DSC 40A - Homework 6

Due: Friday, Nov 25th at 11:59PM

Homeworks are due to Gradescope by 11:59PM on the due date.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homeworks should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it. We encourage you type your solutions in LaTeX, using the Overleaf template on the course website.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of 59 points. The point value of each problem or sub-problem is indicated by the number of avocados shown.

Note: For full credit, make sure to assign pages to questions when you upload your submission to Gradescope. You will lose points if you don't!

Problem 1. Reflection and Feedback Form

Make sure to fill out this Reflection and Feedback Form, linked here, for three points on this homework! This form is primarily for your benefit; research shows that reflecting and summarizing knowledge helps you understand and remember it.

Problem 2. Avi's Bootstraps

Recall from DSC 10 the process of bootstrap resampling. From a population of size n, we draw one random sample of size k, without replacement. Then, we create many bootstrap resamples by sampling k elements from the original sample, with replacement.

Suppose we have a population of 100 stuffed toys, one of which is Avi. From this population, we draw a sample of size 20, without replacement. From this original sample, we create 5 different bootstrap resamples.

a) What is the probability that Avi is included in the original sample?

b) What is the probability that Avi is included in the first resample?

Hint: Don't make any assumptions about whether Avi was included in the original sample.

c) What is the probability that Avi is included in some resample?

Problem 3. Amped Up

Congratulations — you've won a month's supply of energy drinks! Each week, you receive **one prize box with 8 drinks** which contains an assortment of drinks from companies like Monster, Red Bull, Alani etc. Each week's package is a surprise; you don't know which drinks it will contain until you open it. We'll assume for this problem that there are 50 possible drinks, each of which is manufactured in equal quantities, so you're no more likely to get any one drink than any other. We'll also assume that each company only produces one type of energy drink, so (for instance) all drinks produced by Monster are the same, all drinks produced by Red Bull are the same, and so on.

Note: Your finals solutions can include $P(n, k), C(n, k)$ or $\binom{n}{k}$ without simplifying further.

a) Suppose that the content of each week's prize boxes are selected uniformly at random **without replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly 8 different companies?

b) Suppose that the content of each week's prize boxes are selected uniformly at random **without replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly 16 different companies?

c) Suppose that the content of each week's prize boxes are selected uniformly at random **with replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly 16 different companies?

d) Suppose that the content of each week's prize boxes are selected uniformly at random **with replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly **15** different companies?

Problem 4. Pascal's Identity

I have invited 5 friends over for dinner. Suppose I have 11 plates in the cupboard, each of which is a different color, and I want to create a combination of 6 of them. From lecture, we know that this can be done in $\binom{11}{6}$ ways. In this problem, we'll look at another way to arrive at this same result.

a) The purple plate is my favorite. How many ways can I select 6 plates from my collection of 11 plates such that my purple plate is one of the 6 selected?

b) My friend Varun hates purple. How many ways can I select 6 plates from my collection of 11 plates such that my purple plate is **not** one of the 6 selected?

c) Using the results of the previous two parts, how many ways can I select 6 plates from my collection of 11 plates?

Note: You must write your answer in terms of your results to parts (a) and (b); you will get no credit if you write $\binom{11}{6}$.

d) What you've just discovered is an application of Pascal's Identity, which states that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Prove Pascal's identity.

Hint: Start with the left-hand side, use the definition of the binomial coefficient, and try and bring both terms to a common denominator. Another hint — what is $\frac{n}{n!}$?

You may assume $n > k$. *Note however, that in general, there is no restriction on the relative sizes of n and k. If $n < k$, the value of the binomial coefficient is zero and the identity remains valid.*

Side note: Pascal's identity has a close connection to Pascal's triangle.

e) Using Pascal's rule, show Hockey-stick identity:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n-1}{k} + \binom{n-2}{k} + \ldots + \binom{k}{k}$$

The Hockey-stick name comes from the visualized pattern in Pascal's triangle, in which each element's value equals the sum of its left-upper and right-upper neighbors:

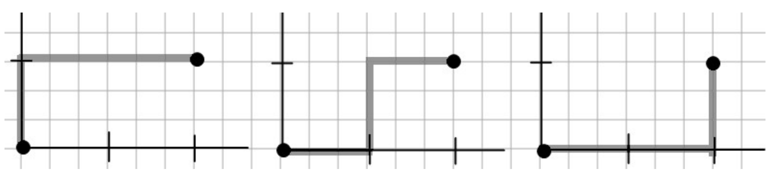
$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&\color{green}{1}&&& \\ &&1&&\color{green}{2}&&1&& \\ &1&&\color{green}{3}&&3&&1& \\ 1&&4&&\color{red}{6}&&4&&1 \end{array}$$

$$\color{red}{\binom{4}{2}} = \color{green}{\binom{3}{1}} + \color{green}{\binom{2}{1}} + \color{green}{\binom{1}{1}}$$

Problem 5. Paths

Suppose we are given two nonnegative integers n and m. We want to find all the different paths from $(0, 0)$ to (n, m) that use only upward and rightward steps of length one.

For example, if $n = 2$ and $m = 1$, the goal is to find all paths from the origin to $(2, 1)$. There are three such paths, as shown below.



We can represent each path as a sequence of up and right arrows. In the example above, these sequences would be ↑→→ for the leftmost path, →↑→ for the middle path, and →→↑ for the rightmost path.

a) Find all paths from the origin to $(3, 2)$. You can either draw them out or describe them using up and right arrows. No explanation needed.

b) For nonnegative integers n and m, let $P(n, m)$ be the number of paths from $(0, 0)$ to (n, m) using upward and rightward steps of length 1. Explain why $P(n, m)$ satisfies each of the following:

$$P(0, m) = 1 \tag{1}$$
$$P(n, 0) = 1 \tag{2}$$
$$P(n, m) = P(n-1, m) + P(n, m-1) \quad \text{for } n > 0 \text{ and } m > 0. \tag{3}$$

c) In the supplementary Jupyter notebook (linked), fill in the blanks (marked by ...) in the provided partially complete `paths` function. This function should take as input two nonnegative integers n and m return an array containing all the different paths from $(0, 0)$ to (n, m) that use only upward and rightward steps of length one.

d) For nonnegative integers n and m, how many paths are there from $(0, 0)$ to (n, m) using upward and rightward steps of length 1? Give your answer as a formula involving n and m, and justify your answer.

Problem 6. House of cards

You have a standard deck of card containing 52 cards. There are 13 cards in each of 4 suits (hearts $\heartsuit$, spades $\spadesuit$, diamonds $\diamondsuit$, and clubs $\clubsuit$.) Within a suit, the 13 cards each have a different rank. In ascending order, these ranks are 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace.

You are playing a four-player card game using two regular decks of cards. Each player will be dealt 26 cards. Two identical cards, with the same rank and same suit, are called a pair. A pair can only be outranked (beaten) by a pair of higher ranked cards within the same suit.

a) Suppose both decks of cards are first shuffled together and then dealt out to the four players. What is the probability that you get a pair of Kings of Hearts?

Use $P(n, k)$ notation and interpret the solution in terms of ordered choices, or permutations. Next, rewrite the expression using $C(n, k)$ notation and interpret the solution in terms of unordered choices, or combinations.

b) Suppose both decks of cards are first shuffled together and then dealt out to the four players. You are dealt two Kings of Hearts. What is the probability that some other player has a pair of Aces of Hearts?

c) Suppose both decks of cards are first shuffled together and then dealt out to the four players. You are dealt two Kings of Hearts and you do not have any Aces of Hearts. What is the probability that some other player has a pair of Aces of Hearts?

d) Suppose both decks of cards are first shuffled together and then dealt out to the four players. What is the probability that you get either a pair of Kings of Hearts or a pair of Aces of Hearts?