problems at practice.dsc40a.com.
  - There are tons of past probability exams, searchable by topic.
  - Check out the advice page for study strategies.
- No formal review session but lots of office hours this week - come through!

Agenda

- Independence.
- Conditional independence.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Independence

Updating probabilities

- Bayes' Theorem describes how to update the probability of one event, given that another event has occurred.

reversing conditional probabilities

old

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B) \cdot \mathbb{P}(A|B)}{\mathbb{P}(A)}$$

new ratio

  - $\mathbb{P}(B)$ can be thought of as the "prior" probability of B occurring, before knowing anything about A.
  - $\mathbb{P}(B|A)$ is sometimes called the "posterior" probability of B occurring, given that A occurred.

- What if knowing that A occurred doesn't change the probability that B occurs? In other words, what if:

new old ratio =1

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

Independent events

- A and B are **independent events** if one event occurring does not affect the chance of the other event occurring.

$$\boxed{\mathbb{P}(B|A) = \mathbb{P}(B)} \overset{\text{equivalent}}{\Longleftrightarrow} \mathbb{P}(A|B) = \mathbb{P}(A)$$

- Otherwise, A and B are **dependent events**.

- Using Bayes' theorem, we can show that if one of the above statements is true, then so is the other.

with extra knowledge of A, $\mathbb{P}(B)$ is unchanged

Suppose $\mathbb{P}(B|A) = \mathbb{P}(B)$. Let's show $\mathbb{P}(A|B) = \mathbb{P}(A)$

Bayes: $\mathbb{P}(B|A) = \dfrac{\mathbb{P}(A|B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)} = \mathbb{P}(B) \Rightarrow \cancel{\mathbb{P}(B)}\,\mathbb{P}(A|B) = \mathbb{P}(A) \cdot \cancel{\mathbb{P}(B)}$

$\mathbb{P}(A|B) = \mathbb{P}(A)$

Independent events

- **Equivalent definition:** A and B are independent events if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- To check if A and B are independent, use whichever is easiest:
  - $\mathbb{P}(B|A) = \mathbb{P}(B)$.
  - $\mathbb{P}(A|B) = \mathbb{P}(A)$.
  - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

general multiplication rule:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) \cdot \mathbb{P}(A | B) = \mathbb{P}(A) \cdot \mathbb{P}(B | A)$$

Question 🤔

Answer at q.dsc40a.com

can't happen together

Mutual exclusivity and independence

Suppose A and B are two events with non-zero probabilities. Is it possible for A and B to be both mutually exclusive and independent?

- A. Yes.
- B. No.

If A, B independent, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

If A, B mutually exclusive: $\mathbb{P}(A \cap B) = 0$

If both: $\mathbb{P}(A) \cdot \mathbb{P}(B) = 0$

one must be 0

but! both nonzero

impossible

Example: Venn diagrams

$\mathbb{P}(A) = a$

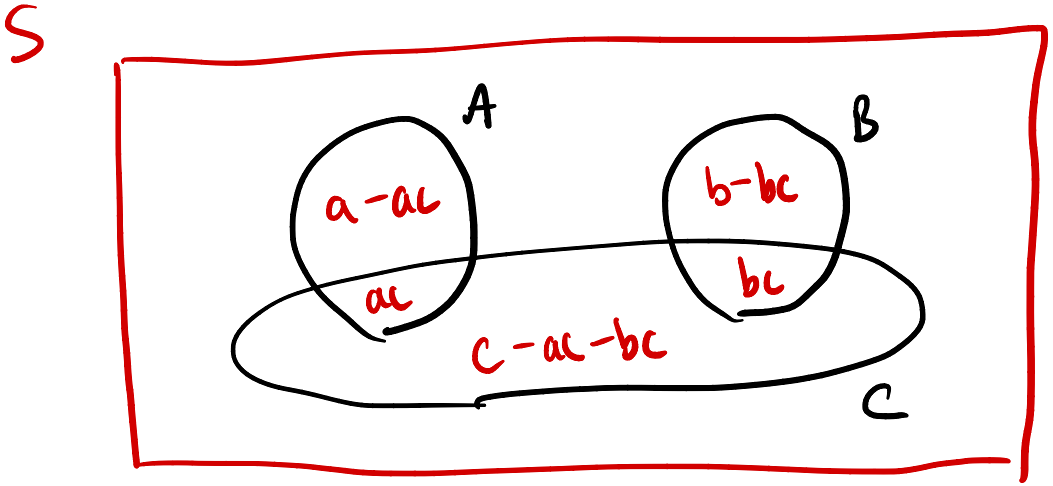
$\mathbb{P}(B) = b$

$\mathbb{P}(C) = c$

For three events A, B, and C, we know that:

- A and C are independent,
- B and C are independent,
- A and B are mutually exclusive,
- $\mathbb{P}(A \cup C) = \frac{2}{3}$, $\mathbb{P}(B \cup C) = \frac{3}{4}$, $\mathbb{P}(A \cup B \cup C) = \frac{11}{12}$.

Find $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(C)$.



3 equations, 3 unknowns:

$\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$ independent

1) $\mathbb{P}(A \cup C) = a + c - ac = \frac{2}{3}$

2) $\mathbb{P}(B \cup C) = b + c - bc = \frac{3}{4}$

3) $\mathbb{P}(A \cup B \cup C) = a + b + c - ac - bc = \frac{11}{12}$

1+2: $a + b + 2c - ac - bc = \frac{2}{3} + \frac{3}{4}$

3: $a + b + c - ac - bc = \frac{11}{12}$

1+2-3: $c = \frac{2}{3} + \frac{3}{4} - \frac{11}{12} = \frac{8 + 9 - 11}{12} = \frac{6}{12} = \boxed{\frac{1}{2}}$

c into 1: $a + \frac{1}{2} - \frac{a}{2} = \frac{2}{3} \Rightarrow \frac{1}{2} a = \frac{1}{6} \Rightarrow \boxed{a = \frac{1}{3}}$

c into 2: $b + \frac{1}{2} - \frac{b}{2} = \frac{3}{4} \Rightarrow \frac{b}{2} = \frac{1}{4} \Rightarrow \boxed{b = \frac{1}{2}}$

$a = \mathbb{P}(A) = \frac{1}{3}$

$b = \mathbb{P}(B) = \frac{1}{2}$

$c = \mathbb{P}(C) = \frac{1}{2}$

Example: Cards

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw two cards, one at a time.
  - A is the event that the first card is a heart.
  - B is the event that the second card is a club.

Yes! $\mathbb{P}(B|A) = \frac{13}{52} = \mathbb{P}(B)$

- If you draw the cards **with** replacement, are A and B independent?
- If you draw the cards **without** replacement, are A and B independent?

No! Once we remove the ♡, the remaining cards are less likely to be ♡ more likely to be other suits. $\mathbb{P}(B|A) = \frac{13}{51}$, $\mathbb{P}(B) = \frac{13}{52}$. Not equal.

Example: Cards

B

A

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Suppose you draw one card from a deck of 52.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent? ✓ Yes!

$\mathbb{P}(A) = 13/52 = 1/4$

$\mathbb{P}(B) = 12/52 = 3/13$

$\mathbb{P}(A \cap B) \overset{?}{=} \mathbb{P}(A) \cdot \mathbb{P}(B) = 3/52 = \frac{1}{4} \cdot \frac{3}{13} = \frac{3}{4 \cdot 13} = \frac{3}{52}$

Another perspective:

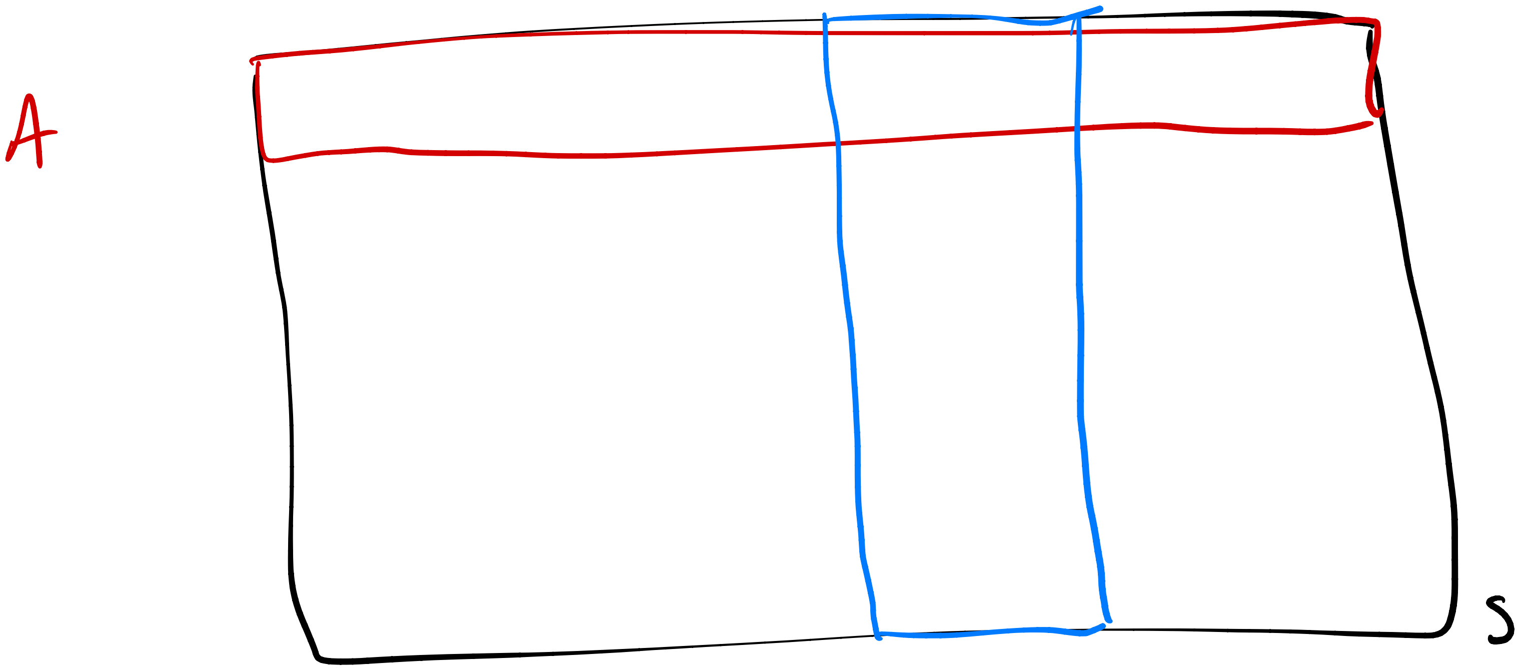
The proportion of face cards w/in A

$\mathbb{P}(B \mid A) = \frac{3}{13}$

equals proportion of face cards in the whole deck

$\mathbb{P}(B) = \frac{12}{52} = \frac{3}{13}$

Visualizing independence when outcomes are equally likely:



$$\mathbb{P}(B \mid A) \overset{\text{independent}}{=\!=} \mathbb{P}(B)$$

proportion of A taken up by B

the proportion of S taken up by B

Assuming independence

- Sometimes we assume that events are independent to make calculations easier.
- Real-world events are almost never exactly independent, but they may be close.

Example: Breakfast

1% of UCSD students are DSC majors. 25% of UCSD students eat avocado toast for breakfast. Assuming that being a DSC major and eating avocado toast for breakfast are independent:

1. What percentage of DSC majors eat avocado toast for breakfast?

$P(\text{avo toast} \mid \text{DSC}) = P(\text{Avocado toast}) = 25\%$

2. What percentage of UCSD students are DSC majors who eat avocado toast for breakfast?

$P(\text{Avo toast} \wedge \text{DSC}) = P(\text{Avo toast}) \cdot P(\text{DSC}) = 0.01 \times 0.25$

$= 0.0025 = \boxed{0.25\%}$

Conditional independence

Conditional independence

- Sometimes, events that are dependent **become** independent, upon learning some new information.
- Or, events that are independent can become dependent, given additional information.

Example: Cards

B A

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, ~~K~~, A

♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K).
- Are A and B independent?

$$\mathbb{P}(B \mid A) = \frac{3}{13}$$

$$\mathbb{P}(B) = \frac{11}{51}$$

not the same!

Another interpretation:

$$\mathbb{P}(A) = \frac{13}{51}, \quad \mathbb{P}(B) = \frac{11}{51}$$

$$\mathbb{P}(A \cap B) = \frac{3}{51} \neq \frac{13}{51} \cdot \frac{11}{51}$$

Example: Cards

A B

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, ~~K~~, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

R → card is red

- Your dog ate the King of Clubs. Suppose you draw one card from a deck of 51.
  - A is the event that the card is a heart.
  - B is the event that the card is a face card (J, Q, K). $\mathbb{P}(A \mid R)$, $\mathbb{P}(B \mid R)$
- Suppose you learn that the card (given!) is red. Are A and B independent given this new information? Yes! Given R,

$$\underbrace{\mathbb{P}(B|A)}_{B|A,C} = \frac{3}{13} = \frac{6}{26} = \underbrace{\mathbb{P}(B)}_{B|C}$$

W/in red cards, A&B are independent!

Conditional independence

- Recall that A and B are independent if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

old definition but with "given C" everywhere

- A and B are **conditionally independent** given C if:

$$\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

- Given that C occurs, this says that A and B are independent of one another.

Practically: check with

$$\frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} \cdot \frac{\mathbb{P}(B \cap C)}{\mathbb{P}(C)}$$

Assuming conditional independence

- Sometimes we assume that events are conditionally independent to make calculations easier.
- Real-world events are almost never exactly conditionally independent, but they may be close.

Example: Harry Potter and Discord

Suppose that 50% of UCSD students like Harry Potter and 80% of UCSD students use Discord. What is the probability that a random UCSD student likes Harry Potter and uses Discord, assuming that these events are conditionally independent given that a person is a UCSD student?

$$\mathbb{P}\left((\text{HP} \cap \text{Discord}) \mid \text{UCSD} \right) = \mathbb{P}(\text{HP} \mid \text{UCSD}) \cdot \mathbb{P}(\text{use Discord} \mid \text{UCSD})$$

$$= (0.5)(0.8)$$

$$= 0.4 = 40\%$$

Question 🤔

Answer at q.dsc40a.com

- Is it reasonable to assume conditional independence of:
  - liking Harry Potter
  - using Discord

  given that a person is a UCSD student?

Yes

- Is it reasonable to assume independence of these events in general, among all people?

No

Age!

Which assumptions do you think are reasonable?

A. Both.

B. Conditional independence only.

C. Independence (in general) only.

D. Neither.

Independence vs. conditional independence

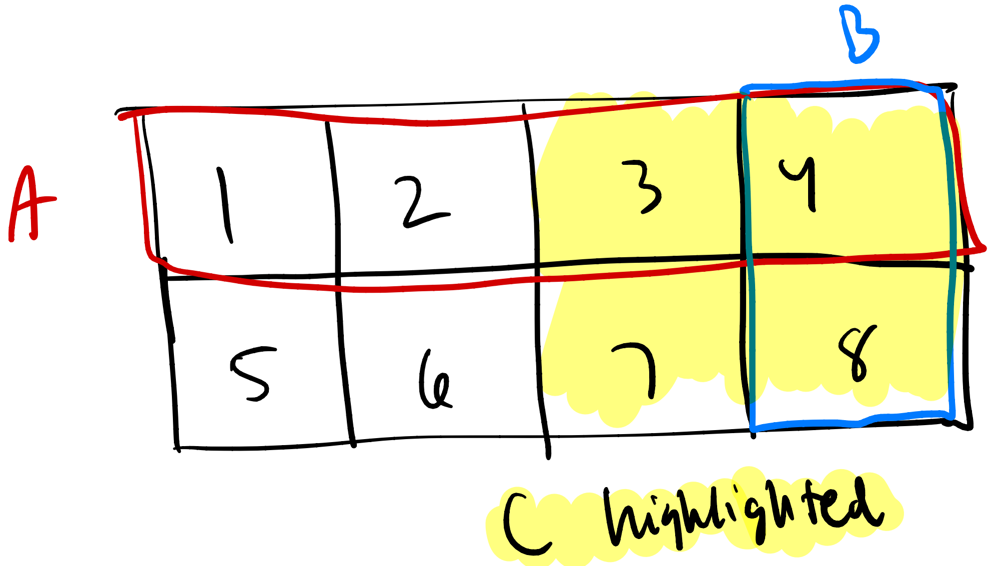
In general, **there is no relationship between independence and conditional independence**. All four scenarios before are possible:

1. A and B **are** independent, and **are** conditionally independent given C.
2. A and B **are** independent, but are **not** conditionally independent given C.
3. A and B are **not** independent, but **are** conditionally independent given C.
4. A and B are **not** independent, and are **not** conditionally independent given C.

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 1: A and B **are** independent, and **are** conditionally independent given C.



A, B indep: $\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{4}$, $\mathbb{P}(A \cap B) = \frac{1}{8}$

A, B, cond. indep given C: $\mathbb{P}(A \cap B \mid C) = \frac{1}{4}$

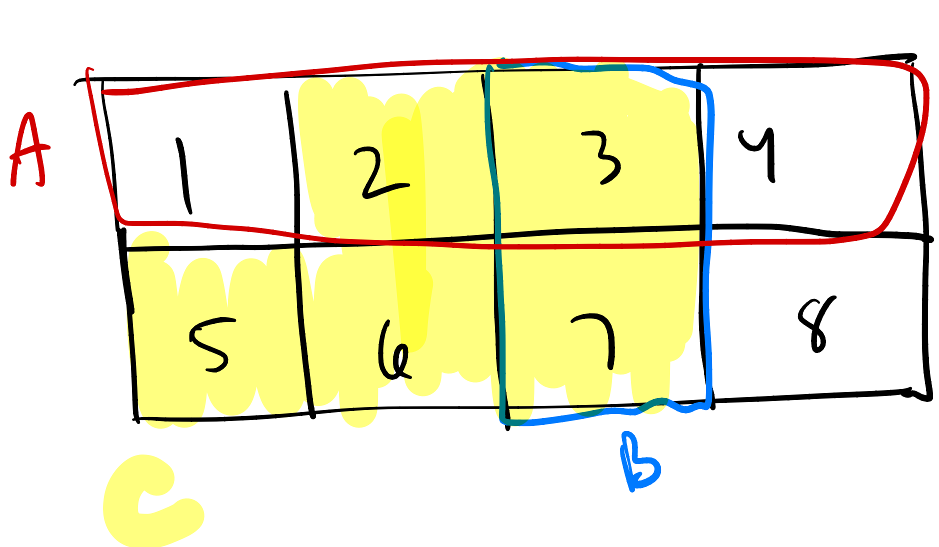
$\mathbb{P}(A \mid C) = \frac{1}{2}$

$\mathbb{P}(B \mid C) = \frac{1}{2}$

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 2: A and B **are** independent, but are **not** conditionally independent given C.



A, B ind ✓

A, B not cond. ind given C

$\mathbb{P}(A \cap B \mid C) = \frac{1}{5}$

$\mathbb{P}(A \mid C) = \frac{2}{5}$

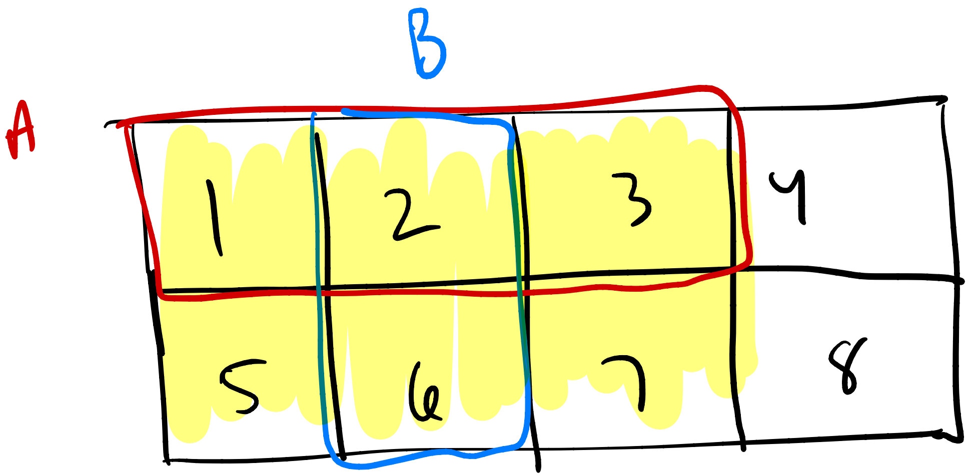
$\mathbb{P}(B \mid C) = \frac{2}{5}$

$\frac{2}{5} \cdot \frac{2}{5} \neq \frac{1}{5}$

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 3: A and B are **not** independent, but **are** conditionally independent given C.



A, B not indp $\quad \mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$

$$\frac{1}{8} \neq \frac{3}{8} \quad \frac{1}{4}$$

A, B cond inp. given C $\quad \mathbb{P}(A \cap B \mid C) = \mathbb{P}(A \mid C) \cdot \mathbb{P}(B \mid C)$

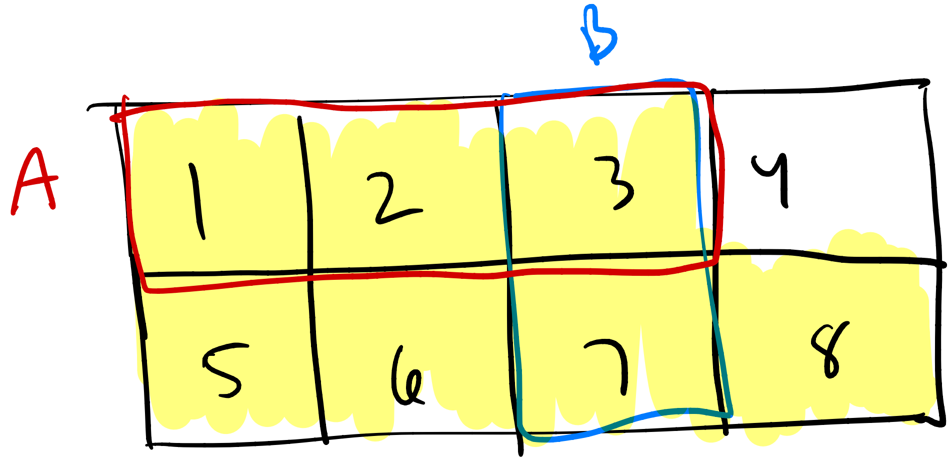
$$\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$$

✓

Example: Constructing events

- Consider a sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ where all outcomes are equally likely.
- Specify events A, B, and C that satisfy the given conditions (e.g. $A = \{2, 5, 6\}$).
- Choose events that are neither impossible nor certain, i.e. $0 < \mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) < 1$.

Scenario 4: A and B are **not** independent, and are **not** conditionally independent given C.



A, B not independent:

$$\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$$

$$\frac{1}{8} \neq \frac{3}{8} \cdot \frac{1}{4}$$

A, B not cond. independent

$$\mathbb{P}(A \cap B \mid C) \neq \mathbb{P}(A \mid C) \cdot \mathbb{P}(B \mid C)$$

$$\frac{1}{7} \neq \frac{3}{7} \cdot \frac{2}{7}$$

Summary

Summary

- Two events A and B are **independent** when knowledge of one event does not change the probability of the other event.
  - Equivalent conditions: $\mathbb{P}(B|A) = \mathbb{P}(B)$, $\mathbb{P}(A|B) = \mathbb{P}(A)$, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.
- Two events A and B are **conditionally independent** given a third event, C, if they are independent given knowledge of event C.
  - Condition: $\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$.
- In general, there is no relationship between independence and conditional independence.
- **Next time**: Using Bayes' Theorem and conditional independence to solve the **classification problem** in machine learning.