DSC 40B

Theoretical Foundations II

Lecture 2 | Part 1

News

News

- Lab 01 posted
  - Due Friday @ 11:59 pm PST on Gradescope.
- Homework 01 posted
  - Due Monday @ 11:59 pm PST on Gradescope.
  - LaTeX template available.
- Coffee with a prof, etc.

Agenda

1. Analyzing nested loops.
2. What is Θ notation, really?

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Lecture 2 | Part 2

Warm Up

Exercise

Write an algorithm for finding the maximum of an array of n numbers. What is its time complexity?

```python
def maximum(numbers):
    current_max = -float('inf')
    for x in numbers:
        if x > current_max:
            current_max = x
    return current_max
```

Time/exec.	# of execs.
c_1	1
c_3	n
c_4	?
c_2	1

$= \Theta(n)$

Main Idea

Using Big-Theta allows us not to worry about *exactly* how many times each line runs.

By the way...

Approximate timing for various operations on a typical PC:

execute typical instruction	1/1,000,000,000 sec = 1 nanosec
fetch from L1 cache memory	0.5 nanosec
branch misprediction	5 nanosec
fetch from L2 cache memory	7 nanosec
Mutex lock/unlock	25 nanosec
fetch from main memory	100 nanosec
send 2K bytes over 1Gbps network	20,000 nanosec
read 1MB sequentially from memory	250,000 nanosec
fetch from new disk location (seek)	8,000,000 nanosec
read 1MB sequentially from disk	20,000,000 nanosec
send packet US to Europe and back	150 milliseconds = 150,000,000 nanosec

From Peter Norvig's essay, "Teach Yourself Programming in Ten Years"
`http://norvig.com/21-days.html`

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Lecture 2 | Part 3

Nested Loops

Example 1: Interview Problem

Example 1: Interview Problem

- Design an algorithm to solve the following problem...

- Given the heights of n people, what is the height of the tallest doctor you can make by stacking two of them?

Exercise

- What is the time complexity of the brute force solution?
- **Bonus:** what is the **best possible** time complexity of any solution?

The Brute Force Solution

(a, b) (b, a)

- Loop through all possible (ordered) pairs.
  - How many are there?

n^2

- Check height of each.
- Keep the best.

(2, 4) (4, 2)

```python
def tallest_doctor(heights):
    max_height = -float('inf')
    n = len(heights)
    for i in range(n):
        for j in range(n):
            if i == j:
                continue
            height = heights[i] + heights[j]
            if height > max_height:
                max_height = height
    return max_height
```

Time/exec.	# of execs.
c_1	1
c_2	1
c_9	$n+1$
c_{10}	$n(n+1)$
c_4	n^2
c_5	$< n^2$
c_6	n^2
c_7	n^2
c_8	$< n^2$
c_3	1

$$\underbrace{n + n + n + \cdots + n}_{n}$$

$$An^2 + \cdots$$
$$= \Theta(n^2)$$

Time Complexity

- Time complexity of this is $\Theta(n^2)$.
- **TODO**: Can we do better?
- Note: this algorithm considers each pair of people **twice**.
- We'll fix that in a moment.

First: A shortcut

- Making a table is getting tedious.
- Usually we'll find a line that **dominates** time complexity; i.e., yields the leading term of $T(n)$.

Totalling Up

$$\overbrace{n + n + n + \ldots + n}^{n} = n^2$$

```
for i in range(n):
    for j in range(n):
        height = heights[i] + heights[j]
```

- On outer iter. # 1, inner body runs ___n___ times.
- On outer iter. # 2, inner body runs ___n___ times.
- On outer iter. # α, inner body runs ___n___ times.
- The outer loop body runs ___n___ times.

$n^6 - n^5$

```
def f(n):
    for i in range(3*n**3 + 5*n**2 - 100):
        for j in range(n**5, n**6):
            print(i, j)
```

$\Theta(n^3 \times n^6) = \Theta(n^9)$

Example 2: The Median

- **Given:** real numbers $x_1, \ldots, x_n$.
- **Compute:** h minimizing the **total absolute loss**

$$R(h) = \sum_{i=1} |x_i - h|$$

Example 2: The Median

- **Solution**: the **median**.
- That is, a **middle** number.
- But how do we actually **compute** a median?

A Strategy

- **Recall**: one of $x_1, \dots, x_n$ must be a median.

- **Idea**: compute $R(x_1), R(x_2), \dots, R(x_n)$, return x_i that gives the smallest result.

$$R(h) = \sum_{i=1} |x_i - h|$$

- Basically a **brute force** approach.

Exercise

- What is the time complexity of this brute force approach?
- How long will it take to run on an input of size 10,000?

```python
def median(numbers):
    min_h = None
    min_value = float('inf')
    for h in numbers:
        total_abs_loss = 0
        for x in numbers:
            total_abs_loss += abs(x - h)
        if total_abs_loss < min_value:
            min_value = total_abs_loss
            min_h = h
    return min_h
```

$\Theta(n)$

n^2 execs

so

$\Theta(n^2)$

The Median

- The brute force approach has $\Theta(n^2)$ time complexity.

- **TODO**: Is there a better algorithm?

The Median

- The brute force approach has $\Theta(n^2)$ time complexity.

- **TODO**: Is there a better algorithm?
  - It turns out, you can find the median in *linear* time.[1]

[1] Well, *expected* time.

```
In [8]: numbers = list(range(10_000))

In [9]: %time median(numbers)
CPU times: user 7.26 s, sys: 0 ns, total: 7.26 s
Wall time: 7.26 s
Out[9]: 4999

In [10]: %time mystery_median(numbers)
CPU times: user 4.3 ms, sys: 2 µs, total: 4.3 ms
Wall time: 4.3 ms
Out[10]: 4999
```

Careful!

- Not every nested loop has $\Theta(n^2)$ time complexity!

```python
def foo(n):
    for x in range(n):
        for y in range(10):
            print(x + y)
```

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Lecture 2 | Part 4

Dependent Nested Loops

Example 3: Tallest Doctor, Again

- Our previous algorithm for the tallest doctor computed height for each *ordered* pair of people.
  - `i = 3` and `j = 7` is the same as `i = 7` and `j = 3`
- **Idea**: consider each *unordered* pair only once:

```
for i in range(n):
    for j in range(i + 1, n):
```

- What is the time complexity?

Pictorially

```
for i in range(4):
    for j in range(4):
        print(i, j)
```

(0,0)	(0,1)	(0,2)	(0,3)
(1,0)	(1,1)	(1,2)	(1,3)
(2,0)	(2,1)	(2,2)	(2,3)
(3,0)	(3,1)	(3,2)	(3,3)

Pictorially

```
for i in range(4):
    for j in range(i + 1, 4):
        print(i, j)
```

(0,1)	(0,2)	(0,3)
	(1,2)	(1,3)
		(2,3)

```python
1 def tallest_doctor_2(heights):
2     max_height = -float('inf')
3     n = len(heights)
4     for i in range(n):
5         for j in range(i + 1, n):
6             height = heights[i] + height[j]
7             if height > max_height:
8                 max_height = height
```

- **Goal**: How many times does line 6 run in total?
- Now inner nested loop **depends** on outer nested loop.

Independent

```
for i in range(n):
    for j in range(n):
        ...
```

- Inner loop doesn't depend on outer loop iteration #.
- Just multiply: inner body executed n^2 times.

Dependent

```
for i in range(n):
    for j in range(i, n):
        ...
```

- Inner loop depends on outer loop iteration #.
- Can't just multiply: inner body executed ??? times.

Dependent Nested Loops

```
for i in range(n):
    for j in range(i + 1, n):
        height = heights[i] + heights[j]
```

- Idea: find formula $f(\alpha)$ for "number of iterations of inner loop during outer iteration α"[2]
- Then total: $\sum_{\alpha=1}^{n} f(\alpha)$

$f(1) + f(2) + \ldots + f(k)$

[2] Why α and not i? Python starts counting at 0, math starts at 1. Using i would be confusing – does it start at 0 or 1?

```python
for i in range(n):
    for j in range(i + 1, n):
        height = heights[i] + heights[j]
```

range(i+1, n)
range(2, n)

- On outer iter. # 1, inner body runs $\underline{n-1}$ times.

  i=0

- On outer iter. # 2, inner body runs $\underline{n-2}$ times.

  i=1

- On outer iter. # α, inner body runs $\underline{n-\alpha}$ times.

- The outer loop body runs $\underline{n}$ times.

Totalling Up

- On outer iteration α, inner body runs $n - \alpha$ times.
    - That is, $f(\alpha) = n - \alpha$

- There are n outer iterations.

- So we need to calculate:

$$\sum_{\alpha=1}^{n} f(\alpha) = \sum_{\alpha=1}^{n} (n - \alpha)$$

$$= (n-1) + (n-2) + (n-3) + \dots + 3 + 2 + 1 + \overset{0}{(n-n)}$$

$$\sum_{\alpha=1}^{n}(n-\alpha)$$

$$=$$

$$\underbrace{(n-1)}_{\text{1st outer iter}} + \underbrace{(n-2)}_{\text{2nd outer iter}} + \ldots + \underbrace{(n-\alpha)}_{\text{kth outer iter}} + \underbrace{(n-(n-1)}_{\text{(n-1)th outer iter}} + \underbrace{(n-n)}_{\text{nth outer iter}}$$

$$=$$

$$1+2+3+\ldots+(n-3)+(n-2)+(n-1)$$

$$= \frac{n(n-1)}{2}$$

Aside: Arithmetic Sums

- 1 + 2 + 3 + ...+ (n-1) + n is an **arithmetic sum**.
- Formula for total: $n(n + 1)/2$.
- You should memorize it!

$$\frac{n(n-1)}{2} = \Theta(n^2)$$

Time Complexity

- `tallest_doctor_2` has $\Theta(n^2)$ time complexity
- Same as original `tallest_doctor`!
- Should we have been able to guess this? Why?

Reason 1: Number of Pairs

- We're doing constant work for each unordered pair.

- Recall from 40A: number of pairs of n objects is

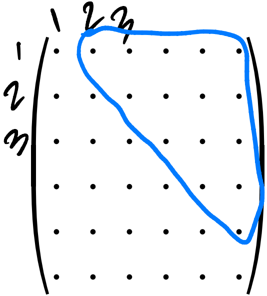
$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

- So $\Theta(n^2)$

Reason 2: Half as much work

- Our new solution does roughly half as much work as the old one.

- But Θ doesn't care about constants: $\frac{1}{2}\Theta(n^2)$ is still $\Theta(n^2)$.



Main Idea

If the loops are dependent, you'll usually need to write down a summation, evaluate.

Main Idea

Halving the work (or thirding, quartering, etc.) doesn't change the time complexity.

Exercise

Design a linear time algorithm for this problem.

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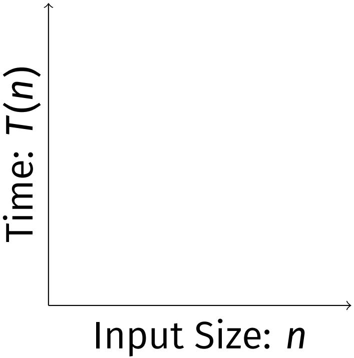
Lecture 2 | Part 5

Growth Rates

Linear vs. Quadratic Scaling



- $T(n) = \Theta(n)$ means "$T(n)$ grows like n"
- $T(n) = \Theta(n^2)$ means "$T(n)$ grows like n^2"

Definition

An algorithm is said to run in **linear time** if $T(n) = \Theta(n)$.

Definition

An algorithm is said to run in **quadratic time** if $T(n) = \Theta(n^2)$.

Linear Growth

- If input size doubles, time *doubles*.
- If code takes 5 seconds on 1,000 points...
- ...on 100,000 data points it takes $\approx$ 500 seconds.
- i.e., 8.3 minutes

Quadratic Growth

- If input size doubles, time *quadruples*.
- If code takes 5 seconds on 1,000 points...
- ...on 100,000 points it takes ≈ 50,000 seconds.
- i.e., ≈ 14 hours

In data science...

- Let's say we have a training set of 10,000 points.
- If model takes **quadratic** time to train, should expect to wait minutes to hours.
- If model takes **linear** time to train, should expect to wait seconds to minutes.
- These are rules of thumb only.

Common Growth Rates

- $\Theta(1)$: **constant**
- $\Theta(\log n)$: **logarithmic**
- $\Theta(n)$: **linear**
- $\Theta(n \log n)$: **linearithmic**
- $\Theta(n^2)$: **quadratic**
- $\Theta(n^3)$: **cubic**
- $\Theta(2^n)$: **exponential**

Exercise

Which grows faster, $n!$ or 2^n?

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

$$2 \cdot 2 \cdot 2 \cdots 2 \cdot 2$$

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Lecture 2 | Part 6

Big Theta, Formalized

So Far

- Time Complexity Analysis: a picture of how an algorithm **scales**.
- Can use Θ-notation to express time complexity.
- Allows us to **ignore** details in a rigorous way.
  - **Saves us work!**
  - **But what exactly can we ignore?**

Now

- A deeper look at **asymptotic notation**:
- What does $\Theta(\cdot)$ mean, exactly?
- Related notations: $O(\cdot)$ and $\Omega(\cdot)$.
- How these notations save us work.

Theta Notation, Informally

- $\Theta(\cdot)$ forgets constant factors, lower-order terms.

$$5n^3 + 3n^2 + 42 = \Theta(n^3)$$

Theta Notation, Informally

- $f(n) = \Theta(g(n))$ if $f(n)$ "grows like" $g(n)$.

$$5n^3 + 3n^2 + 42 = \Theta(n^3)$$

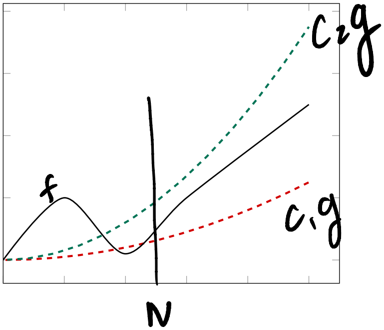
Theta Notation Examples

- $4n^2 + 3n - 20 = \Theta(n^2)$
- $3n + \sin(4\pi n) = \Theta(n)$
- $2^n + 100n = \Theta(2^n)$

Definition

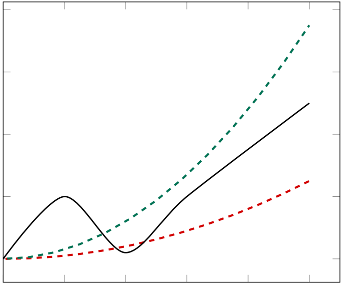
We write $f(n) = \Theta(g(n))$ if there are positive constants N, c_1 and c_2 such that for all $n \geq N$:

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$



Main Idea

If $f(n) = \Theta(g(n))$, then when n is large f is "sandwiched" between copies of g.



Proving Big-Theta

- We can prove that $f(n) = \Theta(g(n))$ by finding these constants.

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \qquad (n \geq N)$$

- Requires an upper bound and a lower bound.

Strategy: Chains of Inequalities

- To show $f(n) \le c_2 g(n)$, we show:

  $$f(n) \le (\text{something}) \le (\text{another thing}) \le \ldots \le c_2 g(n)$$

- At each step:
  - We can do anything to make value **larger**.
  - But the goal is to simplify it to look like $g(n)$.

Example

- Show that $4n^3 - 5n^2 + 50 = \Theta(n^3)$.

- Find constants c_1, c_2, N such that for all $n > N$:

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- They don't have to be the "best" constants! Many solutions!

Example

$$c_1 n^3 \quad \le \quad 4n^3 - 5n^2 + 50 \quad \le \quad c_2 n^3$$

- We want to make $4n^2 - 5n^2 + 50$ “look like” cn^3.
- For the upper bound, can do anything that makes the function **larger**.
- For the lower bound, can do anything that makes the function **smaller**.

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- Upper bound:

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- Lower bound:

Upper-Bounding Tips

- "Promote" lower-order **positive** terms:

$$3n^3 + 5n \le 3n^3 + 5n^3$$

- "Drop" **negative** terms

$$3n^3 - 5n \le 3n^3$$

Lower-Bounding Tips

- "Drop" lower-order **positive** terms:

$$3n^3 + 5n \geq 3n^3$$

- "Promote and cancel" negative lower-order terms if possible:

$$4n^3 - 2n \geq 4n^3 - 2n^3 = 2n^3$$

Lower-Bounding Tips

- "Cancel" negative lower-order terms with big constants by "breaking off" a piece of high term.

$$
\begin{aligned}
4n^3 - 10n^2 &= (3n^3 + n^3) - 10n^2 \\
&= 3n^3 + (n^3 - 10n^2)
\end{aligned}
$$

$$n^3 - 10n^2 \geq 0 \text{ when } n^3 \geq 10n^2 \implies n \geq 10:$$

$$\geq 3n^3 + 0 \qquad (n \geq 10)$$

Caution

- To upper bound a fraction A/B, you must:
  - Upper bound the numerator, A.
  - *Lower* bound the denominator, B.

- And to lower bound a fraction A/B, you must:
  - Lower bound the numerator, A.
  - *Upper* bound the denominator, B.

Exercise

Let $f(n) = [3n + (n\sin(\pi n) + 3)]n$. Which of the following are true?

- $f = \Theta(n)$
- $f = \Theta(n^2)$
- $f = \Theta(n\sin(\pi n))$
- $f = \Theta(n^2(n\sin(\pi n) + 3))$