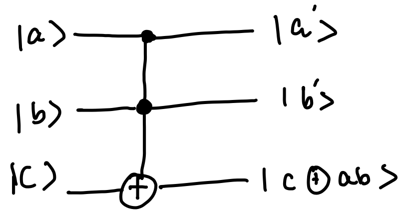
4. Toffoli gate :

or controlled C-NOT



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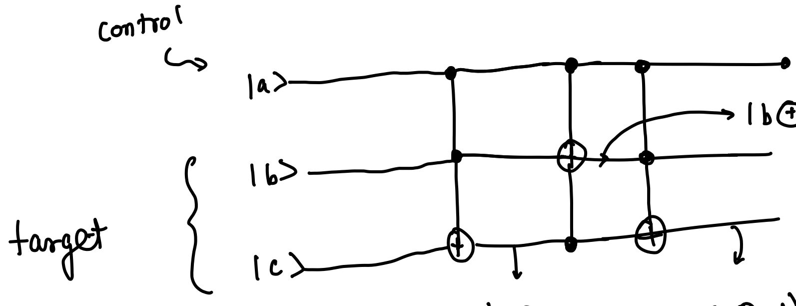
Controlled swap using Toffoli gate :



$|b \oplus (c \oplus ab)\, a\rangle$
$= |b \oplus ac \oplus ab\rangle = |b'\rangle$

$|(c \oplus ab) \oplus ((a)(b \oplus (c \oplus ab)))\rangle$

$|(c \oplus ab) \oplus (ab \oplus ac \oplus ab)\rangle$

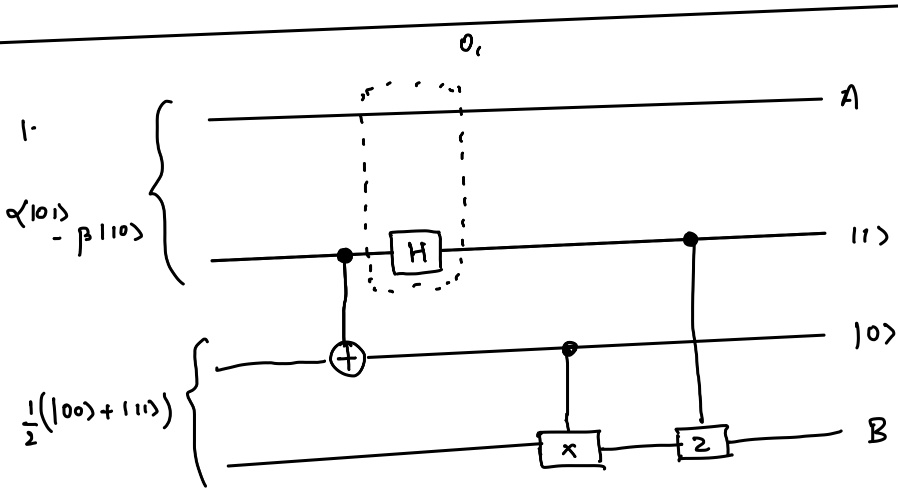
$= |c \oplus ab \oplus ac\rangle = |c'\rangle$

$|a, b, c\rangle \longrightarrow |a, b \oplus ac \oplus ab, c \oplus ac \oplus ab\rangle$

The truth table when control is on.

$\|a\rangle$	$\|b\rangle$	$\|c\rangle$	$\|a'\rangle$	$\|b'\rangle$	$\|c'\rangle$
1	0	0	1	0	0
1	0	1	1	1	0
1	1	0	1	0	1
1	1	1	1	1	1

1.



considering left most bit to represent top.

$$(\alpha|01\rangle - \beta|10\rangle) \otimes \left(\frac{1}{\sqrt{2}}\right)(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|0100\rangle + \alpha|0111\rangle - \beta|1000\rangle - \beta|1011\rangle\right)$$ is the initial state.

on applying CNOT

$$\frac{1}{\sqrt{2}}\left(\alpha|0110\rangle + \alpha|0101\rangle - \beta|1000\rangle - \beta|1011\rangle\right)$$

further

$$\frac{1}{\sqrt{2}}\left(\alpha|0\rangle|-\rangle|10\rangle + \alpha|0\rangle|-\rangle|01\rangle - \beta|1\rangle|+\rangle|00\rangle - \beta|1\rangle|+\rangle|11\rangle\right)$$

on applying H-gate

$$= \frac{1}{2}\left(\alpha|0\rangle(|0\rangle - |1\rangle)(|10\rangle + |01\rangle) - \beta|1\rangle(|0\rangle + |1\rangle)(|00\rangle + |11\rangle)\right)$$

on applying X-gate

$$= \frac{1}{2}\left(\alpha|0\rangle(|0\rangle - |1\rangle)(|11\rangle + |01\rangle) - \beta|1\rangle(|0\rangle + |1\rangle)(|00\rangle + |10\rangle)\right)$$

on applying Z-gate

$$= \frac{1}{2}\left(\alpha|0\rangle(|011\rangle + |001\rangle + |111\rangle + |101\rangle) - \beta|1\rangle(|000\rangle + |010\rangle + |100\rangle + |110\rangle)\right)$$

$$= \frac{1}{2}\left(\alpha(|0011\rangle + |0001\rangle + |1111\rangle + |0101\rangle) - \beta(|1000\rangle + |1010\rangle + |1100\rangle + |1110\rangle)\right)$$

$$= \frac{1}{2}\left(\alpha|0101\rangle - \beta|1100\rangle\right)$$

From here we can conclude that A and B are entangled as can't be represented by tensor product of 2 states.

Thus $|AB\rangle = \frac{1}{2}\left(\alpha|01\rangle - \beta|10\rangle\right)$

2.

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

on applying C-NOT: $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$

Thus, A is $\frac{1}{\sqrt{2}}(\alpha|010\rangle + \beta|011\rangle + \alpha|100\rangle + \beta|101\rangle)$

Further on applying C-NOT with 3rd control bit,

$$\frac{1}{\sqrt{2}}(\alpha|010\rangle + \beta|001\rangle + \alpha|100\rangle + \beta|111\rangle)$$

Applying H-gate to 3rd qubit.

$$\frac{1}{\sqrt{2}}(\alpha|01\rangle|+\rangle + \beta|00\rangle|-\rangle + \alpha|10\rangle|+\rangle + \beta|11\rangle|-\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\alpha}{\sqrt{2}}|01\rangle(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}|00\rangle(|0\rangle - |1\rangle) + \frac{\alpha}{\sqrt{2}}|10\rangle(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}|11\rangle(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}\left(\alpha(|010\rangle + |011\rangle) + \beta(|000\rangle - |001\rangle) + \alpha(|100\rangle + |101\rangle) + \beta(|110\rangle - |111\rangle)\right)$$

$$= \frac{1}{2}\left(\alpha(|010\rangle + |011\rangle + |100\rangle + |101\rangle) + \beta(|000\rangle + |110\rangle - |001\rangle - |111\rangle)\right)$$

$$= \frac{1}{2}\left(\alpha(|010\rangle + |111\rangle + |100\rangle + |001\rangle) + \beta(|000\rangle + |110\rangle - |101\rangle - |011\rangle)\right)$$

$$= \frac{1}{2}\left(\alpha(|010\rangle - |111\rangle + |100\rangle + |001\rangle) + \beta(|000\rangle - |110\rangle - |101\rangle - |011\rangle)\right)$$

Thus state at B is as above.

and $|\phi\rangle = \frac{1}{2}\left(\alpha|1\rangle + \beta|0\rangle\right)$

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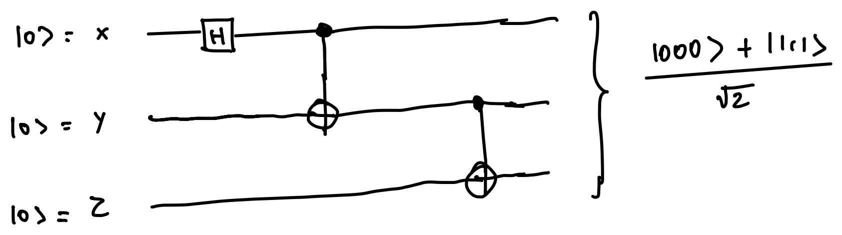
3. To obtain GHZ state, we can use the following circuit:



Here, on applying H-gate

$$|000\rangle \rightarrow |+\rangle|00\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}\left(|000\rangle + |110\rangle\right)$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right)$$