Efficient and Convergent Sequential Pseudo-Likelihood Estimation of Dynamic Discrete Games

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Introduction

- Dynamic discrete games are of interest to economists
  - Canonical application: firm entry/exit
- We focus on games of incomplete information
- Estimating dynamic discrete games is a difficult problem:
  - Significant computational burden to compute all equilibria
  - Renders nested fixed-point algorithm infeasible unless game is small or has special structure

CCP Methods

- As a result, the literature has largely focused on adapting CCP methods, pioneered by Hotz and Miller (1993), to dynamic games:
  - Obtain estimates of CCP's in first stage
  - Use these to approximate equilibrium conditions: $P = \Psi(\theta, P)$
  - Then estimate structural parameters θ
- Issues:
  - Inaccurate first-stage estimates induce finite-sample bias
  - E.g., frequency estimator with sparse data
  - May lose asymptotic efficiency due to two-step estimation

k-NPL

- Aguirregabiria and Mira (2002, 2007) proposed the nested pseudo-likelihood (k-NPL) estimator sequence to address issues finite-sample bias and asymptotic inefficiency of two-step estimation.
- They show the DGP equilibrium satisfies: $P^* = \Psi(\theta^*, P^*)$
- Sketch of k-NPL: Iterate and update choice-probabilities at each new parameter estimate:
  1. $\hat{\theta}_k = \arg\max_\theta N^{-1} \sum_i \ln \Psi(\theta, \hat{P}_{k-1})(s_i)$
  2. $\hat{P}_k = \Psi(\hat{\theta}_k, \hat{P}_{k-1})$
  3. Iterate until convergence to a fixed point ($k \to \infty$)

k-NPL: Single-Agent

- k-NPL has many good properties in single-agent models:
- Each iteration is asymptotically efficient
- Iterations achieve very fast local convergence.
- Convergence produces the finite-sample MLE.
- With linear flow payoffs, estimation of $\hat{\theta}_k$ reduces to:
  1. Solve linear systems to generate "pseudo-regressors"
  2. Use pseudo-regressors in static logit/probit estimation

k-NPL: Dynamic Games

- In dynamic games, only the computational simplicity still holds
- Slow convergence (relative to single-agent case).
- k-NPL fixed point may be unstable, leading to non-convergence or even inconsistency
  - Pesendorfer and Schmidt-Dengler (2010); Kasahara and Shimotsu (2012); Egesdal, Lai, and Su (2015).
  - Key: Ψ mapping is first-order equivalent to a best-response mapping
  - Iterations in estimator sequence behave similarly to best-response iteration
- Not fully efficient, even if consistent.

Minimum Distance

- Pesendorfer and Schmidt-Dengler (2008) propose an efficient minimum distance estimator
- Bugni and Bunting (2020) propose k-step iterated version
  - Call it k-MD
  - Each iteration is asymptotically efficient
  - They focus on *finite k*

k-MD: Issues

- Much greater computational burden than k-NPL
  - Bugni and Bunting's (2020) Monte Carlos use a very small game (2 players, 2 actions, 4 states)
  - They find 1-MD takes about 33% longer than 20-NPL
  - Time disparity likely to grow with the size of the game
- May suffer from same issues as k-NPL with iteration
  - CCP updating is the source of inconsistency/non-convergence for ∞-NPL
  - Also leads to increased finite-sample bias with iteration
  - k-MD updates CCPs in the same way as k-NPL

Research Question

- Can we develop a sequential estimator for games with several good properties?
  - Computational simplicity
  - Consistency and asymptotic efficiency for every k (including $k \to \infty$)
  - Fast, stable convergence as $k \to \infty$
  - Good finite sample properties
- (Spoiler alert: yes.)

Contribution

- We propose a new k-step Efficient Pseudo-Likelihood (k-EPL) sequence of estimators.
    1. Change of variable in equilibrium fixed point conditions for dynamic games.
    2. Implement Newton-like steps on the fixed point equation.
- Newton-like step orthogonalizes second step estimation from first step, yielding efficiency.
- Fixed points are stable, making k-EPL robust.
- k-EPL converges quickly to MLE (locally), with good stability in practice

Related Work

- Ericson and Pakes (1995), Bajari et al. (2007), Pakes et al. (2007)
- Aguirregabiria and Mira (2002, 2007)
- Pesendorfer and Schmidt-Dengler (2008, 2010)
- Kasahara and Shimotsu (2008, 2012)
- Egesdal, Lai, and Su (2015)
- Bugni and Bunting (WP 2020)
- Aguirregabiria and Marcoux (WP 2020)

Motivating k-EPL: Maximum Likelihood with Equality Constraints

$$\max_{(\theta, Y) \in \Theta \times \mathcal{Y}} Q_N(\theta, Y)$$
$$\text{s.t. } G(\theta, Y) = 0$$

- MLE subject to an equality constraint.
  - $Q_N(\theta, Y) = N^{-1} \sum_{i=1}^{N} \ln Pr(s_i | \theta, Y)$
- θ is a finite-dimensional vector of structural parameters.
- Y is a vector of important auxiliary parameters.
  - May include value functions and/or conditional choice probabilities.
- Equality constraint is often derived from a fixed point condition such as $G(\theta, Y) = Y - \Gamma(\theta, Y) = 0$.

Assumptions

Assumption 1

a Observations $\{x_i : i = 1, \ldots, N\}$ are i.i.d. and generated from one equilibrium with true parameters (θ^*, Y^*).

b Θ and $\mathcal{Y}$ are compact and convex and $(\theta^*, Y^*) \in int(\Theta \times \mathcal{Y})$.

c $Q_N(\theta, Y) \overset{a.s.}{\rightarrow} Q^*(\theta, Y)$, both are twice continuously differentiable, and Q^* has a unique maximum in $\Theta \times \mathcal{Y}$ subject to $G(\theta, Y) = 0$, and the maximum occurs at (θ^*, Y^*).

d $G(\theta, Y)$ is thrice continuously differentiable and $\nabla_Y G(\theta^*, Y^*)$ is non-singular.

Motivating k-EPL

- Define $Y(\theta)$ such that $G(\theta, Y(\theta)) = 0$. MLE problem is then

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in \Theta} \quad Q_N(\theta, Y(\theta))$$

- And $\hat{Y}_{MLE} = Y(\hat{\theta}_{MLE})$
- k-NPL maximizes a "pseudo-likelihood" in each iteration:
  - Define $Y \equiv P$, so that $Y(\theta) = P(\theta)$
  - Replace $P(\theta)$ with $\Psi(\theta, \hat{P}_{k-1})$

Motivating k-EPL

$$\hat{\theta}_{MLE} = \arg\max_{\theta \in \Theta} \quad Q_N(\theta, Y(\theta))$$

- k-EPL also maximizes a pseudo-likelihood in each iteration:
  - But replace $Y(\theta)$ with a Newton-like step
  - Uses a different definition of Y (more on this later)
- Useful composite parameter: $\gamma = (\theta, Y)$

Algorithm

k-Step Efficient Pseudo-Likelihood:

- **Step 1:** Obtain strongly $\sqrt{N}$-consistent $\hat{\gamma}_0 = (\hat{\theta}_0, \hat{Y}_0)$.
- **Step 2:** For $k \geq 1$, define

  $$\Upsilon(\theta, \hat{\gamma}_{k-1}) \equiv \hat{Y}_{k-1} - \nabla_Y G(\hat{\theta}_{k-1}, \hat{Y}_{k-1})^{-1} G(\theta, \hat{Y}_{k-1})$$

  and obtain estimates iteratively:

  $$\hat{\theta}_k = \arg\max_{\theta \in \Theta} Q_N\left(\theta, \Upsilon(\theta, \hat{\gamma}_{k-1})\right)$$

  and

  $$\hat{Y}_k = \Upsilon(\hat{\theta}_k, \hat{\gamma}_{k-1}).$$

- **Step 3:** Increment k and repeat Step 2.

Properties of k-EPL: Efficiency

- Efficiency of k-NPL for single-agent models stems from the *zero Jacobian* property:
  - Optimal choice probabilities maximize expected utility.
  - $\nabla_P \Psi(\theta^*, P^*) = 0$ and $\nabla_P \Psi(\hat{\theta}_{MLE}, \hat{P}_{MLE}) = 0$.
- k-EPL restores the zero Jacobian property in dynamic games:
  - Υ is essentially a Newton step and has the zero Jacobian property
  - $\nabla_\gamma \Upsilon(\theta^*, \gamma^*) = 0$ and $\nabla_\gamma \Upsilon(\hat{\theta}_{MLE}, \hat{\gamma}_{MLE}) = 0$
  - Lemma 2 in the paper
- Zero-Jacobian property will imply that EPL is:
  - Efficient for any $k \geq 1$
  - Converges locally to MLE in finite samples
  - Convergence rate is fast

Theorem 1

The k-EPL sequence computed with the algorithm above satisfies the following for all $k \geq 1$:

1. *(Consistency)* $\hat{\gamma}_k = (\hat{\theta}_k, \hat{Y}_k)$ *is a strongly consistent estimator of* (θ^*, Y^*).
2. *(Efficiency)* $\sqrt{N}(\hat{\theta}_k - \theta^*) \xrightarrow{d} \mathcal{N}(0, \Omega_{\theta\theta}^{*-1})$, *where* $\Omega_{\theta\theta}^*$ *is the information matrix of the full MLE problem.*
3. *(Large Sample Convergence) There exists a neighborhood* $\mathcal{B}^*$ *of* $\gamma^* = (\theta^*, Y^*)$ *such that* $\lim_{k \to \infty} \hat{\gamma}_k = \hat{\gamma}_{MLE}$ *almost surely for any* $\hat{\gamma}_0 \in \mathcal{B}^*$.

Theorem 2

(Iterating to finite-sample MLE)

1. $\hat{\gamma}_{MLE}$ *is a fixed point of the k-EPL iterations.*
2. $\hat{\gamma}_k - \hat{\gamma}_{MLE} = O_p(N^{-1/2}||\hat{\gamma}_{k-1} - \hat{\gamma}_{MLE}|| + ||\hat{\gamma}_{k-1} - \hat{\gamma}_{MLE}||^2)$

Discussion of Theorem 2

- Theorem 2 has a couple important implications:
- Fast local convergence of k-EPL iterations to MLE
- Iteration yields higher-order equivalence to finite-sample MLE
  - Suppose $\hat{\gamma}_0 - \hat{\gamma}_{MLE} = O_P(N^{-1/2})$
  - Then, $\hat{\gamma}_k - \hat{\gamma}_{MLE} = O_P(N^{-(k+1)/2})$

A Dynamic Discrete Choice Game

- Firm: $j \in \mathcal{J} = \{1, \ldots, |\mathcal{J}|\}$
- Action: $a \in \mathcal{A} = \{0, \ldots, |\mathcal{A}| - 1\}$
- Observed state: $x \in \mathcal{X} = \{1, \ldots, |\mathcal{X}|\}$
- Private information: $\varepsilon^j(a^j)$
- Period payoff: $\bar{u}^j(x, a^j, a^{-j}; \theta)$
- Conditional choice probabilities (CCPs): $P^j(x, a^j)$ and $P^{-j}(x, a^{-j})$
- Expected period payoff and transition probabilities:

$$u^j\left(a^j, x; P^{-j}, \theta\right) = \sum_{a^{-j}} P^{-j}(x, a^{-j})\bar{u}^j(x, a^j, a^{-j}; \theta)$$

- Discount factor: β

Equilibrium Fixed-Point Equation

- Choice-specific value functions determine CCPs:

$$P^j = \Lambda^j \left(v^j \right)$$

where

$$P^j(x, a^j) = Pr \left(a^j = \arg \max_a \left\{ v^j(x, a) + \varepsilon^j(a) \right\} \right)$$

- In equilibrium, for all (j, x, a^j) we have

$$\begin{aligned} v^j(x, a^j) = u^j & \left(a^j, x; \Lambda^{-j}(v^{-j}), \theta \right) \\ & + \beta \sum_{x'} f^j \left(x' \mid x, a^j; \Lambda^{-j}(v^{-j}) \right) S \left(v^j(x') \right) \end{aligned}$$

- $S(\cdot)$ is McFadden's social surplus function

Equilibrium Fixed-Point Equation

- More compactly:

$$v = \Phi(\theta, v)$$
$$G(\theta, v) \equiv v - \Phi(\theta, v) = 0$$

- Lemma 1 in the paper establishes validity of this equilibrium representation

Properties of k-EPL: Linearity

- If $\bar{u}^j$ (and hence u^j) are linear in θ, then so are G and Υ

$$G(\theta, \hat{v}_{k-1}) = H(\hat{v}_{k-1})\theta + z(\hat{v}_{k-1})$$

$$\begin{aligned}
\Upsilon(\theta, \hat{\gamma}_{k-1}) &\equiv \hat{v}_k - \nabla_v G(\hat{\theta}_{k-1}, \hat{v}_{k-1})^{-1} G(\theta, \hat{v}_{k-1}) \\
&= -\nabla_v G(\hat{\theta}_{k-1}, \hat{v}_{k-1})^{-1} H(\hat{v}_{k-1})\theta \\
&\qquad + \hat{v}_{k-1} - \nabla_v G(\hat{\theta}_{k-1}, \hat{v}_{k-1})^{-1} z(\hat{v}_{k-1}) \\
&= A(\hat{\gamma}_{k-1})\theta + b(\hat{\gamma}_{k-1})
\end{aligned}$$

Properties of k-EPL: Linearity

- v determines CCPs: $Q_N(\theta, v) \equiv Q_N(v)$
- With Gumbel or normal errors, $Q_N(v)$ is concave in v
- Our problem:

$$\max_{\theta \in \Theta} Q_N(\Upsilon(\theta, \hat{\gamma}_{k-1}))$$

- We just showed $\Upsilon(\cdot)$ is linear in θ, so the problem is strictly concave in θ

Properties of k-EPL: Linearity

- Each iteration reduces to solving linear systems + static logit/probit maximization (linear index)
- Linear systems: $O((|\Theta| + 1)|\mathcal{X}|^3|\mathcal{A}|^3|\mathcal{J}|^3)$ vs. $O((|\Theta| + 1)|\mathcal{X}|^3|\mathcal{J}|)$ for NPL
  - Worst-case bounds
  - Sparsity reduces the difference in bounds
  - In our Monte Carlo experiments, actual time difference is much smaller than these bounds suggest
  - Solving via iterative methods can reduce cubes to squares
- Tradeoff: fewer k-EPL iterations should be required due to faster convergence

Summary of k-EPL

- We work in "v space" instead of "P space", and we use Υ instead of Ψ to characterize the equilibrium.
- Newton steps on P would be problematic
  - Can lead outside the simplex
  - Much larger computational burden (detailed explanation in paper)
- Switching to v space and using Υ to characterize the equilibrium restores *efficiency* while preserving *linearity* in θ.

Note on Single-Agent Models

- Can show that k-NPL in single-agent models is a modified form of k-EPL
  - Return to P space: $G(\theta, P) = P - \Psi(\theta, P)$
  - $\nabla_P G(\hat{\theta}_{MLE}, \hat{P}_{MLE}) = I$. So just use I all the time.
  - Then, $\Upsilon(\cdot) = \Psi(\theta, \hat{P}_{k-1})$
  - Expected, since Aguirregabiria and Mira (2002) show that NPL updates are like Newton steps
- Can still use our formulation of fixed-point equation in single-agent models

Monte Carlo Simulations: 2×2 Dynamic Entry Model

- Model of Pesendorfer and Schmidt-Dengler (2008)
- 2 firms: $i \in \{1, 2\}$
- 2 actions: $a^j \in \{0, 1\}$ (exit/enter)
- $x_t = (a^1_{t-1}, a^2_{t-1})$

$$\bar{u}^j(x_t, a^j_t = 1; \theta) = \theta_{\mathrm{M}} + \theta_{\mathrm{C}} a_t^{-j} + \theta_{\mathrm{EC}}(1 - a^j_t)$$
$$\bar{u}^j(x_t, a^j_t = 0; \theta) = \theta_{\mathrm{SV}} a^j_{t-1}.$$

- Fix discount factor $\beta = 0.9$ and $\theta_{\mathrm{SV}} = 0.1$.
- Generate data from $N \in \{250, 1000\}$ i.i.d. markets
- Carry out 1000 replications each.
- Estimate $(\theta_{\mathrm{M}}, \theta_{\mathrm{C}}, \theta_{\mathrm{EC}})$.

Monte Carlo Results

- Game has 3 equilibria
  - (i) is stable for k-NPL
  - (ii) and (iii) are unstable for k-NPL
- We compare 1-NPL, 1-EPL, ∞-NPL, and ∞-EPL.
- Will show (i) and (ii) only; (iii) is qualitatively similar to (ii).
- Local convergence results underscore importance of good starting values:
  - Show small sample size $N = 250$ vs large $N = 1000$.
  - Also try multiple random starting values, instead of consistent first-stage estimates.

Game from Aguirregabiria and Mira (2007)

- Larger-scale, empirically-relevant game
- Basis of many other simulation studies in dynamic games literautre
- 5 firms, 2 actions, 160 states
- Utilities: $\bar{u}^j(x_{it}, a^j_{it} = 0, a^{-j}_{it}; \theta) = 0$

$$\bar{u}^j(x_{it}, a^j_{it} = 1, a^{-j}_{it}; \theta) = \theta^j_{\text{FC}} + \theta_{\text{RS}} s_{it} - \theta_{\text{EC}}(1 - a^j_{i,t-1}) - \theta_{\text{RN}} \ln\left(1 + \sum_{l \neq j} a^l_{it}\right)$$

Game from Aguirregabiria and Mira (2007)

- Sample sizes: $N \in \{1600, 6400\}$
- 1000 replications
- Estimate all parameters
- Vary the true value of $\theta_{\mathrm{RN}} \in \{1, 2.5, 4\}$

Conclusion

- Develop a k-EPL estimator that balances several nice properties:
    - Computational simplicity
    - Consistency and asymptotic efficiency for every k (including $k \to \infty$)
    - Fast, stable convergence as $k \to \infty$
    - Good overall finite sample properties
- Method works well in difficult example models, both small-scale and large-scale