PHYS 581 Assignment 2

100 points total

Due Friday, February 7

Consider again the Chebyshev polynomials $T_n(x)$ explored in Assignment 1, which have the exact form

$$T_n(x) = \cos\left[n\cos^{-1}(x)\right]. \tag{1}$$

The oscillation length of these functions decreases both as n increases and as $|x| \to 1$, which means that capturing their derivatives becomes increasingly challenging using an evenly-spaced spatial grid with a fixed number of points m. On the other hand, $T_n(\cos\theta) = \cos(n\theta)$, so that points in '$\theta$-space' could be chosen to be equidistant instead. A natural choice is

$$\theta_k = \frac{\pi}{m}\left(k - \frac{1}{2}\right), \quad k = 1, 2, \ldots, m,$$

where $m \geq n$. Then

$$x_k = \cos\theta_k = \cos\left[\frac{\pi}{m}\left(k - \frac{1}{2}\right)\right]; \qquad T_n(x_k) = \cos\left[\frac{\pi n}{m}\left(k - \frac{1}{2}\right)\right].$$

I'd suggest choosing $m = 10$ or $m = 11$.

1. (10 points) Write a function / procedure / algorithm in Mathematica to generate all m Lagrange basis polynomials for an arbitrary set of m spatial points (these should all be degree $m - 1$ or less).

2. (50 points) Again using Mathematica,

   (a) (10 points) Using the results obtained in Question (1), obtain explicit expressions for the Lagrange basis polynomials and their first and second derivatives, for the uneven grid positions x_k, $k = 1, 2, \ldots, m$ defined above.

   (b) (10 points) Using the results in part (a), derive mth-order finite-difference approximations for the first and second spatial derivatives near the center of the domain $x \sim 0$ and near one of the boundaries $|x| \sim 1$. Do the stencil coefficients change? Does it make sense?

   (c) (15 points) Again using the results obtained in part (a), obtain interpolated approximations (i.e. the Lagrange interpolating polynomials) to the Chebyshev polynomials $T_n(x)$ and their first and second derivatives $T_n'(x)$ and $T_n''(x)$ for all $1 \leq n \leq m$, using only the exact values of these polynomials sampled at the uneven grid points, and compare with the true polynomials by plotting them. Comment on the results.

   (d) (15 points) Repeat the calculation above but now for an evenly-spaced grid. How have things changed?

3. (40 points) Switching now to C,

   (a) (10 points) Write a function / procedure / algorithm to calculate the weights w_j for an arbitrary set of grid points x_k [c.f. Eq. (1.31) in the course notes].

   (b) (10 points) Using the standard form, Eq. (1.36) in the notes, the uneven grid points defined above, and the weights obtained in (a), obtain interpolated approximations (i.e. the Lagrange interpolating polynomials) to the Chebyshev polynomials $T_n(x)$ (not their derivatives) for all $1 \leq n \leq m$, using only the exact values of these polynomials sampled at the grid points, and compare with the true polynomials and the results obtained above. Comment on the results.

   (c) (20 points) Finally, extend the expression (1.36) in the notes to obtain the first derivative of an arbitrary function $f'(x)$ in terms of the weights w_j, the grid points x_j, and the amplitude of the function a_j evaluated at the grid points. You can use Mathematica to help you derive this expression. Use this to obtain a numerical approximation to the first derivative $T_n'(x)$ in C, and compare to the results above.