Climate Dynamics

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1 Thermodynamics

1.1 Ideal gases

A primitive form of the ideal gas law is

$$p = knT$$

where k is Boltzmann thermodynamic constant and n is the particle number density. Let μ be the mass of a proton (and roughly that of a neutron), and M be the number of protons and neutrons in a molecule of the gas, so that $\rho = nM\mu$. Moreover, define $R^* = k/\mu = 8314.5\,\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}$. Then

$$p = \frac{R^*}{M}\rho T$$

and if we then define a constant $R = R^*/M$ particular to the gas under consideration, we have

$$p = R\rho T$$

For dry air we have $R \approx 287\,\mathrm{m}^2\,\mathrm{s}^{-2}\,\mathrm{K}^{-1}$.

1.2 First and second laws of thermodynamics

Suppose some heating δQ is applied to a parcel. The parcel will (without loss of generality) expand with a volume change $\mathrm{d}V$, such that the work done is $p\,\mathrm{d}V$. We will now assume that all calculations are done per unit mass, such that the work of expansion is $p\,\mathrm{d}\alpha$, where $\alpha = \rho^{-1}$. Moreover, the temperature will change with heating. Assuming the processes are incremental, we can write the energy associated with temperature change as $c_\mathrm{v}\,\mathrm{d}T$, where c_v is the specific heat at constant volume. Thus

$$\begin{aligned}
\delta Q &= c_\mathrm{v}\,\mathrm{d}T + p\,\mathrm{d}\alpha \\
&= c_\mathrm{v}\,\mathrm{d}T + \mathrm{d}(p\alpha) - \alpha\,\mathrm{d}p \\
&= c_\mathrm{v}\,\mathrm{d}T + \mathrm{d}(RT) - \alpha\,\mathrm{d}p \\
&= (c_\mathrm{v} + R)\,\mathrm{d}T - \alpha\,\mathrm{d}p
\end{aligned}$$

and this last relation is an expression of the first law of thermodynamics. We define $c_\mathrm{p} = c_\mathrm{v} + R$, the specific heat at constant pressure. To get an exact differential, we define the entropy s such that

$$\begin{aligned}
\mathrm{d}s = \frac{\delta Q}{T} &= c_\mathrm{p}\left(\frac{\mathrm{d}T}{T} - \frac{R}{c_\mathrm{p}}\frac{\mathrm{d}p}{p}\right) \\
&= c_\mathrm{p}\,\mathrm{d}\left[\ln\left(Tp^{-R/c_\mathrm{p}}\right)\right]
\end{aligned}$$

as can be verified using the chain rule property of the differential operator. Thus we have the entropy

$$s = c_\mathrm{p}\ln\left(Tp^{-R/c_\mathrm{p}}\right)$$

which is a state variable. The second law of thermodynamics states that s cannot decrease when the system is energetically closed. For adiabatic changes of state — those where $\delta Q = 0$ — the entropy is conserved.

1.3 Potential temperature and the dry adiabat

We can define a reference pressure p_0. Then if the system is in some other state (T, p), conservation of entropy shows that the potential temperature

$$\theta = T\left(\frac{p}{p_0}\right)^{-R/c_\mathrm{p}}$$

is the temperature the system would have if brought adiabatically to reference pressure. For a compressible, well-stirred atmosphere, we expect constant θ, such that the temperature profile is

$$T = T_0 \left(\frac{p}{p_0}\right)^{R/c_\mathrm{p}}$$

It is then rather straightforward to check that $\mathrm{d}\ln T/\,\mathrm{d}\ln p = R/c_\mathrm{p}$, and this is one useful definition of the dry adiabatic lapse rate. We can derive the dry adiabat in height coordinates provided we assume the hydrostatic relation $\mathrm{d}p = -\rho g\,\mathrm{d}z$. Then, starting from $c_\mathrm{p}\,\mathrm{d}T = \alpha\,\mathrm{d}p$, we have

$$-\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{g}{c_\mathrm{p}} \approx 9.8\,\mathrm{K\,km^{-1}}$$

which is perhaps the more commonly seen formulation.

1.4 Phase changes and the moist adiabat

We define the latent heat of vaporization L_v as the heat absorbed when a unit mass of water vapor evaporates. Then the Clausius-Clapeyron relation states

$$\frac{\mathrm{d}p_\mathrm{sat}}{\mathrm{d}T} = \frac{L_\mathrm{v}}{R_\mathrm{v}T^2}p_\mathrm{sat}$$

If we make the reasonable assumption that L_v is independent of temperature, we find

$$p_\mathrm{sat}(T) = p_\mathrm{sat}\left(T_0\right)\exp\left\{-\frac{L_\mathrm{v}}{R_\mathrm{v}}\left(\frac{1}{T}-\frac{1}{T_0}\right)\right\}$$

Thus, as a parcel is lifted and and the pressure decreases, so too does the saturation pressure, and so eventually some water vapor will condense out, releasing energy and partially counteracting the temperature decrease. The moist adiabatic lapse rate is thus smaller than the dry rate.

2 Radiative balance

2.1 Energy balance with no atmosphere

We begin by modeling the Earth as a sphere with no atmosphere and uniform temperature T, gaining energy only from the Sun and losing it only to blackbody radiation. The latter goes as σT^4, where $\sigma \approx 5.67\times10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}}$ is the Stefan-Boltzmann constant.

We define by $S_0 \approx 1360\,\mathrm{W\,m^{-2}}$ the solar constant, the amount of energy per unit area reaching the Earth from the Sun. Then the energy received by the Earth (which looks like a disk when projected on the sphere around the Sun) is $\pi a^2 S_0(1-\alpha)$, where a is the radius of the Earth and α is its albedo. Moreover the energy lost by the planet is $4\pi a^2\sigma T^4$, since energy is lost uniformly over the surface. If the Earth is in equilibrium with its surroundings, the gain must equal the loss, such that

$$T = \left(\frac{S_0(1-\alpha)}{4\sigma}\right)^{1/4} \approx 255\,\mathrm{K}$$

where for the numerical approximation we take $\alpha = 0.3$. This value is the *emission temperature* — the temperature the Earth would have to reach to be in equilibrium if it were a uniform blackbody.

2.2 Influence of atmospheric layers

The observed mean surface temperature of the Earth is roughly 288 K, far warmer than the emission temperature derived above. The gap can be explained by the presence of an atmosphere containing greenhouse gases. Dynamical processes also play a role, but we will leave them out for now.

The Earth's atmosphere is largely transparent to incoming solar radiation and somewhat opaque to outgoing longwave radiation. We set up a model with a single atmospheric layer with emmisivity ϵ. Set $F_\mathrm{s} = \sigma T_\mathrm{s}^4$ and $F_\mathrm{a} = \epsilon\sigma T_\mathrm{a}^4$ to be the fluxes per unit area of the surface and atmosphere, respectively. At the top of the atmosphere, we have

$$\frac{S_0(1-\alpha)}{4} = F_\mathrm{a} + (1-\epsilon)F_\mathrm{s}$$

since some of the surface flux is absorbed by the atmosphere. At the surface, we have

$$\frac{S_0(1-\alpha)}{4} + F_\mathrm{a} = F_\mathrm{s}$$

Solving, we have

$$\begin{aligned}\sigma T_\mathrm{s}^4 = F_\mathrm{s} &= \frac{S_0(1-\alpha)}{2(2-\epsilon)} \\ &= \frac{2}{2-\epsilon}\cdot\frac{S_0(1-\alpha)}{4} \\ &= \frac{2}{2-\epsilon}\sigma T_\mathrm{e}^4\end{aligned}$$

where T_e is the emission temperature determined above. It follows that

$$T_\mathrm{s} = \left(\frac{2}{2-\epsilon}\right)^{1/4} T_\mathrm{e}$$

Since $0 \le \epsilon \le 1$, the surface temperature is warmer than the emission temperature, and for a completely opaque atmospheric layer we have $T_\mathrm{s} = 2^{1/4}T_\mathrm{e} \approx 303\,\mathrm{K}$. Alternatively, we could choose ϵ to match the observed temperature difference, finding $\epsilon \approx 0.77$. We also compute

$$T_\mathrm{a} = \left(\frac{1}{2-\epsilon}\right)^{1/4} T_\mathrm{e}$$

and so we have $T_\mathrm{e} \le T_\mathrm{a} < T_\mathrm{s}$, where the first relation is equality exactly when $\epsilon = 1$.

2.3 Implications for global warming

The simple models above illustrate a rough mechanism for how global warming works. As ϵ goes to unity — that is, as the atmosphere becomes more and more opaque to outgoing longwave radiation, as occurs when greenhouse gas concentrations increase — the temperature of the atmosphere decreases towards T_e, allowing the surface temperature to warm even as the whole system stays in balance.

In reality, the atmosphere is better modeled as a large number of stacked layers (and it is straightforward to extend the model above to, say, a two-layer model). In this case, as ϵ increases, the level at which the emission temperature is achieved rises higher in the atmosphere. The surface temperature thus increases, as can be seen simply by projecting the lapse rate down from the emission level.

Note that this mechanism depends on compressibility, for there can only be colder air aloft in a stratified system if the fluid is compressible (and so potential temperature still increases upwards). The greenhouse effect moves the emission level higher, but there must be colder air above for such a shift to be possible.

3 El Niño and the Southern Oscillation

3.1 Characteristics

El Niño is characterized by anomalously high temperatures in the surface waters of the eastern tropic Pacific. SSTs can rise by as much as 6 K compared to neutral years, and the eastern Pacific may be as warm as the west. El Niño events occur irregularly, with intervals ranging from three to seven years, and generally last for several months.

There is a covarying pattern — the Southern Oscillation — in the atmosphere. The pressure difference between Darwin and Tahiti is commonly taken as an index. The (normally westerly) trade winds tend to weaken or even reverse during a strong El Niño event.

3.2 Mechanism

Cold water flows northward along the western coast of South America — this flow is the Humboldt current. At the equator, the trade winds drive the surface waters to the west, and so there is upwelling of cold water off the South American coast to match the wind-driven divergence. As the surface water moves to the west, it is heated by the sun. The end result is that in the basic state, the western tropical Pacific is warmer by about 10 K then the east. Thermocline depth is also (to some extent) correlated with SST, and the deeper thermocline in the west reinforces the gradient by making it more difficult for cold abyssal water to upwell.

The high SSTs in the west cause increased convective activity in the atmosphere, which produces strong updrafts over the western Pacific and a subsiding return flow over the east, with a westerly flow aloft countering the surface trade winds. This cell is the Walker circulation, and it is in positive feedback with the strength of the SST gradient.

The positive feedback means that if the trade winds weaken, so too will the SST gradient, until heating in the east is commensurate with that in the west, leading to an El Niño event. Of course, the feedback does not grow indefinitely; it is damped by heat loss to the atmosphere and by horizontal advection of the warm pool along the coast. The positive feedback mechanism than occurs in reverse to bring the system back to its basic state.

Dynamically, when there is an anomalous weakening of the trade winds, the thermocline depth normally present in the western Pacific is carried eastward by equatorial Kelvin waves and then dissipated along the American coast by coastal Kelvin waves. The off-equator shallow anomaly is carried westward by Rossby waves and then reflected back along the equator as Kelvin waves.