DSC 40B

Theoretical Foundations II

Lecture 3 | Part 1

Big Theta, Formalized

Today in DSC 40B...

- Formally define Θ, O, Ω notation.
- Some useful properties.
- The drawbacks of asymptotic time complexity.
- Best, worst case time complexities.

So Far

- Time Complexity Analysis: a picture of how an algorithm **scales**.
- Can use Θ-notation to express time complexity.
- Allows us to **ignore** details in a rigorous way.
  - **Saves us work!**
  - **But what exactly can we ignore?**

Theta Notation, Informally

- $\Theta(\cdot)$ forgets constant factors, lower-order terms.

$$5n^3 + 3n^2 + 42 = \Theta(n^3)$$

Theta Notation, Informally

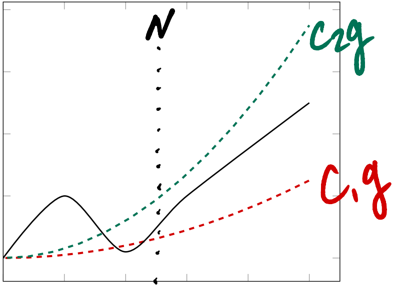
- $f(n) = \Theta(g(n))$ if $f(n)$ "grows like" $g(n)$.

$$5n^3 + 3n^2 + 42 = \Theta(n^3)$$

Definition

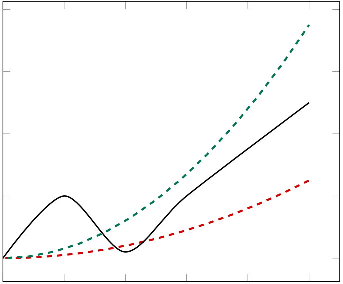
We write $f(n) = \Theta(g(n))$ if there are positive constants N, c_1 and c_2 such that for all $n \geq N$:

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$



Main Idea

If $f(n) = \Theta(g(n))$, then when n is large f is "sandwiched" between copies of g.



Proving Big-Theta

- We can prove that $f(n) = \Theta(g(n))$ by finding these constants.

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \qquad (n \geq N)$$

- Requires an upper bound and a lower bound.

Strategy: Chains of Inequalities

- To show $f(n) \leq c_2 g(n)$, we show:

  $$f(n) \leq (\text{something}) \leq (\text{another thing}) \leq \ldots \leq c_2 g(n)$$

- At each step:
  - We can do anything to make value **larger**.
  - But the goal is to simplify it to look like $g(n)$.

Example

- Show that $4n^3 - 5n^2 + 50 = \Theta(n^3)$.

- Find constants c_1, c_2, N such that for all $n > N$:

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- They don't have to be the "best" constants! Many solutions!

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- We want to make $4n^3 - 5n^2 + 50$ “look like” cn^3.
- For the upper bound, can do anything that makes the function **larger**.
- For the lower bound, can do anything that makes the function **smaller**.

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- Upper bound:

$$\begin{aligned} 4n^3 - 5n^2 + 50 &\leq 4n^3 + 50 \\ &\leq 4n^3 + 50n^3 \quad (n \geq 1) \\ &= \underbrace{54}_{c_2 = 54} n^3 \end{aligned}$$

Upper-Bounding Tips

- "Promote" lower-order **positive** terms:

$$3n^3 + 5n \leq 3n^3 + 5n^3$$

- "Drop" **negative** terms

$$3n^3 - 5n \leq 3n^3$$

Example

When is $n^3 - 5n^2 > 0$?

$$\overset{3}{c_1} n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad \overset{54}{c_2} n^3 \qquad N \geq 5$$

$$n^3 - 5n^2 > 0$$
$$n^3 > 5n^2$$
$$n > 5$$

- Lower bound:

$$\begin{aligned} 4n^3 - 5n^2 + 50 &\geq 4n^3 - 5n^2 \\ &= (3n^3 + n^3) - 5n^2 \\ &= 3n^3 + (n^3 - 5n^2) \\ &\geq 3n^3 \qquad (n > 5) \end{aligned}$$

$c_1 = 3$

Lower-Bounding Tips

- "Drop" lower-order **positive** terms:

$$3n^3 + 5n \geq 3n^3$$

- "Promote and cancel" negative lower-order terms if possible:

$$4n^3 - 2n \geq 4n^3 - 2n^3 = 2n^3$$

Lower-Bounding Tips

- "Cancel" negative lower-order terms with big constants by "breaking off" a piece of high term.

$$\begin{aligned} 4n^3 - 10n^2 &= (3n^3 + n^3) - 10n^2 \\ &= 3n^3 + (n^3 - 10n^2) \end{aligned}$$

$$n^3 - 10n^2 \geq 0 \text{ when } n^3 \geq 10n^2 \implies n \geq 10:$$

$$\geq 3n^3 + 0 \qquad (n \geq 10)$$

Caution

- To upper bound a fraction A/B, you must:
  - Upper bound the numerator, A.
  - *Lower* bound the denominator, B.

- And to lower bound a fraction A/B, you must:
  - Lower bound the numerator, A.
  - *Upper* bound the denominator, B.

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Lecture 3 | Part 2

Big-Oh and Big-Omega

Other Bounds

- $f = \Theta(g)$ means that f is both **upper** and **lower** bounded by factors of g.
- Sometimes we only have (or care about) upper bound or lower bound.
- We have notation for that, too.

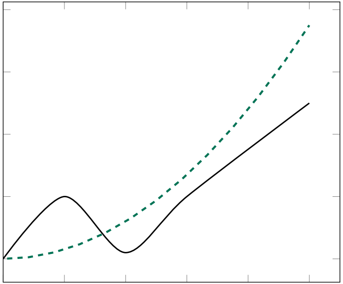
Big-O Notation, Informally

- Sometimes we only care about **upper bound**.
- $f(n) = O(g(n))$ if f "grows at most as fast" as g.
- Examples:
  - $4n^2 = O(n^{100})$
  - $4n^2 = O(n^3)$
  - $4n^2 = O(n^2)$ and $4n^2 = \Theta(n^2)$

Definition

We write $f(n) = O(g(n))$ if there are positive constants N and c such that for all $n \geq N$:

$$f(n) \leq c \cdot g(n)$$



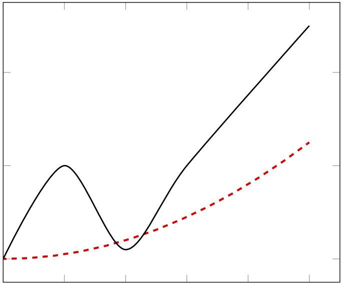
Big-Omega Notation

- Sometimes we only care about **lower bound**.
- Intuitively: $f(n) = \Omega(g(n))$ if f "grows at least as fast" as g.
- Examples:
  - $4n^{100} = \Omega(n^5)$
  - $4n^2 = \Omega(n)$
  - $4n^2 = \Omega(n^2)$ and $4n^2 = \Theta(n^2)$

Definition

We write $f(n) = \Omega(g(n))$ if there are positive constants N and c such that for all $n \geq N$:

$$c_1 \cdot g(n) \leq f(n)$$



FUN FACT

“Omega” in Greek literally means: big O.
So translated, “Big-Omega” means “big big O”.

Theta, Big-O, and Big-Omega

- If $f = \Theta(g)$ then $f = O(g)$ and $f = \Omega(g)$.
- If $f = O(g)$ and $f = \Omega(g)$ then $f = \Theta(g)$.
- Pictorially:
  - $\Theta \implies (O \text{ and } \Omega)$
  - $(O \text{ and } \Omega) \implies \Theta$

Analogies

- Θ is kind of like $=$
- O is kind of like $\leq$ $f \leq g$
- Ω is kind of like $\geq$

Why?

- Laziness.
- Sometimes finding an upper or lower bound would take **too much work**, and/or we don't really care about it anyways.

Big-Oh

- Often used when another part of the code would dominate time complexity anyways.

Exercise

What is the time complexity of `foo`?

```python
def foo(n):
    for a in range(n**4):
        print(a)

    for i in range(n):
        for j in range(i**2):
            print(i + j)
```

$T(n) = \Theta(n^4)$

Example: Big-Oh

```
def foo(n):
    for a in range(n**4):
        print(a)

    for i in range(n):
        for j in range(i**2):
            print(i + j)
```

$\Theta(n^4)$

$\leq n^2$

$O(n^3)$

$1 + 2^2 + 3^2 + 4^2$

Big-Omega

- Often used when the time complexity will be so large that we don’t care what it is, exactly.

Example: Big-Omega

```
best_separation = float('inf')
best_clustering = None

for clustering in all_clusterings(data):
    sep = calculate_separation(clustering)
    if sep < best_separation:
        best_separation = sep
        best_clustering = clustering

print(best_clustering)
```

2^n

$T(n) = \Omega(2^n)$

Other Notations

- $f(n) = o(g(n))$ if f grows “much slower” than g.
  - Whatever c you choose, eventually $f < cg(n)$.
  - Example: $n^2 = o(n^3)$
- $f(n) = \omega(g(n))$ if f grows “much faster” than g
  - Whatever c you choose, eventually $f > cg(n)$.
  - Example: $n^3 = \omega(n^2)$
- We won’t really use these.

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Lecture 3 | Part 3

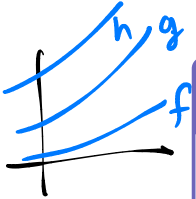
Properties

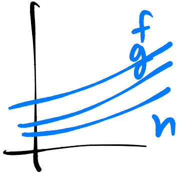
Properties

- We don't usually go back to the definition when using Θ.
- Instead, we use a few basic **properties**.

Properties of Θ

1. **Symmetry**: If $f = \Theta(g)$, then $g = \Theta(f)$.
2. **Transitivity**: If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.
3. **Reflexivity**: $f = \Theta(f)$





Exercise

Which of the following properties are true?

$f \le g$ $g \le h$ $f \le h$

- T or F: If $f = O(g)$ and $g = O(h)$, then $f = O(h)$.

n^3 $h = n$ n^4

- T or F: If $f = \Omega(h)$ and $g = \Omega(h)$, then $f = \Omega(g)$.

$\log n$ $\log n$ n n — and $g_1 = \Omega(g_2)$

- T or F: If $f_1 = \Theta(g_1)$ and $f_2 = O(g_2)$, then $f_1 + f_2 = \Theta(g_1 + g_2)$.
- T or F: If $f_1 = \Theta(g_1)$ and $f_2 = \Theta(g_2)$, then $f_1 \times f_2 = \Theta(g_1 \times g_2)$.

Proving/Disproving Properties

- Start by trying to disprove.
- Easiest way: find a counterexample.
- Example: If $f = \Omega(h)$ and $g = \Omega(h)$, then $f = \Omega(g)$.
  - **False!** Let $f = n^3$, $g = n^5$, and $h = n^2$.

Proving the Property

- If you can't disprove, maybe it is true.
- Example:
  n^2 n^3 n n^4
  - Suppose $f_1 = O(g_1)$ and $f_2 = O(g_2)$.
  - Prove that $f_1 \times f_2 = O(g_1 \times g_2)$.

$n^2 \times n = O(n^3 n^4)$

Step 1: State the assumption

- We know that $f_1 = O(g_1)$ and $f_2 = O(g_2)$.
- So there are constants c_1, c_2, N_1, N_2 so that for all $n \geq N$:

$$\begin{aligned} f_1(n) &\leq c_1 g_1(n) \qquad (n \geq N_1) \\ f_2(n) &\leq c_2 g_2(n) \qquad (n \geq N_2) \end{aligned}$$

Step 2: Use the assumption

- Chain of inequalities, starting with $f_1 \times f_2$, ending with $\leq cg_1 \times g_2$.
- Using the following piece of information:

$$f_1(n) \leq c_1 g_1(n) \qquad (n \geq N_1)$$
$$f_2(n) \leq c_2 g_2(n) \qquad (n \geq N_2)$$

$$f_1(n) \times f_2(n) \leq (c_1 g_1(n)) g_2(n) \qquad (n \geq N_1)$$
$$\leq (c_1 g_1(n))(c_2 g_2(n)) \qquad (n \geq \max\{N_1, N_2\})$$
$$= c\, g_1(n) g_2(n) \quad \text{where } c = c_1 c_2$$

Analyzing Code

- The properties of Θ (and O and Ω) are useful when analyzing code.
- We can analyze pieces, put together the results.

Sums of Theta

- **Property**: If $f_1 = \Theta(g_1)$ and $f_2 = \Theta(g_2)$, then $f_1 + f_2 = \Theta(g_1 + g_2)$
- Used when analyzing **sequential** code.

Example

```python
def foo(n):
    bar(n)
    baz(n)
```

- Say `bar` takes $\Theta(n^3)$, `baz` takes $\Theta(n^4)$.
- `foo` takes $\Theta(n^4 + n^3) = \Theta(n^4)$.
- `baz` is the **bottleneck**.

Products of Theta

- **Property**: If $f_1 = \Theta(g_1)$ and $f_2 = \Theta(g_2)$, then
$$f_1 \cdot f_2 = \Theta(g_1 \cdot g_2)$$
- Useful when analyzing nested **loops**.

Example

```
def foo(n):
    for i in range(3*n + 4, 5n**2 - 2*n + 5):
        for j in range(500*n, n**3):
            print(i, j)
```

Careful!

- If inner loop index depends on outer loop, you have to be more careful.

```python
def foo(n):
    for i in range(n):
        for j in range(i):
            print(i, j)
```

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Lecture 3 | Part 4

Asymptotic Notation Practicalities

In this part...

- Other ways asymptotic notation is used.
- Asymptotic notation *faux pas*.
- Downsides of asymptotic notation.

Not Just for Time Complexity!

- We most often see asymptotic notation used to express time complexity.
- But it can be used to express any type of growth!

Example: Combinatorics

- Recall: $\binom{n}{k}$ is number of ways of choosing k things from a set of n.

- How fast does this grow with n? For fixed k:

$$\binom{n}{k} = \Theta(n^k)$$

- Example: the number of ways of choosing 3 things out of n is $\Theta(n^3)$.

Example: Central Limit Theorem

- Recall: the CLT says that the sample mean has a normal distribution with standard deviation $\sigma_{\text{pop}}/\sqrt{n}$

- The **error** in the sample mean is: $O(1/\sqrt{n})$

Faux Pas

- Asymptotic notation can be used improperly.
  - Might be technically correct, but defeats the purpose.

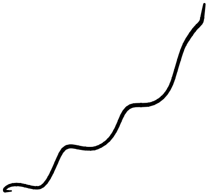
- Don’t do these in, e.g., interviews!

Faux Pas #1

- Don't include constants, lower-order terms in the notation.
- **Bad:** $3n^2 + 2n + 5 = \Theta(3n^2)$.
- **Good:** $3n^2 + 2n + 5 = \Theta(n^2)$.
- It isn't *wrong* to do so, just defeats the purpose.

Faux Pas #2

- Don't include base in logarithm.
- **Bad:** $\Theta(\log_2 n)$
- **Good:** $\Theta(\log n)$
- Why? $\log_2 n = c \cdot \log_3 n = c' \log_4 n = \ldots$



$f(n) = 3n^3 + \sin n$

$= \Theta(n^3)$

Faux Pas #3

- Don't misinterpret meaning of $\Theta(\cdot)$.
- $f(n) = \Theta(n^3)$ does **not** mean that there are constants so that $f(n) = c_3 n^3 + c_2 n^2 + c_1 n + c_0$.

$\Theta(n)$ $\Theta(n^2)$ $\Theta(n^3)$

$\Theta(n^{2.7})$

Faux Pas #4

- Time complexity is not a **complete** measure of efficiency.
- $\Theta(n)$ is not always "better" than $\Theta(n^2)$.
- Why?

Faux Pas #4

- **Why?** Asymptotic notation "hides the constants".
- $T_1(n) = 1{,}000{,}000n = \Theta(n)$
- $T_2(n) = 0.00001n^2 = \Theta(n^2)$
- But $T_1(n)$ is **worse** for all but really large n.

Main Idea

Time complexity is not the **only** way to measure efficiency, and it can be misleading.

Sometimes even a $\Theta(2^n)$ algorithm is better than a $\Theta(n)$ algorithm, if the data size is small.

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Lecture 3 | Part 5

The Movie Problem

The Movie Problem

The Movie Problem

- **Given**: an array `movies` of movie durations, and the flight duration `t`

- **Find**: two movies whose durations add to `t`.
  - If no two movies sum to `t`, return **`None`**.

Exercise

Design a brute force solution to the problem. What is its time complexity?

```python
def find_movies(movies, t):
    n = len(movies)
    for i in range(n):
        for j in range(i + 1, n):
            if movies[i] + movies[j] == t:
                return (i, j)
    return None
```

Time Complexity

- It looks like there is a **best** case and **worst** case.
- How do we formalize this?

For the future...

- Can you come up with a better algorithm?
- What is the *best possible* time complexity?

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Lecture 3 | Part 6

Best and Worst Cases

Example 1: `mean`

```python
def mean(arr):
    total = 0
    for x in arr:
        total += x
    return total / len(arr)
```

Time Complexity of `mean`

- Linear time, $\Theta(n)$.
- Depends **only** on the array's **size**, n, not on its actual elements.

Example 2: Linear Search

- **Given**: an array `arr` of numbers and a target `t`.
- **Find**: the index of `t` in `arr`, or `None` if it is missing.

```python
def linear_search(arr, t):
    for i, x in enumerate(arr):
        if x == t:
            return i
    return None
```

Exercise

What is the time complexity of `linear_search`?

```python
def linear_search(arr, t):
    for i, x in enumerate(arr):
        if x == t:
            return i
    return None
```

Observation

- It looks like there are *two* extreme cases...

The Best Case

- When the target, `t`, is the very first element.
- The loop exits after one iteration.
- $\Theta(1)$ time?

The **Worst** Case

- When the target, `t`, is not in the array at all.
- The loop exits after n iterations.
- $\Theta(n)$ time?

Time Complexity

- `linear_search` can take vastly different amounts of time on two inputs of the **same size**.
  - Depends on **actual elements** as well as size.

- It has no single, overall time complexity.

- Instead we'll report **best** and **worst** case time complexities.

Best Case Time Complexity

- How does the time taken in the **best case** grow as the input gets larger?

Definition

Define $T_{\text{best}}(n)$ to be the **least** time taken by the algorithm on any input of size n.

The asymptotic growth of $T_{\text{best}}(n)$ is the algorithm's **best case time complexity**.

Best Case

- In `linear_search`'s **best case**, $T_{\text{best}}(n) = c$, no matter how large the array is.

- The **best case time complexity** is $\Theta(1)$.

Worst Case Time Complexity

- How does the time taken in the **worst case** grow as the input gets larger?

Definition

Define $T_{\text{worst}}(n)$ to be the **most** time taken by the algorithm on any input of size n.

The asymptotic growth of $T_{\text{worst}}(n)$ is the algorithm's **worst case time complexity**.

Worst Case

- In the worst case, `linear_search` iterates through the entire array.
- The **worst case time complexity** is $\Theta(n)$.

Exercise

What are the best case and worst case time complexities of `find_movies`?

```
def find_movies(movies, t):
    n = len(movies)
    for i in range(n):
        for j in range(i + 1, n):
            if movies[i] + movies[j] == t:
                return (i, j)
    return None
```

Best Case

- Best case occurs when movie 1 and movie 2 add to the target.
- Takes constant time, independent of number of movies.
- Best case time complexity: $\Theta(1)$.

Worst Case

- Worst case occurs when no two movies add to target.
- Has to loop over all $\Theta(n^2)$ pairs.
- Worst case time complexity: $\Theta(n^2)$.

Caution!

- The best case is never: “the input is of size one”.
- The best case is about the **structure** of the input, not its **size**.
- Not always constant time! Example: sorting.

Note

- An algorithm like `linear_search` doesn't have **one single** time complexity.
- An algorithm like `mean` does, since the best and worst case time complexities coincide.

Main Idea

Reporting **best** and **worst** case time complexities gives us a richer of the performance of the algorithm.

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Lecture 3 | Part 7

Appendix: About Notation

A Common Mistake

- You'll sometimes see people equate $O(\cdot)$ with **worst case** and $\Omega(\cdot)$ with **best case**.
- This isn't right!

Why?

- $O(\cdot)$ expresses ignorance about a lower bound.
  - $O(\cdot)$ is like $\leq$
- $\Omega(\cdot)$ expresses ignorance about an upper bound.
  - $\Omega(\cdot)$ is like $\geq$
- Having both bounds is actually important here.

Example

- Suppose we said: “the worst case time complexity of `find_movies` is $O(n^2)$.”
- Technically true, but not precise.
- This is like saying: “I **don’t know** how bad it actually is, but it can’t be worse than quadratic.”
  - It could still be linear!”
- **Better**: the worst case time complexity is $\Theta(n^2)$.

Example

- Suppose we said: “the best case time complexity of `find_movies` is $\Omega(1)$.”

- This is like saying: “I **don’t know** how good it actually is, but it can’t be better than constant.”
  - It could be linear!

- **Correct**: the best case time complexity is $\Theta(1)$.

Put Another Way...

- It isn't **technically wrong** to say worst case for `find_movies` is $O(n^2)$...
- ...but it isn't **technically wrong** to say it is $O(n^{100})$, either!

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Lecture 3 | Part 8

Appendix: Asymptotic Notation and Limits

Limits and Θ, O, Ω

- You might prefer to use limits when reasoning about asymptotic notation.
- **Warning!** There are some tricky subtleties.
- Be able to “fall back” to the formal definitions.

Theta and Limits

- **Claim**: If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$, then $f(n) = \Theta(g(n))$.

- Example: $4n^3 - 5n^2 + 50$.

Warning!

- Converse **isn't true**: if $f(n) = \Theta(g(n))$, it need not be that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$.
- The limit can be **undefined**.
- Example: $5 + \sin(n) = \Theta(1)$, but the limit d.n.e.

Big-O and Limits

- If $\lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = O(g(n))$.
- Namely, the limit can be zero. e.g., $n = O(n^2)$.

Big-O and Limits

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $f(n) = O(g(n))$.
- Namely, the limit can be zero. e.g., $n = O(n^2)$.
- **Warning!** Converse not true. Limit may not exist.

Big-Omega and Limits

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) = \Omega(g(n))$.
- Namely, the limit can be ∞. e.g., $n^2 = \Omega(n)$.

Big-Omega and Limits

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $f(n) = \Omega(g(n))$.
- Namely, the limit can be ∞. e.g., $n^2 = \Omega(n)$.
- **Warning!** Converse not true. Limit may not exist.

Good to Know

- $\log_b n$ grows slower than n^p, as long as $p > 0$.
- Example:

$$\lim_{n \to \infty} \frac{\log_2 n}{n^{0.000001}} = 0$$