Automata Theory Homework 1
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1. (a) Set of all positive integers
   (b) Set of all even integers
   (c) Set of even natural numbers
   (d) Set of all multiples of k
   (e) Set of all binary palindromes
   (f) Empty set

2. (a) $\{1, 10, 100\}$
   (b) $\{n \mid n \in \mathbb{Z} \ \& \ n > 5\}$
   (c) $\{n \mid n \in \mathbb{N} \ \& \ n < 5\}$
   (d) {"aba"}
   (e) $\{\epsilon\}$
   (f) $\emptyset$

3. (a) x + y = 0 is relationship R
   (1) Reflexive
   $x + x = 0$ where $x \in \mathbb{R}$
   $2x = 0$
   Counterexample, let $x = 1$
   $2(1) = 0$
   Therefore, R is NOT Reflexive
   (2) Symmetric
   $x,y \in \mathbb{R}$
   $x + y = 0 = y + x$
   Addition is associative
   Therefore, R is Symmetric
   (3) Anti symmetric
   $x,y \in \mathbb{R}$
   $x + y = 0 = y + x$
   So $x = y$
   False because it is possible $x = -y$ and $y = -x$
   Therefore, R is NOT anti symmetric
   (4) Transitive
   $x,y,z \in \mathbb{R}$
   $x + y = 0$, $y + z = 0$, therefore $x + z = 0$
   Let $y = -x$, then $z = -(-x)$ and assume $x \neq 0$
   Then $x + z = x + -(-x) = 2x \neq 0$
   Therefore, $x + y = 0$ is NOT transitive
   (b) x - y is rational is relationship R
   (1) Reflexive
   Let $x \in \mathbb{R}$
   $x - x = 0$, 0 is rational
   Therefore, R is Reflexive
   (2) Symmetric
   $x,y \in \mathbb{R}$
   $x - y \in \mathbb{Q}$
   $y - x \in \mathbb{Q}$ because if $x - y \in \mathbb{Q}$ then its opposite will produce a result in $\mathbb{Q}$
   Therefore, R is Symmetric

(3) Antisymmetric
Let x,y $\in \mathbb{R}$
x - y $\in \mathbb{Q}$, y - x $\in \mathbb{Q}$, then y = x
y does not have to equal x to produce a rational number(EX: 1 - 2 and 2 - 1 $\in \mathbb{Q}$)
Therefore, R is not Anti symmetric

(4) Transistive
x,y,z $\in \mathbb{R}$
x - y $\in \mathbb{Q}$, y - z $\in \mathbb{Q} \implies$ x - z $\in \mathbb{Q}$
Yes because x - z = x - y + y - z
Therefore, R is Transistive

(c) x = 2y

(1) Reflexive
let x $\in \mathbb{R}$
x = 2x
Counterexample: 2 = 2(2)
Therefore, R is NOT Reflexive

(2) Symmetric
let x,y $\in \mathbb{R}$
x = 2y, y = 2x
let x = 1 and y = $\frac{1}{2}$, then 1 = 2($\frac{1}{2}$) but $\frac{1}{2} \neq 2(1)$
Therefore, R is NOT Symmetric

(3) Antisymmetric
let x,y $\in \mathbb{R}$
x = 2y, y = 2x, therefore y = x
This implies that x = 0 and y = 0
Therefore, R is Anti symmetric

(4) Transistive
x = 2y, y = 2z $\implies$ x = 2z
let x = 4, y = 2, z = 1, then 4 = 2(y), y = 2z, but 4 = 2(1)
Therefore, R is NOT Transistive

(d) xy $\geq 0$

(1) Reflexive
let x $\in \mathbb{R}$
x $*$ x $\geq 0$, true for all x
Therefore, R is Reflexive

(2) Symmetric
let x,y $\in \mathbb{R}$
x $*$ y $\geq 0 \implies$ y $*$ x $\geq 0$
Multiplication is associative.
Therefore, R is Symmetric

(3) Antisymmetric
let x,y $\in \mathbb{R}$
x $*$ y $\geq 0$ & y $*$ x $\geq 0 \implies$ x = y
Counter example: x = 0, y = 2, 2 $*$ 0 $\geq 0$, 0 * 2 $\geq 0$, but y $\neq$ x
Therefore, R is NOT Anti symmetric

(4) Transistive
let x,y,z $\in \mathbb{R}$
x $*$ y $\geq 0$ & y $*$ z $\geq 0 \implies$ x $*$ z $\geq 0$
x,y,z must all have the same sign, therefore will always be greater than or equal to 0
Therefore, R is Transistive

4. (m, n) R (j, k) iff m + k = n + j

   (1) Reflexive
   let m,n $\in \mathbb{R}$
   (m,n) R (m, n)
   m + n = n + m
   Addition is associative
   Therefore, R is Reflexive

   (2) Symmetric
   let m,n,j,k $\in \mathbb{R}$
   (m,n) R (j, k)
   m + k = n + j
   j + n = k + m
   Addition is associative
   Therefore, R is Symmetric

   (3) Transistive
   let m,n,j,k,x,y $\in \mathbb{R}$
   pairs (m,n), (j, k), (x, y)
   m + k = n + j & j + y = k + x $\implies$ m + y = n + x
   y = k + x - j
   m + k + x - j = n + x
   m + k - j = n
   m + k = n + j
   Therefore, R is Transistive

   Thus, R is an equivalence relation since it holds all 3 properties.

5. L = {a,b}$^*$
   (1) $\epsilon \in$ L
   (2) If x, y $\in$ L, then so are axby and bxay
   #(x) = number of symbols x in string

   Basis: $\epsilon \in$ L, #(a) = #(b) = 0
   Inductive Hypothesis(IH): If a string s is in language L, then #(a) = #(b)
   Inductive Step:
   L is of length 2k for some k $\geq$ 0 where k $\in \mathbb{Z}$
   let a string S $\in$ L where length of S = 2k + 2 and S = axby
   by IH, strings x and y hold property #(a) = #(b).
   In S, the number of both a's and b's increment by 1, so #(a) + 1 = #(b) + 1
   Same reasoning for S = bxay.
   By IH, #(a) = #(b) for all strings in language L.

6. (a) Prove that the following sum holds the property: $\sum_{i=1}^{n} n * n! = (n+1)! - 1$

   (b) Basis:
   $\sum_{i=1}^{1} n * n! = (1+1)! - 1$
   1*1! = 2! - 1
   1 = 1

   (c) Inductive Hypothesis(IH):
   $\sum_{i=1}^{n} n * n! = (n+1)! - 1$ for all n $\geq$ 1

(d) Let k $\in \mathbb{Z}$ where k $\geq 0$

$$\begin{aligned}
\sum_{i=1}^{k} k * k! &= (k+1)! - 1 \text{ by IH} \\
1*1! + 2*2! + ... + k*k! &= (k+1)! - 1 \\
1*1! + 2*2! + ... + k*k! + (k+1)(k+1)! &= (k+1)! - 1 + (k+1)(k+1)! \\
&= (k+1)!(k+1+1) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= ((k+1)+1)! - 1
\end{aligned}$$

Therefore by induction, the property holds true since k+1 can be found if k is true.

7. (a) Let L = $\{a\}^*$
   $\{\epsilon\} \neq$ L
   $\{a,b\}^* \neq$ L
   L $\subseteq \{a,b\}^*$
   L = $L^*$ because $\{a\}^* =$ L
   So language L fits these properties

   (b) Let L = $\{a,b\}^+$
   $\{\epsilon\} \notin$ L
   $\{\epsilon\} \in \{a,b\}^*$
   So L $\neq L^*$ and is infinite.