Equilibrium Trade in Automobiles[*]

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September 2021

Abstract

We introduce a computationally tractable dynamic equilibrium model of automobile markets with heterogeneous consumers who choose to keep their car or trade for a different make, model and age. We focus on stationary flow equilibria, where outflows of cars due to accidents and endogenous scrappage equal inflows of new cars. We introduce a fast robust algorithm for computing equilibria and use it to estimate a model with eight household types and four car types using nearly 39 million observations on car ownership transitions from Denmark. The estimated model fits the data well and counterfactual simulations show that Denmark is over the top of the Laffer curve: it could *raise* total tax revenue by reducing the new car registration tax rate. We show that reducing this tax rate while raising the tax rate on fuel increases aggregate welfare, tax revenues, and car ownership, while reducing car ages, driving, and CO_2 emissions.

Keywords: secondary markets, trade, consumer heterogeneity, transactions costs, dynamic programming, extreme value distribution, dynamic discrete choice, multinomial logit model, stationary equilibrium, Markov chains, invariant distributions, doubly nested fixed point estimator (DNFXP)

[*] We dedicate this paper to James A. Berkovec whose contributions to the development of micro-founded equilibrium models of the auto market was far ahead of his time and so inspirational to our own work on this subject. His untimely death at age 52 in 2009 remains a huge loss to the economics profession. We also acknowledge helpful comments from the editor and referees and from Charles Manski and Dmitriy Stolyarov, whose contributions to modeling dynamic equilibrium in auto markets are equally inspiring and important. We received additional feedback and suggestions from Aureo de Paula. We are grateful for funding from the IRUC research project, financed by the Danish Council for Independent Research. Rust acknowledges financial support from the Gallagher Family Chair in Economics at Georgetown University.

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1 Introduction

Modeling the automobile market is particularly challenging due to the trading and substitution possibilities that exist due to the presence of a secondary market where used cars are traded. Not only are there dozens or even hundreds of different makes and models of new cars to choose from in the primary market, consumers have a huge array of used car options as well. Consumers have control over when their used cars are scrapped, and can respond to an increase in new car prices by substituting away from new cars and holding used cars longer. Endogenous scrappage of cars is also of interest for safety and environmental reasons, since it is well documented that used cars become less safe and and pollute more as they age.[1]

We develop a dynamic model of trading in new and used cars that demonstrates how secondary markets lead to significant gains from trade via clear patterns of specialization in the holdings of cars by heterogeneous consumers. The secondary market facilitates a "hand-me-down-chain" for cars where rich consumers buy brand new cars and hold them a few years before selling to other slightly less rich consumers, who hold the car for a few more years before being traded to a an even poorer consumer who may hold the car until it is involved in an accident or voluntarily scrapped.

The secondary market creates substitution possibilities that can limit market power and affect pricing decisions by new car producers in the primary market. If new car producers raise the prices of new cars too much, consumers can respond by holding onto their used cars longer. High government taxation of new cars can also cause consumers to hold their used cars longer, as well as to substitute to the "outside good", i.e. not to own a car. Government sales taxes and vehicle emissions and safety inspections can also interfere with the operation of secondary markets and reduce trade and consumer welfare. Beyond some point, sufficiently high taxation and overly onerous safety/emissions regulations can serve to kill off the secondary market and in the limit drive all consumers into the outside good. Thus there is a "Laffer curve" and the possibility of *increasing* total tax revenues by *decreasing* tax rates.

We use our model to analyze the fiscal and welfare effect of the new car registration tax in Denmark, one of the highest in the world, which in the sample period amounted to 180%

[1] See, for example, Borken-Kleefeld and Chen (2015) and NHTSA (2013). The evidence is less clear on whether a car's fuel efficiency (measured as kilometers per liter) declines with age.

of the new car price even *before* an additional 25% VAT. The Danish government is highly reliant on this tax, which accounts for approximately 4% of all tax revenues (including income taxes) or about 2% of Danish GDP. Our model predicts that the Danish tax rate is "over the top of the Laffer curve" and could raise more tax revenue by reducing the registration tax rate. We show that reducing new car taxes in favor of higher fuel taxes improves aggregate welfare and significantly increases tax revenue and car ownership, while reducing average car ages and per household driving and aggregate CO_2 emissions. Though aggregate consumer surplus and externality adjusted social welfare increase, the new policy is not a Pareto improvement absent offsetting transfers: the change in tax policy reduces the welfare of those with long commute distances at the expense of those with shorter ones.

Our results are made possible by the fact that we can rapidly compute equilibria of the model using a fast and robust Newton-based solution algorithm that can be nested within a maximum likelihood estimation algorithm. Using Danish register data, which records the car ownership and trading decisions of all Danish citizens, we empirically estimate a version of our model with 8 types of households and 4 types of cars and show it provides a good approximation to car holdings and trading in Denmark. Our model provides a simple explanation of a striking zig-zag pattern in scrappage rates of older cars, whereby cars of even ages are scrapped with significantly higher probability compared to cars whose ages are odd numbers. We show that this is consistent with the rigorous bi-annual safety inspections in Denmark. Our estimation results reveal that these inspections have high perceived "hassle costs" so once cars are sufficiently old, most Danes prefer to scrap their vehicles rather than incur the time and expense to repair their vehicles to pass inspection, which is mandatory regardless of whether the car is kept or sold to another consumer.

The primary contribution of this paper is to advance the state of the art for computing equilibria in the primary and secondary markets for automobiles and other durable goods. We introduce a computationally tractable dynamic equilibrium model where new and used vehicles of multiple types (e.g. makes and models) are traded by heterogeneous consumers. The prices and quantities of used cars are determined endogenously to equate supply and demand for all car types and traded vintages, and the ages at which cars are scrapped are also determined endogenously as part of the equilibrium. The model allows for transactions costs, taxes, and flexible specifications of car characteristics, consumer preferences, and heterogeneity. Our framework can be used to address a wide range of research and policy questions relating to the primary and secondary markets for vehicles. We also show how to incorporate a utility-based model of driving into the model, which is crucial for analyzing environmental policies.

We derive market demand from micro aggregation of an individual-level dynamic discrete choice model of ownership and trade of automobiles. Our specification of consumer heterogeneity includes additive idiosyncratic generalized extreme value preference shocks that have a clear economic interpretation as unobserved costs of maintaining an existing car, consumer-specific variations in search/transactions costs, and idiosyncratic variations in transaction prices and other costs involved in trading cars that constitute an important source of gains from trade that explain the existence of secondary markets. By varying the scale of these additive extreme value preference shocks, we show how reductions in consumer heterogeneity reduce gains from trade and ultimately kill off secondary markets when trade frictions are sufficiently large.

The generalized extreme value specification results in logit or nested logit conditional choice probabilities for the decisions to keep or trade different types and ages of vehicles. We show how additional persistent consumer heterogeneity can be added, giving us the flexibility to match rich patterns of trading, consumers who choose not to own cars (i.e. "outside good") and brand loyalty and brand switching behaviors. We show that the choice probabilities, and thus aggregate demand, are smooth functions of car prices which allows us to use fast derivative-based methods such as Newton's method to solve for consumers' dynamic trading strategies and equilibrium prices.

We formulate our model and define equilibrium in an infinite horizon stationary environment. We use the machinery of Markov processes to describe trading behavior and characterize the vehicle holdings of different types of consumers as invariant distributions to certain Markov chains. These Markov chains reflect the trading of vehicles, their aging and the impact of stochastic accidents that result in premature scrappage of some vehicles. Our stationary equilibrium concept results in a very compact and elegant description of equilibrium, but it can also be extended to non-stationary environments with macroeconomic shocks and overlapping generations of consumers with finite lifespans.

We start in Section 2 by reviewing the large theoretical and empirical literature on modeling auto markets and other durable goods on which we build. In Section 3 we set up the basic model with multiple car brands and idiosyncratic consumer heterogeneity. In Section 4 we consider persistent consumer heterogeneity to study equilibrium sorting of cars between different consumer types. Section 5 describes how the model parameters can be structurally estimated by maximum likelihood using a doubly nested fixed point algorithm (DNFXP) that recomputes equilibrium prices, holdings and consumer trading strategies each time the likelihood function is evaluated. In section 6 we estimate the model using data on the Danish auto market and analyze the welfare and environmental impacts of changes in Danish car tax policies. Section 7 concludes.

2 Previous work on modeling automobile markets

A starting point of any discussion of the literature on equilibrium models of automobile markets is the well-known BLP model (Berry, Levinsohn and Pakes, 1995). This influential work focuses on the primary market for new vehicles, but ignores the presence of the secondary market and the substitution possibilities it offers consumers. Rust (1985b) and Esteban and Shum (2007) were the first to tackle the challenging problem of solving for a full equilibrium in both the primary and secondary markets for automobiles. Rust studied the simultaneous determination of price and durability by a monopolist new vehicle producer, while Esteban and Shum studied oligopolistic pricing of competing new vehicle producers. To make progress, both of these studies assumed stationarity and zero transaction costs, which implies that consumers trade each period for their most preferred vehicle in the entire market.[2]

We build on a substantial literature focused on modeling equilibrium in secondary markets for automobiles, taking the price of new vehicles as given. The earliest work that we are aware of in this literature is a series of papers by Manski (1980), Manski and Sherman (1980), and Manski (1983). These papers introduced theoretical models of equilibrium in secondary markets for cars that could be numerically solved for prices and quantities and used for policy forecasting of a wide range of policies of interest.

The next important early contribution was Berkovec (1985), who microeconometrically estimated and numerically solved a large-scale equilibrium model of the new and

[2] Esteban and Shum (2007) also assume quality ladder preferences, which further simplifies the choice problem.

used vehicle markets using a nested logit model. He defined "expected excess demand" by summing estimated discrete choice probabilities for cars of each age and class, net of scrappage.[3] Berkovec computed equilibrium prices using Newton's method to find a zero to a system of 131 nonlinear equations representing the excess demand for the vehicles in his model.[4]

The contributions of Manski, Sherman, and Berkovec were extremely advanced given the computing power at the time, and in many respects represent the closest point of departure for our own work. However, their work was based on short run, *static* equilibrium holding models of the market. Implicit in the static discrete choice formulation is the assumption that consumers only keep their vehicle for a single period, so that at the end of each period consumers trade their current vehicle for their most preferred vehicle. Rust (1985a) formulated the first *dynamic* equilibrium model of automobile trading.[5] He assumed the state of a vehicle is captured by its odometer reading x_t, which evolves according to an exogenous Markov process representing variable usage of cars with transition probability $\Phi(x_{t+1}|x_t)$ that reflects stochastic usage and deterioration of vehicles.[6]

When there are no transaction costs and the economy is in a stationary equilibrium (i.e. no macroeconomic shocks or other time-varying factors altering the market), the optimal trading strategy involves trading every period for the most preferred age/condition of vehicle $x^*(\tau)$, where τ indexes potentially heterogeneous preferences over "newness" of vehicles. However, the assumption of zero transactions costs is unrealistic, and so is the excessive trading behavior it implies. When there are transactions costs (which are distinct from *trading costs*, i.e. the difference between the price of a car x a consumer wishes to buy, $P(x)$, less the price $P(x')$ of the car x' that the consumer wishes

[3]Berkovec used a probabilistic model of vehicle scrappage from Manski and Goldin (1983), where the probability a vehicle is scrapped is a decreasing function of the difference between the second-hand price of the vehicle (net of any repair costs) and an exogenously specified scrap value for the vehicle. This implies that, except for random accidents, there is very little chance that new vehicles are scrapped, but the probability a used vehicle is scrapped increases monotonically with the age of the vehicle.

[4]Berkovec showed that the Jacobian matrix had special structure he called "identity outer product" that enabled him to invert the Jacobian via inverting a smaller 48×48 matrix and doing some additional matrix vector multiplications.

[5]Other dynamic models of vehicle choice appeared around this time such as Mannering and Winston (1985) but their analysis focused on dynamics of utilization, but did not consider dynamics of car trading or equilibrium. In subsequent work Winston and Yan (2021) develop empirically estimable model of dynamics of utilization and trading of cars, but in a partial-equilibrium framework.

[6]Since x_t fully captures the state of a car and is observable by both parties in a transaction, Rust's analysis avoided "lemons problem" information asymmetries of the type analyzed in the seminal work of Akerlof (1970) that can potentially kill off the secondary markets for cars.

to sell), the optimal trading strategy involves less frequent trading and consumers will generally keep cars for multiple periods. The optimal strategy then takes the form of an "(S, s) rule" reminiscent of optimal inventory theory: trade is characterized by two thresholds $(\underline{x}^*(\tau), \overline{x}^*(\tau))$, where $\underline{x}^*(\tau) < \overline{x}^*(\tau)$ and $\underline{x}^*(\tau)$ is the state of the optimal replacement vehicle whenever the consumer trades in for a new one. $\overline{x}^*(\tau)$ is the *replacement threshold* or odometer threshold where it is optimal to trade the current car in condition x for a replacement car in condition $\underline{x}^*(\tau)$. When transactions costs are zero, then $\overline{x}^*(\tau) = \underline{x}^*(\tau) = x^*(\tau)$ and it is optimal to trade for the optimal car $x^*(\tau)$ every period. However, in a homogeneous agent economy, the slightest transaction costs will completely kill off the secondary market, driving all consumers into an autarkic "buy and hold" equilibrium where all consumers buy brand new vehicles whenever they trade (i.e. $\underline{x}^*(\tau) = 0$) and hold them until it is optimal to scrap their current car when the odometer exceeds an optimal replacement threshold $\overline{x}^*(\tau)$.

There are potential gains from trade in a heterogeneous agent economy that enable the existence of secondary market and a wide range of car trading strategies. However establishing the existence of a stationary equilibrium in such an economy in the presence of transactions costs is challenging. Consider a consumer of type τ who desires to buy a vehicle with $\underline{x}^*(\tau) > 0$. When there are transactions costs there is no guarantee that some other consumer τ' is willing to sell their vehicle at $\underline{x}^*(\tau)$. Using advanced methods from functional analysis (e.g. the Fan-Glicksburg fixed point theorem), Konishi and Sandfort (2002) established the existence of a stationary equilibrium in the presence of transactions costs under certain conditions. Their proof shows that it is possible for the equilibrium price function $P(x)$ to adjust to prevent such coordination failures. However to our knowledge, there has been no work actually calculating equilibria with transactions costs in this infinite-dimensional setting.

Stolyarov (2002) advanced the literature by assuming that the state of a vehicle can be summarized by its age a, which can take only a finite number of values, $a = 0, 1, 2, \dots, \bar{a}$, where $\bar{a}$ is age when cars are scrapped. Stolyarov introduced a continuous uni-dimensional parameterization of consumer heterogeneity with quasi-linear preferences, and computed equilibria in the presence of stochastic transactions costs using a fixed point formulation of the problem. Gavazza, Lizzeri and Roketskiy (2014) extended Stolyarov's approach by allowing households to own up to two vehicles using a two-dimensional specification of consumer heterogeneity. They find that transaction costs have a large effect on equilibrium trade.[7]

Our model can be thought of as combining Stolyarov (2002) with the earlier work by Manski and Berkovec by using a multi-dimensional extreme value specification of consumer heterogeneity. We use a hierarchical specification of heterogeneity that includes both time-varying idiosyncratic preference shocks (i.e. the extreme value error terms in the model) as well as flexible specifications for persistent heterogeneity and fixed consumer types τ. The extreme value distribution allows for continuous formulas for choice probabilities even in the case where there is no other time-invariant heterogeneity, and this continuity permits us to demonstrate the existence of equilibrium via the Brouwer fixed point theorem. More importantly, we show that the excess demand function for used cars in our model, $ED(P)$, is a continuously differentiable function of P that enables us to rapidly and accurately calculate equilibrium prices by solving the system of nonlinear equations $ED(P) = 0$ by Newton's method. This makes our approach very attractive for use in empirical work and policy modeling.

3 Equilibrium with Idiosyncratic Consumer Heterogeneity

In this section we introduce a dynamic model of equilibrium trade in the automobile market. We use the concept of *stationary flow equilibrium* in the market of stochastically deteriorating durable goods from Rust (1985a) but adapt it for the discrete goods trade in presence of flexible transactions costs. We start by considering equilibrium with J different makes/models of cars and a unit mass of consumers whose preferences for cars as well as the outside option, are *idiosyncratically heterogeneous.* We adopt a generalized extreme value (GEV) specification of consumer heterogeneity that results in a nested logit

[7] There is a close connection between models of automobile trading that incorporate transactions costs and models that emphasize information asymmetries, such as Akerlof (1970). House and Leahy (2004) show how adjustment costs of the (S, s) variety discussed above "arise endogenously from adverse selection in the secondary market." (p. 582). For example, there are "lemons laws" in many countries that require sellers to compensate buyers for defects or problems in a car that were not disclosed and negotiated on at the time of sale. Dealers typically perform inspections and repair cars before selling, and often provide a limited term warranty, all of which mitigate the informational asymmetries and result in transactions costs that are often borne by the dealer. As a result, it is not clear that informational asymmetries seriously inhibit trade in used vehicles, but they would be expected to show up in transaction costs. Hendel and Lizzeri (1999) study equilibria in auto markets with and without asymmetric information and find that adverse selection does not necessarily kill off the secondary market. They find it difficult to empirically distinguish between predictions of models with asymmetric information and those with transaction costs, and argue that, for Fords and Hondas at least, the evidence does not support adverse selection as the primary reason for steeper price declines of Fords as the vehicles age. In light of this, we use transaction costs to capture various trade frictions in auto markets including informational ones.

specification for choice probabilities similar to Berkovec (1985). In subsequent section we extend the framework to persistent heterogeneity in consumer preferences.

3.1 Key assumptions and restrictions

We consider a stationary equilibrium in an infinite horizon economy where cars are initially sold as new in the primary market and then traded in used car markets called "secondary markets". Consumers make purchase, replacement, trading and scrapping decisions to maximize expected discounted utility with a common discount factor $\beta \in (0, 1)$. We focus on a stationary environment and do not allow for any "macro shocks" that could lead to time-varying fuel prices or prices of new cars.[8]

Our concept of equilibrium results in endogenous determination of a vector of equilibrium prices P with typical element P_{ja}, where $j \in \{1, \ldots, J\}$ indexes makes/models, and $a \in \{1, \ldots, \bar{a} - 1\}$ indexes the ages of the traded cars. When the cars reach the upper bound $\bar{a}$, they are no longer safe to drive and are not allowed to be kept or traded and must be scrapped.[9] We treat the model as a "small open economy" where new car prices are determined in the world market with an infinitely elastic supply of new cars at prices $\overline{P}_j$.[10] We assume there is an infinitely elastic demand for cars *at any age* including $\bar{a}$ for their scrap value $\underline{P}_j$ which normally results in $P_{ja} \geq \underline{P}_j$, $\forall j, a$, provided that the level of transactions costs is not too high.

In our framework all persistent differences between the cars are captured by the make/model $j \in \{1, \ldots, J\}$, and all time-varying characteristics of cars are reflected by the car age $a \in \{1, \ldots, \bar{a}\}$. The unit mass of cars in the economy is distributed among $J\bar{a}$ types given by the combination of car make/model and age (j, a). Clearly used cars of the same age and type have idiosyncratic features, such as odometer reading which we ignore, making it inconsistent with a single common price P_{ja} for all used cars of age a and make/model j. This is partially accounted for in our framework but the stochastic GEV shocks which not only reflect idiosyncratic heterogeneity in consumer preferences,

[8]Although it is possible to extend our framework to allow for macro shocks, this fundamentally changes the definition of the equilibrium. We defer this extension to the future work due to the vastly greater computational challenges that it presents, as noted in the work of Krusell and Smith (1998) and **?**.

[9]The same upper bound is assumed to hold for the cars of all makes/models without loss of generality to simplify exposition. It is straightforward to allow upper bound to be j-specific with more complicated notation.

[10]However, our framework can be used for modeling competition in the primary market for new cars, where $\overline{P}_j$ can be set taking into account the substitution effects not only between different types of *new cars* but also between *new and used* cars.

but also idiosyncratic features of different used cars of the same age and type. Thus, we can interpret P_{ja} as the average price of a car of type j and age a, and components of the idiosyncratic shocks reflect customer and car-specific deviations in these prices from the market average prices that are determined endogenously in equilibrium.

Following the literature, we assume that the net of the idiosyncratic component consumers' preferences are characterized by a common quasi-linear utility function

$$U(\cdot) = u(j, a) - \mu\big[\text{operating costs} + \text{trade and transaction costs}\big],$$

where the first term captures the utility of owning and using a car and the second term accounts for the monetary costs of ownership and trade. We assume that the marginal utility of an additional car is sufficiently small such that no consumer would want to own more than a single car.[11] The parameter $\mu > 0$ is a simple way to capture income/wealth effects in the model. High values of μ can be interpreted as "being poor" because the cost of buying a new car will involve a high opportunity cost in terms of forgone consumption of other goods. We envisage the function $u(j, a)$ to be non-increasing in a for all j, and will later show how it captures the utility of driving (utilization) as well as the expected non-monetary cost of maintaining a car of age a.

Trade costs consist of the difference in prices of traded cars, with the addition of transaction costs. Let (i, a) be the make/model and age of the existing car, and (j, d) denote the car the household purchases. Transactions costs are given by an function $T(i, a, j, d)$ which depends on both the traded cars and on the whole set of prices $\{\overline{P}_j, \underline{P}_j\} \cup \{P_{ja}\}$, $a \in \{1, \dots, \bar{a} - 1\}$, $j \in \{1, \dots, J\}$.[12] We assume that the transactions costs are borne by both buyers and sellers. Even though it would be possible to work with general non-separable specifications for transactions costs $T(i, a, j, d)$, for simplicity we assume that these costs are additively separable into two components denoted $T_b(j, d)$ and $T_s(i, a)$. The first component is associated with searching for and buying another car, the second — with undertaking repairs and improvements to make the car of age a that the consumer is trying to sell acceptable to potential buyers. We do not make further restrictions on the

[11] This is a reasonable assumption for a country like Denmark, where most households only own a single car. It also greatly simplifies notation and the presentation of our model. The assumption could be relaxed to extend the model to countries like the United States, where most households own multiple cars.

[12] All quantities in the consumer choice problem depend on these prices, but for clarity we do not write this explicitly until Section 3.4.

function form of the transaction costs, therefore allowing for both fixed and proportional costs, i.e. sales taxes and registration fees.

The total trade and transaction costs are given by $P_{jd} - P_{ia} + T_b(j,d) + T_s(i,a)$. If a consumer chooses to "purge" their car and choose the outside option of not owning a car, the buyer-side components depending on (j,d) disappear from the expression, and such consumer only faces the seller-side transaction cost. Similarly, if a person without a car decides to purchase one, the seller-side components depending on (i,a) disappear from the trade costs. Finally, we assume that the trade-in price $\underline{P}_i$ for the cars being scrapped already includes all the costs associated with de-registering and transporting the clunker to the scrap yard for recycling/disposal. That is we normalize the seller-side transaction cost of scrapping to zero.

3.2 Consumer states and choices

The *state* of a consumer in any period t is given by the vector (i, a, ϵ) where $i \in \{\o, 1, \ldots, J\}$ denotes the make/model of car the consumer owns at the start of the period, and $a \in \{\o, 1, \ldots, \bar{a}\}$ denotes its age. We use the special symbol $\o$ to denote the state of not owning a car. The random component of the state vector ϵ incorporates the (idiosyncratic) heterogeneity in cars and consumers.

We assume that at the start of each period a consumer who owns an existing car (i,a) can choose whether to keep it, trade it for another car of make/model (j,d), or choose the outside option of not having a car at all. We assume trade occurs instantaneously at the start of each period, and thus the cars (j,d) that households hold after trading are utilized until the end of the period. Cars deterministically age from d to $d+1$, but may be involved in total loss accidents which we model by stochastic transition to the terminal age $\bar{a}$ with probability $\alpha(j,d) \in [0,1)$. The realized state of the car constitutes the car state at the start of period $t+1$. Then instantaneous trading occurs and the process repeats this way for the infinite future.

The cars which reach the terminal age $\bar{a}$ by either natural aging or as a result of an accident are *exogenously* scrapped and removed from the market during the trading stage. In addition, unless the existing car (i,a) is kept, it can be scrapped *endogenously* instead of being sold on the secondary market by its owner. It would seem that all consumers would prefer to sell their existing used car in the market rather than scrap it, however there are transactions costs that a seller must incur, and the net value that a consumer might receive from selling a sufficiently old used car may be lower than the value from simply scrapping it. Our model allows consumers to choose whether to sell or scrap their existing car depending on which option they prefer, which can also include unobserved idiosyncratic inspection/repair costs that are incorporated in the GEV shocks we describe below.

Let $C(i, a)$ denote the choice set for the consumer who enters the period with the existing model i car that is a years old. If the consumer has no car ($a = \emptyset$) they can choose to remain in the no-car state ($d = \emptyset$), buy a new car ($d = 0$) of any type $j \in \{1, \ldots, J\}$, or one of the vintages available for sale in the secondary market. If the consumer already owns a car model i of age $a < \bar{a}$, they have an additional option of keeping it which we denote $d = \kappa$. However, once a car reaches the terminal age $\bar{a}$ it is no longer possible to keep it according to our assumption of exogenous scrappage.[13] Every time an existing car is traded, the consumer chooses to either sell it on the secondary market which we denote by $s = 0_s$, or to take it to the scrap yard we denote by $s = 1_s$. The set of feasible choices for a consumer in a car state (i, a) is thereby summarized as follows[14]:

$$\begin{aligned} C(\emptyset) = C(i, \bar{a}) &= \{\emptyset\} \, \cup \, \{1, \ldots, J\} \times \{0, 1, \ldots, \bar{a} - 1\} \, , \forall j \\ C(i, a) &= \{(\emptyset, 1_s), (\emptyset, 0_s), \kappa\} \, \cup \, \{1, \ldots, J\} \times \{0, 1, \ldots, \bar{a} - 1\} \times \{1_s, 0_s\} \, , \forall j, a < \bar{a}, \end{aligned} \tag{1}$$

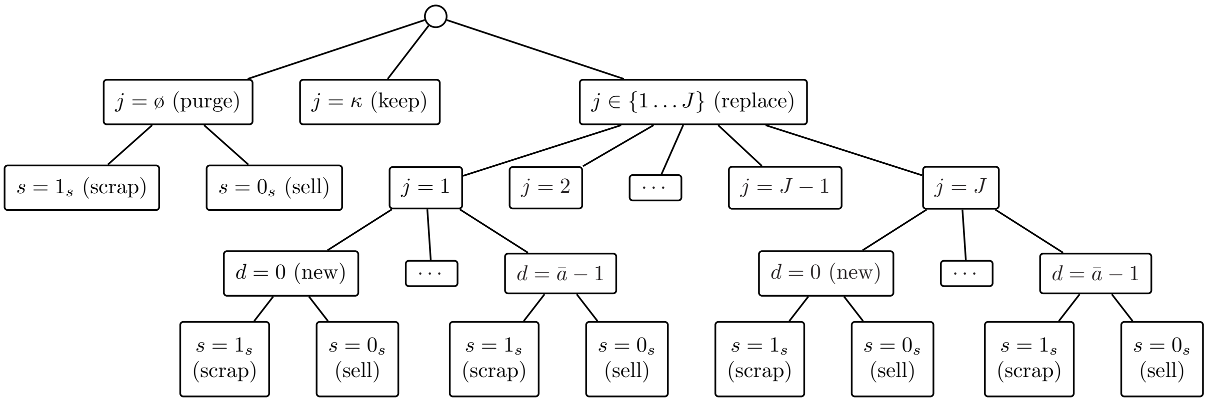
Because of our assumption of exogenous scrappage as soon as cars reach the upper bound on age $\bar{a}$, such cars are not traded, and therefore the oldest car that is possible to buy in the secondary market is $\bar{a} - 1$ old.

The random component ϵ is a vector with the number of elements equal to the number of state dependent discrete choices a consumer has. We assume that ϵ has a multivariate GEV distribution which allows for flexible dependence between its elements, but that the vectors are drawn independently between time periods and individuals, capturing the idiosyncratic heterogeneity between them. The elements of the vector ϵ in turn capture the differences between the discrete choices available to each individual, such as

[13] The difference between the decision $d = \kappa$ to keep the current car of age a and the decision $d = a$ to trade for another car of the same age and model $j = i$ is in the incurred transaction costs.

[14] To simplify notation here and throughout the paper we use single $\emptyset$ symbol for the no car state, i.e. $C(\emptyset)$ instead of $C(\emptyset, \emptyset)$.

Figure 1: Example of consumer choice tree



Notes: The figure presents an example of the choice tree for a consumer who owns a car under the nested logit specification of the GEV distribution of the idiosyncratic heterogeneity term ϵ. Note three choice nests at the top level (to purge, keep or trade the existing car), two intermediate levels of nesting in the case of trading, and a additional scrappage choice of the existing car in the cases when it is traded.

maintenance expenditures, search costs, and variability in the prices of traded cars that reflect their idiosyncratic features.[15]

The GEV distribution we apply in our framework generalizes the standard multivariate extreme value distribution and results in the choice probabilities that take the form that McFadden (1981) called the *nested multinomial logit* (NMNL) rather than standard multinomial logit. Under this specification it is possible to control for correlation patterns in different subsets of the overall choice set $C(i, a)$, and the choice itself can be represented as a sequence of choices from a nested subsets (filtration) of $C(i, a)$. However, even with this representation, all decisions are made *simultaneously* and instantaneously at the beginning of each period as described above. The patters of interdependence of the GEV distribution can be illustrated by a *choice tree,* a directed acyclical graph such as the one shown in Figure 1.

The choice tree illustrated in Figure 1 is one of the many possible ways to introduce dependence patterns into the consumer choice. The tree is drawn for a consumer who owns a car, and has four levels: the top level consists of the choices to purge ($j = \emptyset$), keep ($j = \kappa$) or replace ($j \in \{1, \ldots, J\}$) the current car. Conditional on the decision to replace, the second level is the choice of the make/model j of the replacement vehicle.

[15]The components of ϵ should be interpreted in an *ex ante* sense, as the idiosyncratic utility/disutility the consumer can expect from undertaking a search for a used car of a given type. We do not explicitly model the sequential search process in this paper, nor do we model the "microstructure" of auto dealers and other places consumers go to search for and buy used cars.

The third level contains the choice of the age d of the car to buy, a new car $d = 0$. Finally, the fourth level contains the choice of whether to sell or scrap the existing car if it is not kept.[16] Each level choice, apart from top choice $j = \kappa$ in our example, there is a subtree of lower level choices eventually leading to a distinct *leaf* of the tree corresponding to exactly one alternative in the choice sets defined in equation (1).

Dependence patterns in the distribution of the elements of ϵ are determined by a set of scale parameters which can be defined individually in each choice subset in each separate subtree of choice tree. For the example in Figure 1, parameter σ controls the scale of idiosyncratic shocks at the top level choice set $\{\emptyset, \kappa, \{1, \ldots, J\}\}$ involving the decision to have no car, keep the current car, or trade for some other new or used car. The parameter σ_r controls the degree of similarity in the unobserved shocks affecting the choice of one of the J different types of cars, i.e. idiosyncratic heterogeneity in "brand effects". The parameters σ_j, $j \in \{1, \ldots, J\}$ control the scale of idiosyncratic shocks affecting the choices of different ages of cars of a given make/model. Finally the parameter σ_s controls the scale of idiosyncratic shocks reflecting unobserved components of transaction, inspection and repair costs involved in selling the current car versus scrapping it. McFadden (1981) showed that in order for the GEV distribution to be a valid multivariate probability distribution the similarity parameters must form a non-increasing sequence along any particular branch. In our example this implies that $\sigma \geq \sigma_r \geq \sigma_j \geq \sigma_s$ for all $j \in \{1, \ldots, J\}$. As any of the similarity parameters approaches zero, the choice in the corresponding choice subset becomes deterministic, as do the choices in the subtrees below. When $\sigma = \sigma_r = \sigma_j = \sigma_s$ the choice model reverts to the MNL model with all alternatives in $C(i, a)$ defined in (1).

3.3 Consumer dynamic choice model

The optimal trading/holding strategy for cars is given by the solution of their infinite horizon expected utility maximization problem, which constitutes a discrete choice dynamic programming problem (Rust, 1987, 1994). Let $V(i, a, \epsilon)$ be the value function for a consumer in state (i, a, ϵ), $i \in \{\emptyset, 1, \ldots, J\}$, $a \in \{\emptyset, 1, \ldots, \bar{a}\}$. For a consumer who does

[16] The forth level scrappage choice appears on level two for the top level decision to purge without loss of generality.

not own a car it is given by:

$$V(\text{ø}, \epsilon) = \max \left[v(\text{ø}, \text{ø}) + \epsilon(\text{ø}); \max_{\substack{j \in \{1,\dots,J\}, \\ d \in \{0,1,\dots,\bar{a}-1\}}} \big[v(\text{ø}, j, d) + \epsilon(j, d) \big] \right], \tag{2}$$

where the *choice specific value functions* of remaining in the no car state and of leaving the no car state to buy a car of type j age d respectively are given by:

$$\begin{aligned} v(\text{ø}, \text{ø}) &= u(\text{ø}) + \beta EV(\text{ø}), \\ v(\text{ø}, j, d) &= u(j, d) - \mu \big[P_{jd} + T_b(j, d) \big] \\ &\quad + \beta \big(1 - \alpha(j, d) \big) EV(j, d+1) + \beta \alpha(j, d) EV(j, \bar{a}). \end{aligned} \tag{3}$$

The *expected value functions* $EV(\text{ø})$ and $EV(j, a)$ provide the conditional expected values of starting the next period respectively without a car, and with car of type i and age a, and are given by:

$$EV(\text{ø}) = \int_\epsilon V(\text{ø}, \epsilon) f(\epsilon | \text{ø}) d\epsilon, \quad EV(i, a) = \int_\epsilon V(i, a, \epsilon) f(\epsilon | i, a) d\epsilon, \tag{4}$$

where $f(\epsilon|\cdot)$ is the corresponding probability density function of a GEV distribution for the idiosyncratic shocks ϵ. Implicit in these formulas is the assumption that idiosyncratic shocks affecting the consumer's choice are independent of their past realizations. This implies that the $EV(\cdot)$ functions only depend on the car that has been driven and aged during the current period and constitutes the car state in the beginning of the next period. Under our assumption of GEV distribution of idiosyncratic shocks ϵ, the integrals in (4) can be expressed in closed-form. The formulas depend on the assumed nesting structure of the choices. Later in this section, we provide the analytic formulas for $EV(\cdot)$ corresponding to the nesting structure in Figure 1.

The value function for a consumer who starts the period with a car of terminal age $\bar{a}$ is similarly given by:

$$V(i, \bar{a}, \epsilon) = \max \left[v(i, \bar{a}, \text{ø}) + \epsilon(\text{ø}); \max_{\substack{j \in \{1,\dots,J\}, \\ d \in \{0,1,\dots,\bar{a}-1\}}} \big[v(i, \bar{a}, j, d) + \epsilon(j, d) \big] \right], \tag{5}$$

where again the first component corresponds to the decision to choose the outside option of not owning a car, and the second to the purchase of a new car of type j and age d. Because at terminal age $\bar{a}$ the car is scrapped exogenously, the consumer does not have the option of keeping their car, and also does not have an additional choice of endogenous scrappage of the existing car $(i, \bar{a})$. This is why the the Bellman equation (5) looks very similar to that of the consumers who do not own a car (2). The difference, however, is in the included scrap price $\underline{P}_i$ that is paid by the scrap yard. Therefore the relevant choice specific value functions are given by:

$$\begin{aligned} v(i,\bar{a},\text{ø}) &= u(\text{ø}) + \mu \underline{P}_i + \beta EV(\text{ø}), \\ v(i,\bar{a},j,d) &= u(j,d) - \mu\big[P_{jd} - \underline{P}_i + T_b(j,d)\big] \\ &\qquad + \beta\big(1-\alpha(j,d)\big)EV(j,d+1) + \beta\alpha(j,d)EV(j,\bar{a}). \end{aligned} \tag{6}$$

Finally, the value function for a consumer who starts the period with a car of type i and age a is given by:

$$V(i,a,\epsilon) = \max \begin{bmatrix} v(i,a,\kappa) + \epsilon(\kappa); \\ \max_{s\in\{1_s,0_s\}} \big[v(i,a,\text{ø},s) + \epsilon(\text{ø},s)\big]; \\ \max_{\substack{j\in\{1,\dots,J\},\\ d\in\{0,1,\dots,\bar{a}-1\},\\ s\in\{1_s,0_s\}}} \big[v(i,a,j,d,s) + \epsilon(j,d,s)\big] \end{bmatrix}. \tag{7}$$

In this case the endogenous scrappage choice $s \in \{1_s, 0_s\}$ has to be accounted for, so the complete set of choice specific value functions which correspond to all the alternatives in the choice set $C(i, a)$ defined in (1), is given by:

$$\begin{aligned} v(i,a,\text{ø},1_s) &= u(\text{ø}) + \mu \underline{P}_i + \beta EV(\text{ø}), \\ v(i,a,\text{ø},0_s) &= u(\text{ø}) + \mu[P_{ia} - T_s(i,a)] + \beta EV(\text{ø}), \\ v(i,a,\kappa) &= u(i,a) + \beta\big(1-\alpha(i,a)\big)EV(i,a+1) + \beta\alpha(i,a)EV(i,\bar{a}), \\ v(i,a,j,d,1_s) &= u(j,d) - \mu[P_{jd} - \underline{P}_i + T_b(j,d)] \\ &\qquad + \beta\big(1-\alpha(j,d)\big)EV(j,d+1) + \beta\alpha(j,d)EV(j,\bar{a}), \\ v(i,a,j,d,0_s) &= u(j,d) - \mu[P_{jd} - P_{ia} + T_s(i,a) + T_b(j,d)] \\ &\qquad + \beta\big(1-\alpha(j,d)\big)EV(j,d+1) + \beta\alpha(j,d)EV(j,\bar{a}). \end{aligned} \tag{8}$$

Here $v(i, a, \text{ø}, 0_s)$ is the value of selling the car on the market and not replacing it,

$v(i, a, \emptyset, 1_s)$ is the value of scrapping the car and not replacing it — in both of these cases the customer has no car to drive and ends up in the no car state in the next period.[17] The value of keeping the existing car is given by $v(i, a, \kappa)$, and the values of trading the existing car (i, a) to a replacement car (j, d) is denoted $v(i, a, j, d, 0_s)$ and $v(i, a, j, d, 1_s)$, respectively, depending on whether the existing car is sold or scrapped.

Combined, the value functions defined in (2), (5) and (7) cover the whole state space of the problem (i, a, ϵ), $i \in \{\emptyset, 1, \dots, J\}$, $a \in \{\emptyset, 1, \dots, \bar{a}\}$. With their corresponding choice specific values, and the general formula for the expected value function (4), we can define the Bellman equation for the consumer choice problem as a mapping of the space of value functions $V(i, a, \epsilon)$ to itself, and standard contraction mapping arguments guarantee that V is the unique fixed point to the "Bellman operator". However, the problem can be solved in a computationally much easier fashion in terms of the "projection" $EV(i, a)$ which as we noted above is a much lower dimensional object since it does not depend on the continuously distributed idiosyncratic state variables ϵ.[18] EV is just a finite dimensional vector, $EV \in \mathbb{R}^{J\bar{a}+1}$, whose elements are the expected values of starting the next period in car state (i, a), $i \in \{1, \dots, J\}$, $a \in \{1, \dots, \bar{a}\}$, or with no car.[19] Applying equations (4) for each element of vector EV, plugging in the expressions (2), (5), (7) and their corresponding choice specific value functions, we derive the system of $J\bar{a} + 1$ nonlinear equations

$$EV = \Gamma(EV), \tag{9}$$

whose solution enables us to reconstruct V and characterize optimal trading behavior. Here Γ is the *smoothed Bellman operator* that constitutes a contraction mapping and hence has a unique fixed point. Further, Γ is a smooth mapping from $\mathbb{R}^{J\bar{a}+1}$ to $\mathbb{R}^{J\bar{a}+1}$, which enables us to use Newton's method in combination with the method of successive approximations to rapidly compute this unique finite-dimensional fixed point and thus solve the consumer choice problem for any set of car prices.[20]

[17]Recall that by our assumption the transaction cost of selling to the scrap yard is normalized to zero, i.e. included into $\underline{P}_i$.

[18]Another though inferior possibility is to formulate and solve the Bellman equation in the space of choice specific value functions, which depend on both state and choice variables, and thus constitute even higher dimensional object than the value functions themselves. All three ways to set up the fixed point problem are equivalent in the sense that they lead to the same solution, and the corresponding functional mappings are contractions (Ma and Stachurski, 2021).

[19]Recall that due to our timing assumption it is not possible to start the period with a new car ($a = 0$): all new cars purchased during the trading stage become one year old cars by the start of the next period.

[20]See Lemma L2 for details and proof.

Under the GEV distributional assumption, the integrals in the equation (4) have closed form expressions, further contributing to the computational tractability of the problem. The analytic form of these depends on the assumed nesting of choices in the decision process, as in the example illustrated in Figure 1, and the implied dependency structure of the elements of ϵ. Generally, the choice specific values $v(i,a,\cdot)$ which correspond to the bottom layer of the nodes in the tree, are combined together following the tree structure with the help of McFadden's (1981) *log-sum* (smoothed max) function $\mathcal{I}(\cdot)$, defined as:

$$\mathcal{I}(\lambda, \chi_1, \dots, \chi_n) = \lambda \log \left(\exp \tfrac{\chi_1}{\lambda} + \dots + \exp \tfrac{\chi_n}{\lambda}\right), \tag{10}$$

where λ takes the value of the scale parameter in each particular grouping of alternatives (nest). Using the example choice tree in Figure 1, we have:

$$EV(i,a) = \mathcal{I}\Big(\sigma, \underbrace{\mathcal{I}\big(\sigma_s, v(i,a,\varnothing,1_s), v(i,a,\varnothing,0_s)\big)}_{\text{purge}}, \underbrace{v(i,a,\kappa)}_{\text{keep}}, \underbrace{\mathcal{I}\big(\sigma_r, I_1, \dots, I_J\big)}_{\text{replace}}\Big), \tag{11}$$

where I_j are the *inclusive values* of trading to a car make/model j, which are found by the application of the log-sum function (10) to the further nests as:

$$I_j = \mathcal{I}\big(\sigma_j, \underbrace{I(i,a,j,0,)}_{\text{new car}}, \dots, I(i,a,j,\bar{a}-1)\big),$$

where $I(i,a,j,d) = \mathcal{I}\big(\sigma_s, v(i,a,j,d,1_s), v(i,a,j,d,0_s)\big)$ is the inclusive value of the nested scrappage decision corresponding to the choice of car j of age $d \in \{0,\dots,\bar{a}-1\}$, including the new car. When scale parameters are equalized, $\sigma = \sigma_r = \sigma_j = \sigma_s$, the above nested log-sum functions collapse such that the expected value can be found by a single application of the log-sum formula to the choice specific values corresponding to all alternatives in the choice set $C(i,a)$ defined in (1). The expected values $EV(\varnothing)$ and $EV(i,\bar{a})$ can be computed with analogous closed-form formulas.

Let $\Pi(j,d,s|i,a)$ be the conditional probability of choosing a feasible alternative (j,d,s) from the choice set $C(i,a)$ by a consumer in a given car state (i,a). Under the GEV assumption these choice probabilities take NMNL closed form expressions which also depend on the structure of the choice tree (McFadden, 1981). For example, under the choice structure in Figure 1 the top level probability of keeping the existing car follows directly from (11):

$$\Pi(\kappa|i,a) = \tag{12}$$
$$\frac{\exp(v(i,a,\kappa)/\sigma)}{\exp(\mathcal{I}(\sigma_s, v(i,a,\emptyset,1_s), v(i,a,\emptyset,0_s))/\sigma) + \exp(v(i,a,\kappa)/\sigma) + \exp(\mathcal{I}(\sigma_r, I_1,\dots,I_J)/\sigma)}$$

For more complicated nested choices such as the choice of scrapping/selling the existing car (i,a) and replacing it with car (j,d) following the choice tree in Figure 1 the choice probability $\Pi(j,d,s|i,a)$ can be decomposed into products of conditional probabilities as:

$$\Pi(j,d,s|i,a) = \Pi(\text{replace}|i,a)\cdot\Pi(j|\{1,\dots,J\},i,a)\cdot\Pi(d|j,i,a)\cdot\Pi(s|j,d,i,a), \tag{13}$$

where $\Pi(\text{replace}|i,a)$ denotes the probability of trading for *some* type of car, the choice probability $\Pi(j|\{1,\dots,J\},i,a)$ corresponds to choosing a particular make/model j conditional on having decided to replace the existing car (i,a), $\Pi(d|j,i,a)$ is the choice probability for a particular age d, and $\Pi(s|j,d,i,a)$ is the probability of the choice of scrapping the existing car (i,a) or selling it in the secondary market. These probabilities are given by the following expressions:

$$\Pi(\text{replace}|i,a) = \tag{14}$$
$$\frac{\exp(\mathcal{I}(\sigma_r, I_1,\dots,I_J)/\sigma)}{\exp(\mathcal{I}(\sigma_s, v(i,a,\emptyset,1_s), v(i,a,\emptyset,0_s))/\sigma) + \exp(v(i,a,\kappa)/\sigma) + \exp(\mathcal{I}(\sigma_r, I_1,\dots,I_J)/\sigma)},$$

$$\Pi(j|\{1,\dots,J\},i,a) = \frac{\exp(I_j/\sigma_r)}{\exp(I_1/\sigma_r)+\cdots+\exp(I_J/\sigma_r)}, \tag{15}$$
$$\Pi(d|j,i,a) = \frac{\exp(I(i,a,j,d)/\sigma_j)}{\exp(I(i,a,j,0)/\sigma_j)+\cdots+\exp(I(i,a,j,\bar{a}-1)/\sigma_j)}, \tag{16}$$
$$\Pi(s|j,d,i,a) = \frac{\exp(v(i,a,j,d,s)/\sigma_s)}{\exp(v(i,a,j,d,1_s)/\sigma_s)+\exp(v(i,a,j,d,1_s)/\sigma_s)}. \tag{17}$$

The remaining choice probabilities for all feasible alternatives in the choice sets (1) corresponding to all other states in the model (including having no car $i=\emptyset$ or having the car of terminal age $a=\bar{a}$), have similar closed form expressions which combine the corresponding choice-specific value functions implied by the assumed nesting structure of choices.

Due to additively separable transaction costs, and provided that the scale parameter σ_s

is the same in all nests of the choice tree where the decision is relevant, the choice between scrapping and selling the existing car is independent of the choice of the replacement car and has no implications for the future periods. This implies that the scrap/sell decision is static, so $\Pi(s|j,d,i,a) = \Pi(s|i,a)$ and we can generally factor any choice probability as $\Pi(j,d,s|i,a) = \Pi(j,d|i,a)\Pi(s|i,a)$, where we refer to $\Pi(j,d|i,a)$ as the trading choice probability, and $\Pi(s|i,a)$, $s \in \{1_s, 0_s\}$ as the endogenous scrappage probability. It is readily verified that the scrappage probability is the same for both the case of purging and replacing existing car (i,a), $i \in \{1,\dots,J\}$, $a \in \{1,\dots,\bar{a}-1\}$, and is given by:

$$\Pi(1_s|i,a) = \Big(1 + \exp\left(\tfrac{\mu}{\sigma_s}[P_{ia} - T_s(i,a) - \underline{P}_i]\right)\Big)^{-1}. \tag{18}$$

In other words, car owners choose to scrap or sell their existing car based on the difference between the market price net of seller transactions cost and the scrap value of their car, conditional the marginal utility of money μ and the scale parameter σ_s.

3.4 Equilibrium with idiosyncratic consumer heterogeneity

With the consumer dynamic choice problem fully described, in this section we turn to the definition of stationary equilibrium in the secondary market for automobiles. Recall that we assume that the prices of the new cars $\overline{P}_j$, $j \in \{1,\dots,J\}$ are fixed, and that the supply of new cars is infinitely elastic. Similarly we assume that there is an infinitely elastic demand for cars for their scrap value $\underline{P}_j$, and so we also treat the scrap value of a car of each type j as fixed. Recall that the cars have to scrapped at the upper bound of the age $\bar{a}$.

The used cars of age from $a = 1$ to $a = \bar{a} - 1$ of each make/model are traded in the secondary market. Supply and demand of these $J(\bar{a}-1)$ tradable goods are balanced by $J(\bar{a}-1)$ prices which we combine into the J-block price vector

$$P = \big(P_1,\dots,P_J\big) = \Big(\big(P_{1,1},\dots,P_{1,\bar{a}-1}\big),\dots,\big(P_{J,1},\dots,P_{J,\bar{a}-1}\big)\Big) \in \mathbb{R}^{J(\bar{a}-1)}. \tag{19}$$

The value functions and choice probabilities derived in the previous subsection are all implicit function of P, though we suppressed their dependence on P so as not to overload the notation.

Let $0 \leq q_{ia} \leq 1$ denote the fraction of the unit mass of households in car state

(i, a), namely those who own the car of make/model i of age a at the start of the period before the trading phase. Let $0 \leq q_{\o} \leq 1$ be the fraction of households without a car. The *ownership distribution* vector q summarizes the distribution of the unit mass of consumers in the economy over all possible car states:

$$q = \big(q_1, \ldots, q_J, q_{\o}\big) = \Big((q_{1,1}, \ldots, q_{1,\bar{a}}), \ldots, (q_{J,1}, \ldots, q_{J,\bar{a}}), q_{\o}\Big) \in \mathbb{R}^{J\bar{a}+1}. \tag{20}$$

The ownership distribution q is a proper probability vector (its elements sum up to 1), and thus belongs to the $J\bar{a}$-dimensional unit simplex. The subvectors of the ownership distribution q_i correspond to the particular makes/models of the car, and do not represent a proper distribution unless normalized. The distribution of all cars in the economy is a vector with one less element than q, and can be constructed by using its first $J\bar{a}$ elements, multiplied with the normalization constant $1/(1 - q_{\o})$.

Though we have a continuum of consumers, we are studying an economy with a finite number of goods, so our concept of equilibrium involves the traditional approach of finding a vector P that equates supply and demand for all used cars in the secondary market. However with a continuum of consumers, we will define supply and demand in terms of the *fraction* of the total population of consumers who wish to sell and to buy a car of a given type and age, (j, d).[21] Let $D_{jd}(P, q)$ be the demand for make/model j cars of age d. Conditional on the ownership distribution of consumers q, for $j \in \{1, \ldots, J\}$, $d \in \{1, \ldots, \bar{a} - 1\}$, it is given by

$$D_{jd}(P, q) = \Pi(j, d|\o, P)q_{\o} + \sum_{i=1}^{J} \sum_{a=1}^{\bar{a}} \Pi(j, d|i, a, P)q_{ia}, \tag{21}$$

where we now include P as an argument of the choice probabilities $\Pi(j, d|\cdot, P)$ to emphasize their dependence on market prices.

The supply of used cars to the secondary market are those which are not kept and not scrapped. The corresponding fractions of consumers are given by the complements to the choice probability of keeping, $\Pi(\kappa|i, a, P)$ and scrapping, $\Pi(1_s|i, a, P)$. Let $S_{jd}(P, q)$ be the supply for the cars of make/model j of age d, $j \in \{1, \ldots, J\}$, $d \in \{1, \ldots, \bar{a} - 1\}$.

[21] Each consumer in the economy chooses the alternative that maximizes their payoff conditional on their independent draw of the random component ϵ; by the Law of Large Numbers, these *deterministic* individual choices aggregate to population shares that are defined in terms of q and choice probabilities.

It is given by

$$S_{jd}(P, q) = \big(1 - \Pi(\kappa|j, d, P)\big)\big(1 - \Pi(1_s|j, d, P)\big)q_{jd}. \tag{22}$$

Supply and demand of the cars of all make/models and all ages can be stacked in the same way as price vector (19) to form the J-block vectors of demand and supply, $D(P, q)$ and $S(P, q)$. We then define the vector of *excess demand* as

$$ED(P, q) = \big(D_{11}(P, q) - S_{11}(P, q), \ldots, D_{J,\bar{a}-1}(P, q) - S_{J,\bar{a}-1}(P, q)\big) \in \mathbb{R}^{J(\bar{a}-1)} \tag{23}$$

In equilibrium the prices equate supply and demand of used cars resulting in zero excess demand, so equilibrium prices are a solution to the non-linear system of $J(\bar{a}-1)$ equations given by $ED(P, q) = 0$ with $J(\bar{a} - 1)$ unknown prices P.

Besides the market clearing condition which has to hold in each time period, in a stationary flow equilibrium we require the ownership distribution q to be time-invariant. The evolution of q can be broken into two stages: 1) an instantaneous trading phase in the beginning of each period, and 2) the rest of the period when car utilization takes place. After the trading phase ownership of cars changes due to trade between households. Also old cars are scrapped and new cars purchased. Then between periods t and $t+1$ the state of cars change as they either become one period older or are involved in an accident.

To describe the two phases of the evolution of the ownership distribution we rely on the tools from Markov chain theory, and describe car ownership and state transitions using two transition probability matrices Q and $\Omega(P)$ defined below. Let $\Omega(P)$ be the $(J\bar{a} + 1) \times (J\bar{a} + 1)$ *trade transition probability matrix* given by

$$\Omega(P) = \begin{bmatrix} \Delta_{11}(P) + \Lambda_1(P) & \Delta_{12}(P) & \ldots & \Delta_{1J}(P) & \Delta_{1\varnothing}(P) \\ \Delta_{21}(P) & \Delta_{22}(P) + \Lambda_2(P) & \ldots & \Delta_{2J}(P) & \Delta_{2\varnothing}(P) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{J1}(P) & \Delta_{J2}(P) & \ldots & \Delta_{JJ}(P) + \Lambda_J(P) & \Delta_{J\varnothing}(P) \\ \Delta_{\varnothing 1}(P) & \Delta_{\varnothing 2}(P) & \ldots & \Delta_{\varnothing J}(P) & \Pi(\varnothing|\varnothing, P) \end{bmatrix}, \tag{24}$$

where the typical $\bar{a} \times \bar{a}$ block of *replacing* choice probabilities $\Delta_{ij}(P)$ is given by:

$$\Delta_{ij}(P) = \begin{bmatrix} \Pi(j,1|i,1,P) & \dots & \Pi(j,\bar{a}-1|i,1,P) & \Pi(j,0|i,1,P) \\ \vdots & \ddots & \vdots & \vdots \\ \Pi(j,1|i,\bar{a}-1,P) & \dots & \Pi(j,\bar{a}-1|i,\bar{a}-1,P) & \Pi(j,0|i,\bar{a}-1,P) \\ \Pi(j,1|i,\bar{a},P) & \dots & \Pi(j,\bar{a}-1|i,\bar{a},P) & \Pi(j,0|i,\bar{a},P) \end{bmatrix}, \tag{25}$$

and the typical $\bar{a} \times \bar{a}$ block of *keeping* choice probabilities $\Lambda_i(P)$ is given by:

$$\Lambda_i(P) = \begin{bmatrix} \Pi(\kappa|i,1,P) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \Pi(\kappa|i,\bar{a}-1,P) & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}. \tag{26}$$

In addition, the bottom row and the rightmost column blocks in (24) are given by:

$$\Delta_{\o j}(P) = \Big[\Pi(j,1|\o,P),\dots,\Pi(j,\bar{a}-1|\o,P),\Pi(j,0|\o,P)\Big], \quad \Delta_{i\o}(P) = \begin{bmatrix} \Pi(\o|i,1,P) \\ \vdots \\ \Pi(\o|i,\bar{a},P) \end{bmatrix}.$$

Each $\bar{a} \times \bar{a}$ block in the trade transition probability matrix $\Omega(P)$ refers to the cars of each make/model. The trade probabilities $\Pi(j,d|i,a,P)$ are strictly positive for non degenerate GEV distributions for ϵ, form the bulk of the interior of $\Omega(P)$, and the probabilities of keeping $\Pi(\kappa|i,a,P)$ appear on the diagonal. The bottom row contains the probability of buying a car $\Pi(j,d|\o,P)$ by households who don't have one. The last column contains the probability $\Pi(\o|i,a,P)$ of choosing the no car state, and the bottom left corner element is the probability of remaining in the no car state $\Pi(\o|\o,P)$.

Note that because the cars of the terminal age $\bar{a}$ cannot be traded, we use the last column of each block $\Delta_{ij}(P)$, the sub-transition probabilities for trading car i for car j, to hold the choice probabilities $\Pi(j,0|i,a,P)$ corresponding to buying a new car of type j. Similarly, the last diagonal element in the each block $\Lambda_i(P)$ is zero because the cars of age $\bar{a}$ can not be kept, and instead have to be scrapped during the trading phase.

The structure of the trade transition probability matrix $\Omega(P)$ corresponds to the block structure of the ownership distribution q in (20). The matrix product $q\Omega(P)$

represents the distribution of car ownership in the economy after the trading phase, assuming that all demand is satisfied (which is true in equilibrium). The result is the *post-trade* holdings distribution $q\Omega(P)$, which reflects the distribution of car holdings after the instantaneous trading phase has occurred where new cars are delivered to households who demand them, and used cars which households choose to scrap (both endogenous and exogenous scrappage) are removed from the economy. Due to the special arrangement of the columns in the trade transition probability matrix, the elements in the post-trade holdings distribution are reordered such that the fraction of owners of new cars in $q\Omega(P)$ is the last element in each subvector of length $\bar{a}$.

After the instantaneous trading phase, households own and drive their cars and the aging and accidents in these cars is governed by the $(J\bar{a}+1) \times (J\bar{a}+1)$ block-diagonal stochastic matrix that we refer to as *physical transition probability matrix* Q

$$Q = \begin{bmatrix} Q_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \\ 0 & \dots & Q_J & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}, \text{ where} \tag{27}$$

$$Q_j = \begin{bmatrix} 0 & 1-\alpha(j,1) & \dots & 0 & \alpha(j,1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\alpha(j,\bar{a}-2) & \alpha(j,\bar{a}-2) \\ 0 & 0 & \dots & 0 & 1 \\ 1-\alpha(j,0) & 0 & \dots & 0 & \alpha(j,0) \end{bmatrix}. \tag{28}$$

Each $\bar{a} \times \bar{a}$ block Q_j forms a transition probability matrix which governs the evolution of cars of make/model j. The first $\bar{a}-1$ rows of the matrix describe the joint effect of deterministic aging and stochastic exogenous scrappage. As described in the previous section, the latter is modeled as a direct transition to the terminal age $\bar{a}$ with probability $\alpha(j,d) \in [0,1)$, resulting in a compulsory scrappage next period. Cars of age $\bar{a}-1$ reach the terminal age $\bar{a}$ with certainty as both aging and exogenous scrappage lead to the same outcome $\bar{a}$. The last governs aging and accidents of new cars. Finally, the bottom right corner element of Q denotes the transition by households who choose the no-car state.

It follows immediately that the product of the post trade ownership distribution and the physical transition matrix matrix, $q\Omega(P)Q$, gives the ownership distribution in the beginning of the next period. It is then clear that the stationary ownership distribution is simply given by an invariant distribution of the matrix $\Omega(P)Q$, which the following theorem shows is unique for any P.

Theorem 1. *Let $\sigma \geq \sigma_r \geq \sigma_j \geq \sigma_s > 0$. Then for any vector of prices P there is a unique invariant distribution q that satisfies the stationarity condition $q = q\Omega(P)Q$, and is a continuously differentiable function of P.*

Proof. See Appendix A.1 (page 64). □

Uniqueness of the stationary ownership distribution in Theorem 1 simply follows from the fundamental theorem of Markov chains, once we realize that with positive GEV scale parameters the choice probabilities have full support, and therefore transition matrix $\Omega(P)Q$ is irreducible and aperiodic. However, to show differentiability of $q(P)$ with respect to P we need a more involved argument given in the Appendix.

We are now in position to formally define and prove existence of equilibrium in the automobile market that extends the stationary flow equilibrium concept of Rust (1985a) to economies with positive transactions costs and discrete goods.

Definition D1 (Stationary equilibrium in the automobile market)**.** *A stationary equilibrium in the economy with a unit mass of consumers and cars of J makes/models and ages bounded above by $\bar{a}$, is given by the price vector and the ownership distribution probability vector $(P, q) \in \mathbb{R}^{J(\bar{a}-1)} \times \mathbb{R}^{J\bar{a}+1}$, such that the following conditions are satisfied:*

(a) Consumers follow their optimal trading strategies that arise from the solution of the dynamic problem (2)-(8);

(b) The market clearing conditions are satisfied: the excess demand in (23) is zero;

(c) The ownership distribution q is time invariant;

(d) New cars are supplied at fixed prices $\overline{P}_i$ and scrapped at prices $\underline{P}_i$, $i \in \{1, \ldots, J\}$, infinitely elastically.

Theorem 2. *The stationary equilibrium in the economy without persistent consumer heterogeneity defined in Definition D1 exists. In equilibrium the ownership distribution q satisfies the stationarity condition below, and the equilibrium prices P satisfy the market* *clearing condition:*

$$q\Omega(P)Q = q, \tag{29}$$

$$ED(P, q) = 0. \tag{30}$$

Proof. See Appendix A.2 (page 65). □

The roadmap of the proof is the following. First, by Theorem 1 the equilibrium distribution q given by (29) is a smooth function of market prices P. The same is true for all major components of the model, namely the expected values EV which constitute the fixed point of the Bellman operator Γ in (9), all choice probabilities, and the excess demand $ED(P, q)$ given in (23). Then, given that the excess demand $ED(P, q)$ in our framework is bounded to the $(-1, 1)$ hypercube, we construct a continuous map satisfying the conditions of the Brouwer fixed point theorem, which establishes the existence of equilibrium. A number of intermediate results that the proof of Theorem 2 relies on, and turn out to be very useful for our computational framework, are formulated as separate lemmas in Appendix A.

It follows directly from the stationarity of the ownership distribution q that the fraction of population without cars is also time-invariant. Algebraically it can be seen from comparing the last elements in the left and right hand sides of the stationarity condition (29). Because the last column in Q has only one non-zero element, it follows that

$$\sum_{i=1}^{J} \sum_{a=1}^{\bar{a}} \Pi(\varnothing | i, a, P) q_{ia} + \Pi(\varnothing | \varnothing, P) q_{\varnothing} = q_{\varnothing}. \tag{31}$$

Thus, a consequence of the stationarity of the equilibrium is that the fraction of consumers who demand the outside good, i.e. choose not to have a car, in the left hand side of (31) equals the "supply the outside good".

Another consequence of Definition D1 and the two conditions in Theorem 2 is that the economy is in *stationary flow equilibrium*, i.e. it exhibits the *steady flow* property that the outflow of cars due to endogenous and exogenous scrappage equals the inflow of new cars of each make/model $j \in \{1, \dots, J\}$. If the economy were not in a stationary flow equilibrium there would either be a continual increase or decrease in the total stock of cars of each type j in the economy over time.

Theorem 3. *In a stationary equilibrium under the conditions of Theorem 2 the steady flow property is satisfied for each car make/model $j \in \{1, \dots, J\}$*

$$\underbrace{\sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)\big(1-\Pi(\kappa|j,a,P)\big)q_{ja} + q_{j\bar{a}}}_{\textit{outflow of scrapped cars of make/model } j} = \underbrace{\sum_{i=1}^{J}\sum_{a=1}^{\bar{a}} \Pi(j,0|i,a,P)q_{ia} + \Pi(j,0|\o,P)q_{\o}}_{\textit{inflow of new cars of make/model } j}. \tag{32}$$

Proof. See Appendix A.3 (page 65). □

With the result of Theorem 3, it is clear that Definition D1 defines a *stationary flow equilibrium* in the market of new and used cars. Note that Theorem 2 only establishes the existence of the stationary flow equilibrium, but is silent about its uniqueness. Uniqueness of the equilibrium ownership distribution under any market prices P is established by Theorem 1, however, we have been unable to find high level conditions that guarantee uniqueness of the equilibrium price vector P. Despite this, we have computed many equilibrium solutions and have never encountered an issue of multiplicity of equilibria for a variety of utility function specifications and parameter values. Thus, we conjecture that there are conditions under which a stationary flow equilibrium not only exists, but is unique.

3.5 Numerical implementation

The key to success for our numerical implementation is the possibility to use the efficient Newton-based methods for finding the fixed point of the smooth Bellman operator Γ in the dynamic programming part of the model (9), and when solving the nonlinear system of equations (30) to find the equilibrium price vector P.

As is well known, Newton's method has a quadratic convergence rate when initiated from a sufficiently close starting point in a domain of attraction of the solution. In the dynamic programming part of the algorithm we rely on the globally convergent method of successive approximations before switching to Newton-Kantorovich iterations, in the same way it is done in the nested fixed point estimator (NFXP) in Rust (1994). In the equilibrium price search we initialize the Newton solver at the equilibrium prices of a similar model without consumer heterogeneity and transaction costs that can be computed as a solution to a system of linear equations as shown in Appendix C.

Appendix A contains several lemmas that establish the prerequisite smoothness properties, and specify the analytical formulas for the required gradients. We first note that EV is a smooth function of P that is implicitly defined by the fixed point condition, $EV = \Gamma(EV, P)$. We prove this in Lemma L2 using the Implicit Function Theorem. First we prove Lemma L1 which shows that $\nabla_{EV}\Gamma$, the $(J\bar{a}+1) \times (J\bar{a}+1)$ Jacobian matrix of $\Gamma(EV, P)$ with respect to EV, has matrix norm equal to β. This implies that $I - \nabla_{EV}\Gamma(EV, P)$ is invertible, and thus the Implicit Function Theorem guarantees that $EV(P)$ is a continuously differentiable function of P. With this result, we can sequentially show, via the Chain Rule, that all value functions and all choice probabilities are also continuously differentiable in P. This in turn guarantees that the excess demand function $ED(P, q)$ is also a continuously differentiable function of P for any q.

Uniqueness and differentiability of $q(P)$ as an implicit function of P is established by Theorem 1. Using the chain rule, we obtain the Jacobian matrix for $ED\big(P, q(P)\big)$, and solve the market clearing conditions $ED\big(P, q(P)\big) = 0$ as a system of $J(\bar{a}-1)$ non-linear equations in prices. We can use Newton's method for this, but also using the chain rule to compute the *total derivative* of $ED(P, q(P))$ with respect to P. The computational algorithm involves the following steps:

1. For a given vector of market prices P, solve the Bellman equation (9) for the fixed point $EV(P)$ using Newton-Kantorovich iterations;
2. Compute choice probabilities and form the trading and physical transition probability matrices $\Omega(P)$ and Q;
3. Compute the ownership distribution $q(P)$ as an invariant distribution of $\Omega(P)Q$;
4. Calculate excess demand and update prices via Newton's method
   $P' = P - \; [\nabla_P ED(P, q(P))]^{-1} \; ED(P, q(P))$;
5. Exit if convergence criterion for $ED\big(P', q(P')\big) = 0$ is satisfied, otherwise replace P by P' and return to step 1.

Thus, it is possible to use the gradient-based Newton's method in all steps of our numerical implementation, resulting in a fast algorithm for computing the stationary flow equilibrium in the automobile market.[22] Given how quickly the equilibria can be computed for various parameter values and specifications of the model, it can be nested within other algorithms such as a maximum likelihood estimator that we develop in Section 5.

[22]Implementation code will be available through a public repository.

4 Equilibrium with Persistent Consumer Heterogeneity

The economy with idiosyncratically heterogeneous households analyzed in the previous section is sufficient to support trade in presence of transactions costs. However, when we allow for more persistent consumer heterogeneity there are even larger gains from trade that result in equilibrium sorting of different ages of cars by different types of households.

4.1 Time invariant heterogeneity

We start with the superficially opposite case to the idiosyncratic heterogeneity considered in the previous section, namely the economy with permanent household types. However, the key to tractability of our framework is that we introduce persistent heterogeneity *in addition* to the idiosyncratic heterogeneity due to GEV random components. Later we add an intermediate form of time varying heterogeneity as well, which in the end gives us the best of both worlds: more realistic flexible forms of consumer heterogeneity, while retaining the elegance and computational tractability from the nested logit GEV specification introduced in the previous section. It is essentially a form of "mixed logit" that has been so useful in empirical work.

We introduce the symbol $\tau \in \{1, \ldots, N_\tau\}$ to denote households of different permanent *types*, and denote f_τ the fraction of households of type τ. The structure of the household decision problem (2)-(5) is identical for all households, but due to arbitrary differences in preferences the solution becomes type-specific. Demand and supply from all types are aggregated in market equilibrium, and households endogenously specialize in their holdings of different makes/models and ages of automobiles in response to market prices and differences in their preferences. We allow for essentially unlimited flexibility in how the preferences of households of different types τ differ, as long as each type conforms to the general structure introduced in Section 3.1.

Let $u_\tau(i, a)$ be the utility for owning a car of make/model i and age a by households of type τ. Solving the Bellman equations (2)-(5) N_τ times we obtain the decision-specific value functions $v_\tau(i, a, j, d, s)$, expected value functions $EV_\tau(j, d)$, and choice probabilities $\Pi_\tau(j, d, s|i, a, P)$ for each household type τ, for all states $a \in \{1, \ldots, \bar{a}, \o\}$, $i \in \{1, \ldots, J\}$ and for all choices in $C(i, a)$ defined in (1).

Denote q_τ the ownership distribution of type τ households which we define similarly to (20) as a proper stochastic vector in $\mathbb{R}^{J(\bar{a}+1)}$. The full ownership distribution in the economy can be written as $q = (q_1 f_1, \ldots, q_{N_\tau} f_{N_\tau}) \in \mathbb{R}^{N_\tau J(\bar{a}+1)}$, but it is sufficient to work with its type-specific subvectors. Repeating the definitions of supply and demand from Section 3.4, we can derive τ type-specific excess demand functions $ED_\tau(P, q_\tau)$.

To extended Definition D1 for stationary equilibrium to the economy with permanent household types, note that conditions (a) and (d) remain unchanged, and we only have to modify the market clearing and stationarity conditions. Bearing in mind that trade is allowed between household types, $ED_\tau(P, q_\tau)$ does not need to be zero for each τ, instead the *integrated* demand has to clear, leading to the following condition

$$ED(P, q) = \sum_{\tau=1}^{N_\tau} ED_\tau(P, q_\tau) f_\tau = 0. \tag{33}$$

Further, with multiple types of households we require stationarity of the ownership distribution for each household type

$$q_\tau = q_\tau \Omega_\tau(P) Q_\tau, \; \forall \tau, \tag{34}$$

and thus stationarity of the ownership distribution in the whole economy. By making the aging transition probability matrix Q_τ household type specific in (34), we allow scrappage probabilities to vary by household type.

Theorem 4. *The stationary equilibrium in the economy with $\tau \in \{1, \ldots, N_\tau\}$ time invariant household types in addition to idiosyncratic heterogeneity, see Definition D1, exists. In equilibrium the ownership distribution $q \in \mathbb{R}^{N_\tau J(\bar{a}+1)}$ is composed of type shares weighted subvectors q_τ, each of which satisfies the stationarity condition* (34)*, and equilibrium prices P satisfy the market clearing condition* (33)*. Steady flow property of the equilibrium continues to hold.*

We omit the proof of Theorem 4 because it follows from straightforward modifications of the proofs of Theorems 1, 2 and 3 of section 3. Fully detailed proofs are available on request. But to provide a rough idea of how the proof works, first note that we can use Theorem 1 to prove that for each consumer type τ there is a unique invariant distribution $q_\tau = q_\tau \Omega_\tau(P) Q_\tau$ and this q_τ is continuously differentiable function of P. Then it follows from Lemma L2 that $ED(P, q)$ given in equation (33) is a smooth function of P. Then following the proof of Theorem 2 we can appeal to the Brouwer Fixed Point Theorem to prove the existence of an equilibrium with persistent heterogeneity. Finally, the steady flow equilibrium condition must also hold (otherwise the stock of cars would be continually increasing or decreasing over time) and this result can be proven via straightforward extension of the proof in Theorem 3.

Overall, the stationary equilibrium is exactly as described in Section 3: the only additional step is aggregation of τ-specific excess demands. Otherwise, a price vector P sets excess demand to zero per equation (33) subject to the constraint that the ownership distributions for all types τ are stationary per equation (34). Moreover, most of the theoretical results from Section 3 apply directly for each household type, one by one.

The computational approach from Section 3.5 does not change much at all: we compute the equilibrium by first solving equation (34) for $q_\tau(P)$ which is a smooth implicit function of P, and repeat this calculation N_τ times for every τ. Then the functions $q_\tau(P)$ are jointly substituted into the excess demand, and the corresponding non-linear system of equations in prices is solved, again with Newton's method. Therefore, as one part of the solution algorithm is repeated for each household type, and the other does not depend on N_τ, we conclude that the solver is only linearly more computationally costly.

4.2 Time varying and hybrid heterogeneity

Now consider the case of time varying types. Consider an exogenous Markov process with state space $\mathcal{Y}$ and transition density $\rho(y_{t+1}|y_t)$ for some time varying variable y_t that is household-specific, e.g. income, and which evolves independently for each household. Assuming y enters the utility for cars, $u(j, a, y)$ or the marginal utility of money, $\mu(y)$, the dynamic problem becomes more complex since the household now has to account for stochastic variation in the y_t state variable when considering his/her optimal car trading strategy. An unexpected negative income shock may induce the household to keep their older car and delay replacement, or conversely a positive income shock may induce them to trade their existing car and buy a new one, or upgrade to a different car make/model.

The Bellman equations (2)-(7), which describe the optimal trading strategy, needs to be altered to account for the extra state variable y_t, and need to include an extra integration with respect to the transition density $\rho(y_{t+1}|y_t)$.[23] Assume that $\{y_t\}$ is ergodic

[23]Since these extensions are straightforward we omit the Bellman equations to save space.

and has an invariant distribution $\lambda(y)$ satisfying

$$\lambda(y') = \int_{\mathcal{Y}} \rho(y'|y)\lambda(y)dy. \tag{35}$$

In the case when $\{y_t\}$ is a finite state Markov chain instead of a continuous state process, ρ is simply a transition probability matrix, and (35) can be written as $\lambda = \lambda\rho$, where λ is a probability distribution vector completely analogous to the distribution of permanent types $(f_1, \dots, f_{N_\tau})$ in the previous section.

With time-varying heterogeneity, the value functions, choice probabilities and ownership transition probability matrices are indexed by y similar to the way they were indexed by τ in the time invariant case. Let q_y denote the ownership distribution conditional on y, which we again define similarly to (20) as a proper stochastic vector in $\mathbb{R}^{J(\bar{a}+1)}$. Its typical element q_{jay} is the share of households who own car type j of age a while in income state y.

Continuing with the analogy, let $ED_y(P, q_y)$ denote the excess demand function for a household whose income state is y. Though y changes over time for different households, there is a stationary cross-sectional distribution of y given by the invariant density $\lambda(y)$ defined above, and there is a stationary joint density of car ownership states and y given by $q_y(P)$. So, to extend the Definition D1 for stationary equilibrium to the economy with time variant household types, we modify the market clearing condition to

$$\int_{\mathcal{Y}} ED_y(P, q_y)\lambda(y)dy = 0, \tag{36}$$

which is still the system of $J(\bar{a} - 1)$ non-linear equations in prices and ownership distribution. However the latter is pinpointed by the modified stationarity condition that takes into account the stochastic evolution of types according to the transition density $\rho(y'|y)$, namely

$$q_{y'} = \int_{\mathcal{Y}} q_y \Omega_y(P) Q_y \rho(y'|y)\lambda(y)dy. \tag{37}$$

Theorem 5. *The stationary equilibrium in the economy with time varying household types given by an exogenous ergodic Markov process $\{y_t\} \in \mathcal{Y}$ with transition density $\rho(y'|y)$ and stationary distribution $\lambda(y)$, defined in Definition D1, exists. In equilibrium the joint ownership-type distribution is given by the $\lambda(y)$ and q_y that satisfy the the sta-*

tionarity condition (37)*, and the equilibrium prices* P *satisfy the market clearing condition* (36)*. Steady flow property of the equilibrium continues to hold.*

This proof is also very similar to the proof of Theorem 4 and will be omitted for brevity, though a full detailed proof is available on request. It is also possible to layer combinations of time-invariant and time-varying heterogeneity and these cases can be handled by combining Theorems 4 and 5. For example we could have a finite number of types τ with different transition densities $\rho_\tau(y'|y)$. We can extend the equilibrium conditions by integrating excess demand $ED_{\tau y}(P, q_{\tau y})$ over all y for each type τ and then sum over types. This requires computing stationary ownership distributions $q_{\tau y}$ for each (τ, y) combination using a τ-specific analogue of (37), and stationary distributions λ_τ for each time invariant type τ. We can then substitute these invariant distributions (taken as smooth implicit functions of P) into the formula for excess demand $ED_{\tau y}(P, q_{\tau y})$, and compute the equilibrium prices by searching for a vector P that solves the system of nonlinear equations

$$ED(P, q) = \sum_{\tau=1}^{N_\tau} f_\tau \int_{\mathcal{Y}} ED_{\tau y}(P, q_{\tau y}) \lambda(y) dy = 0 \tag{38}$$

formed by integrating excess demand over all time-varying and invariant household types.

4.3 Illustrative example: sorting in stationary flow equilibrium

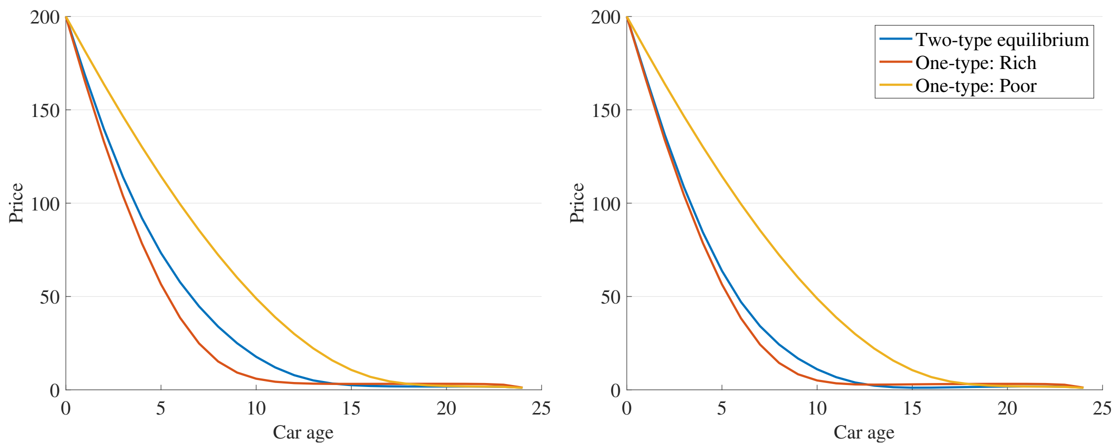
Figure 2 illustrates the stationary flow equilibrium in a heterogeneous agent economy with two permanent types of households who differ in their marginal utility of money, $\mu_2 = 0.3 > 0.1 = \mu_1$. The households who have a lower marginal utility of money are the rich households in this economy. The utility of the outside good is set to 0 for both households types. We assume this economy has 50% rich and 50% poor households.

In this example we collapse the GEV structure of random components to a simple extreme value EV1 distribution with common scale parameter $\sigma = 1$. Consumers also have the same discount factor $\beta = .95$ and the utility function is $u(a) = 10 - 0.5a$. There is a single car make/model $J = 1$ traded in this economy, with new car price $\overline{P} = 200$ and scrap value $\underline{P} = 1$.

We also illustrate the effect of transactions costs by computing an equilibrium with buyer side transactions costs of $T_b(P, d) = 0$ and $T_s(P, a) = 0$ (low transaction cost), as

Figure 2: Equilibrium price functions in a two household type economy

(a) Low transaction costs (b) High transaction costs



Notes: Both panels show the equilibrium price functions of the heterogeneous agent economy as well as homogeneous economies without transaction costs where all households are of either type.

well as an equilibrium with high transactions costs, $T_b(P, d) = 10$.

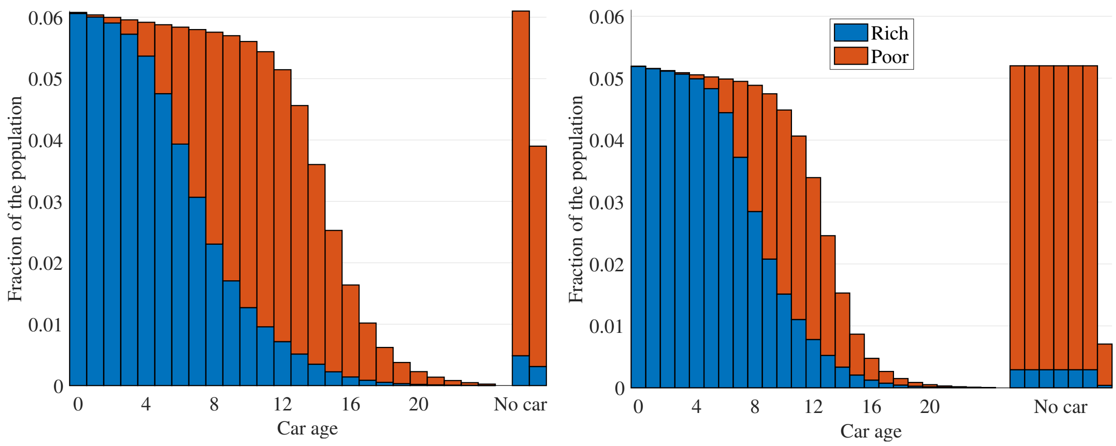
It seems reasonable to conjecture that in equilibrium a hand-me-down chain will emerge in which the rich are more likely to buy brand new cars whereas the poor households will buy the used cars previously owned by the rich. However it is not clear *a priori* what relative fractions of the two types of households will select into what fractions of the new and used cars, and which fraction choose to have no car at all. These questions and the effect of transaction costs on holdings can be answered by numerically computing the equilibrium prices and ownership distributions.

Figures 2 and 3, plots prices and holdings for two equilibria corresponding to the low and high transactions cost cases, respectively. For comparison, we also plot the prices that arise in the economies with single household types, one where all households have high marginal utility of money, and one where it is low. When transactions cost are high, equilibrium prices are closer to prices in the one-type economy where all consumers are rich because many poor consumers are now driven out of the market. Moreover, cars are scrapped earlier. Thus the higher transactions costs limits the gains from trade and partially "kills off" the market for used cars.

There is clear evidence of sorting of households into the ages of the car in panel (a) of Figure 3, which shows the post-trade ownership distribution $q\Omega(P)$. The rich households hold the newer cars and in particular are much more likely to buy new cars than the

Figure 3: Equilibrium ownership in a two household type economy

(a) Low transaction costs (b) High transaction costs



Notes: The total area of the bars in each panel adds up to one, presenting both the distribution of households over car ages, and the share of outside good. There are equal numbers of households of each type. Both panels show the post-trade ownership distributions. Since the density in the no-car state is much higher than each car-by-age category, we have split the no-car bar into several bars placed horizontally next to each other.

poor households. In addition, the fraction of poor households who do not to own a car is much higher. Overall, we see that poor households are driven out of the market to a much larger extent than the rich households when transactions costs increase.

These findings confirm our conjecture that a "hand-me-down-chain" arises endogenously in equilibrium, created by type-specific *specialization in holdings* that facilitates gains from trade between the two types of households. The rich households buy brand new cars and hold them for several years and then sell them to poor households who also hold them for several years, trading the cars over a succession of poor owners until the car is scrapped. Thus, in this example the rich households are net suppliers of older cars, and the poor households are net demanders of older cars. Most of the trade between rich and poor households occurs for cars of roughly middle ages: the rich supply their middle aged cars to the poor households, and market clears in aggregate, but not for each type.

5 Model Estimation

We have introduced a dynamic model of trade in automobiles that allows for rich specifications of observed and unobserved consumer heterogeneity whose equilibrium can be rapidly computed. We have shown that equilibrium prices, quantities, and choice probabilities are implicit functions of parameters of the model and derived analytical formulas for their gradients with respect to these parameters, which we can also rapidly compute. This enables us to extend the model in various directions, such as incorporating Bertrand-Nash price-setting by multi-product new car producers in the primary market. We can also use the model in empirical applications by showing how the parameters of the model can be estimated econometrically. We focus on the latter in this section and defer a discussion of equilibrium in the primary market to a future paper, since in our empirical application to the Danish automobile market in Section 6 we appeal to the "small open economy" assumption that new car prices are exogenously determined via competition in the world automobile market given that all new cars sold in Denmark are imported.

5.1 Maximum Likelihood Estimation

In Theorem 1 we have shown that for any equilibrium price vector P there is a unique distribution of "quantities" $q(P)$ that are equivalent to "market shares" (population shares) for the steady state holdings of each make/model and age of car traded in the used car market as well as the fraction of households that do not own a car. Given data on these shares, one strategy for estimating model parameters involves finding parameters that make the predicted market shares from the model as close as possible to the market shares we observe in the data. However a problem with this approach is that market prices P are endogenously determined and will be functions of "unobserved product characteristics" of different types and ages of cars traded in the market. A common way to deal with the endogeneity problem is the well-known "BLP" method (Berry, Levinsohn and Pakes, 1995). This requires numerical inversion of a mapping from the expected discounted utility that consumers obtain from keeping or purchasing each traded used car (i, a) to its aggregate market share q_{ia}. Assuming this inversion is possible, BLP obtain regression equations where the right hand side variables of the regression involve the value functions and other terms including the price-sensitivity parameters μ_τ multiplied by the net trade and transactions costs that include the equilibrium price P_{ia} of car (i, a). However BLP note that there is a problem of endogeneity in this regression, since equilibrium prices P_{ia} will be functions of *unobserved characteristics of cars* which they denote by ξ_{ia}. So the BLP approach relies on the existence of instrumental variables to consistently estimate the underlying structural parameters from these regressions, assuming all relevant parameters of the model are identifiable from these regressions.

Unfortunately it is not clear that BLP is applicable in our context for several reasons. First, it is not clear that the market shares q, which are the solution to the fixed point condition $q_\tau = q_\tau \Omega_\tau(P) Q_\tau$, can be inverted to obtain the *differenced* value functions, i.e. contrasted to the value of the outside good) for each type of consumer τ as it was possible in BLP's original static formulation of equilibrium. The latter ignored consumer heterogeneity in price-response parameters μ_τ, evaluation of the unobserved characteristics of cars, and the impact of trading in the used car market. Secondly, in our empirical application in section 6 we do not observe the secondary market prices P_{ia} that consumers trade at, so these constitute part of the unknowns we must estimate along with the structural parameters θ. Finally we do not directly observe accidents in the Danish register data, only scrappages. Since scrappages can be either "forced scrappages" due to accidents or "endogenous scrappages" due to voluntary choices by households (and the division between these two types of scrappages is important for the predictions of the model and the welfare effects of policy changes), we need a different econometric approach that can handle these complications.

Fortunately we show that structural estimation of the model by maximum likelihood constitutes an *instrument-free* estimation approach that can handle all of the problems outlined above. We do this using our equilibrium framework that *directly models the endogeneity of prices by capturing the functional dependence of equilibrium prices on (heterogeneous) consumer preferences for their observed and unobserved characteristics.* In particular, we can estimate models with agnostic and flexible specifications of consumer preferences that allow for heterogeneity in the way different consumers evaluate observed and unobserved car characteristics instead of assuming that all consumers have a common evaluation of the utility of these characteristics, as represented by the ξ_{ia} term in the BLP estimation approach. In addition, we show that it is possible to identify heterogeneous preferences and car-specific accident rates even when prices $P_{i,a}$ and accidents are *unobserved by the econometrician.* Our model delivers *predictions* of secondary market prices and accident rates, and as it will be clear in Section 6, they are quite reasonable. Thus, direct full information maximum likelihood estimation with nested numerical solution of the equilibrium for each trial value of the parameter vector θ enables us to overcome serious econometric challenges that other estimation approaches such as BLP are unable to deal with.

However the drawback of our fully structural equilibrium approach to estimation is the computational burden of the full solution estimator which following Rust (1987) we refer to as the *doubly nested fixed point* or DNFXP algorithm. The extra nesting level is due to the fact that the solution algorithm described in Section 3.5 requires a nested loop for calculation of the equilibrium prices P after the expected value functions $EV_\tau(P,\theta)$ and the corresponding choice probabilities are found as a result of solving consumers' dynamic programming problems. Fortunately, this algorithm is extremely fast and robust, so the DNFXP estimator is feasible even on ordinary laptop computers.

Another drawback is that the maximum likelihood estimator requires much more than market level data — a feature that contributes to the popularity of the BLP estimator. Instead the maximum likelihood estimator requires data on *car ownership transitions.* We show below that market share data alone are insufficient to identify the structural parameters whereas data on ownership transitions strongly overidentifies model parameters. Thus, our estimation approach generally requires disaggregate micro data, though we show below that if all heterogeneity in the model is observed, the model can be estimated using the empirical ownership transition probabilities for each observed type of consumer in the model (cell-based data). We make use of this in our empirical analysis in section 6 to not only speed up the estimation of the model by representing nearly 39 million observations over the period 1997 to 2008 in a much lower number of transition-specific cells, but to guarantee confidentiality so the data can be published along with our code.

Let θ be the vector of parameters characterizing consumer preferences, car transactions costs and car type specific accident rates that we wish to estimate. We now derive the likelihood function $L(\theta)$. We assume that the true data generating process is stochastically stationary, so the cross-sectional distribution of data observed for households is the unique invariant distribution of the Markov transition probability matrix for the true data generating process. To simplify exposition, we also assume that all heterogeneity among households is observed, with a finite number of types as we have described in the paper, though it will be clear from the discussion below how the likelihood can be extended to cover the case where there is unobserved heterogeneity as well. Thus, we will let τ index the observed type of a household and assume that for each household, its type τ is time-invariant. We will let x_{ht} denote the observed car ownership state of household h at time t, and for notational simplicity, we will assume that the data generating process involves a stationary Markov chain with transition probability $\pi(x_{ht+1}|x_{ht}, \theta, \tau)$ where the support or set of all possible values for the household state x_{ht} is finite and the same for all household types τ. Let $\pi_\tau(\theta)$ be an $n \times n$ Markov transition probability matrix, where n is the number of possible values that x_{ht} can take. In terms of the notation of the previous sections $\pi_\tau(\theta) = \Omega_\tau(P(\theta), \theta)Q_\tau(\theta)$. By our assumption of stationarity of the true data generating process, we assume there is a unique invariant distribution $q_\tau(\theta)$ (a $1 \times n$ row vector) that is a probability distribution over the n possible states that satisfies $q_\tau(\theta) = q_\tau(\theta)\pi_\tau(\theta)$.

Now, suppose we observe a random sample of H households indexed by $h = 1, \ldots, H$ where each household is followed over an unbroken sequence of periods $t \in \{\underline{t}_h, \ldots, \bar{t}_h\}$ so we have potentially an unbalanced panel of households. The log-likelihood is

$$L_H(\theta) = \frac{1}{H} \sum_{h=1}^{H} \sum_{t=\underline{t}_h}^{\bar{t}_h} \log\left(\pi(x_{h,t}|x_{h,t-1}, \tau_h, \theta)\right), \tag{39}$$

where τ_h is the observed type of household h.[24]

Let $H_{x'x\tau} = \sum_{h=1}^{H} \sum_{t=\underline{t}_h}^{\bar{t}_h} I\left\{x_{h,t} = x', x_{h,t-1} = x, \tau_h = \tau\right\}$ be the total number of observations on household car state transitions such that the next period state is x', the current state is x and the household type is τ. We then have $H = \sum_\tau \sum_{x'} \sum_x H_{x'x\tau}$, and can rewrite the log-likelihood function $L_H(\theta)$ in a "cell based" form (log-transition probabilities weighted by empirical transition frequencies) as follows

$$L(\theta) = \sum_\tau \sum_{x'} \sum_x \frac{H_{x'x\tau}}{H} \log\left(\pi(x'|x, \tau, \theta)\right). \tag{40}$$

Assuming the model is correctly specified and the population distribution of states is given by the stationary distribution $q_{\tau x}(\theta^*)$, $x \in X$, where θ^* is the true parameter vector and X is the finite set of n possible car states that a household could be in, it is not hard

[24]If there is unobserved heterogeneity, we can let τ represent additional parameters that characterize this unobserved heterogeneity and $f_\tau(\theta)$ be the probability of these unobserved types, where θ now includes the additional parameters for each unobserved type of consumer and the parameters characterizing $f_\tau(\theta)$. Then the likelihood $L_H(\theta)$ can be easily modified to allow the unobserved heterogeneity by including an extra summation over thet unobserved types τ and the addition of an extra terms $\log(f_\tau(\theta))$ on the right hand side of equation (39).

to show that an ergodic theorem holds and so with probaility 1 we have

$$\lim_{H\to\infty} \frac{H_{x'x\tau}}{H} = \pi(x'|x,\theta^*,\tau)q_{\tau x}(\theta^*)f(\tau), \tag{41}$$

where $f(\tau)$ is the fraction of the household population that is of type τ. If we let θ be any other generic true parameter value, the smoothness of $\pi(x'|x,\theta,\tau)$ as a function of θ implies that uniformly with probability 1 we have

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} |L_H(\theta) - L(\theta)| = 0, \tag{42}$$

where $L(\theta)$ is given by

$$L(\theta) = \sum_{\tau}\sum_{x'}\sum_{x} \log\left(\pi(x'|x,\theta,\tau)\right)\pi(x'|x,\tau)q_\tau(x)f(\tau). \tag{43}$$

Adapting the argument of White (1982) to this context, we can show that under weak regularity conditions the maximum likelihood estimator $\hat{\theta}_H$ defined by

$$\hat{\theta}_H = \underset{\theta\in\Theta}{argmax}\, L_H(\theta) \tag{44}$$

converges with probability 1 to θ^* where θ^* is the value that minimizes the Kullback-Leibler distance $D(\theta)$ between the parametric model of car state transitions $\pi_\tau(\theta)$ and the true data generating process with transition probability π_τ. Specifically, we have

$$D(\theta) \equiv \sum_{\tau}\sum_{x'}\sum_{x} \left[\log\left(\pi(x'|x,\tau)\right) - \log\left(\pi(x'|x,\theta,\tau)\right)\right]\pi(x'|x,\tau)q_\tau(x)f(\tau). \tag{45}$$

Using Jensen's Inequality, we can show that $D(\theta) \geq 0$, but if the model is correctly specified and identified, i.e. if the true transition probability is $\pi(x'|x,\theta^*,\tau)$ for some $\theta^* \in \Theta$ and for all (x', x, τ), then $D(\theta^*) = 0$ and the maximum likelihood esimator $\hat{\theta}$ will be a consistent and asymptotically efficient estimator of θ^*.

Now consider how the likelihood needs to be modified when secondary market prices and accidents are unobserved. The log-likelihood for the case of unobserved accidents can be written using *post-decision state transition probability* $\pi(\delta_{t+1}|\delta_t,\theta,\tau)$ where δ_t denotes the household's *choice* (post-decision state) at time t which differs from the *pre-decision state* x_t since a household can trade or purge their current car if $x_t = (i, a)$ or buy a car if

$x_t = \emptyset$, so the post-decision state will not be the same as the pre-decision state unless the household chooses to keep their car or remain in the no car state. For households who own cars, we also let δ_t include whether the incoming car was scrapped (either voluntarily or a forced scrappage due to an accident) and our formula for the transition probability $\pi(\delta_{t+1}|\delta_t, \tau, \theta)$ "integrates out" whether the scrappage was voluntary or forced due to an accident. This likelihood has the same form as equation (43) above except that we replace x' by δ' and x by δ. For details on its derivation, see Appendix D.

Finally, when secondary prices P are unobserved, we can use *computed prices* $P(\theta)$ implied by the model instead. This makes secondary prices in effect, additional "unknowns" that are estimated by maximum likelihood. But since they are functions of the underlying structural parameters, we can use the Delta Method to compute the standard errors of the estimated prices.

However since $P(\theta)$ is an implicit function of θ and enters the likelihood, to calculate the gradient of the likelihood $\nabla_\theta L(\theta)$ we need to use the chain rule of calculus and calculate $\nabla_\theta P(\theta)$. Our computer code for calculating equilibrium includes these gradients as a by-product and our maximum likelihood estimation algorithm accounts for the chain rule so that the effect of θ on $P(\theta)$ is accounted for in the gradient of the log-likelihood function $L(\theta)$. This enables us to use fast quasi-Newton algorithms such as the BHHH algorithm of Berndt, Hall, Hall and Hausman (1974) with accurate analytic gradients, without the need to compute Hessians of $L(\theta)$ which is even more tedious.[25]

5.2 Model Identification

It is straightforward to show that we cannot identify the structural parameters using only market share data, i.e. q, alone. This is simply due to the fact that data on aggregate market shares provides only $J(\bar{a}-1)$ moments, but θ will typically have far more than $J(\bar{a}-1)$ parameters. For example, in our empirical analysis in Section 6 there are 131 unknown parameters in θ but we have data only 96 independent market shares in q since $J = 4$ and $\bar{a} = 25$. However there are a total of $[J(\bar{a}-1)]^2$ moments in each of the

[25]Because we use gradient based methods in all three nested loops of the DNFXP estimator and use the chain rule of calculus when computing these gradients, our framework is compatible with many different specifications of preferences and can easily accommodate additional structural parameters. The computational cost of adding parameters is small — there is practically no additional time spent to compute the derivatives with respect to these parameters, and so the run time for a single evaluation of likelihood function and its gradient hardly changes.

household-type specific transition probabilities P_τ, and an additional $J(\bar{a}-1)$ moments for the age-specific scrappage probabilities for each car type. Further, we have these moments broken down by each of the observable household types, so we will typically have vastly many more moments than structural parameters in our model. For example in the empirical analysis in Section 6 we see that our structural model is actually rather parsimonious, since we achieve a very good fit to our data set on Danish auto holdings and trading with 74,496 probabilities or "moments" using only 131 parameters.

However the reader may be skeptical about the identification of the key price sensitivity parameters μ_τ since we noted that we do not observe the prices P_{ia} that Danish consumers bought or sold their cars for. There are two key reasons why we are able to identify μ_τ by treating $P_{ia}(\theta)$ as additional unknown parameters that are implicit functions of the underlying structural parameters θ. First, we do have data on the price of new cars of each type, $\overline{P}_i$ and scrap prices $\underline{P}_i$. This already helps provide information for helping us to identify secondary prices since under ordinary circumstances (e.g. where all consumers prefer newer cars to older ones), secondary prices P_{ia} will be decreasing in a and lie in the interval $[\underline{P}_i, \overline{P}_i]$. Secondly, our assumption of flexible yet parametric preferences for cars described in the previous subsection together with the assumption that the market is in equilibrium imposes strong additional restrictions on car ownership transitions, holdings, and prices, which we showed are an implicit function of consumer preferences. Even if we do not observe P there is more than enough information on the holdings and car trading behavior of the different types of consumers in our data set to identify all of the structural parameters of the model including all of the secondary market prices P. A final source of identification comes from the addition of data on driving of cars and will be discussed further in Section 6.

As we noted above, by appealing to White (1982) it follows that asymptotically the maximum likelihood estimator converges with probability 1 to a value θ^* that minimizes Kullback-Leibler distance $D(\theta)$ between the parametric transition probabilities $P(\delta'|\delta,\theta,\tau)$ implied by our model and the true transition probabilities $P(\delta'|\delta,\tau)$ that can be estimated non-parametrically given sufficient data. Acknowledging the potential for model misspecification (as White does in his analysis) it will generally not be possible for a model with only 131 parameters to fit all 74,496 probabilities $\pi(\delta'|\delta,\tau)$ that we can estimate non-parametrically using Danish register data for the 8 consumers types and 4 car types. It follows that $\inf_\theta D(\theta) > 0$. However we can define a "psuedo-true" parameter θ^* as the value of θ that minimizes the Kullback-Leibler distance between our parametric model and the true data generating process that is captured by the non-parametric estimates $\pi(\delta'|\delta, \tau)$. In direct analogy with the analysis of identification in the case where the model is correctly specified (in which case $\inf_\theta D(\theta) = 0$), we follow Rothenberg (1971) and establish local identification of θ^* by showing that the Hessian of $D(\theta)$, $H(\theta^*) = \nabla^2_\theta D(\theta^*)$ has full rank. In the case where the model is correctly specified, so $\inf_\theta D(\theta) = 0$, this is equivalent to the invertibility of the information matrix $I(\theta^*)$ using the well known information equality.

While it is much more difficult to prove global identification, local identification is straightforward to establish by showing that the Hessian of the log-likelihood $L(\theta)$ has full rank at the maximum likelihood estimates, θ^*. Thus local identification of the model is an immediate consequence of our ability to calculate misspecification-robust standard errors for the parameters per the formula in White (1982). In the next section we demonstrate that our structural model is in fact locally identified even though we do not observe secondary market prices or accidents.

6 Analysis of Danish Car Tax Policy

In this section we use the DNFXP maximum likelihood estimator to structurally estimate a version of our model with 8 types of households and 4 different car types of 25 ages using a data set that follows the car holdings of all Danish households from 1997 to 2008 provided by Statistics Denmark. This data set contains nearly 39 million observations, yet we show that the 131 structural parameters of the model can be estimated in a matter of minutes using an ordinary laptop computer. We then show how the estimated model can be used to make counterfactual predictions that may be crucial for analysis of car tax policy in Denmark. We also extend the model by incorporating driving. This also enables us to study the effects of additional taxes such as fuel taxes and account for environmental and congestion impacts of hypothetical policy changes.

We simulate the effects of reforms similar to ones that have been under consideration in Denmark, which shift taxation from the purchase of new cars to the use of cars by increasing the fuel tax. We compare the predictions from our estimated equilibrium model to those obtained from a model that does not account for equilibrium in the used car market, and instead assumes a proportional change in prices of new and used cars.

We show that predictions from more *ad hoc* non-equilibrium models fail to accurately capture behavioral responses to this policy change. In particular, they overestimate the change in fleet composition compared to an equilibrium analysis where used car prices, scrappage rates, and household holdings of cars are endogenously determined. Ultimately, this means that policies based on a non-equilibrium model are too extreme, causing policy makers to fall short of stated goals such as revenue equivalence. However, we show that it is possible to raise tax revenue and consumer surplus while reducing CO_2 emissions by lowering registration taxes and raising fuel taxes.

6.1 Incorporating Driving

The primary reason to own a car is to drive it and thus far we have ignored this important aspect of car ownership. Let x denote the number of kilometers a consumer chooses to travel in a period, and let p_j denote the price per kilometer traveled for a car of type j. This equals the price of fuel (e.g. kroner per liter) divided by the car's fuel efficiency (kilometers per liter of fuel) for car type j.

Let $u_\tau(j, a, x)$ be the utility a consumer of type τ obtains from owning a car of type j and age a and driving it (on average) for x kilometers during the period. We make a simplifying assumption that the probability of an accident and other physical deterioration in an automobile is independent of driving, x, and is instead only a function of car type j and car age a. The benefit of this assumption is that driving becomes *a static sub-problem* of the consumer's overall dynamic trading problem. The optimal amount of driving $x_\tau(j, a, p_j)$ then simply maximizes the driving utility net of monetary cost, $u_\tau(j, a, x) - \mu_\tau x p_j$. Substituting $x_\tau(j, a, p_j)$ back into the utility function $u_\tau(j, a, x)$ we obtain the *indirect utility* $u_\tau(j, a, p_j)$ for owning a car of age a that incorporates the individual's optimal choice of driving. Assuming that p_j is time-invariant, the resulting model falls within the specification of Section 3.

To allow for discrepancies between the theoretical optimal amount of driving and actual data on kilometers traveled by different cars between (bi-annual) inspections in Denmark, we treat $x_\tau(j, a, p_j)$ as *planned* driving by the consumer at the start of each period. *Actual* driving is subject to *ex post* unexpected events during the period that cannot be predicted exactly. We represent by ζ the net *ex post* effect of these unexpected driving needs on the marginal utility of driving, resulting in an *ex post* utility specification of the following form

$$u_\tau(j,a,x,\zeta) = \psi_\tau(j,a) + \big(\gamma_\tau(j,a)+\zeta\big)x - \tfrac{\phi_{\tau,j}}{2}x^2 - \mu_\tau p_j x, \quad \gamma_\tau(j,a) = \gamma_{\tau,j,0} + \gamma_{\tau,j,1}a, \tag{46}$$

where the first component does not depend on x and can be considered as the utility of owning a car apart from driving it, and therefore only affects trading behavior. We assume that $\psi_\tau(j,a)$ is a quadratic in the age of the car to fit the overall market shares across car types and ages, so we have $\psi_\tau(j,a) = \psi_{\tau,j,0} + \psi_{\tau,j,1}a + \psi_{\tau,j,2}a^2$.

The optimal *ex post* level of driving $x_\tau(j,a,p_j,\zeta)$ implied by this structure is:

$$x_\tau(j,a,p_j,\zeta) = \tfrac{1}{\phi_{\tau,j}}\big[-\mu_\tau p_j + \gamma_{\tau,j,0} + \gamma_{\tau,j,1}a + \zeta\big]. \tag{47}$$

Note that conditional on type-specific coefficient $\phi_{\tau,j}$, the parameters in (47) can be estimated by regressing the observed kilometers traveled to the cost of driving, household and vehicle characteristics. However, the regression only partially identifies a subset of structural parameters as ratios involving the parameter $\phi_{\tau,j}$, the coefficient governing the level of diminishing marginal utility from driving.[26]

When we substitute the expression for optimal *ex post* driving (47) back into the utility function (46) and take expectations over the *ex post* shocks ζ we obtain a specification for the *ex ante* indirect utility of car ownership that is a quadratic in age. By a slight redefinition, the parameters $(\gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \gamma_{\tau,j,2})$ also subsume the first and second moments of the unobserved *ex post* shock to utility ζ. Thus, besides the marginal utility of money parameter μ_τ, there are a total of 6 unknown parameters for each consumer type τ and car type j, which are $\theta_{\tau,j} = (\psi_{\tau,j,0}, \psi_{\tau,j,1}, \psi_{\tau,j,2}, \gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \phi_{\tau,j})$. In Appendix E we show that the 6 parameters in $\theta_{\tau,j}$ are just-identified in terms of the 6 corresponding "semi-reduced-form" parameters, 3 for the linear driving equation, and 3 for the *ex ante* expected indirect utility of owning car (j,a) which is a quadratric in a after taking expectations of the *ex post* preference shock ζ: $E\{u_\tau(j,a,x(j,a,p_j,\zeta))\} = u_{\tau,j,0} + u_{\tau,j,1}a + u_{\tau,j,2}a^2$. This implies that we can estimate the model in two steps: first we estimate separate linear driving regressions for each (τ,j) combination to identify the 3 ratios

[26] We ignore the restriction $x(j,a,p_j,\zeta) \geq 0$ which implies that ζ must be a truncated normal distribution.

$(-\mu_\tau/\phi_{\tau,j}, \gamma_{\tau,j,0}/\phi_{\tau,j}, \gamma_{\tau,j,1}/\phi_{\tau,j})$. Then we use the DNFXP algorithm to estimate and identify the 4 parameters $(\mu_\tau, u_{\tau,j,0}, u_{\tau,j,1}, u_{\tau,j,2})$ of the implied quadratic expected indirect utility function. Using these 7 estimated parameters, Appendix E shows that we can solve for all 7 of the structural parameters $(\mu_\tau, \psi_{\tau,j,0}, \psi_{\tau,j,1}, \psi_{\tau,j,2}, \gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \phi_{\tau,j})$. It is critical to fully identify all the underlying structural parameters in order to make counterfactual predictions involving changes in the fuel price paid by consumers, p, which in turn changes the per kilometer cost of driving, p_j, of the different types of cars $j \in \{1, \ldots, J\}$.

6.2 Estimation Results and Model Fit

Though there are hundreds of different makes and models of cars sold in Denmark, for this analysis we aggregated them into 4 car types differentiated by their fuel economy and pollution levels ("green" for more fuel efficient, environmentally friendly cars and "brown" for others), and based on car weight ("heavy" versus "light"). We divided Danish households into 8 groups τ depending on whether they were a) singles or couples, b) whether the distance to work was short or long, and c) whether the household was rich or poor. The precise criteria for defining these groups are detailed in Appendix F.

We estimated a linear specification for preferences including driving as discussed in the previous section. Household preferences for cars decrease with age but at a diminishing rate, and there is heterogeneity in preferences for the different types of cars. We also estimated household-specific quasi-linear price sensitivity parameters μ_τ for each of the 8 types τ. By dividing the estimated coefficient $u_{\tau,j,0}$ for household τ's utility of a new car of type j by μ_τ, we obtain a measure of willingness to pay for one period's use of a new car in Danish kroner. For example we estimate that a rich couple with low work distance is willing to pay (e.g. rent for one year) a new light brown car for 36,040 DKK (or about US$5,545) compared to 32,256 DKK for a poor household. In general, we find that based on the revealed choices: 1) rich households are willing to pay more for any type of car compared to poor households, 2) all households preferred the heavy cars to the light ones and brown cars to green ones resulting in the following preference ordering: heavy brown $\succ$ heavy green $\succ$ light brown $\succ$ light green, 3) willingness to pay for cars by high work distance households exceeds that of low work distance ones, and 4) couples generally have higher willingness to pay for cars than singles. Given that there are a total of 131 parameters in the model we refer the reader to Appendix F for details on the maximum likelihood parameter estimates and standard errors.

The estimated model also has reasonable implications for driving (Appendix Table 6): households with high work distances drive much more than those with low, and more so for the rich. The estimated model implies fuel price elasticities between -0.10 and -0.60 across households. This is relatively close to Gillingham and Munk-Nielsen (2019), who find an average elasticity of -0.30 using a wide array of regression specifications.

One of the key predictions of the equilibrium model is market shares for the different car types and ages held by the 8 different household types in our model. Figure 4 shows that the estimated model provides a good fit to the data, not only overall for the 4 car types, but also separately for each of the 8 household types (right panel of figure 4). The model successfully captures key features of Danish households: 1) poor households are significantly more likely not to own a car than rich ones, 2) couples are more likely to own cars than singles, and 3) high work distance households are relatively more likely to own cars than those with low work distance. In general, the market shares for the 4 different types of cars are similar, but what really stands out in Denmark is the large fraction of households who are not car owners, 40%. This could be due in part to the excellent public transport infrastructure and the widespread use of bicycles in Denmark, but also due to the high taxation of cars that we will analyze in more detail in the next section.

Figure 5 shows how our model captures the post-trade age distribution of holdings of different cars by different households, including the "hand-me-down-chain" from rich to poor consumers. For example notice that for low work distance singles (the dark blue and red regions at the bottom of the bar graphs) the rich households (colored red) are relatively more concentrated in holding newer cars of each type whereas the poor households are more concentrated in holding older cars. As we noted in section 5.1, our maximum likelihood estimation does not attempt to directly fit the holdings distributions, which we previously denoted by q_τ for each household type τ. Since Figure 5 plots the actual and predicted post-trade ownership distribution, $q\Omega(P)$, it also involves a comparison of the implied stationary distribution from our model, $q_\tau(\hat{\theta})$, to the non-parametric estimates q_τ from the data. Though our model slightly underpredicts holdings of new light cars and overpredicts holdings of new heavy cars, overall we think the model provides a remarkably good overall fit to over 800 non-targeted probabilities shown in

Figure 4: Actual and predicted market shares

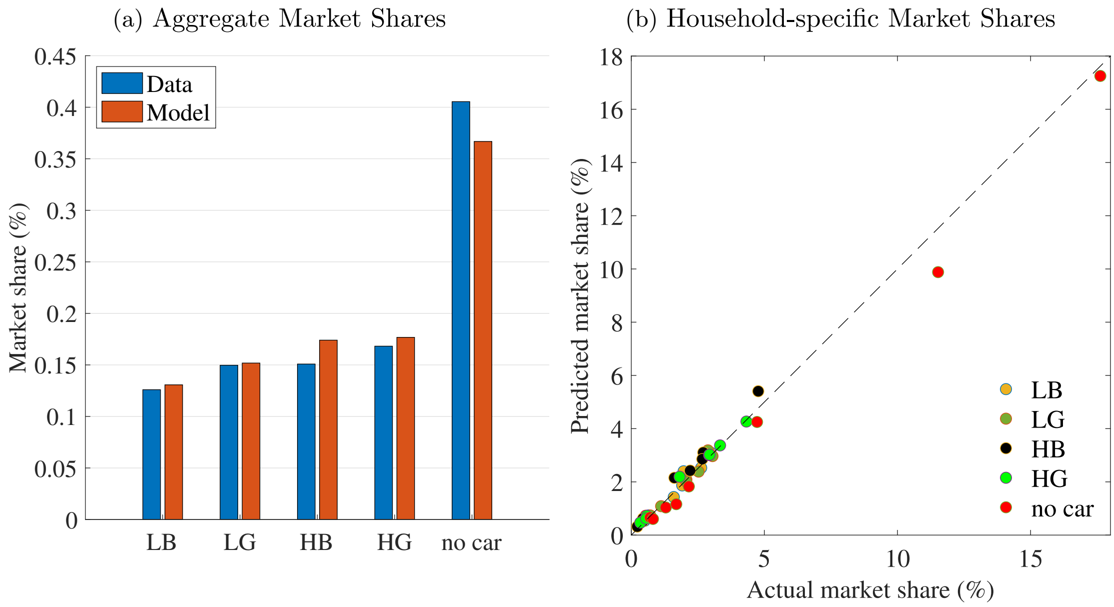


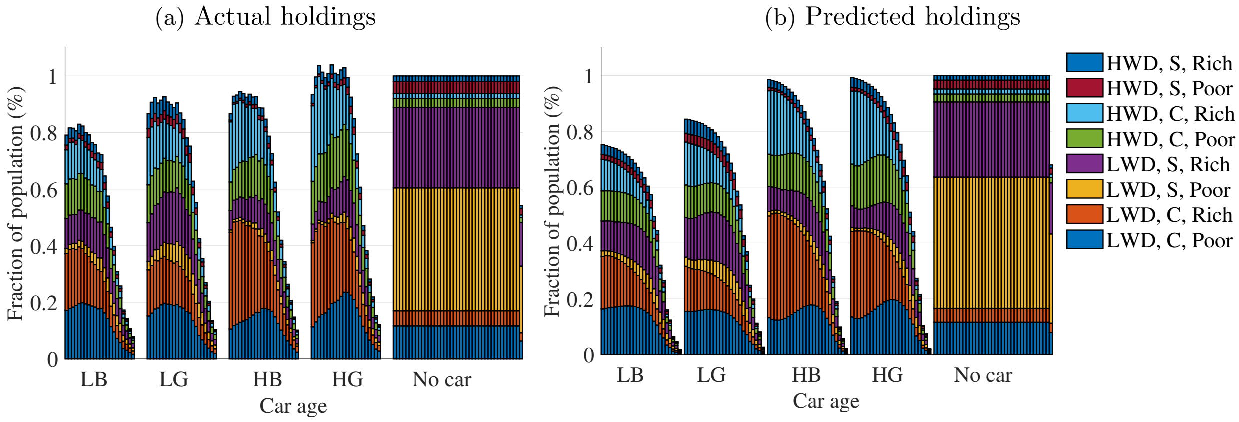
figure 5 using a fairly parsimonious model with 131 parameters.[27]

As we noted above, the Danish register data contain information on nearly 80,000 car ownership transition probabilities, so the next figures provide some information on the model's ability to fit these transitions. Figure 6 illustrates the model's ability to capture the probability of cars purchases as well as the probability of keeping existing car. The left panel of Figure 6 plots the conditional probability that households purchase cars of a given age. The model closely tracks the observed purchase patterns at the aggregate level: households are more likely to buy a new car rather than any of the used ones and purchase probabilities declines as cars approaches the scrap age. When comparing these purchase probabilities for each of the 8 household types (results not shown), the model also closely tracks observed purchase probabilities and mimics the overall pattern from Figure 5 that rich, couple, high work distance households are the types most likely to buy a newer and larger (more expensive) cars.

The right panel of Figure 6 focuses on the 60% of Danish households that do own cars, and plots the conditional probability of keeping their existing as function of its age. The model is generally able to match that the overall level of probability of keeping a

[27]There are relatively few cars that are more than 20 years old in our data set and due to measurement issues relating to the oldest cars discussed in Appendix F we eliminated cars over 22 years old from our estimation sample. Thus the histograms for the data in figure 5 are truncated at age 22.

Figure 5: Actual and predicted holdings, by household type, car type and car age



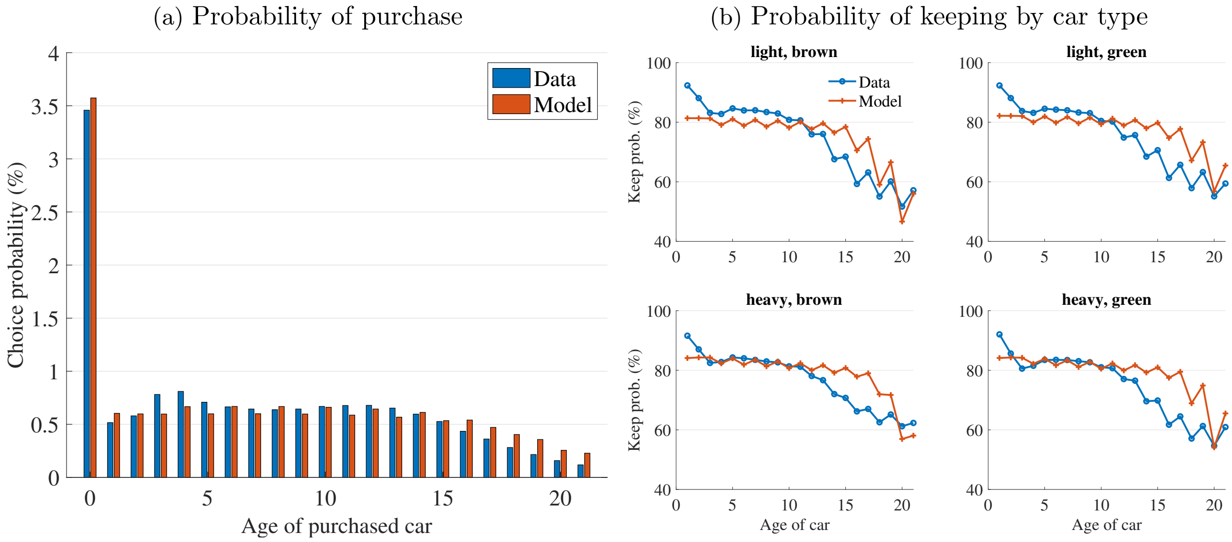
Note: The graphs show the fraction of the population holding each car-age combination as well as the outside good. For the outside good, we have opted to split the bars and put them next to each other since otherwise, that option would dominate the scale of the y axis. The abbreviations are: LB = light brown, LG = light green, HB = heavy brown, HG = heavy green. Within each of the four car types, car ages go from 0 to 24.

car, but also reveals an aspect of the data that our model is unable to capture well: we see that in the data, the probability of keeping a car is very high in the first couple of years and the gradually falls with the age of the existing car whereas our model predicts only a more modest decrease until it drops around age 15.

We conclude our presentation of the estimation results with Figure 7, which illustrate the model's predictions of quantities that we do not directly observe in our data set from Statistics Denmark. As we noted, our data allows us to observe *scrappage* of cars but not directly *accidents leading to scrappage* since another source of scrappages are *voluntary scrappages* by households, such as cars that are still drivable but may require expensive repairs to enable them to pass safety checks that a required before they can be sold to another household. As we described in section 5.1 we are able to identify and estimate accident probabilities using our structural estimation approach even though we do not observe accidents. The estimated parameters shown in Table 4 implies that accidents of the light cars are generally higher than the heavy ones, and accident rates rise quickly with age after cars are 15 years old, but are negligible when cars are new.

The left hand panel of Figure 7 displays the overall probability that a car is scrapped by the age and type of the car. We do directly observe when scrappages occur in the Danish Register Data so in this graph we can compare the model predictions (red lines) to the data (blue lines). Here we note the curious "zig-zag" pattern in scrappage rates that

Figure 6: Actual and predicted probability of keep and purchase

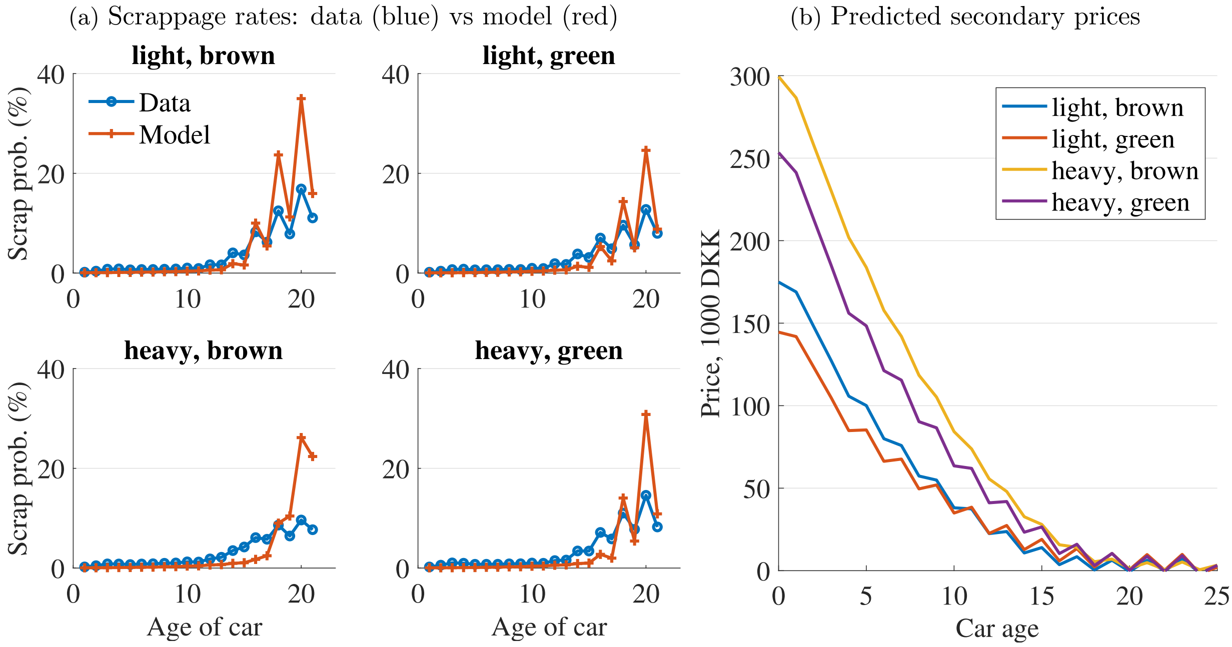


we noted in the introduction: Danish cars are much more likely to be scrapped at even ages than odd ages. We have verified that this effect is real and is not an artifact of how and when scrappages are recorded. It is due to the very strict biannual car inspections in Denmark that occur at even ages once cars are 4 years old or older. If the inspection reveals mechanical, safety or emissions problems, the owner is required to repair them in order to continue driving the car. We believe that Danes find these costs to be onerous and thus they are more likely to scrap rather then keep or sell their cars if they have problems when they are sufficiently old. We capture this effect in our model by including an even age dummy in the utilty of car ownership. The estimation results reveal big negative estimated coefficients for these dummy variables, with an estimated disutility that is typically 15% to 50% as large as the estimated single period utility the household obtains from owning a brand new car of the same type.

Finally, the right hand panel of figure 7 shows the estimated secondary prices of the 4 types of cars in our model. As we noted in section 5.1, even though we do not directly observe these prices, we can compute them for any trial value of the structural parameters, and we can use the substantial number of other moments in the data to identify both the structural parameters θ and the implied secondary prices, $P_{i,a}(\theta)$ for all four car types i and car ages a.[28] We firstly note that the rate of decline of our used car prices is broadly

[28] We do observe prices of new cars, so we use the average new car prices $\overline{P}_i$ of the different makes and models in the 4 aggregate type groupings of cars as data rather than estimating these as additional parameters (see

Figure 7: Zig-Zag Patterns in Scrappage and Predicted Equilibrium Prices



Note: Panel (a) shows the fit of the scrappage rates for each car and age (averaged over households weighted by the equilibrium ownership distribution), and panel (b) shows the equilibrium prices.

consistent with external evidence. We have limited data on suggested annual discount rates for used cars from the Danish Used Car Dealer Association, which suggest that prices should fall by 13% per year on average. Our model solution implies that prices fall on average 14% per year for three of the four cars, and 11% per year for the light green car. Thus, the overall magnitude of depreciation in our results is quite similar to the best data available.

Second, we note that the zig-zag pattern in scrappages is also present in the secondary prices predicted by our model. The effect on secondary market prices is a natural consequence of the estimated disutility our model predicts that Danish households experience during the even aged years when their car is subject to inspection. As mentioned above, independent evidence from the limited data we have an actual used car transactions prices suggest that the zig-zag pattern in secondary prices that our model predicts is a real phenomenon in Denmark.

Appendix Table 3). The observed new car prices help to “tie down” secondary prices.

6.3 Counterfactual Policy Analysis

In this final section we carry out counterfactual predictions using our estimated equilibrium model of the Danish auto market. We focus on the effects of changes in the Danish new car (registration) tax and the fuel tax. As we noted in the introduction, Denmark has one of the highest new car taxes in the world. The tax is progressive with a rate of 105% of the retail price of the car in the first bracket (for the cars priced up to 81,000 DKK, approximately equal $16,000 USD at the time, excluding VAT), and a rate of 180% of the retail price in the second bracket. There is an additional VAT of 25% applied to the price inclusive of the new car tax. Appendix F provides further details about car taxation policy in Denmark, which has been subject to vigorous political debate and a few reforms in recent years.[29]

In order to make the counterfactual predictions, which involve tax policy changes that affect new car prices $\overline{P}_i$ and thus used car prices $P_{i,a}$ as well fuel prices p that affect the price per kilometer driven for different car types, p_i, we need estimates of the "deep structural parameters" described in section 6.1 where we introduced driving into the model. These deep parameters are $\theta_\tau = (\mu_\tau, \{\psi_{\tau,j,0}, \psi_{\tau,j,1}, \psi_{\tau,j,2}, \gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \phi_{\tau,j}\})$ that differ for each household type τ where the other parameters except the marginal utility of money μ_τ also differ by car type j. Recall the ψ parameters reflect the pure utility of ownership (independent of any driving) for different cars whereas the γ and ϕ parameters capture the utility from driving.

In order to identify and recover these deep structural parameters θ_τ, we follow the approach in Appendix E. This approach shows how to recover all of the θ_τ parameters given the "semi-reduced form" linear driving equation:

$$x_{\tau,j} = d_{\tau,j,\tau,0} + d_{\tau,j,1} * a_{\tau,j} + d_{\tau,j,1} p_j + \zeta_{\tau,j} \tag{48}$$

where $a_{\tau,j}$ is the age of a car of type j owned by household τ, $x_{\tau,j}$ is the annual driving by the houshold (in thousands of kilometers), and $\zeta_{\tau,j}$ is a regression residual. Next we estimated the parameters $(\mu_\tau, u_{\tau,j,0}, u_{\tau,j,1}, u_{\tau,j,2})$ from our dynamic equilibrium model of car ownership and trading described in the previous sections. Recall the u coefficients determine household and car type specific quadratic functions for the indirect utility

[29] After the end of our sample, the registration tax rate has been reduced across two separate reforms, in combination with changes to the treatment of electric vehicles (which were not present during our sample period).

obtained from owning or trading for cars of different types and ages. These coefficient estimates implicitly embody the price of fuel p under the *status quo* so to do counterfactual predictions that vary p, we need to recover the deep structural parameters θ_τ for each household type τ.

The beauty of our specification is that the deep structural parameters θ_τ are just identified from the driving regression parameters and the structural estimation of the θ_τ parameters. This means there are no "cross-equation" restrictions that would force us to estimate all the θ_τ parameters in a single joint likelihood that predicts both driving and the dynamic discrete choice model of car ownership decisions. Thus, we estimated the unrestricted driving equation (48) and the preference parameters of the equilibrium model separately in steps 1 and 2 of the estimation process, and finally in step 3 we solved for θ_τ following the equations in Appendix E. All of the parameter estimates from steps 1 and 2 are presented in Appendix F.

Once we have identified the deep structural parameters, we can systematically vary the registration tax rate (which affect the gross of tax new car prices $\overline{P}_i$ and thus also equilibrium secondary prices $P_{i,a}$) as well as the after-tax fuel price p which affects the cost per kilometer driven p_j of the different car types j. For each alternative policy we consider, we calculate the counterfactual equilibrium which also enables us to evaluate consumer welfare as well as the impact on overall tax revenues received by the Danish government. Specifically, we analyze a proposed policy involving cutting the new car registration tax in half, while increasing the fuel tax to offset the revenue loss from reduced new car taxes. Intuitively, this policy shifts taxation from the purchases of cars to their usage, with the intention to offset some of their harmful externalities. We assumed that the social cost of carbon is US \$50/ton to estimate the marginal external social costs of driving, using results from Transport (2010) that also include negative externalities from congestion, accidents, noise and local air pollution.

The overall purpose of this analysis is to illustrate the added value of using an equilibrium model to inform car tax policy. We want to illustrate how a policy maker would design a revenue-neutral reform that shifts taxation away from car registrations and towards fuel and usage. Specifically, we want to compare a sophisticated policy maker to a "naive" policy maker using a non-equilibrium model. As mentioned above, a naive way of handling the used car market in a non-equilibrium framework is to assume a proportional changes in used and new car prices. The baseline tax system for new cars is a two-part linear system with a kink, after which the marginal tax rate increases. Thus, the registration tax is progressive. We analyze a reform where both the low and high rates are cut in half. Assuming full passthrough to new car prices, this results in new car prices falling by between 25.6% and 27.6% for the four car types. In the non-equilibrium setting, which we refer to as *naive, expected,* we assume that the prices of the used cars of all vintages fall by the same percentages relative to the baseline equilibrium. However while it is easier to make predictions, the naive approach fails to account for the endogenous adjustment of car trading and secondary prices to the change in fuel prices and new car prices. Thus, we will consider the following four scenarios:

1. **Baseline:** the model is solved for equilibrium prices and calibrated under *status quo* Danish tax policy as of 2008.
2. **Naive, Expected:** a naive policy maker, assuming that used-car prices will fall by the same proportion as the corresponding new car price for each car type. That is, the market is not in equilibrium. The policy maker raises fuel taxes until revenue is equivalent to the baseline. We calculate individual household welfare using these prices even though the used car market is not in equilibrium.
3. **Naive, Realized:** this is the equilibrium outcome that would result from the fuel tax policy enacted by the naive policy maker above. That is, the market is in equilibrium here and used car prices are set to equate supply and demand, but tax revenue is not equal to the baseline.
4. **Sophisticated:** these are the predictions of a sophisticated policy maker who correctly predicts the endogenous equilibrium responses to tax policy changes. That is, the used-car prices are such that the market is in equilibrium, and fuel taxes are set so that the total tax revenue is equal to the baseline tax policy scenario.

The outcomes under the four different policy scenarios are presented in Table 1, and the resulting car prices are in Figure 8.

In the "Naive, expected" scenario, the policy maker is guided by a naive expectation of proportional passthrough. Lowering registration taxes results in an *increase* in tax revenue, so in order to achieve revenue equivalence, the policy maker increases fuel taxes from 57% of the price at the pump in the the baseline up to 76%.[30] According to this non-

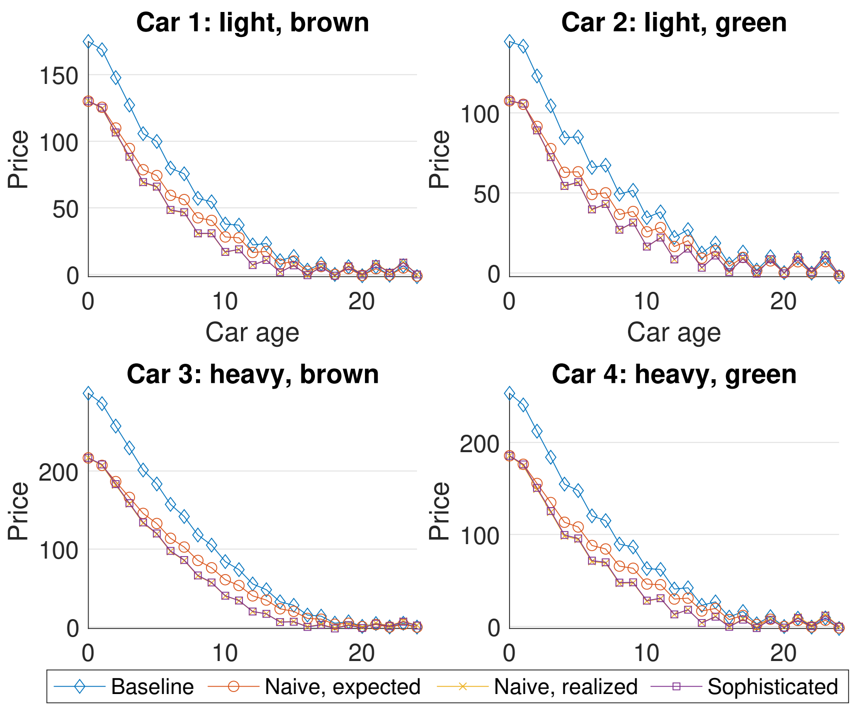
[30] Alternatively, the policy maker could also lower fuel taxes to achieve revenue equivalence. However, the

Table 1: Policy Simulation Results

	Baseline	Naive, expected	Naive, realized	Sophisticated
Policy choice variables				
Registration tax (bottom rate)	1.050	0.525	0.525	0.525
Registration tax (top rate)	1.800	0.900	0.900	0.900
Fuel tax (share of pump price)	0.573	0.760	0.760	0.728
Exogeneous prices				
Price, light, brown (1000 DKK)	174.902	130.110	130.110	130.110
Price, light, green (1000 DKK)	144.551	107.532	107.532	107.532
Price, heavy, brown (1000 DKK)	299.452	216.760	216.760	216.760
Price, heavy, green (1000 DKK)	253.397	185.508	185.508	185.508
Fuel price (DKK/l)	8.322	14.824	14.824	13.054
Outcomes				
Social surplus (1000 DKK)	8.831	10.851	7.705	9.632
Total tax revenue (1000 DKK)	9.302	9.302	7.113	9.302
Fuel tax revenue (1000 DKK)	4.281	5.039	4.788	6.191
Car tax revenue (1000 DKK)	5.021	4.263	2.325	3.111
Non-CO2 externalities (1000 DKK)	6.749	3.312	3.182	4.782
Externalities (1000 DKK)	7.372	3.622	3.476	5.234
Consumer surplus (1000 DKK)	6.901	5.172	4.069	5.564
CO2 (ton)	2.148	1.069	1.016	1.559
VKT (1000 km)	10.858	5.329	5.118	7.693
E(car age)	6.505	2.911	4.228	5.473
Pr(no car)	0.367	0.546	0.548	0.415

Note: In the baseline scenario the institutional parameters conform to the data for 2008. In the column "Naive, expected", the two rates for the new car tax are both cut in half and then fuel taxes are increased until tax revenue is equal to the baseline. Used car prices are assumed to fall by the same percent as new car prices (i.e. 100% passthrough from the new to used car market). In "Naive, realized", the new car taxes and fuel taxes are as in "Naive, expected", but we solve for the equilibrium used car prices. In "Sophisticated", we change the fuel tax so that the total tax revenue is equal to the baseline, each time solving for the used car price equilibrium.

Figure 8: Equilibrium prices



Note: Each panel shows the prices for the corresponding car type where the four lines represent the four different scenarios (see Table 1 for descriptions).

equilibrium model, that should achieve revenue equivalence at 9,302 DKK per household annually, but with a much younger car fleet where the average car age falls from 6.5 to 2.9 years.

However, the scenario "Naive, realized" shows what will actually happen once used car prices adjust to equilibrate the market. Firstly, we note from Figure 8 that used car prices fall by *more* than proportionally, which correspondingly results in car ages falling by less than predicted, only to 4.2 years. In other words, the naive model predicts a much too strong movement towards newer cars, which results in excess demand for newer cars. As a result, registration tax revenue crumbles, resulting in total tax revenue falling short of the intended equivalence target from 9,302 to 7,113 DKK per household.

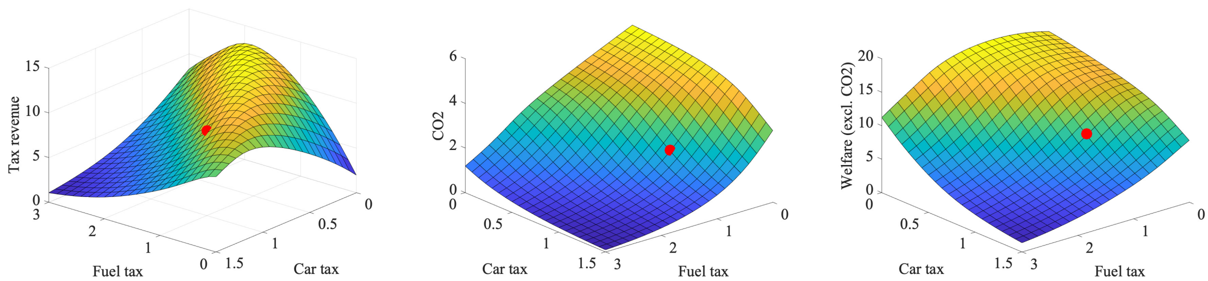
Instead, the column "Sophisticated" shows that the policy maker would only have to increase fuel taxes until they make up 73% of the price at the pump if she takes into account the endogenous responses in the used car market.

The general takeaway message is that a non-equilibrium model can produce much

required reduction implies a virtual abolishment of fuel taxes, which we judge to be less realistic in practice. Nevertheless, this illustrates the complexities in policy design with Laffer curve effects.

Figure 9: The Effects of Varying the Fuel and Registration Tax Rates

(a) Total tax revenue (b) CO_2 Emissions (c) Social Surplus



Note: All three panels have the same x and y axes, namely the tax rate for fuel and car registrations respectively, normalized by the sample values so that the baseline outcomes occur at (1,1). The panels differ in terms of the rotationd and which outcome is on the z axis: tax revenue, CO_2 emissions, and social surplus respectively. Social surplus is the sum of consumer surplus and tax revenue, subtracting the external costs of driving with CO_2 valued at \$50/ton.

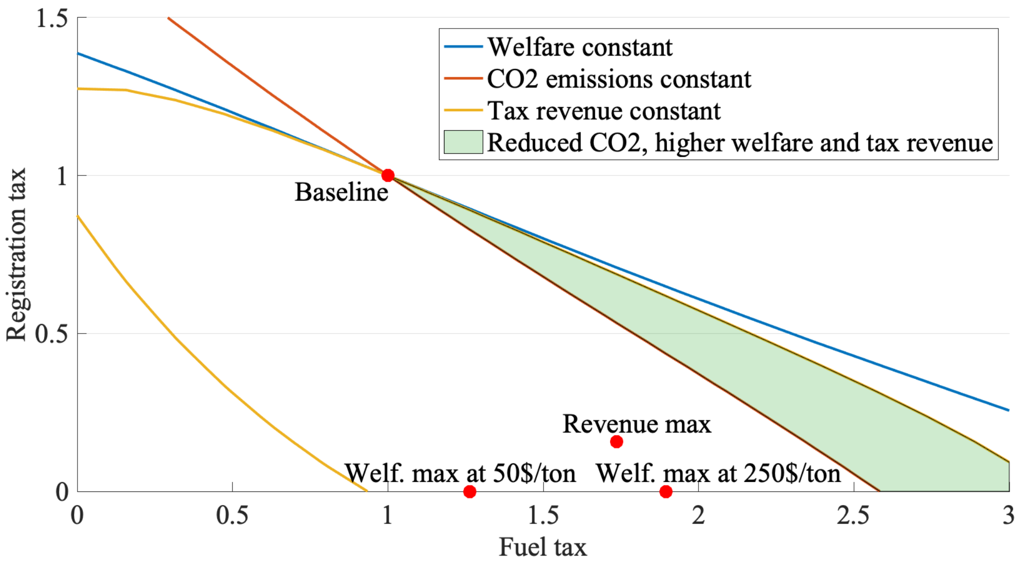
greater movements in new car sales, and thus in registation taxes, than what an equilibrium model can sustain in flow equilibrium. This means that the policy maker will expect greater effects from changes in registration taxes than what will actually come to pass.

The analysis so far has focused on comparing the decisions made under the guidance of a non-equilibrium as opposed to an equilibrium model. Now, we turn instead to the question of what policy our model would suggest. If we consider first the implications of the tax policy in the “Sophisticated” column, we note that the reform succeeds in raising total societal welfare. This is because although consumer surplus falls slightly from 6,901 to 5,564 DKK, driving-related externalities fall by more as total driving falls from 10,858 to 7,693 km annually in response to the much higher cost of driving.

Thus, there are clear welfare differences between revenue equivalent combinations of the two tax rates, so Figure 9 presents three 3D graphs with differing outcomes on the z-axes, but the same x- and y-axes. On the two horizontal axes are the fuel and registration taxes respectively (normalized so that (1,1) is the baseline), and on the vertical axis is total tax revenue. The graph in Panel (a) showing tax revenue clearly illustrates that the top point of the Laffer curve for each tax depends on the level of the other tax. However, welfare and CO_2 emissions are not optimal at that point.

Panels (b) and (c) of Figure 9 show similar graphs but with CO_2 emissions and total social welfare on the vertical axes, respectively. They show that lowering the overall level

Figure 10: Contour Lines



Note: The x and y axes show the tax rates for fuel and car registrations, respectively, normalized so that the baseline is 1. Each line represents the contour lines for one of three outcomes; that is, combinations of the two tax rates where the outcome is kept constant and equal to the value in the baseline configuration, occurring at the point (1,1). The three outcomes are tax revenue, CO_2 emissions, and social welfare (excluding the external costs of CO_2 emissions). Moving from the baseline in the direction of the origin implies an *increase* in all three outcomes, although tax revenue will eventually start to decline again (recall that in the baseline, both tax rates are above the top point of the Laffer curve). Four points are depicted on the graph in red: first the baseline, (1,1). Second, the top point of the Laffer curve, where overall tax revenue is maximized. And third, two points that show the overall social welfare maximizing policies: one under a CO_2 price of \$50/ton, and one at a higher price of \$250/ton (a price recently suggested by the Danish environmental council). Not surprisingly, a higher price results in a higher fuel tax, but still a zero tax on car purchases.

of both taxes (i.e. moving towards the origin) tends to result in higher social welfare (including CO_2, priced at \$50/ton), but at the cost of higher emissions. However, the maximum for social welfare does not occur with zero taxes; rather, it occurs with a zero registration tax and a fuel tax that is 30% higher than the baseline level. At that point, lowering fuel taxes might improve consumer surplus but by less than the marginal external cost of driving. These results clearly highlight that fuel taxes are superior to registration taxes from a welfare standpoint. This is not surprising as they target externalities much better and allow consumers to adjust to the tax at either the margin of car ownership, type choice, or driving. Finally, we consider second-best policy options. In practice, policy makers might be hessitant to choose a policy that drastically reduces tax revenues, which would then have to be recovered through distortionary labor taxes. Furthermore, countries might not want to raise CO_2 emissions and endangering environmental goals. Thus, we consider whether there are policy options that can raise welfare without harming tax revenue or raising emissions.

To do this, Figure 10 shows contour lines for the 3D graphs in Figure 9. The shaded green area to the southeast of the baseline levels, (1,1), represent combinations of the two tax rates that result in lower emissions, higher welfare, and higher tax revenue. The complex interaction between the two car taxes illustrates the importance of jointly modeling the purchase and driving decisions in an equilibrium framework, and the possibility of using such a model to optimize tax policy.

7 Discussion and Conclusions

We have introduced a computationally tractable model of equilibrium in the primary and secondary markets for automobiles that allows for flexible specifications of preferences and consumer heterogeneity and transactions costs. Our work was inspired by the early static discrete choice models of equilibrium in the automobile market pioneered by Manski, Sherman, and Berkovec and the subsequent efforts to extend their models to include dynamics and transactions costs and model equilibrium price setting in the primary market by Rust, Stolyarov, Gavazza, Lizzeri and Roketskiy, and Esteban and Shum. We believe that our framework is promising for empirical applications and policy analysis, and in future work we plan to further extend and apply it in a number of directions.

One of these directions is ongoing work (Gillingham, Iskhakov, Munk-Nielsen, Rust and Schjerning, 2019) to structurally estimate an "overlapping generations" version of our model using Danish register data to allow for a realistic counterfactual analysis of vehicle tax reform in Denmark. Another direction would relax the assumption of stationarity and extend our definition of equilibrium to allow for macroeconomic shocks that can capture the pronounced "waves" often found in the age distribution of vehicles (Adda and Cooper, 2000). We are comparing different solution concepts in terms of computational tractability and empirical realism, including the "temporary equilibrium" concept of Grandmont (1977), the "sufficient statistic" approach of Krusell and Smith (1998), as well as a full blown rational expectations equilibrium that takes into account the entire holdings distribution of cars as a component of the "state variables" that consumers use to predict future prices as in Cao (2016).

A very challenging extension of our model would endogenize the characteristics of vehicles by allowing firms to invest in R&D to produce new vehicle designs. Longer-run competition on attributes will likely require a fundamentally non-stationary framework and raises questions of consumer expectations over future products. Very promising headway into this sort of analysis has been done in the pioneering work of Goettler and Gordon (2011), and it may be possible to adapt this approach into a more evolutionary model of the automobile market. A final challenging extension would be to incorporate asymmetric information in a more detailed treatment of the "microstructure" of trade in the automobile market, including endogenous intermediation of trade by car dealers as well as direct consumer transactions. Recent studies such as Biglaiser, Li, Murry and Zhou (2020) have provided new empirical insights into the microstructure of trade that are not modeled in our framework, but represent important directions to pursue in the development of more detailed and realistic models of the microstructure of trade in automobiles.

References

Adda, J. and R. Cooper (2000): "Balladurette and Juppette: A Discrete Analysis of Scrapping Subsidies," *Journal of Political Economy*, 108(4), 778–806.

Akerlof, G. A. (1970): "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism," *Quarterly Journal of Economics*, 84, 488–500.

Berkovec, J. (1985): "New Car Sales and Used Car Stocks: A Model of the Automobile Market," *RAND Journal of Economics*, 16(2), 195–214.

Berndt, E., B. Hall, R. Hall and J. Hausman (1974): "Estimation and Inference in Nonlinear Structural Models," *Annals of Economic and Social Measurement*, 3 (4), 653–685.

Berry, S., J. Levinsohn and A. Pakes (1995): "Automobile Prices in Market Equilibrium," *Econometrica*, 63(4), 841–890.

Biglaiser, G., F. Li, C. Murry and Y. Zhou (2020): "Intermediaries and product quality in used car markets," *RAND Journal*, 51(3), 905–933.

Borken-Kleefeld, J. and Y. Chen (2015): "New emission deterioration rates for gasoline cars – Results from long-term measurements," *Atmospheric Environment*, 101, 58–64.

Cao, D. (2016): "Recursive Equilibrium in Krusell and Smith (1998)," *manuscript*.

Esteban, S. and M. Shum (2007): "Durable-goods Oligopoly with Secondary Markets: The Case of Automobiles," *RAND Journal of Economics*, 38(2), 332–354.

Gavazza, A., A. Lizzeri and N. Roketskiy (2014): "A Quantitative Analysis of the Used-Car Market," *American Economic Review*, 104(11), 3668–3700.

Gillingham, K., F. Iskhakov, A. Munk-Nielsen, J. Rust and B. Schjerning (2019): "A Dynamic Model of Vehicle Ownership, Type Choice, and Usage," *manuscript*.

Gillingham, K. and A. Munk-Nielsen (2019): "A tale of two tails: Commuting and the fuel price response in driving," *Journal of Urban Economics*, 109, 27–40.

Goettler, R. L. and B. R. Gordon (2011): "Does AMD Spur Intel to Innovate More?," *Journal of Political Economy*, 119(6), 1141–1200.

Grandmont, J. M. (1977): "Temporary General Equilbrium Theory," *Econometrica*, 45(3), 535–572.

Hendel, I. and A. Lizzeri (1999): "Adverse Selection in Durable Goods Markets," *American Economic Review*, 89(5), 1097–1115.

House, C. L. and J. V. Leahy (2004): "An sS Model with Adverse Selection," *Journal of Political Economy*, 112(3), 581–614.

Konishi, H. and M. Sandfort (2002): "Existence of Stationary Equilibrium in the Markets for New and Used Durable Goods," *Journal of Economic Dynamics and Control*, 26(6), 1029–1052.

Krusell, P. and T. Smith (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106(5), 867–896.

Ma, Q. and J. Stachurski (2021): "Dynamic Programming Deconstructed: Transformations of the Bellman Equation and Computational Efficiency," *Operations Research*.

Mannering, F. and C. Winston (1985): "A Dynamic Empirical Analysis of Household Vehicle Ownership and Utilization," *RAND Journal of Economics*, 16(2), 215–236.

Manski, C. (1980): "Short Run Equilibrium in the Automobile Market," *Falk Institute Discussion Paper 8018*, Hebrew University of Jerusalem.

——— (1983): "Analysis of Equilibrium Automobile Holdings in Israel with Aggregate Discrete Choice Models," *Transportation Research*, 17B (5), 373–389.

Manski, C. and E. Goldin (1983): "An Econometric Analysis of Vehicle Scrappage," *Transportation Science*, 17 (4), 365–375.

Manski, C. and L. Sherman (1980): "Forecasting Equilibrium Motor Vehicle Holdings by Means of Disaggregate Models," *Transportation Research Record*, 764, 96–103.

McFadden, D. (1981): "Econometric Models of Probabilistic Choice," in *Structural Analysis of Discrete Data*, ed. by C. Manski and D. McFadden, pp. 198–272. MIT Press, Cambridge, Massachusetts.

NHTSA (2013): "How Vehicle Age and Model Year Relate to Driver Injury Severity in Fatal Crashes," Discussion paper, U.S. Department of Transportation.

Rothenberg, T. J. (1971): "Identification in Parametric Models," *Econometrica*, 39-3, 570–591.

Rust, J. (1985a): "Stationary Equilibrium in a Market for Durable Assets," *Econometrica*, 53(4), 783–806.

——— (1985b): "When is it Optimal to Kill Off the Market for Used Durable Goods?," *Econometrica*, 54(1), 65–86.

——— (1987): "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher," *Econometrica*, 55(5), 999–1033.

——— (1994): "Structural estimation of markov decision processes," in *Handbook of Econometrics*, vol. 4, pp. 3081 – 3143. Elsevier.

Rust, J., J. Traub and H. Wozniakowski (2002): "Is There a Curse of Dimensionality for Contraction Fixed Points in the Worst Case?," *Econometrica*, 70(1), 285–329.

Stolyarov, D. (2002): "Turnover of Used Durables in a Stationary Equilibrium: Are Older Goods Traded More?," *Journal of Political Economy*, 110(6), 1390–1413.

Transport, D. (2010): "Transportøkonomiske Enhedspriser," Discussion paper, Technical University of Denmark, version 1.1.

White, H. (1982): "Maximum Likelihood Estimation of Misspecifed Models," *Econometrica*, 50(1), 1–25.

Winston, C. and J. Yan (2021): "Vehicle size choice and automobile externalities: A dynamic analysis," *Journal of Econometrics*, 222(1), 196–218.

Appendix

A Proofs

Lemma L1 (Jacobian Matrix of the Smoothed Bellman Operator)**.** *Let EV be the unique fixed point of the smoothed Bellman operator Γ in* (9)*, and let $\nabla_{EV}\Gamma(EV, P)$ be the Jacobian matrix of Γ with respect to EV. The following holds:*

$$\nabla_{EV}\Gamma(EV, P) = \beta\Omega(P)Q, \tag{49}$$

where the matrices $\Omega(P)$ and Q are defined in equations (24) *and* (27)*, respectively. The norm of this matrix is $\|\nabla_{EV}\Gamma(EV, P)\| = \beta \in (0, 1)$.*

Proof. This Lemma can be proved via direct calculation, though the algebra involved is extremely tedious. We sketch a more conceptual proof of the result here. First, note that the operator Γ is a smooth nonlinear and recursively nested function of log-sum functions (11). The log-sum functions are in turn functions of the choice-specific value functions $v(i, a, j, d)$ given in equations (3), (6), and (8) of section 3. These value functions, call them v, are in turn functions of the expected values EV a result we emphasize by writing $\Gamma(EV, P) = \Gamma(v(EP), P)$. Using the chain rule to compute the Jacobian matrix $\nabla_{EV}\Gamma(EV, P)$ with respect to the $J\bar{a} + 1 \times 1$ vector EV, we have

$$\nabla_{EV}\Gamma(EV, P) = \nabla_v\Gamma(v(EV), P)\nabla_{EV}v(EV). \tag{50}$$

The Lemma follows by showing that $\nabla_v\Gamma(v(EV), P)$ equals $\Omega(P)$ and $\nabla_{EV}v(EV)$ equals βQ. The former result follows from the Williams-Daly-Zachary Theorem (see McFadden (1981)), and the fact that the Γ operator can be expressed as the expected maximum of the v functions with the additive GEV errors as shown via the representation of $\Gamma(EV, P)$ as the value functions V given in equations (2) and (5). The Williams-Daly-Zachary Theorem implies that the derivatives of the expected maximum of $v + \epsilon$ with respect to v equals the choice probabilities Π such as in equation (13). When the matrix of values v is arrayed in the same order that EV is arrayed (with first $J\bar{a}$ elements of EV equalling $EV(i, a)$ for $i = 1, \ldots, J$ and $a = 1, \ldots, \bar{a}$ and $J\bar{a} + 1$-st element equal to $EV(\o)$ and we account for the fact that some elements of EV appear in multiple different elements in any given row of the matrix v), it is not difficult to see that $\nabla_v\Gamma(v(EV), P) = Q(P)$, where the latter is a $(J\bar{a} + 1) \times (J\bar{a} + 1)$ Markov transition probability matrix given in equation (24). Further using the formulas for $v(EV)$ in equations (3), (6), and (8) of section 3, it is not hard to see that $\nabla_{EV}v(EV) = \beta Q$, where Q is the accident/aging transition probability given in equation (27). Since Q and $\Omega(P)$ are both Markov transition probability matrices, so is their product, $M = \Omega(P)Q$. Recall the notion of a matrix norm, $\|M\| = \sup_{x \neq 0} \|Mx\|/\|x\|$ where $\|x\|$ is a norm of the vector x (e.g. Euclidean norm or sup-norm). When M is a transition probability matrix, it is easy to see that $\|Mx\| \leq \|x\|$, which implies that $\|M\| \leq 1$. Let e be a vector all of whose elements equal 1. Then $Me = e$ which implies $\|M\| \geq 1$, or $\|M\| = 1$. Also it is easy to see from the definition of a matrix-norm, $\|\beta M\| = \beta\|M\|$. It follows that $\|\beta\Omega(P)Q\| = \beta$. □

Lemma L2 (Differentiability)**.** *The unique fixed point EV of the smoothed Bellman operator in* (9)*, the choice-specific value functions $v(\cdot)$ in* (2)*,* (5) *and* (7)*, the choice probabilities $\Pi(j, d, s|i, a)$ in* (13)*, the trade transition probability matrix $\Omega(P)$ and excess* *demand function $ED(P)$ exist and are continuously differentiable functions of market prices P. The Jacobian matrix of the fixed point EV with respect to the market prices is given by*

$$\nabla_P EV(P) = -[I - \nabla_{EV}\Gamma(EV, P)]^{-1}\nabla_P\Gamma(EV, P), \tag{51}$$

where $\nabla_P\Gamma(EV, P)$ is the $J\bar{a} + 1 \times J(\bar{a} - 1)$ Jacobian matrix of Γ with respect to market prices P.

Proof. Existence of a unique fixed point $EV = \Gamma(EV, P)$ for any P follows because the operator Γ can be shown to be a *quasi-linear, monotone mapping* (see Rust, Traub and Wozniakowski (2002)) and thus is a contraction mapping with a unique fixed point EV. By Lemma L1, Γ is a continuously differentiable function of EV with gradient $\nabla_{EV}\Gamma(EV, P)$. By the Implicit Function Theorem we can express EV as a zero, $F(EV, P) = 0 = EV - \Gamma(EV, P)$ and the solution EV will be a continuously differentiable implicit function of P provided that $\nabla_{EV}F(EV, P)$ is invertible. However from Lemma L1 $\nabla_{EV}F(EV, P) = I - \beta\Omega(P)Q$, and this is invertible with the geometric series representation for its inverse

$$[I - \beta\Omega(P)Q]^{-1} = \sum_{t=0}^{\infty}\beta^t[\Omega(P)Q]^t. \tag{52}$$

Then since $EV(P)$ is a continuously differentiable function of P, it is easy to see from the formulas for v, the conditional choice probabilities Π and the transition probability matrix $\Omega(P)$ are continuously differentiable since they are explicit smooth functions of $EV(P)$. The formula for $\nabla_P EV(P)$ in equation (51) is a consequence of total differentiation of the identity $\Gamma(EV(P), P) = 0$ with respect to P and solving for $\nabla_P EV(P)$.

□

A.1 Proof of Theorem 1 (page 24)

Proof. When scale parameters of GEV distribution of random components ϵ are positive, $\sigma \geq \sigma_r \geq \sigma_j \geq \sigma_s > 0$, the choice probabilities are bounded away from zero for all choices and for any price vector P. Thus, the transition probability matrix $\Omega(P)Q$ is irreducible and aperiodic. Uniqueness of the stationary distribution q that satisfies $q = q\Omega(P)Q$ follows from the fundamental theorem of Markov chains.

To show continuous differentiability of the stationary distribution q given as an implicit function of P by $q = q\Omega(P)Q$, it would be enough to apply the Implicit Function Theorem, but unfortunately the prerequisite invertibility condition fails in our case. Indeed, q can be treated as a left zero of the matrix $I - \Omega(P)Q$, where I is the identity matrix of the appropriate size. In other words, $q(P)$ can be written as a zero the non-linear mapping $F(q, P) = q\big(I - \Omega(P)Q\big) = 0$. However when q is ergodic, Appendix B provides an explicit solution for $q(P)$ as the inverse of a bordered matrix for which $I - \Omega(P)Q$ is an upper left $(J\bar{a} + 1) \times (J\bar{a} + 1)$ submatrix. It also provides a formula for $\nabla_P q(P)$ in terms of gradients of the matrix $\Omega(P)$ with respect to P. This proves that $q(P)$ is uniquely defined and continuously differentiable function of P.

□

A.2 Proof of Theorem 2 (page 24)

Proof. The proof of existence of an equilibrium follows from Brouwer's Fixed Point Theorem by defining a mapping $\Psi(P) : R^{J(\bar{a}-1)} \to R^{J(\bar{a}-1)}$ where $\Psi(P) = P + ED(P)$, $J(\bar{a}-1)$ is the dimension of the price vector P and number of used cars traded in secondary markets with J types of cars sold in the primary market and $\bar{a}$ the oldest tradeable car age in each market, and $ED(P)$ is defined in (23). From Lemma L2 it follows that ED and thus Ψ is a continuous mapping from $R^{J(\bar{a}-1)} \to R^{J(\bar{a}-1)}$. Note also that for any P the components of $ED(P)$ lie in the interval $[-1, 1]$. Thus, when prices are sufficiently high, a vanishing number of consumers will wish to buy any new car but nearly all consumers will want to sell their cars, so $ED(P)$ will be close to a vector with all its components equal to -1. Similarly, for a sufficiently low set of prices (possibly negative), nearly all consumers will wish to buy used cars and very few will want to sell their vehicles at such low prices. So for such prices $ED(P)$ will be close to a vector with all of its components equal to $+1$. It follows that we can define a compact box B in $R^{J(\bar{a}-1)}$ where Ψ satisfies an "inward pointing" property on the boundaries of this box, so it follows that $\Psi : B \to B$. Since Ψ is a continuous mapping and B is a compact, convex set, the Brouwer fixed point theorem implies that a fixed point of Ψ exists, and it is clear that any such fixed point satisfies $ED(P) = 0$.

$\square$

A.3 Proof of Theorem 3 (page 26)

Proof. First, note that for any P the $J(\bar{a}-1) \times J(\bar{a}-1)$ trade-transition probability matrix has the decomposition

$$\Omega(P) = \Delta(P) + \Lambda(P) \tag{53}$$

where $\Delta(P)$ it a transition sub-probability matrix for trading the currently owned car and $\Lambda(P)$ is a diagonal matrix with the probabilities of keeping the current car on the diagonal (though the last element of $\Lambda(P)$ is zero, since the probability of keeping the current car is zero in the no-car state). See equations (24), (25) and (26) of section 3. Let $q\Delta(P)_{j,a}$ be the $(j,a)^{\text{th}}$ component of the vector $q\Delta(P)$. It is not hard to see that $q\Delta(P)_{j,a} = D(j,a)$, the aggregate demand for car type j and age a by all households in the economy. Similarly $q\Delta(P)_{\o}$ is the fraction of the consumers who demand the outside good. Thus, it is easy to see that equation (31) can be written in matrix form as $q\Delta(P)_{\o} = q_{\o}$. Next, by the way we ordered the columns of the $\Delta(P)$ matrix (putting the probabilities of buying new cars in the final columns of the $\Delta_{ij}(P)$ blocks see equation (25) of section 3), it follows that $D_{j,0}(P)$, the demand for new cars of type j, is the last component $(j, \bar{a})$ of the vector $q\Delta(P)$, i.e. $D_{j,0}(P) = q\Delta(P)_{j,\bar{a}}$. We can write the excess demand for car (j,d), $ED_{j,d}(P)$, as

$$ED_{j,d}(P) = [q\Delta(P)]_{j,d} - [1 - \Pi(1_s|j,d,P)][1 - \Pi(\kappa|j,d,P)]q_{j,d}, \quad d \in \{1,\ldots,\bar{a}\}. \tag{54}$$

where $\Pi(1_s|j,d,P)$ is the probability the household chooses to scrap their current car (j,d). Since $q\Omega(P) = q[\Delta(P) + \kappa(P)]$ is a probability distribution and thus sums to 1, it follows that aggregate demand for new cars is given by

$$\sum_{j=1}^{J} D_{j,0}(P) = \sum_{j=1}^{J}[q(P)\Delta(P)]_{j,\bar{a}} = 1-[q(P)\Delta(P)]_{\o} - \sum_{j=1}^{J}\sum_{a=1}^{\bar{a}-1} q(P)[\Delta(P)+\kappa(P)]_{j,a}, \quad (55)$$

where we have used the condition that the diagonal elements of $\kappa(P)$ corresponding to new cars and the outside good are zero (since there is no option to "keep" a brand new car, nor an alternative to "keep" the outside good – the latter is captured by the $\Pi(\o|\o, P)$, the probability of choosing to remain in the no car state which is in the lower $(\o, \o)$ element of the $\Delta(P)$ matrix).

In a stationary equilibrium we have $ED_{j,d}(P) = 0$ for $d \in \{1, \ldots, \bar{a}-1\}$, so using equation (54) and equation (31) ($q\Delta_{\o} = q_{\o}$), it follows that

$$\begin{aligned} q(P)[\Delta(P)+\Lambda(P)]_{j,d} &= [1-\Pi(1_s|j,d,P)][1-\Pi(\kappa|j,d,P)]q_{j,d} + \Pi(\kappa|j,d,P)q_{j,d} \\ &= q_{j,d} - [\Pi(1_s|j,d,P)[1-\Pi(\kappa|k,d,P)]]\, q_{j,d} \end{aligned} \quad (56)$$

Substituting equation (56) into equation (55) we can write the aggregate demand for new cars as

$$\sum_{j=1}^{J} D_{j,0}(P) = 1 - q_{\o}(P) - \sum_{j=1}^{J}\sum_{a=1}^{\bar{a}-1} q_{j,a}(P) + \sum_{j=1}^{J}\sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)[1-\Pi(\kappa|j,a,P)]q_{j,a}(P). \quad (57)$$

Since $q(P)$ is a probability vector, we have $1 - q_{\o}(P) = \sum_{j=1}^{J}\sum_{a=1}^{\bar{a}} q_{j,a}(P)$ and we can rewrite the aggregated flow equilibrium condition (57) as

$$\sum_{j=1}^{J} D_{j,0}(P) = \sum_{j=1}^{J} q_{j,\bar{a}}(P) + \sum_{j=1}^{J}\sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)[1-\Pi(\kappa|j,a,P)]q_{j,a}(P). \quad (58)$$

It is clear that this is the sum of the car type specific flow equilibrium conditions in equation (32) of Theorem 3. So we have shown that flow equilibrium must hold in aggregate over all car types.

However flow equilibrium must hold for each car type j individually, since if $D_{j,0}(P) > q_{j,\bar{a}}(P) + \sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)[1-\Pi(\kappa|j,a,P)]q_{j,a}(P)$, then there would be a net inflow of cars of type j into the economy. However this would be incompatible with a stationary market share for cars of type j, which does hold in a stationary equilibrium. Similarly, if the inequality above were reversed, there would be a net outflow of cars of type j from the economy, and this is also incompatible with a stationary and positive market share of cars of type j in the economy. We conclude that the flow equilibrium condition (32) must hold for each car type $j \in \{1, \ldots, J\}$.

To show this mathematically, we repeat a similar argument to above, but now we focus on a single car type j. Let $q_j(P)$ denote the subvector of the distribution $q(P)\Omega(P)$ corresponding to ownership of some age of car type j

$$q_j(P) \equiv ([q(P)\Omega(P)]_{j,1}, \ldots, [q(P)\Omega(P)]_{j,\bar{a}}). \quad (59)$$

and let $\overline{q}_j$ be the sum of these probabilities

$$\overline{q}_j = \sum_{a=1}^{\bar{a}} [q(P)\Omega(P)]_{j,a}. \tag{60}$$

Then following the same argument as above we can write

$$D_{j,0}(P) = [q(P)\Delta(P)]_{j,\bar{a}} = \overline{q}_j - \sum_{a=1}^{\bar{a}-1} [q(P)\Omega(P)]_{j,a}. \tag{61}$$

Following the same approach of substituting from the excess demand equations $ED_{j,a}(P) = 0$ for $a \in \{1, \ldots, \bar{a} - 1\}$ we get

$$D_{j,0}(P) = \overline{q}_j - \sum_{a=1}^{\bar{a}-1} q_{j,a}(P) + \sum_{a=1}^{\bar{a}-1} \Pi(1_s|j, a, P)[1 - \Pi(\kappa|j, a, P)]q_{j,a}(P). \tag{62}$$

However, from the definition of $\overline{q}_j$, the first term becomes

$$\overline{q}_j - \sum_{a=1}^{\bar{a}-1} q_{j,a}(P) = q_{j,\bar{a}}(P), \tag{63}$$

so substituting this in the equation for $D_{j,0}(P)$ in (62) we obtain the flow equilibrium condition in equation (32). So we conclude that flow equilibrium not only has to hold in aggregate, but it must hold for each type of car sold in the market in any stationary equilibrium. $\square$

B Gradient of Invariant Distribution

In this appendix we consider the general problem of calculating the derivative of an invariant distribution with respect to parameters affecting a Markov transition matrix. Let the parameters be θ (in our application θ is a vector of prices of cars in a secondary market equilibrium) and consider a Markov transition probability matrix $P(\theta)$ that depends on these parameters in a continuously differentiable fashion. Thus, we assume that the mapping $\nabla_\theta P(\theta)$ from R^k to R^{k*n*n} (where the latter can be interpreted as the space of k-tuples of $n \times n$ matrices) exists and is a continuous function of θ. To make things easier to understand, assume initially that $k = 1$ so we are considering $P(\theta)$ and $h(\theta)$ as functions of a single parameter θ. If θ has k components (i.e. $\theta \in R^k$) we simply "stack" the formulas we provide below in the univariate case into a k-tuple.

We are interested in determining the conditions under which $h(\theta)$, the unique invariant distribution of $P(\theta)$, is a continuously differentiable function of θ and, if so, to find an expression for $\nabla_\theta q(\theta)$. The invariant distribution $h(\theta)$ satisfies the equation

$$h(\theta) = h(\theta)P(\theta), \tag{64}$$

which can be recast as $h(\theta)$ being a left zero of the matrix $I - P(\theta)$, $h(\theta)[I - P(\theta)] = 0$. The usual application of the Implicit Function Theorem applies when $h(\theta)$ can be written as a zero of some continuously differentiable nonlinear mapping $F(h, \theta) = 0$ with the added condition that $\nabla_h F(h,\theta)$ is nonsingular at a zero of F. Then the Implicit Function Theorem guarantees that there is a continuously differentiable function $h(\theta)$ in a neighborhood of this zero, and we have

$$\nabla_\theta h(\theta) = -[\nabla_h F(h(\theta),\theta)]^{-1}\nabla_\theta F(h(\theta),\theta). \tag{65}$$

However this usual application of the Implicit Function Theorem is inapplicable because in this case $\nabla_h F(h,\theta) = I - P(\theta)$ and this matrix is singular (note that if e is a vector of ones, then $[I - P(\theta)]e = 0$ where 0 is a vector of zeros). Thus, we have to approach this problem from a different angle.

When the invariant distribution is unique, it can be shown that $h(\theta)'$, the $n \times 1$ transpose of $h(\theta)$, is the unique solution to the expanded $(n+1)\times(n+1)$ linear system given by

$$\begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix} \begin{bmatrix} h(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ 2 \end{bmatrix} \tag{66}$$

where e is an $n \times 1$ vector all of whose elements equal 1. Thus, the matrix on the left hand side of equation (66) is invertible and we can write

$$\begin{bmatrix} h(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 2 \end{bmatrix}. \tag{67}$$

Let $A(\theta)$ be the $(n+1)\times(n+1)$ matrix on the left hand side of equation (66). Then we have that $\nabla_\theta h(\theta)'$ is the upper left $n \times n$ submatrix of the product of $\nabla_\theta A^{-1}(\theta)$ times the vector $(e'\ 2)'$. Further, we use the following formula for the gradient of $A^{-1}(\theta)$ with respect to θ

$$\nabla_\theta A^{-1}(\theta) = -A^{-1}(\theta)\,[\nabla_\theta A(\theta)]\,A^{-1}(\theta). \tag{68}$$

Now, to relate these results to calculate the gradient of the equilibrium holding distribution with respect to P, we see to compute $\nabla_P q(P)$ where $q(P) = q(P)\Omega(P)Q$, using the result above, each component $\nabla_{P(i,a)} q(P)'$ is the upper left $n \times n$ submatrix of the product of $\nabla_\theta A^{-1}(\theta)$ times the vector $(e'\ 2)'$, where $n = \bar{a}$ and dimension of P is $\bar{a} - J$, and we identify θ as the price vector P and $A(\theta)$ as the $(n+1)\times(n+1)$ matrix on the left side of equation (66) where $P(\theta) = \Omega(P)Q$. Thus, the gradient $\nabla_\theta A(\theta)$ is also an $(n+1)\times(n+1)$ matrix given by

$$\nabla_\theta A(\theta) \equiv \nabla_{P(i,a)} A(P) = \begin{bmatrix} -\nabla_{P(i,a)}\Omega(P)Q' & 0 \\ 0' & 0 \end{bmatrix}. \tag{69}$$

Instead of prices P, we may also be interested in computing $\nabla_\theta q(\theta)$ where θ are parameters of the utility functions of consumers. Then we have $q(\theta) = q(\theta)\Omega(\theta)Q$ and to compute $\nabla_\theta q(\theta)$ we use the same formula as above for $\nabla_P q(P)$, except that in equation (69) we use $\nabla_{\theta_k}\Omega(\theta)$ instead of $\nabla_{P(i,a)}\Omega(P)$ in the upper $n \times n$ block of the matrix. Finally for parameters such as accident probabilities α that enter both Ω and Q, we compute $\nabla_{\alpha(i,a)} q(\alpha)$ in the same way too, except in equation (69) the upper $n \times n$ block is $-[\nabla_{\alpha(i,a)}\Omega(\alpha)Q(\alpha) + \Omega(\alpha)\nabla_{\alpha(i,a)}Q(\alpha)]$.

C Solving the Homogeneous Consumer Economy

The limiting case of our model when $\sigma \to 0$ constitutes the *discrete product market* version of Rust (1985a). In this appendix we lay out a simple and efficient numerical solution algorithm for this limiting case, which constitutes the source of precise starting values for the main numerical algorithm in Section 3. We have proven that the following results from Rust (1985a) continue to hold in our discrete setting. Proofs for these results are available on request.

Theorem 1 (Equilibrium in homogeneous consumer economy)**.** *Consider the primary and secondary market for automobiles with one car make/model and homogenous consumers ($\sigma = 0$). Assume infinitely elastic supply of new cars at price $\overline{P}$ and infinitely elastic demand for scrapped cars at price $\underline{P}$. The unique stationary equilibrium $\{q, P, a^\star\}$ on this market exists, and is composed of:*

1. *Ownership distribution q, which is the unique invariant distribution corresponding to the physical transition probability matrix Q defined in (27);*
2. *Common scrappage age $a^\star$ equal to the optimal replacement age in the social planner's problem of optimal car replacement (in the absence of secondary market, or equivalently within the class of "buy and hold" strategies);*
3. *Non-increasing price function $P(a)$ defined by*

$$P(a) = \begin{cases} \overline{P} - \frac{1}{\mu}\big(W(0) - W(a)\big), & a \in \{1, \dots, a^\star - 1\}, \\ \underline{P}, & a \geq a^\star, \end{cases} \tag{70}$$

where $W(a)$ is the unique fixed point of the Bellman operator in the forementioned social planner's replacement problem.

Corollary C1. *In the equilibrium of the homogeneous consumer economy with no transaction costs defined in Theorem 1, consumers are indifferent between replacing their existing car with the car of any age available in the economy. This holds for owners of all ages of cars with positive shares in the stationary fleet age distribution.*

Corollary C2. *The equilibrium in the homogeneous consumer economy with no transaction costs defined in Theorem 1 is welfare maximizing, in particular the discounted expected utility of all consumers is equal to maximum attainable welfare, $V(a) = W(a)$, for all cars with positive shares in the stationary fleet age distribution.*

The main idea of the fast solution algorithm for the homogeneous consumer economy is to express the indifference condition from Corollary C1 as the system of $a^\star - 1$ linear equations to determine the unrestricted prices $P(a)$, $a \in \{1, \dots, a^\star - 1\}$. Because by Corollary C1 consumers are effectively indifferent between any dynamic trading strategies, the strategy of perpetual replacing an existing car of age a results in the maximum attainable expected discounted utility $V(a)$. We have for $a \in \{1, \dots, a^\star - 1\}$

$$V(a) = \tfrac{1}{1-\beta}\Big(u(a) - \beta\mu\big[P(a) - \big(1 - \alpha(a)\big)P(a+1) - \alpha(a)\underline{P}\big]\Big). \tag{71}$$

Let $V(0)$ denote the value of having a new car which is measured right after trading instead of the beginning of the period. Then (71) also holds for $a = 0$, and $V(0) = W(0)$. Thus, using $V(a) - V(a+1) = W(a) - W(a+1)$ (Corollary C2) and the definition of the price function (70), we have for $a \in \{0, \dots, a^\star - 2\}$

$$V(a) - \mu P(a) = V(a+1) - \mu P(a+1), \tag{72}$$

which leads to the following linear equation in prices $(P(a), P(a+1), P(a+2))$:

$$\begin{aligned}\mu P(a) + \mu\big(\alpha(a)\beta - \beta - 1\big)P(a+1) + \mu\beta\big(1 - \alpha(a+1)\big)P(a+2) = \\ = u(a) - u(a+1) + \beta\mu\underline{P}\big(\alpha(a) - \alpha(a+1)\big).\end{aligned} \tag{73}$$

The collection of equations (73) for $a \in \{0, \dots, a^\star - 2\}$ forms the system of $a^\star - 1$ equations with $a^\star - 1$ unknowns, and can be easily solved numerically under our assumption that $u(a+1) \leq u(a)$ for $a \geq 0$. The system be written in matrix form as

$$X \cdot P = Y, \tag{74}$$

where P is the column vector of prices $\{P(a)\}_{a \in \{1, \dots, a^\star - 1\}}$,

$$X = \begin{bmatrix} \alpha(0)\beta - \beta - 1 & \beta - \alpha(1)\beta & 0 & 0 & \cdots & 0 & 0 \\ 1 & \alpha(1)\beta - \beta - 1 & \beta - \alpha(2)\beta & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha(2)\beta - \beta - 1 & \beta - \alpha(3)\beta & \cdots & 0 & 0 \\ 0 & 0 & 1 & \alpha(3)\beta - \beta - 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha(a^\star - 3)\beta - \beta - 1 & \beta - \alpha(a^\star - 2)\beta \\ 0 & 0 & 0 & 0 & \cdots & 1 & \alpha(a^\star - 2)\beta - \beta - 1 \end{bmatrix},$$

$$Y = \begin{bmatrix} \big(u(0) - u(1)\big)/\mu + \beta\underline{P}\big(\alpha(0) - \alpha(1)\big) - \overline{P} \\ \big(u(1) - u(2)\big)/\mu + \beta\underline{P}\big(\alpha(1) - \alpha(2)\big) \\ \big(u(2) - u(3)\big)/\mu + \beta\underline{P}\big(\alpha(2) - \alpha(3)\big) \\ \vdots \\ \big(u(a^\star - 3) - u(a^\star - 2)\big)/\mu + \beta\underline{P}\big(\alpha(a^\star - 3) - \alpha(a^\star - 2)\big) \\ \big(u(a^\star - 2) - u(a^\star - 1)\big)/\mu + \beta\underline{P}\big(\alpha(a^\star - 2) - 1\big) \end{bmatrix}.$$

Note that system (74) is well defined for any $a^\star \geq 2$, and consequently a solution to the system denoted $P(a, a^\star)$ can be computed for any value of $a^\star$. We show that the solution of the social planner's problem $W(a)$ corresponds one-to-one to the solution of the linear system (74) under the condition on the price vector $\forall a \in \{1, \dots, a^\star\} : \underline{P} \leq P(a, a^\star) \leq \overline{P}$ and $P(a^\star, a^\star + 1) < \underline{P}$. Because solving a system of linear equations implies much lower computational cost than finding a fixed point $W(a)$, and in particular because the dimensionality of the social planner's problem is larger than that of the linear system, an iterative algorithm that solves (74) for various $a^\star$ to ensure the conditions above hold, is the most efficient numerically.

D Likelihood when accidents are unobserved

In this appendix we derive the expression for the log-likelihood function when accidents are unobserved, i.e. a likelihood similar to the "full information" likelihood in equation (43) but using the post-decision state δ_t and integrating out with respect to accidents. So consider the transition probability in a partial information situation where we do not directly observe whether a given car is involved in an accident that leads to its scrappage. As a result we do not fully observe the state x_t representing car holdings of a given household, since for any car state $x_t = (i_t, a_t)$ where $a_t = \overline{a}_{i_t}$ (i.e. cars that have reached the mandatory scrappage state/age $\overline{a}_{i_t}$) we cannot distinguish which of those car are in that state due of an accident (since accidents are not directly observed) and which are in that state not due to an accident, but due to being age $\overline{a}_{i_t} - 1$ in period $t-1$. However we do observe whether the previously chosen car (j_{t-1}, d_{t-1}) was scrapped or not, so this implies we fully observe δ_t so our approach to inference in the partial information case will be based on the transition probability $P(\delta_{t+1}|\delta_t, \tau, \theta)$. In the case where $\delta_t = (\emptyset, s_t)$ (either a decision to enter the no car state via a purge decision with an associated scrappage decision s_t, or a decision to remain in the no car state if $x_t = \emptyset$, in which case the scrappage decision s_t is not relevant), $P(\delta_{t+1}|\delta_t, \tau, \theta)$ is just the conditional choice probability given in equation (13). However if the household did choose to either trade for a car (i_t, a_t) or keep their car (i_t, a_t) at time t, since we do not observe whether that car was involved in an accident between t and $t+1$, the relevant choice probability $P(\delta_{t+1}|\delta_t, \tau, \tau)$ is a mixture of two choice probabilities, depending on whether the car was involved in an accident or not. Thus, when a scrappage is observed for the car the household chose at t, then $s_{t+1} = 1$, (here we let the scrappage indicator, s_{t+1}, equal 1 if the car the household chose at t was scrappage by period $t+1$, and 0 otherwise), the transition probability $P(\delta_{t+1}|\delta_t, x_t, \tau, \theta)$ is given by

$$\begin{aligned} P(j_{t+1}, d_{t+1}, s_{t+1} = 1|i_t, a_t, s_t, \tau, \theta) &= P(j_{t+1}, d_{t+1}|i_t, \overline{a}_{i_t}, s_t, \tau, \theta)\alpha(i_t, a_t) + \\ P(j_{t+1}, d_{t+1}|i_t, a_t + 1, s_t, \tau, \theta)P(s_{t+1} &= 1|j_{t+1}, d_{t+1}, i_t, a_t + 1, s_t, \tau, \theta)(1 - \alpha(i_t, a_t)), \end{aligned} \tag{75}$$

where $P(j_{t+1}, d_{t+1}|i_t, \overline{a}_{i_t}, s_t, \tau, \theta)$ is the conditional choice probability for a household holding a "clunker" which requires a forced scrappage of the vehicle by our modeling assumptions. Thus, in the event of an accident of the car the houseold chose at time t, we represent it as a transition to the clunker state $\bar{a}$ and in this choice there is no choice over whether or not to sell or scrap the car, so $P(s_{t+1} = 1|j_{t+1}, d_{t+1}, i_t, \bar{a}) = 1$. If there is no accident, then the household does have a choice of whether to scrap or sell the car, so in this case we do include the conditional probability of the scrappage choice, $P(s_{t+1} = 1|j_{t+1}, d_{t+1}, i_t, a_t + 1, s_t, \tau, \theta)$ to calculate the overall transition probability in (76) in the event of an observed scrappage of the chosen car between period t and $t+1$.

If there was no scrappge of the car the household chose at time t between periods t and $t+1$, which we denote by $s_{t+1} = 0$, then $P(\delta_{t+1}|\delta_t, x_t, \tau, \theta)$ is given by

$$\begin{aligned} &P(j_{t+1}, d_{t+1}, s_{t+1} = 0|i_t, a_t, s_t, \tau, \theta) = \\ &P(j_{t+1}, d_{t+1}|i_t, a_t + 1, s_t, \tau, \theta)(1 - P(s_{t+1} = 1|j_{t+1}, d_{t+1}, i_t, a_t + 1, \tau, \theta))(1 - \alpha(i_t, a_t)), \end{aligned} \tag{76}$$

i.e. it is the probability of choosing to trade the existing car but also choosing not to scrap at $t+1$ conditioning on the event no accident occurred between t and $t+1$ either.

In the event of a choice to keep the current car at time $t+1$, which we have denoted earlier in the paper via the special symbol $(j_{t+1}, d_{t+1}) = (\kappa, \kappa)$ to distinguish it from a decision to trade the current car (i_t, a_t) for another car with the same time and age, i.e. $j_{t+1} = i_t$ and $d_{t+1} = a_t$, we can conclude that no accident and no voluntary scrappage of the previously chosen vehicle could have occurred. Since the keep decision obviates any choice about selling or scrapping the existing car, the transition probability for the keep decision is given by

$$P(j_{t+1} = \kappa, d_{t+1} = \kappa|i_t, a_t, s_t, \tau, \theta) = P(\kappa, \kappa|i_t, a_t + 1, s_t, \tau, \theta)(1 - \alpha(i_t, a_t)). \tag{77}$$

We include the previous car ownership state x_t as a conditioning variable in the transition probability $P(\delta_{t+1}|\delta_t, x_t, \tau, \theta)$ just as we did in the full information case above since when a household decides to keep their current car, we need the information in the incoming car state $x_t = (i_t, a_t)$ to determine the coditional probabilities of a scrappage decision and whether the car will have an accident.

The Kullback-Leibler distance in the case of unobserved accidents is given below

$$D(\theta) \equiv \tag{78}$$
$$\sum_\tau \sum_x \sum_\delta \sum_{\delta'} \left[\log\left(P(\delta'|\delta, x, \tau)\right) - \log\left(P(\delta'|\delta, x, \tau, \theta)\right)\right] P(\delta'|\delta, x, \tau) P(\delta|x, \tau) q_\tau(x) f(\tau).$$

However since the x state variable is not fully observed, the formula above is not directly operational. We assume that via direct observation of households car choices, δ_t and δ_{t+1}, it is possible to non-parametrically estimate the transition probability $P(\delta'|\delta, x, \tau)$ for each observed household type τ. Note that though x is not always fully observed, it is observed when a household chooses to keep their car. It is not observed for all decisions involving trading the previous car, however the identity of the new car captured in δ is a sufficient statistic for determining the probability distribution of δ' and in the case of a trade, $P(\delta'|\delta, x, \tau)$ is independent of x given δ. It follows that $P(\delta'|\delta, x, \tau)$ can be non-parametrically estimated in the case where accidents are unobserved and thus the state x is not fully observed.

It is also possible to non-parametrically estimate the cross-sectional distribution of car ownership choices $q_\tau(\delta)$ for each household type τ. This is the stationary distribution of car ownership states, prior to accounting for accidents, whereas $q_\tau(x)$ is the actual incoming stationary distribution of car ownership states, accounting for accidents. It is not hard to show that if $q_\tau(x)$ is the invariant distribution of car ownership states at the start of any period t for household type τ, then $q_\tau(\delta) = \sum_x P(\delta|x, \tau) q_\tau(x)$ is an invariant distribution of household car choices, where $P(\delta|x, \tau)$ is the conditional probability that a household of type τ makes a car ownership decision δ at time $t + 1$ conditional on their car state being x at time t, and where we have added the probability of keeping the current car $x = (i, a)$, denoted by $\delta = (\kappa, \kappa)$, to the conditional probability of trading for a car of type/age $x = (i, a)$. That is, when $\delta = x$ we define $q_\tau(\delta)$ as $\sum_{x'} P(\delta|x', \tau) q_\tau(x') + P(1|x, \tau) q_\tau(x)$. With this redefined δ variable it is no longer necessary to condition on x when writing the conditional choice probability of choosing a car (and scrapping the existing one) at time $t + 1$, we can now simply express it as $P(\delta'|\delta, \tau)$. Then using the invariant distribution over car ownership decisions, $q_\tau(\delta)$, we can write the Kullback-Leibler distance in the case where accidents are not observed as follows

$$D(\theta) \equiv \sum_\tau \sum_\delta \sum_{\delta'} \left[\log\left(P(\delta'|\delta, \tau)\right) - \log\left(P(\delta'|\delta, \tau, \theta)\right)\right] P(\delta'|\delta, \tau) q_\tau(\delta) f(\tau). \tag{79}$$

In summary, in the full information case the relevant transition probability that we use as a basis for estimation is $P(x_{t+1}|x_t, \tau, \theta)$, i.e. the transition probability for the fully observed car ownership state of the household, expressed as a product of the conditional choice probability for the household's decision over next period car state, and an accident probability that gives the final realized car state x_{t+1} at the start of the next period $t+1$. However in the case where we do not observe accidents, the relevant transition probability that we use is $P(\delta_{t+1}|\delta_t, \tau, \theta)$, i.e. the transition probability for the household's choices of cars, which is necessitated by the fact that we do not fully observe the car state x_t at each time period due to the fact that accidents are unobserved.

E Notes on the identification of the model with driving

This appendix is a short note on the identification of the model when we allow for driving, with a linear specification for the predicted optimal driving that can depend on the age of the car. Consider a utility function of the form

$$u(a,x) = \psi_0 + \psi_1 a + \psi_2 a^2 + (\gamma_0 + \gamma_1 a)x - \mu p x + \frac{\phi}{2}x^2 \tag{80}$$

where a is the age of the car, x is vehicle kilometers driven each period, and the parameter vector is $\theta = (\psi_0, \psi_1, \psi_2, \gamma_0, \gamma_1, \mu, \phi)$. We can consider the sum of the first three terms on the right hand side of equation (80) to be the utility of ownership *per se,* i.e. the utility a consumer gets from the pure ownership (and option value to drive) even if the household does not do any driving, $x = 0$. The remaining components are the net utility from driving, after deducting the cost of driving (translated into utility terms by multiplying by the marginal utility of money, μ). There are seven parameters to be identified for each consumer type/car type in the model, and for notational simplicity we have suppressed the dependence of all θ parameters except for μ (the marginal utility of money or price-responsiveness coefficient) since we assume that μ depends on consumer type τ but not on the car type j.

We want to consider the identification of these parameters from observations on: 1) consumer trading of automobiles, and 2) observed driving. The symbol p is the per kilometer cost of fuel plus any taxes, and in our model there is *no variation in this price, either over time or across consumers.* However there is variation in p across car types due to different fuel efficiency of different types and ages of cars. However we really cannot use variation in p as a source of identification of the model parameters if we are being fully faithful to the model, which currently does not allow any time series or cross sectional variation in p except over car types as noted above. Fortunately, we now show that we can identify the parameters using the information on car trading, and in a way that is "just identified" so we don't face a trade-off in terms of fitting the model of car driving or the model of car trading: each can be estimated separately and the structural parameters that imply the best-fitting "reduced-form" specifications for driving and car trading are derived below.

We derive the indirect utility function first, by using the utility function above to calculate the optimal level of driving,

$$x^*(p,a) = -\frac{1}{\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]. \tag{81}$$

Equation (81) is the "structural driving equation" implied by the direct utility function $u(x,a)$ in equation (80). But there is a corresponding "reduced form" or unrestricted driving equation which we denote by

$$x = d_0 + d_1 a + d_2 p. \tag{82}$$

Though there is no identifying variation in p we do have identifying variation in a, so we can separately identify the constant term in the driving equation in (81) and the age coefficient. So we can treat the age coefficient γ_1/ϕ as known given knowledge of observed driving $x^*(p,a)$ and also the constant term $[\gamma_0 - \mu p]/\phi$, though at this point we cannot separately identify all the parameters from only these two identified coefficients from the "driving equation." Note that optimal driving must be positive, so this implies an additional inequality restriction on the parameters,

$$\gamma_0 + \gamma_1 a \geq \mu p, \quad a = \{0, 1, \ldots, \bar{a} - 1\} \tag{83}$$

and if, as we expect, $\gamma_1 < 0$, then the set of restrictions below can be reduced to this single inequality restriction at the last age consumers are allowed to own cars

$$\gamma_0 + \gamma_1(\bar{a} - 1) \geq \mu p \tag{84}$$

but if this inequality is violated, then it is easy to see that $x^*(p,a) = 0$ and the consumer gets zero utility from owning a car (and hence absent extreme value shocks would not buy one since the utility from the outside good is normalized to zero, and there are purchase price and transactions costs involved in buying a car).

Now plugging the optimal driving back into utility, we can derive the indirect utility function,

$$\begin{aligned} u(a, x^*(p,a)) = v(a,p,\tau) &= \psi_0 + \psi_1 a + \psi_2 a^2 - \frac{1}{2\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]^2 && (85) \\ &= u_0 + u_1 a + u_2 a^2 && (86) \end{aligned}$$

where

$$\begin{aligned} u_0 &= \psi_0 - \frac{1}{2\phi}\left[\gamma_0 - \mu p\right]^2 \\ u_1 &= \psi_1 - \frac{\gamma_1}{\phi}\left[\gamma_0 - \mu p\right] \\ u_2 &= \psi_2 - \frac{1}{2\phi}\left[\gamma_1^2\right]. \end{aligned} \tag{87}$$

We can consider the coefficients (u_0, u_1, u_2) and the marginal utility of money as identified from an unrestricted or "reduced form" dynamic discrete choice model of car trading. We now argue that the seven parameter specification of utility $u(x, a, \theta)$ given in equation (80) is just identified from unrestricted estimation of the reduced-form driving equation (82) and the dynamic discrete choice model of car trading. First, assume we can identify the marginal utility of money, μ, from estimation of the latter model. Then there are only 6 remaining structural parameters to be identified, $(\psi_0, \psi_1, \psi_2, \gamma_0, \gamma_1, \phi)$ and these are determined from the following 6 equations that provide enough flexibility to ensure perfect unrestricted fit of both the reduced-form driving equation (parameters (d_0, d_1, d_2)) and the reduced-form dynamic discrete choice model (parameters (u_0, u_1, u_2) plus the marginal utility parameter μ).

$$\begin{aligned} d_0 &= -\frac{\gamma_0}{\phi} \\ d_1 &= -\frac{\gamma_1}{\phi} \\ d_2 &= \frac{\mu}{\phi} \end{aligned} \tag{88}$$

Thus, given the estimate of the marginal utility of money $\hat{\mu}$ from the dynamic discrete choice model, we can back out $\hat{\phi}$ from the last equation of (88), and thus also $(\hat{\gamma}_0, \hat{\gamma}_1)$. Then given these parameter estimates we can determine the parameters $(\hat{\psi}_0, \hat{\psi}_1, \hat{\psi}_2)$ from equation (87) in a way that entails no restrictions on the estimation of the coefficients (d_0, d_1, d_2) of the reduced-form driving model (82) or the coefficients (μ, u_0, u_1, u_2) of the reduced-form dynamic discrete choice model of car trading.

To derive (86) we assumed that inequality (84) holds so that the consumer would want to do some positive driving at all possible ages. However if the inequality does not hold at all ages, then $u(a, x^*(p, a))$ is given by (86) only for ages a satisfying inequality (83) and for all higher ages $a \geq \hat{a}$ (where $\hat{a}$ is the largest age for which inequality (86) holds), then $x^*(x, p) = u(a, x^*(p, a)) = 0$ for all $a > \hat{a}$. Notice that $u_0 \geq 0$ and $u_2 \geq 0$ and if $\gamma_1 < 0$, then $u_1 \leq 0$. Thus, consumer preferences over cars of different ages are expected to be decreasing and convex in the age of the car. The strict convexity in age is required in order to imply a finite amount of driving: i.e. if $\phi = 0$ (so indirect utility is linear in a) then $x^*(p, a)$ is predicted to be infinite for any a where the inequality restriction (83) is strict.

Now let's suppose that we could identify the four parameters (u_0, u_1, u_2, μ) by estimating a model of automobile trading that ignored driving, but with a utility function that is quadratic in the age of the car as in equation (86). We presume that the marginal utility of money parameter μ is identified by the variation in trading cars of different ages using known prices of used cars, $P(a)$, $a = 0, \ldots, P(\bar{a} - 1)$ that are treated as data. Even though these prices are the same for all consumers, there is variation over ages of different cars and combined with the scrappage decision, we will assume these coefficients can be identified.

Now given the additional observation on driving are the other coefficients $(\gamma_0, \gamma_1, \phi)$ identified? The answer is yes. As we noted above, the ratio γ_1/ϕ is identified from the driving equation (81), and the coefficient γ_1^2/ϕ is identified as the quadratic coefficient u_2 from the dynamic discrete choice model of vehicle trading with the quadratic in age utility function given in (86). Thus γ_1 is identified as the ratio of these two coefficient estimates. Next, given γ_1 we can identify ϕ from either the linear term of the driving equation, $-\gamma_1/\phi$ or the coefficient u_2 on the a^2 term of the utility of the dynamic discrete choice model, $u_2 = -\gamma_1^2/(2\phi)$. Then using the equation for u_1, the coefficient on a in the dynamic discrete choice model, we can identify γ_0. Thus the parameter vector $\theta = (\gamma_0, \gamma_1, \mu, \phi)$ is in fact overidentified since there are additional restrictions implied by the parameters via the constant term in the driving equation, $x^*(0, p)$ and the constant term u_0 in the utility function in the dynamic discrete choice model.

These parameters could be estimated by jointly by maximum likelihood, if we treat observed driving as affected by measurement error. But doing this would require estimation of an additional parameter for the variance of the measurement error in driving. We would also have to take seriously the inequality restriction (83) and make sure that we don't ignore it and get strange results from squaring negative predicted driving, rather than carefully obeying the inequality restriction that implies cars provide zero direct utility when predicted driving is negative.

To get started, I would favor an "indirect least squares" sort of approach where we just estimate the coefficients (u_0, u_1, u_2, μ) of the dynamic discrete choice ingoring driving and make sure we can fit the pattern of trading in cars well. Then with these estimated, we can estimate a linear driving equation and "back out" the implied $(\gamma_0, \gamma_1, \phi)$ coefficients via an "indirect least squares" approach and check that they are reasonable. This would be the "second step". Finally for more efficient parameter estimates, we could estimate the model "structurally" (either by ML or by MM) that impose the "cross equation restrictions" from the observations on driving and observations on car trading using the indirect least squares parameters as starting points and then recasting the parameters of the model directly in terms of the "deep structural parameters" that allow for driving, $\theta = (\gamma_0, \gamma_1, \mu, \phi)$.

F Estimation details

F.1 Data and Institutional Details

The data comes from the Danish demographic registers and covers the period 1996 to 2008. The dataset covers the universe of all Danish households and all cars owned by private individuals. Driving information is obtained from odometer readings that occur when cars are taken to mandatory driving inspections biannually starting from a car age of four.[31] Fuel prices come from eof.dk and are a country-level average.

Car types: Cars are aggregated into four discrete types: we first split cars based on whether the car's weight is above or below the median for that car's vintage cohort, and then within each of those two subsamples, we further split cars based on whether they are above or below the median fuel efficiency. This way, we get four classes of cars, that we name "green" or "brown" for high and low fuel efficiency respectively, and "heavy" and "light" for high and low weights, respectively. Making the splits separately by vintage has the benefit that the distribution of car types is roughly constant over time, but has the drawback that car attributes for the four classes are not constant.[32] Therefore, we simply take the average of the car attributes for each of the four types in our model. Finally, we split cars into 25 age groups from brand new to 24 years old, where the last age category captures cars of 24 years or older. Table 3 presents key summary statistics for our four car types, aggregated over all the years of our sample as well as all the different car ages.

Household types: Households are split into 8 different types based on whether the household is a couple or single, has high or low work distance, and whether income is high or low based. Income splits are based on the median income in the demographic cell. Work distance comes from the Danish tax deduction for travel distance to work. This deduction is only applicable for full-time workers living further than 12 km from their work place (each way), and slightly under half of Danes have high work distance by this definition, although it differs quite a bit by cohabitation status. Table 2 presents summary statistics, where we have computed the weighted average over the years of our

[31] That is, a car is inspected at ages 4, 6, 8, 10, and so forth.

[32] In reality, technological progress implies that car attributes are improving over time as engines can drive further per liter of fuel. To accomodate our model's stationary nature, we ignore this "attribute inflation".

data for each household type to simplify the exposition. We also present averages for the number of kids in the household.

The registration tax paid upon the purchase of a new car in Denmark is among the highest in the world. It is a linear tax that has a kink, K, with one rate, τ_1, below K and a higher rate, $\tau_2 > \tau_1$, applying to any price above K. Finally, 25% VAT is paid of the price including the tax. So if the price before the registration fee is given by P, then the registration fee to be paid is given by $T(P) = 1.25[\tau_1 \min(P, K) + \tau_2 \times \max(P - K, 0)]$. In 2008, K was 81,000 DKK (around 16,000 USD), τ_1 was 105% and τ_2 was 180%. In our counterfactuals, we lower τ_1 and τ_2 to half their initial values. There are also annual taxes for car ownership as well as mandatory insurance costs, which we abstract from.

We use a social cost of carbon of US$50/ton (290 DKK) and the other external costs per kilometer travelled are valued at 0.6216 DKK/km and they consist of noise, accidents, congestion and local air pollution, as measured by Transport (2010).

We do not observe scrappage in our data *per se*. Instead, we define a vehicle as having been scrapped if an ownership spell ends and no other ownership spell ever begins afterwards. Since our extract of the ownership register comes from September of 2011 while our last sample year is 2008, this means that a car should have been without owner for 3 years, which typically means it has been scrapped. Note also that we do not observe whether a car was involved in an accident in the data, although our model will make a distinction between accidents and voluntary scrappage decisions.

F.2 Estimation

As explained in Section 6, estimation is composed of three steps:

1. Estimate the reduced form driving parameters from (82) using linear regression: the estimates are in Table 6.
2. Estimate the reduced form dynamic discrete choice parameters from (86) using Maximum Likelihood: the estimates are in Tables 4, 5 and 7 to 10.
3. Back out the "deep structural parameters" $\theta_{\tau,j} = (\psi_{\tau,j,0}, \psi_{\tau,j,1}, \psi_{\tau,j,2}, \gamma_{\tau,j,0}, \gamma_{\tau,j,1}, \phi_{\tau,j})$ for each of the 8 consumer types τ and 4 car types j using equations (87) and (88) of Appendix E.

Table 2: Summary Statistics for Households

τ	Name	N	Income	1(Single)	Work distance	Age	1(Urban)	No. kids
1	Low WD, Couple, Poor	6,500,464	311.68	0.00	0.00	55.03	0.22	0.48
2	Low WD, Couple, Rich	6,352,821	777.19	0.00	0.00	46.38	0.21	1.03
3	Low WD, Single, Poor	7,906,100	109.92	1.00	0.00	54.21	0.35	0.11
4	Low WD, Single, Rich	7,666,452	301.15	1.00	0.00	48.21	0.33	0.20
5	High WD, Couple, Poor	4,031,412	494.61	0.00	34.63	40.58	0.12	0.99
6	High WD, Couple, Rich	3,862,441	862.43	0.00	42.13	43.57	0.12	1.21
7	High WD, Single, Poor	1,217,611	215.04	1.00	26.71	33.85	0.25	0.22
8	High WD, Single, Rich	1,171,919	413.24	1.00	32.98	41.14	0.22	0.24

Note: The column "N" denotes the observations of each household type available across all the years, 1996–2008. The remaining variables are all weighted averages of the corresponding variables with the annual observation counts as weights. Household types are defined based on splitting the sample into cells based on single/couple status, whether work distance is zero or positive, and finally splitting households in two depending on income within the cell is above or below the median. Work distance is based on a travel tax deduction, and it is only positive if one of the household members has more than 12 km to work (each way), and so it is naturally zero for unemployed. The urban dummy is equal to one for the six largest cities in Denmark: Copenhagen, Frederiksberg, Aarhus, Aalborg, and Odense.

Table 3: Summary Statistics for Cars

	No car	light, brown	light, green	heavy, brown	heavy, green
Obs.	16,895,290	4,683,737	5,594,897	5,351,904	6,183,392
Fuel efficiency (km/l)	–	12.84	15.06	9.89	12.63
Weight (tons)	–	1.42	1.28	1.96	1.64
Price, new (1000 DKK)	–	142.33	110.09	275.84	205.25
Diesel share	–	0.00	0.08	0.14	0.21
Depreciation Factor	–	0.87	0.87	0.87	0.87

Note: The four car categories are defined by first splitting cars into two groups based on weight, and then on fuel efficiency within each weight sub-group. The splits are made separately for every car vintage, implying that the attributes of, say, a "light, green" car is changing over time. The variable "Depreciation Factor" is a suggested annual depreciation factor set by the Danish Automobile Dealer Association. The rate varies across cars but not over time, implying that the association uses a constant exponential discounting rule.

Table 4: Maximum Likelihood Estimates of Equilibrium Model

$\alpha_{\tau,j,2}$: coefficients of binary logit model of accident rates				
	light, brown	light, green	heavy, brown	heavy, green
intercept	-5.6248	-6.0443	-5.6728	-5.7826
	(0.0119)	(0.0105)	(0.0095)	(0.0088)
age slope	0.1804	0.2216	0.2020	0.2048
	(0.0015)	(0.0011)	(0.0009)	(0.0009)

Table 5: Maximum Likelihood Estimates of Equilibrium Model

utility costs of transacting		
	Estimate	Standard Error
σ_s	0.3454	(0.3454)
sales transaction cost	0.9106	(0.9106)
sales transaction cost (inspection year)	-2.1929	(-2.1929)

Table 6: Driving Regression Parameter Estimates

Dependent variable: driving (1,000 km per year)			
γ_0	Intercept	19.07	(0.58)
$\hat{\gamma}_1^a/\phi_\tau$	Car age	-0.1325	(0.03)
$\hat{\gamma}_2^a/\phi_\tau$	Car age squared	-0.001975	(0.00)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Couple, Rich	4.43	(0.70)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Poor	-3.752	(1.24)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Rich	-0.0325	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Poor	9.825	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Rich	12.33	(0.74)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Poor	6.436	(1.52)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Rich	12.23	(1.27)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: light, green	-1.994	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, brown	4.345	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, green	3.606	(0.14)
μ/ϕ	Price (common)	-7.074	(0.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Couple, Rich	-4.111	(1.02)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Poor	4.732	(1.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Rich	0.2781	(1.16)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Poor	-6.41	(1.17)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Rich	-9.892	(1.09)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Poor	-1.714	(2.29)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Rich	-9.007	(1.91)
N	Driving periods	19,635,940	

Note: The parameters are estimated as described in Section 6.2. The first stage uses a rich dataset of driving and is a linear regression of driving on covariates. Each observation is a driving period, where the outcome is the driving in 1,000 km/year, computed using the first difference in odometer measurements from one safety inspection to the next divided by time between. Standard errors in parentheses. The mean of the outcome in our sample is 17.6 (1,000 km/year), and the mean of the price is 0.67 (DKK/km).

Table 7: Maximum Likelihood Estimates of Equilibrium Model

μ_τ: marginal utility of money	
Low WD, Couple, Poor	0.1131
	(0.0007)
Low WD, Couple, Rich	0.1119
	(0.0007)
Low WD, Single, Poor	0.0941
	(0.0007)
Low WD, Single, Rich	0.1077
	(0.0007)
High WD, Couple, Poor	0.1036
	(0.0006)
High WD, Couple, Rich	0.1155
	(0.0007)
High WD, Single, Poor	0.0920
	(0.0008)
High WD, Single, Rich	0.1081
	(0.0008)

Table 8: Maximum Likelihood Estimates of Equilibrium Model

$u_{\tau,j,0}$: intercept in quadratic indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Couple, Poor	3.6490	3.1132	5.1535	4.7760
	(0.0178)	(0.0179)	(0.0281)	(0.0257)
Low WD, Couple, Rich	4.0324	3.4657	5.7492	5.3478
	(0.0172)	(0.0173)	(0.0271)	(0.0247)
Low WD, Single, Poor	2.4042	2.1504	3.5823	3.2068
	(0.0191)	(0.0178)	(0.0298)	(0.0274)
Low WD, Single, Rich	3.2454	2.8222	4.6934	4.2829
	(0.0175)	(0.0174)	(0.0275)	(0.0252)
High WD, Couple, Poor	3.8821	3.4351	5.3199	5.0561
	(0.0164)	(0.0164)	(0.0258)	(0.0235)
High WD, Couple, Rich	4.7620	4.3185	6.5617	6.2375
	(0.0182)	(0.0183)	(0.0285)	(0.0260)
High WD, Single, Poor	2.6685	2.4290	3.7554	3.5129
	(0.0206)	(0.0188)	(0.0324)	(0.0295)
High WD, Single, Rich	3.5538	3.1741	4.9946	4.6985
	(0.0204)	(0.0194)	(0.0319)	(0.0291)

Table 9: Maximum Likelihood Estimates of Equilibrium Model

$u_{\tau,j,1}$: coefficient of age in quadratic indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Couple, Poor	-0.1459	-0.0922	-0.2196	-0.1717
	(0.0009)	(0.0011)	(0.0014)	(0.0012)
Low WD, Couple, Rich	-0.1586	-0.0985	-0.2411	-0.1953
	(0.0009)	(0.0011)	(0.0013)	(0.0012)
Low WD, Single, Poor	-0.0984	-0.0615	-0.1600	-0.1128
	(0.0009)	(0.0010)	(0.0014)	(0.0012)
Low WD, Single, Rich	-0.1308	-0.0841	-0.2056	-0.1544
	(0.0009)	(0.0011)	(0.0014)	(0.0011)
High WD, Couple, Poor	-0.1390	-0.0825	-0.2134	-0.1696
	(0.0008)	(0.0010)	(0.0013)	(0.0011)
High WD, Couple, Rich	-0.1642	-0.1069	-0.2551	-0.2074
	(0.0009)	(0.0011)	(0.0014)	(0.0012)
High WD, Single, Poor	-0.1108	-0.0707	-0.1732	-0.1273
	(0.0010)	(0.0010)	(0.0016)	(0.0013)
High WD, Single, Rich	-0.1493	-0.1007	-0.2208	-0.1735
	(0.0010)	(0.0011)	(0.0016)	(0.0013)

Table 10: Maximum Likelihood Estimates of Equilibrium Model

utility costs of transacting		
	common	no car
Low WD, Couple, Poor	6.5944	1.7899
	(0.0226)	(0.0029)
Low WD, Couple, Rich	6.4425	1.0719
	(0.0229)	(0.0030)
Low WD, Single, Poor	6.5457	3.0816
	(0.0186)	(0.0045)
Low WD, Single, Rich	6.6036	2.5769
	(0.0212)	(0.0031)
High WD, Couple, Poor	6.2644	0.7793
	(0.0207)	(0.0036)
High WD, Couple, Rich	6.4237	0.1742
	(0.0237)	(0.0045)
High WD, Single, Poor	6.0303	2.3383
	(0.0182)	(0.0066)
High WD, Single, Rich	6.2786	1.7884
	(0.0220)	(0.0064)