Assignment #3: Projectile motion under air resistance (**total 10 points**), due by 3:00 pm Friday, 03 March 2023

Most introductory physics courses spend a considerable amount of time studying the motion of projectiles but often ignore the air resistance that inevitably impacts the motion of these objects. In many problems, this is an excellent approximation; in others, air resistance is obviously very important and we need to know how to account for it. In this computational lab, you will investigate air-resistance effects in the motion of falling objects and projectiles. By solving second Newton's law numerically, you will be able to test how good or how crude an approximation it is to neglect the influence of air resistance. Air resistance is dependent on the velocity with which the object moves, i.e. the higher the velocity, the higher the resistance. Mathematically, the resistive force represented by the vector $\vec{F}$ given by $\vec{F} = -f(v)\hat{u}$, where $\hat{u} = \vec{v}/\mid v \mid$ is the unit vector along the direction of the velocity $\vec{v}$. The minus sign guarantees that the air resistance acting on the projectile is always opposite to the direction of its velocity. The function $f(v)$ is a positive quantity that describes how the magnitude of the air-resistance force depends on the magnitude of the velocity v. It is often a good approximation to write $f(v)$ as $f(v) = bv + cv^2$. The coefficients b and c depend on the size and shape of the object. In the case of spherical objects, $b = BD$ and $c = CD^2$, where D denotes the diameter of the spherical object and the coefficients B and C depend on the nature of the medium. For a spherical projectile in air $B = 1.6 \times 10^{-4}$ Ns/m^2 and $C = 0.25$ Ns^2/m^4.

1 How does the air resistance scale with the velocity?

Note that depending on the diameter and velocity of the projectile we may simplify the function $f(v)$ by neglecting the linear or the quadratic term. In order to establish whether the linear or the quadratic terms can be neglected, you should:

(a) **IMPORTANT:** Write a simple code that plots the function $f(v)$ as a function of the velocity magnitude. Note that the function $f(v)$ actually scales with the product Dv, so it is instructive to plot the separate contributions to $f(Dv)$ as a function of Dv. In other words, plot the quantities bv and cv^2, both as a function of $x = Dv$. By comparing their relative magnitudes, establish the range of values of Dv for which the linear term can be neglected, the range for which the quadratic term becomes negligible and the range for which both terms must be included. **(0.2 points)**

(b) **IMPORTANT:** Identify the ideal form for $f(v)$ in the case of a baseball of diameter $D = 7$ cm traveling at a speed of $v = 5$ m/s; of a tiny drop of oil ($D = 1.5 \times 10^{-6}$ m) moving very slowly ($v = 5 \times 10^{-5}$ m/s); and of a raindrop of diameter $D = 1$ mm traveling at a speed of $v = 1$ m/s. **(0.2 points)**

(c) **IMPORTANT:** Upload the codes generated for items 1(a) and 1(b) in addition to your report. Include the plots for items 1(a) and 1(b) in the report in the section 'Results and analysis' and discuss your results. Note that the points for items 1(a) and 1(b) can only be considered if their respective codes are uploaded. **(0.2 points)**

2 Vertical motion under the action of air resistance

In the case of a spherical grain of dust of mass density of 2×10^3 kg/m^3 and diameter $D = 10^{-4}$ m that is released from rest, how do we decide which approximation to take for the air resistance? The maximum velocity reached by the particle is given by the terminal velocity v_T, which is the velocity for which the magnitude of the air resistance equals the weight force. **Note that all items in this section 2 are worth points. Include plots (if applicable) and discussions for each item in the 'Results and analysis' section to be considered for the points.**

(a) Compare the value of the product Dv_T with the ranges you obtained in problem 1 to convince yourself that, in this case, it is a good approximation to neglect the quadratic contribution to the air resistance, i.e. to assume that $c = 0$. In this case, Newton's law can be expressed as

$$\frac{dv_y}{dt} = g - \frac{b}{m} v_y \tag{1}$$

where v_y represents the vertical velocity of the particle and g is the acceleration due to gravity. The derivative on the left is the acceleration whereas on the right-hand side of the equation, both the weight and the air-resistance forces are divided by the mass m. **(0.2 points)**

(b) Write a simple code (and upload it to Gradescope) that obtains how the vertical velocity v_y of a spherical object varies with time t as it is released from rest. The key to write codes of this type is to divide your time into small intervals Δt and assume that in the limit when $\Delta t \to 0$, all the relevant quantities are constant. In other words, you replace the differential equation above with finite difference elements as

$$\Delta v_y = v_y(t + \Delta t) - v_y(t) = g\Delta t - \frac{b}{m} v_y \Delta t \tag{2}$$

and assume that all quantities on the right-hand side of the equation are constant within the time interval Δt (remember of the Euler's method for numerical integration in assignment #2?). It is as if the interval Δt is so small that there is not much time for the quantities on the right side to vary! This will give you the change in velocity Δv_y, which you will then use to update the velocity v_y. This has to be done repeatedly, always increasing the time t in steps of Δt and the velocity in steps of Δv_y. The suggested pseudocode below shows the algorithm that will help you to script your code. **(0.3 points)**

```
set parameters g, B, D, etc.
set initial conditions t = 0; vy = 0
block repeat-until t reaches maximum value of tmax:
    dvy = g*dt - (b/m)*vy*dt
    vy += dvy
    t += dt
    store (t,vy) values using pre-defined lists
end block repeat-unit
plot (t,vy)
```

(c) You should then plot graphs showing v_y as a function of time t. Repeat this procedure for grains of different masses (you can choose the range of masses). What happens when the mass gets very large? Furthermore, verify that results for larger masses are similar to the cases of smaller resistances. In other words, increases in m are similar to reductions in the coefficient b. *Note:* You may need to set `D = 1` and vary b within $[0, 1/2]$ to verify this particular behaviour. **(0.3 points)**

(d) It turns out that there is an analytical solution to equation (1) and it is given by

$$v_y = \frac{mg}{b}\left(1 - e^{-bt/m}\right) \tag{3}$$

Compare the results obtained with your code with those obtained by the analytical expression above. Plot a graph of the error (absolute or relative) and how it evolves with time. What can you do to improve the accuracy of your computer-generated results? **(0.3 points)**

(e) Now that you have investigated how the velocity increases with time, you should then describe how the position of your projectile varies with time. Knowing that $v_y = dy/dt$, simply manipulate the $v_y \times t$ algorithm to find how the position y varies with time. Imagine that the grain is released from a height of $H = 5$ m and calculate the time it takes for the grain to reach the ground. In the following, show that this time depends on the mass of the grain. Plot a graph of the time to reach the ground as a function of the mass of the object. What can you say about the often-quoted statement that all objects fall together with the same acceleration regardless of their masses? When is this a good approximation? **(0.3 points)**

(f) Upload the code generated for item 2(d) in addition to your report. **(0.5 points)**

3 Projectile motion under air resistance - Part A

Consider now a spherical object launched with a velocity v forming an angle θ with the horizontal ground. In the absence of air resistance, the trajectory followed by this projectile is known to be a parabola. This follows from writing Newton's law separately for the horizontal and vertical coordinates. The former scales linearly with time whereas the latter varies quadratically. Therefore, when time is eliminated, we are left with a quadratic equation that gives rise to a parabolic trajectory. Let's see how the trajectory changes when air resistance is no longer neglected. In the case of a resistive force that grows linearly with velocity ($c = 0$), we can still separate the motion between horizontal and vertical coordinates. Second Newton's law for both the horizontal and vertical coordinates become

$$\frac{dv_x}{dt} = -\frac{b}{m}v_x \tag{4}$$

$$\frac{dv_y}{dt} = -g - \frac{b}{m}v_y \tag{5}$$

The code written earlier can be applied to both directions separately, the difference being that gravity acts on the vertical direction (y-axis) but not on the horizontal one (x-axis). Once again, you will have results relating the coordinates x and y with the time t.

(a) Plot $y(t)$ versus $x(t)$, which will give you the trajectory followed by the object under the action of air resistance. Superimpose this trajectory with the one which you would obtain in vacuum to see how different the two cases are. **(0.5 points)**

(b) Another well-known fact, often derived in introductory physics courses, is that the launching angle of 45^o leads to the maximum horizontal displacement in a projectile motion. This is the case in the absence of air resistance. The question we now pose is whether this is also the case when air resistance is not neglected. You can now use your code to determine what the optimum launching angle is. How does that depend on the mass m? Plot θ-optimum as a function of m. **(0.5 points)**

(c) **IMPORTANT:** Attach the `.ipynb` file you generated for item 3(b) in addition to your report **(0.5 points)**.

4 Projectile motion under air resistance - Part B

Imagine now that you are in a situation where the air resistance depends quadratically on the velocity. In this case, $b = 0$. Obtain the trajectory of a spherical object of diameter D launched with a speed v_0 that forms an angle θ with the horizontal ground. Attention to the fact that now the differential equations describing the time evolution of the velocity components are no longer decoupled. In other words, the differential equation for the horizontal component v_x depends on the vertical component v_y, and vice versa. Mathematically, we have

$$\frac{dv_x}{dt} = -\frac{c}{m}\sqrt{\left(v_x^2 + v_y^2\right)}\ v_x \tag{6}$$

$$\frac{dv_y}{dt} = -g - \frac{c}{m}\sqrt{\left(v_x^2 + v_y^2\right)}\ v_y \tag{7}$$

(a) Adapt your code to account for the quadratic dependence of the air resistance. Once again, plot the trajectory of the projectile, this time superimposing it with the trajectory you would obtain for the linearly-dependent air resistance and with the case of no air resistance at all. This should give you three different trajectories. Do that for a few different values of masses, launching angles and initial velocities and interpret your results. **(0.5 points)**

(b) **IMPORTANT:** Attach the `.ipynb` file you generated for item 4(a) in addition to your report **(0.5 points)**.

5 Report writing guide (5 points)

After generating your codes and results for the exercises above, you will summarize them in the form of a written report. Please, follow the instructions provided in the report template located in the sub-module of the assignment to write your report. All margins, font style/size, line spacing, and page limits set on the template need to be respected. **We will deduct a significant amount of points from reports that do not align with the template provided or we will not be able to assess the report, meaning we will have to deduct -5/10 points from the total marks.**

There is a `docx` file (generated in MS Word editor) available in the sub-module of the assignment that can, in principle, be used and exported to other text editors. However, some of its pre-defined macros may not be read correctly by other editors. Moreover, inserting text on the editable `docx` file may shift or duplicate some of the original headers. Make sure your written content obeys the format set in the read-only `PHYS 381 Assignment 3 Template.pdf` file. In case you have problems with the `docx` file provided, a suggestion is to start a blank document in the text editor of your choice and set the layout as below:

- Page size: 8.5" $\times$ 11" (letter);
- Margins: 2.54 cm (top, bottom, left, right);
- Font type and size: Times New Roman, 12 pt, non-italic, non-bold;
- Line spacing: between 1.0 and 1.15 pt;
- Figures must be numbered and contain captions. Axes must carry names and their quantities must be expressed with the proper units whenever required. Do not insert too large figures just to cover space. Choose figure sizes typically seen in textbooks and scientific manuscripts.

- Follow the exact page configuration and order as described in the report template, e.g., title page on page 1, abstract on page 2, introduction on page 3, etc. The title page needs to contain the exact same information as in the report template.

5.1 Submission

The due date for the submission of your report is depicted on the first page of the assignment. Once you conclude the writing of your report in accordance with the template provided, generate a `pdf` file and name it as `report_assignment_3.pdf` (since this is report #3). You will upload the `pdf` file of your report plus your Notebook `.ipynb` files requested in the assignment to the Gradescope platform. **It is important that you log in to Gradescope.ca and not Gradescope.com.** You can upload multiple files to the Gradescope platform and you can resubmit your work until the due date. We will test-run all your submitted codes to test for errors. We will also check if the results/figures presented in your report match the output of your codes.

For Groups: If you worked in a group, remember to include your group members in Gradescope after you submit all the files on behalf of the group. You will see an option "Add Group Member" after you upload all files to Gradescope. Students can work in groups of up to 4 members.

Autograder option: Ignore the autograder results that may appear after you upload files to Gradescope. There is no autograder configured for this assignment since everything will be evaluated manually by the PHYS 381 Team.

Please, remember to add comments in all your submitted Notebook codes! Pure code lines without explanatory comments or written Markdown cells will have reduced marks.

Standard `.py` Python codes are not accepted in this submission. Work only with Notebook environments. If any member of the group is experiencing problems with their coding environment, please, contact the instructor immediately.

ONLY `pdf` (for the written report) AND `.ipynb` (for codes) FILES ARE ACCEPTED.

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