PHYS 581 Assignment 1

90 points total

Due Tuesday, January 28

Consider Chebyshev polynomials $T_n(x)$, which are defined by the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{1}$$

subject to $T_0(x) = 1$ and $T_1(x) = x$. These polynomials are valid on the domain $x \in [-1, 1]$. The first few are (check this analytically):

$$T_0(x) = 1;\; T_1(x) = x;\; T_2(x) = 2x^2 - 1;\; T_3(x) = 4x^3 - 3x. \tag{2}$$

These polynomials have a closed-form representation in terms of trigonometric functions:

$$T_n(x) = \cos\left[n \cos^{-1}(x)\right], \tag{3}$$

so that $T_n(\cos\theta) = \cos(n\theta)$.

1. (10 points) Use Mathematica to obtain the explicit expressions for the Chebyshev polynomials up to $n = 10$ using

   (a) (5 points) the recurrence relation,

   (b) (5 points) the closed-form expression,

   and verify that they coincide (you can also verify that they coincide with Mathematica's built-in Chebyshev function if you're inclined).

2. (5 points) Use Mathematica to tabulate expressions for the exact first and second spatial derivatives of the Chebyshev polynomials for all $n \in [1, 10]$.

3. (20 points) In C, store only the $n = 10$ Chebyshev polynomial $T_{10}(x)$ (feel free to use the closed-form expression) on an equally-spaced grid with m points located between $x = -1$ and $x = 1$, inclusive. Also store the exact first and second derivatives, $T'_{10}(x)$ and $T''_{10}(x)$ using the Mathematica results obtained above.

4. (20 points) Again in C, obtain the approximate first and second derivatives of the list data representing $T_{10}(x)$ obtained above, $\tilde{T}'_{10}(x)$ and $\tilde{T}''_{10}(x)$, using the lowest-order finite-difference expressions in the notes. i.e. Eqs. (1.2), (1.4), and (1.5). For the second derivative, be sure to use forward and backward differences at the boundaries!

5. (20 points) Again in C, obtain the errors in evaluating the first and second derivatives using the finite-difference approximations, by summing over the absolute values of the differences between the exact and approximate values. Repeat all the calculations (starting again at Question 3) a few times by choosing different values of m. Tabulate the errors as a function of the point spacing a.

6. (15 points) Back in Mathematica, obtain the best fit of the errors calculated above as a function of a. Does the dependence of the error on a follow expectations?