Stat 111: Problem Set 5

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1. (a) Given $Y_j \sim \text{Unif}(0, \theta)$, $\frac{Y_j}{\theta} \sim \text{Unif}(0, 1)$. Because if $U_1, ..., U_j \sim \text{Unif}(0, 1)$, $U_{(n)} \sim \text{Beta}(n, 1)$, we have $\frac{y_{(n)}}{\theta} = \hat{\theta}_{\text{MLE}} \sim \text{Beta}(n, 1)$. Thus,

$$P\left(F^{-1}_{\text{Beta}(n,1)}\left(\frac{0.05}{2}\right) \leq \frac{\hat{\theta}_{\text{MLE}}}{\theta} \leq F^{-1}_{\text{Beta}(n,1)}\left(1 - \frac{0.05}{2}\right)\right) = 0.95$$

$$P\left(0.8575 \leq \frac{\hat{\theta}_{\text{MLE}}}{\theta} \leq 0.9989\right) = 0.95$$

$$P\left(\frac{\hat{\theta}_{\text{MLE}}}{0.9989} \leq \theta \leq \frac{\hat{\theta}_{\text{MLE}}}{0.8575}\right) = 0.95$$

As a result,

$$C(\boldsymbol{Y}) = [83.59, 97.37]$$

(b) Given $\sqrt{n}(\hat{\theta}_{\text{MoM}} - \theta) \xrightarrow{d} \mathcal{N}\left(0, \frac{\theta^2}{3}\right)$,

$$\frac{\hat{\theta}_{\text{MoM}} - \theta}{\theta/\sqrt{3n}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Thus,

$$P\left(Q_{\mathcal{N}(0,1)}(0.025) \leq \frac{\hat{\theta}_{\text{MoM}} - \theta}{\theta/\sqrt{3n}} \leq Q_{\mathcal{N}(0,1)}(0.975)\right) = 0.95$$

$$P\left(\frac{-1.96}{\sqrt{3n}} \leq \frac{\hat{\theta}_{\text{MoM}} - \theta}{\theta} \leq \frac{1.96}{\sqrt{3n}}\right) = 0.95$$

$$P\left(\frac{-1.96}{\sqrt{3n}} \leq \frac{\hat{\theta}_{\text{MoM}}}{\theta} - 1 \leq \frac{1.96}{\sqrt{3n}}\right) = 0.95$$

$$P\left(\frac{\hat{\theta}_{\text{MoM}}}{1 + \frac{1.96}{\sqrt{3n}}} \leq \theta \leq \frac{\hat{\theta}_{\text{MoM}}}{1 - \frac{1.96}{\sqrt{3n}}}\right) = 0.95$$

As a result,

$$C(\boldsymbol{Y}) = \left[\frac{\hat{\theta}_{\text{MoM}}}{1 + \frac{1.96}{\sqrt{3n}}}, \frac{\hat{\theta}_{\text{MoM}}}{1 - \frac{1.96}{\sqrt{3n}}}\right]$$
$$C(\boldsymbol{Y}) = [69.1, 100]$$

(c) The confidence interval based on $\hat{\theta}_{\text{MLE}}$ is shorter than that based on $\hat{\theta}_{\text{MoM}}$.

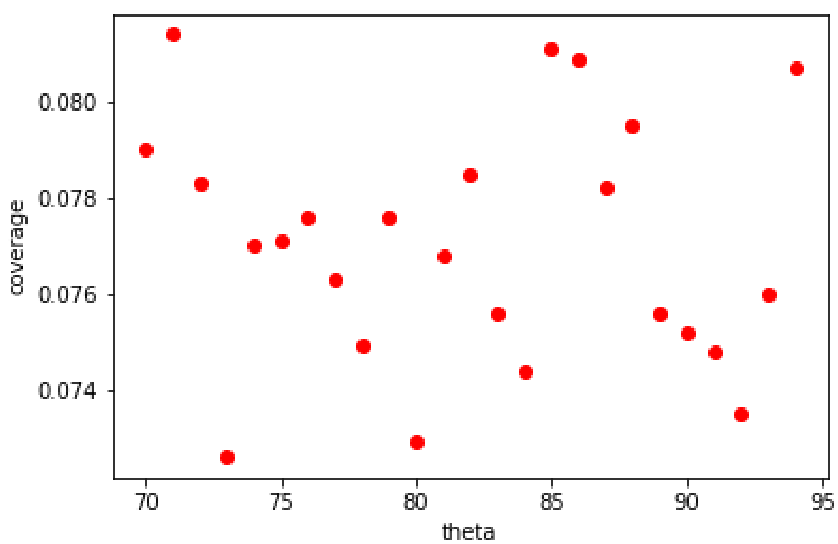
(d) The following code was used to produce the subsequent graphic for this problem.

```
theta = np.arange(70,95,1)
B = 10000
n = 24

def determineCoverage(theta):
    Y = [np.random.uniform(0, theta) for i in range(n)]
    th = 2 * np.mean(Y)
    lower = th / (1 + 1.96 / np.sqrt(3 * B))
    upper = th / (1 - 1.96 / np.sqrt(3 * B))
    return (theta >= lower and theta <= upper)

res = [[determineCoverage(t) for i in range(B)] for t in theta]
coverage = [1/B * sum(row) for row in res]

plt.plot(theta, coverage, 'or')
plt.xlabel("theta")
plt.ylabel("coverage")
plt.savefig("1d.png")
plt.show()
```



(e) No, this is not true here.

2. (a) According to example 5.4.2,

$$C(\boldsymbol{Y}) = \left[\bar{Y} - Q_{t_{n-1}}(0.975)\frac{\hat{\sigma}}{\sqrt{n}}, \bar{Y} + Q_{t_{n-1}}(0.975)\frac{\hat{\sigma}}{\sqrt{n}}\right]$$

$$C(\boldsymbol{Y}) = \left[4.2 - 2.20 \cdot \frac{3.4}{\sqrt{12}}, 4.2 + 2.20 \cdot \frac{3.4}{12}\right]$$

$$C(\boldsymbol{Y}) = [2.0, 6.4]$$

(b)

$$P\left(Q_{\mathcal{N}(0,1)}(0.025) \leq \frac{(\bar{Y} - \theta)}{\sqrt{S^2/n}} \leq Q_{\mathcal{N}(0,1)}(0.975)\right) = 0.95$$

$$P\left(\bar{Y} - 1.96\sqrt{S^2/n} \leq \theta \leq \bar{Y} + 1.96\sqrt{S^2/n}\right) = 0.95$$

Thus,

$$C(\boldsymbol{Y}) = [2.28, 6.12]$$

(c) Gary has a narrower confidence interval.

(d) As $n \to \infty$, both Gary and Abigail's confidence intervals will shrink at the rate of $1/\sqrt{n}$, closing in around the sample mean $\bar{y}$. In addition, Abigail's non-Gaussian confidence interval will approach Gary's as $n \to \infty$, since the t distribution approaches the Gaussian distribution as $n \to \infty$.

(e) Yes, the convergence of Abigail's confidence interval to Gary's as n gets large does imply that Abigail's methods have asymptotically correct coverage probabilities if the data is not Gaussian and the sample size is large.

3. Given Stat 111 Theorem 4.7.1,

$$\sqrt{n}(\hat{\theta}_{\text{MLE}} - \theta) \xrightarrow{d} \mathcal{N}(0, I_{Y_1}^{-1}(\theta))$$

Thus,

$$P\left(Q_{\mathcal{N}(0,1)}(0.025) \leq \frac{\sqrt{n}}{\sqrt{\frac{1}{I_{Y_1}(\theta)}}}(\hat{\theta}_{\text{MLE}} - \theta) \leq Q_{\mathcal{N}(0,1)}(0.975)\right) = 0.95$$

$$P\left(-1.96\frac{\sqrt{\frac{1}{I_{Y_1}(\theta)}}}{\sqrt{n}} \leq \hat{\theta}_{\text{MLE}} - \theta \leq 1.96\frac{\sqrt{\frac{1}{I_{Y_1}(\theta)}}}{\sqrt{n}}\right) = 0.95$$

$$P\left(\hat{\theta}_{\text{MLE}} - 1.96\frac{\sqrt{\frac{1}{I_{Y_1}(\theta)}}}{\sqrt{n}} \leq \theta \leq \hat{\theta}_{\text{MLE}} + 1.96\frac{\sqrt{\frac{1}{I_{Y_1}(\theta)}}}{\sqrt{n}}\right) = 0.95$$

Substituting in the approximate values for n, $\hat{\theta}_{\text{MLE}}$ and $\sqrt{\frac{1}{I_{Y_1}(\theta)}}$,

$$C(\boldsymbol{Y}) = \left[1.603 - 1.96\frac{0.168}{\sqrt{164}}, 1.603 + 1.96\frac{0.168}{\sqrt{164}}\right]$$

$$C(\boldsymbol{Y}) = [1.577, 1.629]$$

4. (a) By the Central Limit Theorem,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, \text{Var}(h(Y_1)))$$

By a location-scale transformation,

$$\hat{\theta} \xrightarrow{d} \mathcal{N}\left(\theta, \frac{\text{Var}(h(Y_1))}{n}\right)$$

(b) By the definition of variance,

$$\frac{1}{n}\text{Var}(h(Y_1)) = \frac{1}{n}\left(E[h(Y_1)^2] - E[h(Y_1)]^2\right)$$

By the Method of Moments,

$$\frac{1}{n}\text{Var}(h(Y_1)) \approx \frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right)$$

Since by the Law of Large Numbers $\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 \xrightarrow{p} E[h(Y_1)^2]$, and by the Continuous Mapping Theorem $\hat{\theta}^2 \xrightarrow{p} E[h(Y_1)]^2$, we can consistently estimate the asymptotic variance of $\hat{\theta}$ from the data using the estimator defined above. That is to say,

$$\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right) \xrightarrow{p} \frac{1}{n}\text{Var}(h(Y_1))$$

(c) By a location-scale transformation of the result from part (a),

$$\frac{\hat{\theta} - \theta}{\sqrt{\frac{\text{Var}(h(Y_1))}{n}}} \xrightarrow{d} \mathcal{N}(0, 1)$$

By the result from part (b),

$$\frac{\hat{\theta} - \theta}{\sqrt{\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right)}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Thus,

$$P\left(\hat{\theta} - 1.96\sqrt{\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right)} \leq \theta \leq \hat{\theta} + 1.96\sqrt{\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right)}\right) = 0.95$$

This leaves

$$C(\boldsymbol{Y}) = \left[\hat{\theta} - 1.96\sqrt{\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right)}, \hat{\theta} + 1.96\sqrt{\frac{1}{n}\left(\frac{1}{n}\sum_{j=1}^{n} h(Y_j)^2 - \hat{\theta}^2\right)}\right]$$