DSC 40B

Theoretical Foundations II

Lecture 2 | Part 1

News

News

- Lab 01 posted on Gradescope
  - Due Friday @ 11:59 pm PST on Gradescope.

- Homework 01 will be posted at dsc40b.com
  - Due next Tuesday @ 11:59 pm PST on Gradescope.
  - LaTeX template available.

Agenda

1. Analyzing nested loops.
2. What is Θ notation, really?

DSC 40B

Theoretical Foundations II

Lecture 2 | Part 2

Nested Loops

Example 1: Interview Problem

Example 1: Interview Problem

- Design an algorithm to solve the following problem...

- Given the heights of n people, what is the height of the tallest doctor you can make by stacking two of them?

Exercise

- What is the time complexity of the brute force solution?
- **Bonus:** what is the **best possible** time complexity of any solution?

The Brute Force Solution

- Loop through all possible (ordered) pairs.
  - How many are there?
- Check height of each.
- Keep the best.

$(3, 5)$

```python
def tallest_doctor(heights):
    max_height = -float('inf')
    n = len(heights)
    for i in range(n):
        for j in range(n):
            if i == j:
                continue
            height = heights[i] + heights[j]
            if height > max_height:
                max_height = height
    return max_height
```

Time/exec.	# of execs.
c_1	1
c_2	1
c_3	$n+1$
c_4	$n(n+1)$
c_5	n^2
c_6	n
c_7	$(n-1)n$
c_8	$(n-1)n$
c_9	$\leq (n-1)n$
c_{10}	1

$$T(n) = (c_1)(1) + (c_2)(1) + (c_3)(n+1) + \cdots$$

$$= \Theta(n^2)$$

Time Complexity

- Time complexity of this is $\Theta(n^2)$.
- **TODO**: Can we do better?
- Note: this algorithm considers each pair of people **twice**.
- We'll fix that in a moment.

First: A shortcut

- Making a table is getting tedious.
- Usually, find a chunk that **dominates** time complexity; i.e., yields the leading term of $T(n)$.
- **Observation:** If each line takes constant time to execute once, the line that runs the most **dominates** the time complexity.

Totalling Up

$$\overbrace{n + n + n + \cdots + n}^{n}$$

```
for i in range(n):
    for j in range(n):
        height = heights[i] + heights[j] # <- count execs.
```

- On outer iter. # 1, inner body runs $\underline{\quad n \quad}$ times.
- On outer iter. # 2, inner body runs $\underline{\quad n \quad}$ times.
- On outer iter. # α, inner body runs $\underline{\quad n \quad}$ times.
- The outer loop runs $\underline{\quad n \quad}$ times.
- Total number of executions: $\underline{\quad n^2 \quad}$

Outer iter #1: $n^6 - n^5 - 1$

range(1, n)

#2: $n^6 - n^5 - 1$

⋮

α : $n^6 - n^5 - 1$

```python
def f(n):
    for i in range(3*n**3 + 5*n**2 - 100):
        for j in range(n**5, n**6):
            print(i, j)
```

Total # execs: $(n^6 - n^5 - 1)(3n^3 + 5n^2 - 100)$

$T(n) = \Theta(n^9)$

Example 2: The Median

- **Given:** real numbers $x_1, \ldots, x_n$.
- **Compute:** h minimizing the **total absolute loss**

$$R(h) = \sum_{i=1} |x_i - h|$$

Example 2: The Median

- **Solution**: the **median**.
- That is, a **middle** number.
- But how do we actually **compute** a median?

3 7 12 20

A Strategy

- **Recall**: one of $x_1, \ldots, x_n$ must be a median.
- **Idea**: compute $R(x_1), R(x_2), \ldots, R(x_n)$, return x_i that gives the smallest result.

$$R(h) = \sum_{i=1} |x_i - h|$$

- Basically a **brute force** approach.

Exercise

- What is the time complexity of this brute force approach?
- How long will it take to run on an input of size 10,000?

```python
def median(numbers):
    min_h = None
    min_value = float('inf')
    for h in numbers:
        total_abs_loss = 0
        for x in numbers:
            total_abs_loss += abs(x - h)
        if total_abs_loss < min_value:
            min_value = total_abs_loss
            min_h = h
    return min_h
```

$R(h)$

n^2 execs

$\Theta(n^2)$

The Median

- The brute force approach has $\Theta(n^2)$ time complexity.
- **TODO**: Is there a better algorithm?

The Median

- The brute force approach has $\Theta(n^2)$ time complexity.
- **TODO**: Is there a better algorithm?
  - It turns out, you can find the median in *linear* time.[1]

[1] Well, *expected* time.

```
In [8]: numbers = list(range(10_000))

In [9]: %time median(numbers)
CPU times: user 7.26 s, sys: 0 ns, total: 7.26 s
Wall time: 7.26 s
Out[9]: 4999

In [10]: %time mystery_median(numbers)
CPU times: user 4.3 ms, sys: 2 µs, total: 4.3 ms
Wall time: 4.3 ms
Out[10]: 4999
```

Careful!

- Not every nested loop has $\Theta(n^2)$ time complexity!

```
def foo(n):
    for x in range(n):
        for y in range(10):
            print(x + y)
```

$\Theta(n)$

10n execs

DSC 40B

Theoretical Foundations II

Lecture 2 | Part 3

Dependent Nested Loops

Example 3: Tallest Doctor, Again

- Our previous algorithm for the tallest doctor computed height for each *ordered* pair of people.
  - `i = 3` and `j = 7` is the same as `i = 7` and `j = 3`
- **Idea**: consider each *unordered* pair only once:

```
for i in range(n):
    for j in range(i + 1, n):
```

- What is the time complexity?

Pictorially

```
for i in range(4):
    for j in range(4):
        print(i, j)
```

(0,0)	(0,1)	(0,2)	(0,3)
(1,0)	(1,1)	(1,2)	(1,3)
(2,0)	(2,1)	(2,2)	(2,3)
(3,0)	(3,1)	(3,2)	(3,3)

Pictorially

```
for i in range(4):
    for j in range(i + 1, 4):
        print(i, j)
```

(0,1)	(0,2)	(0,3)
	(1,2)	(1,3)
		(2,3)

```python
1 def tallest_doctor_2(heights):
2     max_height = -float('inf')
3     n = len(heights)
4     for i in range(n):
5         for j in range(i + 1, n):
6             height = heights[i] + height[j]
7             if height > max_height:
8                 max_height = height
```

- **Goal**: How many times does line 6 run in total?
- Now inner nested loop **depends** on outer nested loop.

Independent

```
for i in range(n):
    for j in range(n):
        ...
```

- Inner loop doesn't depend on outer loop iteration #.
- Just multiply: inner body executed $n \times n = n^2$ times.

Dependent

```
for i in range(n):
    for j in range(i, n):
        ...
```

- Inner loop depends on outer loop iteration #.
- Can't just multiply: inner body executed ??? times.

Dependent Nested Loops

```
for i in range(n):
    for j in range(i + 1, n):
        height = heights[i] + heights[j]
```

- Idea: find formula $f(\alpha)$ for "number of iterations of inner loop during outer iteration α"[2]
- Then total: $\sum_{\alpha=1}^{n} f(\alpha)$

[2] Why α and not i? Python starts counting at 0, math starts at 1. Using i would be confusing – does it start at 0 or 1?

```python
for i in range(n):
    for j in range(i + 1, n):
        height = heights[i] + heights[j]
```

$0+1=1$

- On outer iter. # 1, inner body runs $n-1$ times.

  $i=0$ range(1,n)

- On outer iter. # 2, inner body runs $n-2$ times.

  $i=1$ range(2,n)

- On outer iter. # α, inner body runs $n-\alpha$ times.

  #n-1   1

  #n   0

- The outer loop runs n times.

Totalling Up

- On outer iteration α, inner body runs $n - \alpha$ times.
  - That is, $f(\alpha) = n - \alpha$

- There are n outer iterations.

- So we need to calculate:

$$\sum_{\alpha=1}^{n} f(\alpha) = \sum_{\alpha=1}^{n} (n - \alpha)$$

$$\sum_{\alpha=1}^{n}(n-\alpha)$$

$$=$$

$$\underbrace{(n-1)}_{\text{1st outer iter}} + \underbrace{(n-2)}_{\text{2nd outer iter}} + \ldots + \underbrace{(n-\alpha)}_{\text{kth outer iter}} + \ldots + \underbrace{(n-(n-1)}_{\text{(n-1)th outer iter}} + \underbrace{(n-n)}_{\text{nth outer iter}}$$

$$=$$

$$1+2+3+\ldots+(n-3)+(n-2)+(n-1)$$

$$=$$

$$\frac{n(n-1)}{2} = \Theta(n^2)$$

$$\frac{n(n+1)}{2} - n = \frac{n(n-1)}{2}$$

Aside: Arithmetic Sums

- 1 + 2 + 3 + ...+ (n-1) + n is an **arithmetic sum**.
- Formula for total: $n(n+1)/2$.
- You should memorize it!

$$S = 1 + 2 + 3 + 4 + \dots + 98 + 99 + 100$$

$$S = 100 + 99 + 98 + 97 + \dots + 3 + 2 + 1$$

$$2S = \underbrace{101 + 101 + 101 + 101 + \dots + 101 + 101 + 101}_{100}$$

$$= 100 \times 101$$

$$2S = (100)(101) \Rightarrow S = \frac{101 \times 100}{2}$$

Time Complexity

- `tallest_doctor_2` has $\Theta(n^2)$ time complexity
- Same as original `tallest_doctor`!
- Should we have been able to guess this? Why?

Reason 1: Number of Pairs

- We're doing constant work for each unordered pair.

- Recall from 40A: number of pairs of n objects is

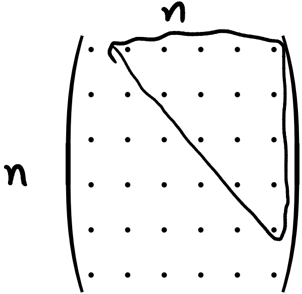
$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

- So $\Theta(n^2)$

Reason 2: Half as much work

- Our new solution does roughly half as much work as the old one.

- But Θ doesn't care about constants: $\frac{1}{2}\Theta(n^2)$ is still $\Theta(n^2)$.



$$\frac{1}{2} n \times n = \frac{1}{2} n^2$$
$$= \Theta(n^2)$$

Main Idea

If the loops are dependent, you'll usually need to write down a summation, evaluate.

Main Idea

Halving the work (or thirding, quartering, etc.) doesn't change the time complexity.

Exercise

Design a linear time algorithm for this problem.

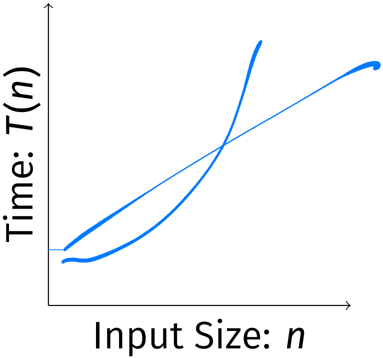
DSC 40B

Theoretical Foundations II

Lecture 2 | Part 4

Growth Rates

Linear vs. Quadratic Scaling



- $T(n) = \Theta(n)$ means "$T(n)$ grows like n"
- $T(n) = \Theta(n^2)$ means "$T(n)$ grows like n^2"

Definition

An algorithm is said to run in **linear time** if $T(n) = \Theta(n)$.

Definition

An algorithm is said to run in **quadratic time** if $T(n) = \Theta(n^2)$.

Linear Growth

- If input size doubles, time roughly *doubles*.
- If code takes 5 seconds on 1,000 points...
- ...on 100,000 data points it takes ≈ 500 seconds.
- i.e., 8.3 minutes

Quadratic Growth

- If input size doubles, time roughly *quadruples*.
- If code takes 5 seconds on 1,000 points…
- …on 100,000 points it takes ≈ 50,000 seconds.
- i.e., ≈ 14 hours

In data science…

- Let's say we have a training set of 10,000 points.
- If model takes **quadratic** time to train, should expect to wait minutes to hours.
- If model takes **linear** time to train, should expect to wait seconds to minutes.
- These are rules of thumb only.

Exponential Growth

- Increasing input size by one *doubles* (triples, etc.) time taken.
- Grows very quickly!
- **Example:** brute force search of 2^n subsets.

```
for subset in all_subsets(things):
    print(subset)
```

Logarithmic Growth

- To increase time taken by one unit, must *double* (triple, etc.) the input size.
- Grows very slowly!
- $\log n$ grows slower than n^{α} for *any* $\alpha > 0$
  - I.e., $\log n$ grows slower than n, $\sqrt{n}$, $n^{1/1{,}000}$, etc.

Exercise

What is the asymptotic time complexity of the code below as a function of *n*?

```
i = 1
while i <= n
    i = i * 2
```

$\Theta(\log_2 n)$

Solution

- Same general strategy as before: "how many times does loop body run?"

```
i = 1
while i <= n
    i = i * 2
```

n	# iters.
1	
2	
3	
4	
5	
6	
7	
8	

Common Growth Rates

- $\Theta(1)$: **constant**
- $\Theta(\log n)$: **logarithmic**
- $\Theta(n)$: **linear**
- $\Theta(n \log n)$: **linearithmic**
- $\Theta(n^2)$: **quadratic**
- $\Theta(n^3)$: **cubic**
- $\Theta(2^n)$: **exponential**

Exercise

Which grows faster, $n!$ or 2^n?

DSC 40B

Theoretical Foundations II

Lecture 2 | Part 5

Big Theta, Formalized

So Far

- Time Complexity Analysis: a picture of how an algorithm **scales**.
- Can use Θ-notation to express time complexity.
- Allows us to **ignore** details in a rigorous way.
  - **Saves us work!**
  - **But what exactly can we ignore?**

Now

- A deeper look at **asymptotic notation**:
- What does $\Theta(\cdot)$ mean, exactly?
- Related notations: $O(\cdot)$ and $\Omega(\cdot)$.
- How these notations save us work.

Theta Notation, Informally

- $\Theta(\cdot)$ forgets constant factors, lower-order terms.

$$5n^3 + 3n^2 + 42 = \Theta(n^3)$$

Theta Notation, Informally

- $f(n) = \Theta(g(n))$ if $f(n)$ "grows like" $g(n)$.

$$5n^3 + 3n^2 + 42 = \Theta(n^3)$$

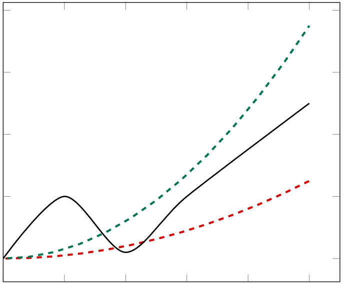
Theta Notation Examples

- $4n^2 + 3n - 20 = \Theta(n^2)$
- $3n + \sin(4\pi n) = \Theta(n)$
- $2^n + 100n = \Theta(2^n)$

Definition

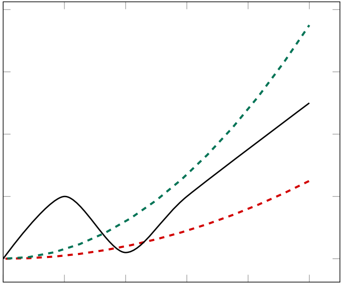
We write $f(n) = \Theta(g(n))$ if there are positive constants N, c_1 and c_2 such that for all $n \geq N$:

$$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$



Main Idea

If $f(n) = \Theta(g(n))$, then when n is large f is "sandwiched" between copies of g.



Proving Big-Theta

- We can prove that $f(n) = \Theta(g(n))$ by finding these constants.

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \qquad (n \geq N)$$

- Requires an upper bound and a lower bound.

Strategy: Chains of Inequalities

- To show $f(n) \leq c_2 g(n)$, we show:

  $$f(n) \leq (\text{something}) \leq (\text{another thing}) \leq \ldots \leq c_2 g(n)$$

- At each step:
  - We can do anything to make value **larger**.
  - But the goal is to simplify it to look like $g(n)$.

Example

- Show that $4n^3 - 5n^2 + 50 = \Theta(n^3)$.
- Find constants c_1, c_2, N such that for all $n > N$:

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- They don't have to be the "best" constants! Many solutions!

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- We want to make $4n^2 - 5n^2 + 50$ “look like” cn^3.
- For the upper bound, can do anything that makes the function **larger**.
- For the lower bound, can do anything that makes the function **smaller**.

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- Upper bound:

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- Lower bound:

Example

$$c_1 n^3 \quad \leq \quad 4n^3 - 5n^2 + 50 \quad \leq \quad c_2 n^3$$

- All together:

Upper-Bounding Tips

- "Promote" lower-order **positive** terms:

$$3n^3 + 5n \leq 3n^3 + 5n^3$$

- "Drop" **negative** terms

$$3n^3 - 5n \leq 3n^3$$

Lower-Bounding Tips

- "Drop" lower-order **positive** terms:

$$3n^3 + 5n \geq 3n^3$$

- "Promote and cancel" negative lower-order terms if possible:

$$4n^3 - 2n \geq 4n^3 - 2n^3 = 2n^3$$

Lower-Bounding Tips

- "Cancel" negative lower-order terms with big constants by "breaking off" a piece of high term.

$$
\begin{aligned}
4n^3 - 10n^2 &= (3n^3 + n^3) - 10n^2 \\
&= 3n^3 + (n^3 - 10n^2)
\end{aligned}
$$

$$n^3 - 10n^2 \geq 0 \text{ when } n^3 \geq 10n^2 \implies n \geq 10:$$

$$\geq 3n^3 + 0 \qquad (n \geq 10)$$

Caution

- To upper bound a fraction A/B, you must:
  - Upper bound the numerator, A.
  - *Lower* bound the denominator, B.

- And to lower bound a fraction A/B, you must:
  - Lower bound the numerator, A.
  - *Upper* bound the denominator, B.

Exercise

Let $f(n) = [3n + (n\sin(\pi n) + 3)]n$. Which one of the following is true?

- $f = \Theta(n)$
- $f = \Theta(n^2)$
- $f = \Theta(n\sin(\pi n))$