DSC 40B

Theoretical Foundations II

Lecture 6 | Part 1

Binary Search Recurrence

Binary Search

$$T(n) = \Theta(1) + T(n/2)$$

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

$\Theta(1)$

Binary Search

- What is the time complexity of `binary_search`?
- Best case: $\Theta(1)$.
- Worst case:

$$T(n) = \begin{cases} T(n/2) + \Theta(1), & n \geq 2 \\ \Theta(1), & n = 1 \end{cases}$$

Simplification

- When solving, we can replace $\Theta(f(n))$ with $f(n)$:

$$T(n) = \begin{cases} T(n/2) + 1, & n \geq 2 \\ 1, & n = 1 \end{cases}$$

- As long as we state final answer using Θ notation!

Another Simplification

- When solving, we can assume n is a power of 2.

Step 1: Unroll several times

$$T(n) = \begin{cases} T(n/2) + 1, & n \geq 2 \\ 1, & n = 1 \end{cases}$$

$$T(n) = T(n/2) + 1$$
$$= [T(n/4) + 1] + 1$$
$$= T(n/4) + 2$$
$$= [T(n/8) + 1] + 2 = T(n/8) + 3$$

$$T(n/2) = T((n/2)/2) + 1$$
$$= T(n/4) + 1$$

$$T(n/4) = T(n/8) + 1$$

Step 2: Find general formula

$$\begin{aligned} T(n) &= T(n/2) + 1 && \text{step } 1 \\ &= T(n/4) + 2 && 2 \\ &= T(n/8) + 3 && 3 \end{aligned}$$

On step k:

$$T(n) = T\left(n/2^k\right) + k$$

Step 3: Find step # of base case

- On step k, $T(n) = T(n/2^k) + k$

- When do we see $T(1)$?

$\frac{n}{2^k} = 1$ $n = 2^k \Rightarrow k = \log_2 n$

$$2^{\log_2 n} = n$$

Step 4: Plug into general formula

- $T(n) = T(n/2^k) + k$

$$T(n) = \Theta(\log n)$$

- Base case of $T(1)$ reached when $k = \log_2 n$.
- So: $T(n) = T\left(\frac{n}{2^{\log_2 n}}\right) + \log_2 n$

$$= T\left(\frac{n}{n}\right) + \log_2 n = T(1) + \log_2 n$$

$$= 1 + \log_2 n$$

$\Theta(1)$

Note

- Remember: $\log_b x = (\log_a x)/(\log_a b)$
- So we don't write $\Theta(\log_2 n)$
- Instead, just: $\Theta(\log n)$

Time Complexity of Binary Search

- Best case: $\Theta(1)$
- Worst case: $\Theta(\log n)$

Is binary search fast?

- Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- Goal: find a particular grain.
- Assume one basic operation takes 1 nanosecond.

Is binary search fast?

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- Linear search: 317 years.

Is binary search fast?

- Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- Goal: find a particular grain.
- Assume one basic operation takes 1 nanosecond.
- Linear search: 317 years.
- Binary search: $\approx$ 60 nanoseconds.

Exercise

Binary search seems so much faster than linear search. What’s the caveat?

Caveat

- The array must be **sorted**.
- This takes $\Omega(n)$ time.

Why use binary search?

- If data is **not sorted**, sorting + binary search takes longer than linear search.
- But if doing **multiple queries**, looking for nearby elements, sort once and use binary search after.

Theoretical Lower Bounds

- A lower bound for searching a sorted list is $\Omega(\log n)$.
- This means that binary search has **optimal** worst case time complexity.

Databases

- Some database servers will **sort** by key, use binary search for queries.
- Often instead of sorting, **B-Tree indexes** are used.
- But sorting + binary search still used when space is limited.

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Theoretical Foundations II

Lecture 6 | Part 2

Selection Sort and Loop Invariants

Sorting

- Sorting is a very common operation.
- But why is it important?

Sorting

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- But why is it important?
- `A e s t h e t i c` reasons?

Sorting

- Sorting is a very common operation.
- But why is it important?
- A e s t h e t i c reasons?
- Sorting makes some problems easier to solve.

Today

- How do we sort?
- How fast can we sort?
- How do we use sorted structure to write faster algorithms?

Today

- **Also:** how to understand complex loops with **loop invariants**.

Selection Sort

- Repeatedly remove smallest element.
- Put it at beginning of new list.

Example: `arr = [5, 6, 3, 2, 1]`

1 2 3 5 6

In-place Selection Sort

- We don’t need a separate list.
  - We can swap elements until sorted.
- Store “new” list at the beginning of input list.
- Separate the old and new with a **barrier**.

Example: `arr = [5, 6, 3, 2, 1]`

1 2 3 5 | 6

1 | 6 3 2 5

↑

barrier_ix

arr[i], arr[j] = arr[j], arr[i]

```python
def selection_sort(arr):
    """In-place selection sort."""
    n = len(arr)
    if n <= 1:
        return
    for barrier_ix in range(n-1):
        # find index of min in arr[start:]
        min_ix = find_minimum(arr, start=barrier_ix)
        #swap
        arr[barrier_ix], arr[min_ix] = (
                arr[min_ix], arr[barrier_ix]
        )
```

```python
def find_minimum(arr, start):
    """Finds index of minimum. Assumes non-empty."""
    n = len(arr)
    min_value = arr[start]
    min_ix = start
    for i in range(start + 1, n):
        if arr[i] < min_value:
            min_value = arr[i]
            min_ix = i
    return min_ix
```

Loop Invariants

- How we understand an iterative algorithm?
- A **loop invariant** is a statement that is true after every iteration.
  - And before the loop begins!

Loop Invariant(s)

After the αth iteration of selection sort, each of the first α elements is ≤ each of the remaining elements.

Example: `arr = [5, 6, 3, 2, 1]`

Loop Invariant(s)

After the αth iteration, the first α elements are sorted.

Example: `arr = [5, 6, 3, 2, 1]`

Loop Invariants

- Plug the total number of iterations into the loop invariant to learn about the result.
    - `selection_sort` makes $n - 1$ iterations:
    - After the $(n - 1)$th iteration, the first $(n - 1)$ elements are sorted.
    - After the $(n - 1)$th iteration, each of the first $(n - 1)$ elements is ≤ each of the remaining elements.

Time Complexity

```python
def selection_sort(arr):
    """In-place selection sort."""
    n = len(arr)
    if n <= 1:
        return
    for barrier_ix in range(n-1):
        # find index of min in arr[barrier_ix:]
        min_value = arr[barrier_ix]
        min_ix = barrier_ix
        for i in range(barrier_ix + 1, n):
            if arr[i] < min_value:
                min_value = arr[i]
                min_ix = i
        #swap
        arr[barrier_ix], arr[min_ix] = (
                arr[min_ix], arr[barrier_ix]
        )
```

$$(n-1)+(n-2)+(n-3) + \cdots + 3+2+1$$

$$= \Theta(n^2)$$

Time Complexity

- Selection sort takes $\Theta(n^2)$ time.

Exercise

Modify `selection_sort` so that it computes a **median** of the input array. What is the time complexity?

$x = n/2$

```
def selection_sort(arr):
    """In-place selection sort."""
    n = len(arr)
    if n <= 1:
        return
                              n/2
    for barrier_ix in range(n-1):
        # find index of min in arr[start:]
        min_ix = find_minimum(arr, start=barrier_ix)
        #swap
        arr[barrier_ix], arr[min_ix] = (
                arr[min_ix], arr[barrier_ix]
        )
```

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Theoretical Foundations II

Lecture 6 | Part 3

Mergesort

Can we sort faster?

- The tight theoretical lower bound for **comparison** sorting is $\Theta(n \log n)$.
- Selection sort is quadratic.
- How do we sort in $\Theta(n \log n)$ time?

Mergesort

- Mergesort is a fast sorting algorithm.
- Has **best possible** (worst-case) time complexity: $\Theta(n \log n)$.
- Implements **divide/conquer/recombine** strategy.

The Idea

- **Divide**: split the array into halves
  - `[6,1,9,2,4,3]` → `[6,1,9]`, `[2,4,3]`
- **Conquer**: sort each half, recursively
  - `[6,1,9]` → `[1,6,9]` and `[2,4,3]` → `[2,3,4]`
- **Combine**: merge sorted halves together
  - `[1,6,9]`, `[2,3,4]` → `[1,2,3,4,6,9]`

Aside: splitting arrays

middle

- Splitting an array in half by **slicing**:

```
>>> arr = [9, 1, 4, 2, 5]
>>> middle = math.floor(len(arr) / 2)
>>> arr[:middle]
[9, 1]
>>> arr[middle:]
[4, 2, 5]
```

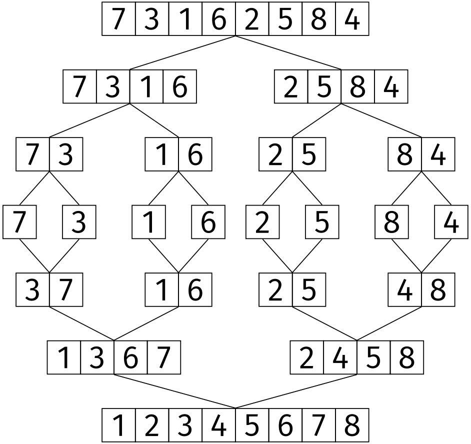
2

- **Warning!** Creates a copy!

Mergesort

```python
def mergesort(arr):
    """Sort array in-place."""
    if len(arr) > 1:
        middle = math.floor(len(arr) / 2)
        left = arr[:middle]
        right = arr[middle:]
        mergesort(left)
        mergesort(right)
        merge(left, right, arr)
```

The Idea



7 3 1 6 2 5 8 4
7 3 1 6
2 5 8 4
7 3
1 6
2 5
8 4
7
3
1
6
2
5
8
4
3 7
1 6
2 5
4 8
1 3 6 7
2 4 5 8
1 2 3 4 5 6 7 8

Understanding Mergesort

1. What is the base case?

2. Are the recursive problems smaller?

3. Assuming the recursive calls work, does the whole algorithm work?

1. Base Case: $n = 1$

- Arrays of size one are trivially sorted.
- Returns immediately. **Correct!**

2. Smaller Problems?

- Are `arr[:middle]` and `arr[middle:]` always smaller than `arr`?
- Try it for `len(arr) == 2`.

3. Does it Work?

- Assume `mergesort` works on arrays of size $< n$.
- Does it work on arrays of size n?

Mergesort

```python
def mergesort(arr):
    """Sort array in-place."""
    if len(arr) > 1:
        middle = math.floor(len(arr) / 2)
        left = arr[:middle]
        right = arr[middle:]
        mergesort(left)
        mergesort(right)
        merge(left, right, arr)
```

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Theoretical Foundations II

Lecture 6 | Part 4

Merge

Merging

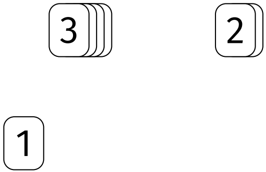
- We have sorted each half.
- Now we need to **merge** together.

Merging

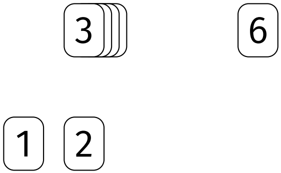
- We have sorted each half.
- Now we need to **merge** together.
- **Note:** this is an example of a problem that is made easier by sorting.

merge

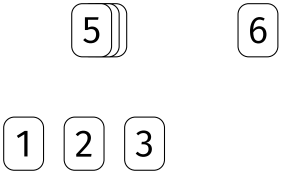
merge



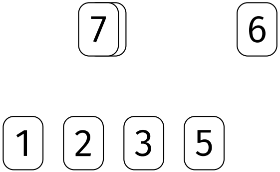
merge



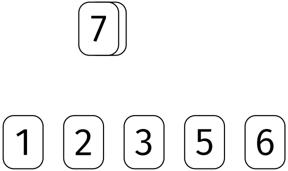
merge



merge



merge



`merge`

8

1 2 3 5 6 7

`merge`

$\boxed{7}$

$\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{5}$ $\boxed{6}$ $\boxed{8}$

merge

```python
def merge(left, right, out):
    """Merge sorted arrays, store in out."""
    left.append(float('inf'))
    right.append(float('inf'))
    left_ix = 0
    right_ix = 0

    for ix in range(len(out)):
        if left[left_ix] < right[right_ix]:
            out[ix] = left[left_ix]
            left_ix += 1
        else:
            out[ix] = right[right_ix]
            right_ix += 1
```

Loop Invariant

- Assume `left` and `right` are sorted.
- **Loop invariant**: After αth iteration, `out[:`α`] == sorted(left + right)[:`α`]`

Key of `mergesort`

- `merge` is where the **actual sorting** happens.
- Example: `merge([3], [1], ...)` results in `[1,3]`

Time Complexity of `merge`

n/2 n/2 n

```
def merge(left, right, out):
    """Merge sorted arrays, store in out."""
    left.append(float('inf'))
    right.append(float('inf'))
    left_ix = 0
    right_ix = 0

    for ix in range(len(out)):
        if left[left_ix] < right[right_ix]:
            out[ix] = left[left_ix]
            left_ix += 1
        else:
            out[ix] = right[right_ix]
            right_ix += 1
```

n times

$\Theta(n)$

$\Theta(1)$

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Theoretical Foundations II

Lecture 6 | Part 5

Time Complexity of Mergesort

$$T(n) = \Theta(1) + \Theta(n) + 2T(n/2)$$

Time Complexity

```
def mergesort(arr):
    """Sort array in-place."""
    if len(arr) > 1:
        middle = math.floor(len(arr) / 2)
        left = arr[:middle]
        right = arr[middle:]
        mergesort(left)
        mergesort(right)
        merge(left, right, arr)
```

$\Theta(1)$

$\Theta(n)$

$\Theta(n)$

Aside: Copying

- What is `arr[:middle]` doing “under the hood”?
- What is the time complexity?

The Recurrence

```python
def mergesort(arr):
    """Sort array in-place."""
    if len(arr) > 1:
        middle = math.floor(len(arr) / 2)
        left = arr[:middle]
        right = arr[middle:]
        mergesort(left)
        mergesort(right)
        merge(left, right, arr)
```

Solving the Recurrence

$T(n) = 2T(n/2) + \cancel{\Theta(}n\cancel{)}$

$= 2\left[2T\left(\frac{n}{4}\right)+\frac{n}{2}\right]+n$

$= 4T\left(\frac{n}{4}\right)+n+n$

$= 4T\left(\frac{n}{4}\right)+2n$

$= 8T\left(\frac{n}{8}\right)+3n$

$T(n/2) = 2T\left(\frac{n}{4}\right)+\frac{n}{2}$

$2^k T\left(\frac{n}{2^k}\right)+kn$

$\frac{n}{2^k}=1 \Rightarrow k=\log_2 n$

$\Downarrow$

$2^{\log n} T(1)+(\log_2 n)n$

$= nT(1) + n\log_2 n$

Optimality

- **Theorem**: Any (comparison) sorting algorithm's worst-case time complexity must be $\Omega(n \log n)$.

- Mergesort is **optimal**!

Be Careful!

- It is possible for a sorting algorithm to have a **best case** time complexity smaller than $n \log n$.
  - Insertion sort, for example.
- Mergesort has best case time complexity of $\Theta(n \log n)$.
- Mergesort is **sub-optimal** in this sense!

Be Careful!

- The $\Theta(n \log n)$ lower-bound is for **comparison sorting**.
- It is possible to sort in worst-case $\Theta(n)$ time without comparing.[1]

[1] Bucket sort, radix sort, etc.

What if?

- **Divide**: split the array into halves
- **Conquer**: sort each half **using selection sort**
- **Combine**: merge sorted halves together

mergeselectionsort

```python
def mergeselectionsort(arr):
    """Sort array in-place."""
    if len(arr) > 1:
        middle = math.floor(len(arr) / 2)
        left = arr[:middle]
        right = arr[middle:]
        selection_sort(left)
        selection_sort(right)
        merge(left, right, arr)
```

Exercise

What is the time complexity of this algorithm?

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Theoretical Foundations II

Lecture 6 | Part 6

Using Sorted Structure

Sorted structure is useful

- Some problems become **much easier** if input is sorted.
  - For example, median, minimum, maximum.
- Sorting is useful as a **preprocessing** step.

Recall: The Movie Problem

- You're on a flight that will last D minutes.
- You want to pick two movies to watch.
- You want the total time of the two movies to be **as close as possible** to D.

The Movie Problem

- Brute force algorithm: $\Theta(n^2)$
- We can do better, if movie times are **sorted.**

Example

- Flight duration $D = 155$
- Movie times: $60, 80, 90, 120, 130$

	60	80	90	120	130
60					
80					
90					
120					
130					

Best pair:

The Algorithm

- Keep index of shortest and longest remaining.
- On every iteration, pair the shortest and longest.
- If this pair is too long, remove longest movie; otherwise remove shortest.
  - If times are **sorted**, finding new longest/shortest movie takes $\Theta(1)$ time!

60, 80, 90, 120, 130

The Algorithm

```python
def optimize_entertainment(times, target):
    """assume times is sorted."""
    shortest = 0
    longest = len(times) - 1

    best_pair = (shortest, longest)
    best_objective = None

    for i in range(len(times) - 1):
        total_time = times[shortest] + times[longest]

        if abs(total_time - target) < best_objective:
            best_objective = abs(total_time - target)
            best_pair = (shortest, longest)

        if total_time == target:
            return (shortest, longest)
        elif total_time < target:
            shortest += 1
        else: # total_time > target
            longest -= 1

    return best_pair
```

Main Idea

Sorted structure allows you to rule out possibilities without explicitly checking them. But, it requires you to spend the time sorting first.

Tip: when designing an algorithm, think about sorting the input first.