

Lecture 6

Dot Products and Projections

DSC 40A, Summer 2024

Announcements

- Homework 2 is due **tonight**. Remember that using the Overleaf template is required for Homework 2 (and only Homework 2).
- Check out the [new FAQs page](#) and the [tutor-created supplemental resources](#) on the course website.
 - The proof that we were going to cover last class (that $R_{\text{sq}}(w_0^*, w_1^*) = \sigma_y^2(1 - r^2)$) is now in the [FAQs page, under Week 3](#).

Agenda

- Why?
- Dot products.
- Spans and projections.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions"
link in the top right corner of dsc40a.com.

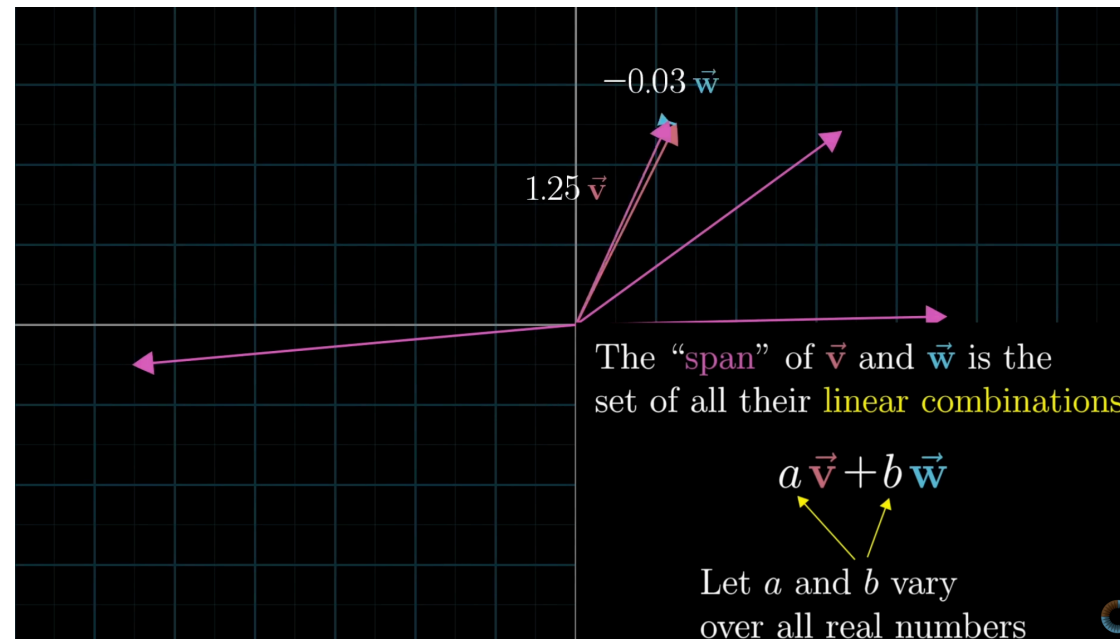
Dot products

Wait... why do we need linear algebra?

- Soon, we'll want to make predictions using more than one feature.
 - Example: Predicting commute times using departure hour and temperature.
- Thinking about linear regression in terms of **matrices and vectors** will allow us to find hypothesis functions that:
 - Use multiple features (input variables).
 - Are non-linear in the features, e.g. $H(x) = w_0 + w_1x + w_2x^2$.
- Before we dive in, let's review.

Spans of vectors

- One of the most important ideas you'll need to remember from linear algebra is the concept of the **span** of one or more vectors.
- To jump start our review of linear algebra, let's start by watching 🎥 [this video by 3blue1brown](#).



Warning

- We're **not** going to cover every single detail from your linear algebra course.
- There will be facts that you're expected to remember that we won't explicitly say.
 - For example, if A and B are two matrices, then $AB \neq BA$.
 - This is the kind of fact that we will only mention explicitly if it's directly relevant to what we're studying.
 - But you still need to know it, and it may come up in homework questions.
- We **will** review the topics that you really need to know well.

Vectors

- A **vector** in \mathbb{R}^n is an **ordered collection of n numbers**.
- We use lower-case letters with an arrow on top to represent vectors, and we usually write vectors as **columns**.

$$\vec{v} = \begin{bmatrix} 8 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

- Another way of writing the above vector is $\vec{v} = [8, 3, -2, 5]^\top$.
- Since \vec{v} has four **components**, we say $\vec{v} \in \mathbb{R}^4$.

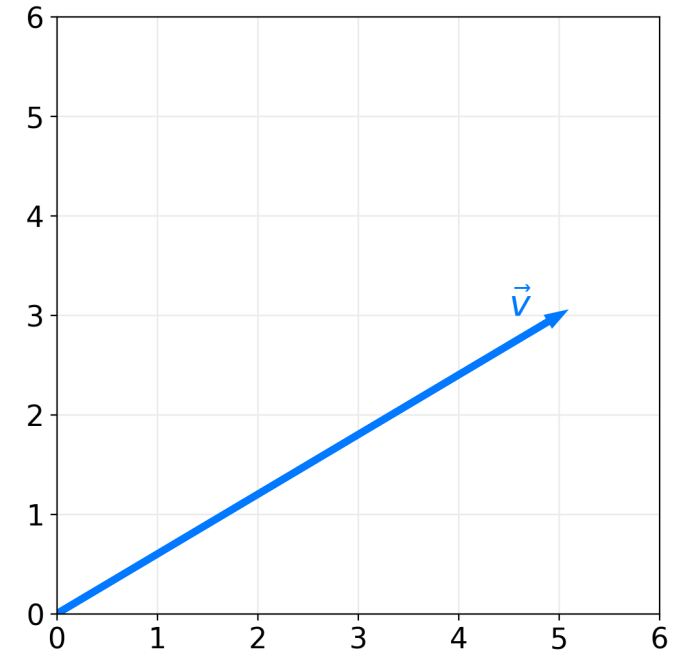
The geometric interpretation of a vector

- A vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is an arrow to the point (v_1, v_2, \dots, v_n) from the origin.

- The **length**, or L_2 **norm**, of \vec{v} is:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

- A vector is sometimes described as an object with a **magnitude/length** and **direction**.



Dot product: coordinate definition

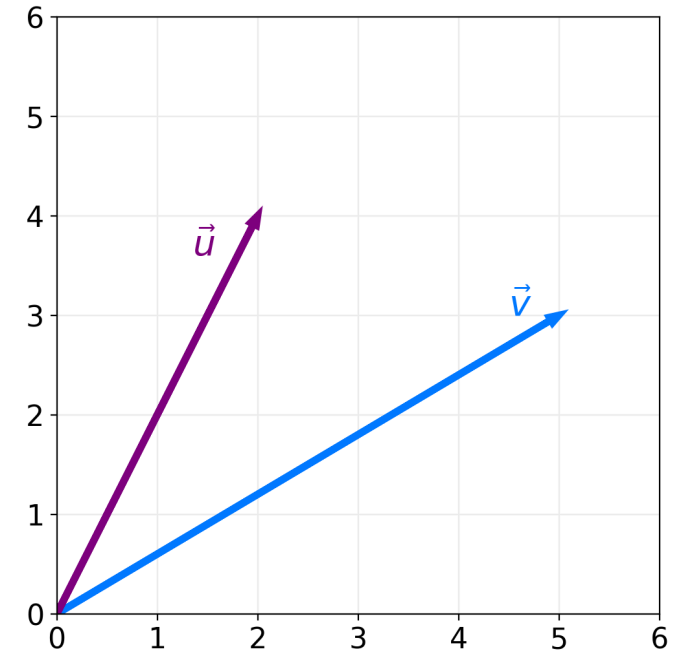
- The **dot product** of two vectors \vec{u} and \vec{v} in \mathbb{R}^n is written as:

$$\vec{u} \cdot \vec{v} = \vec{u}^\top \vec{v}$$

- The computational definition of the dot product:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

- The result is a **scalar**, i.e. a single number.



Question 🤔

Answer at q.dsc40a.com

Which of these is another expression for the length of \vec{v} ?

- A. $\vec{v} \cdot \vec{v}$
- B. $\sqrt{\vec{v}^2}$
- C. $\sqrt{\vec{v} \cdot \vec{v}}$
- D. \vec{v}^2
- E. More than one of the above.

Dot product: geometric definition

- The computational definition of the dot product:

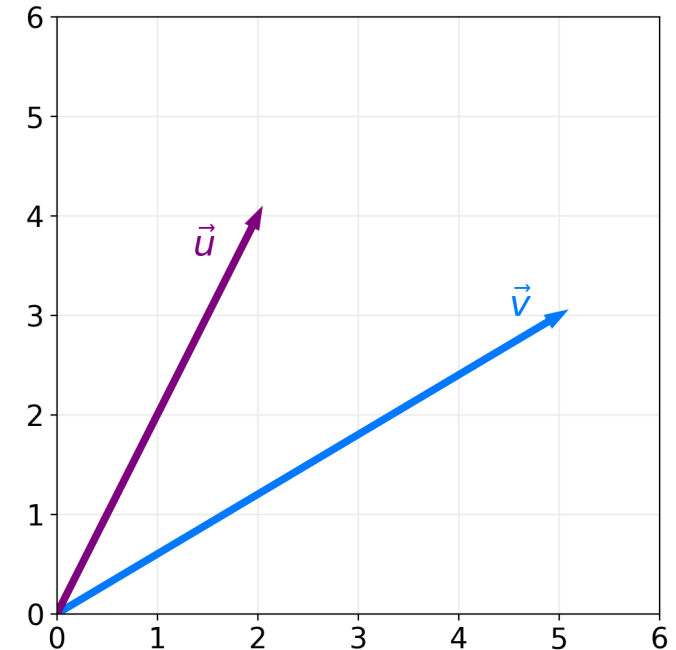
$$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

- The geometric definition of the dot product:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

where θ is the angle between \vec{u} and \vec{v} .

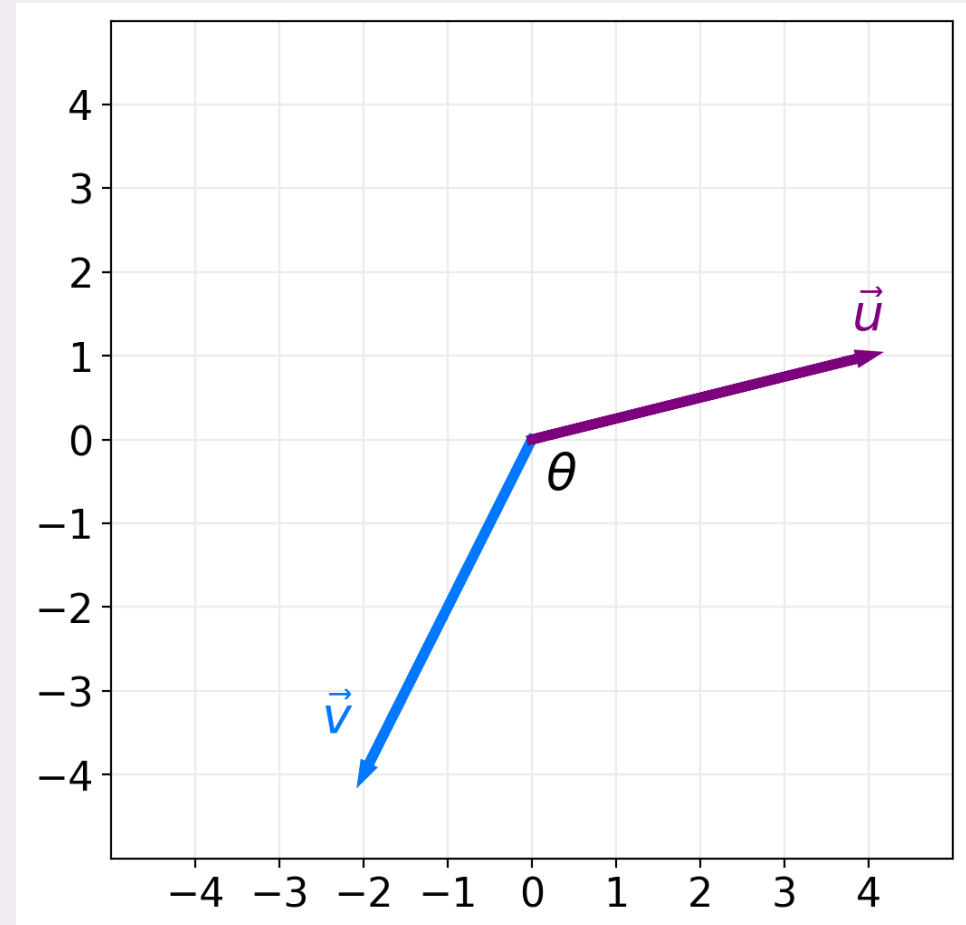
- The two definitions are equivalent! This equivalence allows us to find the angle θ between two vectors.



Question 🤔

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What is the value of θ in the plot to the right?



Orthogonal vectors

- Recall: $\cos 90^\circ = 0$.
- Since $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$, if the angle between two vectors is 90° , their dot product is $\|\vec{u}\| \|\vec{v}\| \cos 90^\circ = 0$.
- If the angle between two vectors is 90° , we say they are perpendicular, or more generally, **orthogonal**.
- **Key idea:**

$$\text{two vectors are } \mathbf{orthogonal} \iff \vec{u} \cdot \vec{v} = 0$$

Exercise

Find a non-zero vector in \mathbb{R}^3 orthogonal to:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -8 \end{bmatrix}$$

Spans and projections

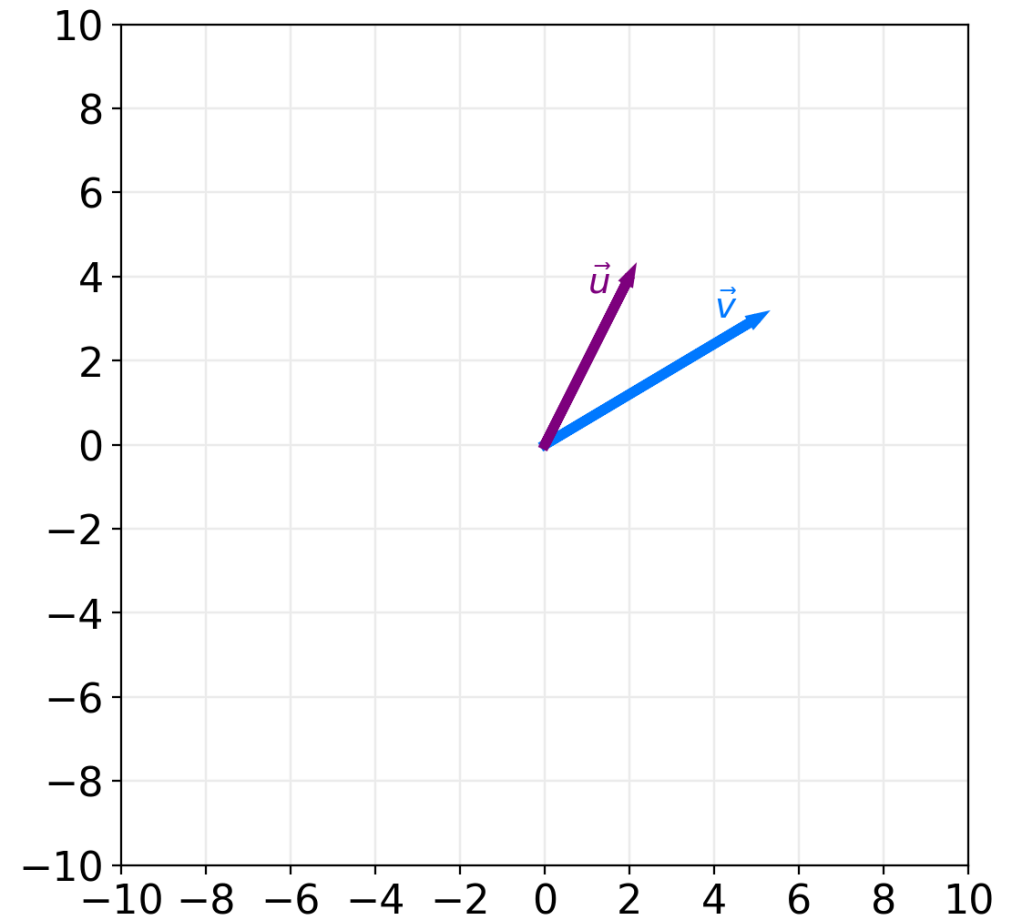
Adding and scaling vectors

- The sum of two vectors \vec{u} and \vec{v} in \mathbb{R}^n is the **element-wise sum** of their components:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

- If c is a scalar, then:

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$



Linear combinations

- Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ all be vectors in \mathbb{R}^n .
- A **linear combination** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ is any vector of the form:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_d\vec{v}_d$$

where a_1, a_2, \dots, a_d are all scalars.

Span

- Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ all be vectors in \mathbb{R}^n .
- The **span** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ is the set of all vectors that can be created using linear combinations of those vectors.
- Formal definition:

$$\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_d\vec{v}_d : a_1, a_2, \dots, a_n \in \mathbb{R}\}$$

Exercise

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and let $\vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Is $\vec{y} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2)$?

If so, write \vec{y} as a linear combination of \vec{v}_1 and \vec{v}_2 .

To whoever teaches this in the future: the slides afterwards were improved in Lecture 7, so you may want to bring some of that material to this lecture.

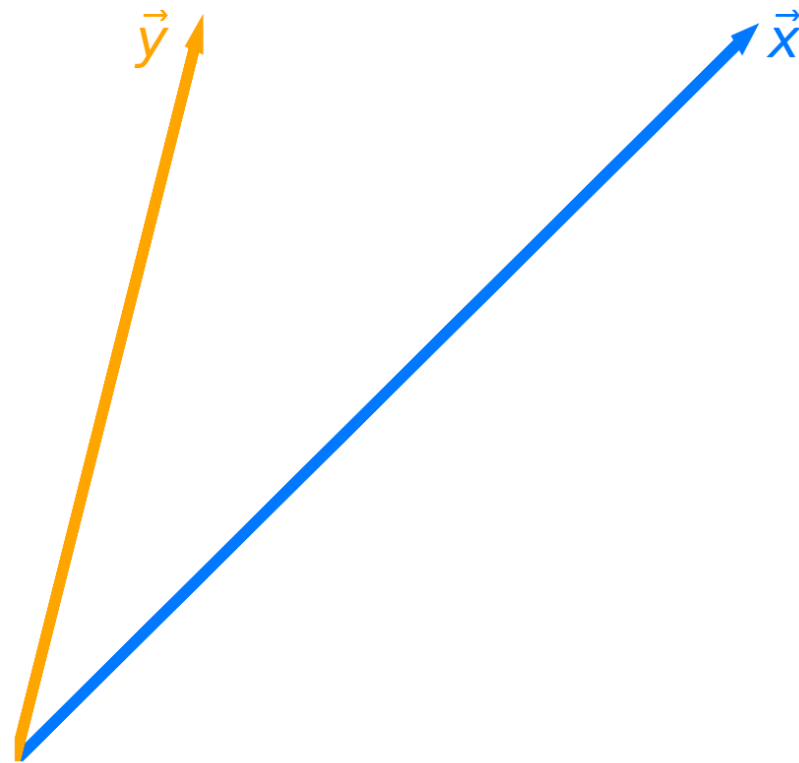
Projecting onto a single vector

- Let \vec{x} and \vec{y} be two vectors in \mathbb{R}^n .
- The span of \vec{x} is the set of all vectors of the form:

$$w\vec{x}$$

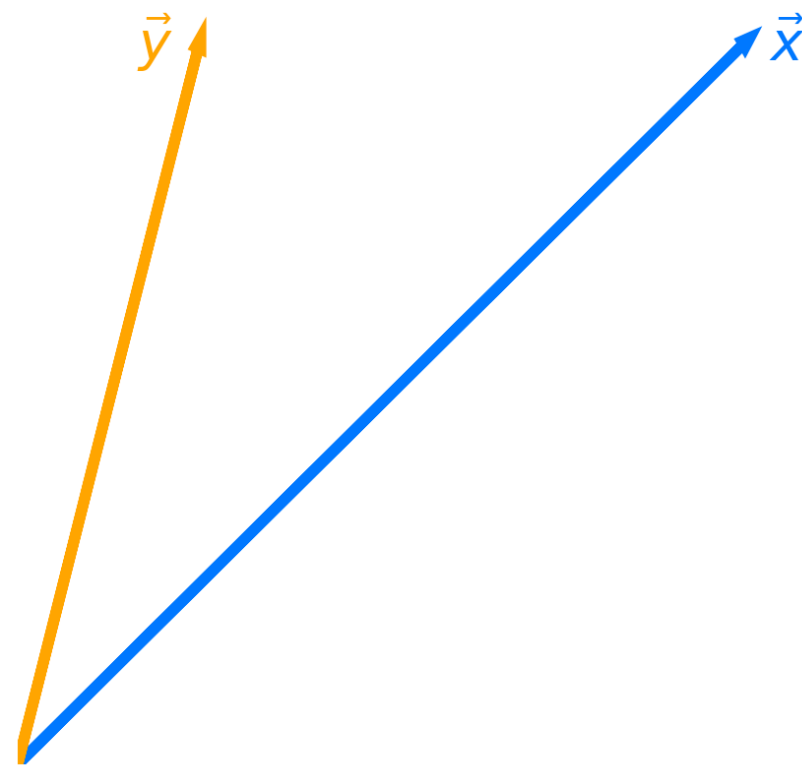
where $w \in \mathbb{R}$ is a scalar.

- **Question:** What vector in $\text{span}(\vec{x})$ is closest to \vec{y} ?
- The vector in $\text{span}(\vec{x})$ that is closest to \vec{y} is the **projection of \vec{y} onto $\text{span}(\vec{x})$** .



Projection error

- Let $\vec{e} = \vec{y} - w\vec{x}$ be the **projection error**: that is, the vector that connects \vec{y} to $\text{span}(\vec{x})$.
- **Goal**: Find the w that makes \vec{e} as short as possible.
 - That is, minimize:
$$\|\vec{e}\|$$
 - Equivalently, minimize:
$$\|\vec{y} - w\vec{x}\|$$
- **Idea**: To make \vec{e} as short as possible, it should be **orthogonal** to $w\vec{x}$.



Minimizing projection error

- **Goal:** Find the w that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible.
- **Idea:** To make \vec{e} as short as possible, it should be **orthogonal** to $w\vec{x}$.
- Can we prove that making \vec{e} orthogonal to $w\vec{x}$ minimizes $\|\vec{e}\|$?

Minimizing projection error

- Goal: Find the w that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible.
- Now we know that to minimize $\|\vec{e}\|$, \vec{e} must be orthogonal to $w\vec{x}$.
- Given this fact, how can we solve for w ?

Orthogonal projection

- **Question:** What vector in $\text{span}(\vec{x})$ is closest to \vec{y} ?
- **Answer:** It is the vector $w^* \vec{x}$, where:

$$w^* = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}$$

- Note that w^* is the solution to a minimization problem, specifically, this one:

$$\text{error}(w) = \|\vec{e}\| = \|\vec{y} - w\vec{x}\|$$

- We call $w^* \vec{x}$ the **orthogonal projection of \vec{y} onto $\text{span}(\vec{x})$** .
 - Think of $w^* \vec{x}$ as the "shadow" of \vec{y} .


Exercise

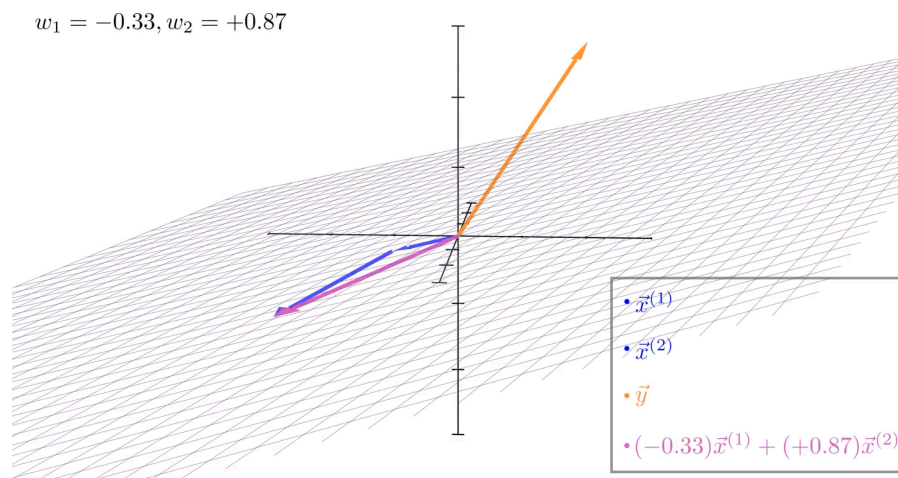
Let $\vec{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$.

What is the orthogonal projection of \vec{a} onto $\text{span}(\vec{b})$?

Your answer should be of the form $w^* \vec{b}$, where w^* is a scalar.

Moving to multiple dimensions

- Let's now consider three vectors, \vec{y} , $\vec{x}^{(1)}$, and $\vec{x}^{(2)}$, all in \mathbb{R}^n .
- Question:** What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to \vec{y} ?
 - Vectors in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ are of the form $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}$, where $w_1, w_2 \in \mathbb{R}$ are scalars.
- Before trying to answer, let's watch  [this animation that Jack, one of our tutors, made](#).



Minimizing projection error in multiple dimensions

- **Question:** What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to \vec{y} ?
 - That is, what vector minimizes $\|\vec{e}\|$, where:

$$\vec{e} = \vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}$$

- **Answer:** It's the vector such that $w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)}$ is **orthogonal** to \vec{e} .
- **Issue:** Solving for w_1 and w_2 in the following equation is difficult:

$$\left(w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)} \right) \cdot \underbrace{\left(\vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)} \right)}_{\vec{e}} = 0$$

What's next?

- It's hard for us to solve for w_1 and w_2 in:

$$\left(w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)} \right) \cdot \underbrace{\left(\vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)} \right)}_{\vec{e}} = 0$$

- **Solution:** Combine $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ into a single matrix, X , and express $w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)}$ as a matrix-vector multiplication, $X\vec{w}$.
- **Next time:** Formulate linear regression in terms of matrices and vectors!