DSC 40B

Theoretical Foundations II

Lecture 13 | Part 1

Depth First Search

Visiting the Next Node

- Which node do we process next in a search?
- BFS: the **oldest** pending node.
- DFS (today): the **newest** pending node.
  - Naturally recursive.

```python
def dfs(graph, u, status=None):
    """Start a DFS at `u`."""
    # initialize status if it was not passed
    if status is None:
        status = {node: 'undiscovered' for node in graph.nodes}

    status[u] = 'pending'
    for v in graph.neighbors(u): # explore edge (u, v)
        if status[v] == 'undiscovered':
            dfs(graph, v, status)
    status[u] = 'visited'
```

Example

```python
def dfs(graph, u, status=None):
    """Start a DFS at `u`."""
    ...
    status[u] = 'pending'
    for v in graph.neighbors(u): # explore edge (u, v)
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```
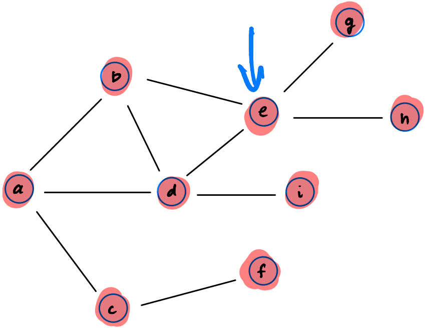



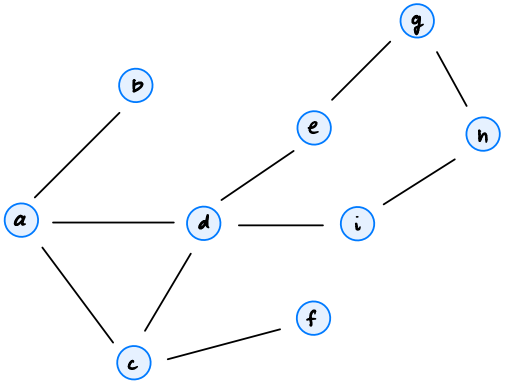
dfs(e)
~~dfs(b)~~
~~dfs(a)~~
~~dfs(c)~~
~~dfs(f)~~
~~dfs(d)~~
~~dfs(i)~~
~~dfs(g)~~
~~dfs(h)~~

Main Idea

We'll see that the structure of the nested function calls gives us useful information about the graph's structure.

Exercise

Write the nested function calls for a DFS on the graph below.



Full DFS

- DFS will visit all nodes reachable from source.
- To visit all nodes in graph, need **full DFS**.

```
def full_dfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered'
            dfs(graph, node, status)
```

```python
def full_dfs(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    for node in graph.nodes:
        if status[node] == 'undiscovered'
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    for v in graph.neighbors(u): # explore edge (u, v)
        if status[v] == 'undiscovered':
            dfs(graph, v, status)
    status[u] = 'visited'
```

$\Theta(V)$ time

$\Theta(E)$ executions

Time Complexity

- In a full DFS:
  - `dfs` called on each node exactly once.
  - Like BFS, each edge is explored exactly:
    - once if directed
    - twice if undirected

- Time: $\Theta(V + E)$, just like BFS.

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Lecture 13 | Part 2

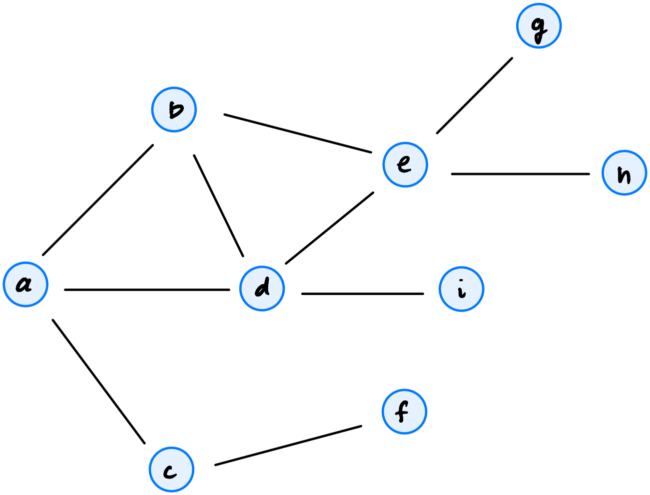
Nesting Properties of DFS

Key Property of DFS (Informal)

- Suppose v is reachable from u, and v is **undiscovered** at the time of `dfs(u)`.

- If there is a path of undiscovered nodes from u to v at the time of `dfs(u)`:
  - `dfs(v)` will be run.
  - v will be marked as **visited**.

Example



Start and Finish Times

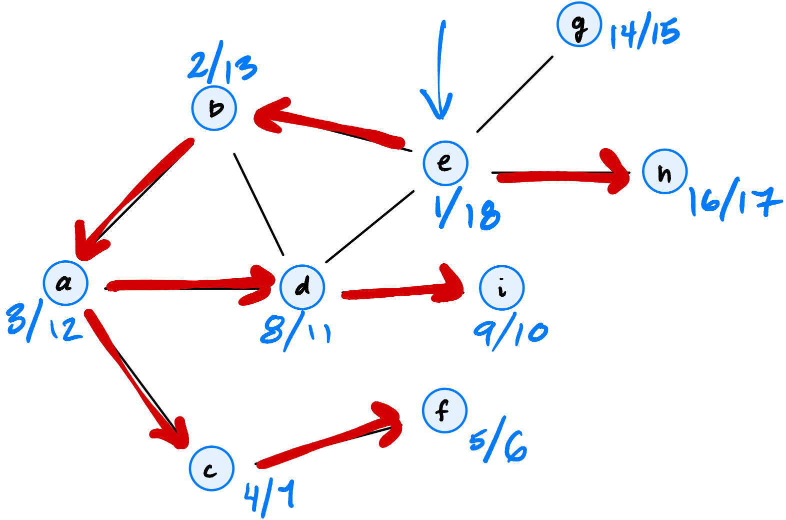
- Keep a running clock (an integer).
- For each node, record
  - **Start time**: time when marked pending
  - **Finish time**: time when marked visited
- Increment clock whenever node is marked pending/visited

```python
from dataclasses import dataclass

@dataclass
class Times:
    clock: int
    start: dict
    finish: dict

def full_dfs_times(graph):
    status = {node: 'undiscovered' for node in graph.nodes}
    predecessor = {node: None for node in graph.nodes}
    times = Times(clock=0, start={}, finish={})
    for u in graph.nodes:
        if status[u] == 'undiscovered':
            dfs_times(graph, u, status, times)
    return times, predecessor

def dfs_times(graph, u, status, predecessor, times):
    times.clock += 1
    times.start[u] = times.clock
    status[u] = 'pending'
    for v in graph.neighbors(u): # explore edge (u, v)
        if status[v] == 'undiscovered':
            predecessor[v] = u
            dfs_times(graph, v, status, times)
    status[u] = 'visited'
    times.clock += 1
    times.finish[u] = times.clock
```

{
'clock': 4,
'start': {
'a': 1,
'b': 2,

$$\frac{18-1+1}{2} \qquad \frac{f-s+1}{2}$$

Example



Key Property

u

- Take any two nodes u and v ($u \neq v$).
- If v is *started* between `start[u]` and `finish[u]`, then v is *finished* between `start[u]` and `finish[u]`.

Key Property

- Take any two nodes *u* and *v* ($u \neq v$).
- Assume for simplicity that `start[u]` < `start[v]`.
- Exactly one of these is true:
  - `start[u]` < `start[v]` < `finish[v]` < `finish[u]`
  - `start[u]` < `finish[u]` < `start[v]` < `finish[v]`

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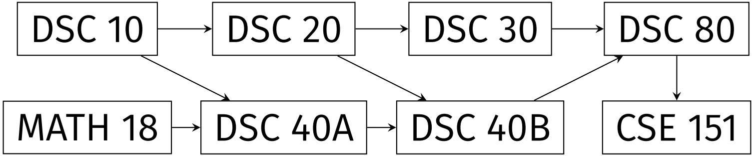
Lecture 13 | Part 3

Topological Sort

Applications of DFS

- Is node v reachable from node u?
- Is the graph connected?
- How many connected components?
- What is the shortest path between u and v? **No**.

Prerequisite Graphs



Goal: find order in which to take classes satisfying prerequisites.

Directed Acyclic Graphs

- A **directed cycle** is a path from a node to itself with at least one edge.
- A **directed acyclic graph (DAG)** is a directed graph with no directed cycles.

Example

- Prerequisite graphs are DAGs.
  - Or at least, they should be!

Topological Sorts

- **Given**: a DAG, $G = (V, E)$.

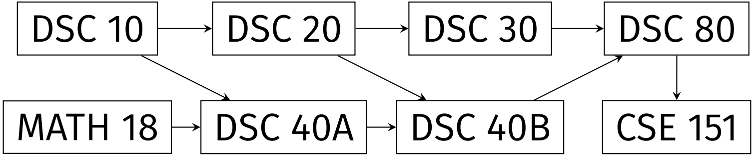
DSC 10 DSC 20

- **Compute**: an ordering of V such that if $(u, v) \in E$, then u comes before v in the ordering

- This is called a **topological sort** of G.

$\{5, 3, 4, 2, 3$ $2 < 3$ $3 < 4$ $4 < 5$

Example



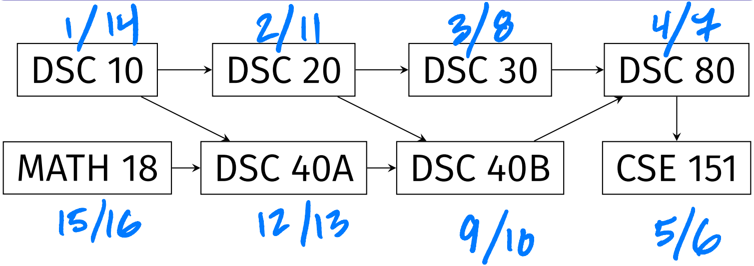
MATH 18, DSC 10, DSC 40A, DSC 20, DSC 40B, DSC 30, DSC 80, CSE 151

Key Property

- Take any two nodes u and v ($u \neq v$).
- Assume the graph is a DAG.
- **Example**: If v is reachable from u, then `finish[v] < finish[u]`.

Exercise

Compute start and finish times using DSC 10 as the source.



18, 10, 40A, 20, 40B, 30, 80, 151

An Algorithm

- **Recall**: If v is reachable from u, then `finish[v]` ≤ `finish[u]`.

- If v is reachable from u, u should come before v.

- Idea: nodes with later finish times should come first.

2 5 7 8 9

0 1 2 3 4 5 6 7 8 9

Algorithm

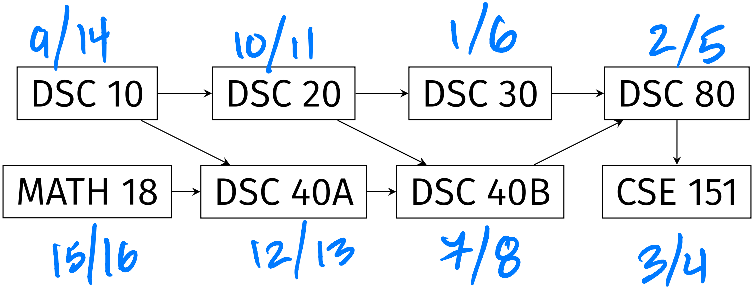
radix

- To find a topological sort (if it exists):
  - Compute times with Full DFS.
  - Sort in **descending** order by finish time.

- Time complexity:

$$\underbrace{\Theta(V+E)}_{\text{Full DFS}} + \underbrace{\Theta(V\log V)}_{\text{merge sort}}$$

Example



18, 10,

Note

- There can be many valid topological sorts!