DSC 40A - Homework 7

due Tuesday, September 3rd at 11:59PM

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59PM on the due date.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homeworks should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it. We encourage you type your solutions in LaTeX, using the Overleaf template on the course website.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of 26 points. The point value of each problem or sub-problem is indicated by the number of avocados shown.

Notes:

- For full credit, make sure to **assign pages to questions** when you upload your submission to Gradescope.
- Throughout this homework, it's fine to leave answers unsimplified, in terms of factorials, exponents, the permutation formula $P(n,k)$, and the binomial coefficient $\binom{n}{k}$.

Problem 1. Reflection and Feedback Form

Make sure to fill out this Reflection and Feedback Form, linked here for three points on this homework! This form is primarily for your benefit; research shows that reflecting and summarizing knowledge helps you understand and remember it.

Problem 2. Amped Up

Congratulations — you've won a month's supply of energy drinks! Each week, you receive **one prize box with 8 drinks** which contains an assortment of drinks from companies like Monster, Red Bull, Alani etc. Each week's package is a surprise; you don't know which drinks it will contain until you open it. We'll assume for this problem that there are 50 possible drinks, each of which is manufactured in equal quantities, so you're no more likely to get any one drink than any other. We'll also assume that each company only produces one type of energy drink, so (for instance) all drinks produced by Monster are the same, all drinks produced by Red Bull are the same, and so on.

a) Suppose that the content of each week's prize boxes are selected uniformly at random **without replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly 8 different companies?

b) Suppose that the content of each week's prize boxes are selected uniformly at random **without replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly 16 different companies?

c) Suppose that the content of each week's prize boxes are selected uniformly at random **with replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly 16 different companies?

d) Suppose that the content of each week's prize boxes are selected uniformly at random **with replacement** from among the 50 possibilities. If you take 2 weeks worth of prize boxes (i.e. 16 total energy drinks), what is the probability that you end up with drinks from exactly **15** different companies?

Problem 3. Combinatorial Proofs

A combinatorial proof is a method of proving that two mathematical expressions are equal by arguing that they both count the number of elements in the same set, i.e. they both count the number of ways to do the same thing. In **Lecture 14**, we gave a combinatorial proof of the fact that $\binom{9}{2} + \binom{9}{3} = \binom{10}{3}$; you may want to review that before proceeding.

Before we set you off to work on your own combinatorial proofs, we'll work through another example together. Here, we'll prove that:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

A good first step is to verify yourself that the equation is true for some small values of n, both so that you understand how both sides of the equation are computed, and to start giving you an intuition for what both sides count. For example, when $n = 3$, the left-hand side is $\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$, which is indeed equal to $2^3 = 8$.

Here's how we'll proceed: we're going to argue that **both sides count the number of subsets of a set of n elements**. As an example, when $n = 3$, a set with n elements might look like $S = \{A, B, C\}$. A subset of S is any other set made up of elements from S. The 8 subsets of S are:

$$\{\}, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \{A, B, C\}$$

Remember, our argument needs to work for any arbitrary positive integer n, not just when $n = 3$. So, let S be a set with n elements. A valid combinatorial proof is then:

- **Left-Hand Side**: A subset of S must either have 0 elements, 1 element, 2 elements, 3 elements, or so on, through n elements. There are $\binom{n}{0}$ subsets with 0 elements (this is just the empty set), $\binom{n}{1}$ subsets with 1 element, $\binom{n}{2}$ subsets with 2 elements, and so on. In general, there are $\binom{n}{k}$ subsets of S with k elements, since there are $\binom{n}{k}$ ways to select k elements from S without replacement such that order doesn't matter (which they don't in sets). So, the total number of subsets of S is $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$, which is the left-hand side.
- **Right-Hand Side**: To construct a subset of S, for each of its n elements, there are 2 choices: include it in the subset, or do not include it in the subset. For element 1, there are 2 options; for element 2, there are 2 options; and so on. So, the total number of subsets of S is:

$$\underbrace{2 \cdot 2 \cdot ... \cdot 2}_{n \text{ times}} = 2^n$$

which is the right-hand side.

This completes our combinatorial proof of the fact that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Now, it's your turn!

a) Give a combinatorial proof of the fact that, that for positive integers m and n, and any integer k such that $0 \leq k \leq \min(m, n)$:

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$$

Hint: Try and come up with a "story" in real-life terms, e.g. "Suppose we want to choose a committee of k students from ..."

b) Evaluate the following expression for $n = 2$, $n = 3$, and $n = 4$, and show your work:

$$\sum_{i=0}^{n} \left(\binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} \right)$$

Then, based on your evaluations for $n = 2$, $n = 3$, and $n = 4$, determine a simplfied formula for the expression for a general value of n.

Hint: It should be a very simple formula.

c) Give a combinatorial proof of your simplified answer from part (b). Your answer to (b) should have been a simple function $f(n)$, so you should be giving a combinatorial proof that:

$$\sum_{i=0}^{n} \left[\binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} \right] = f(n)$$

Note that you won't receive any points for (c) if your $f(n)$ is incorrect. We're happy to confirm in office hours whether you have the right $f(n)$, but only if you show us your work in (b)!

Hint: Suppose I have n shirts and I'm deciding whether to keep, donate, or sell each one. How many ways can I decide what to do?

Problem 4. Ultrapermutations

A flight on Triton Airlines is scheduled for n passengers, all of whom have an assigned seat. Right before boarding, Triton Airlines' data systems malfunction and they lose track of the seat assignments. All n passengers board the flight but sit in a random seat. **What is the probability that nobody is sitting in their correct, originally assigned seat?** This is the question we will ultimately answer in this problem, but it'll take a few steps to get there.

To illustrate, let $n = 3$, and let's say our passengers are numbered 1, 2, and 3, where passenger i was originally supposed to sit in seat i. There are $3! = 6$ possible permutations of our 3 passengers, corresponding to the 6 ways in which they could've sat in the 3 seats on the plane:

$$123 \qquad 132 \qquad 213 \qquad \boxed{231} \qquad \boxed{312} \qquad 321$$

For instance, 123 refers to the permutation where seat 1 is taken by passenger 1, seat 2 is taken by passenger 2, and seat 3 is taken by passenger 3, and 312 refers to the permutation where seat 1 is taken by passenger 3, seat 2 is taken by passenger 1, and seat 3 is taken by passenger 2.

Only 2 of the above permutations were $\boxed{\textbf{ultrapermutations}}$, where none of the 3 passengers were sitting in their originally assigned seat. (If you look at any permutation above that isn't boxed, at least one passenger is sitting in their correctly assigned seat.)

The probability that nobody is sitting in their correct, originally assigned seat is the probability of an ultrapermutation. For $n = 3$, there are 2 ultrapermutations out of 6 total permutations, so we'd say:

$$\mathbb{P}(\text{ultra}_3) = \frac{2}{6} = \frac{1}{3}$$

a) Let's work out one more concrete example. Let $n = 4$. Write out all $4! = 24$ permutations of four passengers, 1, 2, 3, and 4. How many of the 24 permutations are ultrapermutations? What is $\mathbb{P}(\text{ultra}_4)$?

b) Moving forward, let n be any positive integer (that is, not necessarily 3 or 4).

Let A_i be the event that passenger i **is** sitting in their originally assigned seat. Determine the value of $\mathbb{P}(A_i)$. Your answer should involve n.

c) Suppose $1 \leq i < j < k \leq n$. Determine the values of $\mathbb{P}(A_i \cap A_j)$ and $\mathbb{P}(A_i \cap A_j \cap A_k)$. (As in the previous part, your answer should involve n.)

Hint: $\mathbb{P}(A_i \cap A_j)$ *is the probability that two different passengers* ***both are*** *sitting in their correct, originally assigned seat.*

d) Let C_w be defined as follows:

$$C_w = \binom{n}{w} \cdot \mathbb{P}(\text{a particular combination of } w \text{ passengers are all sitting in the correct seat})$$

Note that $\mathbb{P}(\text{a particular combination of 2 passengers are all sitting in the correct seat}) = \mathbb{P}(A_i \cap A_j)$ from part (c), and $\mathbb{P}(\text{a particular combination of 3 passengers are all sitting in the correct seat}) = \mathbb{P}(A_i \cap A_j \cap A_k)$ in part (c).

Show that:

$$C_w = \frac{1}{w!}$$

e) Explain why:

$$\mathbb{P}(A_1 \cup A_2 \cup ... \cup A_n) = C_1 - C_2 + C_3 - C_4 + ...C_n$$

Hint: Refer to Homework 6, Problem 2(c).

f) Show that:

$$\mathbb{P}(\text{ultra}_n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ...\frac{1}{n!}$$

g) $\mathbb{P}(\text{ultra}_n)$ is equal to the sum of the first $n+1$ terms of the Taylor Series of e^x, for a very specific value of $x = x_0$. What is x_0?

To show your work, you should write out the first few and last few terms of $\mathbb{P}(\text{ultra}_n)$ from part (f), the first few and last few terms of the Taylor Series of e^x in general, and the first few and last few terms of the Taylor Series of e^{x_0}, where x_0 is your chosen value of x.

h) Determine the value of:

$$\lim_{n\to\infty} \mathbb{P}(\text{ultra}_n)$$

This is the answer to the original question, "what is the probability that nobody is sitting in their correct, originally assigned seat?" for an arbitrarily large n.

Finding this limit shouldn't require *any* work — it is a direct consequence of your answer to part (g). However, the purpose of this part is to have you take a step back and understand how all 8 parts of this problem tie together.

Problem 5. Avi's music

Avi has a very unique music taste. Whenever he listens to a particular song, he would only enjoy it sometimes, depending on the artist of that song. Avi enjoyed songs by:

- Kendrick Lamar 90
- Taylor Swift 75
- Drake 45
- J Cole 50

a) Avi played a song from one of the above four artists and he really enjoyed the song. You have no idea which of the four artists he listed to, so assume it was equally likely to be any of them. You can also assume that Avi only listens to those 4 artists.

Given that Avi enjoyed the song, what's the probability that it was a song from Kendrick Lamar? Taylor Swift? Drake? J Cole? Show your work.

b) Avi again listens to a song from one of the above 4 artists and enjoys the song. This time, instead of assuming that he's equally likely to listen to all four artists, suppose you know that Avi listens to:

- Kendrick Lamar 20
- Taylor Swift 35
- Drake 15
- J Cole 30

Given that Avi enjoyed the song, what's the probability that the song was by Kendrick Lamar? Taylor Swift? Drake? J Cole? Show your work.

c) Compare your answers to part (a) and part (b) above. Identify which of the four probabilities you computed increased and which decreased, and explain why this makes sense intuitively.

Problem 6. Independence and Conditional Independence

Consider the sample space $S = \{a, b, c, d, e, f\}$ with associated probabilities given in the table below.

outcome	a	b	c	d	e	f
probability	$\frac{4}{42}$	$\frac{4}{42}$	$\frac{2}{42}$	$\frac{10}{42}$	$\frac{16}{42}$	$\frac{6}{42}$

Let $X = \{a, b\}$ and $Y = \{b, e\}$. Remember to show your work for all calculations.

a) Are X and Y independent?

b) In this problem, you will determine whether X and Y are conditionally independent given a third event, Z.

1. Suppose $Z_1 = \{a, b, d, e, f\}$. Are X and Y conditionally independent given Z_1?
2. Suppose $Z_2 = \{a, b, e\}$. Are X and Y conditionally independent given Z_2?
3. Suppose $Z_3 = \{c, d, e, f\}$ Are X and Y conditionally independent given Z_3?