DSC 40A: Theoretical Foundations of Data Science

Lecture 13 Part II

Feature engineering and data transformations

October 27, 2025

Announcements

- Gal is out today so Sawyer is lecturing in her place (hence slightly different slides)
- Your **midterm exam** will take place Monday, Nov. 3rd!
- 50 minutes on paper, no calculators or electronics permitted
- You are allowed to bring a single double-sided page of notes
- Seats are **assigned**; will provide details during Discussion today and Campuswire afterwards
- Content: Lectures 1-13, Homeworks 1-4, Groupworks 1-5
- Prepare by practicing old exam problems on practice.dsc40a.com

Announcements

Varun and Owen will be hosting a midterm review session this Thursday 10/30 from 5pm-7pm in **Ledden Auditorium** (near HSS/APM)

Stay tuned for further details via Campuswire

Recap from last week

On **Friday** you covered a few topics that build on our work with simple linear models and multiple regression:

- Standardizing features $x_{i\ (\text{su})} = \frac{x_i - \overline{x}}{\sigma_x}$,

$$H(x) = w_0 + w_1 x_{1\ (\text{su})} + \ldots + w_d x_{d\ (\text{su})}$$

- Adding polynomial terms to the hypothesis function, e.g.,

$$H(x) = w_0 + w_1 x + w_2 x^2,$$

- Adding terms from combinations of features:

$$H(\mathrm{sqft}, \mathrm{comp}) = \ldots + w_4(\mathrm{sqft} \cdot \mathrm{comp}) + \ldots$$

Question: What does each of these have in common?

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$$H(\mathrm{sqft}, \mathrm{comp}) = ... + w_4(\mathrm{sqft} \cdot \mathrm{comp}) + ...$$

Question: What do each of these have in common?

These are **all linear** in the weights w_i.

What if we want to use a hypothesis function that is **nonlinear in the weights and/or features?**

Example A nonlinear hypothesis function

Consider the following hypothesis function, which depends on a single scalar-valued feature and two weights w_0, w_1:

$$H(x) = w_0 e^{w_1 x}.$$

This function is **nonlinear** in both the weights and the feature x. We can create a new hypothesis function $T(x) = b_0 + b_1 x$, which is linear in the weights b_0, b_1, by applying the transformation

$$T(x) = \ln(H(x)) = \ln(w_0) + w_1 x.$$

The weights are related by the equations $b_0 = \ln(w_0)$ and $b_1 = w_1$.

Example A nonlinear hypothesis function

$$T(x) = \ln(H(x)) = \ln(\mathbf{w_0}) + \mathbf{w_1}x$$

Then, we can fit the linear hypothesis function $T(x) = b_0 + b_1 x$ to data using the normal equations to obtain optimal weights b_0^*, b_1^*. Finally, we can recover the optimal weights for the original hypothesis via

$$\begin{aligned} w_0^* &= e^{b_0^*}, \\ w_1^* &= b_1^*. \end{aligned}$$

We will explore this in an interactive notebook, continuing from last week's example.

Let's do a more detailed example.

Example Drink up!

You operate a beverage bottling plant in the Southwestern US. Recently you collected data over the course of twelve weeks $i = 1, \ldots, 12$, capturing the following statistics:

- $x_i^{(1)}$, the labor-hours of the plant during week i,
- $x_i^{(2)}$, the electricity consumed in the plant during week i,
- $x_i^{(3)}$, the materials input (kg of syrup/concentrate), again during week i.

You would like to model y_i, the liters of finished bottled product during week i, in terms of measurable quantities.

Example Drink up!

After discussing things with your in-house econometrics guru, you arrive at the following hypothesis function:

$$H(\vec{x}) = w_0 (x^{(1)})^{w_1} (x^{(2)})^{w_2} (x^{(3)})^{w_3}.$$

This is an example of a **Cobb-Douglas** production function, commonly used in economics to model output as a function of multiple inputs.

Note that this hypothesis function is **nonlinear** in both the weights w_i and the features $x^{(j)}$.

To fit this model to data, we will need to perform a transformation. By using a logarithm, we can obtain a new hypothesis function $T(\vec{x})$ that is linear in the weights:

$$\begin{aligned} T(\vec{x}) &= \ln(H(\vec{x})) \\ &= \ln(w_0) + w_1 \ln(x^{(1)}) + w_2 \ln(x^{(2)}) + w_3 \ln(x^{(3)}). \end{aligned}$$

Example Drink up!

$$\begin{aligned} T(\vec{x}) &= \ln(H(\vec{x})) \\ &= \ln(\mathbf{w_0}) + \mathbf{w_1} \ln(x^{(1)}) + \mathbf{w_2} \ln(x^{(2)}) + \mathbf{w_3} \ln(x^{(3)}). \end{aligned}$$

We can now fit the linear hypothesis function

$$T(\vec{x}) = b_0 + b_1 z^{(1)} + b_2 z^{(2)} + b_3 z^{(3)},$$

where we have defined the transformed features

$$z^{(j)} = \ln(x^{(j)}), \quad j = 1, 2, 3,$$

using the normal equations to obtain optimal weights $b_0^*, b_1^*, b_2^*, b_3^*$.

Example Drink up!

Finally, we can recover the optimal weights for the original hypothesis via

$$\begin{aligned} w_0^* &= e^{b_0^*}, \\ w_1^* &= b_1^*, \\ w_2^* &= b_2^*, \\ w_3^* &= b_3^*. \end{aligned}$$

Note: unlike before, we need to transform our *features* as well as our *weights*!

Question

Answer at **q.dsc40a.com**.

Which of the following hypothesis functions is **not** linear in the parameters?

(A) $H(\vec{x}) = w_1 \left(x^{(1)} x^{(2)}\right) + \frac{w_2}{x^{(1)}} \sin\left(x^{(2)}\right)$

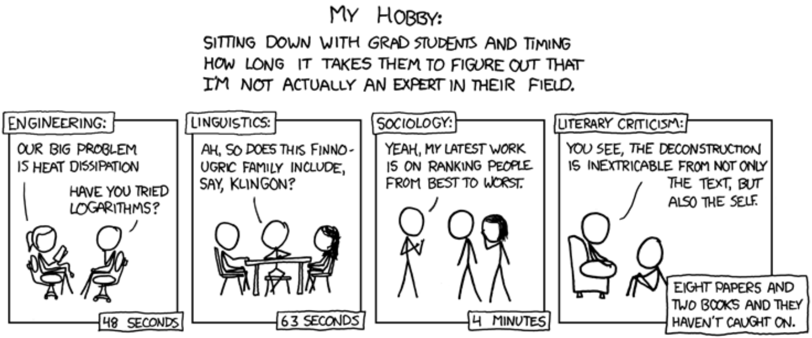
(B) $H(\vec{x}) = 2^{\,w_1}\, x^{(1)}$

(C) $H(\vec{x}) = \vec{w} \cdot \text{Aug}(\vec{x})$

(D) $H(\vec{x}) = w_1 \cos\left(x^{(1)}\right) + w_2\, 2^{\,x^{(2)} \log x^{(3)}}$

(E) More than one of the above.

Have you tried using logarithms?



xkcd #451

Sometimes, it's just not possible to transform a hypothesis function to be linear in terms of some parameters.

In those cases, you'd have to resort to other methods of finding the optimal parameters.

- For example, $H(x) = w_0 \sin(w_1 x)$ *can't* be transformed to be linear.
- But there are other methods of minimizing mean squared error:

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - w_0 \sin(w_1 x)\right)^2.$$

- One method: **gradient descent**, the topic of the next lecture!

Hypothesis functions that are linear in the parameters are much easier to work with.

Roadmap

- This is the end of the content that's in scope for the Midterm Exam.
- Now, we'll introduce **gradient descent**, a technique for minimizing functions that can't be minimized directly using calculus or linear algebra.
- After the Midterm Exam, we'll switch gears to **probability theory**.

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Lecture 14 Part I
Gradient Descent

October 27, 2025

Minimizing empirical risk

- Repeatedly, we've been tasked with **minimizing** the value of empirical risk functions.
  - Why? To help us find the **best** model parameters, h^* or $\vec{w}^*$, which help us make the **best** predictions!
- We've minimized empirical risk functions in various ways.
  - $R_{\text{sq}}(h) = \frac{1}{n}\sum_{i=1}^{n}(y_i - h)^2$ **critical points where $R' = 0$**
  - $R_{\text{abs}}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}|\, y_i - (w_0 + w_1 x)|$ **Brute force (Hw3, P7)**
  - $R_{\text{sq}}(\vec{w}) = \frac{1}{n}\|\vec{y} - X\vec{w}\|^2$ **projections or $\nabla R = \vec{0}$**

Minimizing arbitrary functions

- Assume $f(t)$ is some *differentiable* single-variable function.
- When tasked with minimizing $f(t)$, our general strategy has been to:
  1. Find $\frac{df}{dt}(t)$, the derivative of f.
  2. Find the input t^* such that $\frac{df}{dt}(t^*) = 0$.
  3. Check that $\frac{d^2f}{dt^2}(t^*) > 0$ so that t^* is a true minimizer.
- However, there are cases where we can find $\frac{df}{dt}(t)$, but it is **either difficult or impossible** to solve $\frac{df}{dt}(t^*) = 0$.

$$\frac{df}{dt}(t) = 5t^4 - t^3 - 5t^2 + 2t - 9$$

- Then what?

When we can't directly solve for the minimizer of a function, we can **approximate** the minimizer using an iterative method called **gradient descent**.

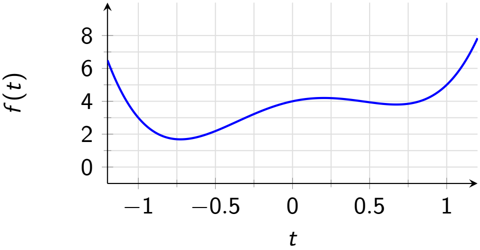
The idea is to start at some initial guess t_0 and then **iteratively improve** our guess by taking steps in the direction of steepest descent (i.e., the negative gradient).

Over time, these steps will (hopefully) lead us to a point close to the true minimizer t^*.

What does the derivative of a function tell us?

Goal: Given a **differentiable** function $f(t)$, find the input t^* that minimizes $f(t)$. First we need to consider the question:

What does $\dfrac{d}{dt}f(t)$ *mean*?

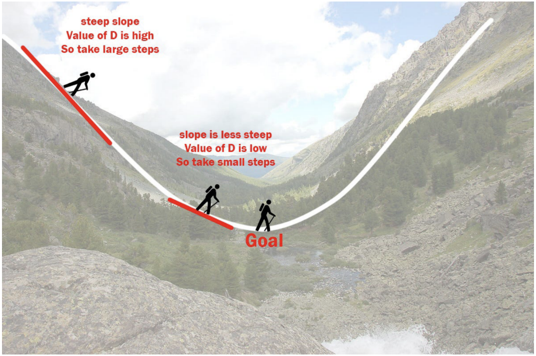


Let's go hiking!

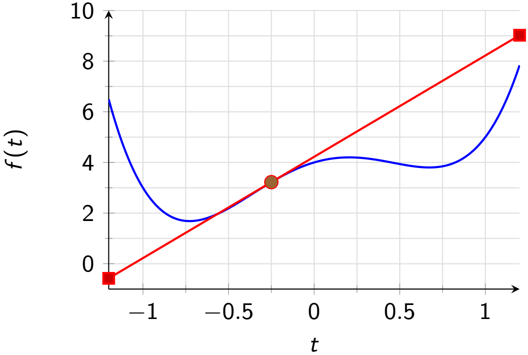
Suppose you're at the top of a mountain and need to get to the **bottom**.

Further, suppose it's really cloudy, meaning you can only see a few feet around you.

How would you get to the bottom?



Searching for the minimum



Suppose we're given an initial *guess* for a value of t that minimizes $f(t)$.

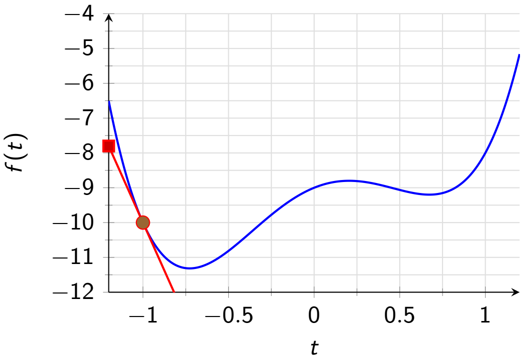
If the slope of the tangent line at $f(t)$ is positive:

Increasing t increases f.

The minimum must be to the left of the point $(t, f(t))$.

Solution: Decrease t.

Searching for the minimum



Suppose we're given an initial *guess* for a value of t that minimizes $f(t)$.

If the slope of the tangent line at $f(t)$ is negative:

Increasing t decreases f.

The minimum must be to the **right** of the point $(t, f(t))$.

Solution: Increase t.

Intuition

To minimize $f(t)$, start with an initial guess t_0.

Where do we go next?

If $\dfrac{df}{dt}(t_0) > 0$, decrease t_0.
If $\dfrac{df}{dt}(t_0) < 0$, increase t_0.

One way to accomplish this:

$$t_1 = t_0 - \frac{df}{dt}(t_0)$$

This is the key ingredient to *gradient descent*! **More on Wednesday!**