Lecture 16: Solving and estimating directional dynamic games

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Roadmap: dynamic games

1. Games in family/life-cycle decision making
   John Rust
2. "Collusion on the beach" a model of leapfrogging investments
   John Rust, Fedor Iskhakov
3. Experiment with the model
   John Rust, Fedor Iskhakov, Bertel Schjerning
4. State recursion algorithm
5. Recursive lexicographical search (RLS) algorithm
6. Full solution for the leapfrogging game
   Fedor Iskhakov
7. Structural estimation of directional dynamic games
   with Nested RLS method

Games in family/life-cycle decision making

- Most empirical work on life-cycle model uses a *single agent* modeling approach — *the unitary model*
- For family decision making, the *collective model* is often used where the family is assumed to act as a single decision maker but it maximizes a weighted average of the husband's and wife's utility functions
- This effectively converts the solution of a multi-agent decision problem to a single agent decision problem
- Other approaches: model family decision making as a *non-cooperative game*
- Other work even models *individual decision making as a non-cooperative game* – as "game against your future selves" or as "games between your alter-egos"

The Collective Model vs Unitary Model

- Introduced by Chiapporri 1992 "in which agents are characterized by their own (possibly altruistic) preferences, and household decisions are only assumed to be Pareto efficient"
- Mazzocco (2007) "The theoretical and empirical literature on household intertemporal decisions has traditionally assumed that households behave as single agents. One of the main drawbacks of this approach is that the effect of intra-household commitment on intertemporal decisions cannot be analyzed and tested."
- The "Unitary Model" — "each household behaves as a single agent independently of the number of decision makers. This is equivalent to the assumption that the utility functions of the individual members can be collapsed into a unique utility function which fully describes the preferences of the entire household."

$$\max_{\{c_t, q_t, s_t\}} E\left\{\sum_{t=0}^{T} \beta^t u(c_t, q_t)\right\} \tag{1}$$

subject to

$$c_t + p_t q_t + s_t \leq y_{h,t} + y_{w,t} + R_t s_{t-1} \tag{2}$$

The Collective Model vs Unitary Model

$$\max_{\{c_{ht}, c_{wt}, q_t, s_t\}} \mu_h(z) E\left\{\sum_{t=0}^{T} u_h(c_{ht}, q_t)\right\} + \mu_w(Z) E\left\{\sum_{t=0}^{T} u_w(c_{wt}, q_t)\right\} \quad (3)$$

subject to

$$c_{ht} + c_{wt} + p_t q_t + s_t \leq y_{ht} + y_{wt} + R_t s_{t-1} \quad (4)$$

- $\mu_h(z)$ and $\mu_w(z)$ are positive "Pareto weights" on the welfare of the husband and wife, respectively, which may depend on time invariant variables z that determine the "type" of the family.
- Can recast the collective model as a unitary model using a *family utility function* $u_f(c, q)$ given by

$$u_f(c, q) = \max_{c_h, c_w} \mu_h(z) u_h(c_h, q) + \mu_w(z) u_w(c_w, q) \quad (5)$$

subject to

$$c_h + c_w \leq c \quad (6)$$

Non-Cooperative Models of Family Behavior

- Eckstein and Lifschitz (2015) *IER* "Household Interaction and the Labor Supply of Married Women"
- Compared three types of households
- *Classical Household* Husband is a Stackelberg leader and wife is a Stackelberg follower
- *Modern Household* Husband and wife are symmetric but make their labor supply decisions independently and simultaneously each period and their choices are a dynamic Nash equilibrium where each member of the couple maximize their own discounted utilties but taking into account the labor supply strategy of their spouse.
- *Cooperative Household* maximizes a collective utility function.
- Note: the Nash equilibria in the Modern Household could be Pareto-inefficient, so the Cooperative Household is able to coordinate and commit to Pareto-efficient decision rules

Non-Cooperative Models of Family Behavior

- Eckstein and Lifchitz estimated their model using PSID data. They treated the *type* of each household as unobserved heterogeneity.
- So they solved the model 3 times under each of the solution concepts for the family described above, and estimated the probability that a household was one of these three types.
- Specifically they used SMM and used logit probabilities that are a function of age and time-invariant household characteristics to determine the simulated proportions of each type of household in their simulations
- They found a model with 3 types can be indentified and results in the best fit to the data. "The estimation results indicate that 57% of the 1983-4 cohort of newlywed couples are of the Classical type, and the hypothesis that all households are Classical is rejected. The proportion of Modern households is 25% and that of Cooperative households is 18%."

Non-Cooperative Models of Family Behavior

- "We find that the labor supply of men is not affected by the type of game, whereas the employment rate for women is lower in Classical households than in Modern households by about 12 percentage points and is higher in Cooperative households than in Modern households by 4 percentage points."
- "In other words, the social norms reflected in a Nash symmetric game and in the collective game lead to an increase in the labor supply of women in Modern and Cooperative households while leaving that of their husbands almost unchanged."
- "The results support the hypothesis that some of the increase in married female labor supply observed in recent decades may be due to changes in social norms that affected the way couples decide on their joint labor supply. "

Dealing with potential multiple equilibria

- In dynamic simultaneous move games there is a possibility of many equilibria. Example: entry games in IO: there is a "coordination problem" and one firm may not want to enter if it believes others are going to enter, but will enter if it believes others will not enter.
- In Eckstein and Lifschitz paper, they did not find multiple Nash equilibria for their Modern households who are playing discrete choice, simultaneous move dynamic games.
- But if multiple equilibria arise, one way to make them go away is model the game as one of *alternating moves* instead of *simultaneous moves*
- Example: Bowlus and Seitz (2006) *IER* "Domestic Violence, Employment and Divorce"
- "Men and women make decisions sequentially in the model. Women make marital status decisions taking into account expectations of abuse given their spouse's characteristics and past behavior, and men decide whether to abuse taking into account the likelihood their wives will divorce them."

Alternating move game with unique equilibrium

- The timing in the model is as follows. Women make decisions in every odd period and men make decisions in every even period.
- Individuals receive a constant level of utility for the period in which they make decisions and for the subsequent period in which their spouses make decisions. One full period for a couple therefore consists of one odd and one even period.
- All agents are single in the first period. All single women meet a potential spouse in every odd period. Women move first and decide whether to work (h) or not (n) and whether to be married (m) or single (s).
- Denote the choice set for women $I = \{sn, sh, mn, mh\}$. After observing their wife's employment choice, the husband decides whether to be abusive (a) or not (na) in the marriage. Denote the choice set for husbands as $J = \{a, na\}$.
- "The evidence presented on the importance of abuse in the divorce decision highlights the fact that many women observed in representative data respond to domestic violence by leaving the relationship."

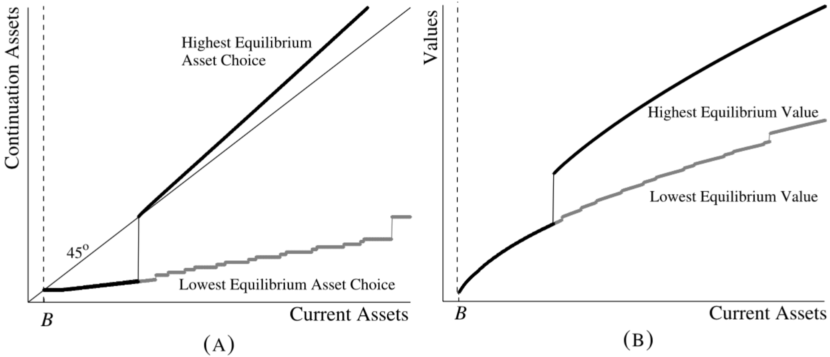
Multiple equilibria for Sophisticated hyperbolic discounters

- Bernheim, Ray and Sveltekin (2015) "Poverty and Self-Control"
- Individuals choose a consumption path to maximize

$$u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t) \tag{7}$$

- Note the *time inconsistency* in planning when $\beta \neq \delta$: the "time t self" puts less weight on the future utility when $\beta < 1$ but assumes that after their momentary "splurge" to consume more today at the expense of their future selves, they will return and behave as a time-consistent planner
- But tomorrow never comes, so at time $t+1$ the time $t+1$ does the same thing and splurges at the expense of their future selves.
- A "sophsticated" hyperbolic discounter realizes this time iconsistency and takes it into account. When they do, this becomes a game with an infinity of players: the "time 0" self, "time 1" self, ...

Multiple equilibria in the game against one's future selves



Unique equilibrium in "dual self" model of self-control

- Fudenberg and Levine (2006) *AER* "A Dual-Self Model of Self-Control"
- "In our model, the patient long-run self and a sequence of myopic short-run selves share the same preferences over stage-game outcomes; they differ only in how they regard the future. Specifically, we imagine that the short-run myopic self has 'baseline preferences' in the stage game that depends only on the outcome in the current stage. That is, the short-run players are completely myopic."
- "The stage game is played in two phases. In the first phase, the long-run self chooses a self-control action that influences the utility function of the myopic self. That is, at some reduction in utility (for both selves), the long-run self can choose preferences other than the baseline preferences."
- "In the second phase of the stage game, after the short-run player preferences have been chosen, the short-run player takes the final decision."

Unique equilibrium in "dual self" model of self-control

- Consumption example: short run self has utility $u(y,0,s) = \log((1-s)y)$ from saving at rate $s \in (0,1)$ when the long run self exercises no self-control cost. In such case the myopic short-run self would save nothing and obtain utility $u(y,0,0) = \log(y)$, but leave nothing for the long run self.
- So the long run self disciplines the short run self by imposing a "mental cost" that incentivizes the short run to choose a positive saving rate s. But doing this costs the long-run self $C(y,s) = \gamma \log(1-s)$.
- The long run self chooses a saving rate s to maximize lifetime utility from the consumptions chosen by the short run selves net of the costs of incentivizing them

$$\max_s \sum_{t=0}^{\infty} \delta^t \left[u(y,0,s) - C(y,s)\right] \tag{8}$$

- The solution is s^* given by $s^* = \delta/(1+\gamma(1-\delta))$.

Bertrand Price Competition with Leapfrogging Investments

- We extend the standard static textbook model of Bertrand price competition by allowing duopolists to undertake cost-reducing investments in discrete time
- Technological progress is exogenous and stochastic
- Each firm has a binary decision to acquire the state of the art production technology
- Even though this is a small extension of the classic static model of Bertrand price competition, surprisingly little is known about Bertrand competition in the presence of production cost uncetainty, especially in dynamic settings
- We show how to compute all equilibria of this game and show that this dynamic model of Bertrand price competition has surprisingly rich, complex, and counter-intuitive equilibrium outcomes.

How do you find *all* Markov Perfect Equilibria?

The Markov Perfect Equilibrium (MPE) concept of Maskin and Tirole (1988) is now a widely used in *empirical IO.* However computing MPE remains a daunting computation problem

Quote (Hörner *et. al. Econometrica* 2011)

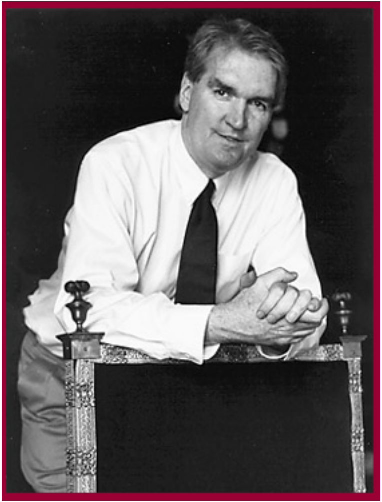
"Dynamic games are difficult to solve. In repeated games, finding some equilibrium is easy, as any repetition of a stage-game Nash equilibrium will do. This is not the case in stochastic games. The characterization of even the most elementary equilibria for such games, namely (stationary) Markov equilibria, in which continuation strategies depend on the current state only, turns out to be often challenging."

Finding even a single MPE is challenging!

- How do people "find" MPEs?
- Theorists: Guess and Verify
- Applied people: Iterate on the player Bellman equations
- Pakes and McGuire (1994): some of the earliest work on computing MPE. Proposed a deterministic, iterative algorithm to compute MPE. Found a curse of dimensionality in trying to solve MPE model of firm dynamics with even moderate numbers of firms
- Pakes and McGuire (2001): Proposed a *stochastic algorithm* to approximate an MPE, in an attempt to break this curse of dimensionality

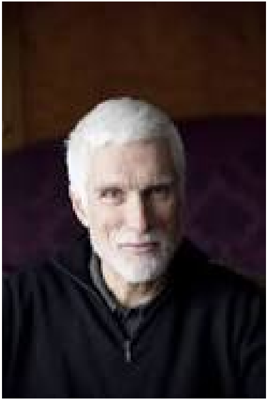
Eric Maskin: taught my game theory class at MIT

One of my classmates: Tim Kehoe



Another classmate: David Levine

Another classmate: Drew Fudenberg



Another classmate: Jean Tirole

Motivation: Collusion on the beach

Peter Brown: Amcor Managing Director

Harry Debney: Visy CEO

Russell Jones: Chairman of Amcor

Richard Pratt: Owner of Visy

The Australian cardboard market

- The Australian market for *cardboard* (CFP) is essentially a duopoly
- Between 2000 and 2005 the two firms, *Visy* and *Amcor* colluded to raise the price of CFP
- I was hired to estimate the damage caused by the collusion, which requires predicting what CFP prices would have been in the absence of collusion
- My opinion was that the "but-for" CFP prices are those predicted by Bertrand price competition in the short run, with *leapfrogging investments* by the two firms over the longer run as they vie for low-cost leadership

Amcor's New B9 Paper Mill

Main Mill Site, Botany Bay Road, Botany Bay NSW

Source: Amcor

B9 is an example of leapfrogging

- Amcor's existing paper plant was over 50 years old
- "The B9 paper machine, so named as it is the ninth paper machine to operate at the company's Botany site, will produce more than 400,000 tonnes of paper annually when operating at full capacity and will deliver significant environmental benefits."
- Cost: $500 million, the largest single investment in Amcor's 144 year history. "Largest and most innovative recycled paper machine of its kind in Australasia"
- "The machine is 330 metres long, and 22 metres high, and produces 1.6 km of paper per minute and reduces water consumption by 26%, energy usage by 34% and the amount of waste sent to landfill by 75%" (Nigel Garrard, Amcor CEO)

But collusion caused B9 to be abandoned

- Amcor had planned B9 back in 1999, and at that time internal studies estimated huge rate of return for this investment because it would enable it to leapfrog Visy to become the low-cost producer of CFP in Australia.
- Amcor and Visy were locked in a price war that started in 1999, around the time the Amcor Board authorized the B9 investment.
- However when Visy and Amcor started to collude in 2000, the B9 project was curiously scrapped. B9 was not actually started until 2011, well after the end of the collusion in 2005. B9 only came online in February 2013.

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing Amcor and Visy before and after collusion
- but there was good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

A cardboard corrugator

Technological progress via cost-reducing investments

- in this industry, Amcor and Visy do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, Amcor and Visy purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

Leapfrogging by Amcor lead to a price war

- the proximate cause of the collusion between Amcor and Visy was a price war in cardboard
- a key input to cardboard is *paper* and Amcor had a severe cost disadvantage relative to Visy due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- Visy on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however Amcor planned to invest in a new paper mill, B9, enabling it to produce CFP at substantially lower costs, thereby leapfrogging Visy to become the low cost leader in Australia

Are price wars evidence of tacit collusion?

- The economic experts defending Amcor and Visy dismissed theory of Bertrand competition and leap frogging investments as naive and out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports a theory *tacit collusion* for repeatedly interacting duopolists
- In particular, they claimed that duopolists could achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- This implies that the damage is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would Amcor and Visy have had any incentive to do illegal explicit collusion?

Paucity of empirical support for tacit collusion

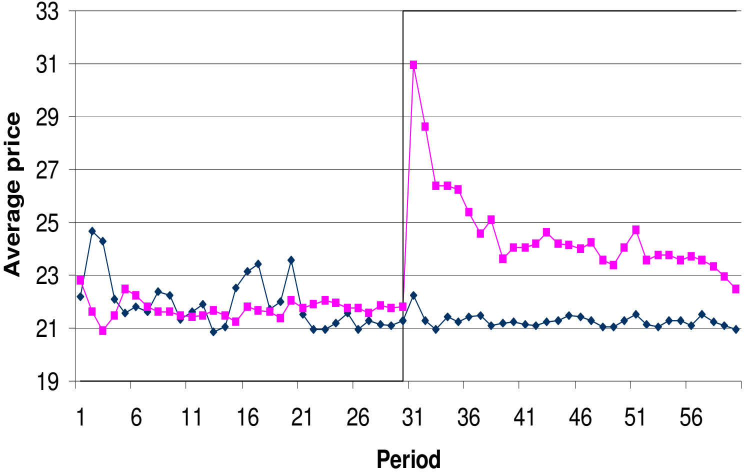
- Tacit collusion is hard to "observe" by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to "grow" tacit collusion in laboratory settings
- There are very few "field studies" that find evidence of tacit collusion outside of Breshnahan's (1987) JIE paper, "Competition and Collusion in the American Automobile Industry: the 1955 Price War"

Conclusions of meta-study of over 500 experiments

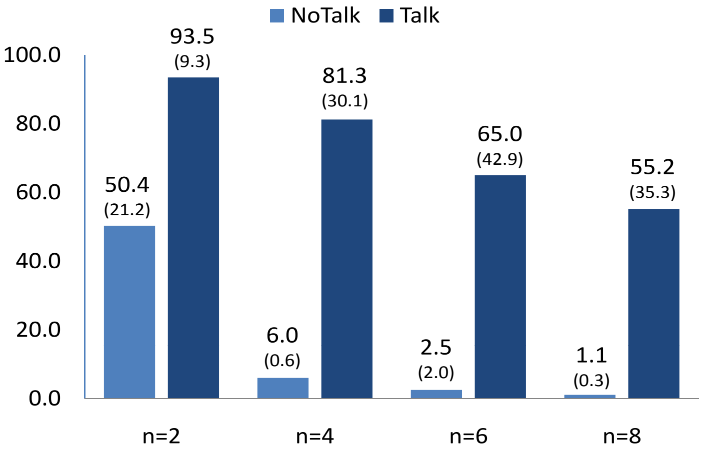
- Christoph Engel (2007) "Tacit Collusion The Neglected Experimental Evidence"
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) "Collusion through price ceilings? A search for a focal point effect"
- "Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude." (p. 2)

Results of a laboratory duopoly

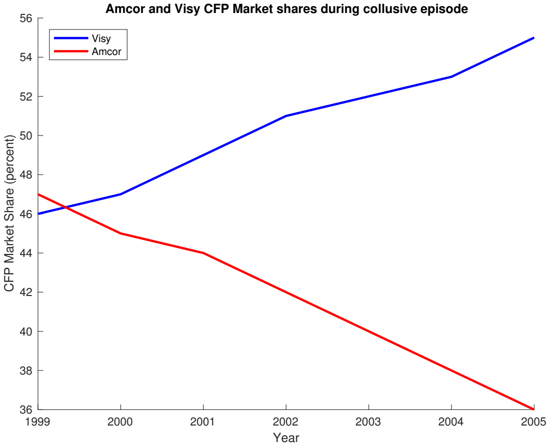
(Note: the Bertrand price is 21, the maximum cartel price is 48 and 28 is the price ceiling)



Explicit communication is necessary for collusion



Amcor and Visy collusive market shares



The Bertrand Investment Paradox

Why should Bertrand competitors undertake cost-reducing investments?

- Suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost c
- Bertrand price competition following these investments will lead to a price of $p = c$ and *zero profits for each firm*
- If each firm earns zero profits *ex post*, why would either have incentive to invest *ex ante*?

The investment stage game is an anti-coordination game. Can the firms dynamically coordinate their investments in equilibrium, in order to avoid "bad" simultaneous investment outcomes?

The Riordan and Salant Conjecture

- In their 1994 *Journal of Industrial Economics* article, Riordan and Salant proved that in continuous time, if duopolists move alternately and technological progress is deterministic, then investment preemption is the only possible equilibrium outcome
- Further, they show this equilibrium is *completely inefficient* due to the excessively frequent investments of the preempting firm, a result they call rent dissipation
- They conjectured that their result does not depend on the alternating move assumption and that preemption (as opposed to leapfrogging) will be the generic equilibrium outcome in models of Bertrand price competition with cost-reducing investements.

Solution to the Bertrand Investment Paradox

We show:

- Endogenous coordination is possible in equilibrium
  - leapfrogging (alternating investments) is possible
  - We show that the Riordan and Salant conjecture is wrong: leapfrogging, not preemption, is the generic outcome
- Price paths are piecewise flat and non-increasing
  - *Price wars* occur when the high cost firm leapfrogs its rival to become the new low-cost leader
  - These price wars lead to *permanent* price declines, unlike the conventional interpretation of price wars as punishments for periodic breakdowns in tacit collusion
- Equilibria are generally inefficient due to overinvestment
  - duplicative investments
  - excessively frequent investments

Computing all equilibria

Our findings are based on the computation of all Markov perfect equilibria of this dynamic game

- New solution approach consisting of:
  1. State recursion algorithm for finding stage equilibria
  2. Recursive Lexicographic equilibrium Search (RLS) algorithm for finding all MPE paths
- Traditional solution approach (value function iterations, i.e. time recursion) fails in this model due to multiplicity of equilibria
  - Implementation of the Bellman operator induces an equilibrium selection rule
  - Not a contraction mapping, convergence is not guaranteed
- Danger of imposing symmetry
  - Most of MPE equilibria we find are asymmetric

Part II: Dynamic Bertrand price competition (leapfrogging) model

Dynamic Bertrand price competition

Stochastic dynamic game

- Two Bertrand competitors, $n = 2$, no entry or exit
- Discrete time, infinite horizon ($t = 1, 2, \ldots, \infty$)
- Each firm maximizes expected discounted profits, common discount factor $\beta \in (0, 1)$
- Each firm has two choices in each period:
  1. Price for the product
  2. Whether or not to buy the state of the art technology

Static Bertrand price competition in each period

- Continuum of consumers make static purchase decision
- No switching costs: buy from the lower price supplier

Cost-reducing investments

State-of-the-art production cost c process

- Initial value c_0, lowest value 0: $0 \leq c \leq c_0$
- Discretized with n points
- Follows exogenous Markov process and only improves
- Markov transition probability $\pi(c_{t+1}|c_t)$
  $\pi(c_{t+1}|c_t) = 0$ if $c_{t+1} > c_t$

Investment choice: dichotomous

- Investment cost of $K(c)$ to obtain marginal cost c
- One period construction time: production with technology obtained at t starts at $t+1$

State space and information structure

Common knowledge

- State of the game: cost structure (c_1, c_2, c)
- State space is $S = (c_1, c_2, c) \subset R^3$: $c_1 \geq c$, $c_2 \geq c$
- Actions are observable

Private information

- In each period each firm incurs additive costs (benefits) from not investing and investing $\eta\epsilon_{i,I}$ and $\eta\epsilon_{i,N}$
- $\epsilon_{i,I}$ and $\epsilon_{i,N}$ are extreme value distributed, independent across choice, time and firms
- $\eta \geq 0$ is a scaling parameter
- Investment choice probabilities have logit form for $\eta > 0$

Timing of moves

Pricing decisions are made simultaneously

Expected one period profit of firm i from Bertrand game ($j \neq i$)

$$r_i(c_1, c_2) = \begin{cases} 0 & \text{if } c_i \geq c_j \\ c_j - c_i & \text{if } c_i < c_j \end{cases}$$

Two versions regarding investment decisions

1. Simultaneous moves:
    - Investment decisions are made simultaneously
2. Alternating moves:
    - The "right to move" state variable $m \in \{1, 2\}$,
    - When $m = i$, only firm i can make a cost reducing investment
    - m follows an own Markov process (deterministic alternation as a special case).

Actions and behavior strategies

Two choices in each period

- $p_i(c_1, c_2, c) = \max(c_1, c_2)$ – Bertrand pricing decision
- $P_i^I(c_1, c_2, c)$ – probability of firm i to invest in state-of-the-art production technology
  $P_i^N(c_1, c_2, c) = 1 - P_i^I(c_1, c_2, c)$ – probability not to invest

Strategy profile

- $\sigma = (\sigma_1, \sigma_2)$ – pair of Markovian *behavior* strategies
  $\sigma_i = \Big(p_i(c_1, c_2, c), P_i^I(c_1, c_2, c)\Big) \in \mathbb{R}_+ \times [0, 1]$
- Pure strategies included as special case

Definition of Markov Perfect Equilibium

Definition (Markov perfect equilibrium (MPE))

MPE of Bertrand investment stochastic game is a pair of

- strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*)$, and
- pair of *value functions* $V(s) = \Big(V_1(s), V_2(s)\Big)$, $V_i : S \to R$,

such that

1. Bellman equations (below) are satisfied for each firm, and
2. strategies σ_1^* and σ_2^* constitute mutual best responses, and assign positive probabilities only to the actions in the set of maximizers of the Bellman equations.

Bellman equations, firm $i = 1$, simultaneous moves

$$V_i(c_1, c_2, c) = \max\left[v_i^I(c_1, c_2, c) + \eta\epsilon_{i,I}, v_i^N(c_1, c_2, c) + \eta\epsilon_{i,N}\right]$$

$$\begin{aligned} v_i^N(c_1, c_2, c) &= r^i(c_1, c_2) + \beta EV_i(c_1, c_2, c|N) \\ v_i^I(c_1, c_2, c) &= r^i(c_1, c_2) - K(c) + \beta EV_i(c_1, c_2, c|I) \end{aligned}$$

With extreme value shocks, the investment probability is

$$P_i^I(c_1, c_2, c) = \frac{\exp\{v_I^I(c_1, c_2, c)/\eta\}}{\exp\{v_i^I(c_1, c_2, c)/\eta\} + \exp\{v_i^N(c_1, c_2, c)/\eta\}}$$

Bellman equations, firm $i = 1$, simultaneous moves

The expected values are given by

$$
\begin{aligned}
EV_i(c_1, c_2, c|N) &= \int_0^c \left[P_j^I(c_1, c_2, c) H_i(c_1, c, c') + \right. \\
&\qquad \left. P_j^N(c_1, c_2, c) H_i(c_1, c_2, c') \right] \pi(dc'|c) \\
EV_i(c_1, c_2, c|I) &= \int_0^c \left[P_j^I(c_1, c_2, c) H_i(c, c, c') + \right. \\
&\qquad \left. P_j^N(c_1, c_2, c) H_i(c, c_2, c') \right] \pi(dc'|c)
\end{aligned}
$$

where

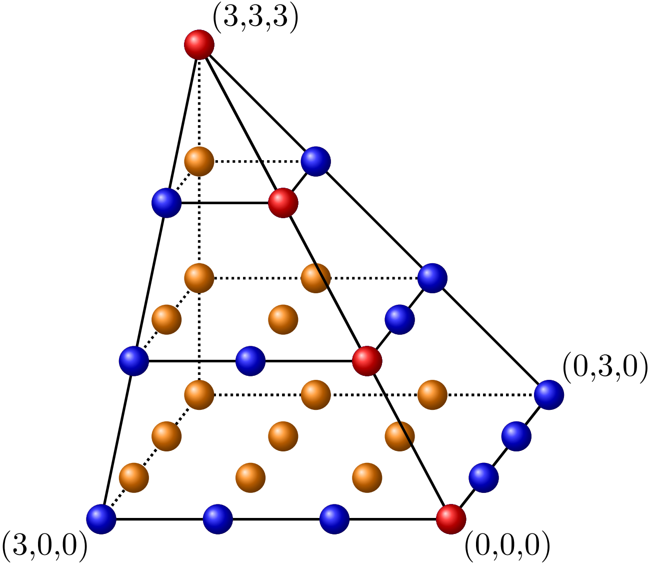
$$H_i(c_1, c_2, c) = \eta \log \left[\exp\left(v_i^N(c_1, c_2, c)/\eta \right) + \exp\left(v_i^I(c_1, c_2, c)/\eta \right) \right]$$

is the "smoothed max" or logsum function

Leapfrogging Game

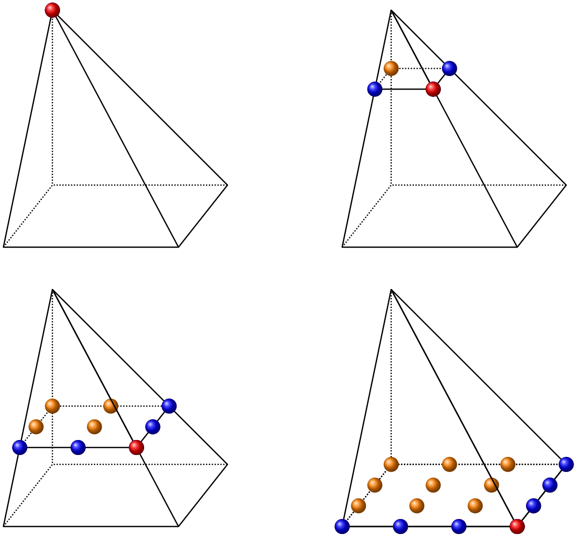
Discretized state space = a "quarter pyramid"

$S = \{(c_1, c_2, c) | c_1 \geq c,\ c_2 \geq c, c \in [0, 3]\},\ n = 4$



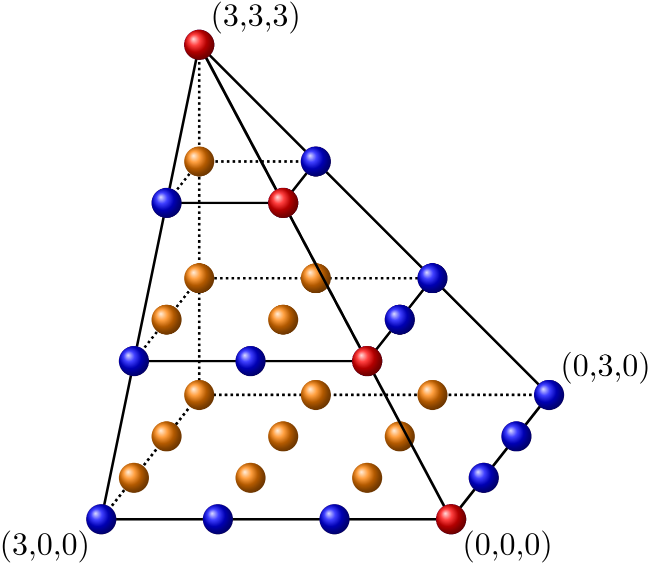
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



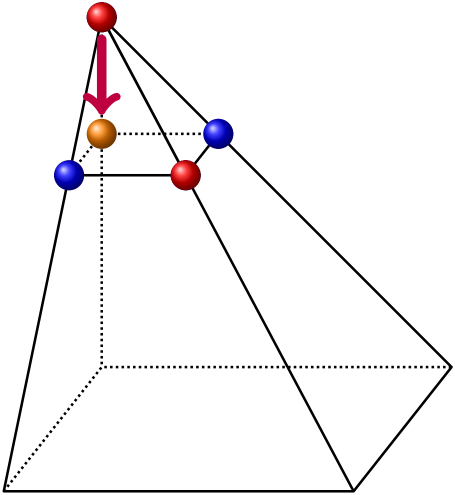
Discretized state space = a "quarter pyramid"

$S = \{(c_1, c_2, c) | c_1 \geq c,\ c_2 \geq c, c \in [0, \overline{c}]\},\ N = 4$



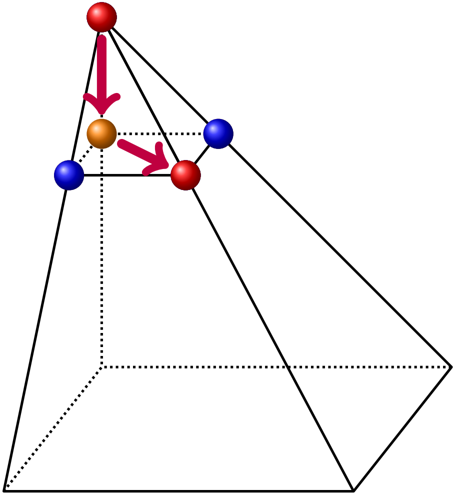
Game dynamics: example

The game starts at the apex, as some point technology improves



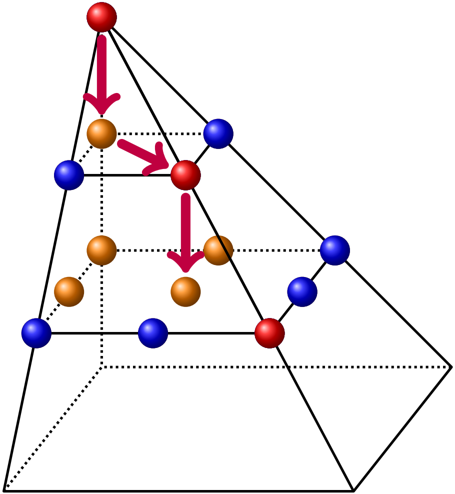
Game dynamics: example

Both firms buy new technology $c = 2 \rightsquigarrow (c_1, c_2, c) = (2, 2, 2)$



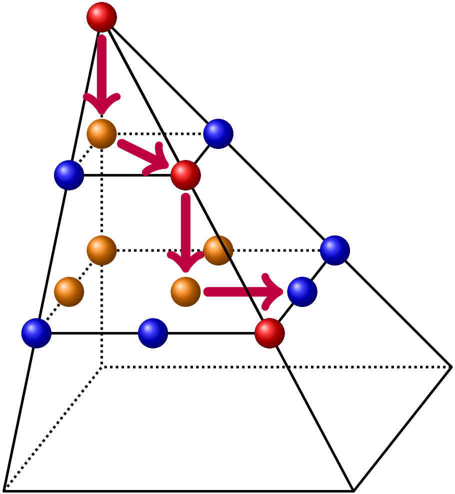
Game dynamics: example

State-of-the-art technology becomes $c = 1 \rightsquigarrow (c_1, c_2, c) = (2, 2, 1)$



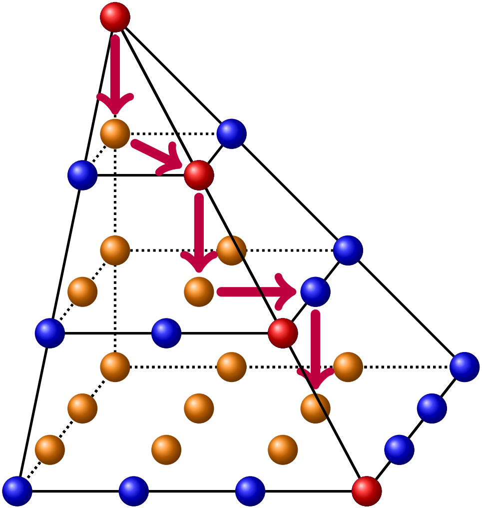
Game dynamics: example

Firm 1 invests and becomes cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 2, 1)$



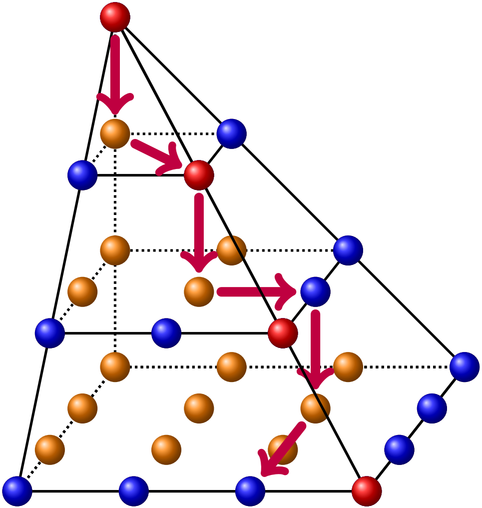
Game dynamics: example

State-of-the-art technology becomes $c = 0 \rightsquigarrow (c_1, c_2, c) = (1, 2, 0)$



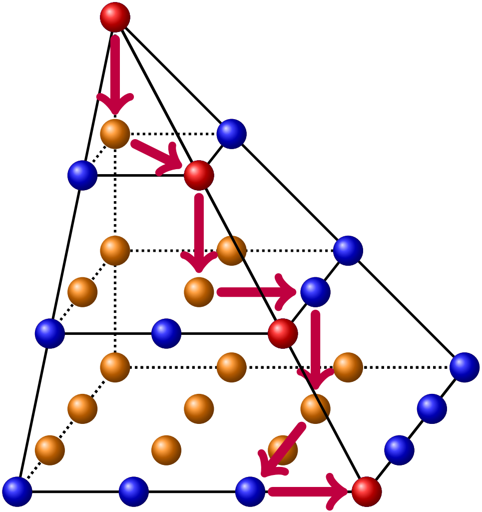
Game dynamics: example

Firm 2 leapfrogs firm 1 to become new cost leader $\rightsquigarrow$ $(c_1, c_2, c) = (1, 0, 0)$



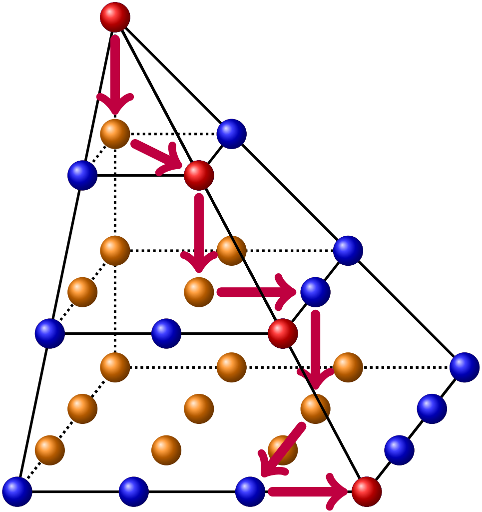
Game dynamics: example

Firm 1 invests, and the game reaches terminal state $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



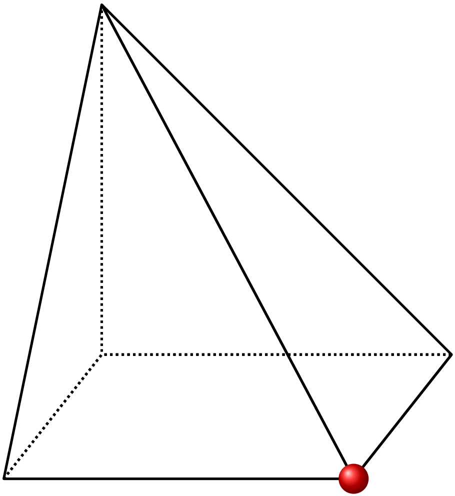
Game dynamics: example

Firm 1 invests, and the game reaches terminal state $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



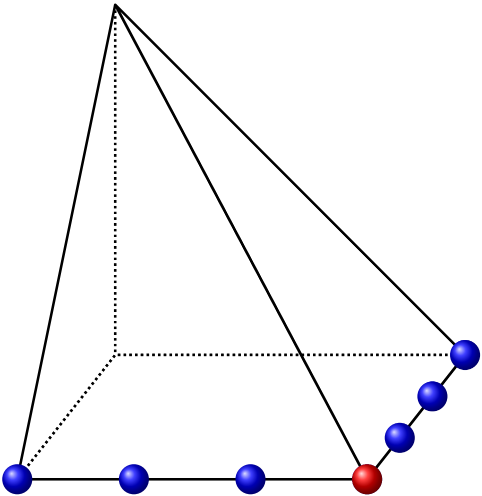
State recursion

Solve terminal state $(c_1, c_2, c) = (0, 0, 0)$



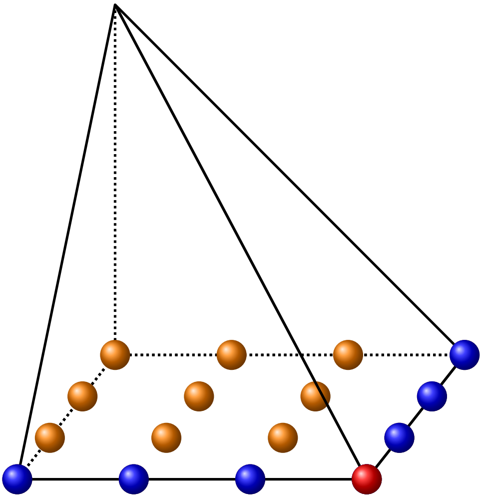
State recursion

Solve bottom layer edges $(c_1, c_2, c) = (0, c_2, 0)$ and $(c_1, 0, 0)$



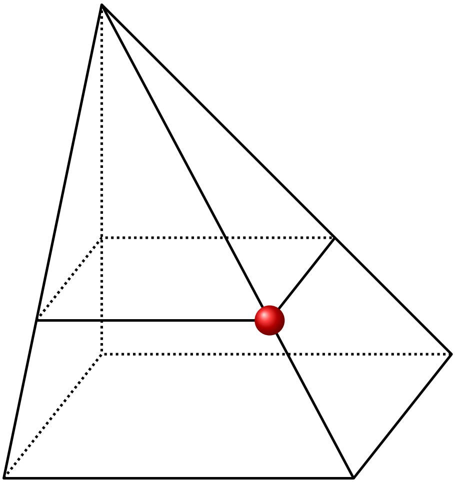
State recursion

Solve bottom layer interior $(c_1, c_2, c) = (c_1, c_2, 0)$



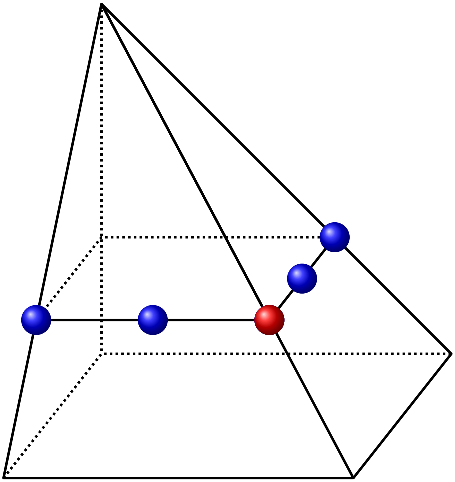
State recursion

Solve the corner $(c_1, c_2, c) = (1, 1, 1)$



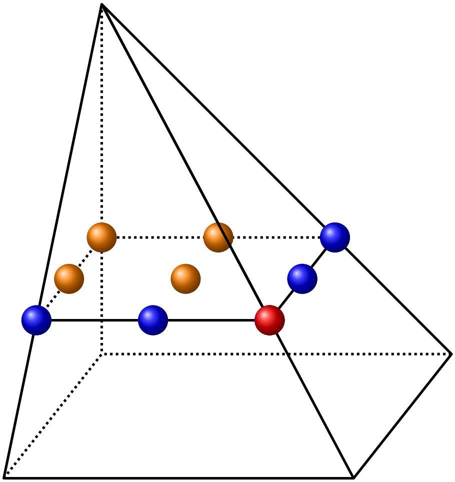
State recursion

Solve edges $(c_1, c_2, c) = (1, c_2, 1)$ and $(c_1, 1, 1)$



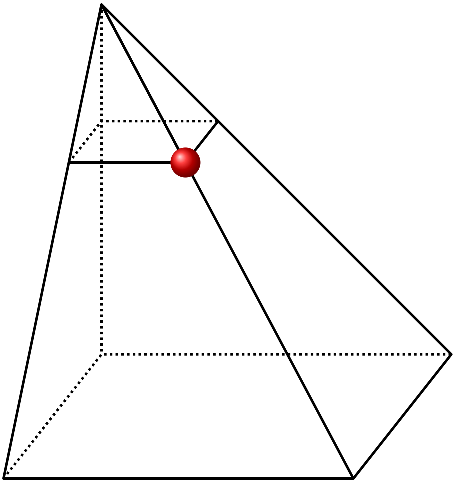
State recursion

Solve interior $(c_1, c_2, c) = (c_1, c_2, 1)$



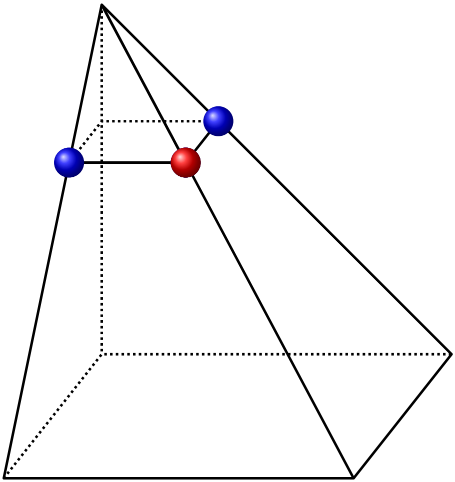
State recursion

Solve the corner $(c_1, c_2, c) = (2, 2, 2)$



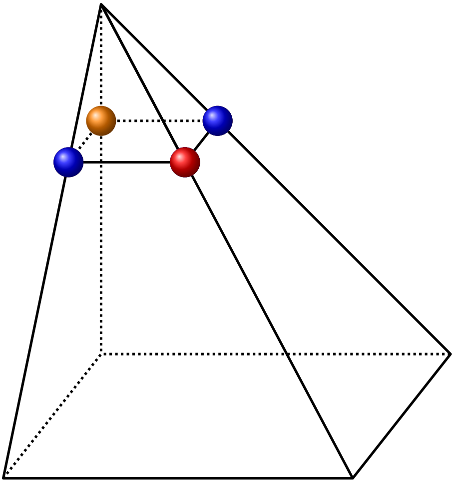
State recursion

Solve edges $(c_1, c_2, c) = (2, c_2, 2)$ and $(c_1, 2, 2)$



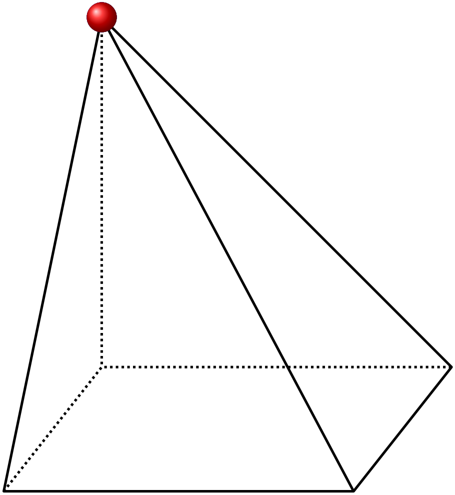
State recursion

Solve interior $(c_1, c_2, c) = (c_1, c_2, 2)$



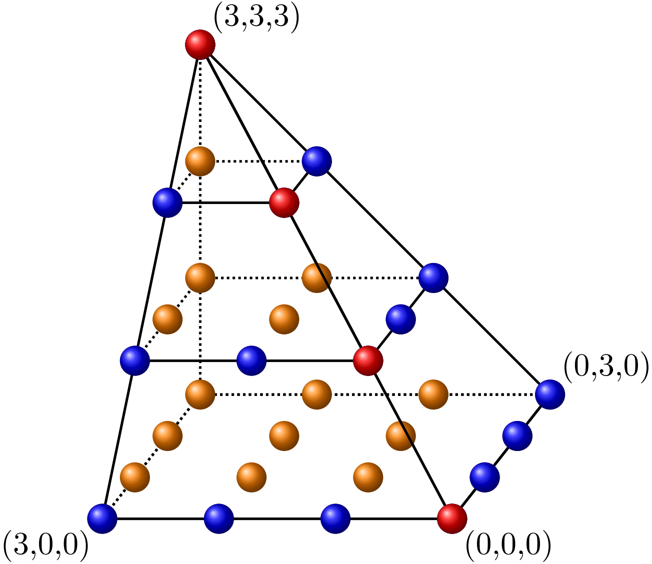
State recursion

Finally solve the apex $(c_1, c_2, c) = (3, 3, 3)$



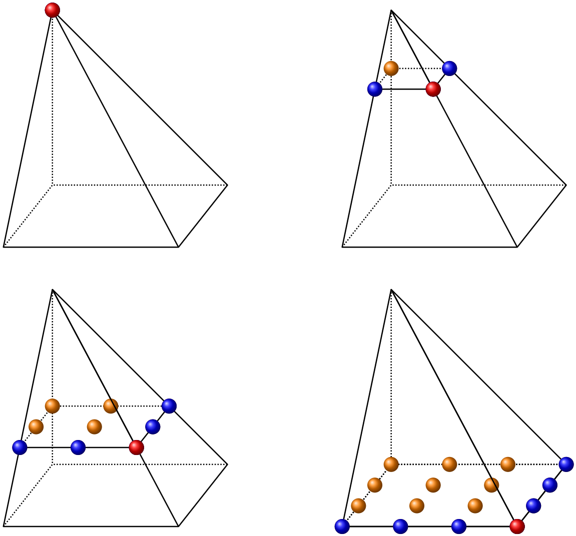
State recursion

The whole game is solved!



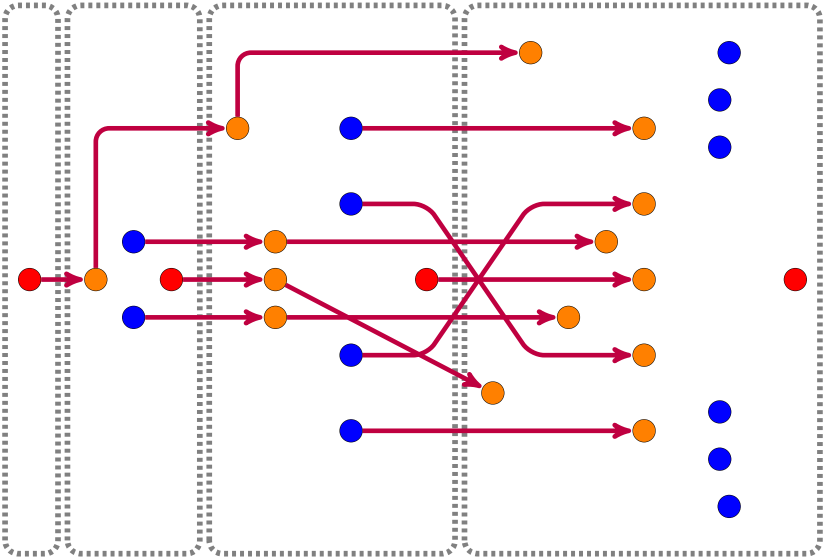
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



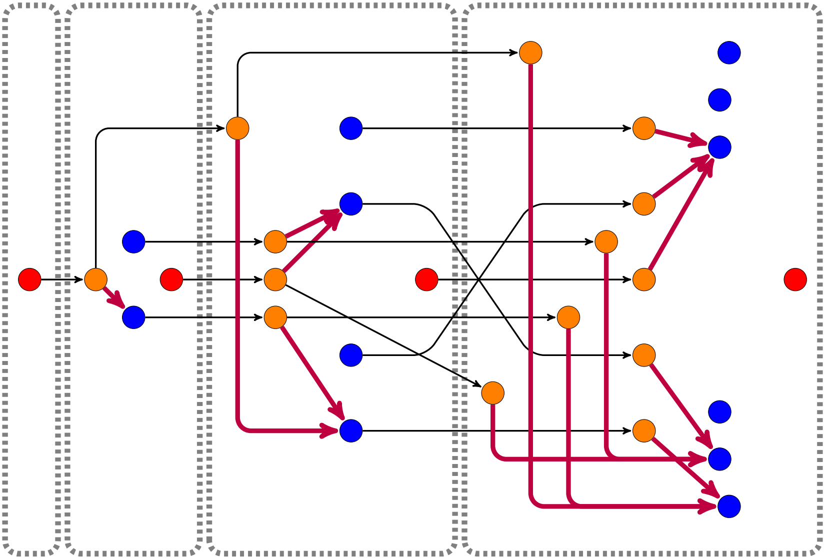
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



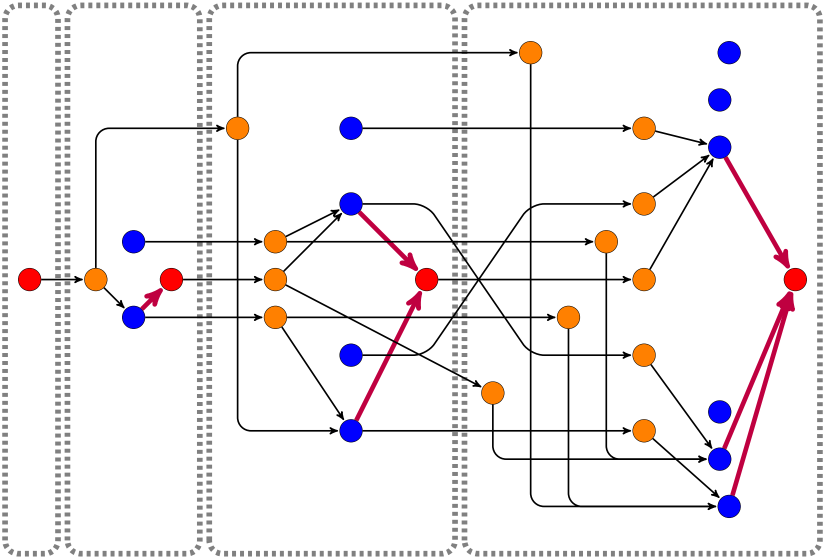
Strategy-specific partial order on S

Strategy σ_1 of firm 1: invest at all interior points



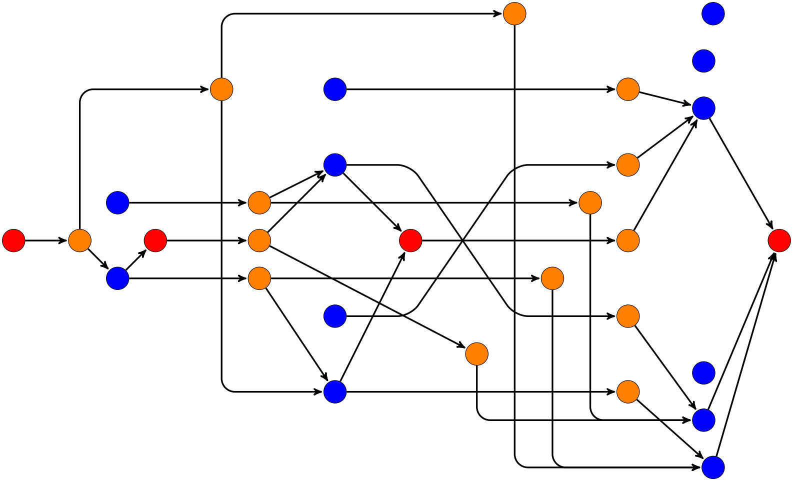
Strategy-specific partial order on S

Strategy σ_2 of firm 2: invest at all edge points



Strategy-specific partial order on S

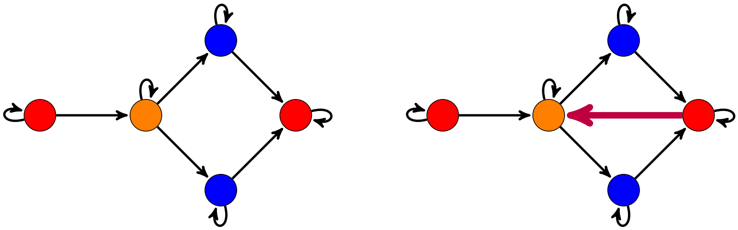
Strategy $\sigma = (\sigma_1, \sigma_2)$ of both firms



No loop (anti-cycling) condition

Hypothetical strategy profile inducing cycles

Self-loops appear when the game remains in the same state for two or more consecutive periods of time

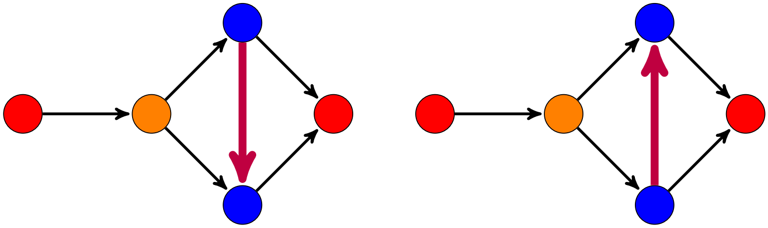


But loops between different states are not allowed

Consistency of strategy specific partial orders

Two hypothetical inconsistent strategies

Two strategies that induce opposite transitions are inconsistent



Note that in both cases the no-loop condition is satisfied

Definition of the Dynamic Directional Games

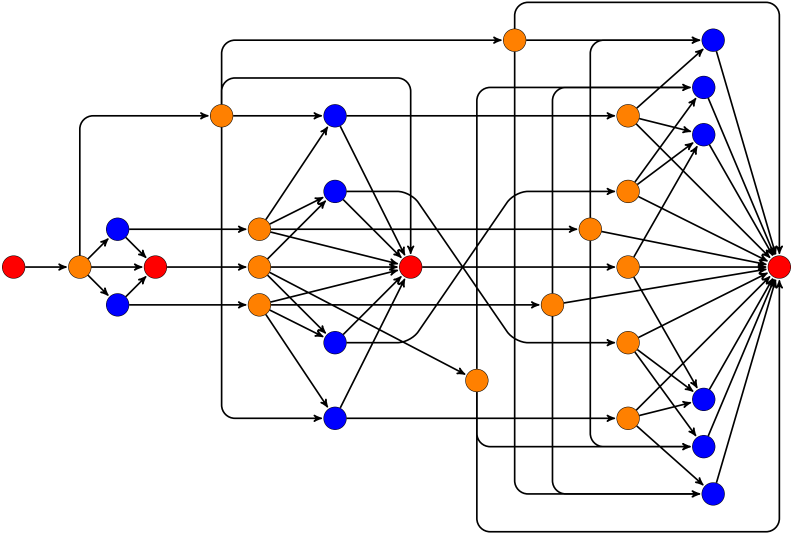
Definition (Dynamic Directional Games, DDG)

Finite state Markovian stochastic game is a DDG if it holds:

1. Every feasible Markovian strategy σ satisfies the no loop condition.
2. Every pair of feasible Markovian strategies σ and σ' induce consistent partial orders on the state space.

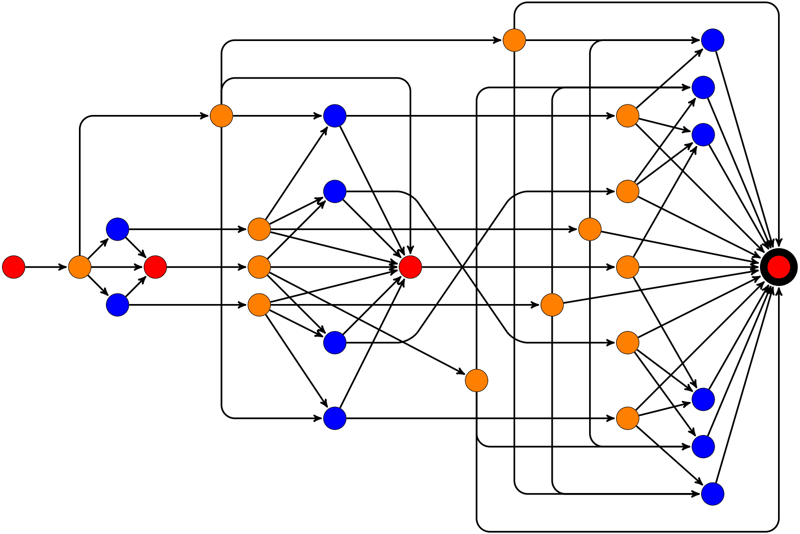
Strategy independent partial order on S

Coarsest common refinement of partial orders induced by all strategies



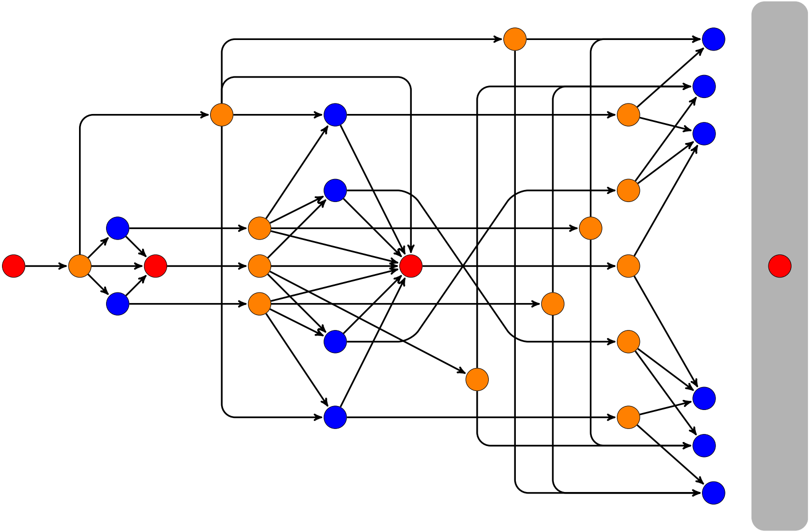
DAG recursion to partition S into stages

Identify terminal states



DAG recursion to partition S into stages

Remove terminal states



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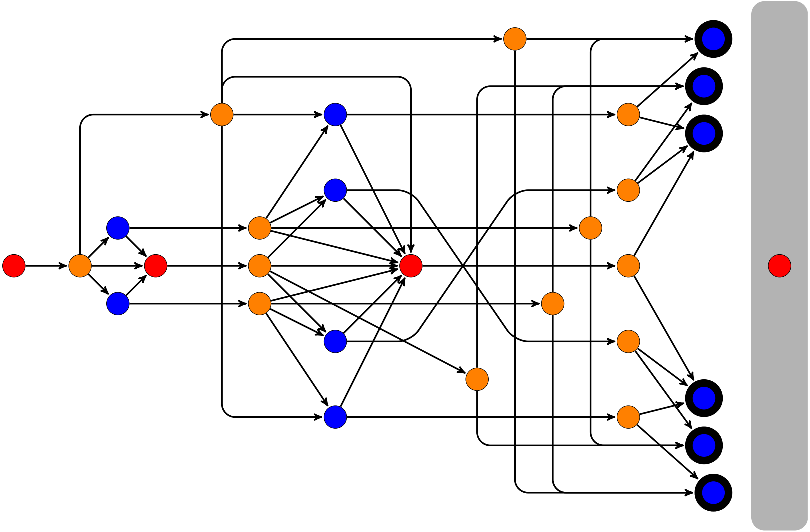
Identify terminal states



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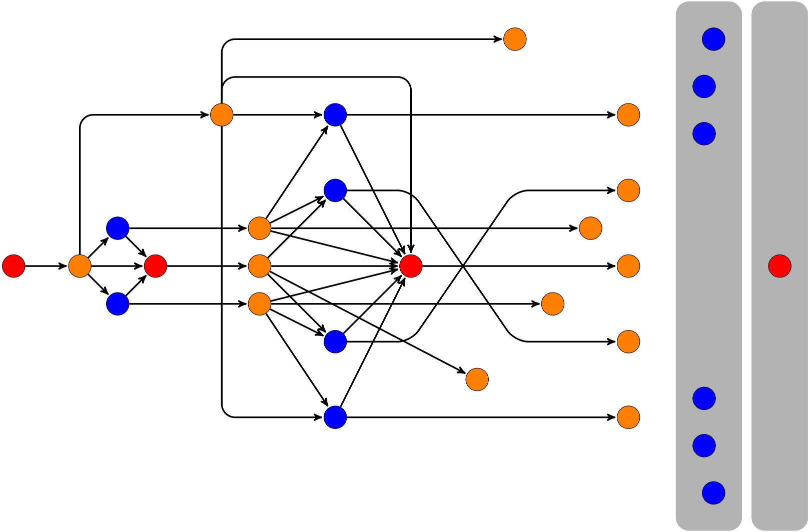

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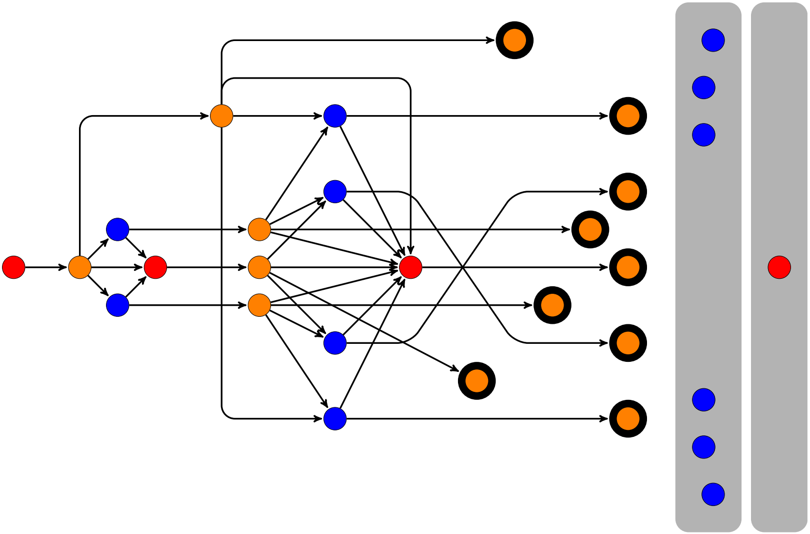

DAG recursion to partition S into stages

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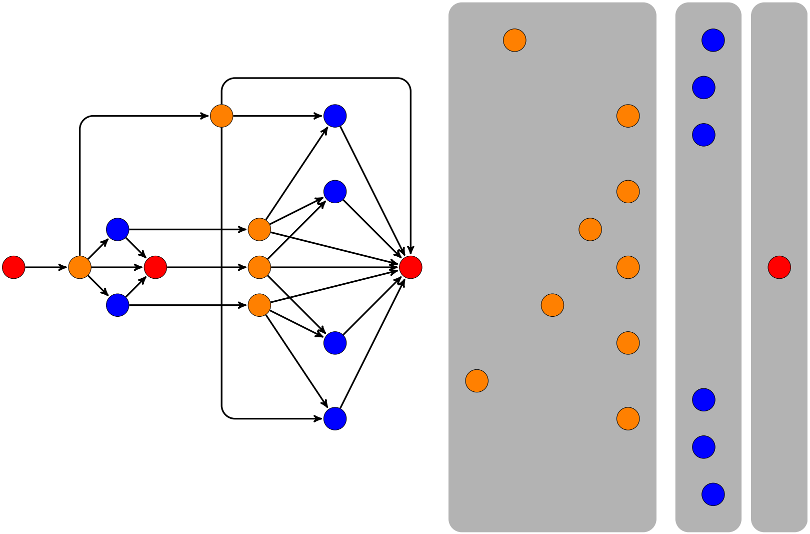

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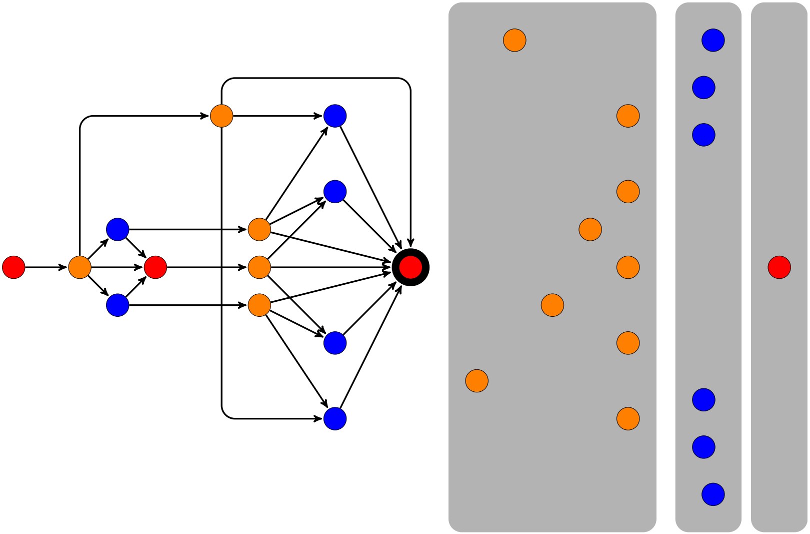
Identify terminal states



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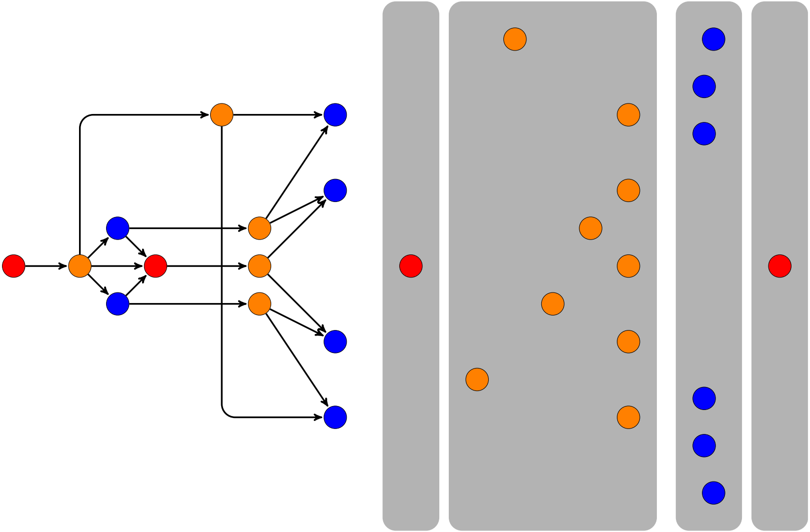
Remove terminal states



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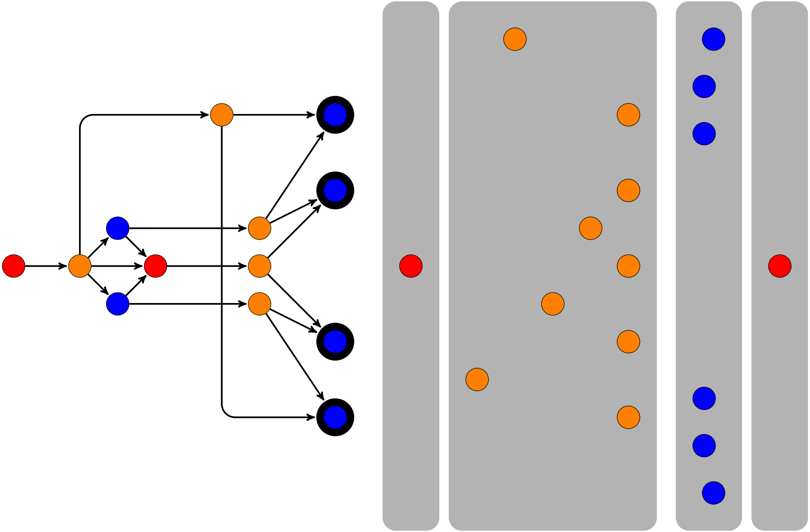
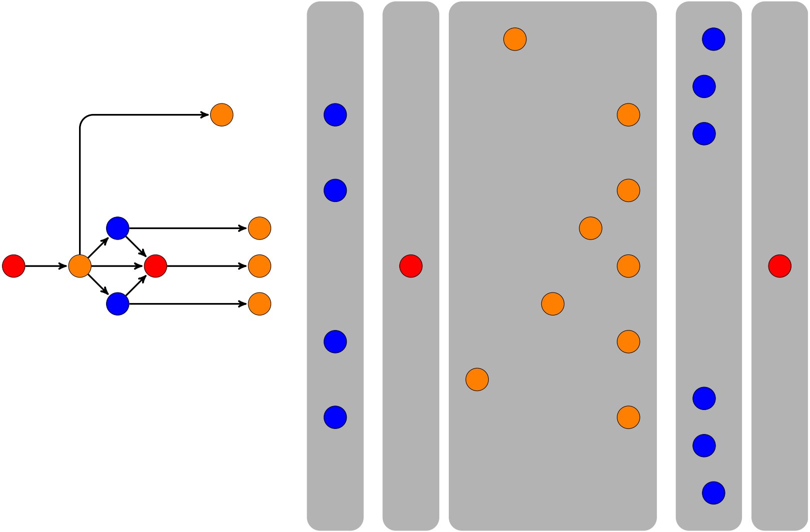


DAG recursion to partition S into stages

Remove terminal states



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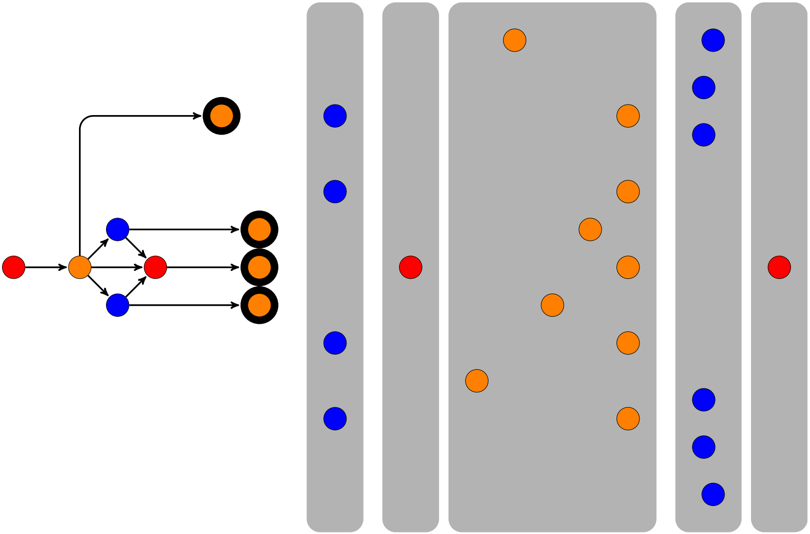
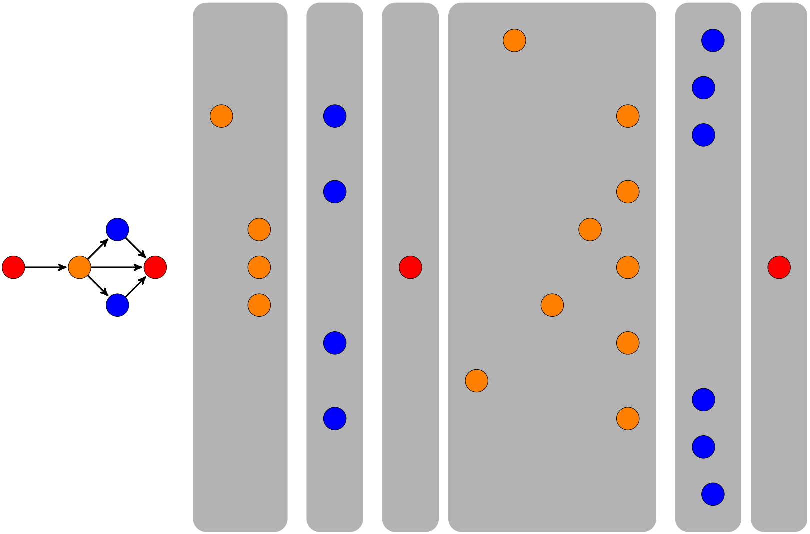


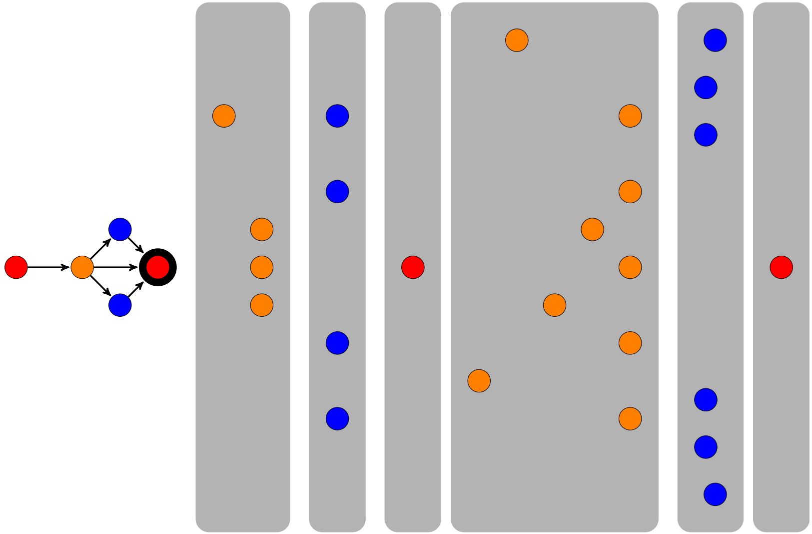
DAG recursion to partition S into stages

Remove terminal states



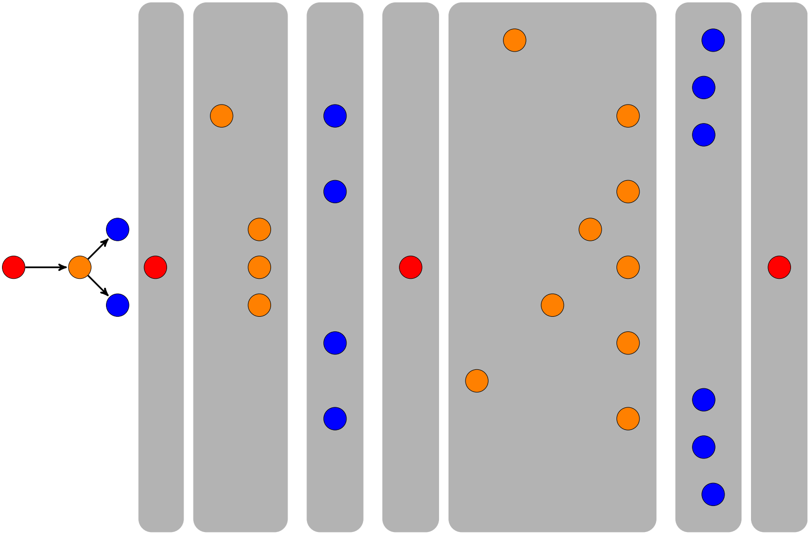
DAG recursion to partition S into stages

Identify terminal states



DAG recursion to partition S into stages

Remove terminal states



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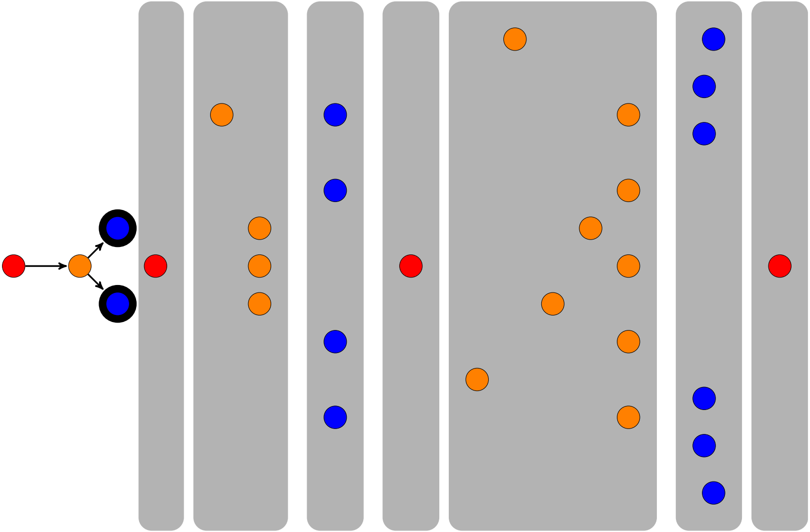
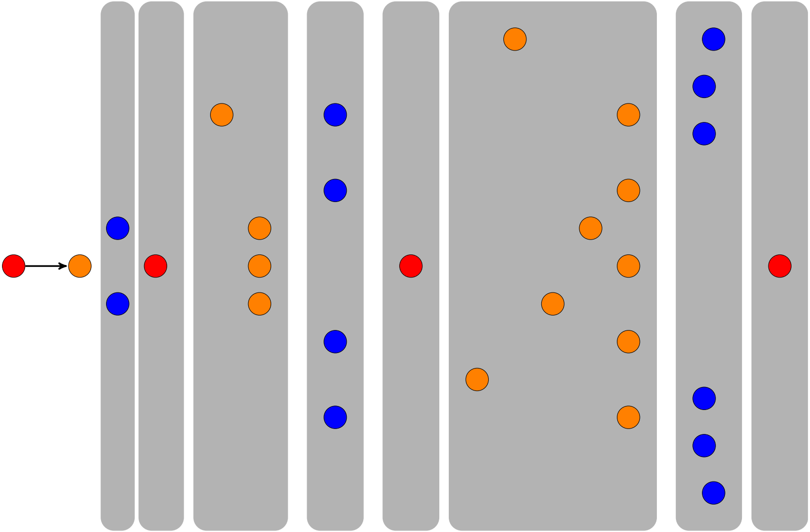


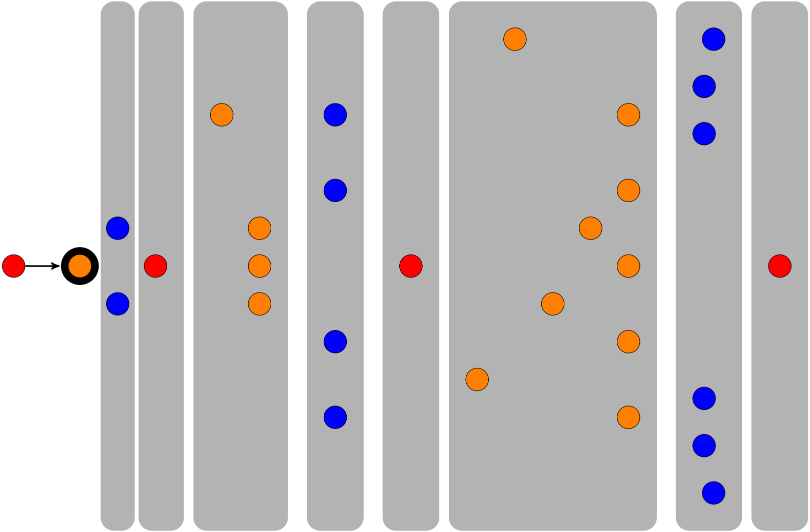
DAG recursion to partition S into stages

Remove terminal states



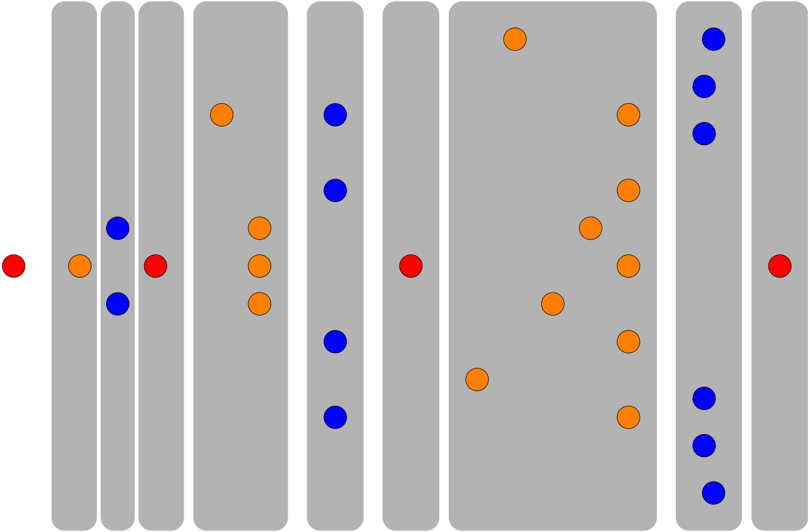
Resulting partition of S into stages

Identify terminal states



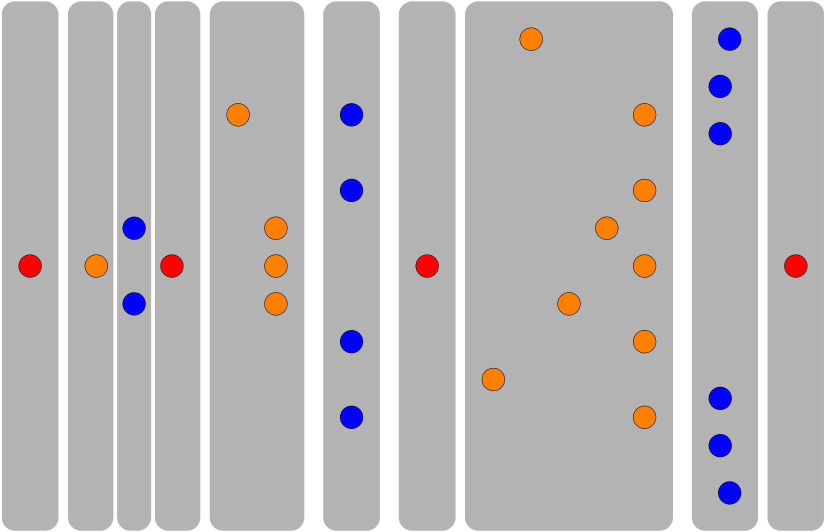
Resulting partition of S into stages

Remove terminal states



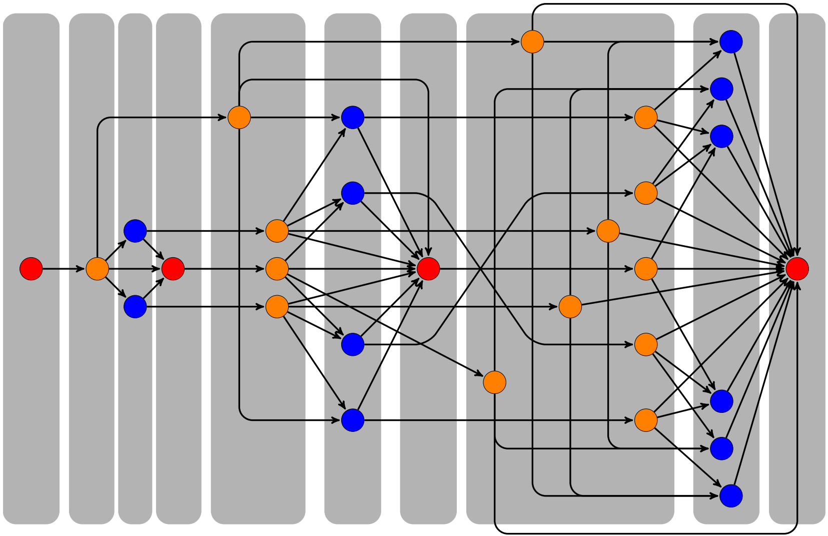
Resulting partition of S into stages

The stages of the game are totally ordered



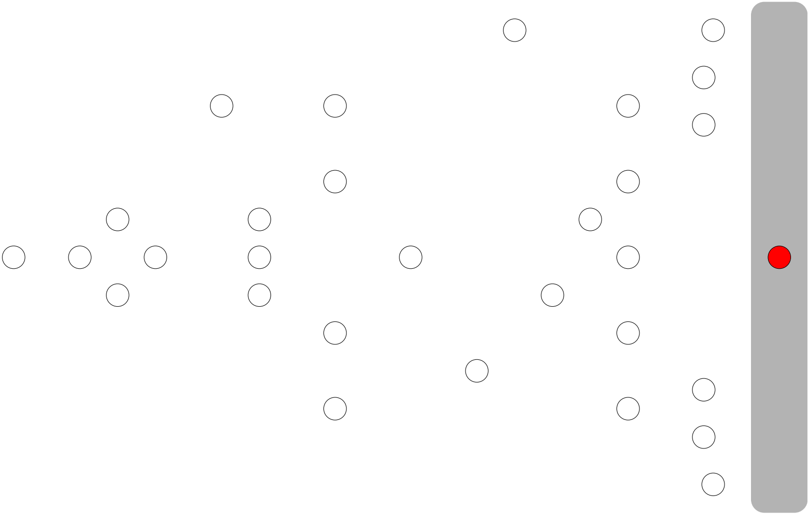
Total order on the set of stages

Subgames of DDG follow the order of stages



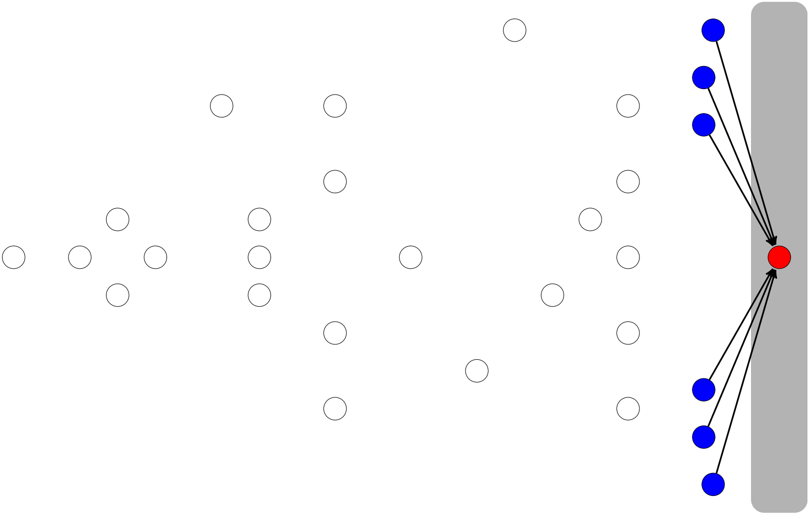
State recursion algorithm

Backward induction on stages of DDG



State recursion algorithm

Backward induction on stages of DDG



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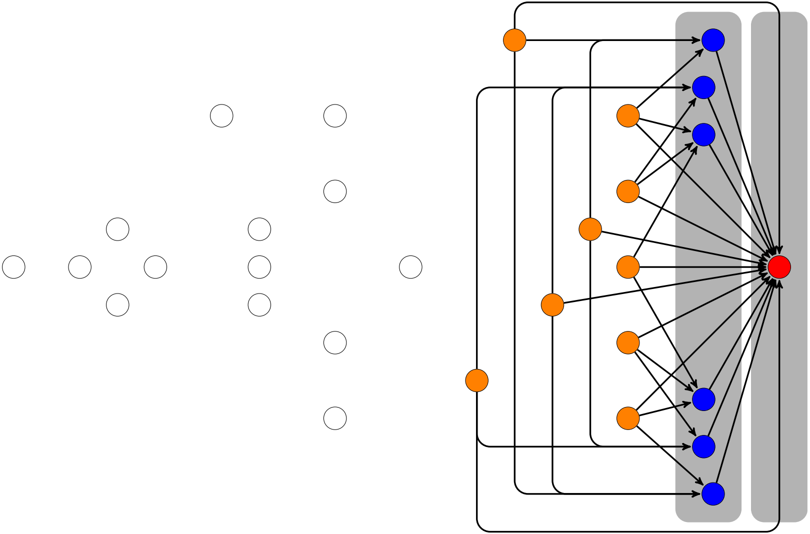
Backward induction on stages of DDG



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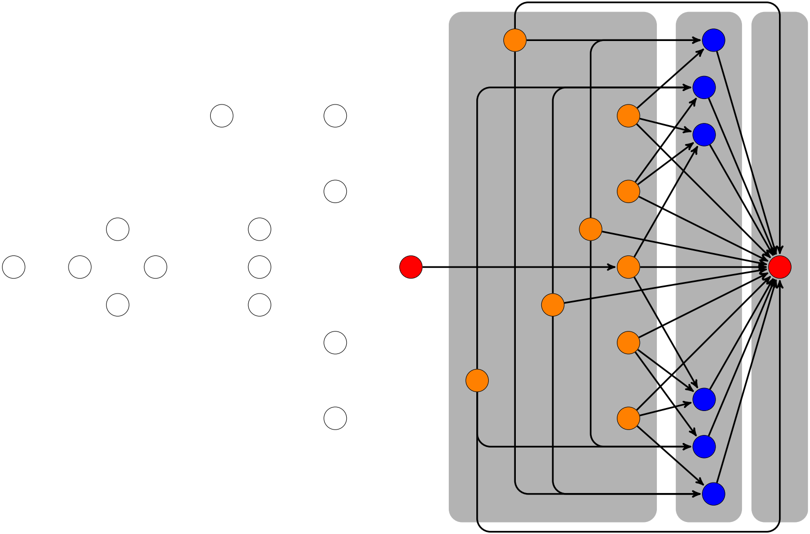

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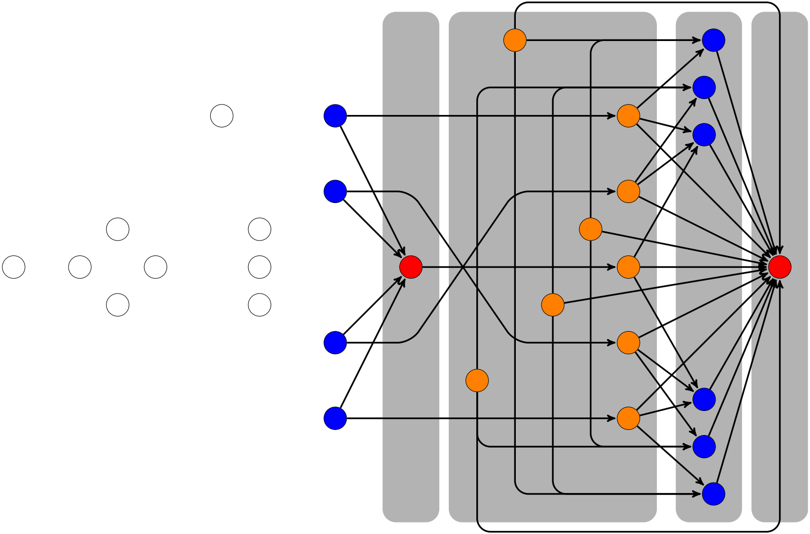

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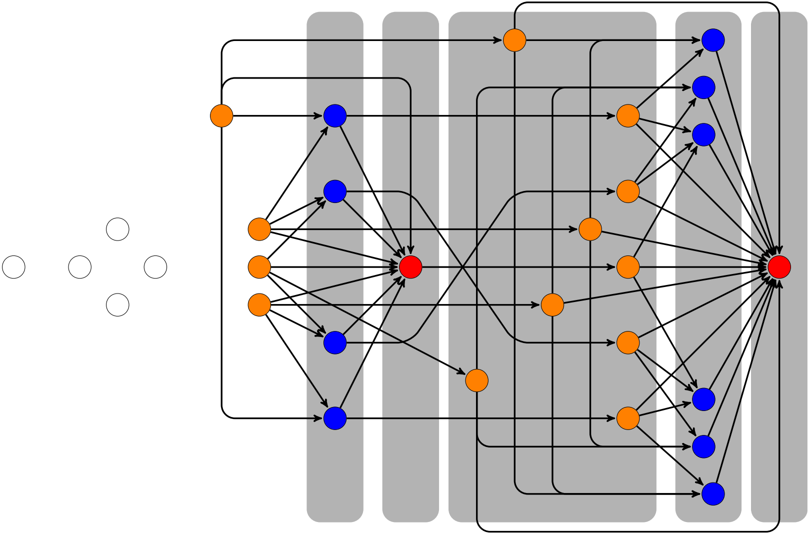
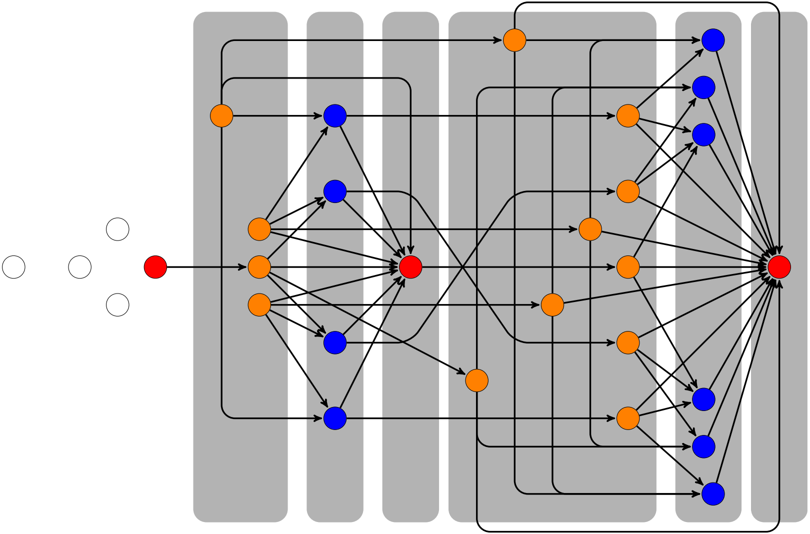
Backward induction on stages of DDG



State recursion algorithm

Backward induction on stages of DDG



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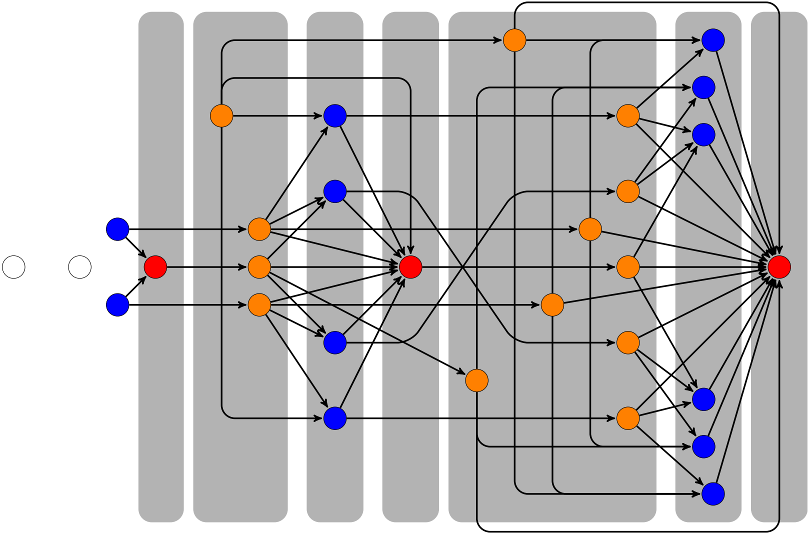
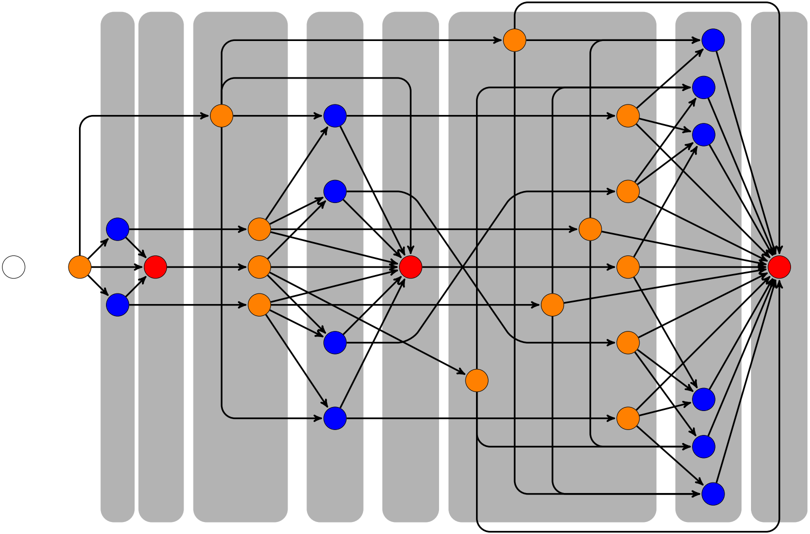


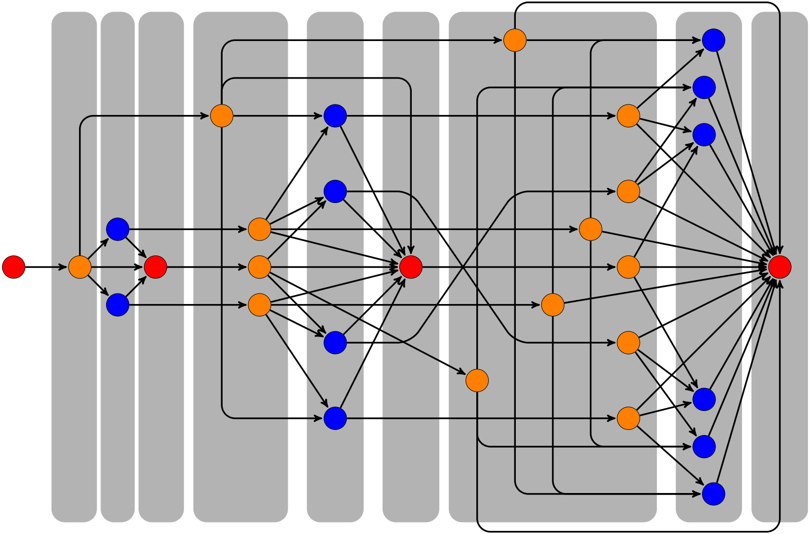
State recursion algorithm

Backward induction on stages of DDG



State recursion algorithm

Backward induction on stages of DDG

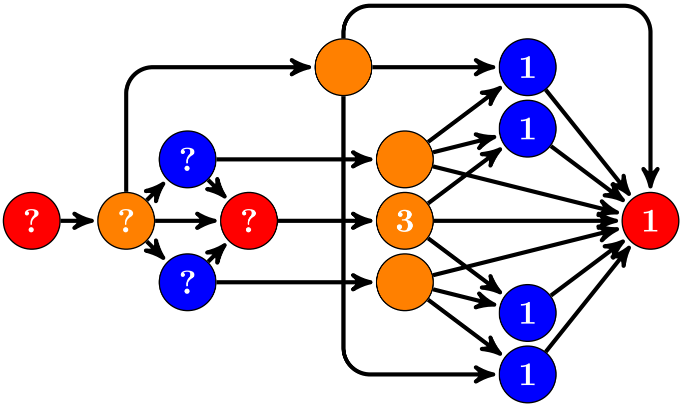


State Recursion versus Backward Induction

- State recursion – generalization of backward induction
- Runs on state space instead of time periods
- Time (t) evolves as $t \to t + 1$ with probability 1
- For stages of state space (τ) transitions are stochastic and not necessarily sequential
- Yet, probability of going $\tau \to \tau'$ is zero when $\tau' < \tau$
- With multiplicity, state recursion is performed conditional of a particular equilibrium selection rule (ESR)

Multiplicity of equibiria

Number of equilibria in the higher stages depends on the selected equilibria



Recursive Lexicographic Search Algorithm

Building blocks of RLS algorithm:

1. State recursion algorithm solves the game conditional on equilibrium selection rule (ESR)
2. RLS algorithm efficiently cycles through all feasible ESRs

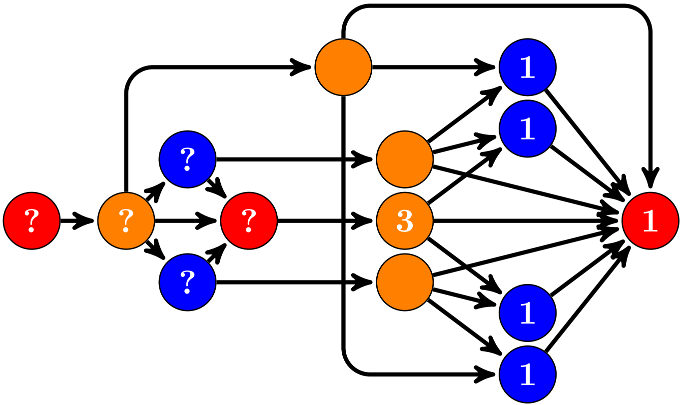
Challenge:

- Choice of a particular MPE for any stage game at any stage
- may alter the set and even the number of stage equilibria at earlier stages

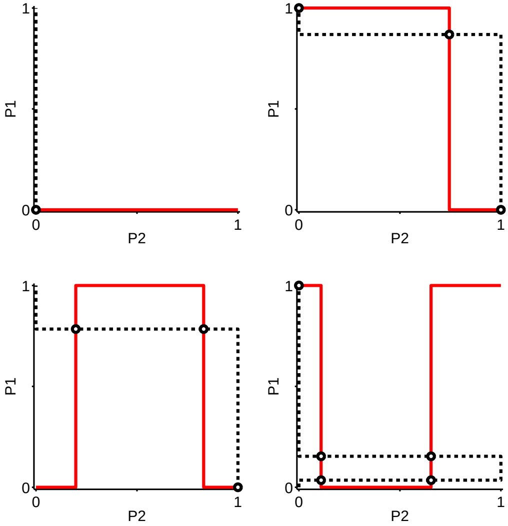
Need to find feasible ESRs

Multiplicity of equibiria

Number of equilibria in the higher stages depends on the selected equilibria



Best response correspondences of the two firms



Represent ESR as string of digits

Use numbers in base-K number system with digits $0, 1, .., K-1$

Dependence preserving property:
Any point of the state space may depend on the points to the left (higher digits) and not the points to the right (lower digits)

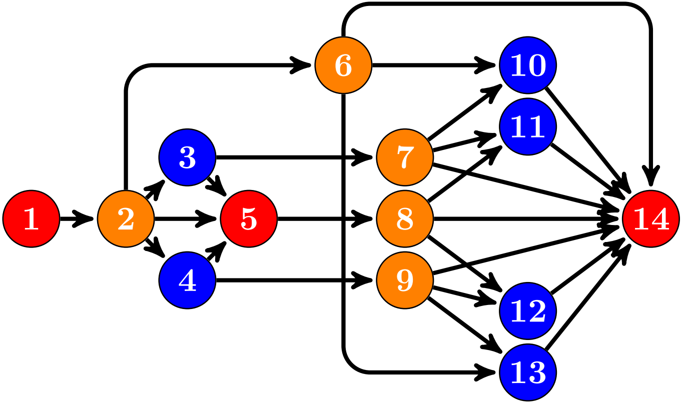
corner

edges interior

	c	e	e	e	e	i	i	i	i	c	e	e	i	c
ESR string	14	13	12	11	10	9	8	7	6	5	4	3	2	1
c	0	0	0	0	0	0	0	0	0	1	1	1	1	2
$c1$	0	0	0	2	1	2	2	1	1	1	1	2	2	2
$c2$	0	2	1	0	0	2	1	2	1	1	2	1	2	2

Digits of the ESR string mapped to the state space

Dependent preserving property



All possible ESR in lexicographic order, $K = 3$

	c	e	e	e	e	i	i	i	i	c	e	e	i	c
ESR string	*14*	*13*	*12*	*11*	*10*	*9*	*8*	*7*	*6*	*5*	*4*	*3*	*2*	*1*
Lexicograph	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	1	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	0	1	2
	0	0	0	0	0	0	0	0	0	0	0	0	2	0
	0	0	0	0	0	0	0	0	0	0	0	0	2	1
	0	0	0	0	0	0	0	0	0	0	0	0	2	2
	0	0	0	0	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	0	0	0	0	1	0	1
														...
	2	2	2	2	2	2	2	2	2	2	2	2	2	0
	2	2	2	2	2	2	2	2	2	2	2	2	2	1
	2	2	2	2	2	2	2	2	2	2	2	2	2	2

4,782,969

Variable base arithmetics

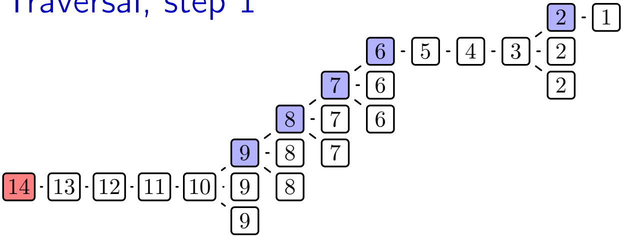
- Replace the base-K ($mod(K)$) arithmetics with variable base arithmetics
- Let (3 1 2) be bases $\rightarrow$
- Allowed digits in the numbers are $\{0, 1, 2\}$, $\{0\}$ and $\{0, 1\}$
- The 3-digit numbers in this system are:

$$
\begin{array}{lll}
0\ 0\ 0 & +1 & \rightarrow \\
0\ 0\ 1 & +1 & \rightarrow \\
1\ 0\ 0 & +1 & \rightarrow \\
1\ 0\ 1 & +1 & \rightarrow \\
2\ 0\ 0 & +1 & \rightarrow \\
2\ 0\ 1 & &
\end{array}
$$

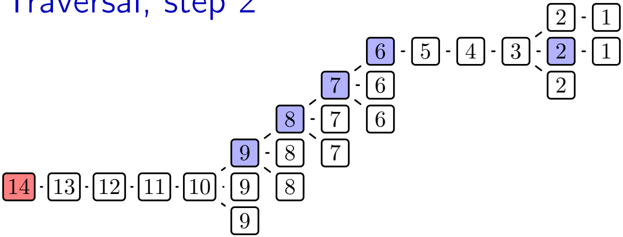
RLS Algorithm: tree traversal

1. Set ESR $= (0, \ldots, 0)$
2. Run State Recursion using the current ESR
3. Save the number of equilibria in every stage game as *ne*(ESR)
4. Add 1 to the ESR in bases *ne*(ESR) to obtain new feasible ESR
5. Stopping rule: successor function exceeds the maximum number with given number of digits
6. Return to step 2

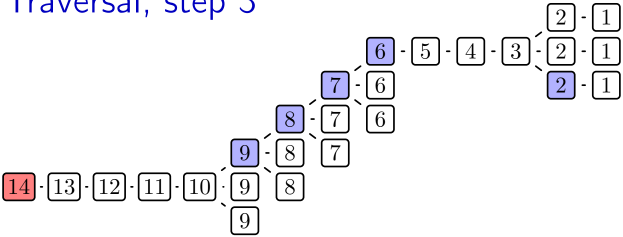
RLS Tree Traversal, step 1



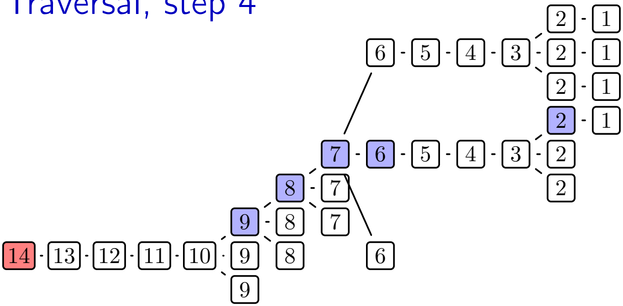
RLS Tree Traversal, step 2



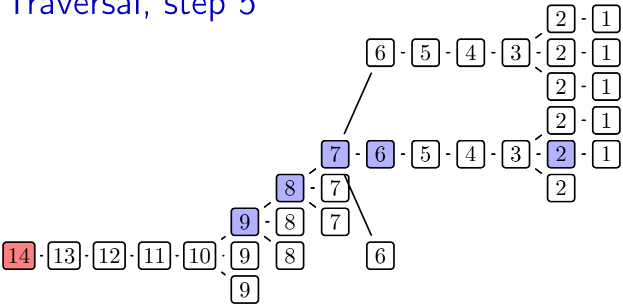
RLS Tree Traversal, step 3



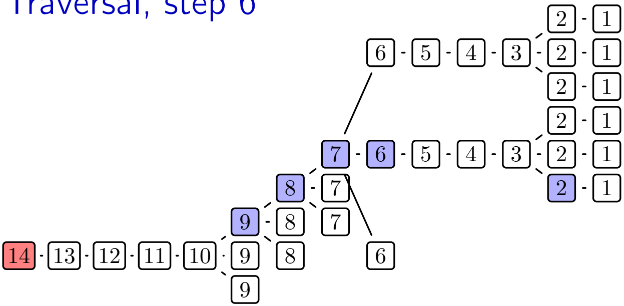
RLS Tree Traversal, step 4



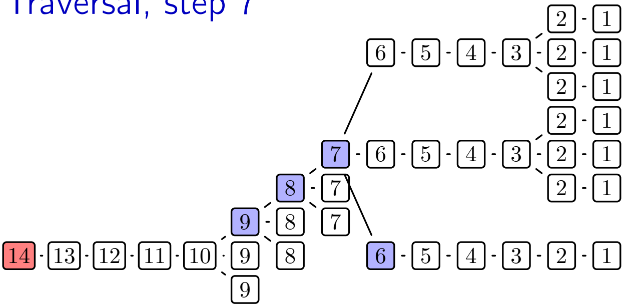
RLS Tree Traversal, step 5



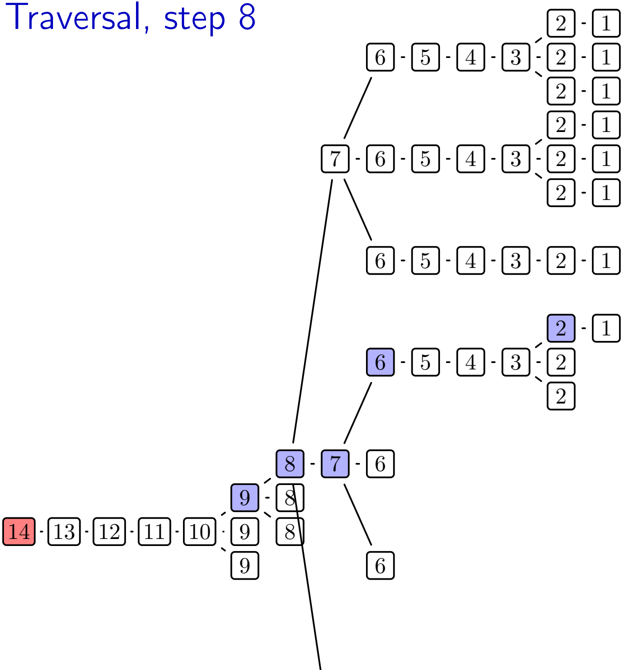
RLS Tree Traversal, step 6



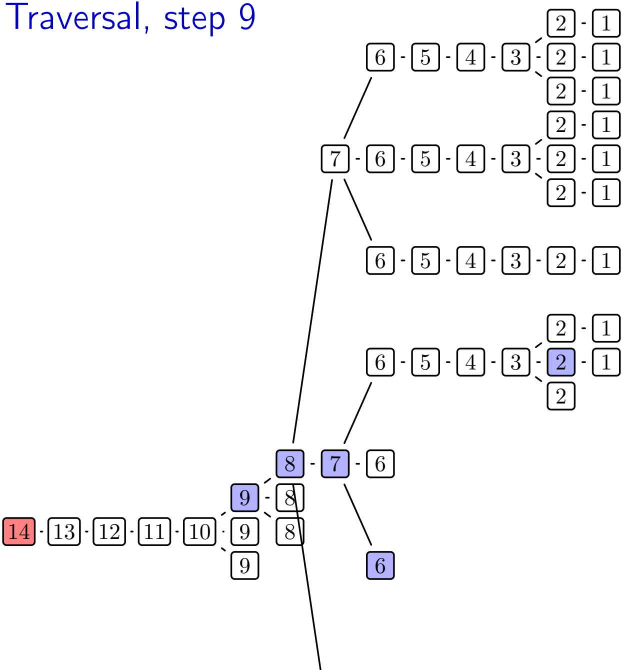
RLS Tree Traversal, step 7



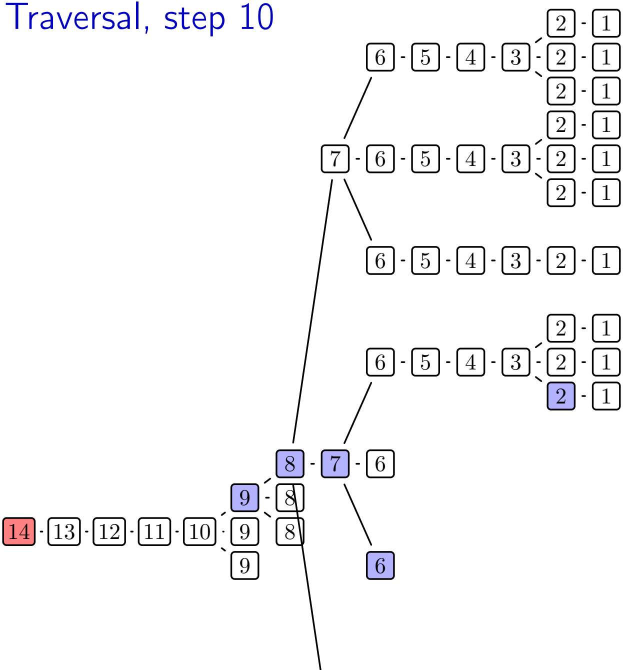
RLS Tree Traversal, step 8



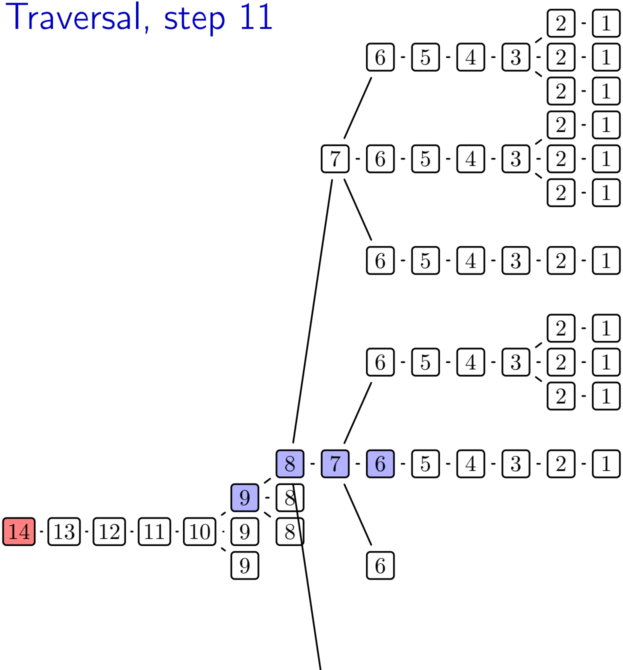
RLS Tree Traversal, step 9



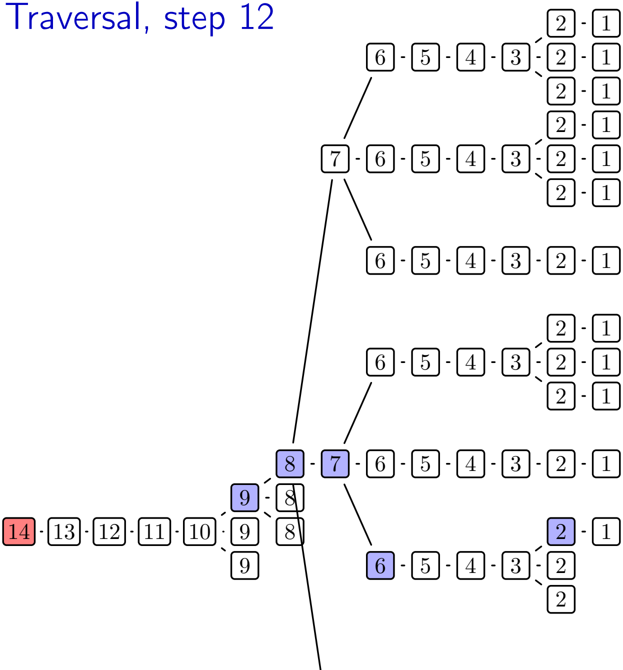
RLS Tree Traversal, step 10



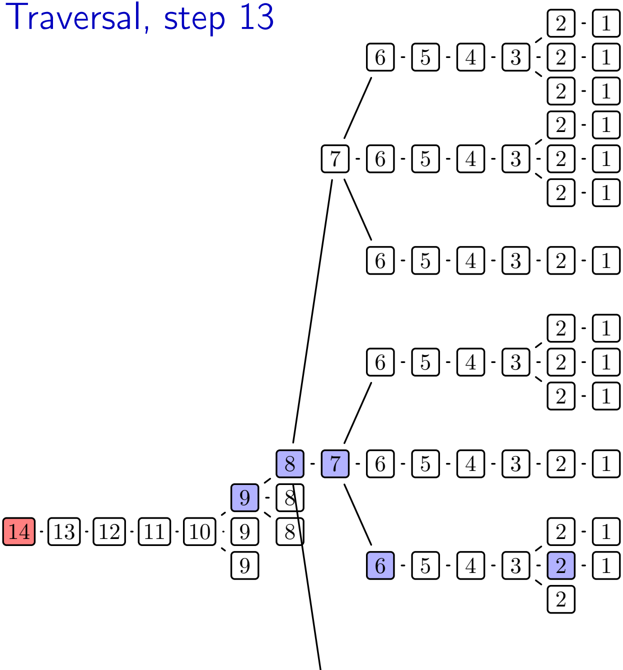
RLS Tree Traversal, step 11



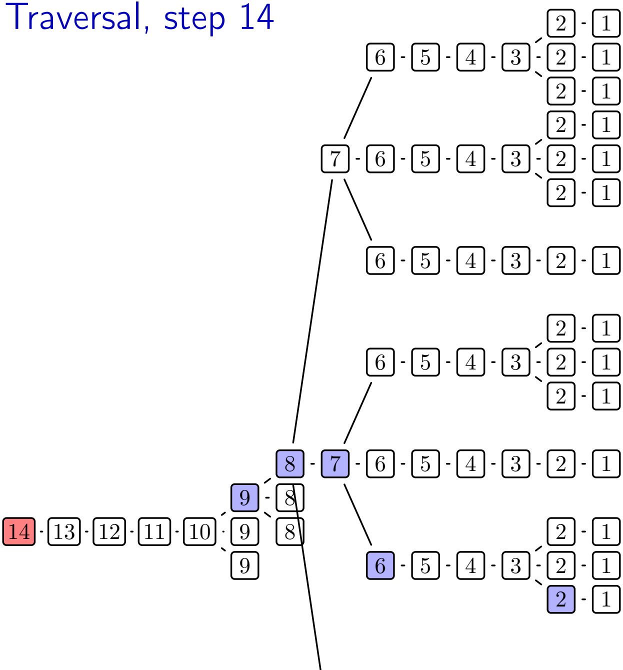
RLS Tree Traversal, step 12



RLS Tree Traversal, step 13



RLS Tree Traversal, step 14



Main result of the RLS Algorithm

Theorem (Decomposition theorem, strong)

Assume there exists an algorithm that can find all MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds all MPE of the DDG in a finite number of steps, which equals the total number of MPE.

Iskhakov, Rust and Schjerning, 2016

Main result of the RLS Algorithm

Theorem (Decomposition theorem, weak)

Assume there exists an algorithm that can at least one MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds some (at least one) MPE of the DDG in a finite number of steps, which does not exceed the total number of MPE.

RLS algorithm: running times

$K = 3$

Simultaneous moves	$n = 3$	$n = 4$
Upper bound on number of MPE	4,782,969	3,948,865,611
Actual number of equilibria	127	46,707
Time used	0.008 sec.	0.334 sec.

Simultaneous moves	$n = 5$
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728
Actual number of equilibria	192,736,405
Time used	45 min.

Alternating moves	$n = 5$
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728
Actual number of equilibria	1
Time used	0.006 sec.

Solving for stage game MPE

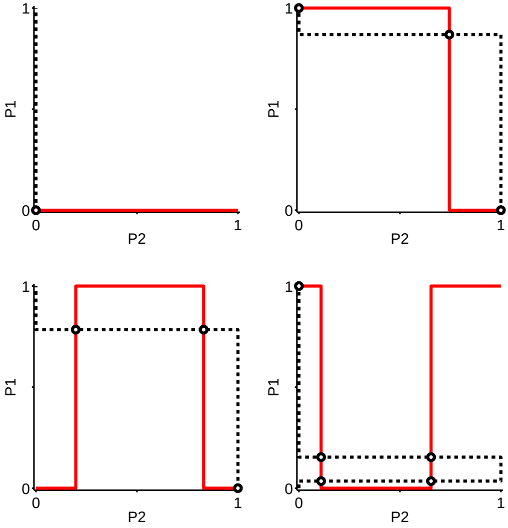
Theorem (1, 3 or 5 MPE in simultaneous move stage games)

When $\eta = 0$ *there exists an exact, non-iterative algorithm that finds all MPE of every d-stage game in the Bertrand pricing and investment game with simultaneous moves.*
The number of MPE in any d-stage game is either 1, 3 or 5.

Theorem (1 or 3 MPE in alternating move stage games)

When $\eta = 0$ *there exists an exact, non-iterative algorithm that finds all MPE of every d-stage game in the Bertrand pricing and investment game with alternating moves.*
The number of MPE in any d-stage game is either 1 or 3.

Best response correspondences of the two firms



Resolutions to Bertrand Investment Paradox

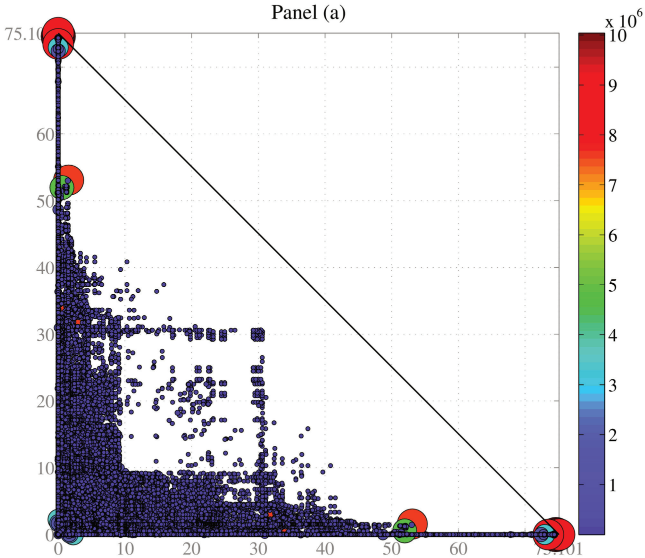
Earlier work:

- Fudenberg et al. (1983 RIE), Reinganum (1985 QJE), Fudenberg and Tirole (1985 ReStud), ...
- Riordan and Salant (1994 JIE): Preemption and rent dissipation (unique equilibrium)

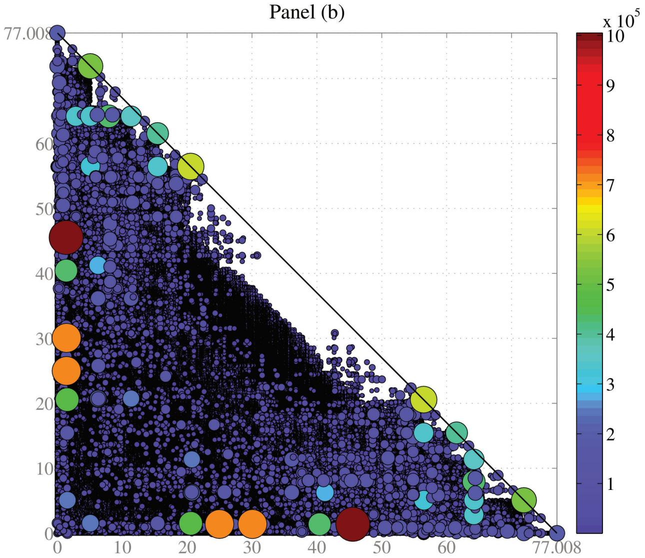
We show:

1. Many types of endog. coordination is possible in equilibrium
    - Leapfrogging (alternating investments)
    - Preemption (investment by cost leader)
    - Duplicative (simultaneous investments)
2. The equilibria are generally inefficient due to over-investment
    - Duplicative or excessively frequent investments

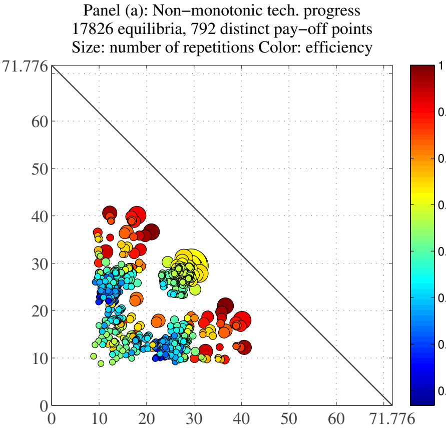
Pay-offs (deterministic tech progress)

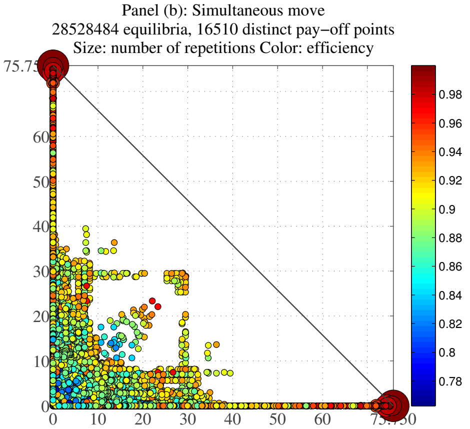


Pay-offs (stochastic tech progress)

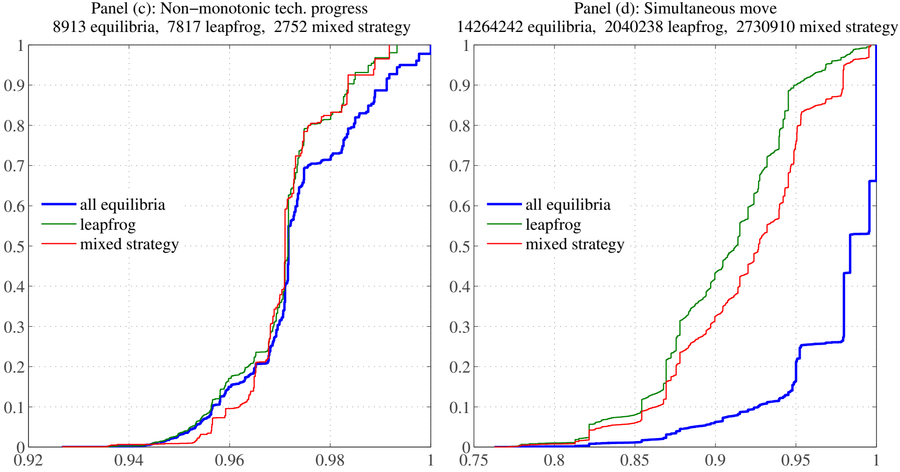


Pay-offs: alternating vs simultaneous move games



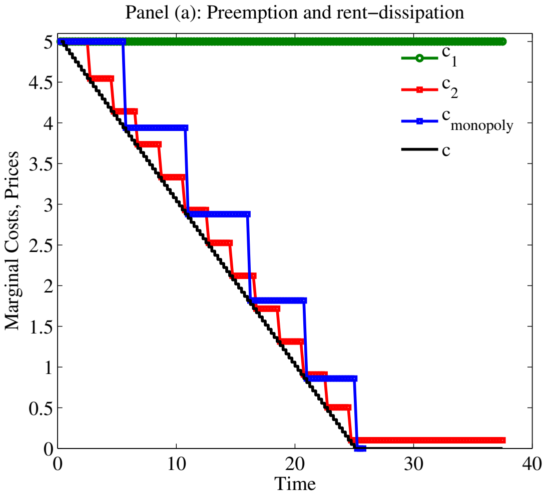


Efficiency: alternating vs simultaneous move games



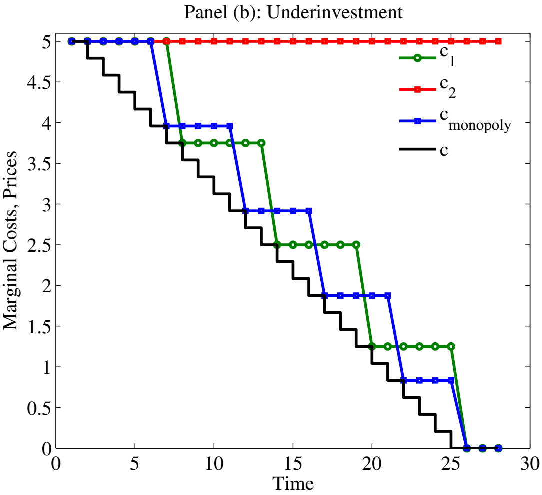
Full preemption and rent dissipation

Confirm R&S the result with high K and small dt



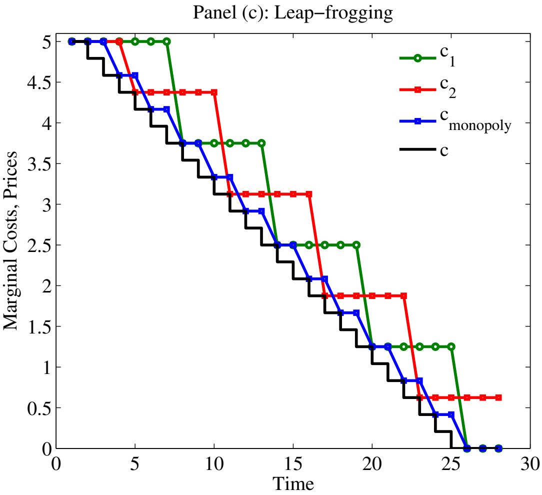
Underinvestment

Rent-dissapation is not a general outcome - disappears when K is low relative dt



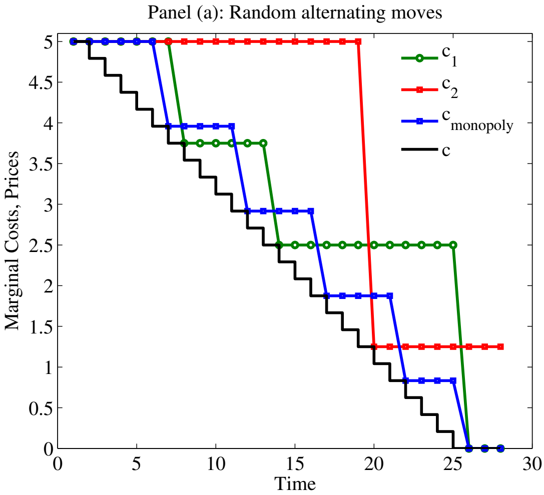
Leap-frogging

Preemption is not the general outcome - disappears when K is even lower



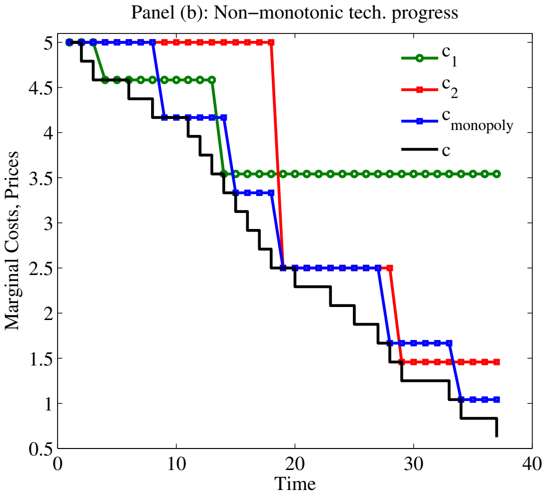
Random alternation → Leapfrogging

Riordan and Salant's result is not robust



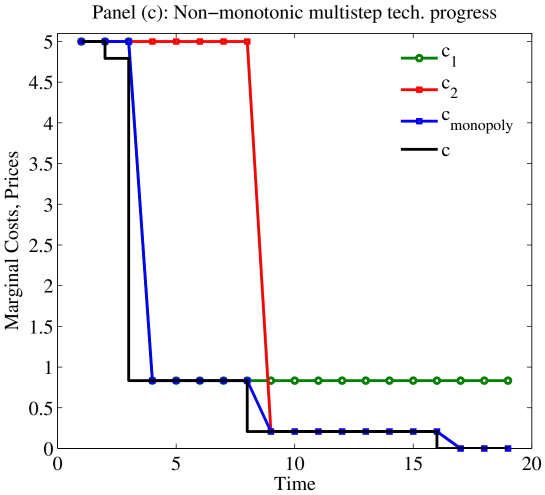
Random onestep technology → Leapfrogging

Riordan and Salant's result is not robust



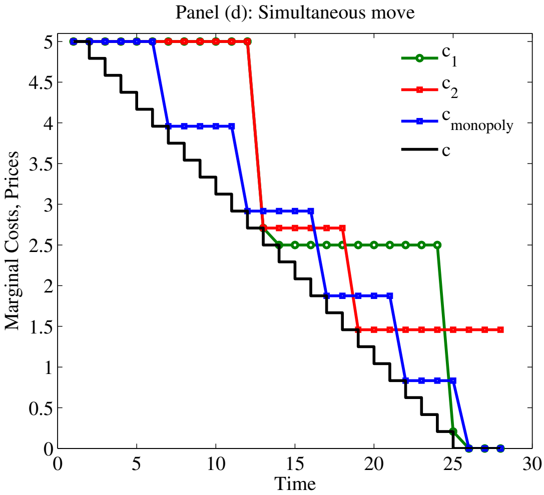
Random multistep technology → Leapfrogging

Riordan and Salant's result is not robust

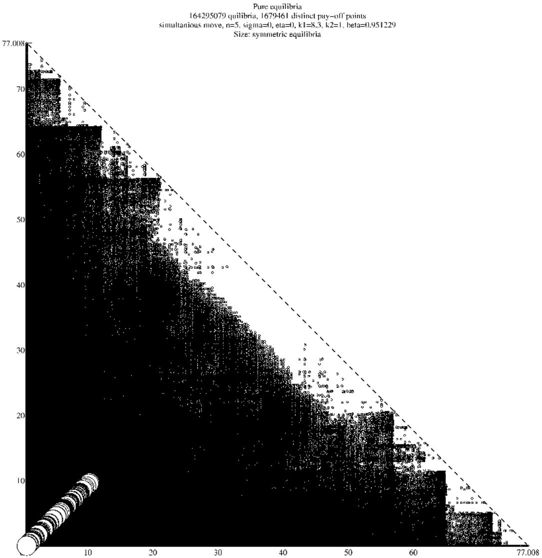


Simultaneous moves: Leapfrogging

Riordan and Salant's conjecture is wrong



Symmetric equilibria: $V_1(c_1, c_2, c) = V_2(c_2, c_1, c)$



Failure of homotopy approach

Homotopy parameter: η

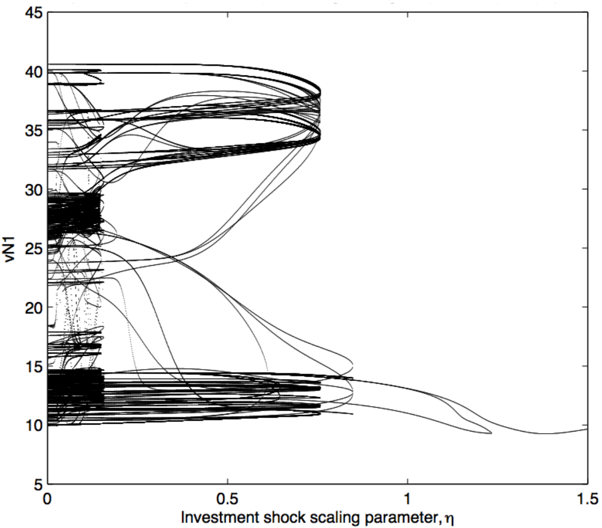
- In each period each firm incurs additive random costs/benefit from not investing and investing
- η is a scaling parameter that index variance of idiosyncratic shocks to investment
- High $\eta \rightarrow$ unique equilibrium $\eta \rightarrow 0 \rightarrow$ multiple equilibria

Problems:

- Multiplicity of equilibria $\rightarrow$ too many bifurcations along the path
- Equilibrium correspondence is not lower hemi-continuous

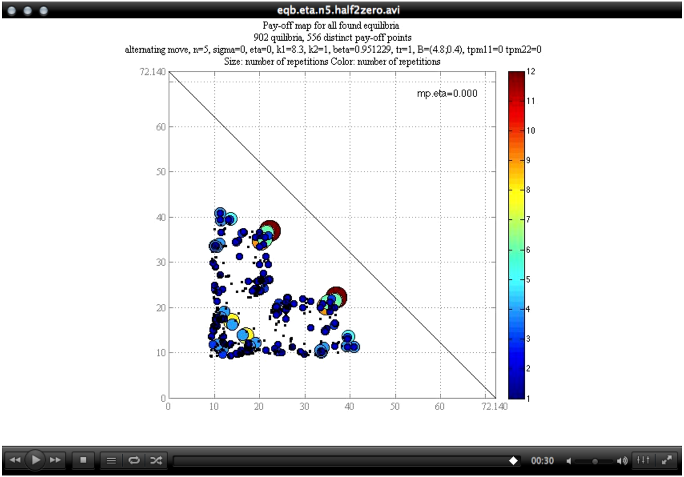
Failure of homotopy approach

Equilibrium correspondance, alternating move game: $V_{N,1}(c_0, c_0, c_0)$ vs. η



Failure of homotopy approach

Video: Set of equilibrium outcomes as variance of shocks decreases to zero



Conclusions: Bertrand investments model

- Many types of endogenous coordination is possible in equilibrium
  - Leapfrogging (alternating investments)
  - Preemption (investment by cost leader)
  - Duplicative (simultaneous investments)
- Full rent dissipation and monopoly outcomes are supported as MPE.
- Numerous MPE equilibria and "Folk theorem"-like result
- The equilibria are generally inefficient due to over-investment
  - Duplicative or excessively frequent investments

Conclusions: Solution of dynamic games

- When equilibrium is not unique the computation algorithm inadvertently acts as an equilibrium selection mechanism
- When directionality in the state space is present, state recursion algorithm is preferred to time iterations
- Plethora of Markov perfect equilibria poses new challenges:
    - How firms manage to coordinate on a particular equilibrium?
    - Increased difficulties for empirical applications.
    - Daunting perspectives for identification of equilibrium selection rule from the data.
- Estimation of dynamic games with multiple equilibria
  Nested Recursive Lexicographical Search (NRLS)

Part III: Full solution estimation

Markov Perfect Equilibrium

- MPE is a pair of strategy profile and value functions:
- **Bellman Optimality**
  Each players solves their Bellman equiation for values V taking other players choice probabilities P into account
- **Bayes-Nash Equilibrium**
  The choice probabilities P are determined by the values V
- In compact notation

$$\begin{aligned} \mathsf{V} &= \Psi^{\mathsf{V}}(\mathsf{V}, \mathsf{P}, \theta) \\ \mathsf{P} &= \Psi^{\mathsf{P}}(\mathsf{V}, \mathsf{P}, \theta) \end{aligned}$$

- Set of all Markov Perfect Equilibria

$$SOL(\Psi, \theta) = \left\{ (\mathsf{P}, \mathsf{V}) \;\middle|\; \begin{array}{l} \mathsf{V} = \Psi^{\mathsf{V}}(\mathsf{V}, \mathsf{P}, \theta) \\ \mathsf{P} = \Psi^{\mathsf{P}}(\mathsf{V}, \mathsf{P}, \theta) \end{array} \right\}$$

Maximum Likelihood

- Data from M independent markets from T periods
  $\mathrm{Z} = \{\bar{\mathrm{a}}^{mt}, \bar{\mathrm{x}}^{mt}\}_{m\in\mathcal{M}, t\in\mathcal{T}}$
  Usually assume only one equilibrium is played in the data.

- For a given θ, let
  $(\mathrm{P}^{\ell}(\theta), \mathrm{V}^{\ell}(\theta)) \in SOL(\Psi, \theta)$ denote the ℓ-the equilibrium

- Log-likelihood function is

$$\mathcal{L}(Z,\theta) = \max_{(\mathrm{P}^{\ell}(\theta), \mathrm{V}^{\ell}(\theta) \in SOL(\Psi,\theta)} \frac{1}{M} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{t=1}^{T} \log P_i^{\ell}(\bar{a}_i{}^{mt} | \bar{\mathrm{x}}^{mt}; \theta)$$

- The ML estimator is $\theta^{ML} = \arg\max_{\theta} \mathcal{L}(Z,\theta)$

Estimation methods for stochastic games

Maximum likelihood estimator

- Efficient, but expensive: need full solution method
- No problem with multiple equilibria

Borkovsky, Doraszelsky and Kryukov (2010) All solution homotopy; Iskhakov, Rust and Schjerning (2016) RLS

Two-step estimators

- Fast, but potentially large finite sample biases

Bajari, Benkard, Levin (2007); Pakes, Ostrovsky, and Berry (2007); Pesendorfer and Schmidt-Dengler (2008)

$$\max_{\theta} \mathcal{L}(Z, \Psi^P(\Gamma(\theta, \hat{P}), \hat{P}, \theta))$$

Estimation methods for stochastic games

Nested psuedo-likelihood (recursive two-step)

- Bridges the gap between efficiency and tractability
- Unstable under multiplicity

Aguirregabiria and Mira (2007); Pesendorfer and Schmidt-Dengler (2010); Kasahara and Shimotsu (2012)

Math Programming with Equilibrium Constraints (MPEC)

- Reformulates ML problem as constrained optimization
- Should not be affected by multiplicity

Su (2013); Egesdal, Lai and Su (2015)

$$\max_{(\theta, \mathsf{P}, \mathsf{V})} \mathcal{L}(\mathsf{Z}, \mathsf{P}) \text{ subject to } \mathsf{V} = \Psi^{\mathsf{V}}(\mathsf{V}, \mathsf{P}, \theta), \mathsf{P} = \Psi^{\mathsf{P}}(\mathsf{V}, \mathsf{P}, \theta)$$

Summary

- Propose robust and computationally feasible MLE estimator for directional dynamic games (DDG), finite state stochastic games with particular transition structure
- Rely of full solution algorithm that provably computes all MPE under certain regularity conditions
- Employ smart discrete programming method to maximize likelihood function over the finite set of equilibria
- Provide Monte Carlo evidence of the performance
- Fully robust to multiplicity of MPE

Nested Recursive Lexicographical Search (NRLS)

1. Outer loop
   Maximization of the likelihood function w.r.t. to structural parameters θ
   $$\theta^{ML} = \arg\max_{\theta} \mathcal{L}(Z, \theta)$$
2. Inner loop
   Maximization of the likelihood function w.r.t. equilibrium selection
   $$\mathcal{L}(Z, \theta) = \max_{(\mathrm{P}^{\ell}(\theta), \mathrm{V}^{\ell}(\theta) \in SOL(\Psi, \theta)} \frac{1}{M} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{t=1}^{T} \log P_i^{\ell}(\bar{a}_i{}^{mt} | \bar{\mathrm{x}}^{mt}; \theta)$$

- Max of a function on a discrete set organized into RLS tree
- Branch and bound optimization method

Battery of MC tests

A

Single equilibrium in the model
Single equilibrium in the data

B

Multiple equilibria in the model
Single equilibrium in the data

C

Multiple equilibria in the model
Multiple equilibria in the data

1. Two-step CCP estimator
2. Nested pseudo-likelihood vs. NRLS estimator
3. Several flavors of MPEC

Battery of MC tests: preliminary results

A

1. Fastest, small sample bias
2. Approaching MLE
3. MLE

B

1. Small sample bias
2. Failing due to multiplicity
3. Local extrema

C

1. Huge data requirements
2. Failing due to multiplicity
3. Curse of dimensionality

1. Two-step CCP estimator
2. Nested pseudo-likelihood
3. MPEC

Monte Carlo setup (A and B)

- $n = 3$ points on the grid on the grid of costs
- 14 points in state space of the model

- 100 random samples from a single equilibrium (one market)
- 10,000 observations per market/equilibrium
- Uniform distribution over state space $\leftrightarrow$ "ideal" data
- Data contains simulated discrete investment choices only

- Estimating one parameter in cost function

Monte Carlo A: no multiplicity

Number of equilibria in the model: 1
Number of equilibria in the data: 1
True value of parameter k1: 1.0

	mle-mpec	pml2step	npl	mle-bnb
Parameter: k1	4.0745	1.0146	1.0203	1.0206
MCSD	3.4974	0.0221	0.0237	0.0212
Bias	3.0745	0.0146	0.0203	0.0206
Log-likelihood	-12,989.73	-12,991.88	-12,987.10	-12,987.37
$\|\|\Psi(P) - P\|\|$	0.00	0.04	0.01	0.00
$\|\|\Gamma(v) - v\|\|$	0.00	0.27	0.15	0.00
Runs converged,	92.00	100.00	100.00	100.00
CPU time, sec	4.03	0.05	0.17	7.22
K-L divergence	4.78	0.00	0.00	0.00
abs deviation	0.38	0.02	0.01	0.00

Monte Carlo B, run 1: moderate multiplicity

Number of equilibria in the model: 5
Number of equilibria in the data: 1
True value of parameter k1: 1.0

	mle-mpec	pml2step	npl	mle-bnb
Parameter: k1	1.1090	0.9985	1.0009	0.9815
MCSD	0.0000	0.0000	0.0000	0.0000
Bias	0.1090	-0.0015	0.0009	-0.0185
Log-likelihood	-11,102.91	-11,102.91	-11,101.19	-11,092.05
$\vert\vert\Psi(P) - P\vert\vert$	0.00	0.03	0.01	0.00
$\vert\vert\Gamma(v) - v\vert\vert$	0.00	0.54	0.13	0.00
Runs converged,	100.00	100.00	100.00	100.00
CPU time, sec	2.35	0.04	0.23	11.13
K-L divergence	0.01	0.01	0.01	0.00
Abs deviation	0.03	0.03	0.04	0.01

Monte Carlo B, run 2: larger multiplicity

Number of equilibria in the model: 95
Number of equilibria in the data: 1
True value of parameter k1: 1.0

	mle-mpec	pml2step	npl	mle-bnb
Parameter: k1	1.0910	0.9948	1.0045	0.9970
MCSD	0.3202	0.0113	0.0094	0.0065
Bias	0.0910	-0.0052	0.0045	-0.0030
Log-likelihood	-6,714.92	-6,714.32	-6,722.46	-6,695.74
$\|\|\Psi(P) - P\|\|$	0.00	0.10	0.12	0.00
$\|\|\Gamma(v) - v\|\|$	0.00	1.22	0.94	0.00
Runs converged,	100.00	100.00	5.00	100.00
CPU time, sec	7.72	0.04	0.32	13.79
Mean K-L divergence	0.27	0.01	0.02	0.00
Mean abs deviation	0.06	0.04	0.06	0.00

Monte Carlo B, run 3: moderate multiplicity, bad start points

Number of equilibria in the model: 5
Number of equilibria in the data: 1
True value of parameter k1: 1.0

	mle-mpec	pml2step	npl	mle-bnb
Parameter: k1	6.7685	0.9948	1.0045	0.9970
MCSD	11.3401	0.0113	0.0094	0.0065
Bias	5.7685	-0.0052	0.0045	-0.0030
Log-likelihood	-6,709.38	-6,714.32	-6,722.46	-6,695.74
$\|\|\Psi(P) - P\|\|$	0.00	0.10	0.12	0.00
$\|\|\Gamma(v) - v\|\|$	0.00	1.22	0.94	0.00
Runs converged,	30.00	100.00	5.00	100.00
CPU time, sec	10.43	0.06	0.37	18.47
Mean K-L divergence	8.63	0.01	0.02	0.00
Mean abs deviation	0.25	0.04	0.06	0.00

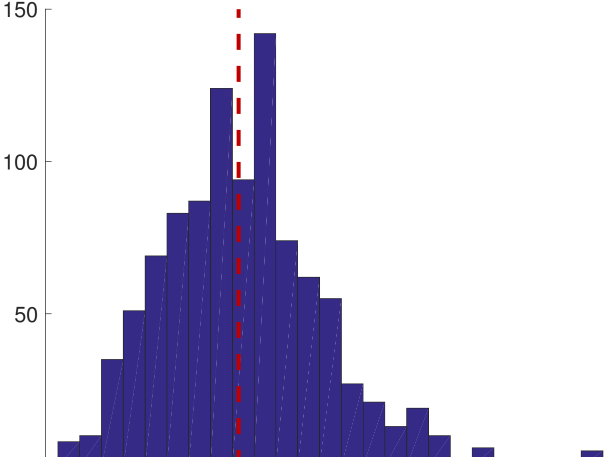
NRLS Monte Carlo setup (C)

- $n = 3$ points on the grid on the grid of costs
- 14 points in state space of the model
- 109 MPE in total

- 1000 random samples from 3 different equilibria (3 markets)
- 100 observations per market/equilibrium
- Uniform distribution over state space $\leftrightarrow$ "ideal" data
- Data contains simulated discrete investment choices only

- Estimating one parameter in cost function

Distribution of estimated k_1 parameter



MC results and numerical performance of NRLS

1. Average bias and RMSE of the estimates of the cost of investment parameter (true value is 10.0)

$$\text{Bias} = 0.0737$$
$$\text{RMSE} = 0.8712$$

2. Average fraction of MPE computed by BnB relative to RLS

   0.321 (std=0.11635)

3. Average fraction of stages solved by BnB relative to RLS

   0.263 (std=0.09725)

4. All 3 MPE correctly identified by BnB in

   98.4% of runs

Distribution of computational reduction factor