# Lecture 16: Equilibrium Trade in Automobile Markets

## 2019 Econometric Society Summer School in Dynamic Structural Econometrics

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University of Chicago
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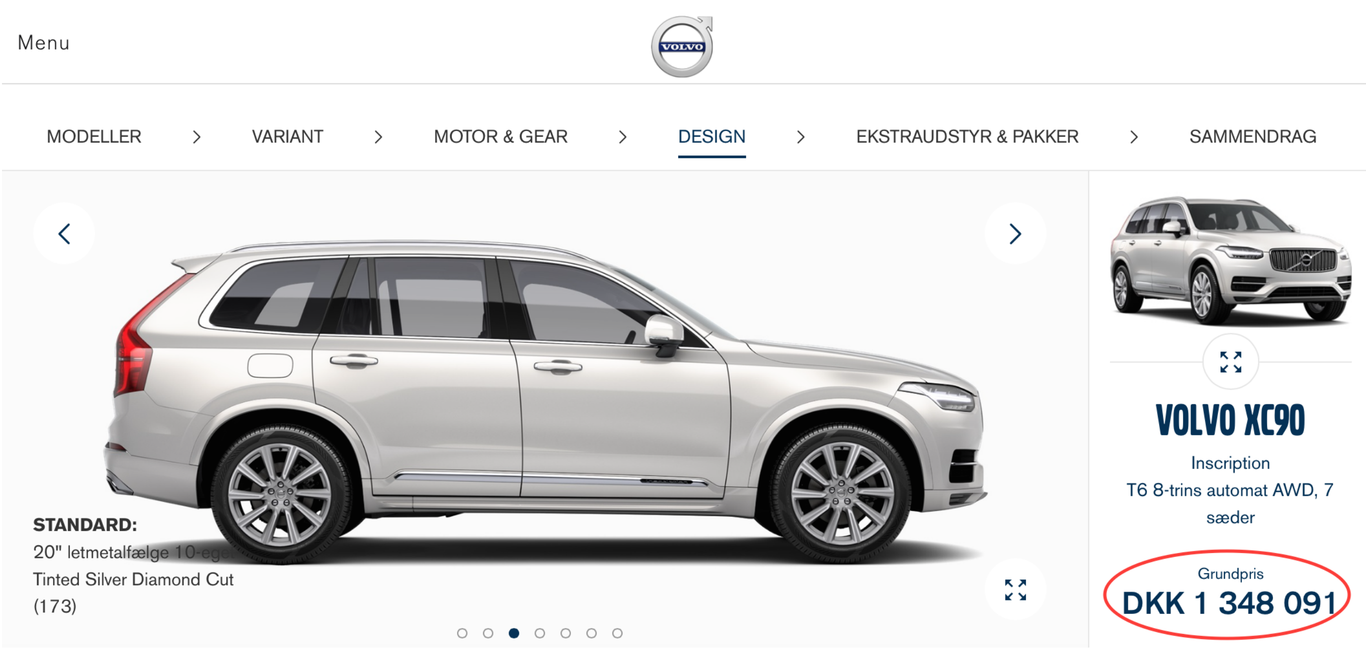
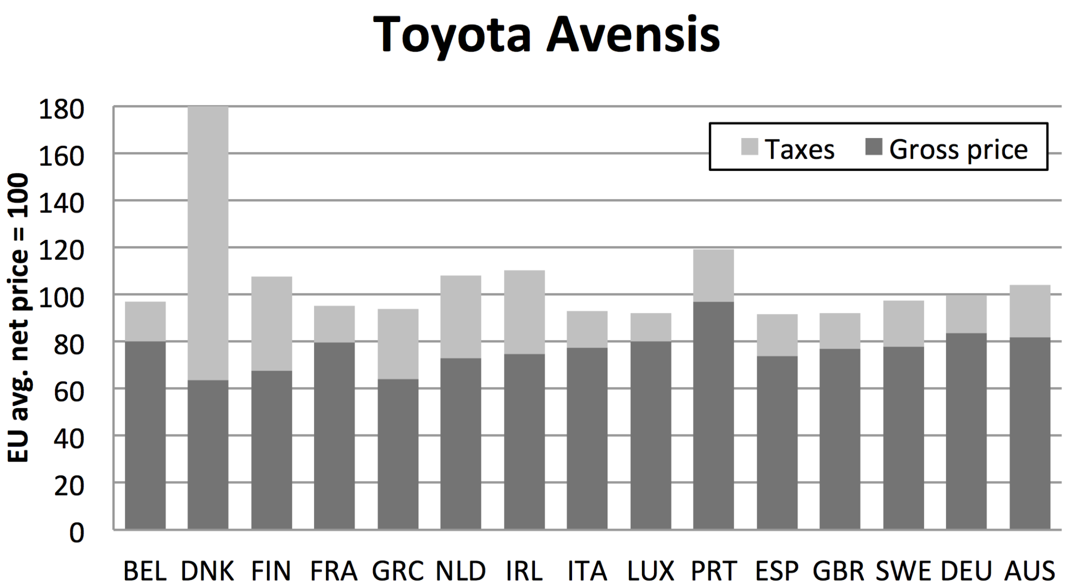
# How much is a Volvo in Denmark?

# 204.816 US Dollars!




MSRP in US: $62.350

# Danish car registration tax: 180%! ..... plus 25% VAT!!!



# Car taxes in Denmark

**Annual Revenue**

- 30–50 billion DKK
- $\cong$ 2–3 pct. of GDP
- $\cong$ 4-7 pct. of total tax revenue
- Most revenue originate from taxation of ovnership and registration of new cars. Car taxes in DK, 1980-2012

It is also widely understood that transport externalities are rarely appropriately priced (e.g., Parry and Small (2005)).

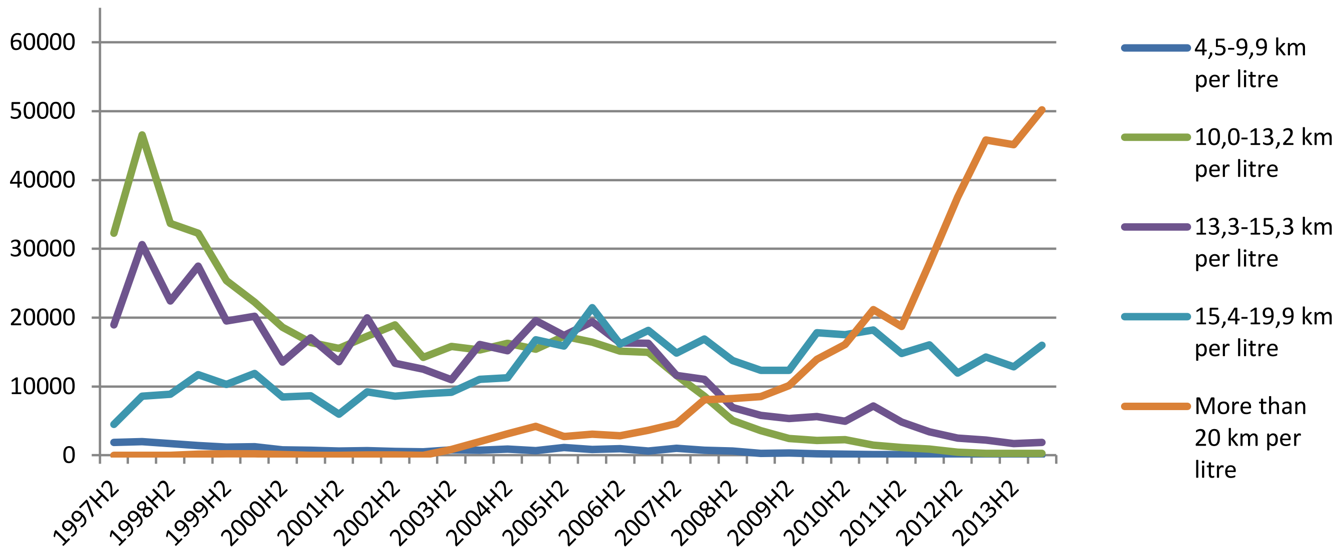
- Underpricing congestion.
- Incorrect taxation of gasoline.

# Car taxes in Denmark

- **Annual green ownership tax** was introduced in 1997 to create incentives for *owning* more energy efficient cars.
  - Reformed in 2007: even stronger incentives to choose smaller, more energy efficient cars including diesels.
  - Green owner tax now ranges from \$3096 for cars getting less than 4.7 km/liter to only \$94 for cars getting more than 20km/liter
- **Vehicle registration tax** was designed to pay for roads and maintenance and to ration the number of cars in Denmark
  - Progressive tax with 180% marginal tax in top bracket (+25% VAT).
  - Reformed in 2007: Discounts registration fee by \$605 per km over 16 km/L gas or 18km/L diesel and penalty of \$152 for cars with efficiencies below these thresholds
  - Reformed in 2017: Marginal tax in top bracket reduced 150%

# Trend in vehicle efficiency in Denmark

*Figure 1*: Newly registered passenger cars, 1997 - 2014




Source: The Danish Statistical Bureau

# Research Question

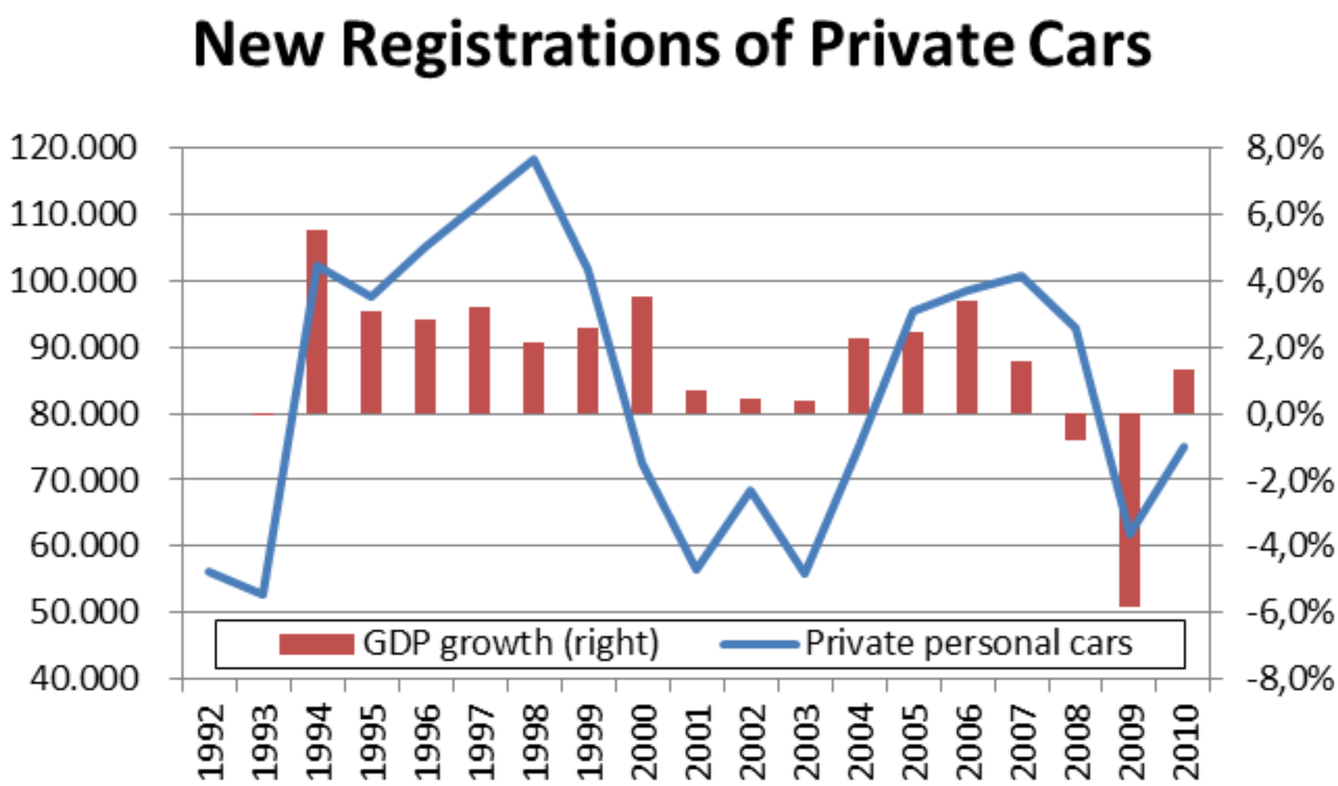
## Potential Policy Option

- Lower registration taxes, and
- higher usage taxes (road charging).

## Outcomes of interest

- Equilibrium dynamics of car ownership and type choice:
  - new car sales and trade in secondary markets
  - fleet age and scrappage,
  - value of the car stock.
- Driving, fuel demand, and emissions.
- Redistribution and welfare.
- Need to capture these effects simultaneously....
  and account for macroeconomics shocks

# Car sales over the business cycle 1992-2010



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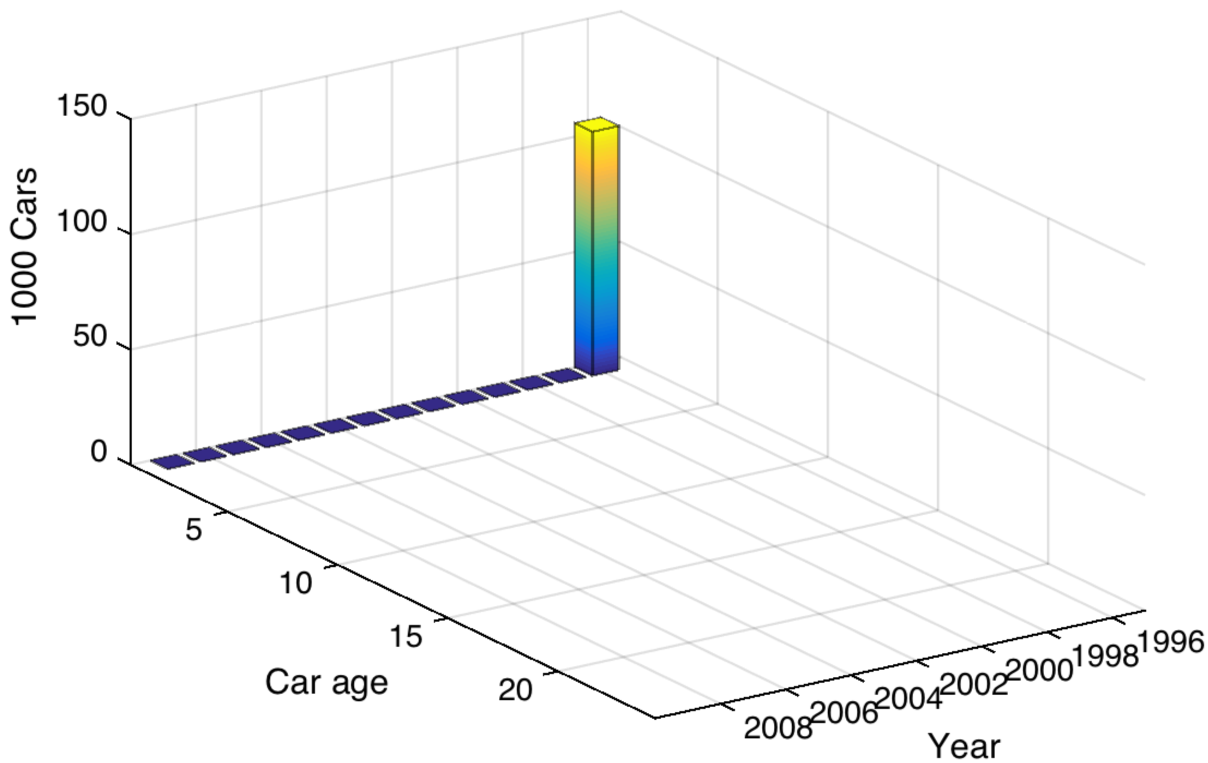

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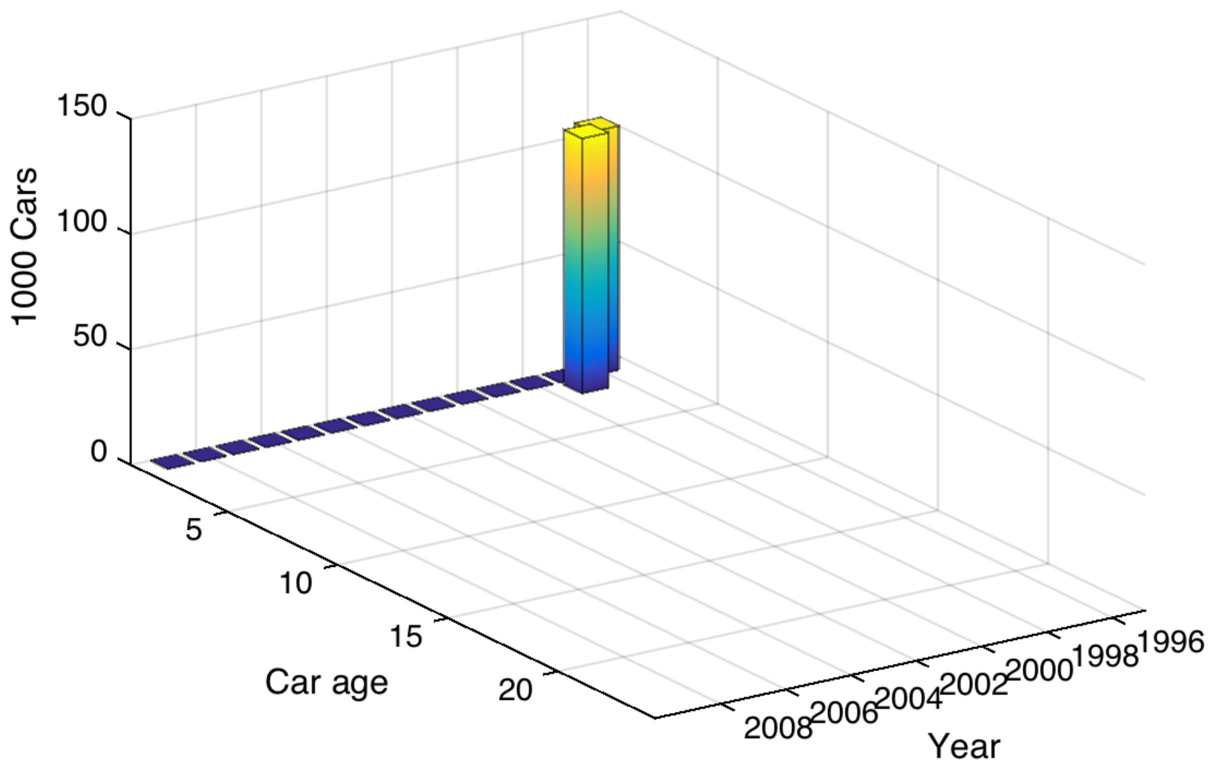

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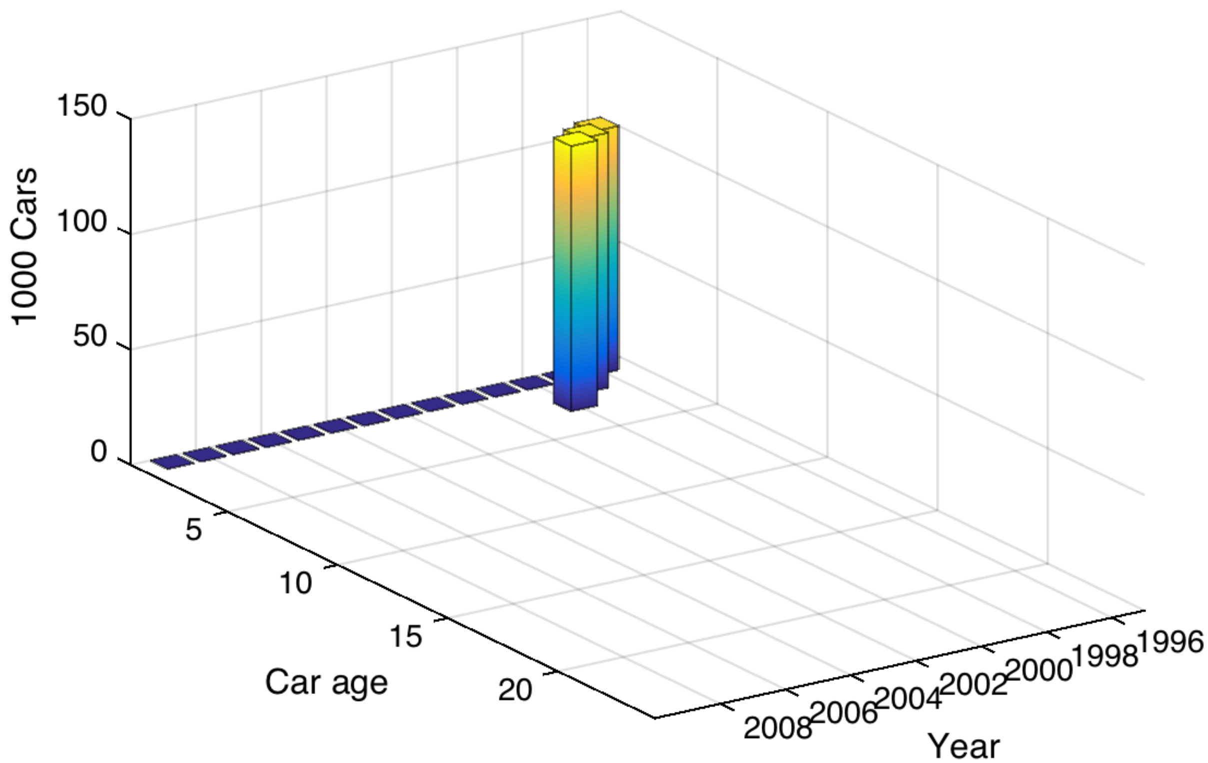

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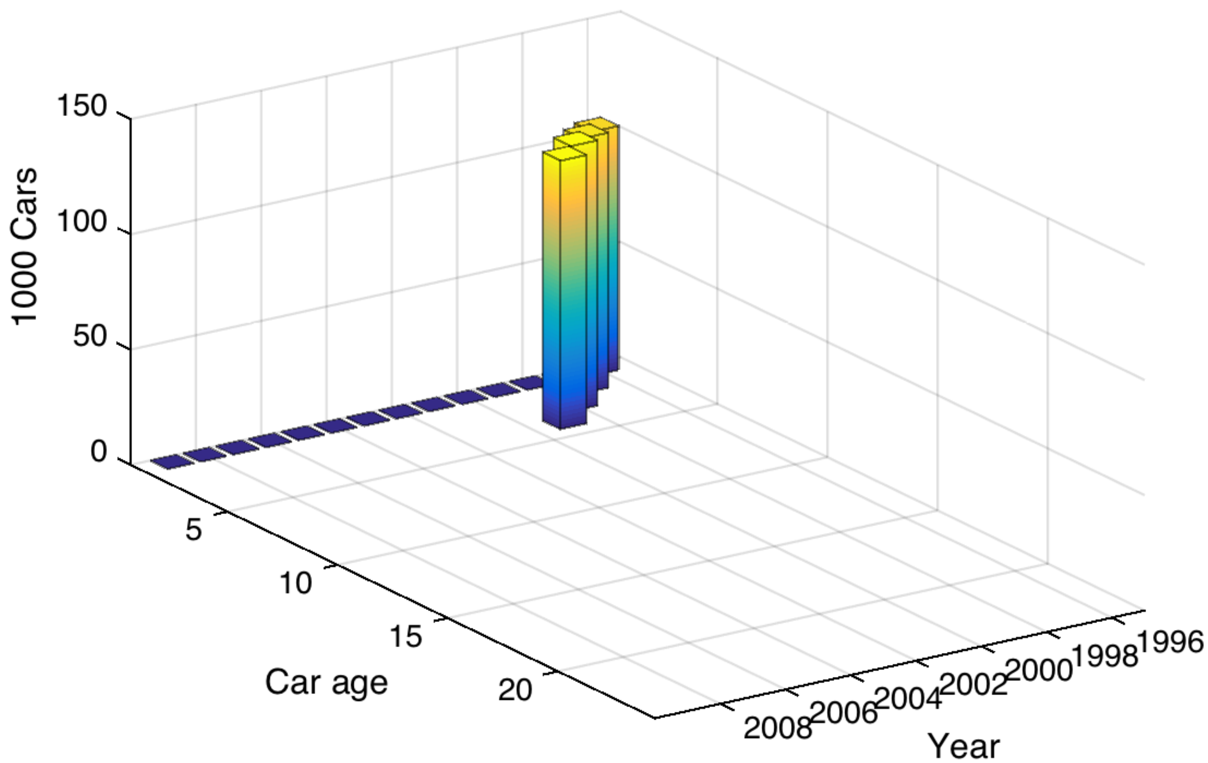

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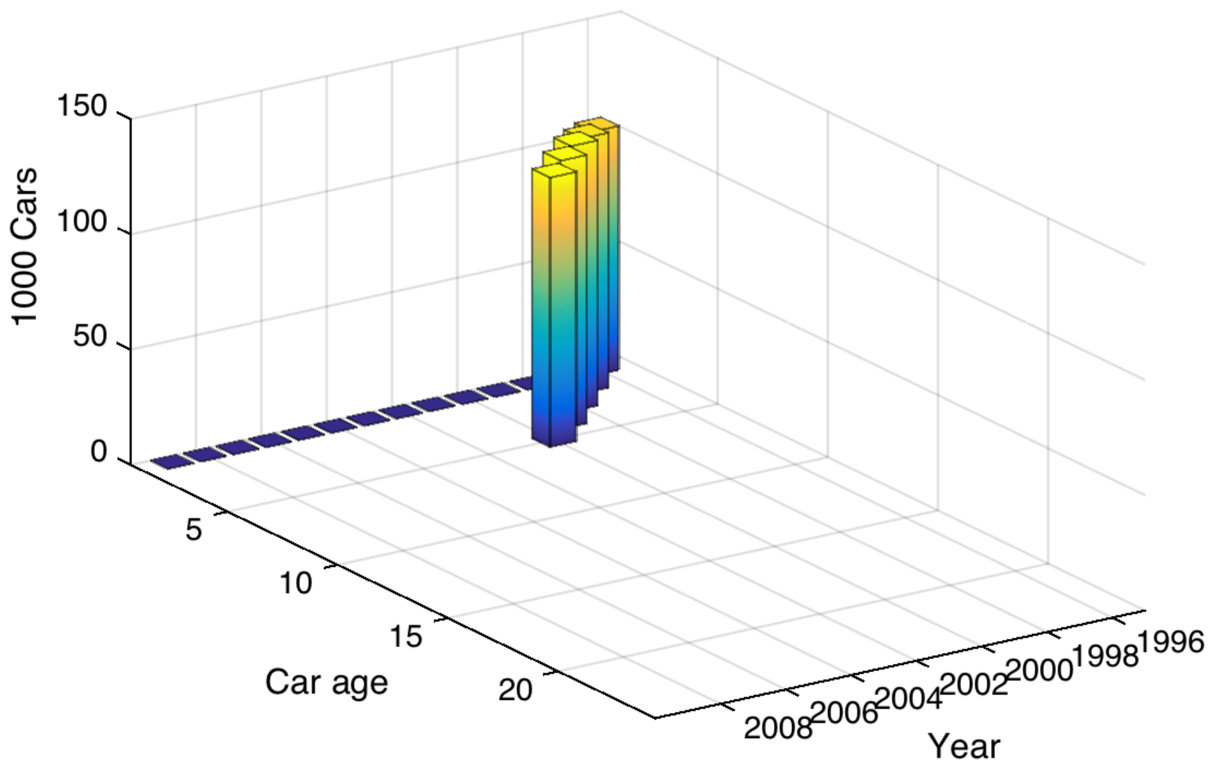

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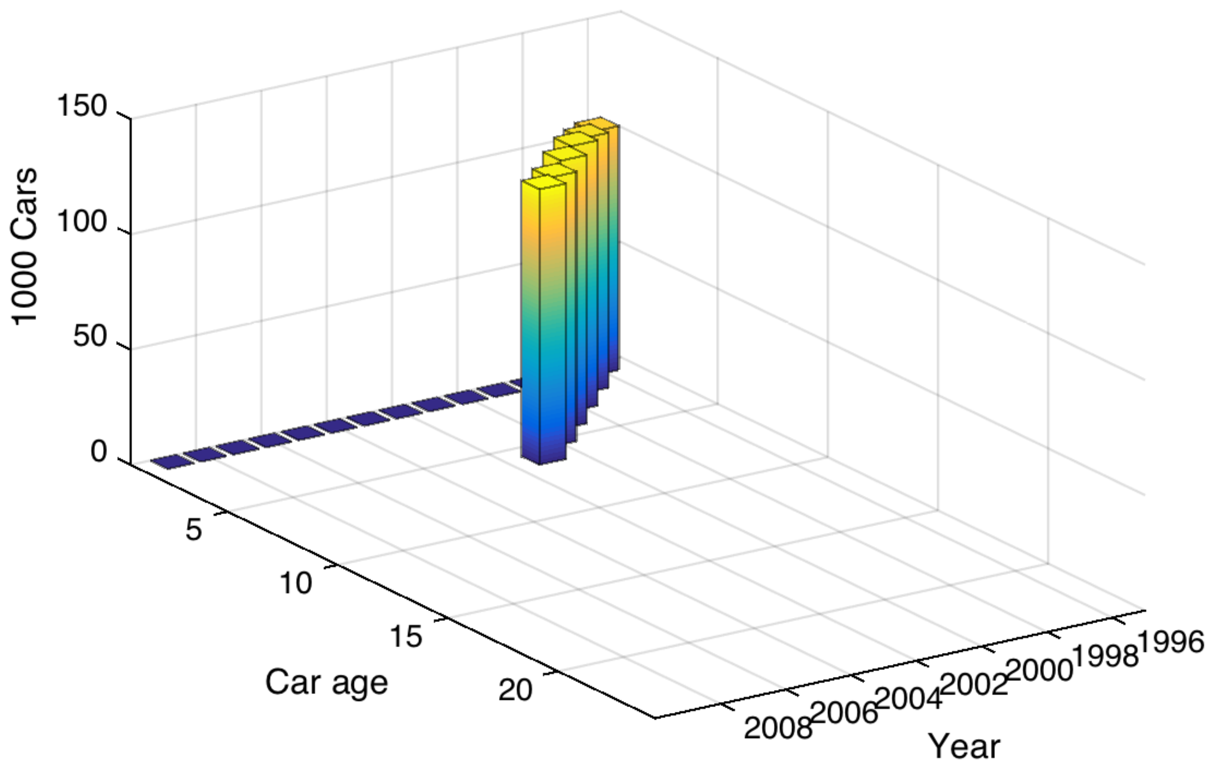

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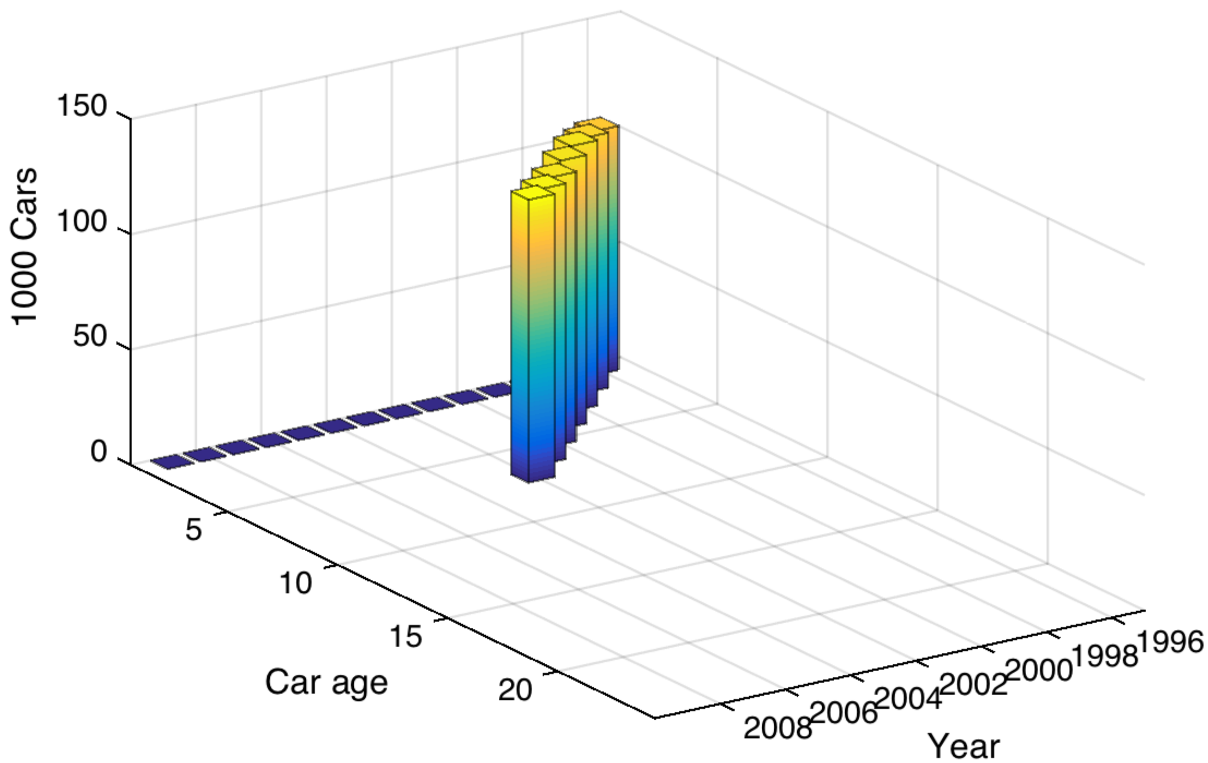

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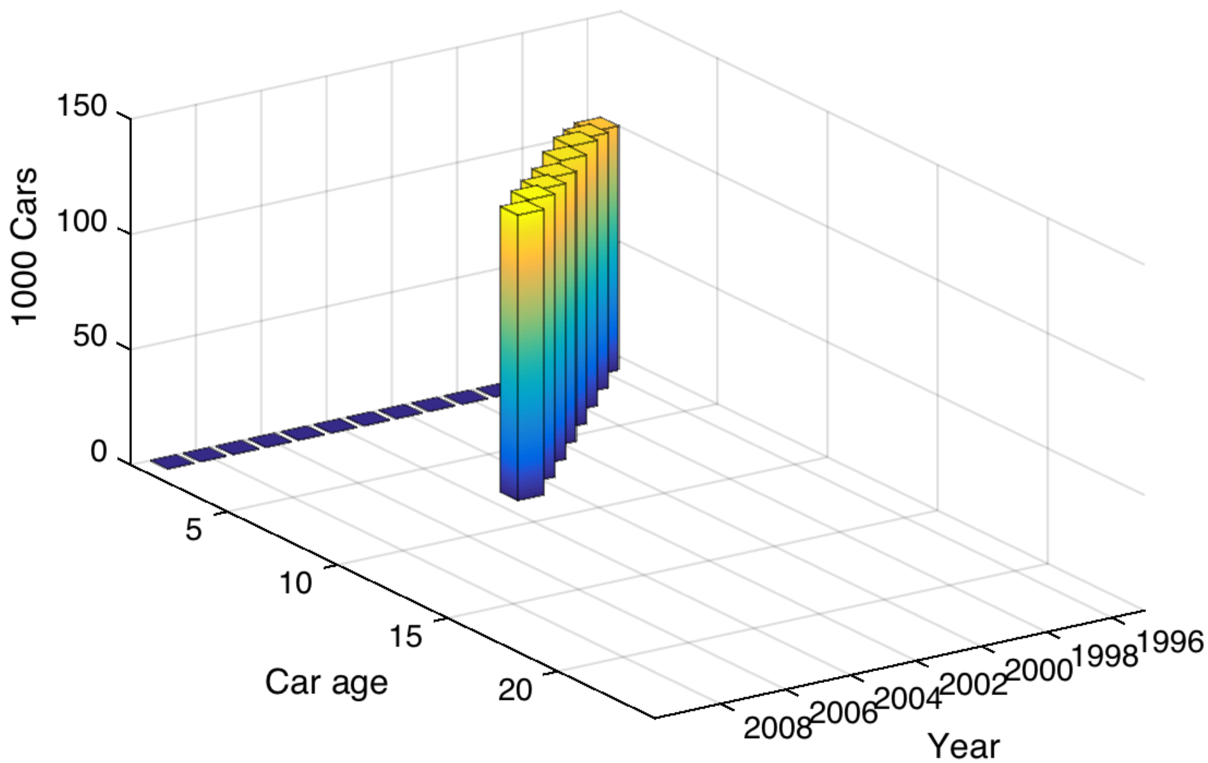

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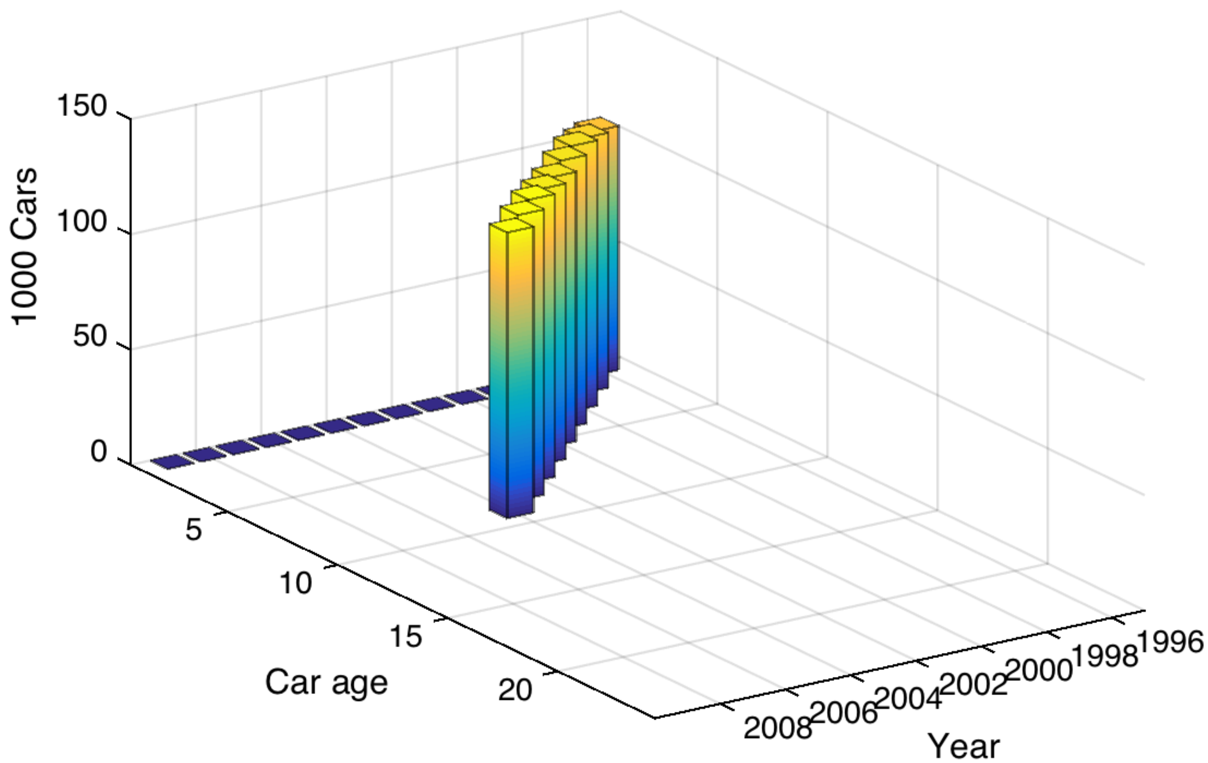

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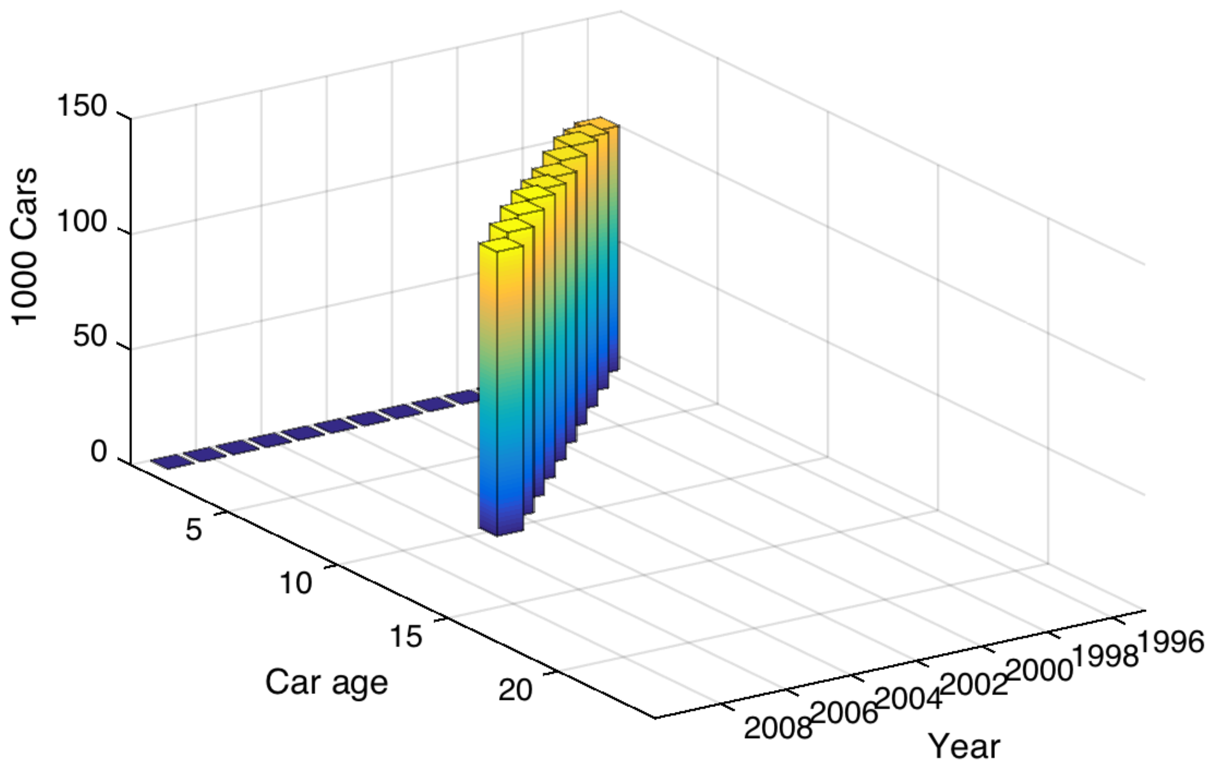

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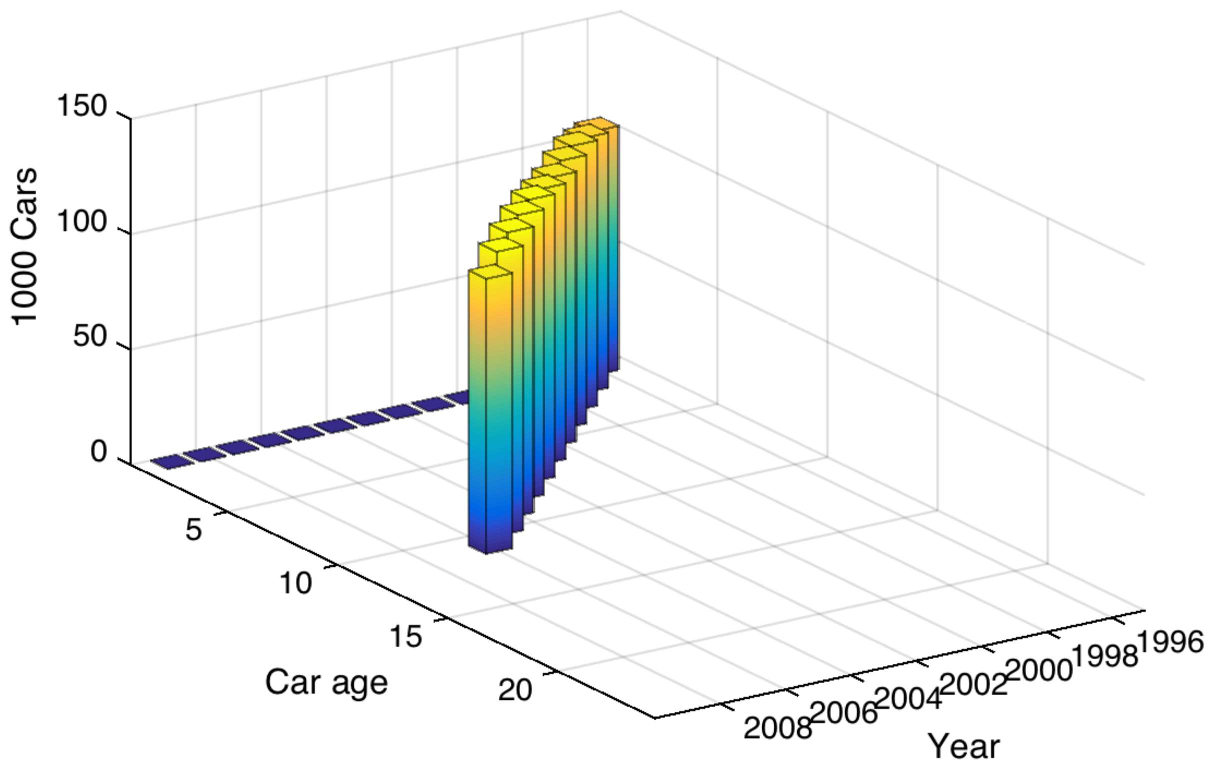

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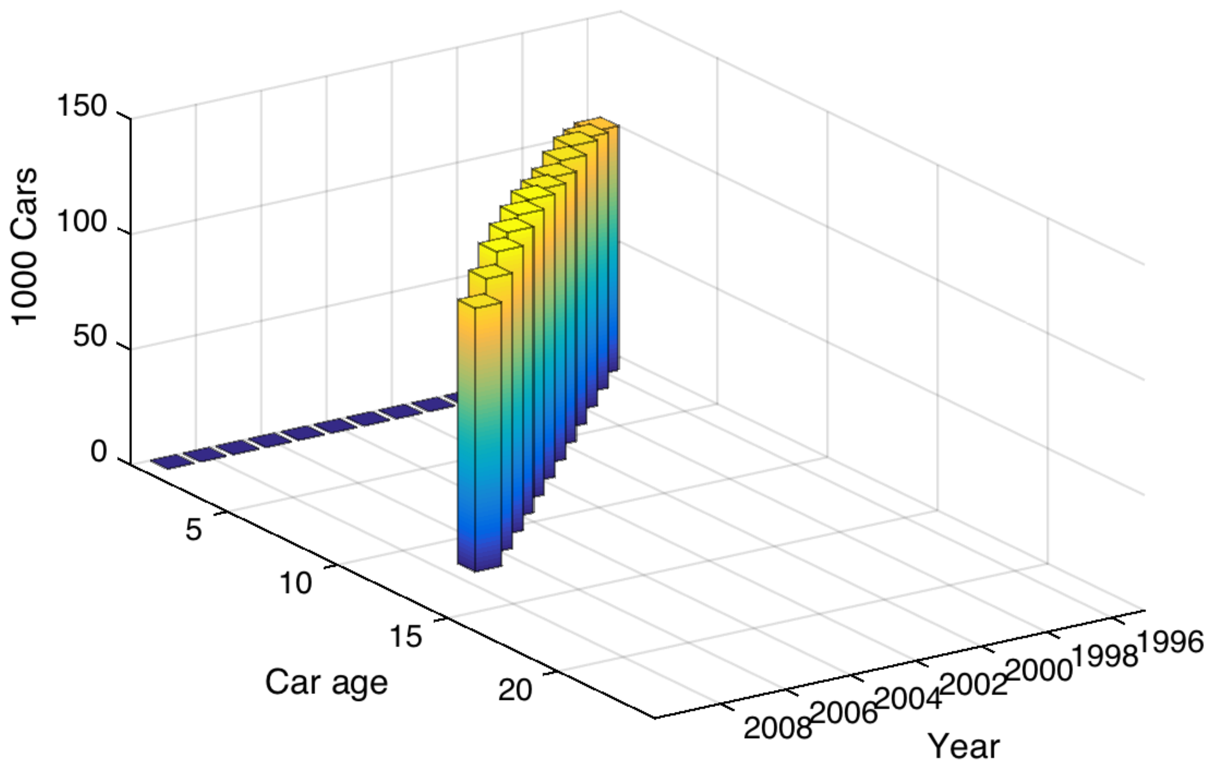

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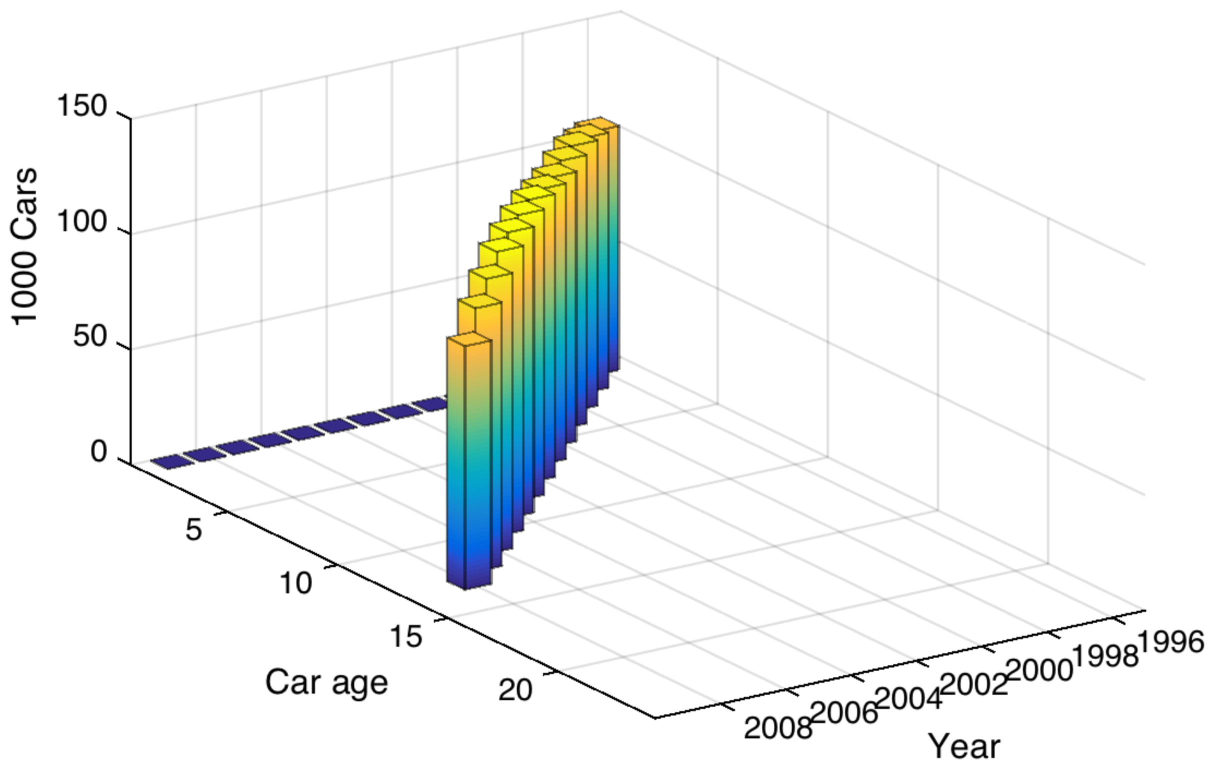

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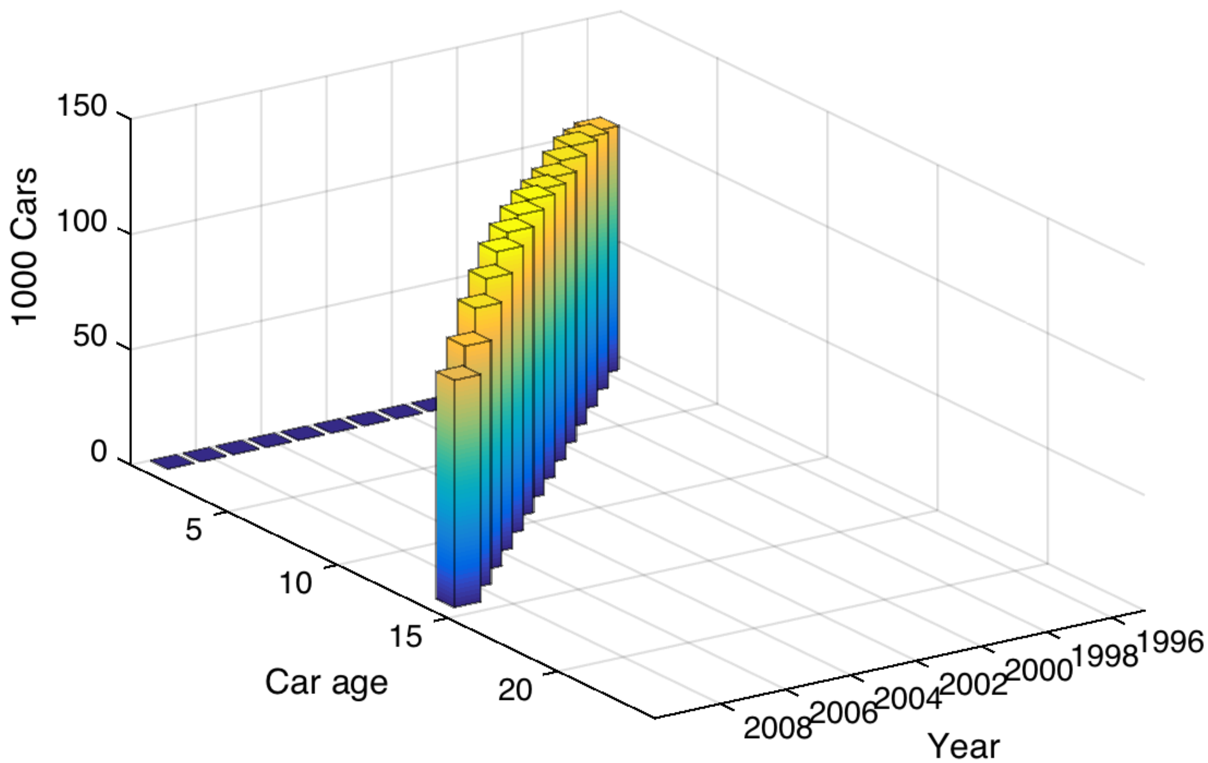

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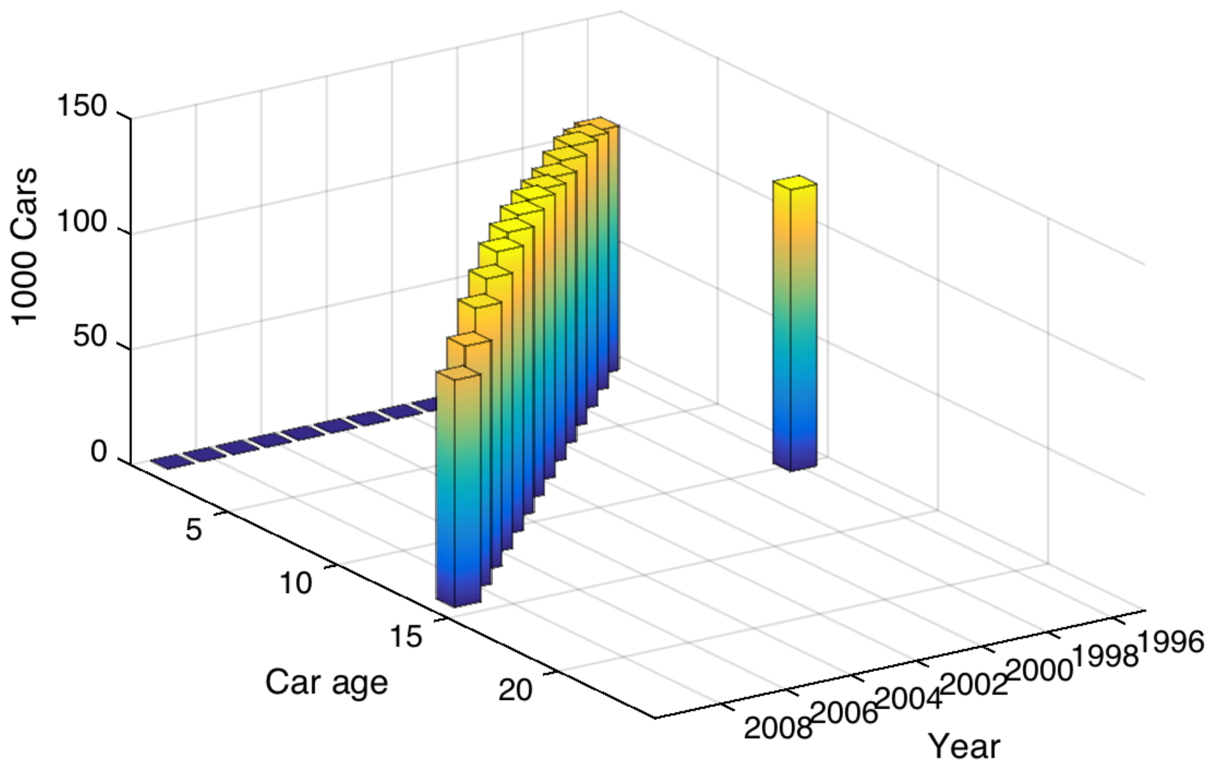

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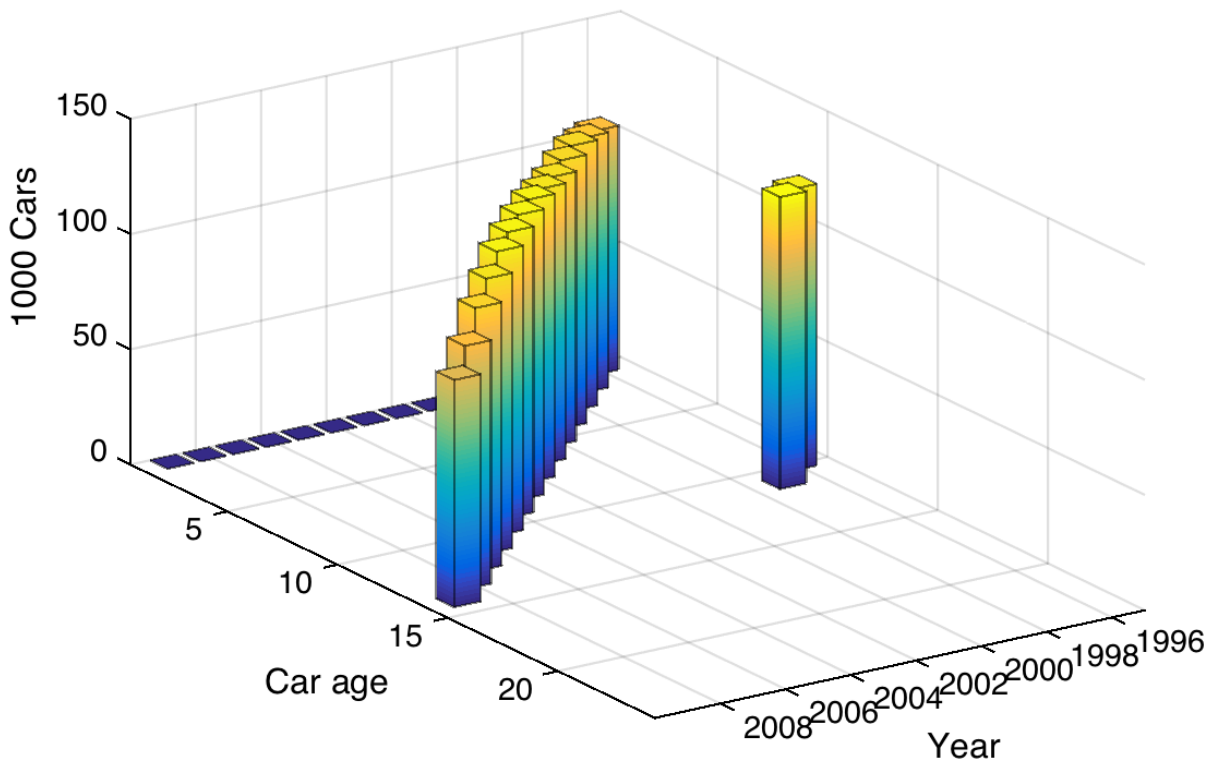

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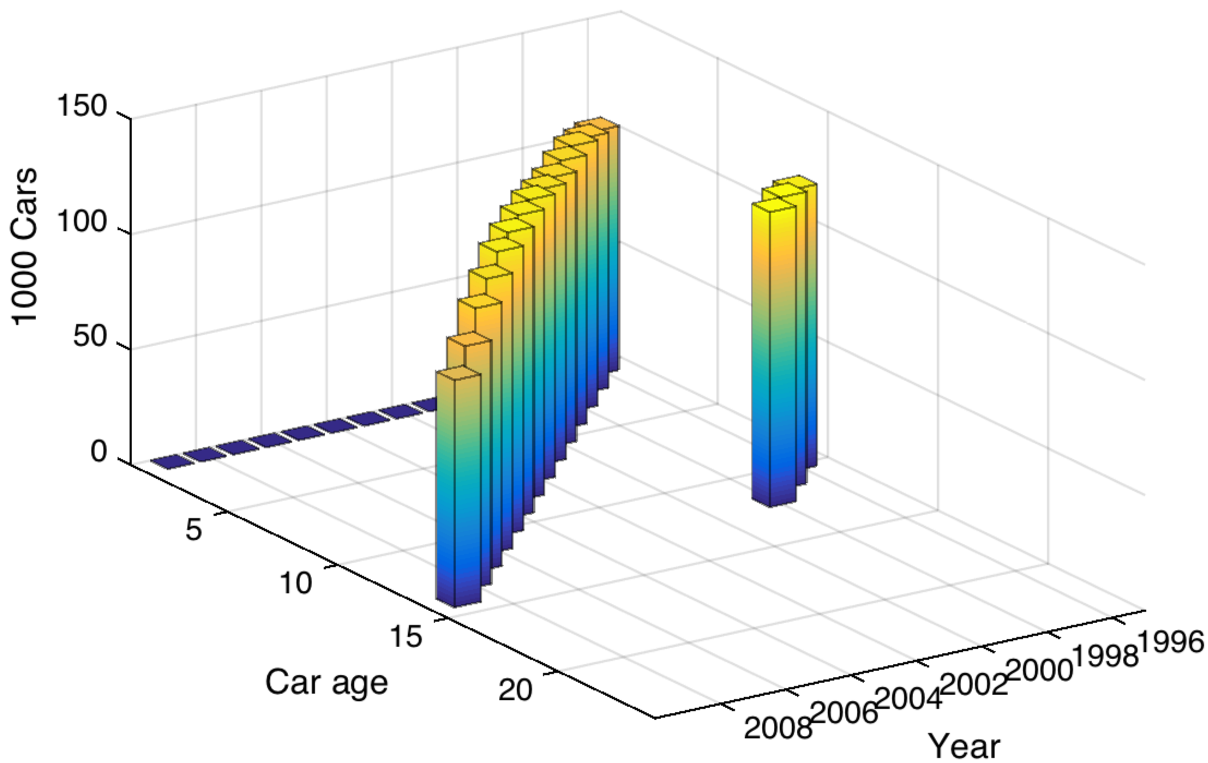

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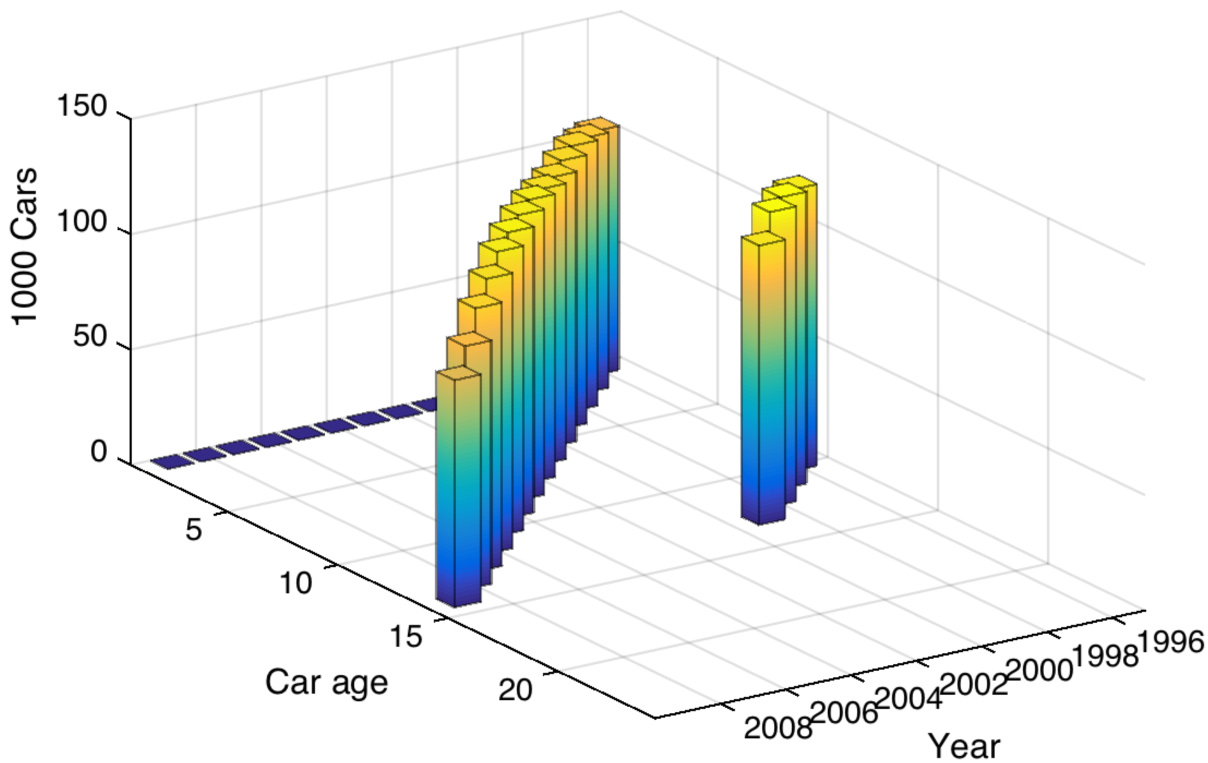

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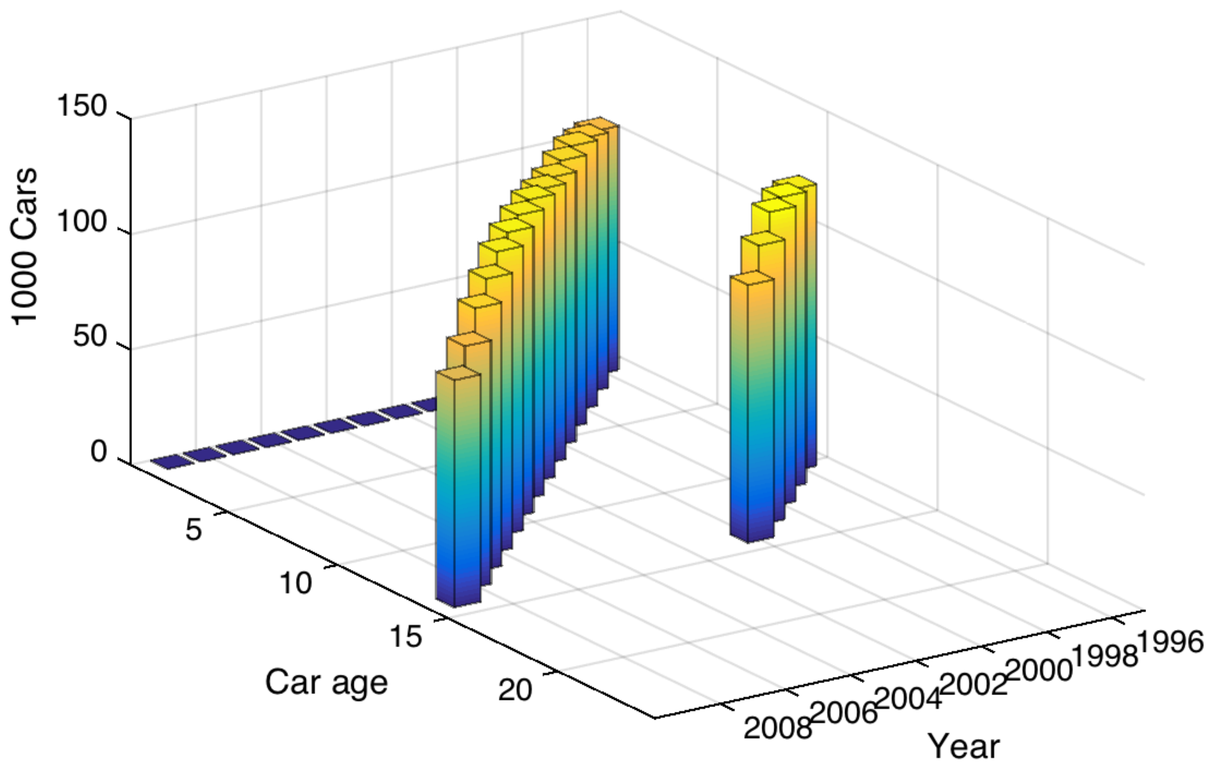

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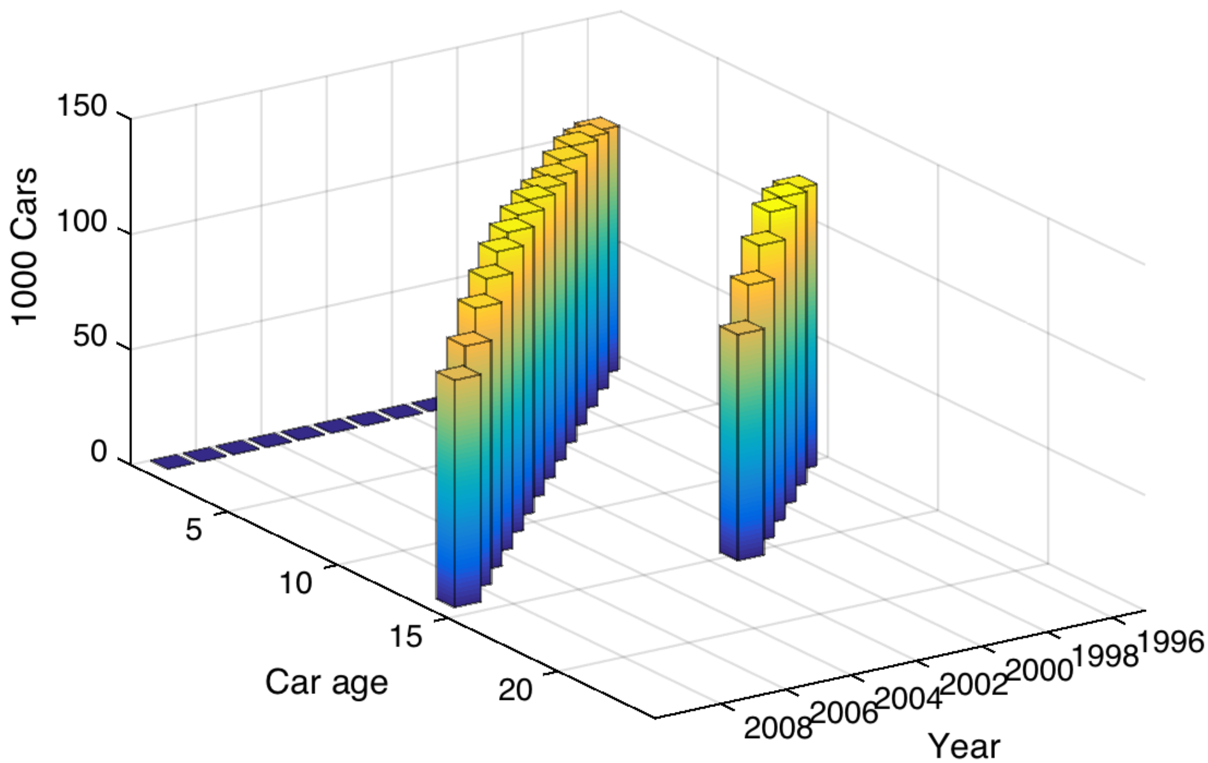

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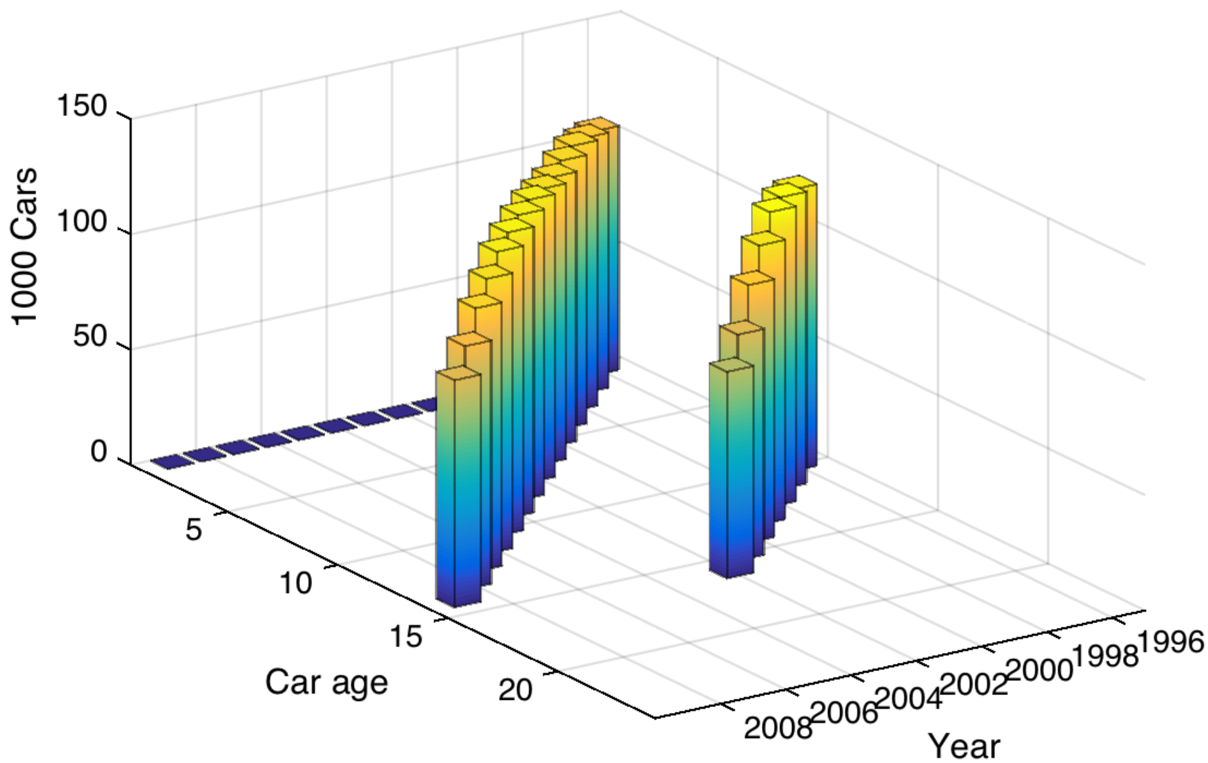

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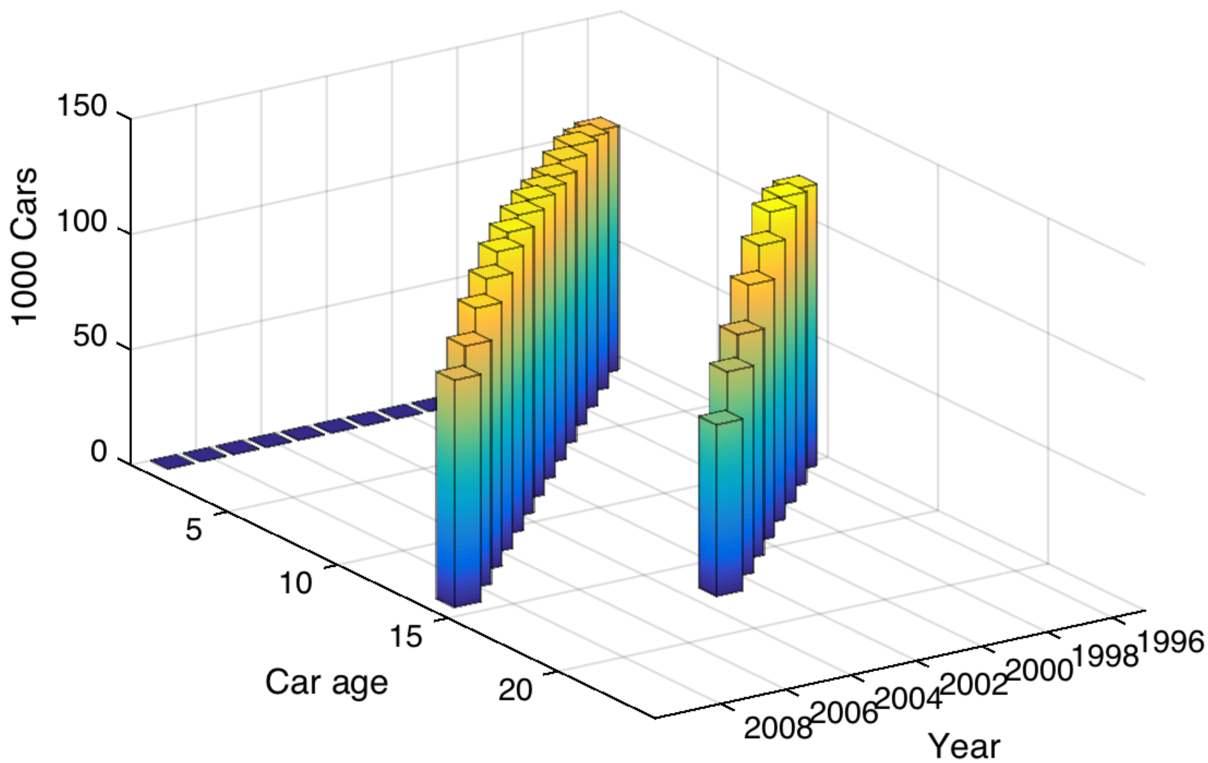

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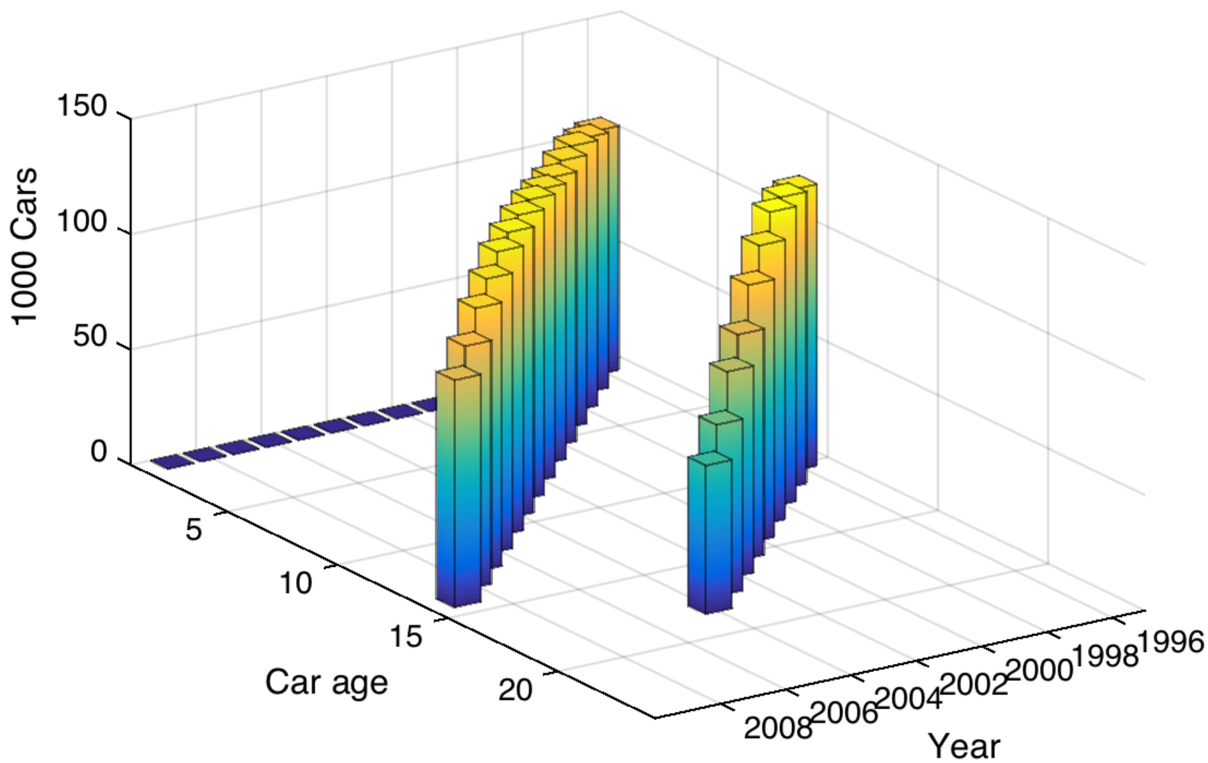

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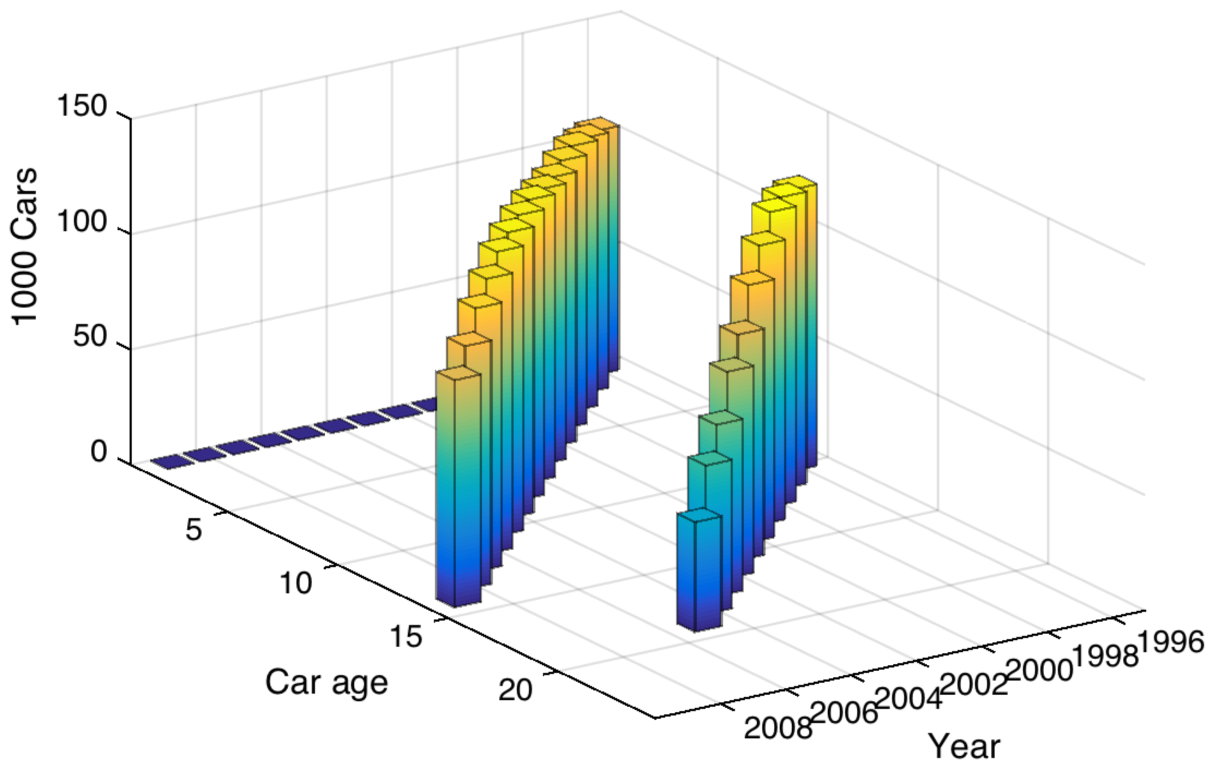

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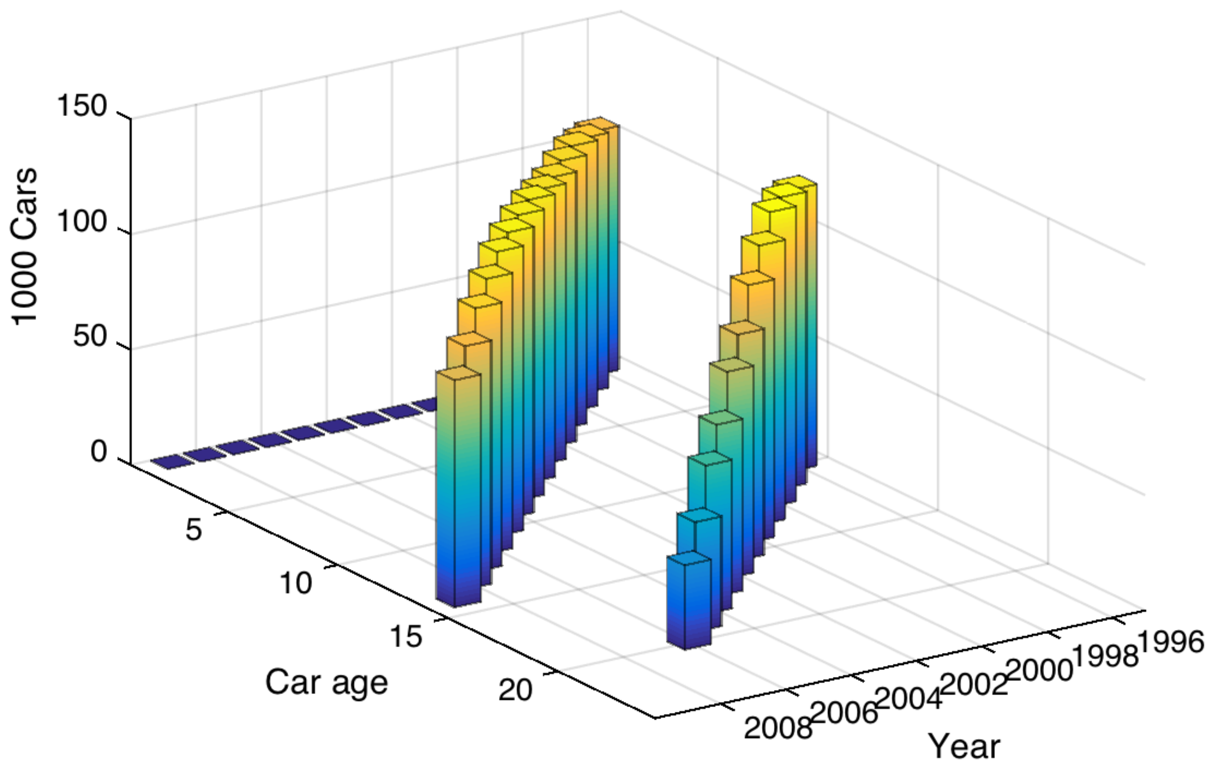

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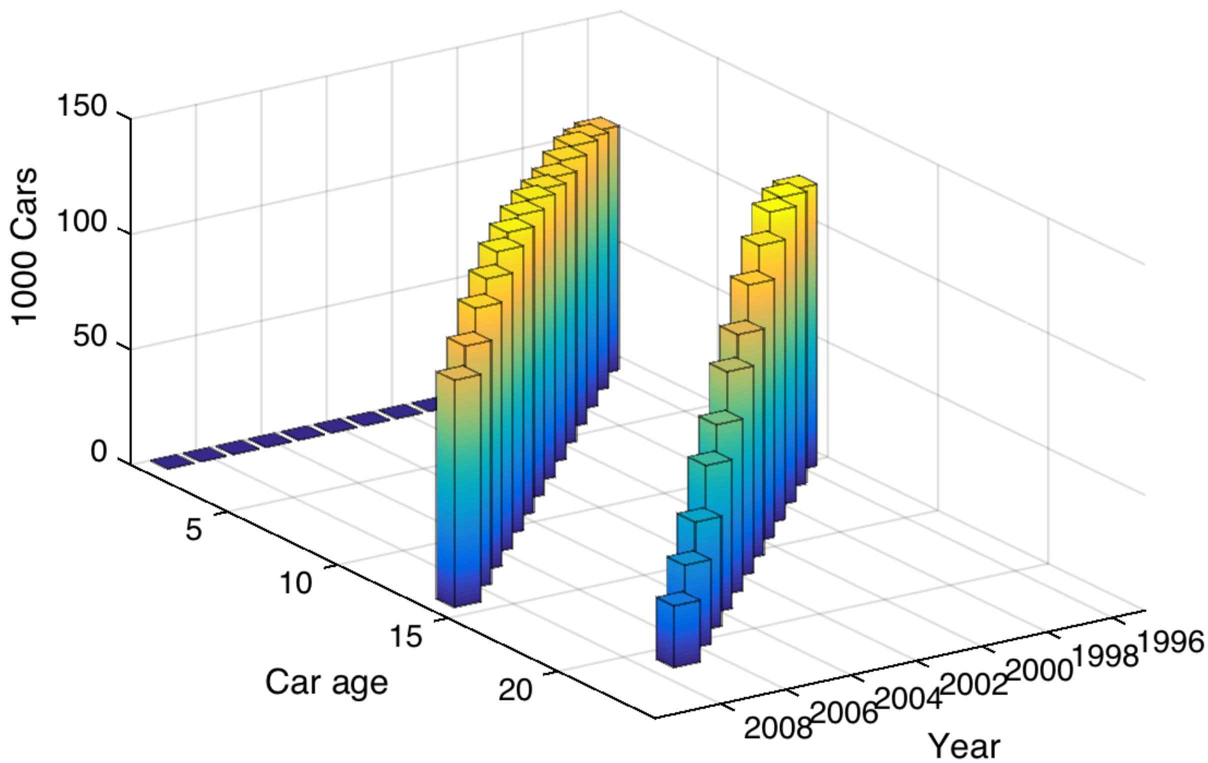

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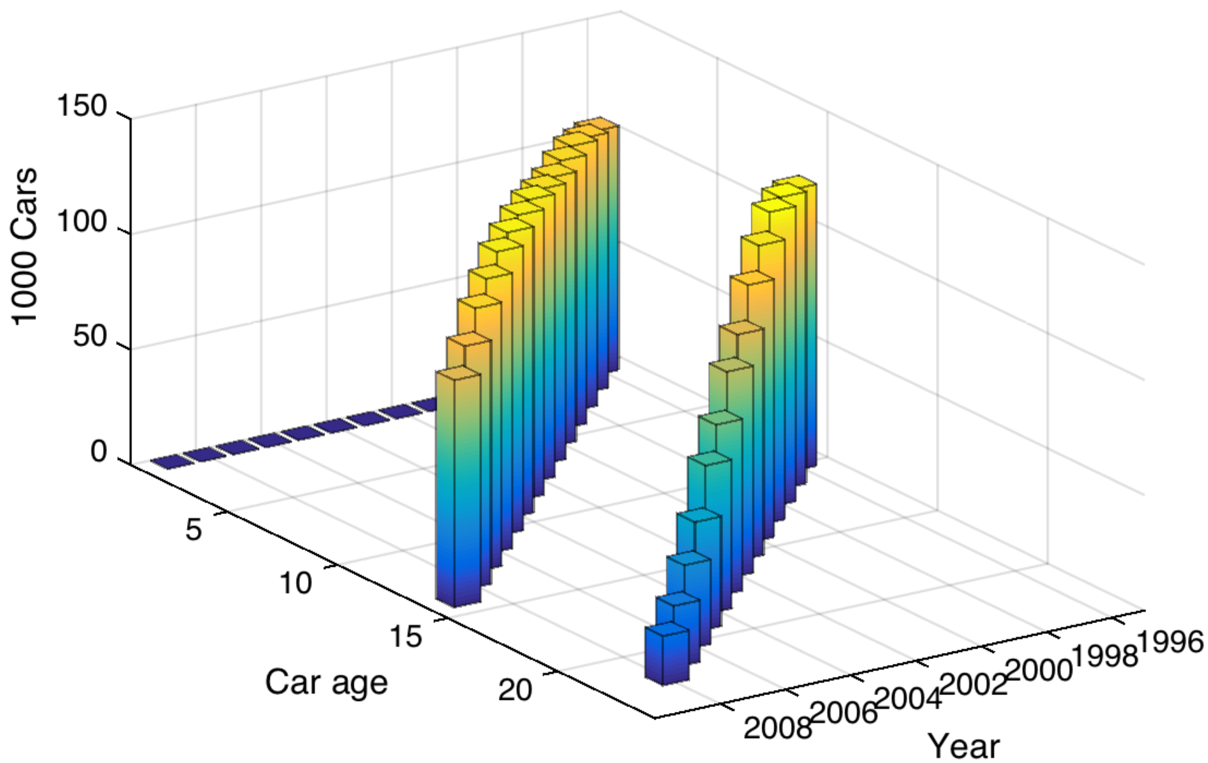

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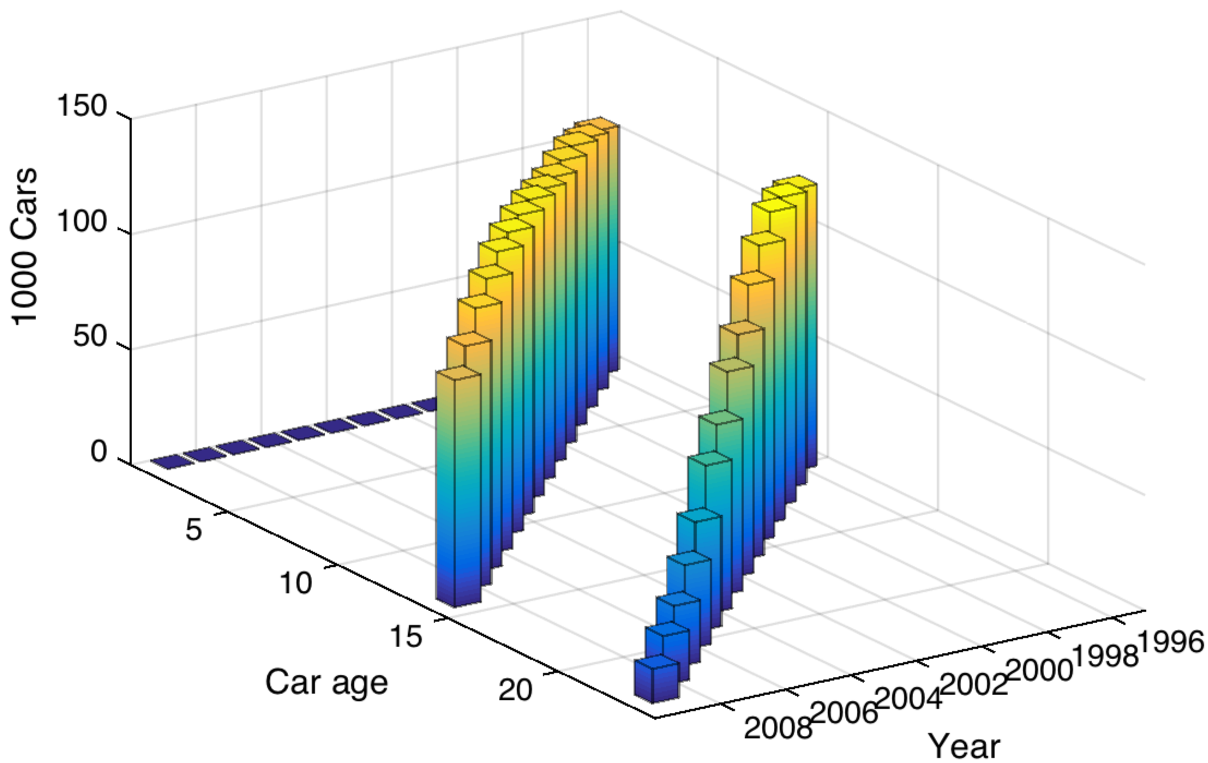
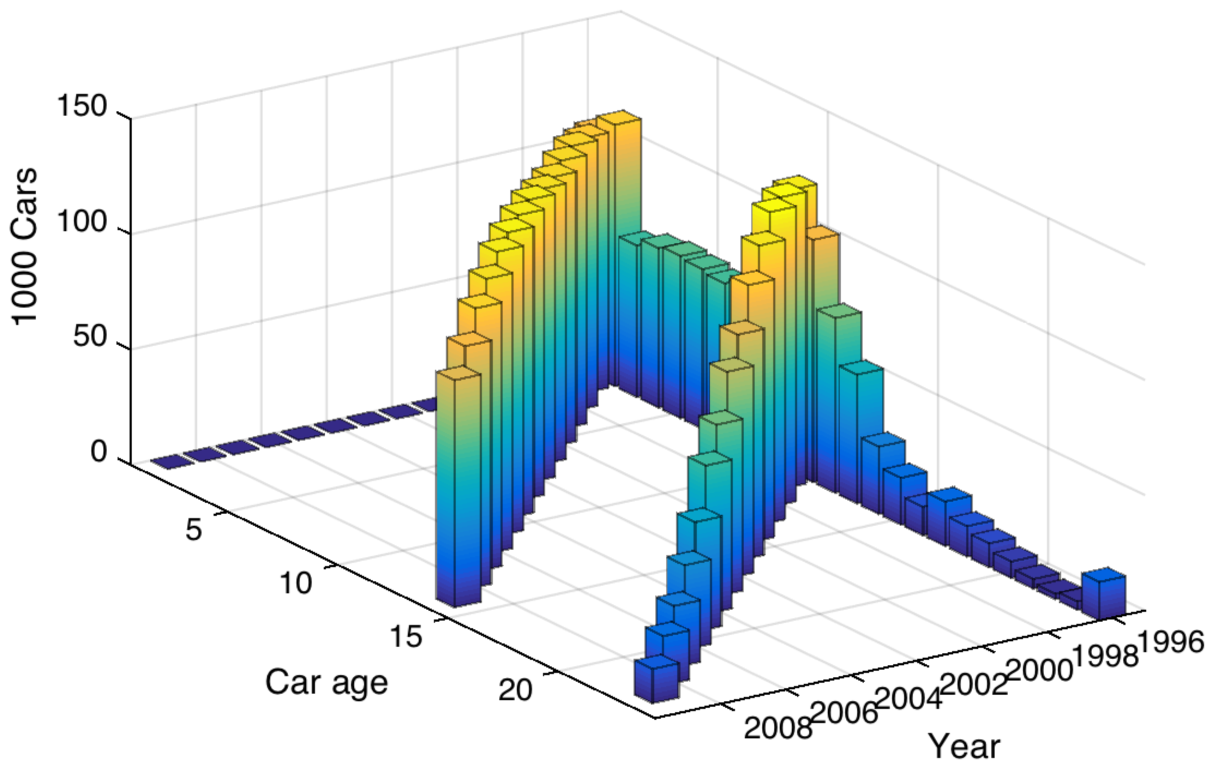


# The Car Age Distribution Over Time



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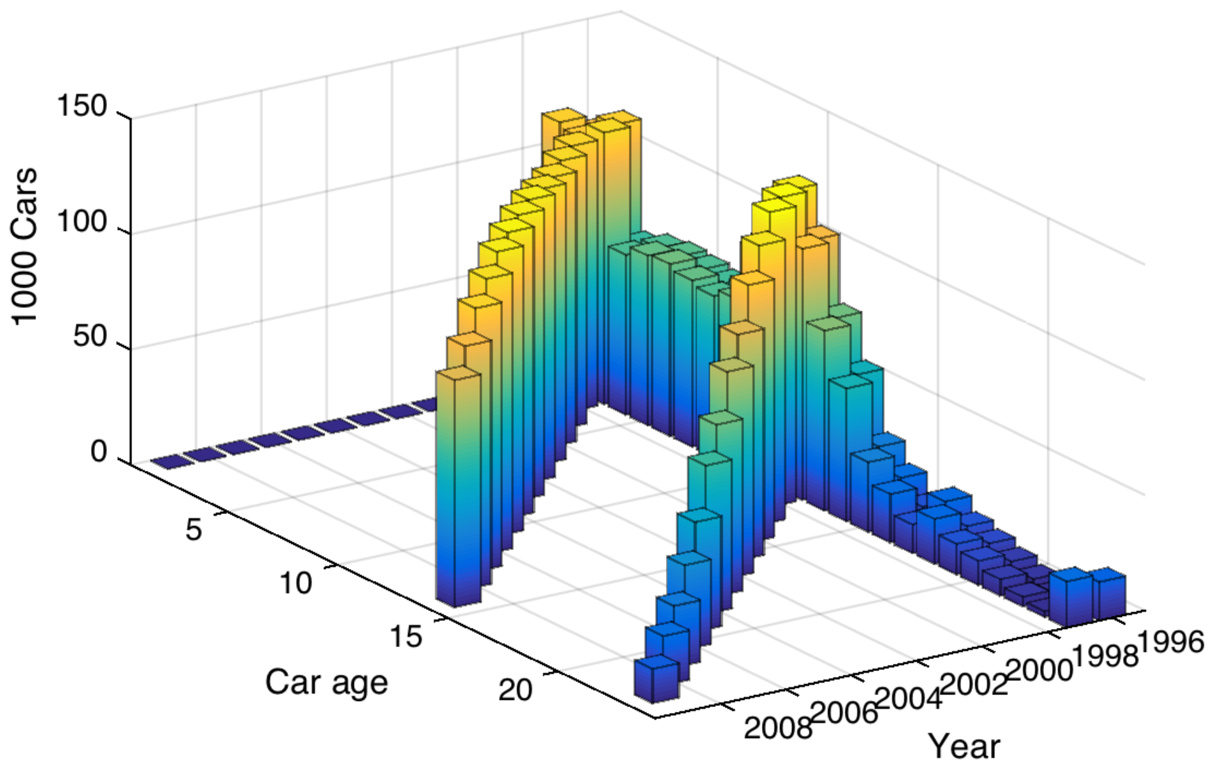

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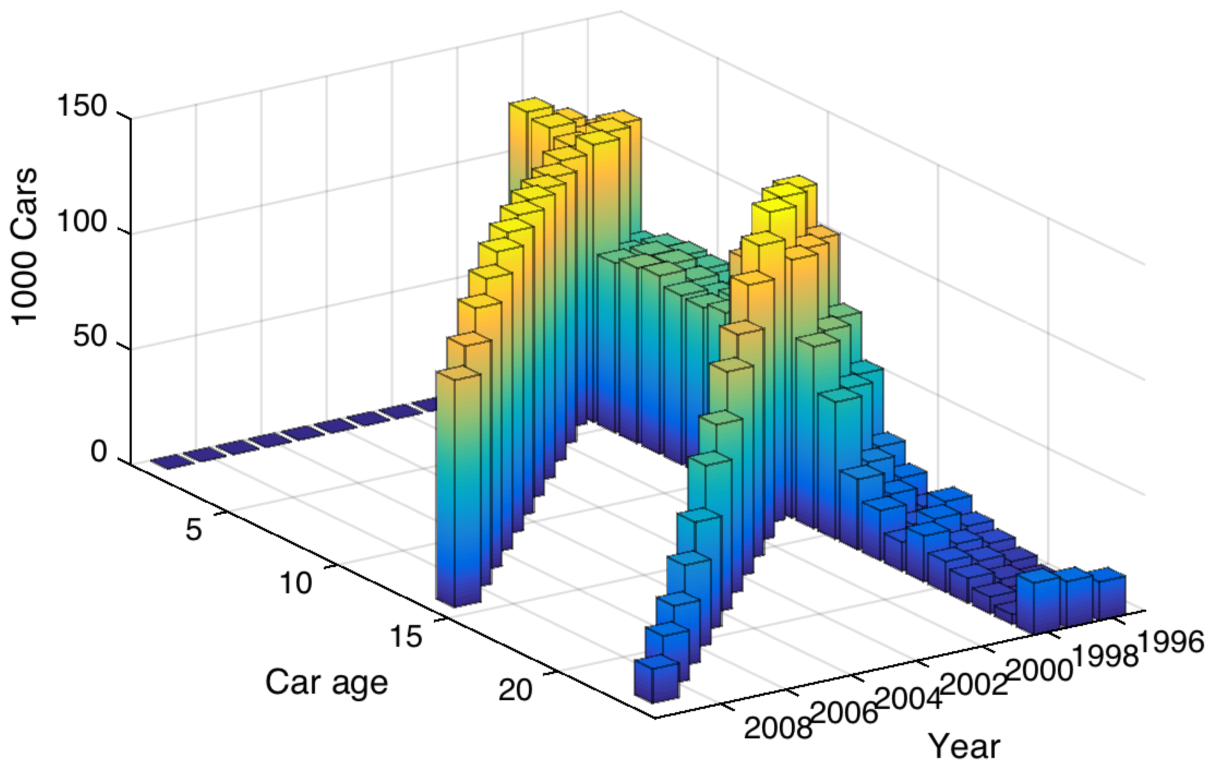

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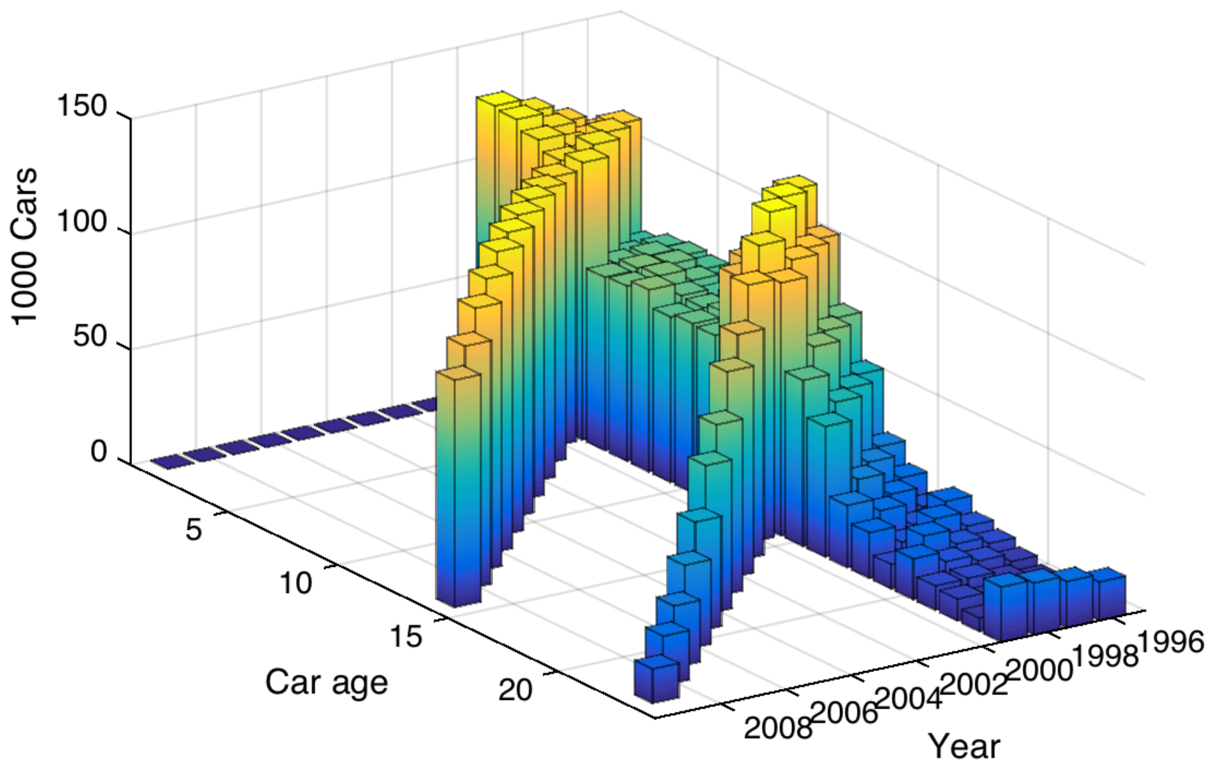

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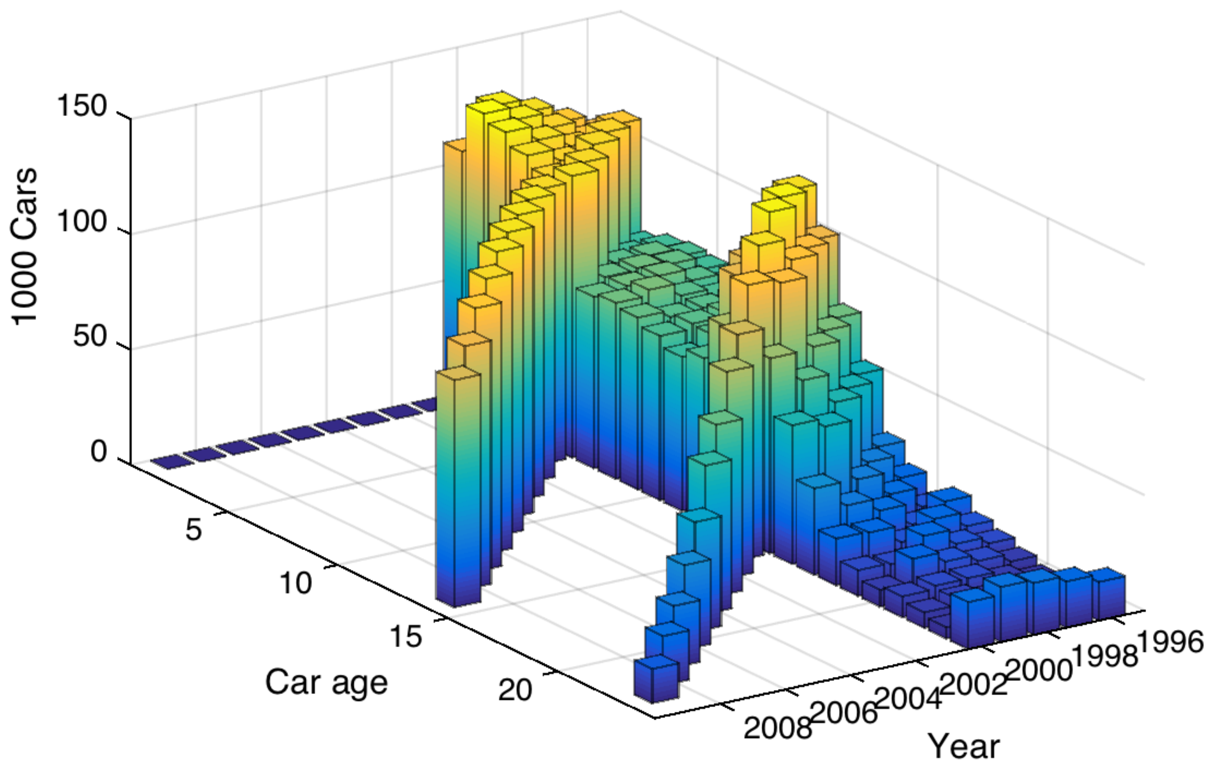

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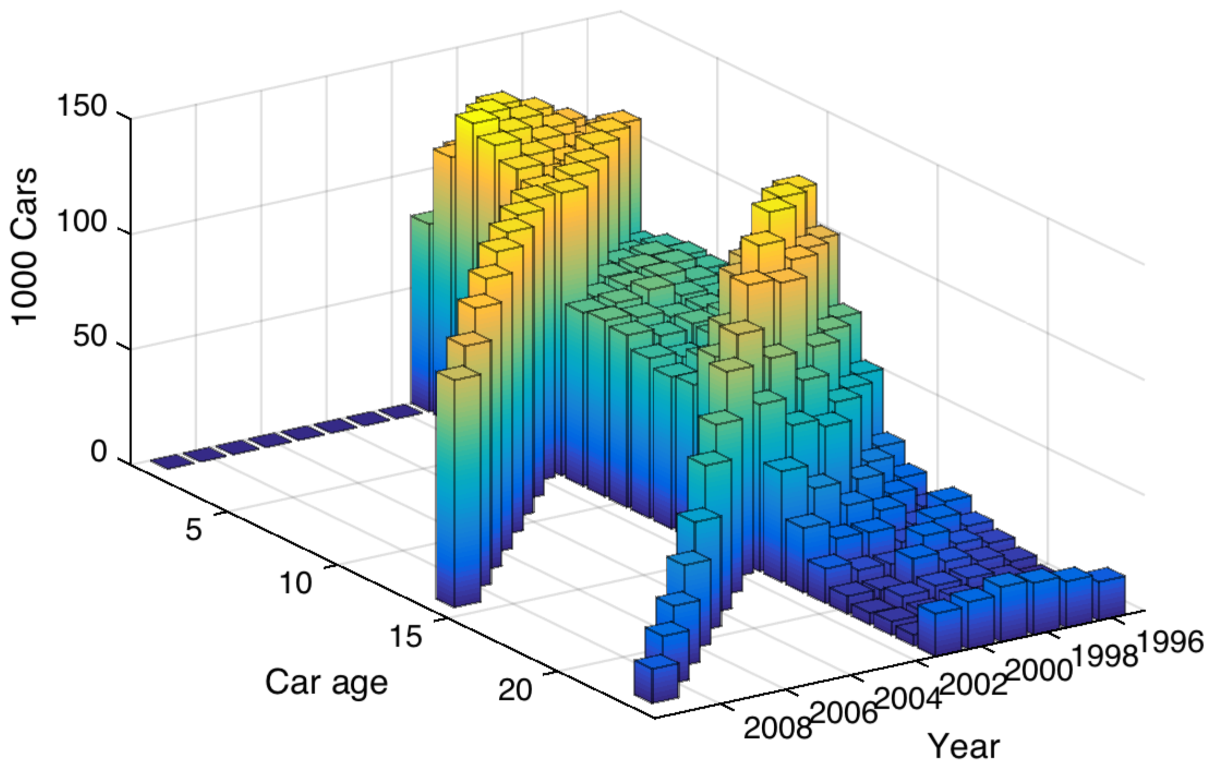

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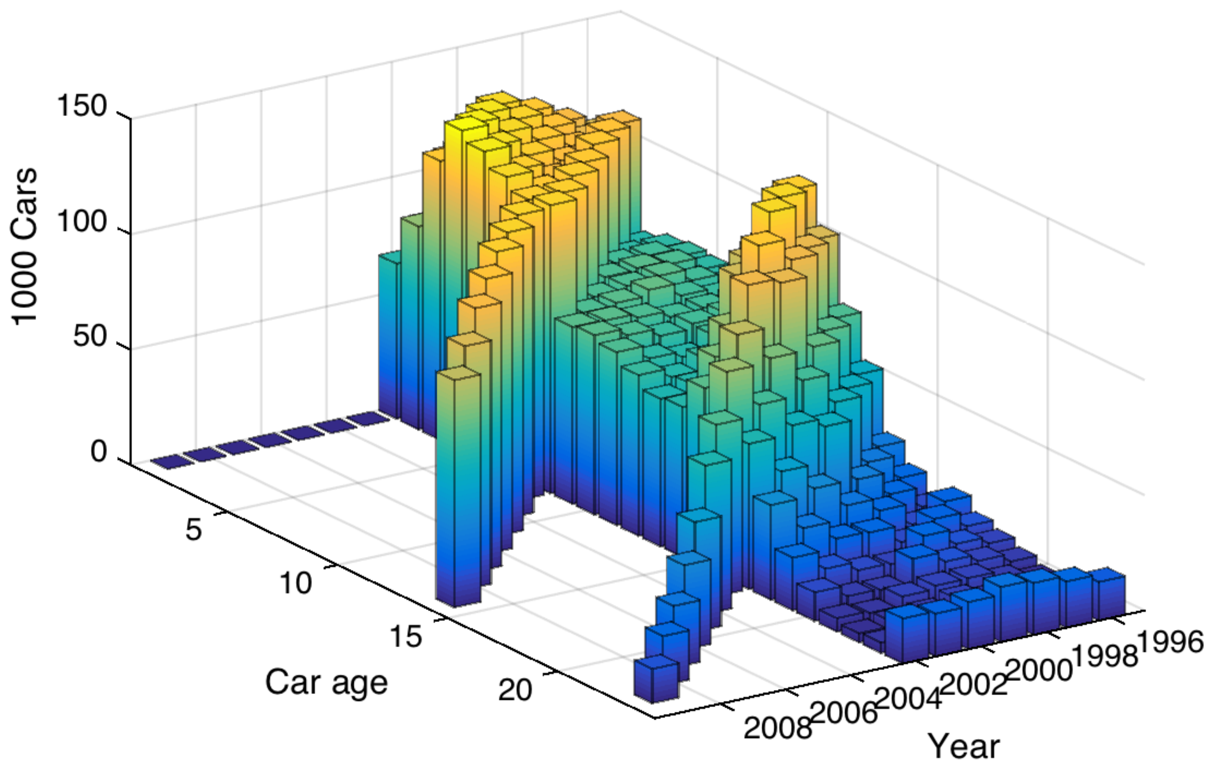

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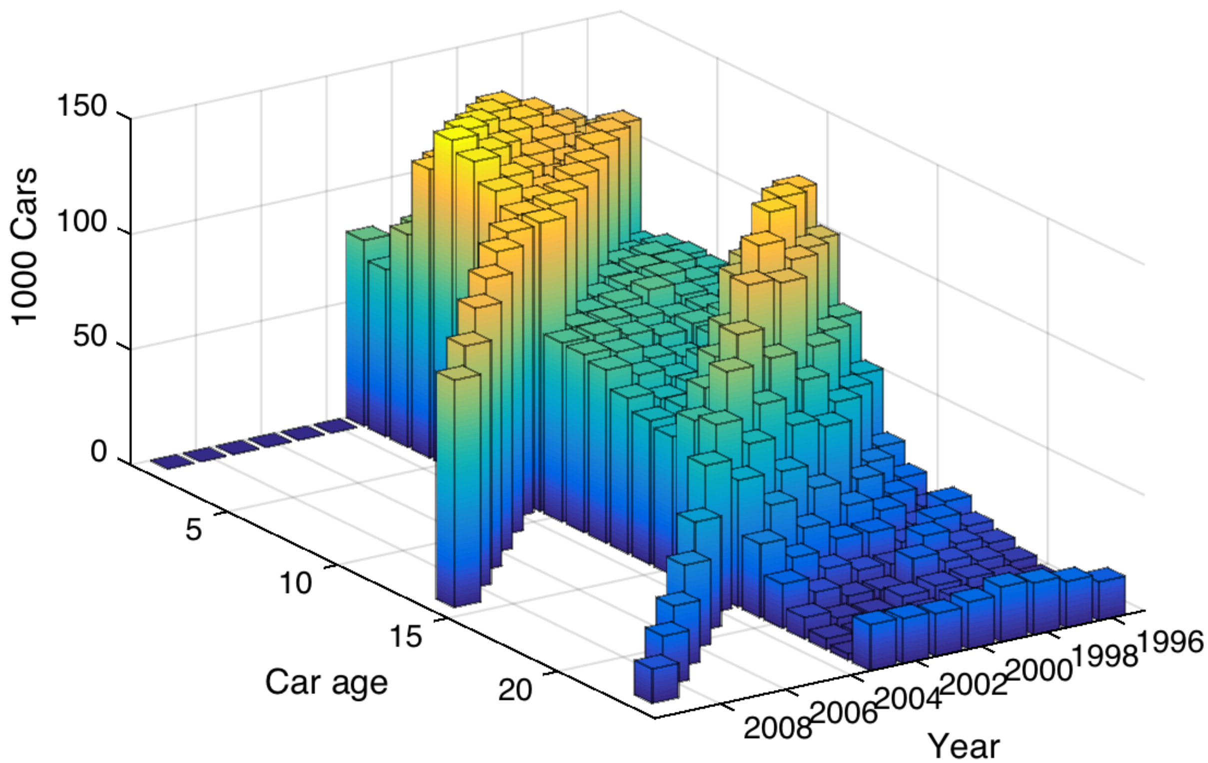

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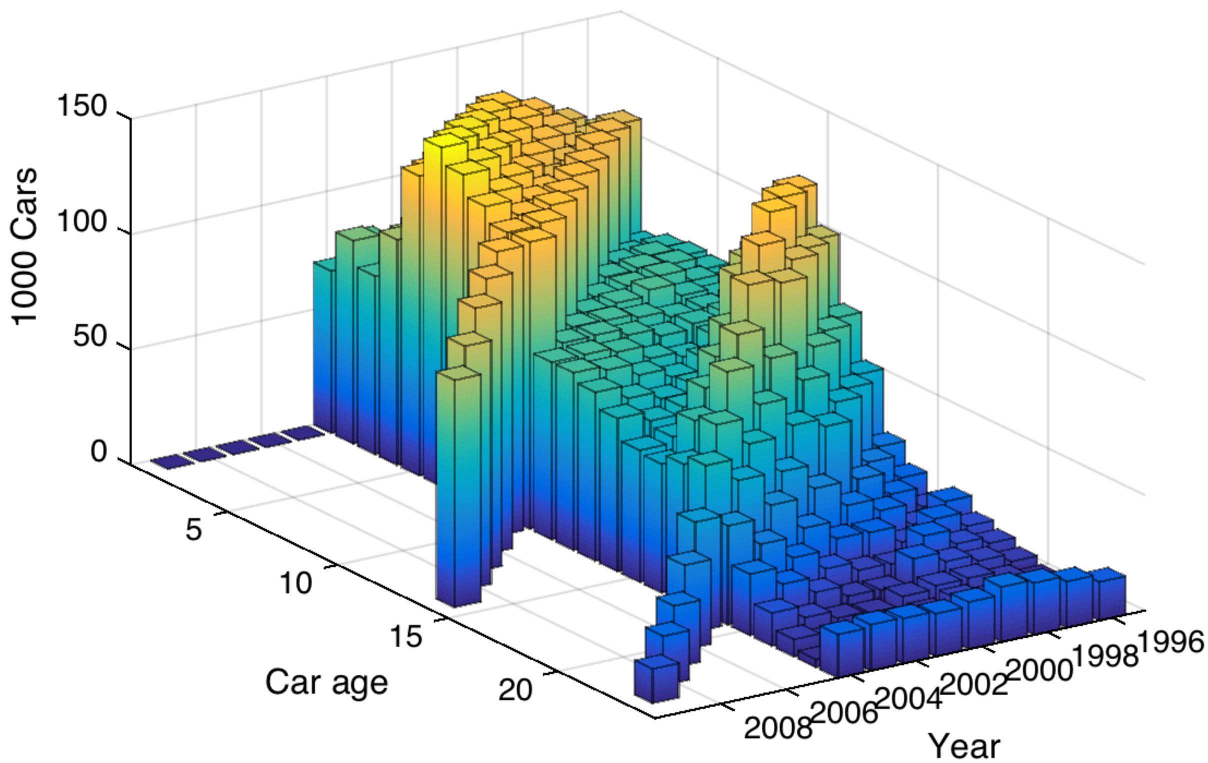

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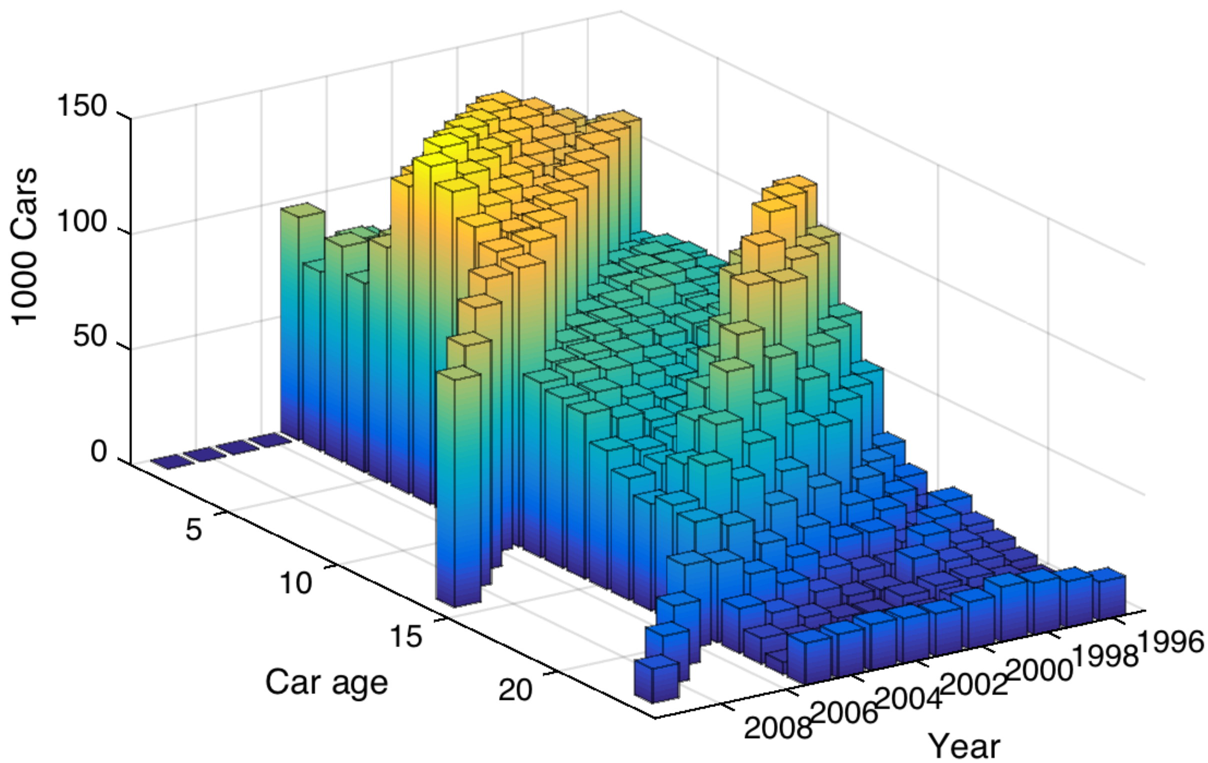

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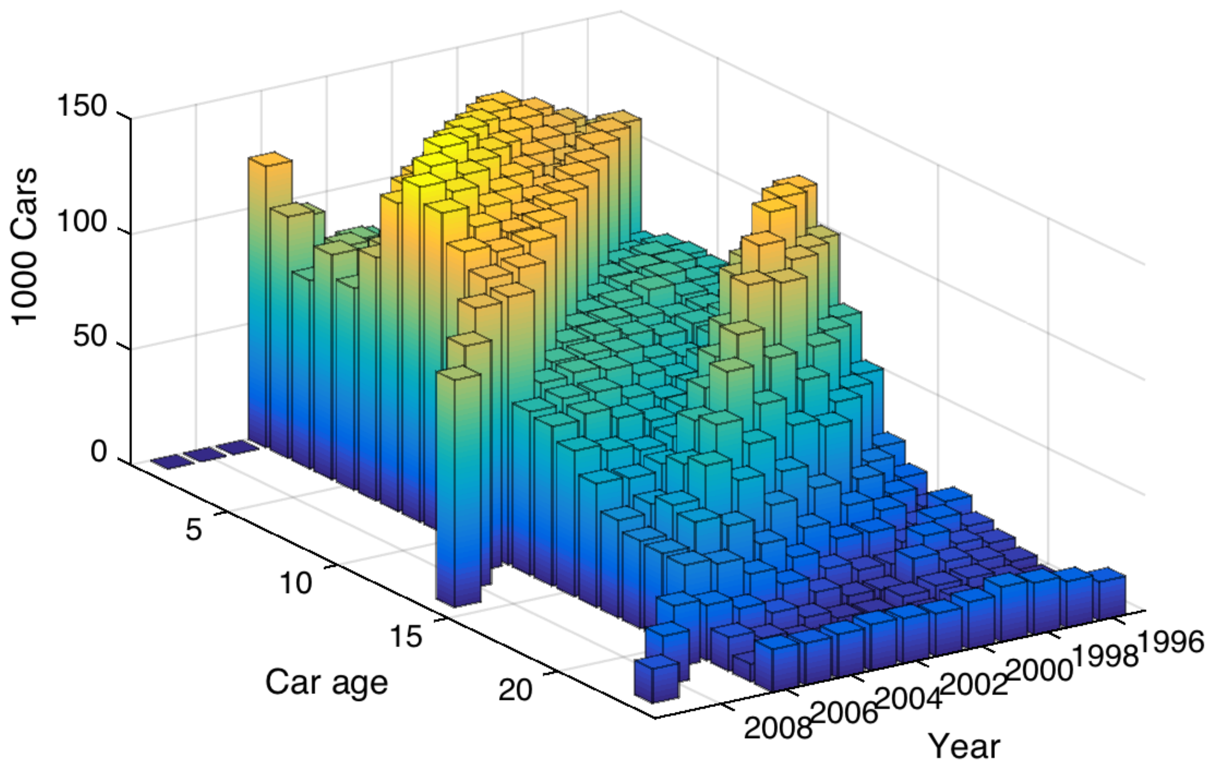

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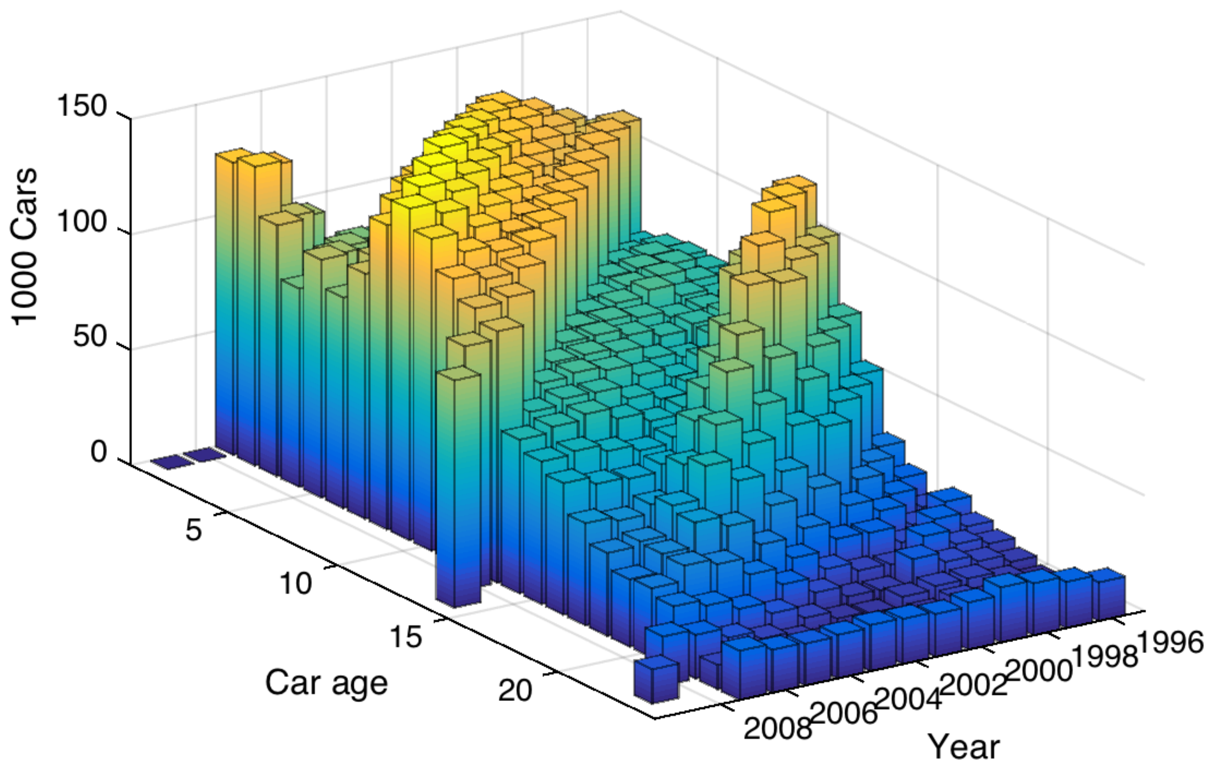

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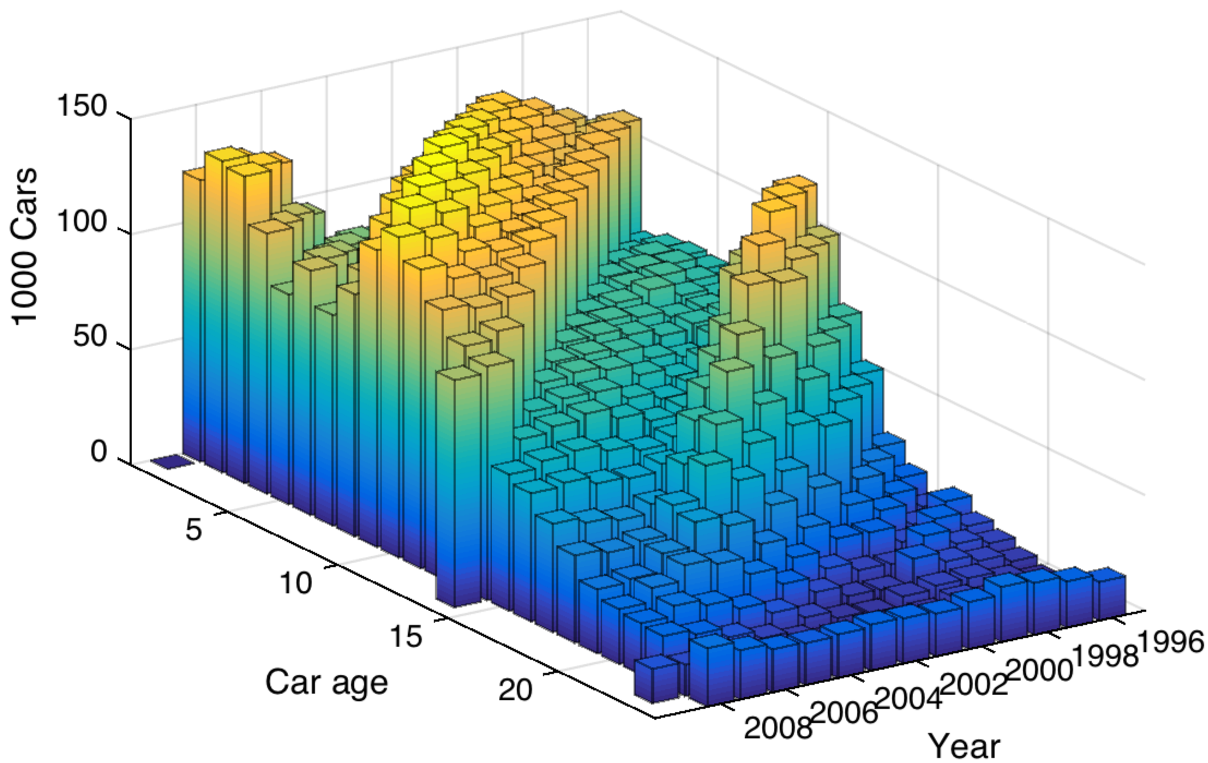

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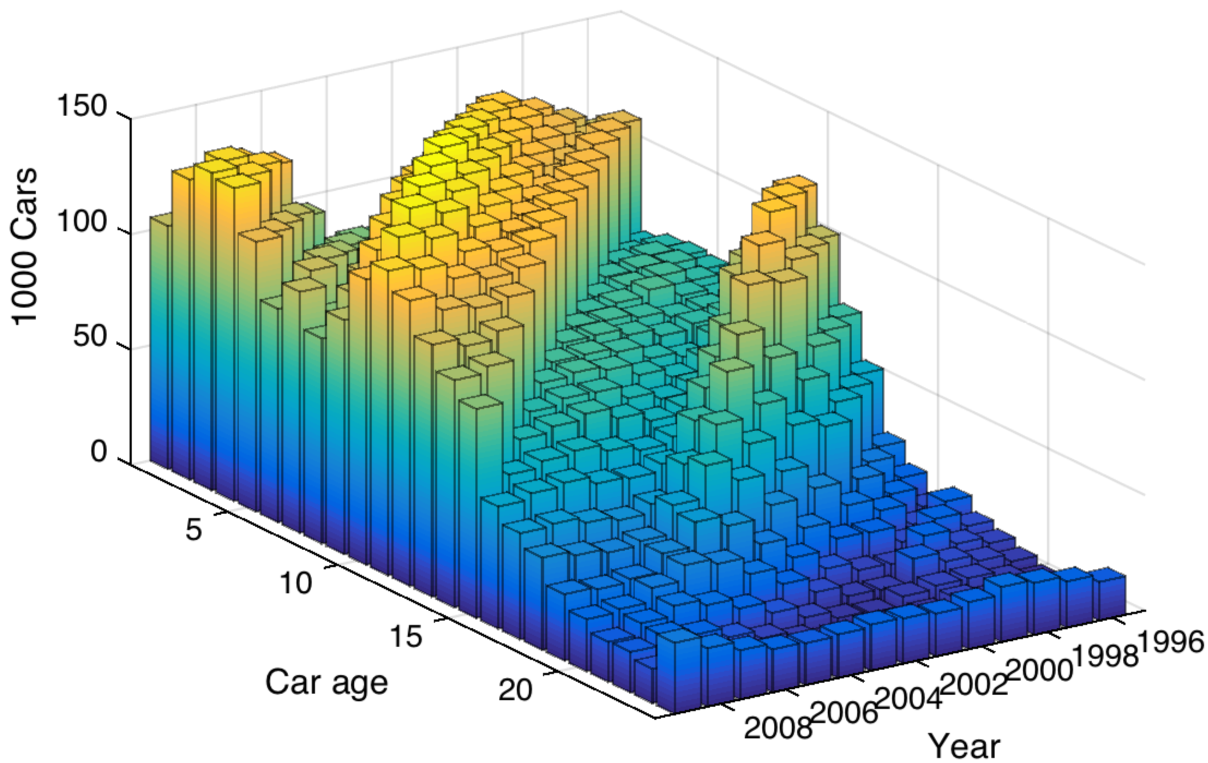
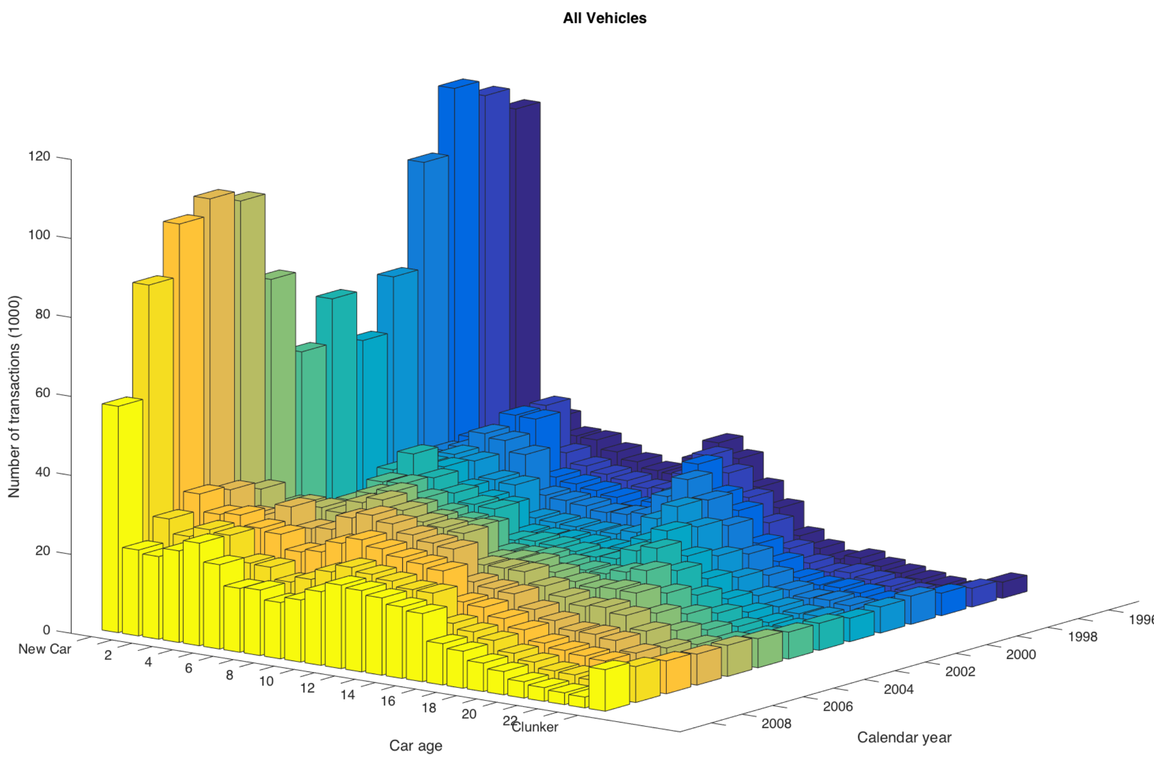


# Waves in car purchases 1995-2009



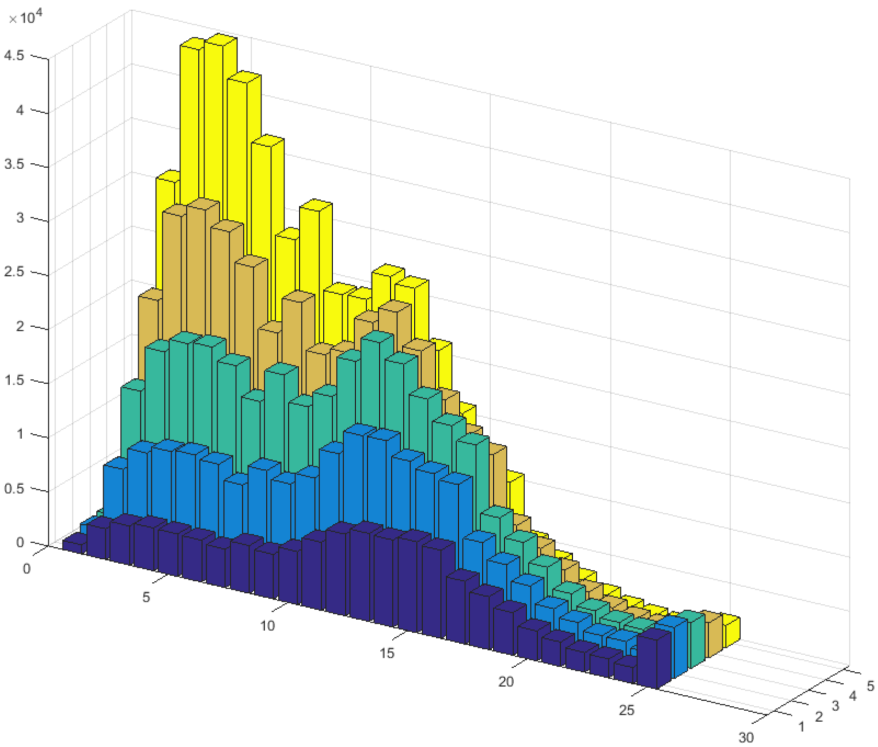
# Gains from trade between rich and poor consumers

Rich mans Volvo

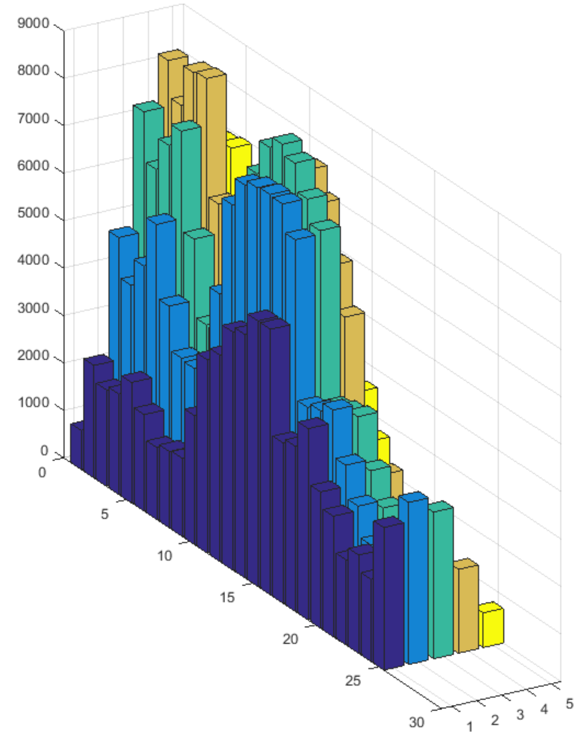

Poor mans Volvo

# Holdings by income quintiles in Denmark

## Heavy vehicles




## Light vehicles



# The challenge of modeling equilibrium in auto markets

1. *Dynamics are crucial:* When deciding to keep your current car, trade for a new one, or get rid of the car you have, consumer take into account that investments in cars are partly irreversible due to transactions cost, technical depreciation and increasing maintenance cost.
2. With dynamics come *expectations:* how much can I sell my current car for, and how much will it cost to buy another one? ... and what is my expected demand for driving?
3. Macro shocks are evidently important: do I want to buy a new car now if the economy is going into recession and I might lose my job?
4. There are many different makes/models and *ages* of cars, so finding an equilibrium is a *high dimensional* numerical problem

# Formulation of the model

**Goal:** Analyze reform that changes the balance between taxes on registration, ownership, fuel, $CO_2$ emissions and road use.

**We would like to develop a dynamic equilibrium model** that simultaneously tracks the following mechanisms

- Scrappage, replacement timing, new vs. used-car tradeoffs
- Type choice, fuel-efficiency and driving
- Trade/sorting of cars between heterogeneous consumers.
- Equilibrium price mechanism in the used-car market
- Business cycle variation in new/used car purchases,
- Life-cycle patterns driving and new/used car purchases

**and structurally estimate this model to do counterfactuals**

- Danish register data: Lots heterogeneity and rich dynamics.
- Simulate the counter-factual equilibrium and evaluate the effects of changing car taxation in Denmark

# The Art of Structural Modelling

Michael Keane's advice: **Start simple!**

> *This "start small" approach may seem obvious. But my experience is that most people try to program up their full-blown model right from the start. This is a big mistake, both from a programming point of view and with respect to developing economic intuition.*

You would think that this advice only applied to beginners.. but we learned the hard way that he was right

# Two papers

- Gillingham, Iskhakov, Munk-Nielsen, Rust, Schjerning
  A Dynamic Model of Vehicle Ownership, Type Choice, and Usage

Structural estimation of the equilibrium model of used car market with comprehensive counterfactual policy simulations of IRUC reform.

- Gillingham, Iskhakov, Munk-Nielsen, Rust, Schjerning
  Equilibrium Trade In Automobile Markets

Theory paper on modeling equilibrium on the used car market with consumer heterogeneity and trading costs.

# The "toy model"

**We develop a tractable computational dynamic equilibrium model** that simultaneously tracks a number of mechanisms:

- New and used cars of multiple types (e.g. makes/models) are traded by heterogeneous consumers,
- Prices and quantities are determined endogenously to equate supply and demand for all car types and vintages, along with the ages at which cars are scrapped,
- The model allows for transactions costs, taxes, flexible specifications of car characteristics, consumer preferences, and heterogeneity $\rightarrow$ sorting into particular cars in equilibrium,
- Competition between producers of news cars with comprehensive model of demand by forward looking consumers.

# Two numerical example applications

1. A revenue-neutral replacement of the new vehicle registration tax and with a higher fuel tax *(in this talk)*
2. A hypothetical "merger to monopoly" in an oligopolistic new car market *(please see the paper)*

We show **substantial consumer welfare gains** from the tax policy change, as well as important effects on government revenues, automobile prices and driving and $CO_2$ emissions.
We show **substantial welfare losses** from the hypothetical merger to monopoly.

# A brief history of equilibrium models of new/used cars

Manski (1982) and Berkovec (1985): developed the first equilibrium models for new and used cars in a static framework Bento, Goulder, Jacobsen and von Haefen (2009) is a recent example of this approach

- James A. Berkovec (1957-2009) "New Car Sales and Used Car Stocks: A Model of the Automobile Market" *RAND* 1985
- Estimated a discrete choice demand model using the *National Transportation Survey* micro data on 1095 households in 1978
- Using estimated demands for cars, solved for equilibrium in the new and used car market, over 131 type/age classes, with 13 types and 10 ages (vintages) from 1969 to 1978
- Used a quasi-Newton method to find the price vector that sets (one period) excess demand to zero, $E(P^*) = 0$

# A brief history of equilibrium models of new/used cars

Rust (1985a and 1985b): developed dynamic equilibrium model
Our papers build on this approach.

- Unit mass of homogeneous consumers,
- Stationary flow equilibrium,
- No transactions costs,
- Continuous quality of goods (odometer reading).

- Rust showed that when *transactions costs are zero,* consumers trade every period for an optimal car, reducing the dynamic problem to a static one

# A brief history of equilibrium models of new/used cars

Konishi and Sandfort (2002): generalized Rust's analysis to allow for positive transaction costs and proved the existence of equilibrium, allowing also for multiple makes/models of cars

Stolyarov (2002): developed a dynamic equilibrium model with *random transactions costs*

Gavazza *et. al.* (2014): used Stolyarov's approach to numerically calculate equilibrium and analyze the impact of varying levels of transaction costs on trade in used cars

Esteban and Shum (2007): dynamic oligopoly model of a differentiated product market with equilibrium production dynamics due to durability of the goods and their active trade in secondary markets

Yurko (2012): heterogeneous agents with multiple vehicle ownership, parametric price function calibrated to minimize excess demand

# Supply side — new vehicles

Bresnahan (1981), Berry, Levinsohn, and Pakes (1995), Goldberg (1995), Petrin (2002): Product differentiation and consumer choice of new vehicles

- General patterns of substitution across differentiated products
- Static models of consumer demand and a Bertrand oligopoly model for automobile supply.
- No *secondary markets* or *dynamics* of consumer decisions

Gillingham (2012), Munk-Nielsen (2015): Type choice and use

- Two period models that take into account consumer's expectation about future use of the product
- Substitution between car types in the new car market
- Rich on type choice, but no secondaty market

# Modeling equilibrium trade in automobile markets

We begin with **Rust (1985)** model for discrete goods:

1. Transaction costs are zero
2. No outside option
3. Continuum of homogeneous consumers

Afterwards we incrementally extend Rust's model to include:

- Outside option, transactions cost, consumer heterogeneity
- Time-invariant (and time-varying) consumer heterogeneity
- Multiple types of cars
- Driving and demand for gasoline
- Endogenous new car prices and oligopolistic competition between multi-product car producers

# Starting point: Homogeneous consumer economy

- Unit mass of *homogeneous* consumers who live forever
- Cars vary only by age $a \in \{0, 1, \ldots, A\}$
- Cars must be scrapped at age $A$
- Consumers can
  1. *trade* their current car for a car of age $d^* \in \{0, 1, \ldots, \bar{a} - 1\}$
  2. *keep* their current car, $(d = \kappa)$ (if $a < A$)
  3. *no out side option*, consumers must own a car every period
- Car utility is a decreasing function $u(a)$ that reflects
  1. decreasing utility of car services
  2. increasing cost of maintenance

# The Bellman equation

The Bellman equation for the owner of car of age $a \in \{1, \dots, A-1\}$

$$\begin{aligned} V(a) &= \max\Big\{ v(a,\kappa); \max_{d \in \{0,\dots,A-1\}} v(a,d) \Big\}, \\ v(a,\kappa) &= u(a) + \beta\big(1-\alpha(a)\big)V(a+1) + \beta\alpha(a)V(A), \\ v(a,d) &= u(d) - \mu\big[P(d) - P(a)\big] \\ &\quad + \beta\big(1-\alpha(d)\big)V(d+1) + \beta\alpha(d)V(A) \end{aligned}$$

$\alpha(a)$ is accident probability
When $a = A$, keep choice is not available $\rightarrow$ for $V(A)$ the first component under max in the right hand side disappears

# Timing of events

1. Consumer enter the period with a car of age $a$
2. Discrete trading/keeping choice, $d \in \{\o, \kappa, 0, 1, \dots, A-1\}$ (immediately at the start of the period)
3. The chosen car $d$ is then utilized during the period, but can be involved in the accident with probability $\alpha(d)$ (or $\alpha(a)$ if consumer choose to keep it's car of age $a$).
4. By the start of the next period:
   - with probability $1 - \alpha(d)$ car has aged $a = d + 1$
   - with probability $\alpha(d)$ car reach scrapage state $a = A$.
5. Hence, it is impossible to start the period with a brand new car $\rightarrow$ the state variable $a$ can only take values from $\{1, \dots, A\}$.

# Implication of Zero Transactions Cost

**Theorem (Consumer choice in absence of transaction cost)**

*Suppose the price of the new car is given by $P(0) = \overline{P}$ and $P(a) = \underline{P}$ for $a \geq \bar{a}$ for some $\bar{a} \leq A$. With no transaction costs, the optimal policy $a^\star(a)$ for $a \in \{1, \dots, A\}$ does not depend on $a$, i.e. $a^\star(a) = a^\star$, and provided that $a^\star \in \{0, \dots, A-2\}$, it holds:*

$$\begin{aligned} V(a^\star) &= \frac{u(a^\star)}{1-\beta} - \frac{\mu\beta}{1-\beta}\Big[P(a^\star) - \big(1-\alpha(a^\star)\big)P(a^\star+1) - \alpha(a^\star)\underline{P}\Big] \\ V(a) &= V(a^\star) - \mu\big[P(a^\star) - P(a)\big] \end{aligned}$$

- With no transaction cost it is always optimal to trade for the preferred car age $d^*$
- Value of preferred car is the discounted sum of utility net of replacement cost

## Consumers must be indifferent in equilibrium

- Otherwise, either consumers are not following *optimal trading strategy* $a^\star(a)$, or the holding distribution collapses to a degenerate distribution after one period of trading, and can not be invariant w.r.t. *car aging transitions* $Q$
- Let $\bar{a}$ be the equilibrium scappage age, such that prices satisfy: $P(0) = \overline{P}$, $P(a) = \underline{P}$ for $a \geq \bar{a}$, and $\underline{P} \leq P(a) \leq \overline{P}$
- Indifference $\rightarrow$ $V(d) = V(d+1)$ for $d \in \{0, 1, \ldots, \bar{a} - 1\}$
- We have a system of $\bar{a} - 1$ linear equations in the $\bar{a} - 1$ unknowns $P(1), \ldots, P(\bar{a} - 1)$

$$X \cdot P = Y$$

where $P$ is the column vector of prices $P(1), \ldots, P(\bar{a} - 1)$, and $X$ and $Y$ are known $\bar{a} - 1 \times \bar{a} - 1$ matrix and $\bar{a} - 1 \times 1$ vector

# Equilibrium linear system

$$X = \begin{bmatrix} -\mu(1+\beta) & \mu\beta & 0 & \cdots & 0 & 0 \\ \mu & -\mu(1+\beta) & \mu\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \mu & -\mu(1+\beta) & \mu\beta \\ 0 & 0 & \cdots & 0 & \mu & -\mu(1+\beta) \end{bmatrix},$$

$$Y = \begin{bmatrix} u(0) - u(1) - \mu\overline{P} \\ u(1) - u(2) \\ \cdots \\ u(\bar{a}-3) - u(\bar{a}-2) \\ u(\bar{a}-2) - u(\bar{a}-1) - \mu\beta\underline{P} \end{bmatrix}.$$

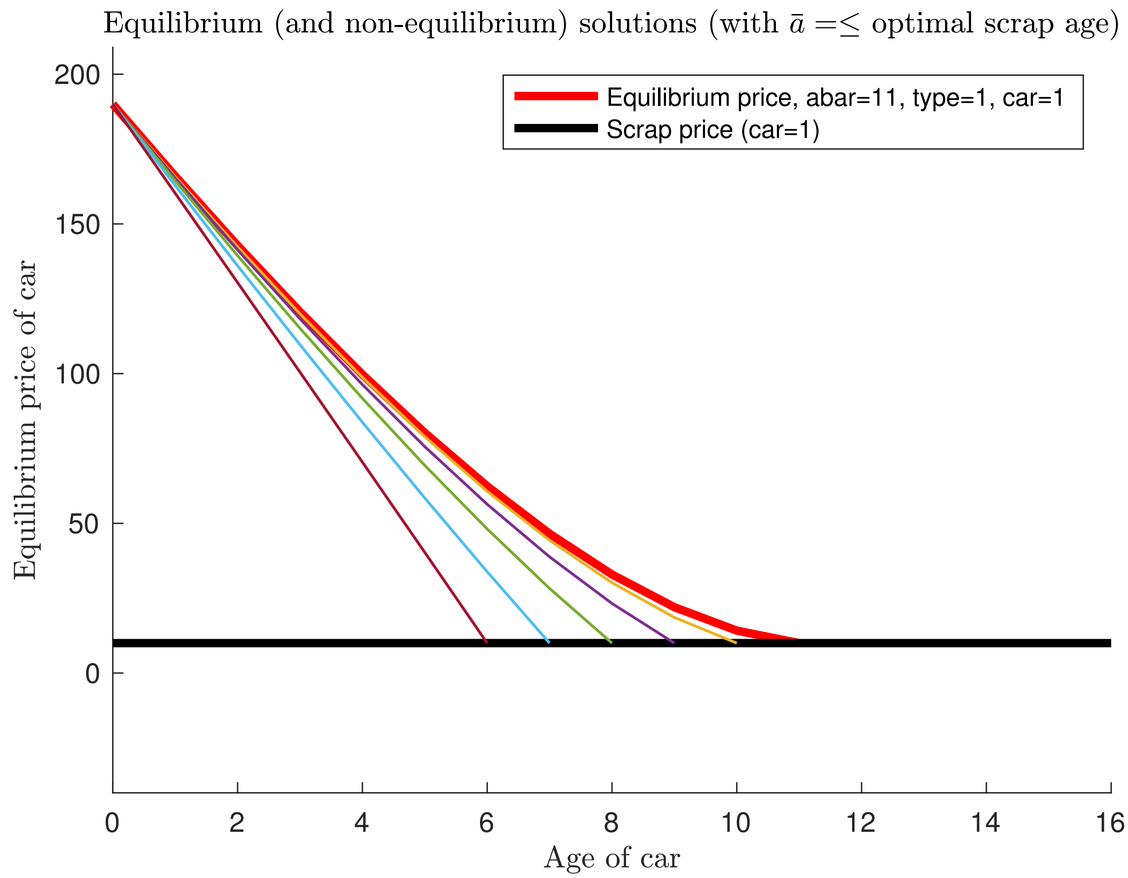
(written here for the case with no accidents, $\alpha(a) = 0$)

# Shape of equilibrium price function

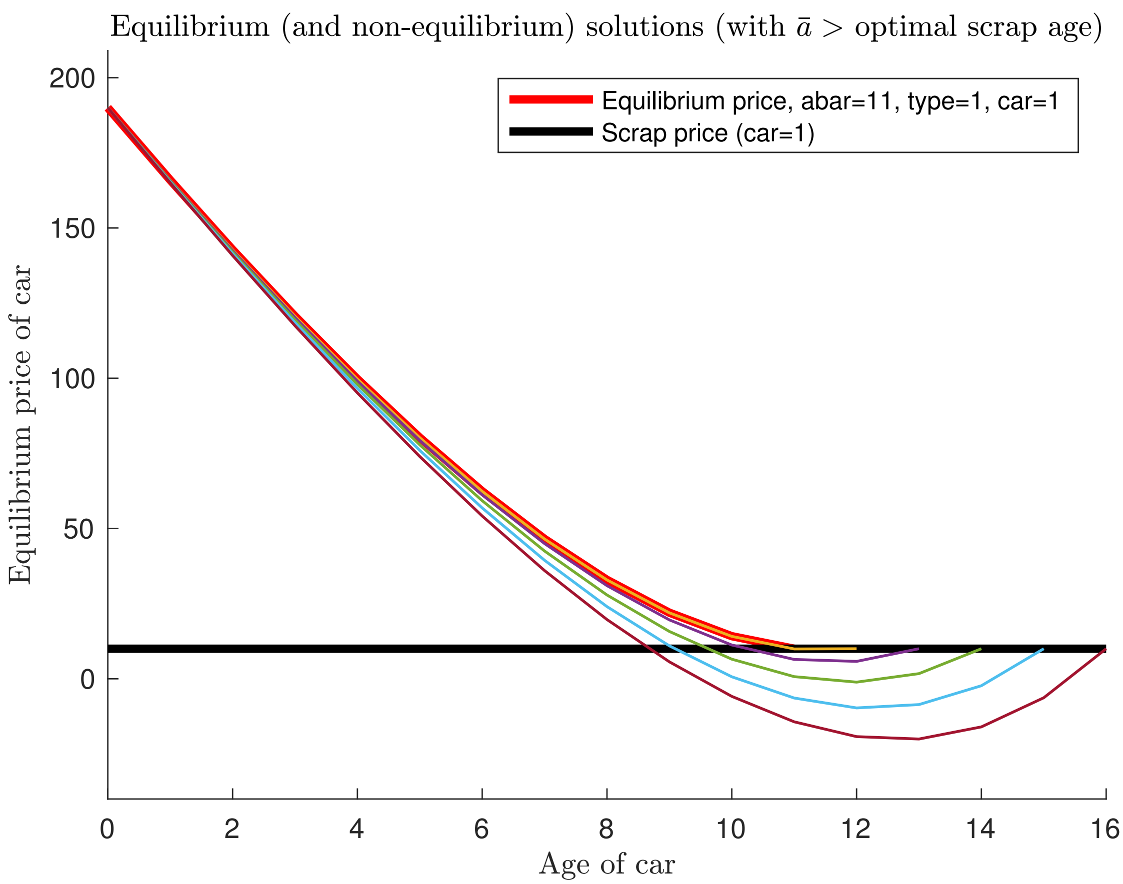
Note:

- The scrappage age $\bar{a}$ is a parameter in the linear system
- The system can typically be solved for many different $\bar{a}$
- There is no requirement of monotonicity of the solution $P(a)$ in $a$, nor is there a requirement $\underline{P} \leq P(a) \leq \overline{P}$
- Thus, main conditions of the Equilibrium definition are satisfied, but we can use the latter to find the "correct" equilibrium price function
- *We prove in the paper that this approach finds the welfare maximizing equilibrium which is the solution to the social planner's optimal replacement problem, similar to Rust (1985)*

# Equilibrium price functions for $\bar{a}$ too small



# Equilibrium price functions for $\bar{a}$ too large



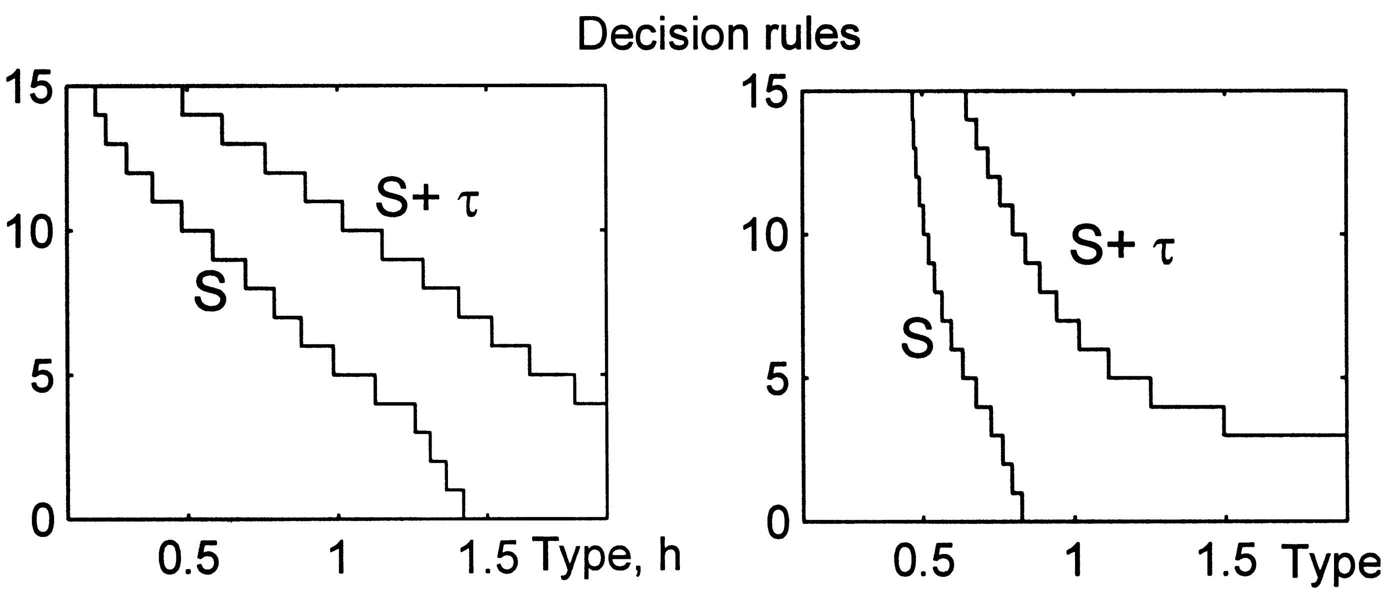
# Heterogenous consumer economy with transactions costs

- Unit mass of *idiosyncratically heterogeneous* consumers
- Discrete product market
- **Extreme value** specification of **time-varying consumer heterogeneity** that results in a logit specification for choice probabilities
- Consumer's choices and timing exactly as before, except the assumption that cars must be scrapped at endogenously determined $\bar{a}$

Now adding to the Rust (1985) setup

1. Transaction costs $T(a, d, P) > 0$
2. Outside good: not owning a car $a = \emptyset$

# The difficulty of transactions costs




Stolyarov, *JPE* (2002) uses random transaction cost
$S$ is optimal purchase age, $\tau$ is optimal holding time

# Consumer's trading problem

- Add extreme value preference shocks → have to redefine consumer problem with **new state space**

$$V(a,\epsilon) = \max\left[v(\kappa,a)+\epsilon(\kappa), \max_{d\in\{\phi,0,1,\dots,\bar{a}-1\}}[v(d,a)+\epsilon(d)]\right]$$

where $v(d,a)$ is the value of trading the current car of age $a \in \{1,2,\dots,\bar{a}\}$ for replacement car car $d \in \{0,1,\dots,\bar{a}-1\}$ or for outside good $d=\phi$

$$V(\phi,\epsilon) = \max\left[v(\phi,\phi)+\epsilon(\phi), \max_{d\in\{0,1,\dots,\bar{a}-1\}}[v(d,\phi)+\epsilon(d)]\right]$$

$$V(\bar{a},\epsilon) = \max_{d\in\{0,1,\dots,\bar{a}-1\}}[v(d,\phi)+\epsilon(d)]$$

# Choice specific value functions

1. Value for keeping a car is

$$\begin{aligned} v(\kappa, a) &= u(a) + \beta\Big[\big(1-\alpha(a)\big)EV(a+1) + \alpha(a)EV(\bar{a})\Big] \\ v(\emptyset, \emptyset) &= u(\emptyset) + \beta EV(\emptyset) \end{aligned}$$

2. Value of trading a car of age $a$ for a car of age $d$ is

$$\begin{aligned} v(d, a) &= u(d) - \mu[P(d) - P(a) + T(P, a, d)] + \\ &+ \beta\Big[\big(1-\alpha(a)\big)EV(d+1) + \alpha(d)EV(\bar{a})\Big] \\ v(d, \emptyset) &= u(d) - \mu[P(d) + T(P, \emptyset, d)] + \\ &+ \beta\left[\big(1-\alpha(a)\big)EV(d+1) + \alpha(d)EV(\bar{a})\right] \end{aligned}$$

# Solving the consumer's trading problem

$$EV(a) = \int_{\epsilon} V(a,\epsilon) f(\epsilon) d\epsilon = \sigma \log\left( \sum_{d \in D(a)} \exp \frac{v(a,d)}{\sigma} \right)$$

$$EV(a) = \Gamma\big(EV(a), P\big)$$

The *EV* vector is a fixed point to a contraction mapping, which can be calculated by the method of successive approximations, though we actually use **Newton's method** (policy iteration) to speed up the numerical solution to this system.

# Implied choice probabilities

Solution to DP problem $(EV(1), \ldots, EV(\bar{a})) \rightarrow$
Choice specific values $v(d, a)$, $d \in D(a) \rightarrow$
Multinomial choice probabilities for $a \in \{\emptyset, 1, \ldots, \bar{a}\}$

$$\Pi(d|a) = \frac{\exp\left\{v(d, a)/\sigma\right\}}{\exp\left\{v(\kappa, a)/\sigma\right\} + \sum_{d \in D(a)} \exp\left\{v(d', a)/\sigma\right\}}$$

$$D(a) = \begin{cases} \{\kappa, 0, 1, \ldots, \bar{a} - 1, \emptyset\}, & a \in \{1, \ldots, \bar{a} - 1\}, \\ \{0, 1, \ldots, \bar{a} - 1, \emptyset\}, & a \in \{\bar{a}, \emptyset\}, \end{cases}$$

# Defining the heterogeneous agent equilibrium

1. Infinitely elastic supply of new cars at price $\overline{P}$
2. Infinitely elastic demand for scrap cars at price $\underline{P}$
3. $\bar{a} - 1$ markets of used cars traded at prices $P(1), \ldots, P(\bar{a} - 1)$
4. Finding an equilibrium requires computing prices to equalize **expected** demand and supply

$$\begin{aligned} \sum_{a=1}^{\bar{a}} \Pi(1|a, P) &= 1 - \Pi(\kappa|1, P) \\ &\vdots \\ \sum_{a=1}^{\bar{a}} \Pi(\bar{a} - 1|a, P) &= 1 - \Pi(\kappa|\bar{a} - 1, P) \end{aligned}$$

# Finding an equilibrium following Berkovec's approach

1. Thus the equilibrium condition is a system of $\bar{a}-1$ smooth nonlinear equations in $\bar{a}-1$ unknowns, $P(a)$, $a \in \{1,\ldots,\bar{a}\}$ which we can expect to have at least one solution under fairly weak restrictions.
2. Write it in a more usual form as a set of prices that sets excess demand in each of the $\bar{a}-1$ second hand markets to zero

$$ED(P) = 0$$

3. Since $E$ is differentiable, we can use Newton's method to find a solution, using the iteration

$$P_{t+1} = P_t - [\nabla ED(P_t)]^{-1} ED(P_t)$$

where $\nabla ED(P_t)$ is the $(\bar{a}-1)\times(\bar{a}-1)$ Jacobian matrix

# Holdings distribution

- Let $0 \leq q(a) \leq 1$ denote the fraction of cars of age $a$ in the economy, $a \in \{1, \ldots, \bar{a}\}$
- **Holdings distribution** $q \in \mathbb{R}^{\bar{a}}$ represents the age distribution of the car fleet
- $q$ changes from period to period due to deterministic aging of cars and stochastic accidents
- $q$ is measured in the beginning of the period before trading, therefore according to timing convention does not have new cars, but has cars of maximum age $\bar{a}$

# Physical transition probability matrix

We can represent the aging of cars in the presence of stochastic age-dependent accidents via an $\bar{a} \times \bar{a}$ transition probability matrix

$$Q = \begin{bmatrix} 0 & 1-\alpha(1) & 0 & \cdots & 0 & \alpha(1) \\ 0 & 0 & 1-\alpha(2) & \cdots & 0 & \alpha(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\alpha(\bar{a}-2) & \alpha(\bar{a}-2) \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1-\alpha(0) & 0 & 0 & \cdots & 0 & \alpha(0) \end{bmatrix}$$

Last row is required for the flow equilibrium, with more conditions to follow to ensure all entering new cars are bought on the market

# The stationary holdings distribution

- In stationary (flow) equilibrium the holdings distribution must be invariant distribution implied by the transition probability matrix $Q$

$$q = q \cdot Q$$

- Without accidents, $Q$ is periodic, and the only invariant distribution is uniform
- With accidents, $Q$ generally has a single invariant distribution, with decreasing by age fractions of cars

# Stationary equilibrium on automobile market

Stationary equilibrium on the market for new and used cars is given by the tuple $\left\{\bar{a}, P(a), q(P, \bar{a})\right\}$ such that:

1. The price function $P(a)$ satisfies the infinite elasticity assumption, so it holds $\underline{P} \leq P(a) \leq \overline{P} = P(0)$ for all $a \in \{1, \ldots, \bar{a} - 1\}$, and $P(a) = \underline{P}$ for all $a \in \{\bar{a}, \ldots, \bar{a}\}$;
2. The stationary holding distribution $q(P, \bar{a})$ is a fixed point of $Q$, so it holds $q(P, \bar{a}) = q(P, \bar{a}) \cdot Q$;
3. Consumers follow their optimal trading strategies;
4. The market clearing conditions $ED(P) = (0, \ldots, 0)$ are satisfied for cars of ages between 1 and $\bar{a} - 1$, and $D(\o, P) = S(\o, P)$, and the demand for new cars equals the supply of scrapped cars $D(0, P) = S(\bar{a}, P)$.

# Stationary ownership distribution

- Without the outside option, $q$ is both age distribution of car fleet *and* distribution of consumers who own cars of different ages
- With outside option, there is an additional element $q(\not{o})$ that reflects the fraction of consumers who do not own a car
- Let **ownership distribution** $p$ be given by

$$p = \big(q[1 - q(\not{o})], q(\not{o})\big) \in \mathbb{R}^{\bar{a}+1}$$

- As holdings distribution $q$, $p$ is measured in the beginning of the period before the trading phase
- In stationary the equilibrium both $q$ and $p$ are time invariant

# Dynamics of ownership distribution

- Dynamics of the ownership distribution is governed by its own block-diagonal physical transition probability matrix

$$\Omega = \begin{bmatrix} & & & 0 \\ & Q & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

- Changes in ownership are reflected in one more Markov transition probability matrix we refer to as **trade transition matrix**

$$\Delta(P) = \Delta K(P) + \Delta T(P)$$

- Keep probabilities $\rightarrow$ **keep** transition matrix $\Delta K(P)$
- Other choice probabilities $\rightarrow$ **trade** transition matrix $\Delta T(P)$

# Trading transition matrix

$$\Delta T(P) =$$

$$\begin{bmatrix} \Pi(1|1,P) & \Pi(2|1,P) & \dots & \Pi(\bar{a}-1|1,P) & \Pi(0|1,P) & \Pi(\emptyset|1,P) \\ \Pi(1|2,P) & \Pi(2|2,P) & \dots & \Pi(\bar{a}-1|2,P) & \Pi(0|2,P) & \Pi(\emptyset|2,P) \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \Pi(1|\bar{a},P) & \Pi(2|\bar{a},P) & \dots & \Pi(\bar{a}-1|\bar{a},P) & \Pi(0|\bar{a},P) & \Pi(\emptyset|\bar{a},P) \\ \Pi(1|\emptyset,P) & \Pi(2|\emptyset,P) & \dots & \Pi(\bar{a}-1|\emptyset,P) & \Pi(0|\emptyset,P) & 0 \end{bmatrix}$$

Note next to last column that reflects the flow equilibrium structure

# Keeping transition matrix

$$\Delta K(P) =$$

$$\begin{bmatrix} \Pi(\kappa|1,P) & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \Pi(\kappa|\bar{a}-1,P) & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & \Pi(\emptyset|\emptyset,P) \end{bmatrix}$$

Note next to last column of zeros that reflects the restriction of keeping choice

# Market clearing conditions

$$\begin{array}{rcl} p \cdot \Delta T(P) & = & \left( \begin{array}{ccc} D(P), & D(0,P), & D(\o,P) \end{array} \right) \\ p \cdot (I - \Delta K(P)) & = & \left( \begin{array}{ccc} S(P), & S(\bar{a},P), & S(\o,P) \end{array} \right) \\ p \cdot (\Delta(P) - I) & = & \left( \begin{array}{ccc} ED(P), & (F), & (N) \end{array} \right) \end{array}$$

- The equilibrium flow condition $(F) = D(0,P) - S(\bar{a},P)$
- Stationarity in outside good share $(N) = D(\o,P) - S(\o,P)$

# Stationary equilibrium with idiosyncratic heterogeneity

**Theorem (Existence of equilibrium with idiosyncratic heterogeneity)**

*The stationary equilibrium for the automobile economy with the idiosyncratically heterogeneous consumers exists, and in equilibrium it holds*

$$p(P, \bar{a}) = p(P, \bar{a}) \cdot \Delta(P) = p(P, \bar{a}) \cdot \Omega = p(P, \bar{a}) \cdot \Delta(P)\Omega$$

- "Double fixed point" point characterization
- $p(P, \bar{a}) \cdot \Delta(P)$ represents the distribution of holdings immediately after trading
- $\Delta(P)\Omega$ is another Markov transition probability matrix that represents the evolution of the holdings distribution
- Only existence, but we see uniqueness in all computations

# Excess demand for the outside good

- One of the equations ensures that the excess demand for the outside good is zero, $S(\o, P) = D(\o, P)$:

$$\left[1 - \Pi(\o|\o, P)\right] q(\o) = \left[1 - q(\o)\right] \sum_{a=1}^{\bar{a}} \Pi(\o|a, P) q(a)$$

- Analytical solution for $q(\o)$ from the linear equation above
- Similarly, one of the equations ensures the *flow equilibrium* condition is satisfied

# Invoking the Impicit Function Theorem (again)

**Lemma (Smoothness)**

*The unique fixed point $EV$ of the smoothed Bellman operator $\Gamma$, the choice-specific value functions $v(\cdot)$, the choice probabilities $\Pi(d|a,P)$ and trade transition probability matrix $\Delta(P)$ are continuously differentiable functions of $P$.*

**Lemma (Derivative of Bellman operator)**

*Let $EV$ be the unique fixed point of the smoothed Bellman operator $\Gamma$ and let $\nabla_{EV}\Gamma(EV,P)$ be the Jacobian matrix of $\Gamma$ with respect to $EV$. It holds:*

$$\nabla_{EV}\Gamma(EV,P) = \beta\Delta(P)\Omega$$

# Computational algorithm

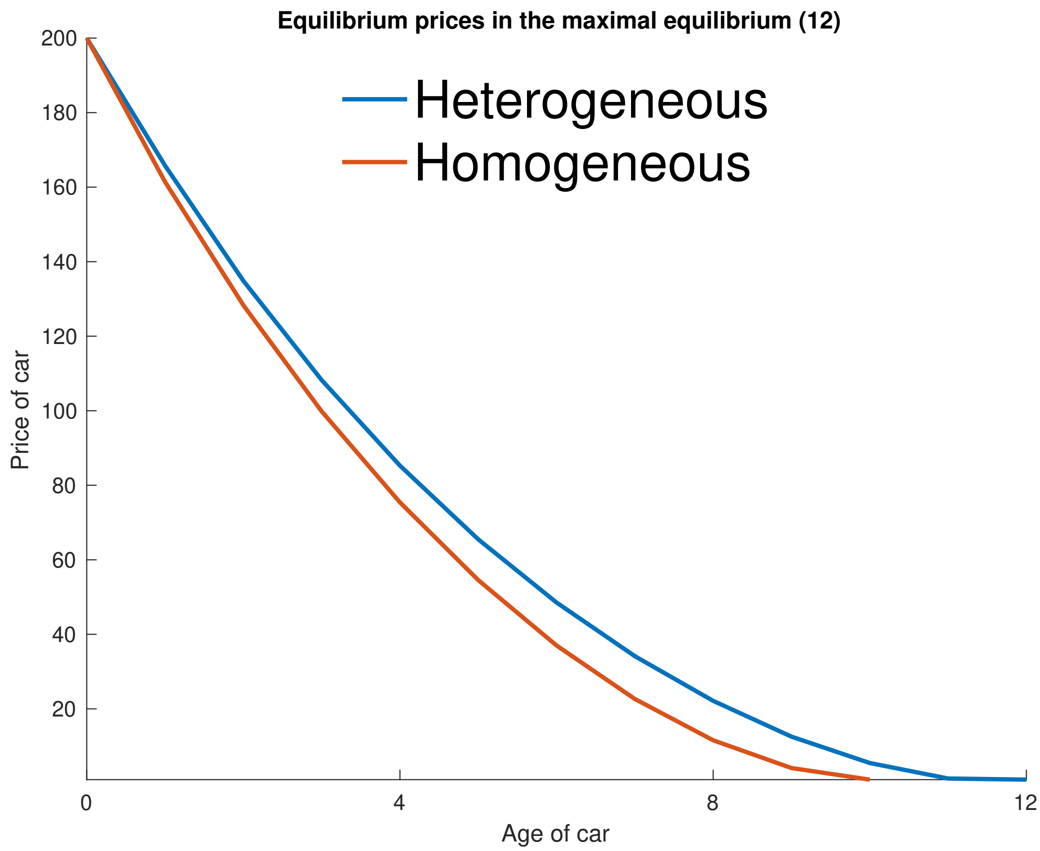
1. Compute accurate starting values for the equilibrium search problem by solving for the homogeneous consumer economy equilibrium without transaction costs
2. Solve for the stationary holdings distribution $q(\bar{a})$ as an invariant distribution of $Q$
3. Using Newton method, solve for the equilibrium prices
   1. Solve for fixed point $EV$ using Newton method;
   2. Form the trade transition matrix $\Delta(P)$ and compute $q(\not{o})$;
   3. Compute the ownership distribution $p(P, \bar{a})$;
   4. Compute excess demand $ED(P)$ in matrix form;
4. Iterate over $\bar{a}$ until finding the maximum $\bar{a}$ such that prices $P(a)$ satisfy definition of equilibrium

Conjecture: similar to homogeneous case without trade costs, **maximal equilibrium** is welfare maximizing

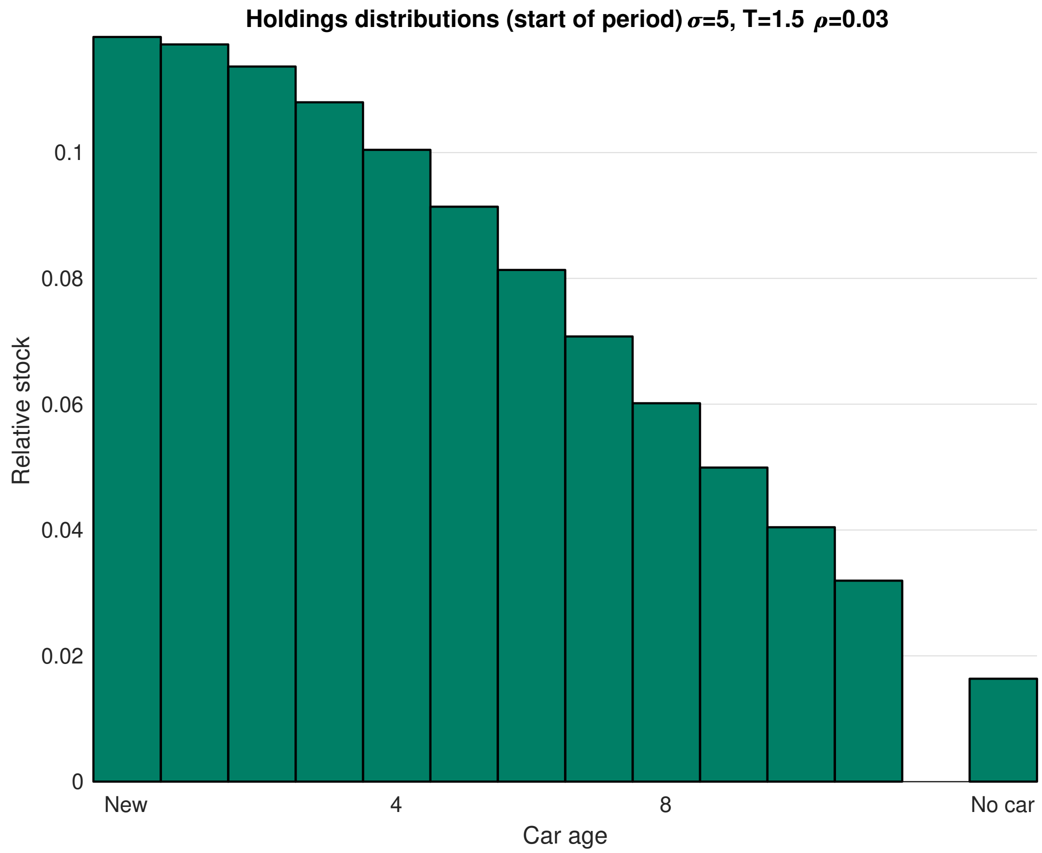
# Example: Equilibrium trade with transactions cost

- Consider the following specification of an example economy consumer who are identicaal apart from extreme value distributed idiosyncratic heterogeneity
- Prices $\overline{P} = 200$, $\underline{P} = 1$
- Utilization utility $u(a) = 60 - 5a$
- Marginal utility of money $\mu = 1$
- Discount factor $\beta = 0.95$
- Scale of EV shocks $\sigma = 5$
- Transaction costs $T(P, a, d) = 1.5 + 0.03P(d)$
- Accident probability $\alpha(a) = 0.01 + 0.02a$

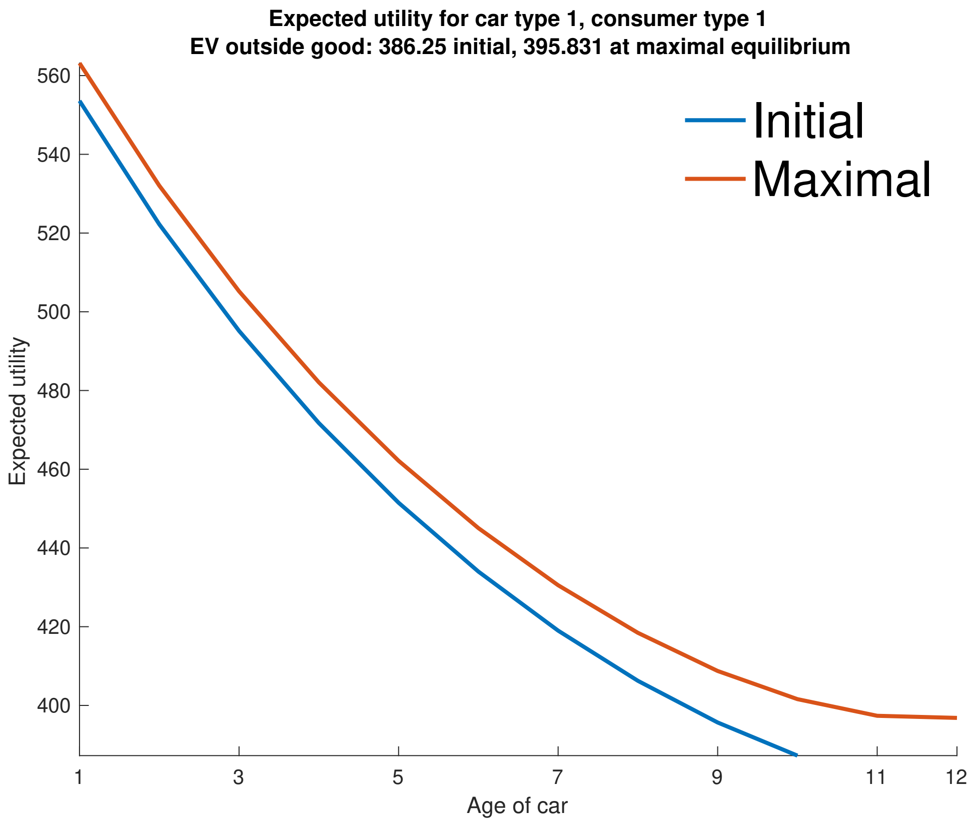
# Heterogeneous agent equilibrium: price function



# Heterogeneous agent equilibrium: ownership distribution



# Heterogeneous agent equilibrium: expected values of cars



# Stationary equilibrium with time-invariant heterogeneity

1. Suppose there are $n$ types of consumers $\tau = 1, \ldots, N_\tau$. Let $f(\tau)$ be the fraction of type $\tau$ consumers in the economy, and let $v(d, a, \tau)$ and $P(d|a, \tau)$ be the value function and choice probability of a type $\tau$ consumer.
2. How to define an equilibrium in this case?
3. Let $\Delta(P, \tau)$ be the demand transition probability matrix for a type $\tau$ consumer.
4. Let $p_\tau$ be the stationary distribution given by
$$p_\tau = p_\tau \Delta(P, \tau)\Omega.$$
5. We call $p_\tau$ the *stationary holdings distribution for type $\tau$ consumers.*

# Aggregate holdings $q$ vs type-specific holdings $p_\tau$

1. Overall holdings is just the type-weighted average of the type-specific holdings distributions $p_\tau$

$$q = \sum_{\tau=1}^{N_\tau} q_\tau f(\tau)$$

2. Will $q$ itself be a stationary holdings distribution? That is, will $q = q\Omega$?
3. **Answer:** Yes when prices are such that market clearing condition $ED(P) = 0$ is satisfied

# Stationary equilibrium with persistent heterogeneity

**Theorem (Equilibrium with time invariant heterogeneity)**

*The stationary equilibrium for the automobile economy with time invariant heterogeneous consumers exists, and the equilibrium holding distribution $q(P, \bar{a})$ is given by*

$$q(P, \bar{a}) = \sum_{\tau=1}^{N_\tau} p_\tau(P, \bar{a}) f(\tau),$$

*where $p_\tau(P, \bar{a})$ is an internal part of the type specific owner distribution $p_\tau(P, \bar{a})$. The latter constitutes the invariant distribution of the transition probability matrix $\Delta_\tau(P)\Omega$:*

$$p_\tau(P, \bar{a}) = p_\tau(P, \bar{a}) \cdot \Delta_\tau(P)\Omega, \; \tau \in \{1, \ldots, N_\tau\}.$$

## Stationary equilibrium with time-invariant heterogeneity

- If follows from the Theorem that in the economy with time invariant heterogeneity then we have

$$q = \sum_{\tau} p_{\tau} f(\tau) = q\Omega$$

- Comparing to the economy with only one type of consumer the condition for "demand equilibrium" $q = q\Delta(P)$ does not hold any more. However the condition that $q = q\Omega = q\Delta(P)\Omega$ holds in both cases.
- Conjecture: similar to homogeneous case without trade costs, **maximal equilibrium** is welfare maximizing

# Challenges to computing stationary equilibrium

Computational algorithm only slightly modified:

- Inner steps have to be performed for each consumer type;
- New step to aggregate type specific quantities;
- Type specific holdings distributions computed as separate fixed points within the inner loop, instead of outside in the idiosyncratic heterogeneity algorithm.
- All other steps remain exactly the same

Key is to verify smoothness of type-specific holdings distributions in $P$: $p_\tau(P) = p_\tau(P)\Delta(P,\tau)\Omega$ is an implicit function of $P$ (Lemma L3 in the paper)

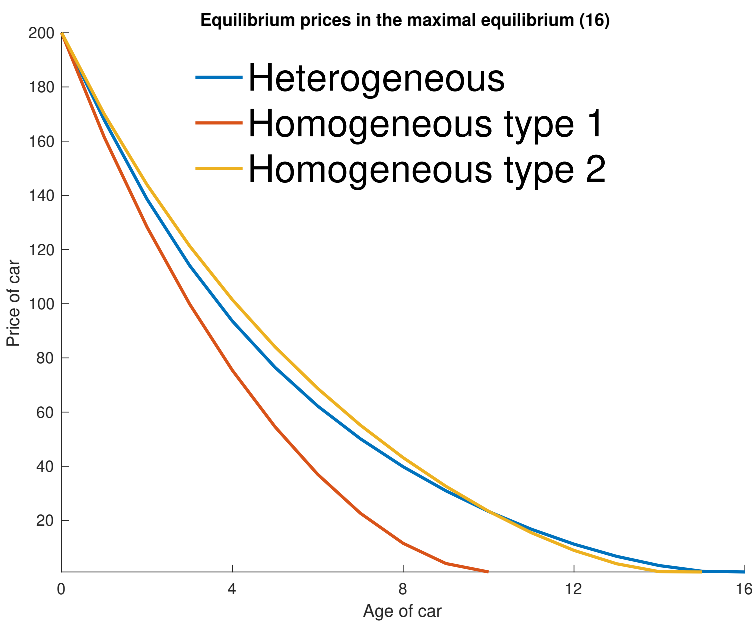
# Illustrative example: Two types of consumers

- Same parameters as before, but $N_\tau = 2$ two types of consumers:
  1. Rich consumers have marginal utility of money $\mu_1 = 1.0$
  2. Poor consumers have marginal utility of money $\mu_1 = 1.75$
- The utility of the outside option is $u(\emptyset) = 0$
- 50% of consumers are rich and 50% are poor

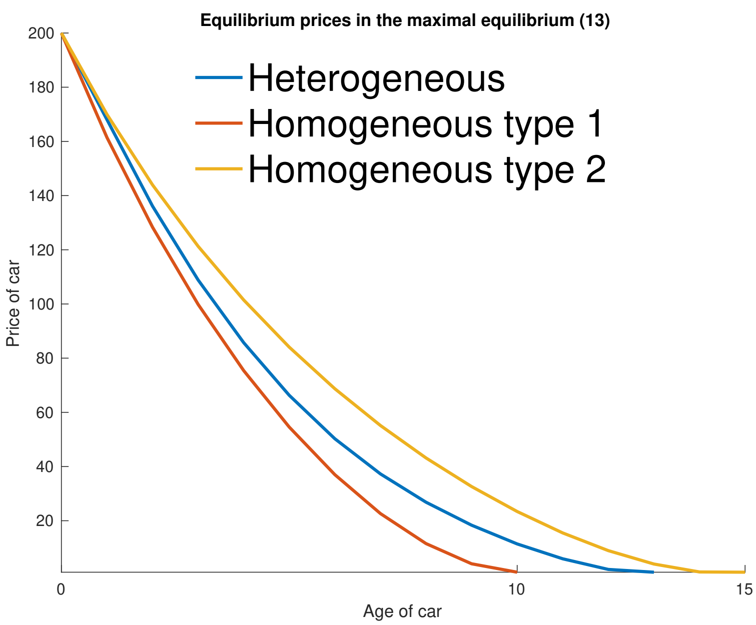
1. Compare heterogeneous equilibrium to homogeneous equilibrium with consumers of each type
2. Compare low and high transactions costs equilibria

# Heterogeneous agent stationary equilibrium: prices

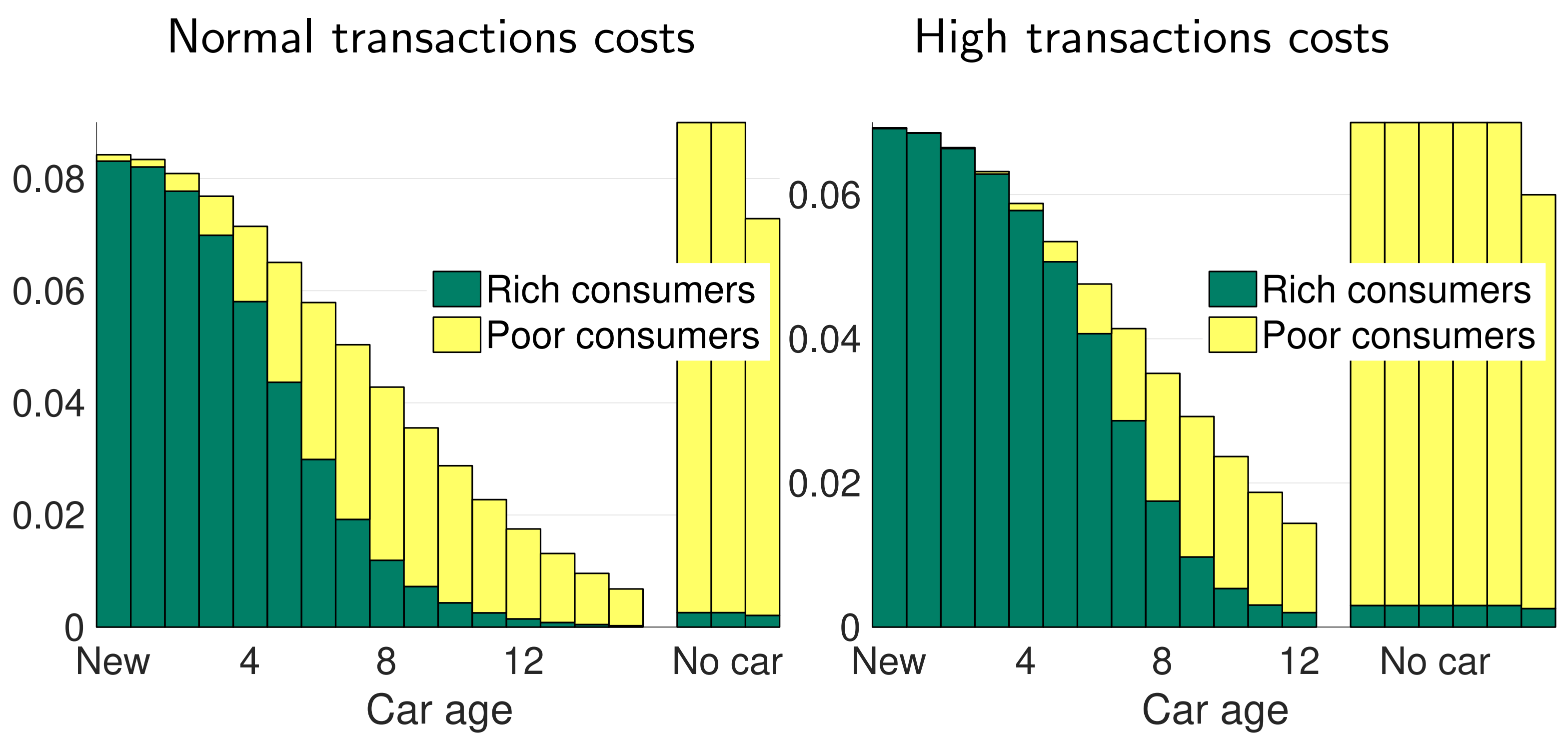
## Normal transactions costs




## High transactions costs



# Heterogeneous agent stationary equilibrium: holdings



# Gains from trade and equilibrium sorting

- Sorting of consumers into the ages of the car:
  1. Rich consumers hold the newer cars, much more likely to buy new cars than the poor consumers
  2. Fraction of poor consumers who do not to own a car is much higher
- *Specialization in holdings* that enables gains from trade
- Cars remain longer on the market
- Market clears in aggregate, but not type-by-type
- High transactions costs suppress trade and lower the scrapping age

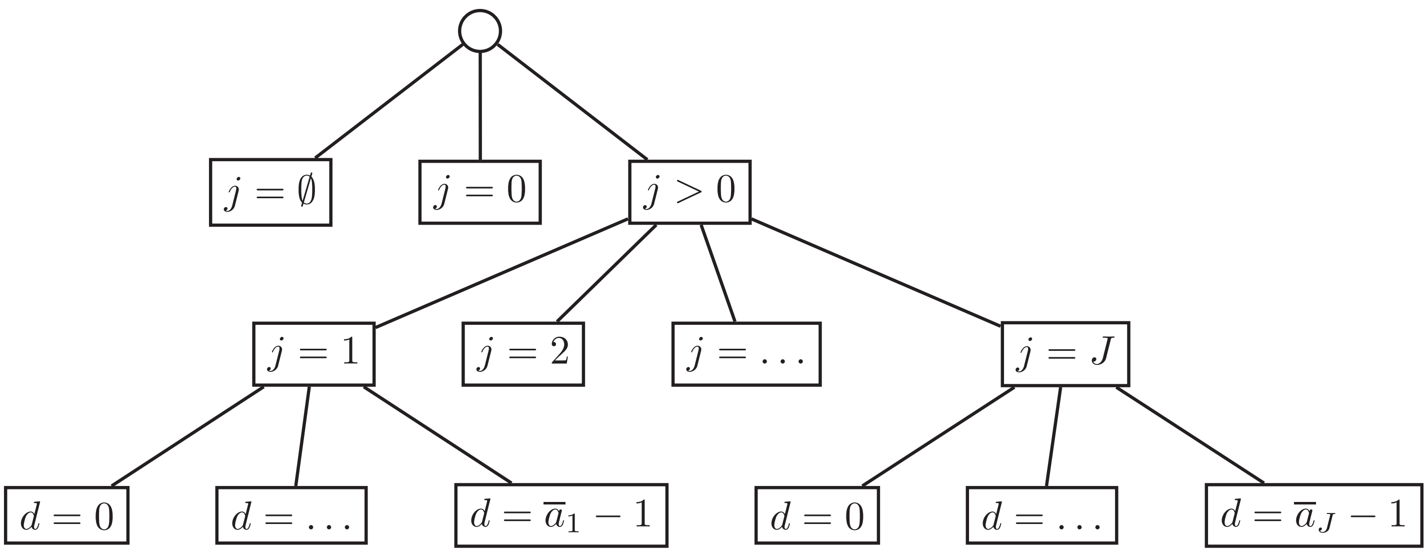
# Multiple types (make/model) of cars

1. Now suppose there are $J$ types (makes/models) of cars. A car is now characterized by a pair $(i, a)$ instead of only age $a$
2. Maintain the infinite elasticity (small open economy) assumption, so new car prices are fixed at $\overline{P} = (\overline{P}_1, \ldots, \overline{P}_J)$
3. For simplicity, assume a common scrap price $\underline{P}$
4. With slight change of notation we let $j = \emptyset$ denote the choice of the outside good (having no car), $j = 0$ denote the decision to keep the current car, and $j > 0$ denote a decision to trade the current car $(j, a)$ for another car $(d, j')$, $j' \in \{1, \ldots, J\}$ and $d \in \{0, \ldots, \bar{a}_{j'} - 1\}$ where $\bar{a}_{j'}$ is the age at which all cars of type $j'$ are scrapped.

# Substitution patterns between makes/models and ages

- The focus now is on the substitution patterns between different cars traded on the used car market
- Simple logit specification of the choice probabilities suffers from the undesirable IIA property
- We adopt a **nested logit** specification for the idiosyncratic utility shocks for different choices of car types, ages, and the choice of the outside good.
- There is a well developed discrete choice theory that we use:
  1. GEV distribution of error terms organized into "nests" of choices allows for flexible modeling of interdependencies between choices
  2. Max-stability property holds, leading to analytical expressions for the choice probabilities

# Three level nested logit specification



# Bellman equation for multiple car types

The model and the theory above generalizes to the multiple car types case, taking into account the larger state space which has 2 variables now
$a \to (j, a)$

$$
\begin{aligned}
V(j,a,\epsilon) &= \max\Big[v(\emptyset,j,a)+\epsilon(\emptyset), v(\kappa,j,a)+\epsilon(\kappa,j), \\
&\qquad \max_{j'\in\{1,\dots,J\}} \max_{d\in\{0,1,\dots,\bar{a}_{j'}-1\}} [v(d,j',j,a)+\epsilon(d,j')]\Big] \\
V(\emptyset,\epsilon) &= \max\Big[v(\emptyset,\emptyset)+\epsilon(\emptyset), \\
&\qquad \max_{j\in\{1,\dots,J\}} \max_{d\in\{0,1,\dots,\bar{a}_{j}-1\}} [v(d,j,\emptyset)+\epsilon(d,j)]\Big]
\end{aligned}
$$

# Equilibrium with multiple car/consumer types

1. Let $\bar{a} = \sum_j \bar{a}(j)$ be the total number of traded age-models
2. Holdings distribution for consumer type $\tau$

$$q_\tau = \big(q_\tau(1), \ldots, q_\tau(J)\big) \in \mathbb{R}^{\bar{a}}$$

3. Ownership distribution

$$\begin{aligned} p_\tau &= \big(q_\tau[1 - q_\tau(\emptyset)], q_\tau(\emptyset)\big) \in \mathbb{R}^{\bar{a}+1} \\ p &= \Big(f(1)p_1, \ldots, f(N_\tau)p_{N_\tau}\Big) \in \mathbb{R}^{N_\tau(\bar{a}+1)} \end{aligned}$$

4. Price function $P(j, a)$ is a vector of $\bar{a} - J$ prices, excluding the prices of new cars, and prices of cars of maximum ages $\bar{a}(j)$

# Equilibrium with multiple car/consumer types

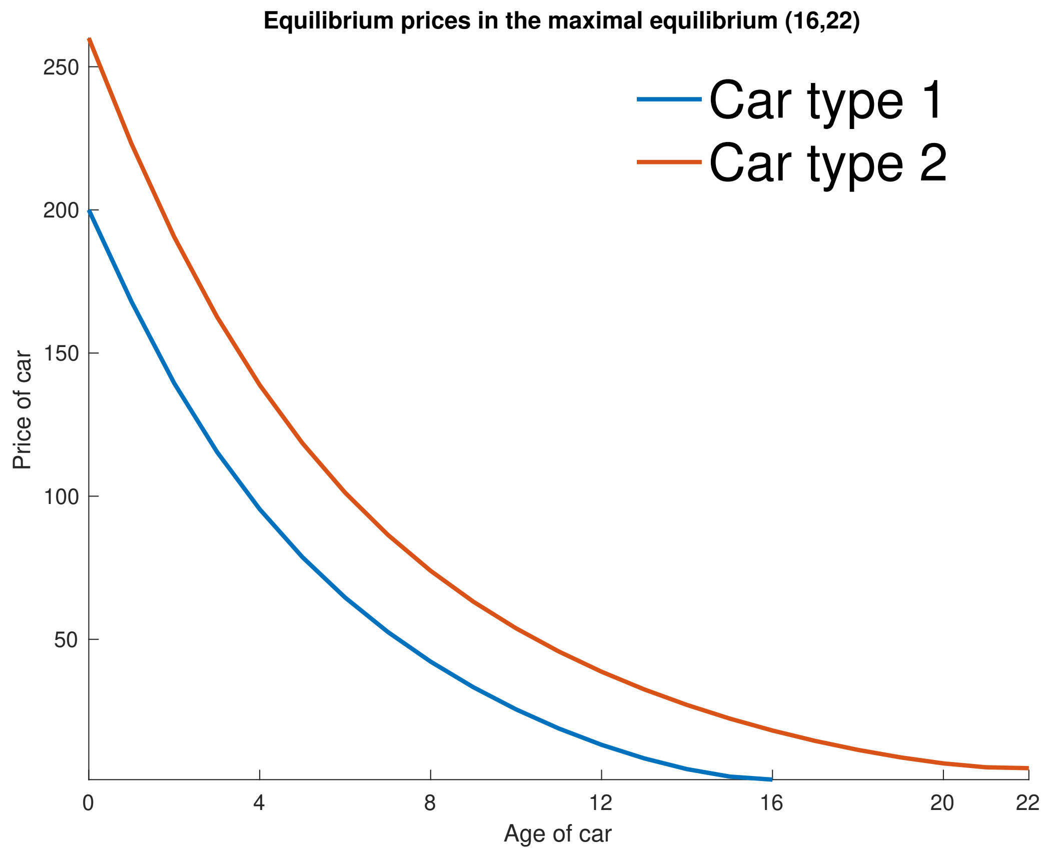
- Physical transition matrix $Q$ is block-diagonal and composed of make/model specific blocks $Q(j)$
- Extended physical transition matrix $\Omega$ as before has an additional 1 on the main diagonal for the outside good.
- Trade transition matrix $\Delta_\tau(P)$ also has an expected block structure
- **The corresponding single or multiple consumer type theory applies directly**
- With one exception: instead of 1 there are $J$ flow equilibrium conditions $D(j,0,P) = S(j,\bar{a},P)$ (all new cars introduced to replace scrapped cars are bought in the market)
- Solution: There are $J-1$ additional unknowns, namely the market shares of makes/models in equilibrium.

# Example: multiple car and consumer types

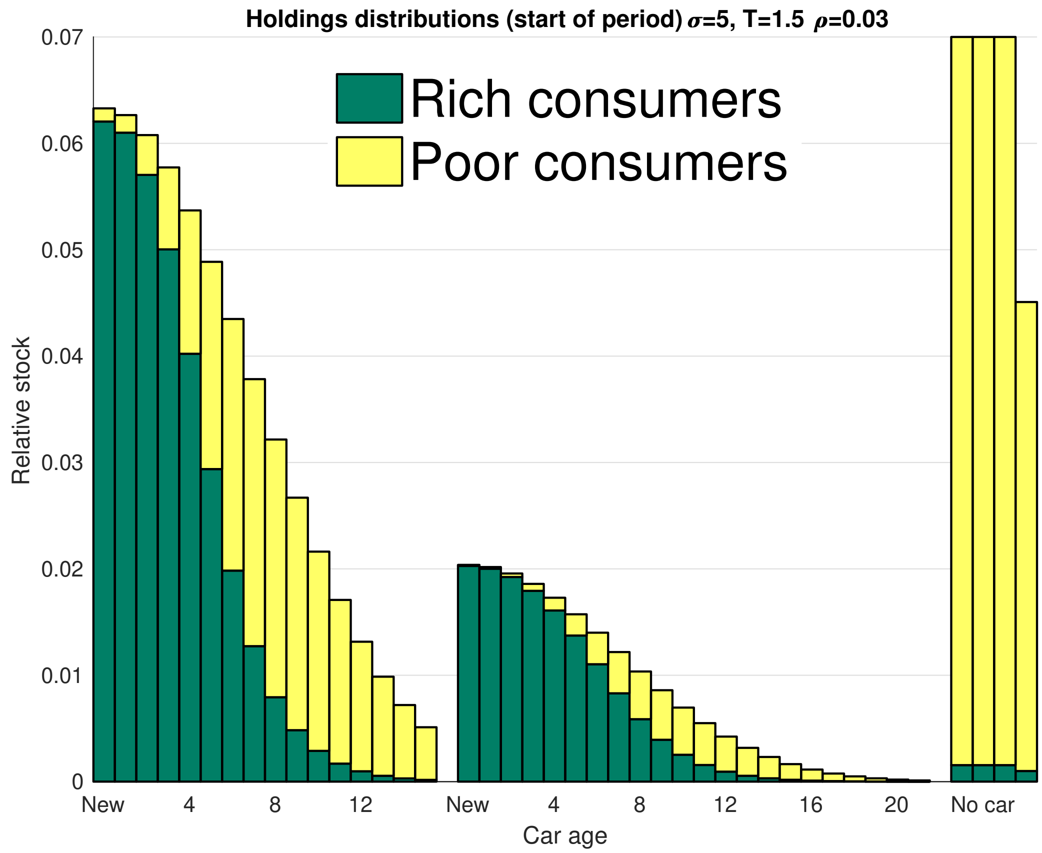
- Consider the example economy with $J = 2$ models of cars and $N_\tau = 2$ two types of consumers
- Consumer types are as in the previous example, poor and rich
- Second car type is a luxury car:
  1. Higher price $\overline{P} = 260$ (30% higher)
  2. Higher scrap value $\underline{P} = 5$
  3. More desirable for the consumers of both types: $u(a) = 65 - 4.75a$ (normal car delivers the utility of $u(a) = 60 - 5a$)
- Both cars have the same transaction costs and accident probabilities

Stronger sorting of consumer type into ages and types of cars

# Equilibrium with 2 consumer and 2 car types: prices



# Equilibrium with 2 consumer and 2 car types: holdings



# Simulating the IRUC reform

Simple back of the envelope simulation of the effects of the Danish IRUC reform

- Lower registration taxes, and
- Higher usage taxes (road charging or gas tax).

Implementation of the counter factual simulation:

1. Calibrate the model to the data
2. Cut the registration tax rates for new vehicles by half
3. Simultaneously increases the fuel tax rate such that revenue is unchanged
4. Compute economic/welfare/environmental implications

# Key assumption for tractability

- **Assumption** *The probability of an accident and other physical deterioration in an automobile is independent of driving, $x$.*
- This implies driving is a static subproblem of the overall DP problem that can be solved independently.

$$x_\tau(j,a,p_f) = \underset{x}{argmax}\left[u_\tau(j,a,x) - \mu_\tau p_f x\right]$$

- We can view $x_\tau(j,a,p_f)$ as the *demand for driving* function
- Substituting back optimal driving into $u$ we get an *indirect utility function* for cars that accounts for optimal driving

# Estimating driving from the data

- We treat $x_\tau(j, a, p_f)$ as *planned* driving
- Assume the following specification for preferences

$$u_\tau(j, a, x) = \big(\gamma(j, \tau) + \xi\big)x - \frac{\phi}{2}x^2, \quad \gamma(j, \tau) = \gamma_j + \gamma_\tau + \gamma_1 a + \gamma_2 a^2,$$

  where $\xi$ is a random variable that represents various unobserved needs for driving
- The optimal level of driving is then:

$$\begin{aligned} x_\tau(j, a, P_j^x) &= \frac{1}{\phi}\big[ -\mu_\tau P_j^x + \gamma_j + \gamma_\tau + \gamma_1 a + \gamma_2 a^2 + \xi \big] \\ &= -\beta_\tau^P P_j^x + \beta_j^{car} + \beta_\tau^{type} + \beta^{a1} a + \beta^{a2} a^2 + \xi. \end{aligned}$$

# Data and driving demand estimation results

- Danish micro data from Gillingham & Munk-Nielsen (2015)
- $J = 2$ car types based on weight: $j = 1$ refers to a smaller, cheaper and slightly more fuel efficient car and $j = 2$ refers to a larger, more durable and expensive car
- $N_\tau = 2$ types of consumers: below and above the median of income distribution

| Coefficient | Notation | 1 | 2 |
|---|---|---|---|
| Fuel price, by car type | $\beta^P$ | -2.6447 | -5.9003 |
| Car type fixed effect | $\beta^{car}$ | 0 | 3.0267 |
| Consumer fixed effect | $\beta^{type}$ | 20.134 | 20.818 |
| Age effect | $\beta^{a1}$ | -0.19906 | |
| Age square effect | $\beta^{a2}$ | 0.0029431 | |

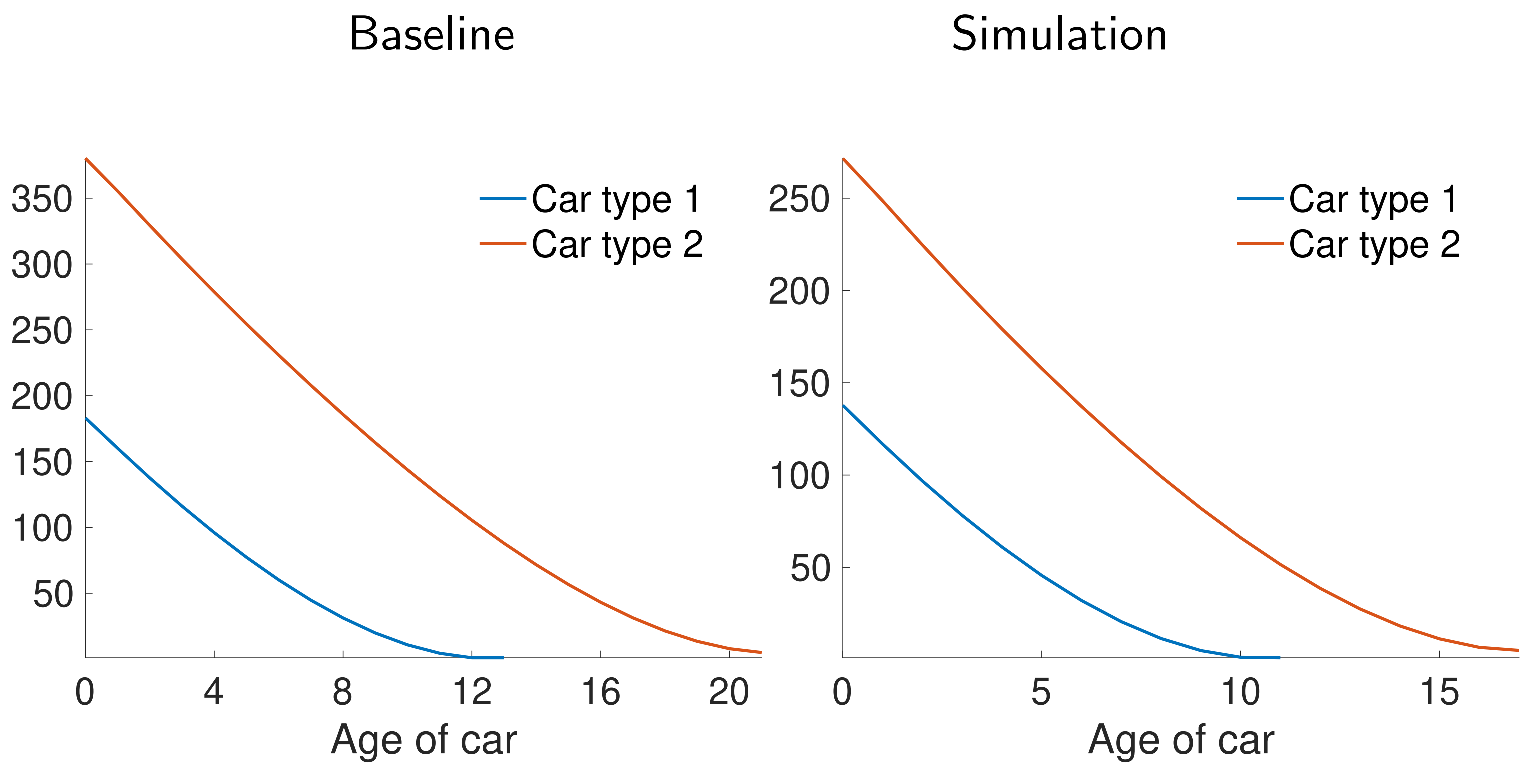
# Further calibration

Given the reduced-form driving estimates of driving we calibrated the marginal utility of money $\mu_\tau$ and an additional car type specific fixed effects to match:

1. fraction of car types by consumer types
2. the fractions of new cars owned by consumers of different types
3. the fraction of cars older that 15 years in the overall car age distribution

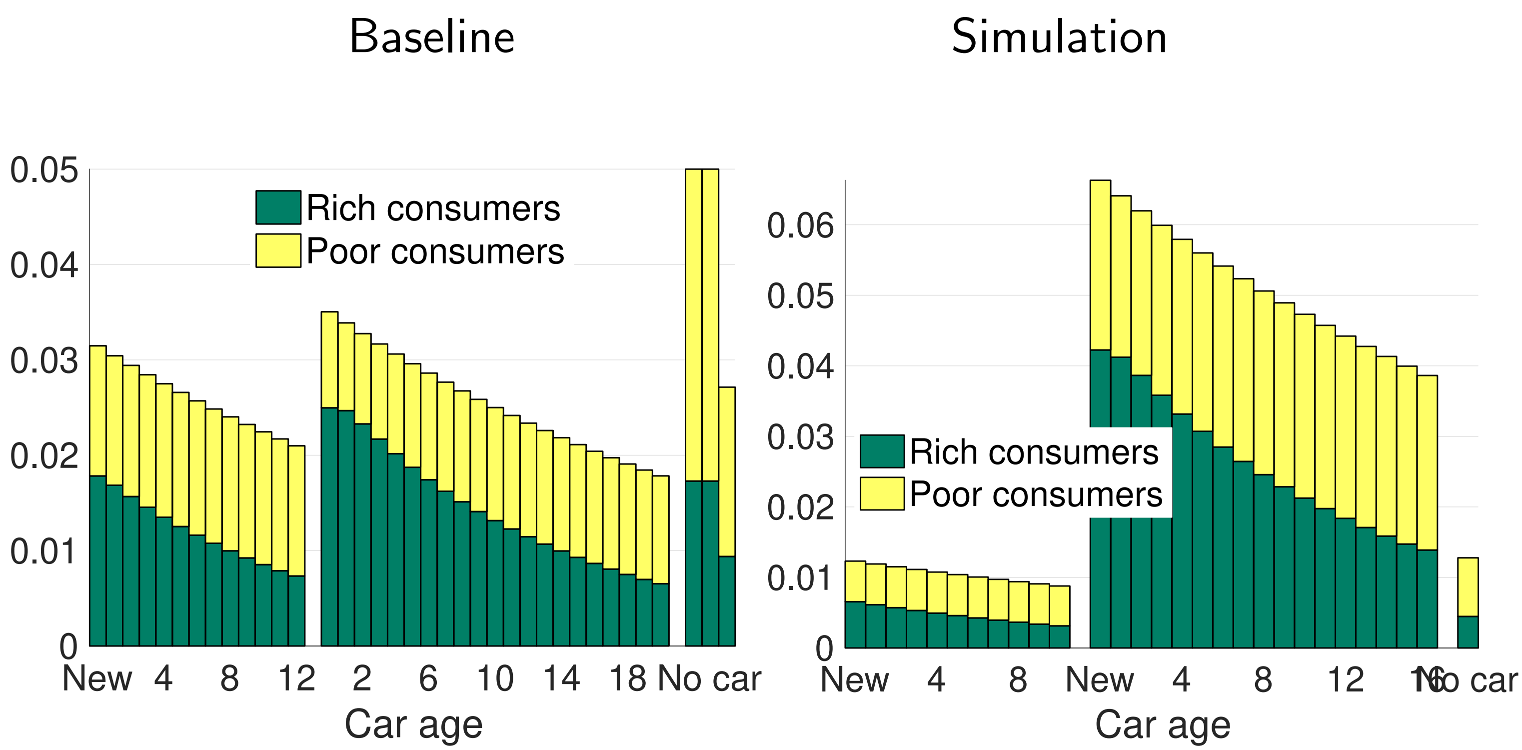
The revenue neutral calculation of halving the registration tax resulted in: increase fuel tax rate by 12% (from 100% to 112%)

# Simulation of IRUC reform: prices

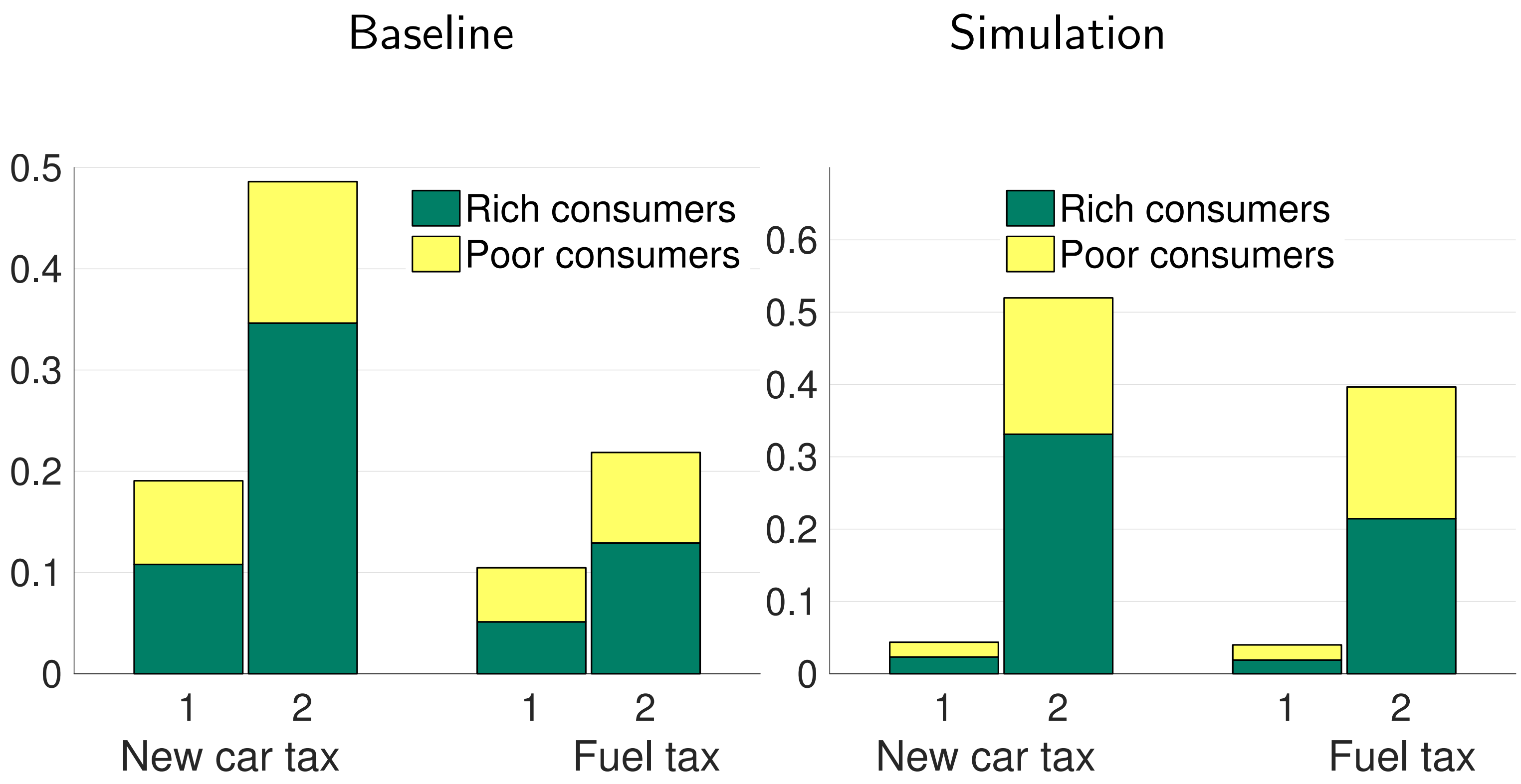



The equilibrium scrappage ages: 13 and 21 vs. 11 and 17

# Simulation of IRUC reform: holdings



# Simulation of IRUC reform: tax revenue



# Effects of IRUC reform in a calibrated example

1. 24.7% and 28.6% reduction in prices of new cars
2. Scrappage age falls by 2 and 4 year for the two car types
3. Car ownership increased from 84% (poor), 92% (rich) to 98% (poor), 100% (rich)
4. Large overall increase in (economic) welfare: 9,302 DKK per capita or 23.3 billion DKK in total, increase for both rich and poor consumers
5. Driving largely stays the same on the intensive margin, but increases in total due to intensive margin from 15,700 to 18,280 km per person per year
6. Fuel consumption increases as people substitute away from a cheaper small car which is also more fuel efficient
7. $CO_2$ emission computed 2.392kg/l increase from 2.65 to 3.20 tonn per person per year

# Endogenous determination of new car prices

- We extend the model to relax the fixed new car price assumstion, and include the market for new cars into consideration.
- Consider a **Bertrand-Nash equilibrium**, where new car producers take into account not only the effect of price changes on substitution to other new cars and to the outside good, but also the effect on equilibrium in the used car market and on the endogenous determination of scrappage ages.
- Thus, we assume new car producers correctly predict the effect of their collective action in setting new car prices on the overall equilibrium in the car market, both in the primary and in the secondary markets.

# New car producers, multi-product firms

1. Let $j \in \{1, \dots, J\}$ index the set of all cars that could potentially be produced by a total of $F$ **firms which we index as** $f \in \{1, \dots, F\}$.
2. Let the sets $(O_1, O_2, \dots, O_F)$ be a partition of the set of possible car types $\{1, \dots, J\}$, so that $O_f$ **denotes the set of car types owned and potentially produced by firm** $f$. We can think of $O_f$ as a collection of patents and production technologies that firm $f$ has for producing its various car types.
3. Let $c_f(\vec{q}_f)$ **be firm** $f$**'s cost of production** of the set of cars that it owns, where $\vec{q}_f = \{q_j | j \in O_f\}$ is the vector of quantities of the various cars that firm $f$ produces

# Bertrand-Nash equilibrium in the new car market

1. Let $\overline{P}_f = \{\overline{P}_j | j \in O_f\}$ be the vector of new car prices that firm $f$ could potentially charge for its own car types $j \in O_f$ in a stationary equilibrium
2. Let $\overline{P}_{-f}$ denote the new car prices of all other types of cars
3. In a Bertrand-Nash equilibrium firm $f$'s prices will be set as a **best response** to the prices charged by its competitors, i.e. we will have

$$\overline{P}_f = \Psi_f(\overline{P}_{-f}),$$

$$\Psi_f(\overline{P}_{-f}) = \underset{\overline{P}_f}{argmax} \sum_{j \in O_f} \overline{P}_j q(\bar{a}_j(\overline{P}_f, \overline{P}_{-f}), j, \overline{P}_f, \overline{P}_{-f})$$
$$- c_f(\{q(\bar{a}_j(\overline{P}_f, \overline{P}_{-f}), j, \overline{P}_f, \overline{P}_{-f}) | j \in O_f\}).$$

# Bertrand-Nash equilibrium in the new car market

1. Each firm $f$ chooses the prices of its car types, $\overline{P}_f$ to maximize the profits
2. In a stationary equilibrium the discounted profits are simply equal to the instantaneous profits scaled with $1/(1-\beta_f)$ where $\beta_f \in (0,1)$ is firm's $f$ discount factor
3. A Bertrand-Nash equilibrium is then any fixed point $(\overline{P}^*_1, \dots, \overline{P}^*_F)$ to the system of best response correspondences for the $F$ firms operating in this market:

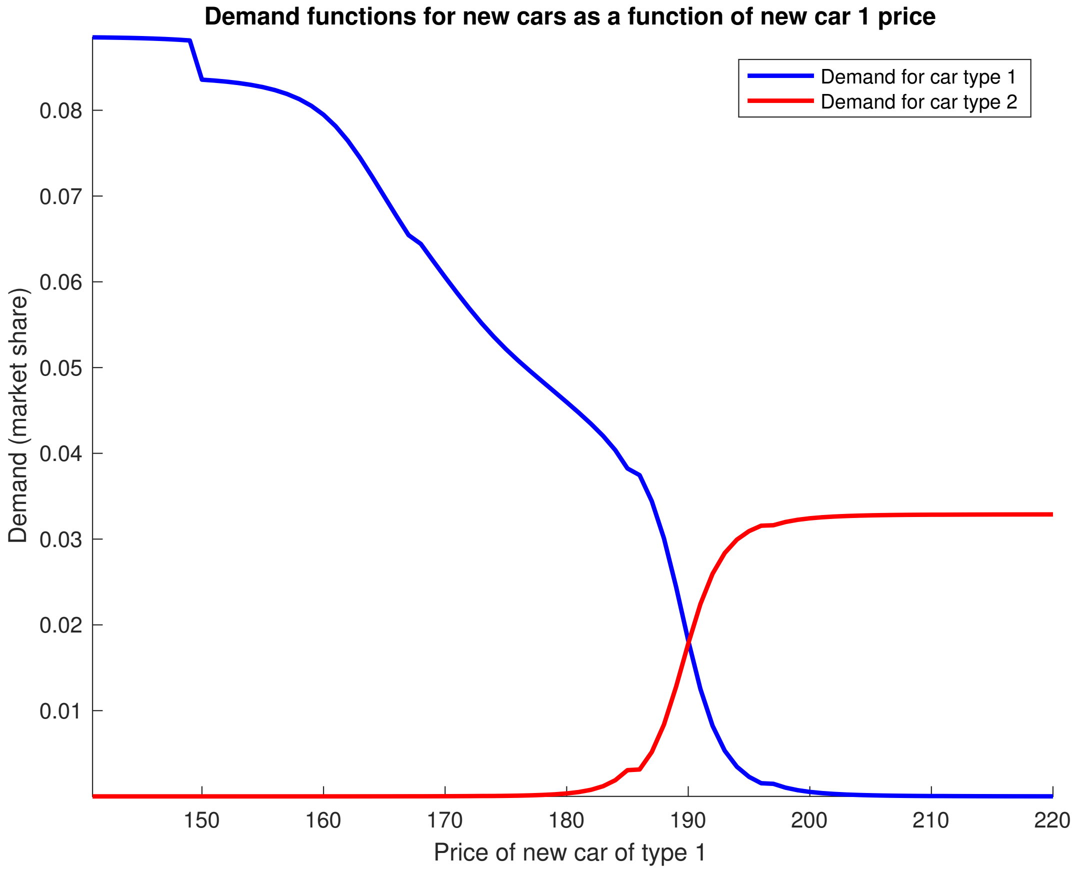
$$(\overline{P}^*_1, \dots, \overline{P}^*_F) \in (\Psi_1(\overline{P}^*_{-1}), \dots, \Psi_F(\overline{P}^*_{-F})).$$

# Example: duopoly equilibrium

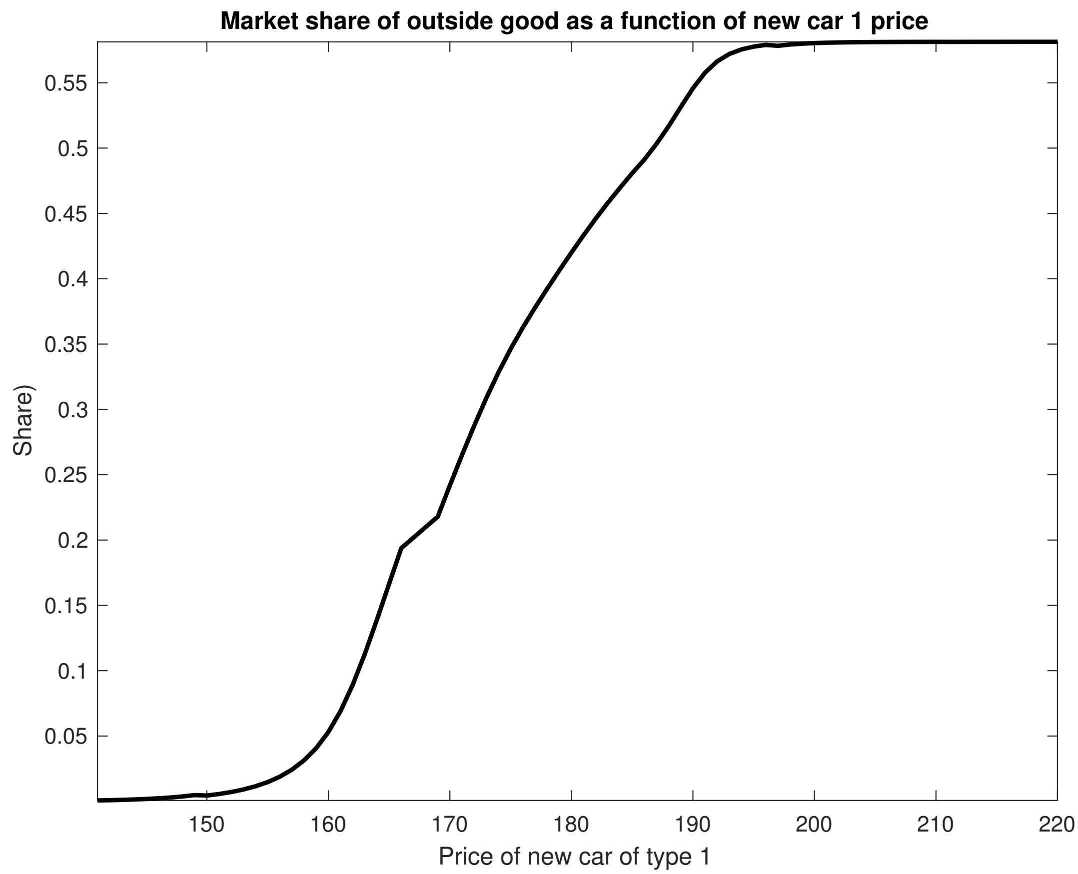
Consider a market with $J = 2$ types of cars and $F = 2$ firms and assume the market is initially a Bertrand-Nash duopoly

1. Computing the two demand curves when firm 1 changes its price and firm 2 keeps its price fixed at $\overline{P}_2 = 220$.
   - Substitution between car models as well as the outside good
2. Find best response of firm 1 to $\overline{P}_2 = 220$, then repeat for many different values of $\overline{P}_2$
3. Perform the same calculation for firm 2 to approximate its best response function.
4. The intersection point then gives the stationary equilibrium on the market of new and used cars

# Primary market demand as a function of type 1 car price

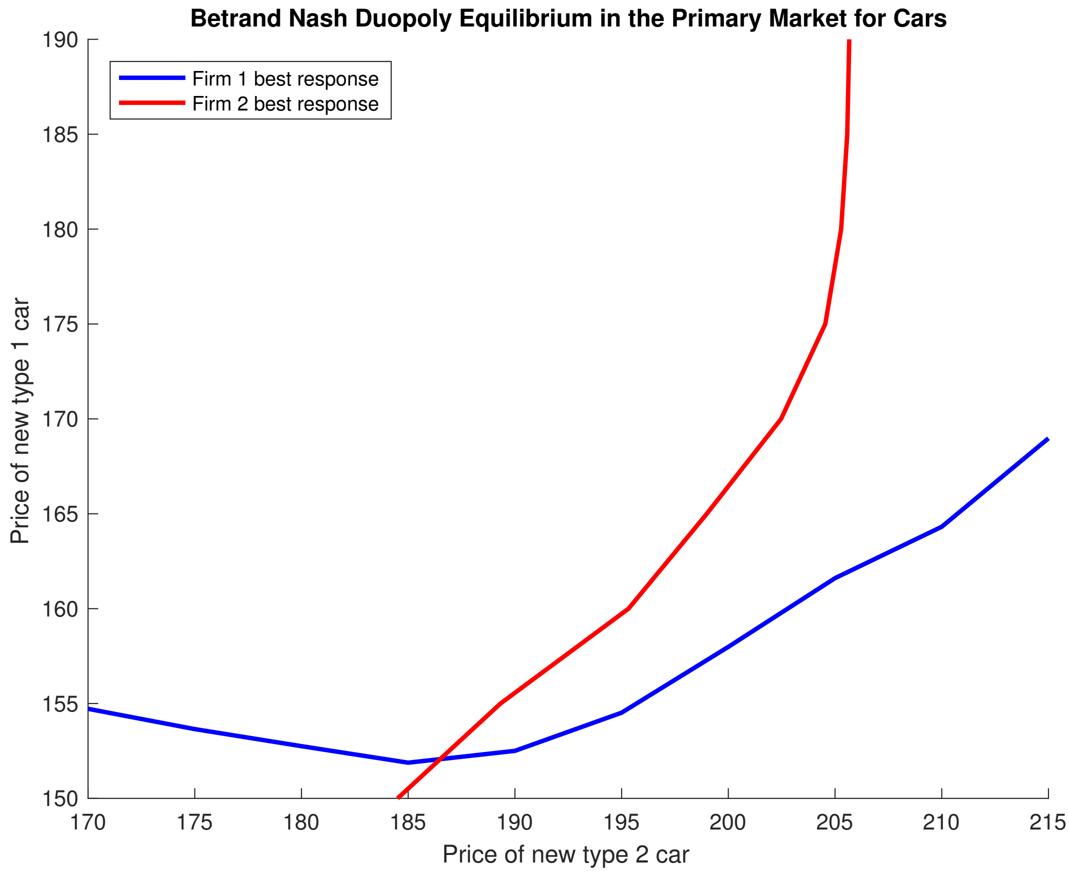


# Market share of outside good



# Scrappage age of type 1 car

# Best response functions and the Bertrand-Nash equilibrium



# Analysis of a merger to monopoly

| Outcome | Monopoly | Duopoly |
|---|---|---|
| Price of car 1 | 374.91 | 151.84 |
| Price of car 2 | 386.93 | 186.27 |
| Profits car 1 | $3 \times 10^{-4}$ | 0.0165 |
| Profits car 2 | 3.35 | 1.09 |
| Total profits | 3.35 | 1.11 |
| Consumer surplus | 39.24 | 74.48 |
| Social surplus | 42.59 | 75.59 |
| Consumer surplus rich, cars 1/2 | 0.002/32.54 | 2.66/26.33 |
| Consumer surplus poor, cars 1/2 | 0.0004/6.69 | 3.23/42.26 |

# Analysis of a merger to monopoly

| Outcome | Monopoly | Duopoly |
|---|---|---|
| New car demand | $1.5 \times 10^{-4}\%/1.5\%$ | 0.9%/6.7% |
| Outside good demand | 77.9% | 0.3% |
| Rich, no car | 2.4% | 0.0% |
| Poor, no car | 96.8% | 0.04% |
| Car 1/2 shares, rich | $7\times 10^{-3}\%/97.6\%$ | 11.5%/88.5% |
| Car 1/2 shares, poor | 0%/3.2% | 10.5%/89.1% |
| Scrappage ages | 18/20 | 15/18 |
| Average age held, poor | 17.15/18.27 | 8.8/9.9 |
| Average age held, rich | 7.80/7.85 | 2.00/2.15 |

# Analysis of a merger to monopoly

- Merger would have devastating impact on the car market: Prices of cars 1 and 2 more than double.
- Monopolist endogenously "overprices" vehicle type 1, causing its market share to fall nearly to zero.
- Nearly 78% of the population chooses to forgo owning a car after the merger.
- Though firm profits triple after the merger, consumer surplus is nearly cut in half and total surplus (consumer surplus plus total profits) falls by 44%.
- Since consumer surplus is vastly greater than firm profits, it seems that allowing a merger would be a very bad idea in this stylized economy.

# Nonstationary equilibrium in the auto market

1. The simulations above was based on a stationary equilibrium.
2. Why care about nonstationarity? **Answer:** because there are constant shocks to actual markets that continually shift prices and quantities in the market
3. Rather that attempting to compute the nonstationary equilirbrium implied by fully rational expectations, we will instead specify a temporary equilibrium relationship in which traders forecast the future at each date as a function of their information on current states of the economy
4. We will operationalize this concept in the car market with a particularly simple/naive adaptive price forecasting function: consumers always believe today's price structure $P$ will persist in the future.

# Nonstationary Temporary equilibria

1. Let $(P_t, q_t)$ be the temporary equilibrium prices and holdings of cars at time $t$.
2. In a stationary equilibrium these same prices and quantities will also be a temporary equilibrium in all future dates, and hence the temporary equilibrium is also a perfect foresight equilibrium and hence a rational expectations equilibrium as well
3. However if the economy is not in stationary equilibrium at time $t$ (or if a shock occurs that affects some aspect of consumer preferences, or income, or new car prices, etc.) then at time $t+1$ the original pair $(P_t, q_t)$ no longer constitutes a temporary equilibrium and some new temporary equilibrium $(P_{t+1}, q_{t+1})$ will arise at $t+1$ to clear the market for used cars.

# Nonstationary Temporary equilibria

1. The result of this is a sequence (or stochastic process) of temporary equilibria $\{(P_t, q_t)\}$ with the property that at each time $t$ $P_t$ clears the market for used cars, i.e. $EED_t(P_t) = 0$.
2. Suppose that at $t = 0$ the economy is in a stationary equilibrium and there is a one time permanent shock to the economy at $t = 1$. Thus if $(P_0, q_0)$ is the initial stationary equilibrium at $t = 0$, then due to the shock at $t = 1$, $(P_0, q_0)$ will not be an equilibrium any longer at $t = 1$.
   ....Instead there will be a sequence of temporary equilibria $\{(P_t, q_t)\}$ originating from this previous shock.
3. Let $(P_\infty, q_\infty)$ denote the new stationary equilibrium corresponding to the post-shock economy. Will it be the case that

$$\lim_{t\to\infty}(P_t, q_t) = (P_\infty, q_\infty)? \tag{1}$$

# Definition of Nonstationary Temporary equilibrium

1. We start by defining $(P_1, q_1)$ and then recursively define temporary equilibria $(P_t, q_t)$ for all $t \geq 1$.
2. At $t = 1$ the market has (permanently shifted) so that $(P_0, q_0)$ is no longer a stationary equilibrium. We assume that prices immediately adjust at $t = 1$ so that supply of cars $a = 1, 2, \ldots, \overline{a} - 1$ in the stationary equilibrium at $t = 0$ equals the (new) demand for these cars in the new regime in periods $t \geq 1$. That is, we assume $P_1$ is a solution to

$$\sum_{\tau} q_{\tau,0}(a) f(\tau) = \sum_{\tau} q_{\tau,0} \Delta_n(P_1, \tau)(a) f(\tau), \quad a \in \{1, 2, \ldots, \overline{a}-1\} \tag{2}$$

where $q_{\tau,0}$ is the stationary holdings distribution for a type $\tau$ consumer at $t = 0$.

# Definition of Nonstationary Temporary equilibrium

1. Note that if $\overline{a}_0$ is the scrappage age in the initial stationary equilibrium $(P_0, q_0)$ there will be no car in the economy older than $\overline{a}_0$. However at the start of period $t = 1$ there will be cars of age $a = \overline{a}_0$ which would be scrapped if the equilibrium continued to remain at the stationary equilibrium $(P_0, q_0)$ but in a new equilibrium it is possible that consumers would decide to keep cars of age $\overline{a}_0$ instead.
2. In that case the largest possible scrap age would be $\overline{a}_0 + 1$ since the absence of any vehicles of this age in the previous equilibrium implies that there is a supply of used cars of age $\overline{a}_0$ but zero supply of cars of age $\overline{a}_0 + 1$.

# Definition of Nonstationary Temporary equilibrium

1. It follows that there can only be trading in used cars up to age $\overline{a}_0$ at period $t = 1$, but there can be no trading in cars of age $\overline{a}_0 + 1$ or older due to the fact that there is no supply of cars of these older ages in the economy at this point.
2. Call the scrappage age from the largest solution to (2) $\overline{a}_1$.
3. Given $P_1$ define $q^0_{\tau,1}$ as follows: for $a \in \{1, 2, \ldots, \overline{a}_1 - 1\}$ we set $q^0_{\tau,1} = q_{\tau,0}(a)$. We define $q^0_{\tau,1}(\emptyset)$, the fraction of type $\tau$ consumers who choose not to own a car by

$$q^0_{\tau,1}(\emptyset) = \Pi(\emptyset|\emptyset, \tau, P_1)q_{\tau,0}(\emptyset) + \sum_{a=1}^{\overline{a}_1-1} \Pi(\emptyset|a, \tau, P_1)q_{\tau,0}(a). \quad (3)$$

# Definition of Nonstationary Temporary equilibrium

1. Finally, let the total quantity of new cars held by type $\tau$ consumers just after trading at the prices $P_1$ at the start of period 1 be given by

$$q^0_{\tau,1}(0) = \Pi(0|\emptyset, \tau, P_1)q_{\tau,0}(\emptyset) + \sum_{a=1}^{\overline{a}_1-1} \Pi(0|a, \tau, P_1)q_{\tau,0}(a). \quad (4)$$

2. Now define $q_{\tau,1} = q^0_{\tau,1}\Omega_n$. This is the distribution of holdings of cars by type $\tau$ consumers at the *end* of period $t = 1$ (thus at the start of period 2, but before trading occurs at prices $P_2$). Finally let $q_1$ be the type-weighted average,

$$q_1 = \sum_{\tau} q_{\tau,1} f(\tau). \quad (5)$$

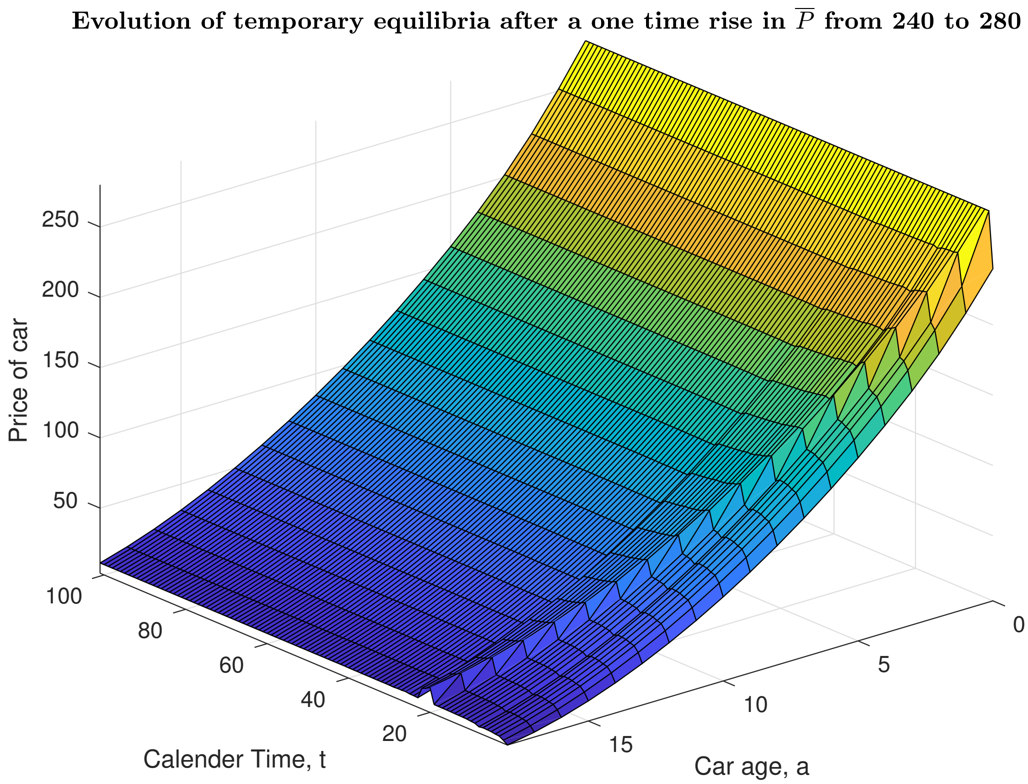
# Definition of Nonstationary Temporary equilibrium

1. Thus, we have defined $(P_1, q_1)$, the temporary equilibrium prices and quantities in the first period after a permanent shock that disturbs the market from its initial stationary equilibrium $(P_0, q_0)$ at $t = 0$.
2. At time $t = 2$ we can follow the same procedure: given the quantities $q_{\tau,1}$ held by each consumer type $\tau$, we use equation (2) with time indices shifted forward one period to $t = 1$ to define the temporary equilibrium price vector $P_2$ and corresponding scrappage age $\overline{a}_2$, where $\overline{a}_2$ is the *largest* scrappage age from the potentially multiple solutions to equation (2) that satisfy a) monotonicity, and b) $P_2(a) \geq \underline{P}_n$, $a \in \{1, 2, \ldots, \overline{a}_2 - 1\}$ but subject to the constraint that $\overline{a}_2 \leq \overline{a}_1 + 1$ where $\overline{a}_1$ is the scrappage age in the temporary equilibrium $(P_1, q_1)$.

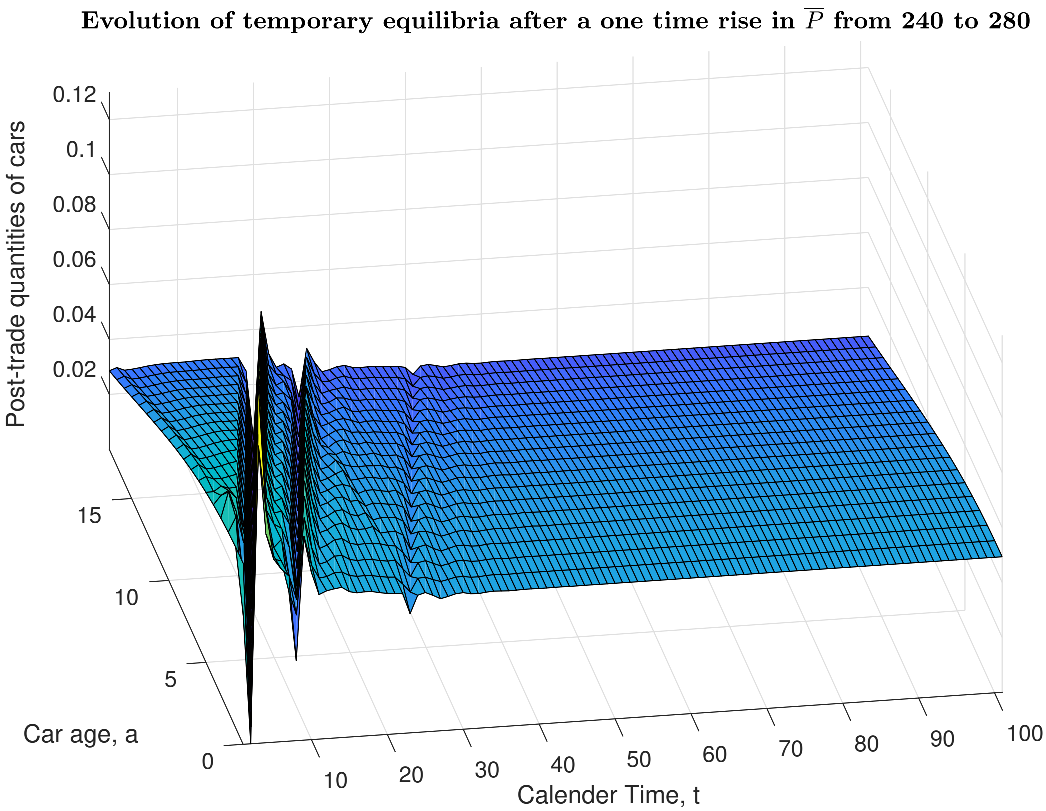
# Definition of Nonstationary Temporary equilibrium

1. $\overline{a}_{t+1}$ can be at most 1 more the scrappage age in the previous TE, $\overline{a}_t$, since there is zero supply of cars that are older than $\overline{a}_t$ in the previous TE.
2. Given $P_2$ we can define $q^0_{\tau,2}$, the post-trade age distribution of cars in period 2, using equations (3) and (4) but with time indices shifted one period ahead. Let $q_{\tau,2} = q^0_{\tau,2}\Omega_n$ be the holdings of cars by type $\tau$ consumers at the *end* of period $t = 2$ (thus at the start of period 3, but before trading occurs at prices $P_3$), and $q_2$ be the type-weighted average,
$$q_2 = \sum_{\tau} q_{\tau,2} f(\tau). \tag{6}$$
3. Thus, we have defined $(P_2, q_2)$, the temporary equilibrium prices and quantities in period $t = 2$, using as initial condition the temporary equilibrium $(P_1, q_1)$ at $t = 1$.

# Impact of one time rise in $\overline{P} = 240$ to $\overline{P} = 280$

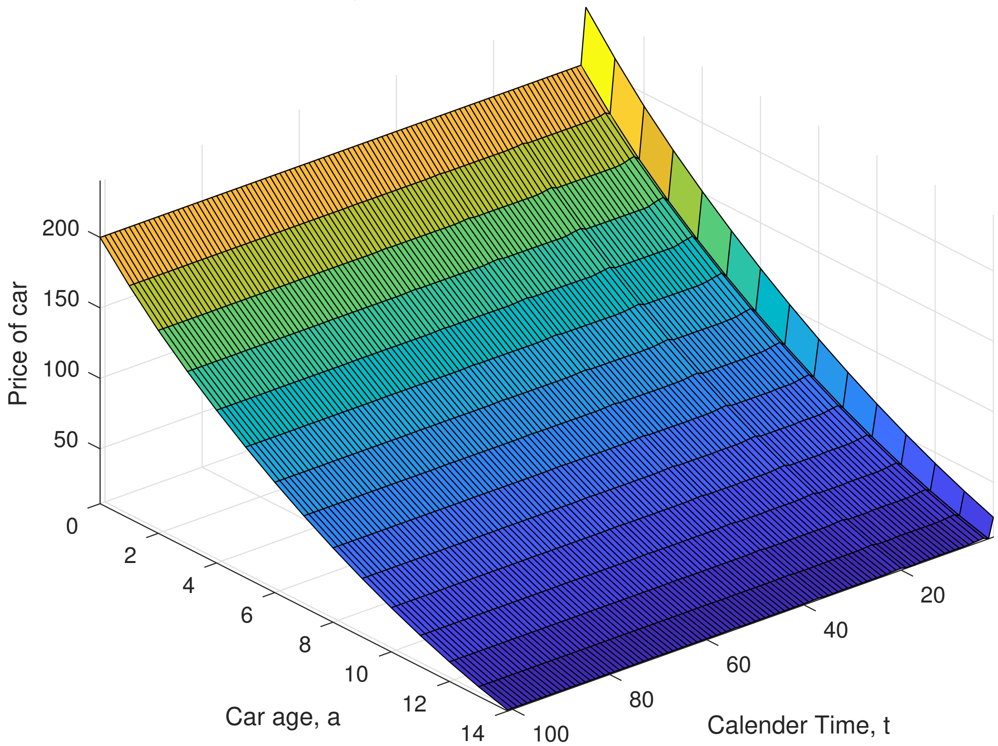


# Impact of one time rise in $\overline{P} = 240$ to $\overline{P} = 280$

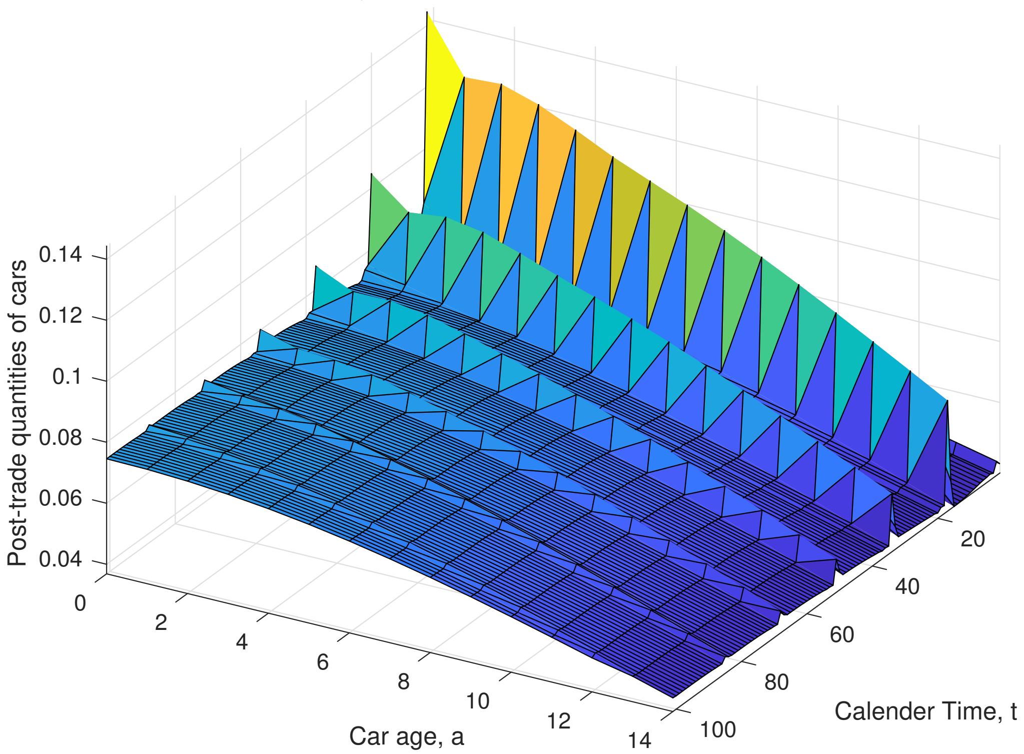


# Impact of one time fall in $\overline{P} = 240$ to $\overline{P} = 200$

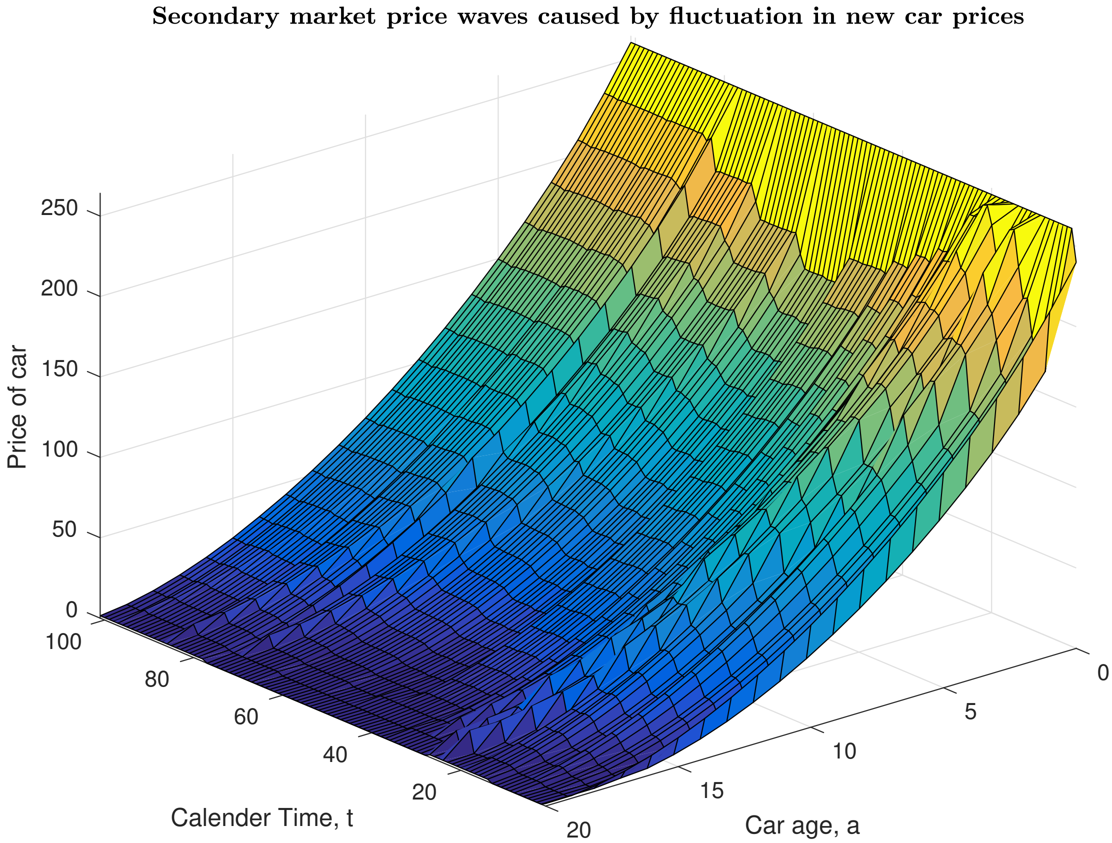
**Evolution of temporary equilibria after a one time fall in $\overline{P}$ from 240 to 200**



# Impact of one time rise in $\overline{P} = 240$ to $\overline{P} = 200$

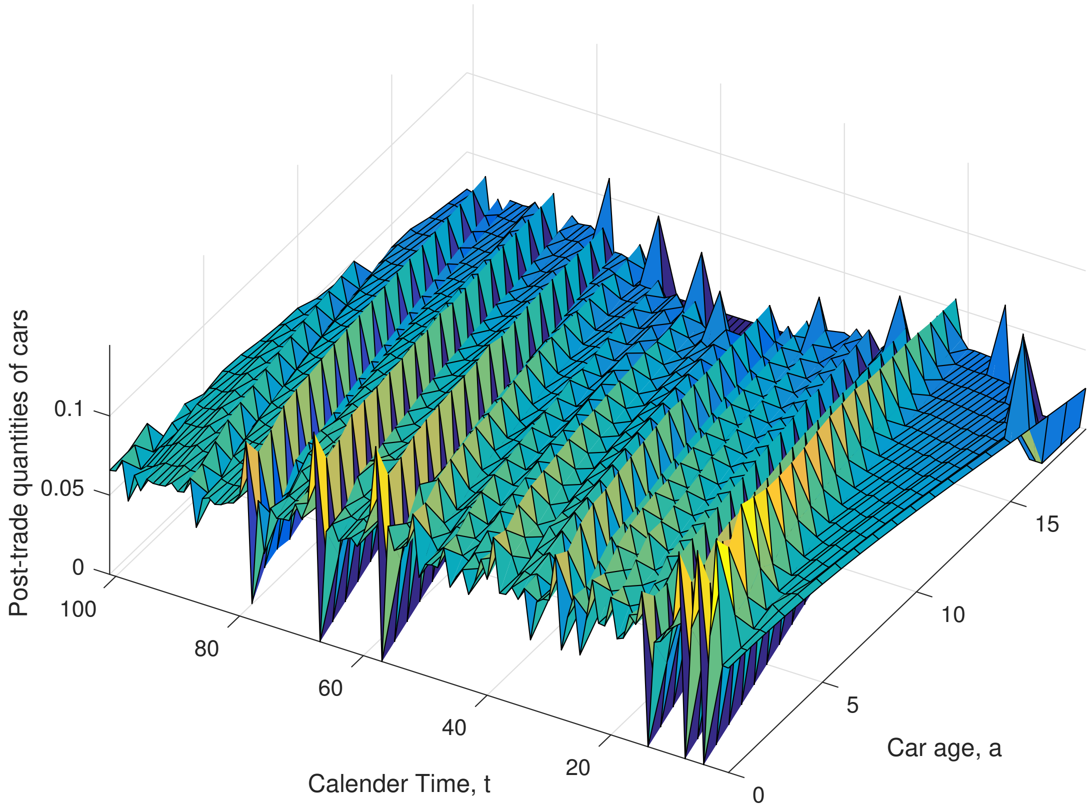


# Impact of repeated fluctuations in $\overline{P}$



# Impact of repeated fluctuations in $\overline{P}$

**Waves in holding distributions for car caused by fluctation in new car prices**



# Ride the wave