Atmospheric Dynamics

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1 Equations of motion

The Navier-Stokes equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{F}$$

where $\nu = \mu/\rho$, and the continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

In this section we discuss various consequences of and approximations to these equations.

1.1 Hydrostasy

If $\mathrm{D}w/\mathrm{D}t$ is small compared to gravity, then the vertical component of Navier-Stokes becomes

$$\frac{\partial p}{\partial z} = -\rho g$$

and this assumption is almost always justified in the atmosphere, except in storms at small scales. If we use the ideal gas law and make the (much less justifiable) assumption that the temperature T is constant, then we find that density varies as

$$\rho(z) = \rho_0 \exp\left\{-\frac{gz}{RT}\right\}$$

which is quantitatively suspect but qualitatively correct.

In the (non-rotating) Boussinesq equations (below), we have

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla \phi$$

so that $\phi \sim U^2$. Moreover, the divergence-free condition $\nabla \cdot \mathbf{v} = 0$ gives

$$W \sim \frac{H}{L} U$$

so that vertical advective terms scale as

$$\frac{\mathrm{D}w}{\mathrm{D}t} \sim \frac{UW}{L} = \frac{U^2 H}{L^2}$$

So the ratio of the advective derivative to the pressure gradient term is

$$\frac{\left(U^2 H\right)/L^2}{U^2/H} = \left(\frac{H}{L}\right)^2$$

such that hydrostatic balance is a good approximation provided the aspect ratio is much less than unity.

1.2 Effects of rotation

The Earth is a rotating frame of reference, and it is genearlly useful to consider the flows as they appear within that frame. It is straightforward to derive that

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathrm{R}} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathrm{I}} - \mathbf{\Omega} \times \mathbf{r}$$

and then, since the velocity is not the same when measured in different frames, we can take time derivatives and do some algebraic manipulations to obtain

$$\left(\frac{\mathrm{d}\mathbf{v}_{\mathrm{R}}}{\mathrm{d}t}\right)_{\mathrm{R}} = \left(\frac{\mathrm{d}\mathbf{v}_{\mathrm{I}}}{\mathrm{d}t}\right)_{\mathrm{I}} - 2\mathbf{\Omega} \times \mathbf{v}_{\mathrm{R}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$$

and it is the first term on the right-hand side that must obey Newton's second law. The second term on the right is the Coriolis force. So (neglecting the centrifugal force, or more properly redefining the body forces to include it) the equations of motion become

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{F}$$

Often, we make the "traditional" approximation and neglect the components of $\boldsymbol{\Omega}$ not aligned with the local vertical, writing the Coriolis term as $\mathbf{f} \times \mathbf{v}$ where $\mathbf{f} = 2\Omega \sin\vartheta$. If we hold the latitude ϑ constant, we arrive at the f-plane approximation, whereas if we write the term to be crossed with $\mathbf{v}$ as $(f_0 + \beta y)\,\mathbf{k}$ where $f_0 = 2\Omega \sin\vartheta_0$ we have the β-plane approximation.

1.3 Boussinesq equations

The Boussinesq approximation involves taking variations of ρ away from a constant background state to be small *except* when multiplied by g, and assuming the background pressure state is in hydrostatic balance. The resulting set of equations is

$$\begin{aligned} 0 &= \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{v} + \nabla\phi - b\mathbf{k} \\ 0 &= \nabla \cdot \mathbf{v} \end{aligned}$$

where

$$\begin{aligned} \phi &= \frac{\delta p}{\rho_0} \\ b &= -\frac{g\delta\rho}{\rho_0} \end{aligned}$$

are the kinematic pressue and buoyancy, respectively.

1.4 Geostrophic balance

The Rossby number is the ratio of the magnitudes of the advective and Coriolis terms in the horizonal momentum equation, namely

$$\mathrm{Ro} = \frac{U^2/L}{fU} = \frac{U}{fL}$$

When the Rossby number is small (as it is in the atmosphere, where a typical value is $\mathrm{Ro} = 0.1$) then the horizonal momentum equation is dominated by the Coriolis and pressure gradient terms. We can define the geostrophic velocity as

$$\mathbf{f} \times \mathbf{u}_\mathrm{g} = -\frac{1}{\rho}\nabla p$$

and in geostrophic balance we have $\mathbf{u} \approx \mathbf{u}_\mathrm{g}$. When the Coriolis force is constant and density varies only in the vertical we can define a geostrophic streamfunction

$$\psi_\mathrm{g} = \frac{p}{f\rho}$$

In geostrophic balance, the kinematic pressure and buoyancy scale as

$$\begin{aligned} \phi &\sim \Phi = fUL \\ b &\sim B = \frac{fUL}{H} \end{aligned}$$

1.5 Thermal wind balance

In the Boussinesq approximation, geostrophy looks like

$$
\begin{aligned}
fu_{\mathrm{g}} &= -\frac{\partial \phi}{\partial y} \\
fv_{\mathrm{g}} &= \frac{\partial \phi}{\partial x}
\end{aligned}
$$

Differentiating with respect to z and using hydrostasy $\phi_z = b$ gives

$$
f\frac{\partial}{\partial z}\left(u_{\mathrm{g}}, v_{\mathrm{g}}\right) = \left(-\frac{\partial b}{\partial y}, \frac{\partial b}{\partial x}\right)
$$

The mechanism is that cold columns have larger hydrostatic pressure variations, which leads to geostrophic wind development at higher levels.

1.6 Static instability

We consider a parcel displaced from z to $z + \delta z$ and conserving its potential density relative to $z + \delta z$. It is straightforward to show that

$$
\delta \rho = -\frac{\partial \tilde{\rho}_\theta}{\partial z}\delta z
$$

If $\delta z > 0$, for stability we need $\delta \rho > 0$ so that the parcel is heavier than its surroundings and will sink back down. Thus we have the condition

$$
\text{stability} \iff \frac{\partial \tilde{\rho}_\theta}{\partial z} < 0
$$

Then we can note that the gravitational acceleration obeys

$$
\frac{\partial^2 \delta z}{\partial t^2} = -N^2 \delta z \text{ where } N^2 = -\frac{g}{\tilde{\rho}_\theta}\left(\frac{\partial \tilde{\rho}_\theta}{\partial z}\right)
$$

In the atmosphere we have $\rho_\theta = p_{\mathrm{ref}}$ and so

$$
N^2 = \frac{g}{\tilde{\theta}}\left(\frac{\partial \tilde{\theta}}{\partial z}\right)
$$

and so the reference point can be freely chosen. In the troposphere, we have $N \sim 0.01\,\mathrm{s}^{-1}$, whereas in the stratosphere, oscillations are a few times faster.

2 Shallow water equations

The shallow water equations are

$$
\begin{aligned}
0 &= \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} + g\nabla \eta \\
0 &= \frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u})
\end{aligned}
$$

Since they are an easy setting in which to consider a few key ideas of geophysical fluid dynamics, we consider them in some detail. We omit a full derivation, but note that

- The momentum equation is derived by assuming density to be constant and integrating the equation for hydrostatic balance.
- The continuity equation is derived by noting that mass flux into a column must be balanced by a change in the column height and applying the divergence theorem.

2.1 Potential vorticity

The shallow water potential vorticity is

$$Q \equiv \frac{\zeta + f}{h}$$

where ζ is the vertical component of the vorticity $\nabla \times \mathbf{u}$. The potential vorticity satisfies

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0$$

and is thus materially conserved, as can be shown by taking the curl of the momentum equation and substituting the result into the mass continuity expression.

2.2 Shallow water waves

Linearizing the rotating shallow water equations around a state of rest gives the dispersion relation

$$\omega^2 = f_0^2 + gH\left(k^2 + \ell^2\right)$$

In the short wave limit, where $k^2 + \ell^2$ is large, the effects of rotation are negligible and we have $\omega \approx \pm K\sqrt{gH}$. In the long wave limit, we have inertial oscillations with $\omega \approx f_0$.

2.3 Geostrophic adjustment

Geostrophic adjustment is the process by which unbalanced disturbances to the flow are radiated away, leaving a flow that is in geostrophic balance. A useful model is the linearized shallow water equations, wherein we have

$$\frac{\partial q}{\partial t} = 0 \text{ where } q = \zeta - f_0 \frac{\eta}{H}$$

Then analytic solutions for the steady state can be obtained from the linearized momentum equations. In a rotating frame, the effects of adjustment are limited by the Rossby radius of deformation

$$L_{\mathrm{d}} = \frac{\sqrt{gH}}{f_0}$$

It also turns out that, in the linear problem, geostrophic balance is the energy-minimizing state for a given linear potential vorticity field q.

3 Geostrophic theory

3.1 Non-dimensional Boussinesq flow

We will present the quasi-geostrophic equations in the continuously stratified Boussinesq system. The equations we start with are

$$\begin{aligned}
\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f} \times \mathbf{u} &= -\nabla\phi' \\
\frac{\partial \phi'}{\partial z} &= b' \\
\frac{\mathrm{D}b'}{\mathrm{D}t} + N^2 w &= 0 \\
\nabla \cdot \mathbf{v} &= 0
\end{aligned}$$

where we have decomposed $b = \tilde{b}(z) + b'$ and defined $N^2 = \tilde{b}_z$. Similarly we have set $\phi = \tilde{\phi}(z) + \phi'$, where $\tilde{\phi}$ is in hydrostatic balance with $\tilde{b}$. Some scaling arguments show

$$\frac{b'_z}{N^2} \sim \mathrm{Ro}\frac{L^2}{L_{\mathrm{d}}^2} \text{ where } L_{\mathrm{d}} = \frac{N_0 H}{f_0}$$

Ultimately, we have the following equations, where all variables are taken to be non-dimensional.

$$\begin{aligned}
\mathrm{Ro}\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f}\times\mathbf{u} &= -\nabla\phi \\
\frac{\partial\phi}{\partial z} &= b \\
\mathrm{Ro}\frac{\mathrm{D}b}{\mathrm{D}t} + \left(\frac{L_\mathrm{d}}{L}\right)^2 N^2 w &= 0 \\
\nabla\cdot\mathbf{v} &= 0
\end{aligned}$$

In the atmosphere the deformation radius is typically $L_\mathrm{d} \approx 1000\,\mathrm{km}$.

3.2 Quasi-geostrophy

For quasi-geostrophic flow, we assume that

- the Rossby number Ro is small.
- the length scale L satisfies $L \sim L_\mathrm{d}$.
- the Coriolis parameter varies slowly such that $\beta y \sim \mathrm{Ro} f_0$.

Thus the ageostrophic component of $\mathbf{u}$ is much smaller than the geostrophic velocity. Nonetheless, the only horizontal contributions to $\nabla\cdot\mathbf{v}$ come from the ageostrophic component, and so w is small.

We can take the curl of the momentum equation, keep $\nabla\cdot\mathbf{u}$ only when it is scaled by f, and neglect cross terms with small terms involving w. Ultimately, we arrive at

$$\frac{\mathrm{D_g}}{\mathrm{D}t}\left(\zeta_\mathrm{g} + f\right) = f_0\frac{\partial w}{\partial z}$$

We can also rewrite the thermodynamic equation with the knowledge that w is small, such that it only advects the basic state, so

$$w = -\frac{1}{N^2}\frac{\mathrm{D_g}b}{\mathrm{D}t}$$

which we can use to eliminate w from the previous equation. We can also define a geostrophic streamffunction $\psi = p/(f_0\rho_0)$ such that

$$\begin{aligned}
\mathbf{u}_\mathrm{g} &= \mathbf{k}\times\nabla\psi \\
\zeta_\mathrm{g} &= \nabla^2\psi \\
b &= f_0\frac{\partial\psi}{\partial z}
\end{aligned}$$

and we have the equation for the quasi-geostrophic equation potential vorticity

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \text{ where } q = \nabla^2\psi + \beta y + f_0^2\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial\psi}{\partial z}\right)$$

3.3 Quasi-geostrophy in other systems

We state without derivation the quasi-geostrophic equations in the shallow water system

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \text{ where } q = \nabla^2\psi + \beta y - \left(\frac{gH}{f_0^2}\right)^{-1}\psi$$

and in the two-layer stratified model

$$\frac{\mathrm{D}q_i}{\mathrm{D}t} = 0 \text{ where } q_i = \nabla^2\psi_i + \beta_y\frac{k_\mathrm{d}^2}{2}\left(\psi_j - \psi_i\right)$$

and $k_\mathrm{d}^2 = 8f_0^2/\left(N^2H^2\right)$.

4 Waves

4.1 The WKB approximation

The WKB method allows approximate solutions of equations of the form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2} + m^2(z)\xi = 0$$

under the assumption that m varies slowly in z, which we can more precisely formulate as

$$\left|\frac{\mathrm{d}m}{\mathrm{d}z}\right| \ll m^2$$

The idea is to propose a solution of the form $\xi = A(z)e^{i\theta(z)}$ and then neglect terms involving $\mathrm{d}^2 A/\mathrm{d}z^2$. The eventual result is

$$\xi(z) = \frac{A_0}{\sqrt{m}} \exp\left\{\pm i \int m\,\mathrm{d}z\right\}$$

where A_0 is a (possibly complex) constant.

4.2 Rossby waves

Rossby waves are generated when a material line is displaced. Parcels along the line that change latitude change their planetary vorticity, and so must change their relative vorticity as well to conserve potential vorticity, which causes a wave to propagate.

It is instructive to consider the case where q is materially conserved and linearize about a basic state that varies only in y and z. Eventually we arrive at

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial \bar{q}}{\partial y} = 0$$

which is a useful equation for the study of Rossby waves. Let us consider this equation in stratified quasi-geostrophic flow, assuming f_0^2/N^2 is independent of z. We linearize the potential vorticity equation about a basic state $\psi = -Uy$ and thus $q = \beta y$. Then the linearized equation above becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla^2\psi' + \frac{f_0^2}{N^2}\frac{\partial^2\psi'}{\partial z^2}\right] + \beta\frac{\partial\psi'}{\partial x} = 0$$

Assuming a plane wave solution, we find

$$\omega = Uk - \frac{\beta k}{k^2 + \ell^2 + \left(f_0^2/N^2\right)m^2}$$

We can solve for the vertical wavenumber and determine a condition for m^2 to be positive — that is, a condition for Rossby wave propagation away from the surface. We find that we need

$$0 < U - \frac{\omega}{k} < \frac{\beta}{k^2 + \ell^2}$$

It is also worth remembering that for infinite deformation radius, we have

$$\omega = Uk - \frac{\beta k}{k^2 + \ell^2}$$

so that, in the absence of a mean flow, the zonal phase speed is always westward.

4.3 Gravity waves

First, we consider the linear Boussinesq equations

$$
\begin{aligned}
\frac{\partial \mathbf{u}'}{\partial t} &= -\nabla \phi' \\
\frac{\partial w'}{\partial t} &= -\frac{\partial \phi'}{\partial z} + b' \\
\nabla \cdot \mathbf{v}' &= 0 \\
0 &= \frac{\partial b'}{\partial t} + w' N^2
\end{aligned}
$$

We can proceed by deriving a single equation for w'. If N^2 is constant we get the dispersion relation

$$
\begin{aligned}
\omega^2 &= \frac{\left(k^2 + \ell^2\right) N^2}{k^2 + \ell^2 + m^2} \\
&= N^2 \cos^2 \vartheta
\end{aligned}
$$

where ϑ is the angle the three-dimensional wave vector makes with the horizontal.

Note that if the perturbations are in hydrostatic balance, we end up with the dispersion relation

$$
\omega^2 = \frac{\left(k^2 + \ell^2\right) N^2}{m^2}
$$

which is now unbounded as the horizontal scale becomes smaller than the vertical. Vallis claims that this shows why convection must be parameterized in models.

If there is a mean zonal flow U, then the same linearization approach simply gives the Doppler-shifted equation

$$
(\omega - Uk)^2 = \frac{K^2 N^2}{K^2 + m^2}
$$

where $K^2 = k^2 + \ell^2$. Note, though, that the vertical group velocity is

$$
\frac{\partial \omega}{\partial m} = \frac{km(U - c)}{K^2 + m^2}
$$

so that the wave propagation slows down near critical levels where $U \approx c$.

Finally, we consider gravity waves in a rotating fluid, using a parcel argument. We consider a parcel that is displaced along a line making an angle ϑ with the vertical. There is a buoyancy restoring force

$$
F_{\mathrm{b}} = -N^2 \cos \vartheta \delta z = -N^2 \cos^2 \vartheta \delta s
$$

and a Coriolis restoring force

$$
F_{\mathrm{C}} = -f^2 \sin \vartheta \delta x = -N^2 \sin^2 \vartheta \delta s
$$

Then, by Newton's second law, the equation of motion is

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2} \delta s = -\left(N^2 \cos^2 \vartheta + f^2 \sin^2 \vartheta\right) \delta s
$$

so that the frequency of oscillation is $\omega^2 = N^2 \cos^2 \vartheta + f^2 \sin^2 \theta$. Writing ϑ in terms of the components of the wave vector and making the straightforward generalization to two horizontal dimensions, we have

$$
\omega^2 = \frac{N^2 \left(k^2 + \ell^2\right) + f^2 m^2}{k^2 + \ell^2 + m^2}
$$

We can also derive this by taking the horizontal divergence and curl of the linearized momentum equations, eliminating vorticity, and positing a wave solution for w'.

5 Instability

An instability is

- **barotropic** if it occurs because of shear in a flow.
- **baroclinic** if it arises due to a horizontal temperature gradient in a rotating, stratified fluid.

Usually, barotropic problems are simpler to analyze, and may provide insights into the baroclinic setting.

5.1 Barotropic instability

Recall that in incompressible two-dimensional Euler flow the vorticity $\zeta = v_x - u_y$ is materially conserved. We linearize about a basic zonal flow $U(y)$, so that the equation for the vorticity perturbation is

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} - v' U_{yy} = 0$$

The natural thing to do is introduce a stream function and posit a solution of the form

$$\psi = \tilde{\psi}(y)e^{ik(x-ct)}$$

From here we can derive a dispersion relation. For concisencess, we include here a more general version that incudes the β-effect. We find

$$(U-c)\left(\tilde{\psi}_{yy} - k^2\tilde{\psi}\right) + \left(\beta - U_{yy}\right)\tilde{\psi} = 0$$

which is the Rayleigh-Kuo equation. We can solve this for piecewise linear mean states by demanding that pressure is continuous across interfaces and that the normal velocity of the interface is consistent with the interface motion.

A necessary condition for instability can be derived by integrating the equation for $\zeta^2/\left(\beta - U_{yy}\right)$ and noting that

$$v\zeta = -\frac{\partial}{\partial y}(uv) + \frac{1}{2}\frac{\partial}{\partial x}\left(v^2 - u^2\right)$$

such that a necessary condition for instability is for $\beta - U_{yy}$ to change sign somewhere in the domain.

5.2 Baroclinic instability

Baroclinic instability can develop in a reasonable model of a stably stratified atmosphere, where $\rho_z < 0$ and $\rho_y > 0$. If a parcel is swapped with another that is both higher and more poleward than it, it will end up both higher than its original altitude and positively buoyant compared to its surroundings.

Quantitatively, we consider the Boussinesq quasi-geostrophic potential vorticity equation and linearize about a mean state $U(y,z)$. The background potential vorticity is then

$$Q = \beta y + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial}{\partial z}F\frac{\partial \Psi}{\partial z}$$

where $F = f_0^2/N^2$, and the linearized equation is

$$\frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} + v\frac{\partial Q}{\partial y} = 0$$

We can seek normal-mode solutions, and we ultimately find the following analog of the Rayleigh equation

$$(U-c)\left(\tilde{\psi}_{yy} + \left(F\tilde{\psi}_z\right)_z - k^2\tilde{\psi}\right) + Q_y\tilde{\psi} = 0$$

We can also multiply the linearized vorticity equation by q and divide by Q_y to arrive at

$$\frac{\partial}{\partial t}\left[\frac{q^2}{Q_y}\right] + \frac{U}{Q_y}\frac{\partial q^2}{\partial x} + 2vq = 0$$

whence we can derive a similar result about Q_y needing to change sign for instability.

5.3 Eady problem

The Eady problem assumes

- motion on an f-plane.
- constant stratification N^2.
- a basic state with uniform shear $U = \Lambda z$ so that the background potential vorticity $Q = 0$.
- rigid, flat boundaries at the top and bottom.

We seek solutions that are waves in x and find

$$(\Lambda z - c)\left(\frac{\partial^2 \tilde{\psi}}{\partial y^2} + \frac{H^2}{L_{\mathrm{d}}^2}\frac{\partial^2 \tilde{\psi}}{\partial z^2} - k^2 \tilde{\psi}\right) = 0$$

It turns out that the maximum growth rate is

$$\sigma = \frac{0.31 \Lambda f}{N}$$

In the atmosphere, the scale of maximum instability is about 4000 km and the growth rate is about $0.26\,\mathrm{d}^{-1}$.

6 Zonal-mean circulation

6.1 General features

In pressure coordinates, thermal wind balance looks like

$$f\frac{\partial u}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y}$$

such that if temperature decreases with pressure, the zonal wind decreases with pressure (and therefore increases with height). Thermal wind balance provides a good approximate description of the zonally-averaged u and T fields.

In the troposphere, temperature decreases with latitude and height, but in the stratosphere, temperature increases with height and may not decrease with latitude. The meridional temperature gradient is larger in the winter because the poles receive little direct insolation, and so the zonal winds are stronger in winter as well. The maxima of the surface zonal winds are poleward of the jets aloft, and unlike the jets must be maintained against dissipation.

Fluid motion in the atmosphere and ocean acts to reduce the zonal wind shear from what it would be in radiative equilibrium. Roughly two thirds of the necessary poleward transport of heat is attributable to the atmosphere, and the remaining third to the ocean.

In each hemisphere, the atmosphere has three cells: the thermally direct Hadley cell in the tropics, the thermally indirect Ferrel cell in the midlatitudes, and a thermally direct cell near the pole that is often weak or absent altogether. The surface winds go from easterly in the tropics to westerly in the midlatitudes to easterly near the pole, though the polar easterlies are again often faint or nonexistent.

6.2 Hadley cell

The Hadley cell is characterized by upwelling near the equator and subsidence at roughly 30°. It is much stronger in the winter hemisphere. We can construct a qualitatively useful axisymmetric model of the Hadley cell by assuming that

- the flow is steady.
- the poleward-moving air in the upper branch conserves angular momentum.

- thermal wind balance holds.

When we take a zonal average and neglect the vertical advection and eddy terms, the momentum equation can be written

$$\left(f+\bar{\zeta}\right)\bar{v}=0 \text{ where } \bar{\zeta}=-\frac{1}{a\cos\vartheta}\frac{\partial}{\partial\vartheta}\left[\bar{u}\cos\vartheta\right]$$

such that, assuming $\bar{v}=0$, we have

$$2\Omega\sin\vartheta=\frac{1}{a}\frac{\partial\bar{u}}{\partial\vartheta}-\frac{\bar{u}\tan\vartheta}{a}$$

If the flow at the equator vanishes, then (dropping overbars) we have

$$u=\Omega a\frac{\sin^2\vartheta}{\cos\vartheta}$$

This functional form is consistent with conservation of angular momentum

$$m=\left(u+\Omega a\cos\vartheta\right)a\cos\vartheta$$

which is probably an easier way to derive the velocity. Thermal wind looks like

$$2\Omega\sin\vartheta\frac{\partial u}{\partial z}=-\frac{1}{a}\frac{\partial b}{\partial\vartheta}$$

where $b=g\delta\theta/\theta_0$. Vertical integration and application of the small-angle approximation gives the average potential temperature as

$$\theta=\theta(0)-\frac{\theta_0\Omega^2y^4}{2gHa^2}$$

but this of course becomes too cold too fast. We can approximate the radiative equilibrium temperature as

$$\theta_{\mathrm{E}}=\theta(0)-\Delta\theta\left(\frac{y}{a}\right)^2$$

and note that the Hadley temperature decreases faster. So the extent of the Hadley cell is where these profiles intersect, giving

$$\frac{y_{\mathrm{H}}}{a}=\left(\frac{2\Delta\theta gH}{\Omega^2a^2\theta_0}\right)^{1/2}$$

7 Jet maintenance

In this section we provide some simple explanations for the observed westerly jets in the midlatitudes, which must be maintained in the presence of dissipation. Though the midlatitudes are quite baroclinic, the dynamical mechanisms we discuss will mostly be set in an idealized barotropic context. However, at several points we will invoke some kind of "stirring" to drive momentum flux propagation, and here we are generally referencing baroclinic instabilities as occur in the real atmomsphere.

7.1 Mechanism

Consider the polar cap circumscribed by the line C of constant latitude ϑ. If there is stirring equatorward, lower-vorticity fluid will enter the polar cap, and the area integral of $\boldsymbol{\omega}_{\mathrm{abs}}$ will therefore fall. The integral along the line of constant latitude of the absolute velocity must similarly decrease by Stokes's theorem, so

$$\oint_C\left(u\left(t_{\mathrm{f}}\right)+\Omega a\cos\vartheta\right)\mathrm{d}\ell<\oint_C\left(u\left(t_{\mathrm{i}}\right)+\Omega a\cos\vartheta\right)\mathrm{d}\ell$$

Since the second term in each integral is constant and time and the first is proportional to the zonal average, poleward of the disturbance we have

$$\bar{u}\left(t_{\mathrm{f}}\right)<\bar{u}\left(t_{\mathrm{i}}\right)$$

such that the stirring drives a decrease in the zonal flow poleward — and, by an analogous argument, equatorward — of its source. If the angular momentum of the global flow is to be conserved, we expect stronger westerly flow at the latitude of the stirring, and a jet should therefore form.

7.2 Dynamics

Note that for divergence-free two-dimensional flow the zonal momentum equation can be written

$$\frac{\partial u}{\partial t} + \frac{\partial\left(u^2\right)}{\partial x} + \frac{\partial(uy)}{\partial y} - fv = -\frac{\partial \phi}{\partial x} - D$$

where D is any dissipation. We decompose the flow variables into their zonal means and deviations therefrom, so that the zonally-averaged form above the equation is

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y}\overline{u'v'} - D$$

where we have taken $\bar{v} = 0$. We will thus focus on the eddy momentum flux tendency $-\partial_y \overline{u'v'}$. Suppose Rossby waves are produced by stirring in the midlatitudes. So long as propagation is linear, the deviation streamfunction can be written

$$\psi' = \mathrm{Re}\, A e^{i(kx+\ell y-\omega t)}$$

for some constant A, and thus we have

$$\begin{aligned}
u' &= \mathrm{Re}\left[-i\ell A e^{i(kx+\ell y-\omega t)}\right] \\
&= A\ell \sin(kx+\ell y-\omega t) \\
v' &= \mathrm{Re}\left[ikAe^{i(kx+\ell y-\omega t)}\right] \\
&= -Ak\sin(kx+\ell y-\omega t)
\end{aligned}$$

so that, since the zonal average involves averaging over a period of the trigonometric term, we have

$$\overline{u'v'} = -\frac{1}{2}A^2 k\ell$$

Now, the Rossby wave dispersion relation in the presence of a mean flow is

$$\omega = \bar{u}k - \frac{\beta k}{k^2+\ell^2}$$

from which we can compute the meridional group velocity

$$\frac{\partial \omega}{\partial \ell} = \frac{2\beta k\ell}{\left(k^2+\ell^2\right)^2}$$

The sign of the group velocity, determined by the sign of $k\ell$, should be positive north of the stirring and negative south of it, since we expect the waves to carry energy away from their source. Thus we have $\overline{u'v'} > 0$ to the south and $\overline{u'v'} < 0$ to the north, so that $-\partial_y \overline{u'v'} > 0$ at the level of the stirring. The eddy momentum flux tendency is therefore positive and accelerates the jet.

As an aside, it is useful to remember that the vorticity flux can be written

$$\begin{aligned}
v\zeta &= v\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} \\
&= \frac{1}{2}\frac{\partial\left(v^2\right)}{\partial x} - \left(\frac{\partial(uv)}{\partial y} - u\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{2}\frac{\partial\left(v^2\right)}{\partial x} - u\frac{\partial u}{\partial x} - \frac{\partial(uv)}{\partial y} \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(v^2-u^2\right) - \frac{\partial(uv)}{\partial y}
\end{aligned}$$

In the zonal average, all terms except the eddy flux vanish on the left-hand side because $\bar{v} = 0$, so

$$\overline{v'\zeta'} = -\frac{\partial}{\partial y}\overline{u'v'}$$

and thus ultimately the momentum equation can be written

$$\frac{\partial \bar{u}}{\partial t} = \overline{v'\zeta'} - D$$