DSC 40B

Theoretical Foundations II

Lecture 5 | Part 1

Searching a Database

Today in DSC 40B...

- How do we analyze the time complexity of **recursive** algorithms?
- How do we know that our recursive code is **correct**?

Databases

- Large data sets are often stored in **databases**.

PID	FullName	Level
A1843	Wan Xuegang	SR
A8293	Deveron Greer	SR
A9821	Vinod Seth	FR
A8172	Aleix Bilbao	JR
A2882	Kayden Sutton	SO
A1829	Raghu Mahanta	FR
A9772	Cui Zemin	SR
⋮	⋮	⋮

Query

- What is the name of the student with PID `A8172`?

Linear Search

- We *could* answer this with a linear search.
- Recall worst-case time complexity: $\Theta(n)$.
- Is there a better way?

Theoretical Lower Bounds

- **Given**: an array `arr` and a target `t`, determine the index of `t` in the array.

- Lower bound: $\Omega(n)$
  - `linear_search` has the best possible worst-case complexity!

Theoretical Lower Bounds

- **Given**: an **sorted** array `arr` and a target `t`, determine the index of `t` in the array.
- This is an **easier** problem.
- Lower bound: $\Omega(?)$

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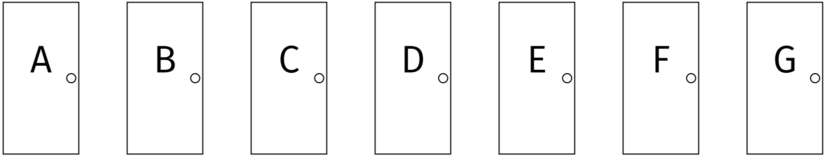
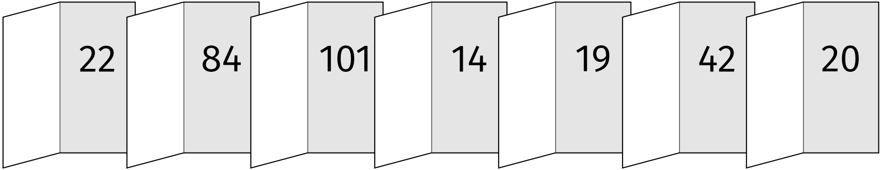
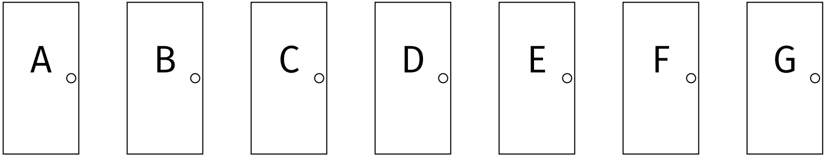
Theoretical Foundations II

Lecture 5 | Part 2

Binary Search


A
B
C
D
E
F
G


22
84
101
14
19
42
20


A
B
C
D
E
F
G

Game Show

- **Goal**: guess the door with number 42 behind it.
- **Caution**: with every wrong guess, your winnings are reduced.

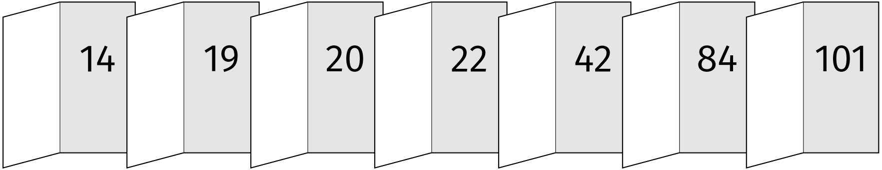
Strategy

- Can’t do much better than linear search.
  - “Is it door A?”
  - “OK, is it door B?”
  - “Door C?”

- After an incorrect first guess, the right door could be any of the other n – 1 doors!

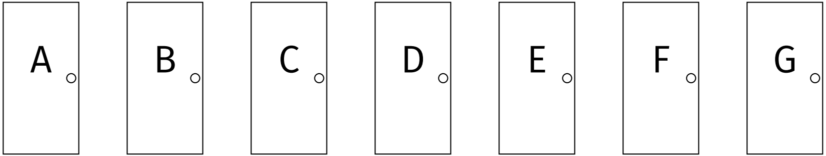
But now...

- Suppose the host tells you that the numbers are **sorted** in increasing order.



Exercise

Which door do you pick first?


A
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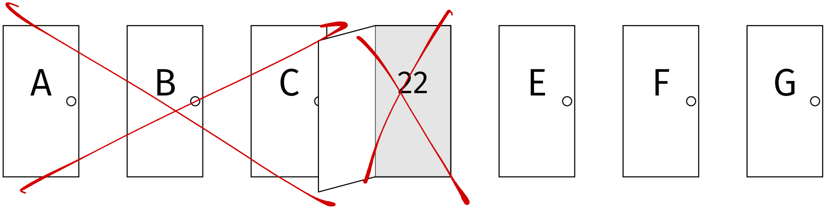
A
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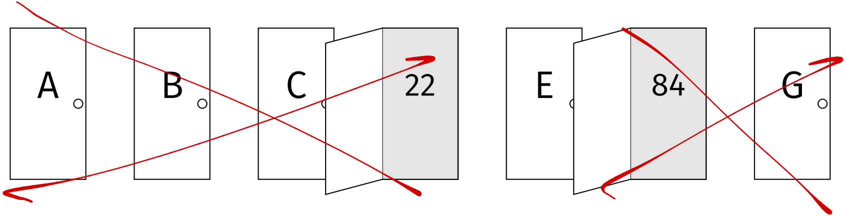
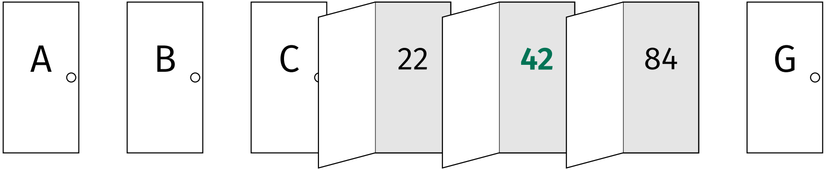

A
B
C
22
E
84
G


A
B
C
22
42
84
G

Strategy

- First pick the middle door.
- Allows you to rule out half of the other doors.
- Pick door in the middle of what remains.
- Repeat, **recursively**.

[ ]

Binary Search in Code

```python
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = ___________ # index of the middle element
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, ______, ______)
    else:
        return binary_search(arr, t, ______, ______)
```

0 1 2 3 4 5

arr[1:4]

Exercise

Fill in the blanks:

```
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = ___________ # index of the middle element
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)      look left
    else:
        return binary_search(arr, t, middle+1, stop)     look right
```

$$\frac{(stop - 1) + start}{2} \qquad \left\lfloor \frac{start + stop}{2} \right\rfloor$$

The Middle Element

- What is the index of the middle element of `arr[start:stop]`?

-10	-6	-3	1	2	5	12	21	33	35	42
0	1	2	3	4	5	6	7	8	9	10

start=1

stop=7

$x \,//\, 2$

Definition

The **floor** of a real number x, denoted $\lfloor x \rfloor$, is the *largest* integer that is $\leq x$.

Examples: $\lfloor 3.14 \rfloor = 3$ $\lfloor -4.5 \rfloor = -5$ $\lfloor 10 \rfloor = 10$

In LaTeX, $\lfloor x \rfloor$ is written: "`\lfloor x \rfloor`".

Definition

The **ceiling** of a real number x, denoted $\lceil x \rceil$, is the *smallest* integer that is $\geq x$.

Examples: $\lceil 3.14 \rceil = 4$ $\lceil -4.5 \rceil = -4$ $\lceil 10 \rceil = 10$

In LaTeX, $\lceil x \rceil$ is written: "`\lceil x \rceil`".

Binary Search

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

start = 0, stop = 11
m = 5
start = 6, stop = 11
m = 8
start = 6 , stop = 8
m = 7
return 7

t = 21

-10	-6	-3	1	2	5	12	21	33	35	42
0	1	2	3	4	5	6	7	8	9	10

start (6) stop (8)

enum

Aside: Default Arguments

ARR_END = object()

```
import math
def binary_search(arr, t, start=0, stop=None):
    if stop is None:
        stop = len(arr)
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

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Lecture 5 | Part 3

Thinking Inductively

Recursion

- Recursive algorithms can almost look like **magic**.
- How can we be sure that `binary_search` works?

Tips

1. Make sure algorithm works in the **base case**.

2. Check that all recursive calls are on **smaller problems**.

3. **Assuming** that the recursive calls work, does the whole algorithm work?

Base Case

- Smallest input for which you can easily see that the algorithm works.
- Recursion works by making problem smaller until base case is reached.
- Usually $n = 0$ or $n = 1$ (or even both!)

Base Case: $n = 0$

- Suppose `arr[start:stop]` is empty.
- In this case, the function returns `None`.
  - **Correct!**

Base Case: $n = 1$

- Suppose `arr[start:stop]` has one element.
- If that element is the target, the algorithm will find it.
  - **Correct!**
- If it isn't, the algorithm will recurse on a problem of size 0 and return `None`.
  - **Correct!**

Recursive Calls

- Recursive calls must be on **smaller problems**.
  - Otherwise, base case never reached. Infinite recursion!

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

stop-start

middle - start

stop-(middle + 1)

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

- Is `arr[start:middle]` smaller than `arr[start:stop]`?
- Is `arr[middle+1:stop]` smaller than `arr[start:stop]`?

Leap of Faith

- **Assume** the recursive calls work.
- Does the overall algorithm work, then?

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

Exercise

Does this code work? Why or why not?

```
import math
def summation(numbers):
    n = len(numbers)
    if n == 0:
        return 0
    middle = math.floor(n / 2)
    return (
        summation(numbers[:middle])
        +
        summation(numbers[middle:])
    )
```

[4]

m=0

[:0]

[0:]

Induction

- These steps can be turned into a formal proof by **induction**.
- For us, less necessary to prove to other people.
- Instead, prove to **yourself** that your code works.
- We won't be doing formal inductive proofs.

Why does this work?

- Show that it works for size 1 (base case).
- $\implies$ will work for size 2 (inductive step).
- $\implies$ will work for sizes 3, 4 (inductive step).
- $\implies$ will work for sizes 5, 6, 7, 8 (inductive step).

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Theoretical Foundations II

Lecture 5 | Part 4

Recurrence Relations

Time Complexity of Binary Search

- What is the time complexity of `binary_search`?
- No loops!

Best Case

$\Theta(1)$

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

Worst Case

Let $T(n)$ be worst case time on input of size n.

```python
import math
def binary_search(arr, t, start, stop):
    """
    Searches arr[start:stop] for t.
    Assumes arr is sorted.
    """
    if stop - start <= 0:
        return None
    middle = math.floor((start + stop)/2)
    if arr[middle] == t:
        return middle
    elif arr[middle] > t:
        return binary_search(arr, t, start, middle)
    else:
        return binary_search(arr, t, middle+1, stop)
```

$T(n) = c + T(n/2)$

$\Theta(1)$

c

$T(n/2)$

Recurrence Relations

- We found

$$T(n) = \begin{cases} T(n/2) + \Theta(1), & n \geq 2 \\ \Theta(1), & n = 1 \end{cases}$$

- This is a **recurrence relation**.

$T(n) = \cancel{\Theta(n^2)}$

$T(n) = T(n/2) + \Theta(1)$

Solving Recurrences

- We want simple, non-recursive formula for $T(n)$ so we can see how fast $T(n)$ grows.
  - Is it $\Theta(n)$? $\Theta(n^2)$? Something else?

- Obtaining a simple formula is called **solving** the recurrence.

Example: Getting Rich

- Suppose on day 1 of job, you are paid \$3.
- Each day thereafter, your pay is doubled.
- Let $S(n)$ be your pay on day n:

$$S(n) = \begin{cases} 2 \cdot S(n-1), & n \geq 2 \\ 3, & n = 1 \end{cases}$$

Example: Unrolling

$S(n) =$

$$S(n) = \begin{cases} 2 \cdot S(n-1), & n \geq 2 \\ 3, & n = 1 \end{cases}$$

- Take $n = 4$.

$S(2) = 2S(1)$

$$S(4) = 2S(3)$$
$$= 2[2S(2)]$$
$$= 2[2[2S(1)]]$$
$$= 2 \times 2 \times 2 \times 3$$

Solving Recurrences

We'll use a four-step process to solve recurrences:

1. "Unroll" several times to find a pattern.
2. Write general formula for kth unroll.
3. Solve for # of unrolls needed to reach base case.
4. Plug this number into general formula.

Step 1: Unroll several times

$$S(n) = \begin{cases} 2 \cdot S(n-1), & n \geq 2 \\ 3, & n = 1 \end{cases}$$

$$\begin{aligned} S(n) &= 2\,S(n-1) \\ &= 2 \cdot 2\,S(n-2) \\ &= 2 \cdot 2 \cdot 2\,S(n-3) \\ &= 2 \cdot 2 \cdot 2 \cdot 2\,S(n-4) \end{aligned}$$

$$S(n-1) = 2\,S(n-2)$$

$$S(n-2) = 2\,S(n-3)$$

Step 2: Find general formula

$$
\begin{aligned}
S(n) &= 2 \cdot S(n-1) && k=1 \\
&= 2 \cdot 2 \cdot S(n-2) && k=2 \\
&= 2 \cdot 2 \cdot 2 \cdot S(n-3) && k=3
\end{aligned}
$$

On step k:

$$S(n) = 2^k S(n-k)$$

Step 3: Find step # of base case

- On step k, $S(n) = 2^k \cdot S(n-k)$.
- When do we see S(1)?

$n-k=1 \Rightarrow k=n-1$

Step 4: Plug into general formula

- From step 2: $S(n) = 2^k \cdot S(n-k)$.
- From step 3: Base case of $S(1)$ reached when $k = n - 1$.
- So: $S(n) = 2^{n-1} \cdot S(n-(n-1))$

$$= 2^{n-1} \cdot S(1)$$

$$= 3 \times 2^{n-1}$$

Solving the Recurrence

- We have **solved** the recurrence[1]:

$$S(n) = 3 \cdot 2^{n-1}$$

- This is the **exact** solution. The **asymptotic** solution is $S(n) = \Theta(2^n)$.

- We'll call this method "solving by unrolling".

[1] On day 20, you'll be paid ≈1.5 million dollars.

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Lecture 5 | Part 5

Binary Search Recurrence

Binary Search

- What is the time complexity of `binary_search`?
- Best case: $\Theta(1)$.
- Worst case:

$$T(n) = \begin{cases} T(n/2) + \Theta(1), & n \geq 2 \\ \Theta(1), & n = 1 \end{cases}$$

Simplification

- When solving, we can replace $\Theta(f(n))$ with $f(n)$:

$$T(n) = \begin{cases} T(n/2) + 1, & n \geq 2 \\ 1, & n = 1 \end{cases}$$

- As long as we state final answer using Θ notation!

$\Omega \quad \geq \quad \geq 1 \quad n \quad n^2$

Θ

Another Simplification

- When solving, we can assume n is a power of 2.

Step 1: Unroll several times

$$T(n) = \begin{cases} T(n/2) + 1, & n \geq 2 \\ 1, & n = 1 \end{cases}$$

Step 2: Find general formula

$$\begin{aligned} T(n) &= T(n/2) + 1 \\ &= T(n/4) + 2 \\ &= T(n/8) + 3 \end{aligned}$$

On step k:

Step 3: Find step # of base case

- On step k, $T(n) = T(n/2^k) + k$
- When do we see $T(1)$?

Step 4: Plug into general formula

- $T(n) = T(n/2^k) + k$
- Base case of $T(1)$ reached when $k = \log_2 n$.
- So:

Note

- Remember: $\log_b x = (\log_a x)/(\log_a b)$
- So we don't write $\Theta(\log_2 n)$
- Instead, just: $\Theta(\log n)$

Time Complexity of Binary Search

- Best case: $\Theta(1)$
- Worst case: $\Theta(\log n)$

Is binary search fast?

- Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- Goal: find a particular grain.
- Assume one basic operation takes 1 nanosecond.

Is binary search fast?

- Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- Goal: find a particular grain.
- Assume one basic operation takes 1 nanosecond.
- Linear search: 317 years.

Is binary search fast?

- Suppose all 10^{19} grains of sand are assigned a unique number, sorted from least to greatest.
- Goal: find a particular grain.
- Assume one basic operation takes 1 nanosecond.
- Linear search: 317 years.
- Binary search: $\approx$ 60 nanoseconds.

Exercise

Binary search seems so much faster than linear search. What’s the caveat?

Caveat

- The array must be **sorted**.
- This takes $\Omega(n)$ time.

Why use binary search?

- If data is **not sorted**, sorting + binary search takes longer than linear search.
- But if doing **multiple queries**, looking for nearby elements, sort once and use binary search after.

Theoretical Lower Bounds

- A lower bound for searching a sorted list is $\Omega(\log n)$.
- This means that binary search has **optimal** worst case time complexity.

Databases

- Some database servers will **sort** by key, use binary search for queries.
- Often instead of sorting, **B-Tree indexes** are used.
- But sorting + binary search still used when space is limited.