Partial Differential Equations

David Connelly
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1 Transport equation

The inhomogeneous transport equation with constant coefficients is

$$\begin{aligned} u_t + \mathbf{a}\cdot\nabla u &= f \text{ on } \mathbb{R}^n \times (0,\infty) \\ u &= g \text{ on } \mathbb{R}^n \times \{0\} \end{aligned}$$

To solve, we fix a point $(\mathbf{x}, t)$ and define

$$z(s) = u(x + s\mathbf{a}, t + s)$$

so that

$$\begin{aligned} \frac{\partial z}{\partial s} &= \mathbf{a}\cdot\nabla u + u_t \\ &= f \end{aligned}$$

Note that if the original equation was homogeneous then z is constant in s. In any case, we have

$$\begin{aligned} u(\mathbf{x}, t) - g(\mathbf{x} - t\mathbf{a}) &= z(0) - z(-t) \\ &= \int_{-t}^{0} \frac{\partial z}{\partial s}\,\mathrm{d}s \\ &= \int_{-t}^{0} f(\mathbf{x} + s\mathbf{a}, t + s)\,\mathrm{d}s \\ &= \int_{0}^{t} f(\mathbf{x} + (s - t)\mathbf{a}, s)\,\mathrm{d}s \end{aligned}$$

so we can conclude

$$\boxed{u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{a}) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{a}, s)\,\mathrm{d}s}$$

2 Wave equation

The inhomogeneous wave equation is

$$u_{tt} - \nabla^2 u = f \text{ on } U \times (0,\infty)$$

with appropriate boundary data, where $U \subseteq \mathbb{R}^n$ is open.

2.1 On the real line

In one spatial dimension the homogeneous equation on the whole real line is $u_{tt} = u_{xx}$, with initial conditions $u = g$ and $u_t = h$ at $t = 0$. We note that the equation can be expressed as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u = 0$$

which decomposes the problem into two transport equations. Applying the results from the previous section and using the initial data when relevant ultimately leads to d'Alembert's formula

$$\boxed{u(x, t) = \frac{g(x + t) + g(x - t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y}$$

2.2 On the half-line

We now solve the homogeneous wave equation on the half-line $x > 0$ with the additional condition $u(0,t) = 0$. We proceed by defining the odd reflection

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & x \geq 0 \\ -u(-x,t) & x < 0 \end{cases}$$

and similarly for $\tilde{g}$ and $\tilde{h}$. Then $\tilde{u}$ can be solved with d'Alembert's formula given above. We would like to extract a solution for u for positive x, but there are references to $\tilde{g}$ and $\tilde{h}$ in the solution for $\tilde{u}$, so we end up with two cases.

$$\boxed{u(x,t) = \begin{cases} \dfrac{g(x+t)+g(x-t)}{2} + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} h(y)\,\mathrm{d}y & x \geq t \\ \dfrac{g(x+t)-g(t-x)}{2} + \dfrac{1}{2}\displaystyle\int_{t-x}^{x+t} h(y)\,\mathrm{d}y & x < t \end{cases}}$$

2.3 In higher dimensions

Finally, we solve the homogeneous equation where the spatial domain is all of $\mathbb{R}^n$ — we will restrict our study to $n = 3$ and then $n = 2$. We fix $\mathbf{x}$ and define the average over the sphere

$$U(r,t) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)\,\mathrm{d}S(\mathbf{y})$$

Define G and H analogously. Immediately, we see that $U = G$ and $U_t = H$ at $t = 0$. Moreover, recall that the the Laplacian operator in n-dimensional spherical coordinates takes the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$$

when the function does not depend on the angle. It turns out that U satisfies the PDE

$$U_{tt} - \left(U_{rr} + \frac{n-1}{r}U_r\right) = 0$$

Let us now take $n = 3$ and set $\tilde{U} = rU$. Then

$$\begin{aligned} \tilde{U}_{tt} &= rU_{tt} \\ &= r\left(U_{rr} + \frac{2}{r}U_r\right) \\ &= rU_{rr} + 2U_r \\ &= (U + rU_r)_r = \tilde{U}_{rr} \end{aligned}$$

so that $\tilde{U}$ satisfies a one-dimensional wave equation on the half-line $r > 0$. We will ultimately be interested in sending $r \to 0$, so we use the case when $r < t$ from the previous section to write

$$\tilde{U}(r,t) = \frac{\tilde{G}(r+t) - \tilde{G}(t-r)}{2} + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

Then we have

$$\begin{aligned} u(x,t) &= \lim_{r\to 0^+} \frac{\tilde{U}}{r} = \tilde{G}'(t) + \tilde{H}(t) \\ &= \frac{\partial}{\partial t}\left(t\fint_{\partial B(\mathbf{x},t)} g(\mathbf{y})\,\mathrm{d}S(\mathbf{y})\right) + t\fint_{\partial B(\mathbf{x},t)} h(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \end{aligned}$$

We can note that

$$\begin{aligned}\frac{\partial}{\partial t} \fint_{\partial B(\mathbf{x},t)} g(y)\,\mathrm{d}S(\mathbf{y}) &= \frac{\partial}{\partial t} \fint_{\partial B(0,1)} g(\mathbf{x}+t\mathbf{z})\,\mathrm{d}S(\mathbf{z}) \\ &= \fint_{\partial B(0,1)} \nabla g(\mathbf{x}+t\mathbf{z}) \cdot \mathbf{z}\,\mathrm{d}S(\mathbf{z}) \\ &= \fint_{\partial B(\mathbf{x},t)} \nabla g(\mathbf{y}) \cdot \frac{\mathbf{y}-\mathbf{x}}{t}\,\mathrm{d}S(\mathbf{y})\end{aligned}$$

so that at long last we have Kirchhoff's formula

$$\boxed{u(\mathbf{x},t) = \fint_{\partial B(\mathbf{x},t)} \Big(g(\mathbf{y}) + \nabla g(\mathbf{y}) \cdot (\mathbf{y}-\mathbf{x}) + t h(\mathbf{y}) \Big)\,\mathrm{d}S(\mathbf{y})}$$

The situation is more complicated for $n = 2$, but the approach is to add a dummy third spatial dimension and proceed somewhat similarly. One qualitative point to remember is that the formula ends up of the form

$$u(x,t) = \frac{1}{2} \fint_{B(\mathbf{x},t)} (\cdots)\,\mathrm{d}\mathbf{y}$$

so that the value of u depends on the whole expanding ball and not just its surface.

3 Fundamental solutions and Green's functions

Before covering the remaining two main equations, we make some statements about fundamental solutions and Green's functions. Let $\mathcal{L}$ be a linear differential operator. Then the *fundamental solution* is a function $\Phi\colon \mathbb{R}^n \to \mathbb{R}$ satisfying

$$(\mathcal{L}\Phi)(\mathbf{x}) = \delta(\mathbf{x})$$

where δ is the Dirac delta. The utility of the fundamental solution becomes clear if we wish to solve the inhomogeneous equation $\mathcal{L}u = f$ on $\mathbb{R}^n$. We claim that the convolution

$$u(\mathbf{x}) = (\Phi * f) \equiv \int f(\mathbf{y}) \Phi(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}$$

furnishes a solution. Indeed, applying $\mathcal{L}$ and moving inside the integral, we have

$$\begin{aligned}(\mathcal{L}u)(\mathbf{x}) &= \int f(\mathbf{y}) (\mathcal{L}\Phi)(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \int f(\mathbf{y}) \delta(\mathbf{x}-\mathbf{y})\,\mathrm{d}y \\ &= f(\mathbf{x})\end{aligned}$$

Such an approach suffices if there are no boundary conditions. If there are, they can be included by way of a Green's function. Here we limit our discussion to $\mathcal{L} = -\nabla^2$, which will be useful for Poisson's equation and for the heat equation. Suppose we wish to solve

$$\begin{aligned}-\nabla^2 u &= f \text{ on } \Omega \\ u &= g \text{ on } \partial\Omega\end{aligned}$$

An easy application of the divergence theorem shows that, for *any* functions u and G, we have

$$\int_\Omega \Big(u \nabla^2 G - G \nabla^2 u \Big) = \int_{\partial\Omega} \Big(u \nabla G - G \nabla u \Big) \cdot \mathbf{n}$$

Fix an $\mathbf{x}$. If a G can be found that satisfies

$$-\nabla^2 G(\mathbf{y}) = \delta(\mathbf{y}-\mathbf{x}) \text{ for } \mathbf{y} \in \Omega$$
$$G(\mathbf{y}) = 0 \text{ for } \mathbf{y} \in \partial\Omega$$

and u solves Poisson's equation, then the identity becomes

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{y}) f(\mathbf{y}) - \int_{\partial\Omega} g(\mathbf{y}) \nabla G(\mathbf{y}) \cdot \mathbf{n}$$

which provides a closed-form solution for u if G is determined.

How can such a Green's function be found? The idea is to let Φ be the fundamental solution for the operator $-\nabla^2$ and construct $G = \Phi(\mathbf{x}-\mathbf{y}) - \eta(\mathbf{y})$, where η solves

$$\nabla^2 \eta(\mathbf{y}) = 0 \text{ for } \mathbf{y} \in \Omega$$
$$\eta(\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) \text{ for } \mathbf{y} \in \partial\Omega$$

In general finding such an η is quite involved and may not even be possible, but we will see that for certain domains the problem will be analytically tractable for the Laplacian.

4 Poisson's equation

Poisson's equation is

$$-\nabla^2 u = f$$

on some $\bar{U} \subseteq R^n$ where U is open. If $f = 0$ then the equation is Laplace's equation, but we will speak solely of Poisson's equation unless homogeneity is important.

4.1 Fundamental solutions

We begin by noting the useful fact that Laplace's equation is "invariant under rotation", such that if $u(\mathbf{x})$ is a solution, then $v(\mathbf{x}) \equiv u(A\mathbf{x})$ also satisfies $\nabla^2 v = 0$ provided A is orthogonal. Such invariance can be shown by straightforward expansion of the relevant derivatives in coordinate form and use of the fact that A has orthogonal rows.

It is thus natural to guess that we might have $u(\mathbf{x}) = v(r)$. It can be checked that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

so that

$$\frac{\partial u}{\partial x_i} = \frac{x_i}{r} v'(r)$$
$$\frac{\partial^2 u}{\partial x_i^2} = \frac{x_i^2}{r^2} v''(r) + \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) v'(r)$$

and subsequently

$$0 = \nabla^2 u = v''(r) + \frac{n-1}{r} v'(r)$$

If $v' \neq 0$ we may write

$$\left(\log v'\right)' = \frac{v''}{v'}$$
$$= \frac{1-n}{r}$$

so that $\log v' = \log r^{1-n}$ and so $v' = cr^{1-n}$. One further integration gives

$$v(r) = \begin{cases} c_1 \log r + c_2 & n = 2 \\ \dfrac{c_1}{r^{n-2}} + c_2 & n \geq 3 \end{cases}$$

Motivated by the calculation above, we define

$$\boxed{\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha(n)|x|^{n-2}} & n \geq 3 \end{cases}}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$, and this Φ is the fundamental solution of Laplace's equation. (Remember that the volume of a ball of radius r is $r^n \alpha(n)$, and its surface area is $nr^{n-1}\alpha(n)$.) Thus $-\nabla^2 u = f$ is solved by

$$u(\mathbf{x}) = \int \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

when the domain is the entirety of $\mathbb{R}^n$.

4.2 Mean value properties

Suppose that u is harmonic, so $\nabla^2 u = 0$ on the interior of its domain U. Then fix $\mathbf{x}$ and define

$$\phi(r) = \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}) \, \mathrm{d}S(\mathbf{y}) = \fint_{\partial B(0,1)} u(\mathbf{x} + r\mathbf{z}) \, \mathrm{d}S(\mathbf{z})$$

It follows that

$$\begin{aligned} \phi'(r) &= \fint_{\partial B(0,1)} \nabla u(\mathbf{x} + r\mathbf{z}) \cdot \mathbf{z} \, \mathrm{d}S(\mathbf{z}) \\ &= \fint_{\partial B(\mathbf{x}, r)} \nabla u(\mathbf{y}) \cdot \frac{\mathbf{y} - \mathbf{x}}{r} \, \mathrm{d}S(\mathbf{y}) \\ &= \fint_{\partial B(\mathbf{x}, r)} \nabla u(\mathbf{y}) \cdot \mathbf{n} \, \mathrm{d}S(\mathbf{y}) \\ &\propto \fint_{B(\mathbf{x}, r)} \nabla^2 u(\mathbf{y}) \, \mathrm{d}\mathbf{y} = 0 \end{aligned}$$

So ϕ is constant, such that

$$\phi(r) = \lim_{t \to 0^+} \phi(t) = u(\mathbf{x})$$

Thus we have the property

$$\boxed{u(\mathbf{x}) = \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}) \, \mathrm{d}S(\mathbf{y}) = \fint_{B(\mathbf{x}, r)} u(\mathbf{y}) \, \mathrm{d}\mathbf{y}}$$

for any $\mathbf{x}$ and r such that $B(\mathbf{x}, r) \subset U$. The second equality follows rather simply by taking the volume integral as an integral over spherical shells.

4.3 Maximum principles

Let u be harmonic on U a connected domain, and define $M = \max_{\bar{U}} u$. Say there is an $\mathbf{x} \in \operatorname{int} U$ with $u(x) = M$. For any appropriate r, we have

$$M = u(\mathbf{x}) = \fint_{B(\mathbf{x}, r)} u(\mathbf{y}) \, \mathrm{d}\mathbf{y} \leq M$$

so that we must have $u(\mathbf{y}) = M$ for all $\mathbf{y} \in B(\mathbf{x}, r)$. So the set $\{\mathbf{x} \in U \mid u(\mathbf{x}) = M\}$ is open, and it is also (relatively) closed because it is the inverse image of the closed set $\{M\}$ under the continuous function u. So the set must be the entirety of U, and U is constant. Thus we have two useful properties.

- A harmonic u achieves its maximum on the boundary of its domain.
- If u achieves its maximum in the interior, it must be constant.

4.4 Other important properties

Here we state without proof some properties of solutions to Poisson's equation.

- The solution to the Poisson boundary problem is unique.
- Harmonic functions are infinitely differentiable.
- Harmonic functions are analytic.

4.5 Green's function for the half-space

We define the half-space

$$\mathbb{R}^n_+ = \left\{\mathbf{x} \in R^n \mid x_n > 0\right\}$$

and the reflection operator

$$\tilde{\mathbf{x}} = (x_1, \ldots, x_{n-1}, -x_n)$$

Then the right Green's function is

$$G(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) - \Phi(\mathbf{y} - \tilde{\mathbf{x}})$$

Now if $\mathbf{y}$ is in the boundary of the half-space — that is, if $y_n = 0$, then $\Phi(\mathbf{y} - \mathbf{x}) = \Phi(\mathbf{y} - \tilde{\mathbf{x}})$ because Φ is radial and only depends on the norm of its argument. Now, if we wish to solve Laplace's equation on the half-space with boundary data g, we can compute $\nabla G \cdot \mathbf{n}$ on the boundary and use the representation formula to write

$$\boxed{u(\mathbf{x}) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} \, \mathrm{d}\mathbf{y}}$$

4.6 Green's function for the unit ball

We will now seek a Green's function for the unit ball. In this context reflection is defined as

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2}$$

It turns out that the right thing to do is to define

$$G(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) - \Phi\left(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}})\right)$$

and then, after some algebra, we have the representation formula

$$\boxed{u(\mathbf{x}) = \frac{1 - |\mathbf{x}|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} \, \mathrm{d}\mathbf{y}}$$

for solutions to Laplace's equation in the unit ball with data g on the spherical boundary.

5 Heat equation

The heat equation is

$$u_t - \nabla^2 u = f \text{ on } \bar{U} \times (0, \infty)$$
$$\text{initial data on } \bar{U} \times \{0\}$$

where $U \subseteq \mathbb{R}^n$ is open.

5.1 Fundamental solution

Motivated by scale invariance, we guess

$$u(\mathbf{x},t) = t^{-\alpha} v(\mathbf{y}) \text{ where } \mathbf{y} = t^{-\beta}\mathbf{x}$$

Then substituting this form into the homogeneous heat equation, taking $\beta = 1/2$ to allow some terms to simplify, and, and guessing $v(\mathbf{y}) = w(|\mathbf{y}|)$ gives

$$\alpha w + \frac{1}{2}w' + w'' + \frac{n-1}{r}w' = 0$$

Choosing $\alpha = n/2$, inverting the chain rule, and demanding that $w \to 0$ as $r \to \infty$ leads to

$$w = ce^{-r^2/4}$$

for some constant c. Thus we define

$$\boxed{\Phi(\mathbf{x},t) = \frac{1}{(4\pi t)^{n/2}} \exp\left\{-\frac{|\mathbf{x}|^2}{4t}\right\}}$$

the fundamental solution of the heat equation. The constant has been chosen such that integrating Φ over the entire spatial domain yields unity.

If we wish to solve the Cauchy problem — wherein we are given initial data $u(\mathbf{x}, 0) = g(\mathbf{x})$ — we have the representation formula

$$\boxed{u(\mathbf{x},t) = \frac{1}{(4\pi t)^{n/2}} \int \exp\left\{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right\} g(\mathbf{y})\,\mathrm{d}\mathbf{y}}$$

which solves the homogeneous heat equation and is infinitely differentiable for $t > 0$. Moreover the solution approaches the initial data as $t \to 0$, which must be proved.

5.2 Maximum principles

The *parabolic boundary* of a domain $\bar{U} \times [0,T]$ on which u solves the heat equation is

$$\Gamma = \bar{U} \times \{0\} \cup \partial U \times (0,T]$$

which is the set on which initial data is given taken together with the boundary of the domain at later times. We state without proof two maximum principles, analogous to those for solutions to Laplace's equation.

- The maximum of u over all of $\bar{U} \times [0,T]$ is attained on Γ.
- If there is a point $(\mathbf{x},t) \notin \Gamma$ where that maximum is obtained, then u is constant on $\bar{U} \times [0,t]$.

5.3 Energy methods

Consider the initial value problem

$$\begin{aligned} u_t - \nabla^2 u &= f \text{ on } U \times (0,T] \\ u &= g \text{ on } \Gamma \end{aligned}$$

The maximum principles can be used to show uniqueness of the solution, but we can also use an energy method. Let w be the difference between u and another solution, and define

$$E(t) = \int_U w^2\,\mathrm{d}\mathbf{x}$$

Then

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= 2\int_U w w_t \,\mathrm{d}\mathbf{x} \\ &= 2\int_U w\nabla^2 w \,\mathrm{d}\mathbf{x} \\ &= -2\int_U |\nabla w|^2 \,\mathrm{d}\mathbf{x} \le 0\end{aligned}$$

so that the energy is non-increasing. But since the energy of w is initially zero and clearly cannot be negative, we have $E(t) = 0$ for all t, and so $w = 0$ identically.

6 Nonlinear equations

6.1 Characteristics

Given a nonlinear first-order equation

$$\begin{aligned}F(\nabla u, u, \mathbf{x}) &= 0 \text{ on } U \\ u &= g \text{ on } \Gamma \subseteq \partial U\end{aligned}$$

we consider a parametric curve $\mathbf{x}(s)$ and define

$$\begin{aligned}z(s) &= u\Big(\mathbf{x}(s)\Big) \\ \mathbf{p}(s) &= \nabla u\Big(\mathbf{x}(s)\Big)\end{aligned}$$

By differentiating $\mathbf{p}$ as well as F and choosing $\mathbf{x}$ correctly, we find that the right system of characteristics is

$$\boxed{\begin{aligned}&\partial_s \mathbf{p} = -\partial_{\mathbf{x}} F - (\partial_z F)\,\mathbf{p} \\ &\partial_s z = \partial_{\mathbf{p}} F \cdot \mathbf{p} \\ &\partial_s \mathbf{x} = \partial_{\mathbf{p}} F\end{aligned}}$$

where in each right-hand side we mean $F(\mathbf{p}, z, \mathbf{x})$. If we solve this system, we have $F(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0$.

6.2 Conservation laws and shocks

We consider the conservation law in one spatial dimension

$$\begin{aligned}u_t + F(u)_x &= 0 \text{ on } \mathbb{R} \times (0, \infty) \\ u &= g \text{ on } \mathbb{R} \times \{0\}\end{aligned}$$

We say that u is a weak solution if, for any smooth test function v on $\mathbb{R} \times [0, \infty)$ with compact support

$$\int_0^\infty \int_{\mathbb{R}} \big(u v_t + F(u) v_x\big)\,\mathrm{d}x\,\mathrm{d}t + \left(\int_{\mathbb{R}} g v\,\mathrm{d}x\right)\bigg|_{t=0} = 0$$

which is obtained by integrating by parts.

Suppose that u is smooth on either side of a smooth curve C. By choosing v to have compact support contained on one side of C or the other, we can show that $u_t + F(u)_x$ away from the shock. Now, suppose C is parameterized as $\{(x, t) \mid x = s(t)\}$. Then by choosing v to have support on both sides of C and carefully integrating, we find that for any weak solution u we must have the Rankine-Hugoniot condition

$$\boxed{\big[F(u)\big] = \frac{\mathrm{d}s}{\mathrm{d}t}\big[u\big]}$$

where the brackets indicate the jump in a quantity across C.

7 Fourier transform solutions

Recall that the Fourier transform of a function $u(x,t))$ on the real line is

$$\mathcal{F}[u](k,t) = \int_{\mathbb{R}} u(x,t)e^{-ikx}\,\mathrm{d}x$$

and the inverse transform is

$$\mathcal{F}^{-1}\left[\hat{u}\right](x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{u}(k,t)e^{ikx}\,\mathrm{d}k$$

In this section we provide a few examples of how the Fourier transform can be used to solve partial differential equations. We will in some cases use the notation $\hat{u} = F[u]$.

7.1 Wave equation

It is easy to check that taking Fourier transforms of both sides of the equation gives

$$\hat{u}_{tt}(k,t) + k^2\hat{u}(k,t) = 0$$

For each k, this is an ordinary differential equation with solutions

$$\hat{u}(k,t) = \hat{f}(k)e^{ikt} + \hat{g}(k)e^{-ikt}$$

so that we have

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}\left[\hat{u}\right](x,t) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\hat{f}(k)e^{ikt} + \hat{g}(k)e^{-ikt}\right)e^{ikx}\,\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k)e^{ik(x+t)}\,\mathrm{d}x + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(k)e^{ik(x-t)}\,\mathrm{d}x \\
&= f(x+t) + g(x-t)
\end{aligned}$$

which is the general form of the solution to the wave equation we are familiar with.