DSC 40A - Homework 6

Due: Tuesday, May 23 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59PM on the due date. You can use a slip day to extend the deadline by 24 hours; you have four slip days to use in total throughout the quarter.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homeworks should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it. We encourage you type your solutions in LaTeX, using the Overleaf template on the course website.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of 52 points. The point value of each problem or sub-problem is indicated by the number of avocados shown.

Notes:

- For full credit, make sure to **assign pages to questions** when you upload your submission to Gradescope.
- Throughout this homework, it's fine to leave answers unsimplified, in terms of factorials, exponents, the permutation formula $P(n, k)$, and the binomial coefficient $\binom{n}{k}$.

Problem 1. Reflection and Feedback Form

Make sure to fill out this Reflection and Feedback Form, linked here for three points on this homework! This form is primarily for your benefit; research shows that reflecting and summarizing knowledge helps you understand and remember it.

Problem 2. Probability Rules for Three Events

a) The multiplication rule for two events says:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

Use the multiplication rule for two events to prove the multiplication rule for three events:

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|(A \cap B))$$

Hint: You can think of $A \cap B \cap C$ as $(A \cap B) \cap C$.

b) Suppose E, F, and G are events. Explain in words why:

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$$

Intuitively, the relationship between $\cap$ and $\cup$ is similar to the relationship between multiplication and addition; if e, f, g are numbers, then $(e + f) \cdot g = e \cdot g + f \cdot g$ as well.

c) The general addition rule for any two events says:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Use the general addition rule for two events, along with the result of part (b), to prove the general addition rule for three events:

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

This is often called the "Principle of Inclusion-Exclusion."

d) To identify what students find most important in DSC 10, we want to administer a survey to the students in DSC 20, DSC 30, and DSC 40A. Consider the following information:

- There are 300 students taking at least one of DSC 20, DSC 30, or DSC 40A right now.
- 200 students are taking DSC 20 right now, and 50 students taking DSC 30 right now. There are no students taking both DSC 20 and DSC 30 right now.
- 50 students are taking both DSC 20 and DSC 40A right now, and 30 students are taking both DSC 30 and DSC 40A right now.

Suppose I choose a single student uniformly at random from the population of students taking at least one of DSC 20, DSC 30, and DSC 40A. What is the probability that they are enrolled in DSC 40A? Simplify your answer.

Problem 3. TritonGPT

To help promote the new TritonGPT bot among students, UCSD's Information Technology (IT) department has decided* to offer a lottery among all 187 students taking DSC 40A this quarter. Each student will be randomly assigned a lottery ticket numbered $1, 2, \ldots, 187$. The IT department will then randomly generate a winning number, and the student with that same number on their lottery ticket will win free lifetime access to ChatGPT Plus.

The IT department announces that the winning number is 25.

a) If you only look at the first (leftmost) digit of your lottery number and see that it's a 2, what is the probability that you've won the lottery?

b) If you glance at your lottery number and see that it contains a 2 somewhere, what is the probability that you've won the lottery?

c) If you glance at your lottery number and see that it contains exactly one 2, what is the probability that you've won the lottery?

**This isn't actually happening, unfortunately.*

Problem 4. YogurtWorld

The tutors in DSC 40A love getting frozen yogurt ("froyo") at YogurtWorld, but recently, they've been disappointed. That's because each time someone visits YogurtWorld, there is a 0.03% chance that the froyo machine is broken. In this problem, we'll assume that YogurtWorld only has one froyo machine.

a) Suppose you visit YogurtWorld v times, where v is a positive integer. What is the probability that the froyo machine is broken at least once?

b) Suppose you visit YogurtWorld v times, where v is a positive integer. What is probability that your vth visit is the first visit at which the froyo machine is broken?

c) Suppose you visit YogurtWorld 12 times. What is the probability that the ice cream machine is broken exactly 5 times?

Problem 5. Cavocado

The tutors in DSC 40A also like eating at Cava, a Mediterranean restaurant off-campus. They also like eating avocados.

In this problem, we will consider various permutations of the string **CAVOCADO**. Unless specified, a permutation of CAVOCADO will be of the same length as CAVOCADO, i.e. also of length 8.

a) Prove that the number of permutations of CAVOCADO is:

$$\frac{8!}{2!2!2!}$$

Note: This is really meant to be the problem "How many permutations of CAVOCADO are there?" but we've framed it as a "proof" so that you know what the answer is supposed to be.

b) How many permutations of CAVOCADO start with a C?

c) How many permutations of CAVOCADO contain the letters CAD next to each other, in that order?

d) How many permutations of CAVOCADO contain the letters CAD next to each other, in any order?

e) How many permutations of CAVOCADO contain all of the vowels **before** all of the consonants? (A and O are vowels, and C, D, and V are consonants.)

f) Prove that the number of permutations of CAVOCADO of **length 4** is 354.

Hint: Break the problem into three cases.

Problem 6. Prime Factorization

We've recorded a hint/walkthrough video for this problem — we highly encourage you to watch it: https://youtu.be/Jx1GrwQ44HU

Recall, a **prime number** is a natural number greater than 1 that cannot be written as a product of smaller natural numbers. For instance, 2 and 3 are prime, while 6 and 24 are not (because $6 = 2 \cdot 3$ and $24 = 2 \cdot 12 = 3 \cdot 8 = ...$). Natural numbers that are not prime (and not equal to 1) are called **composite**.

The **Fundamental Theorem of Arithmetic** states that every natural number greater than 1 can be written as the product of prime factors, and that every natural number has a unique representation in this form. (If a is a factor of b, then $\frac{b}{a}$ is a whole number.)

For example, 12 can be written as $2^2 \cdot 3$, and 1200 can be written as $2^4 \cdot 3 \cdot 5^2$.

To compute the prime factorization of a number, one strategy is to repeatedly divide by the smallest prime factor that divides that number. For instance, here's the process of prime factoring 1200 ("pf" stands for "prime factor"):

$$
\begin{aligned}
1200 &= 1200 \\
&= 2 \cdot 600 \quad \text{(2 is the smallest pf of 1200)} \\
&= 2 \cdot 2 \cdot 300 \quad \text{(2 is the smallest pf of 600)} \\
&= 2 \cdot 2 \cdot 2 \cdot 150 \quad \text{(2 is the smallest pf of 300)} \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 75 \quad \text{(2 is the smallest pf of 150)} \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 25 \quad \text{(3 is the smallest pf of 75)} \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \quad \text{(5 is the smallest pf of 25)} \\
&= 2^4 \cdot 3 \cdot 5^2
\end{aligned}
$$

A handy shortcut is that if the number we are prime factoring ends in a 0, we know that is divisible by 10 and hence we can pull out a factor of $2 \cdot 5$.

You may be thinking, **what does this have anything to do with combinatorics?** Good question — we're getting there.

Suppose we now want to determine the number of factors that 1200 has. One rather painful way to do it would be to enumerate them manually — 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, etc. Of course, there's a better way.

Each factor of 1200 will be made up of some number of 2s, some number of 3s, and some number of 5s, all multiplied together. If you don't believe this, write out a few factors of 1200, and you will see that each of them is a product of 2s, 3s, and 5s. For example:

$$
\begin{aligned}
24 &= 2^3 \cdot 3 \\
30 &= 2 \cdot 3 \cdot 5 \\
120 &= 2^5 \cdot 5
\end{aligned}
$$

There are 5 options for the number of 2s a factor could have: 0, 1, 2, 3 or 4 (meaning a factor of 1200 could either have a factor of 2^0, or 2^1, or 2^2, or 2^3, or 2^4). Similarly, there are 2 options for the number of 3s a factor could have (either 0 or 1) and 3 options for the number of 5s (0, 1, or 2).

In other words, each factor of 1200 will look like

$$2^a \cdot 3^b \cdot 5^c$$

where

$$0 \leq a \leq 4 \qquad 0 \leq b \leq 1 \qquad 0 \leq c \leq 2$$

Since we're making three successive choices, we take the product of the number of choices at each step, yielding

$$\text{number of factors of } 1200 = (4+1)(1+1)(2+1) = 30$$

Note that this counts both 1 ($2^0 \cdot 3^0 \cdot 5^0$) and 1200 ($2^4 \cdot 3^1 \cdot 5^2$) as factors of 1200.

In this problem, you *must* use this approach to find the number of factors, and you must show your work.

a) Let's try out a few examples.

1. Compute the prime factorization of 2500, and use it to determine the number of factors of 2500.

2. Compute the prime factorization of 7260, and use it to determine the number of factors of 7260.

In both subparts, manually listing out all factors will not receive credit; you must use the counting technique outlined in the question.

b) How many factors of 2500 are multiples of 50?

Hint: Think about how this condition restricts the number of options for each prime factor.

c) How many factors of 7260 are multiples of 2 but not multiples of 121?

Again, think about the possibilities for the exponents on each of the prime factors.