PHYS 381 – Computational Physics I (Winter 2023)

Assignment #2: The Pendulum Problem

Due date: 13 February 2023, 3 pm

Group members (add your name and UCID below):

Member #1: David (Scott) Salmon - 30093320

Member #2: Gradyn Roberts - 30113797

Member #3: Gabriel Komo - 30164690

Please, only add text in this report template using Times New Roman font, non-italic, non-bold, font size 12. Equations can be typed using the default font of the equation editor of your choice.

Authors' contributions:

Provide a brief description of how the work of this assignment was distributed among group members. Assignment workload distribution must be balanced among group members, i.e., all group members should be involved equally with coding and written materials for the report. **Authors' contribution** description cannot go over page 1 of this template.

We all worked together on all of the parts of the assignment. We worked together in and outside of class on the code, and then we all typed up parts of the assignment. Gabe mostly worked on the introduction and abstract, Gradyn worked on the Methods section and on the conclusion, and Scott worked on the Results and Analysis section, the code workflow section and also helped on the conclusion section.

Abstract:
Provide a summary of your assignment work and findings. The abstract is a paragraph where you summarize what you studied in this computational lab and what are your main findings. Please, stay on the limit size of up to 300 words max.

Abstract (0.5 points):

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In this lab, numerical methods for solving the motion of pendulum systems were explored and utilized. While for a simple pendulum system with small angles a linear approximation can be made, in the lab numerical methods are used to solve the non-approximated equations.

To do this, general numerical methods for solving differential equations must be employed. The first method used is the simple Euler method, also referred to as the trapezoidal method, which takes the Taylor expansion of the function that essentially breaks the function into several trapezoids of base dt, where we want dt to be small for accurate results. This method can find the area under the graph with the area of the trapezoids. The next method is the Runge-Kutta method. Similar to the simple Euler method, it uses trapezoids to approximate the differential equations solution. This method is better, however, because it uses a weighted average of four slopes across each dt, providing a more accurate answer at the cost of being more computationally expensive.

Our main findings from the lab are that the Runge-Kutta method is better for approximating the solution to ODE's than the trapezoidal method is, and as is explored later in the report, periodic motion can be maintained despite additions of dampening and driving forces, but as the amplitude of the driving force increases, the more chaotic the motion becomes.

Introduction (0.5 points):

Provide an overview of the topics studied in the assignment. Concentrate on the 'physics' part, in this case, the pendulum system and its distinct dynamics (linear, nonlinear, damped, driven, periodic or chaotic motion, etc.). **Introduction** cannot go over page 3 of this template.

This assignment focuses on the motion of the pendulum system with various complications. For a basic pendulum system, in which the only external force acting upon the system is the force of gravity, a linear relation can be found between the angle (of the pendulum to the normal of the ground) and the acceleration of the pendulum. This is assuming small angles. Of course, the small angle approximation is not necessary, however it complicates the problem to where one would have to solve the motion numerically rather than analytically.

The pendulum system can become more complicated when adding in damping and driving forces. A damping force such as friction will be linearly related to the angular velocity, while a driving force is related to the cosine of time multiplied by the driving forces angular frequency. If it is just the additional force of friction present, then the pendulum will come to a stop. However, with both forces present, chaotic motion can arise. If the frequency of the driving force is not an integer multiple of the frequency of motion, then not be periodic. Rather, it would be chaotic.

Due to the nature of the differential equations used in the equations for the motions of pendulums (other than the simplest cases), numerical methods are necessary. Computationally, there are several ways to achieve this, as will be explored in this lab. The methods used here are useful in several branches of physics that deal with solving differential equations. This is merely an example of such a set of systems.

Methods (1 point):

Describe the used numerical methods. Provide your own explanations about the two numerical methods used in this assignment: trapezoid (page 4) and Runge-Kutta (page 5). **Methods** cannot go over page 5 of this template.

The first numerical method used in this assignment for solving ordinary differential equations (ODE's) was the trapezoidal approximation.

This approximation finds the area under a by picking two points on a function, and calculating the area of a trapezoid that fits under a region connecting those two points. The smaller the distance between the two points chosen, the higher the accuracy of the trapezoidal method. This is because the trapezoidal method uses a linear approximation between the two points. These linear approximations better represent the curve over smaller distances, and lose accuracy over larger distances.

The main advantage of this approximation is that it is easy to understand and compute, assuming that an accurate function is given for all values between the two points chosen. The largest downfall of this approximation is that it lacks accuracy compared to the Runge-Kutta method which is discussed on the next page. This is because the linear approximations of the trapezoidal method will be consistently inaccurate unless the trapezoids are infinitesimally small; this case would result in an integral, but these approximations are used when integrating a function is not feasible.

Overall, this method is simplistic in nature and easy to understand, but lacks the accuracy of other numerical methods in approximating the solutions to ordinary differential equations.

Methods:

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The second numerical method of approximating the solution to ordinary differential equations was the Runge-Kutta method. This method is specifically most useful to solve initial value problems, where it can find approximations given a starting point.

In this case, we will specifically be referring to the fourth order Runge-Kutta method, as it is a more accurate and standard approximation than the second order Runge-Kutta method. This method takes a starting point, and adds the weighted average of four different values over an interval to get to the next point. The first value is given by the linear approximation of the function at the starting point. The second and third values are then given by taking a midpoint approximation of the point before it, and then the final point is found using the approximation of the previous value over the entire interval. The smaller the interval chosen, the more accurate the average of the slopes will be. This average is weighted because the second and third values are more accurate; their approximations are used over half the interval that both the first and fourth points are used.

The largest advantage of this method is that the errors in each of these approximations are minimized by each other and provide a high degree of accuracy for solving ordinary differential equations. This method is similar to the trapezoidal approximation in that it also increases in accuracy given a smaller interval. However, this method struggles to approximate functions that are not smooth. If the function is not differentiable at a point, then not every value can be found for slopes, and the approximations can not be made.

This method is also much more complicated and mathematically intensive than the trapezoidal method, in addition to each approximation requiring four different computed values, rather than the two required for the trapezoidal method.

Overall, this method is highly accurate for smooth functions when compared to the trapezoid method. This method requires more computational work, which makes it more appropriate for work done by computers, rather than being done by hand.

Code workflow (1 point):

Explain how the code you used in exercise 9(d) works. Do not simply copy and paste lines from your code. Use your own words to explain its procedures and algorithm. **Code workflow** cannot go over page 6 of this template.

- The code in exercise 9(d) is essentially the culmination of all the other exercises, each step adding a little bit of complexity to the program. In 9(d), we are modeling the motion of a nonlinear, damped-driven pendulum using equation (11) from the lab manual [1] as follows:

$$f(\theta, \omega, t) \;=\; -\frac{g}{L}sin(\theta) \;-\; k\omega \;+\; Acos(\phi t)$$

  - Most of these variables are set to constants. g=L=1, k=0.5, and ϕ=0.6667. A is the constant that gets changed before running the code to produce different results. For the Figures produced in the Results and Analysis section below, A = 1.07 and 1.47 respectively.
- This function above is made into a function in the code for quick calculations in the algorithm later.

- Next, initial conditions are designated for Θ,ω and t. (we used 3.0, 0.0 and 0.0 respectively). Also, any additional parameters used later in the code are set to their predetermined values (explained later). Empty lists that are going to be used to store all the time, Θ, and ω values are also created.

- Next, the Runge-Kutta algorithm is utilized via a for loop. The mathematics behind this are explained in the previous section, but the code we used that models the mathematics is directly from the lab manual [1].
  - After each iteration of the Runge-Kutta algorithm, a few steps occur. First, the time is iterated by a predetermined Δt. The Δt we used was 0.01. This means that every iteration, the time increases by 0.01 (arb. units).
  - Next, the calculated Θ value from the RK algorithm is corrected if it is larger or equal to π+Q, (with Q being a predetermined shift parameter) or if it is smaller or equal to -π+Q.
  - After Θ is corrected, it is checked if the t value is greater than the predetermined "transient" value so that there is no "tail" lines that result from the transient motion. If "t" is greater than the transient value, then the values of Θ and ω are recorded.

- After this whole process is complete, the graph is then created with matplotlib commands. The graph is Θ(t) vs. ω(t), with the angular velocity being the y-axis and the angle being the x-axis. This creates a phase portrait plot.

Results and analysis (1 point):

Present the plots requested in items 4(i), 5(f), 7(d), 8(b), and 9(e). Discuss and explain the plots and their results. **Results and analysis** cannot go over page 10 of this template. Do not insert too large figure panels just to cover space. Choose figure sizes typically seen in textbooks and scientific manuscripts.

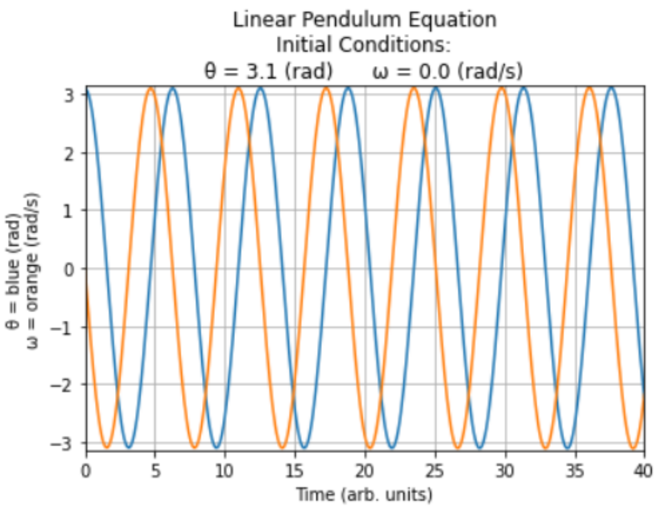

Figure 1: Linear Pendulum Equation

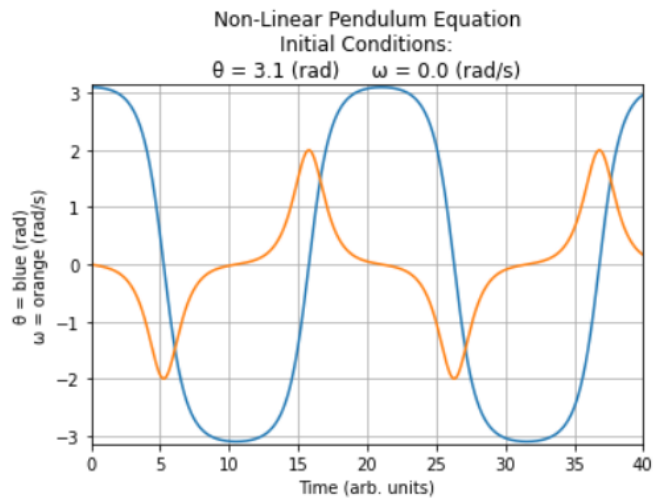

Figure 2: Non-Linear Pendulum Equation

- Above are the graphs generated in section 4(i) and 5(f) respectively. Figure 1 represents the linear pendulum equation (i.e. $\sin(\theta)$ is assumed to be $\approx \theta$), whereas Figure 2 represents the nonlinear pendulum equation (i.e. that assumption is not made). As can be seen, this assumption changes the results dramatically.
- In Figure 1, ω and θ are very close in magnitude and are continuously oscillating fairly rapidly, with θ following ω .
- In Figure 2, θ is oscillating fairly similarly on the second graph to how it did on the first graph, but it is stretched and elongated in the x-direction. ω is also still oscillating, but its shape has changed from perfect sinusoidal waves to exponential shaped curves that approach -2 to 0 to 2 and so on. Again, it is also elongated and stretched along the x-axis in the second graph compared to the first graph. It appears the behavior of ω reaching its local max and θ reaching its local max that was apparent in the first figure is also in the second figure.

Results and analysis:

Present the plots requested in items 4(i), 5(f), 7(d), 8(b), and 9(e). Discuss and explain the plots and their results. **Results and analysis** cannot go over page 10 of this template. Do not insert too large figure panels just to cover space. Choose figure sizes typically seen in textbooks and scientific manuscripts.

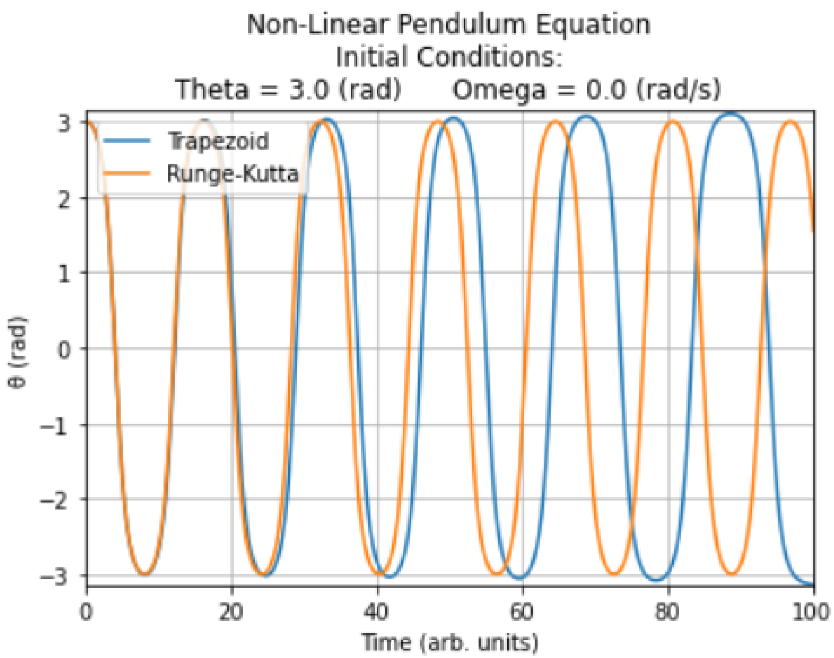

Figure 3: Trapezoid Method vs. Runge Kutta Algorithm

- This is the graph that was generated in 7(d). It is a θ as a function of time graph, with θ at t=0 being 3.0 radians, and ω at t=0 being 0.0 radians per second, as outlined in the lab manual.
- The point of this graph is that it can be used as a direct comparison of the Trapezoid and Runge-Kutta methods of numerical integration. As can be seen, both methods produce very similar results, but as time elapses eventually they begin to differentiate and produce wildly different results. Over a long period of time, this will change results dramatically. For example, at t=80 we can see that they're producing nearly opposite results

Results and analysis:

Present the plots requested in items 4(i), 5(f), 7(d), 8(b), and 9(e). Discuss and explain the plots and their results. **Results and analysis** cannot go over page 10 of this template. Do not insert too large figure panels just to cover space. Choose figure sizes typically seen in textbooks and scientific manuscripts.

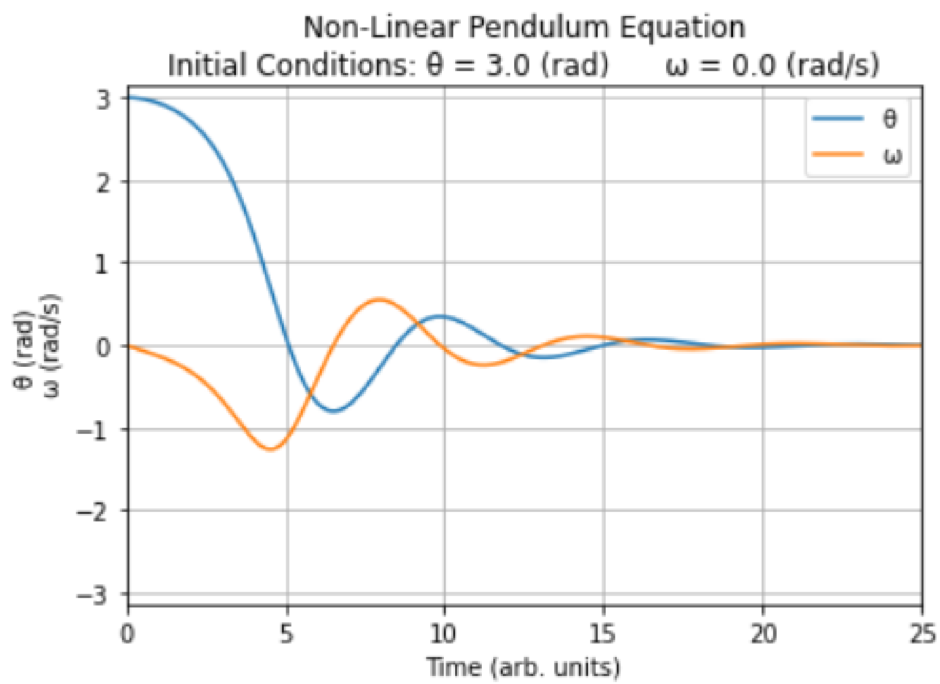

Figure 4: Damped Non-Linear Pendulum

- This is the graph generated in 8(b). It is a θ(t) and ω(t) as a function of time graph, with θ at t=0 being 3.0 radians, and ω at t=0 being 0.0 radians per second, as outlined in the lab manual.
- What this graph is showing is the nonlinear pendulum's motion, but with damping included (i.e. friction, air resistance, etc.). This graph should be compared to Figure 2 to see the difference between damping and no damping.
- The shape of the curves for ω and θ start off similarly to how they did in Figure 2, however they quickly begin to lose magnitude every oscillation. This makes sense in a more realistic setting, as energy from the pendulum would be lost every oscillation to the environment, so there would be less energy in the pendulum as it continues to swing. Eventually, the oscillations become very difficult to see as the pendulum is presumably coming closer and closer to rest.

Results and analysis:

Present the plots requested in items 4(i), 5(f), 7(d), 8(b), and 9(e). Discuss and explain the plots and their results. **Results and analysis** cannot go over page 10 of this template. Do not insert too large figure panels just to cover space. Choose figure sizes typically seen in textbooks and scientific manuscripts.

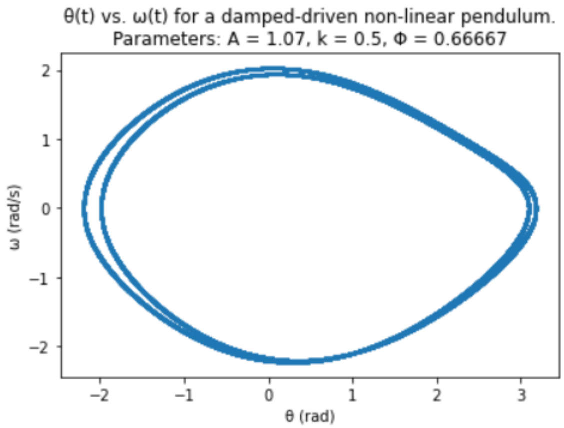

Figure 5: Linear Pendulum Equation

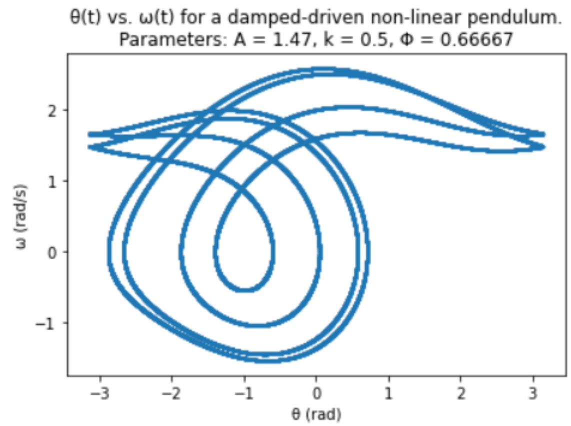

Figure 6: Non-Linear Pendulum Equation

- Both Figures 5 and 6 are generated in 9(e). They are θ vs. ω graphs. The initial parameters for both graphs when t=0 are as follows: θ = 3.0, ω = 0.0, transient = 6500, g=L=1.0, Φ = 0.6667. Then for Figure 5, A = 1.07 and Q = 0.50, and for Figure 6, A=1.47 and Q = 0.0.
    - The transient value is used to get rid of the "tail" lines that are created through transient motion. θ and ω values are not recorded when t is less than the transient value.
    - The Q value is used to shift the domain range. In Figure 5, there is a little bit of the graph that is greater than 2π, so the Q value is 0.50 so that the values greater than 2π are not shifted to the left part of the figure. This shift is not required in Figure 6, so the Q value is 0.0.

- Both of these figures are taking damped non-linear pendulum force as explored in section 8, and now adding an external (i.e. driven) force, that is represented in the variable A. Now, because of this extra force in combination with the damping that was already occurring, the motion of the pendulum is now much more complex to plot, and may become chaotic and/or non-periodic. This is why instead of graphing the motion as a function of time, theta versus the angular velocity is graphed instead. This kind of graph is also commonly called a phase portrait, or a phase space plot.
    - The amplitude of the driving force (i.e. variable A) is set to 1.07 in Figure 5, and it is set to 1.47 in Figure 6. As can be seen, this increase in the driving force complicates the motion greatly.
    - The motion of the pendulum in both figures is periodic, however that is much easier to observe in Figure 5 compared to Figure 6. What this means is that eventually if you continue to increase the amplitude of the driving force (and thus the complexity of the motion of the pendulum), there will come a point where there is virtually no periodic motion.

Conclusions (0.5 points):

Wrap up the assignment by highlighting your main findings and results. Please, stay on the limit size of up to 300 words max. **Conclusions** cannot go over page 11 of this template.

The purpose(s) of this laboratory "is to investigate the dynamics of a system which have nonlinear force laws"[1], and to learn how to solve ODE's in a python environment. Throughout the lab, we had to go through multiple exercises that added complexities to the equation of motion for a pendulum. First we had to depict linear motion, and then nonlinear motion, then damped nonlinear motion, and then finally driven-damped nonlinear motion of a pendulum. We also had to compare the "trapezoid" method and the Runge-Kutta algorithm, and determine which was more accurate when it came to solving ODE's, and which to continue to use moving forwards.

We concluded that the Runge-Kutta method was more accurate for approximating ODE's because it used four weighted slopes that minimized the error between them. This method requires more mathematical computation, but is preferred for its high level of accuracy. This is contrasted by the linear approximations given by the trapezoid method, which only uses two points to approximate the value of an ODE.

In the final step when adding driven force, we were also able to determine that the motion is periodic, but also that when the amplitude of the driving force is increased, the motion's complexity also increases. Eventually, if the amplitude is high enough, there will come a point where it appears that there is virtually no periodic motion.

References:

Include any citations or references used during the preparation of this report and codes. Consulted webpages should also be cited as well as our assignment manuals. Choose a citation style of your preference and adopt it consistently throughout. There is no page limit for the **References** section.

1. Assignment #2 Manual: The Pendulum Problem - PHYS 381 Winter term 2023 - Retrieved from https://d2l.ucalgary.ca/d2l/le/content/497542/viewContent/5751025/View
2. In Class Demo: pendulum_linear_trapezoid1, *Claudia Gomes da Rocha* - Retreived from https://d2l.ucalgary.ca/d2l/le/content/497542/viewContent/5753929/View

Other (0.5 points):

This page is to be filled by the instructor or TAs ONLY.

Remaining 0.5 points are granted for following the template and overall quality of the report.

Was the assignment submitted before the due date (mark one option)? YES NO