Lecture 15
Modelling Equilibrium in Durable Goods Markets

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Road Map: Equilibrium models

1. Introduction and literature overview
   John Rust

2. Stationary equilibrium with transaction cost and consumer heterogeneity
   Fedor Iskhakov

3. Modelling Danish secondary market of automobiles
   Bertel Schjerning

Equilibrium models in human capital literature

- Chao Fu (2014) *JPE* "Equilibrium Tuition, Applications, Admissions, and Enrollment in the College Market"
- Formulates a model of applications and admissions to college in the US and estimates the structural model by SMM
    1. Stage 1: Colleges choose tuition and financial aid policies
    2. Stage 2: Students choose *subsets* of the set of all colleges to apply
    3. Stage 3: Colleges choose *subsets* of the set of applicants to admit
    4. Stage 4: Students choose the college they will attend from the set of colleges they are admitted to
- "Colleges, observing noisy measures of student ability, choose tuition and admissions policies to compete for better students. Tuition, applications, admissions, and enrollment are joint equilibrium outcomes. I estimate the model using the NLSY97 via a three-step procedure to deal with potential multiple equilibria. I use the model to examine the extent to which college enrollment can be increased by expanding college supply and to assess the importance of various measures of student ability."

A brief history of auto models

1. Manski (1982) and Berkovec (1985): were the first to develop models of equilibrium in the used car market by microaggregation, but used static discrete choice models
2. Rust (1985): provided a dynamic framework for equilibrium prices and quantities, and showed that when *transactions costs are zero*, consumers trade every period for an optimal car, so the dynamic problem reduces to a static one
3. Konishi and Sandfort (2002): generalized Rust's analysis to allow for positive transaction costs and proved the existence of equilibrium, allowing also for multiple makes/models of cars
4. Gavazza *et. al.* (2014): numerically calculated equilibrium with one car type and discrete ages/qualities of cars and analyzed the impact of the secondary market with varying levels of transaction costs

James A. Berkovec 1957-2009

1. "New Car Sales and Used Car Stocks: A Model of the Automobile Market" *RAND* 1985.
2. Estimated a discrete choice model of demand for cars using the *National Transportation Survey* – micro data on 1095 households in 1978
3. Using estimated demands for cars, he solved for equilibrium in the new and used car market, over 131 type/age classes, with 13 types and 10 ages (vintages) from 1969 to 1978 plus a residual (131st) category of all cars produced prior to 1969 (clunkers)

Berkovec's approach to calculating equilibrium

1. Let P be the 131×1 vector of prices of the 131 car type/age classes.
2. Let $E(P)$ be the 131×1 vector of excess demands for these 131 vehicles implied by the price vector P
3. An equilibrium in the auto market is a vector P^* satisfying $E(P^*) = 0$.
4. Berkovec defined $E(P) = D(P) - S(P) - Q - S(P)$ where Q is the vector of stocks of cars, and $S(P)$ is the number cars scrapped
5. Berkovec used a quasi-Newton method to find a P^* that satisfies $E(P^*) = 0$

$$P' = P - \lambda[\nabla E(P)]^{-1} E(P) \tag{1}$$

where $\lambda \geq 0$ is a scalar *step size.*

A Dynamic Model of the Auto Market: Rust (1985)

1. Rust (1985) *Econometrica* "Stationary Equilibrium in a Market for Durable Goods" introduced a *durable asset pricing model* that includes particular durable goods such as cars that trade both on a *primary market* (new cars) and a *secondary market* (used cars)
2. "the essential benefit of a secondary market is to create dynamic trading opportunities similar to a securities market: the consumer can hold the current durable for any desired length of time and has the opportunity of trading it for a new asset or choosing from an array of used assets of different ages and physical conditions." (Rust, 1985, p. 783)

Equilibrium with a continuum of goods/consumers

1. Assume an infinitely elastic supply of new cars at price $\overline{P}$ and an infinite demand for cars by *scrappers* at price $\underline{P}$.
2. Suppose the condition car is summarized by its odometer x (no *lemon's problems*). Let $\phi(y|x)$ be the probability density of next period odometer given this period's value x. Example: $\phi(y|x) = \lambda \exp\{-\lambda(y-x)\}$ if $y \geq x$, 0 otherwise.
3. Consumers are *heterogeneous, infinitely lived,* must always own 1 car, and have *quasi-linear utility functions*

   $$U(x, I, \tau) = a(\tau)I + q(x, \tau)$$

   where τ is the *type* of consumer, I is income, and x is the age/odometer of the car.
4. Let $m(x, \tau)$ be the maintenance costs incurred by consumer τ.

Equivalent cost-minimization formulation

1. Let $P(x)$ be an *equilibrium price function.* It should satisfy $P(0) = \overline{P}$ and $P(x) = \underline{P}$ where $x \geq \gamma$ where γ is the *scrapping threshold*. Let $T(x)$ be the *transactions cost* that a customer incurs when they sell a car of odometer x.
2. We can reframe the *car trading problem*, the consumer's dynamic utility maximization problem for owning an infinite sequence of autos, as a cost minimization problem using the "cost function" $G_\tau(x, d)$ given by

$$G_\tau(x, d) \quad = \quad \begin{cases} M(x, \tau) & \text{if} \quad d = x \\ M(z, \tau) + P(z) - P(x) + T(x) & \text{if} \quad d = z \end{cases}$$

where $M(x, \tau) = m(x, \tau) - q(x, \tau)/a(\tau)$ is the disutility of owning a car of odometer x expressed as a "generalized maintenance cost"

Bellman equation for the consumer

1. Let $V_\tau(x)$ be consumer τ's optimal discounted cost of owning a car of odometer x. The Bellman equation is given by

$$
\begin{aligned}
V_\tau(x) \quad = \quad & \min \Big[M(x,\tau) + \beta E V_\tau(x), \\
& \inf_{0 \le z \le \gamma} M(z,\tau) + P(z) - P(x) + T(x) + \beta E V_\tau(z) \Big]
\end{aligned}
$$

   where

$$
EV_\tau(x) = \int_x^\infty V_\tau(y)\phi(y|x)dy.
$$

2. **Theorem** *Assume that transactions cost are zero, $T(x) = 0\ \forall x \ge 0$. Then it is optimal for each consumer τ to trade each period for their optimal car $z^*(\tau)$ given by*

$$
z^*(\tau) = \underset{0 \le z \le \gamma}{argmin} \left[M(z,\tau) + P(z) + \beta E V_\tau(z) \right].
$$

Implication of Zero Transactions Cost

1. **Lemma** If transactions costs are zero, $T(x) = 0$, then we have

   $$V_\tau(x) = \left[M(x,\tau) + P(z^*(\tau)) - \beta EP(z^*(\tau))\right]/(1-\beta) - P(x)$$

   where $z^*(\tau)$ is given by

   $$z^*(\tau) = \underset{0 \leq z \leq \gamma}{argmin}\left[M(z,\tau) + P(z) - \beta EP(z)\right]. \tag{2}$$

2. What is no secondary market existed? Consumer's problem is now a *regenerative optimal stopping problem*

   $$V_\tau(x) = \min\left[M(x,\tau) + \beta EV_\tau(x), M(0,\tau) + \overline{P} - \underline{P} + \beta EV_\tau(0)\right].$$

   Optimal strategy is to keep the car x unless $x > \gamma(\tau)$ where $\gamma(\tau)$ is the solution to

   $$\overline{P} - \underline{P} + V_\tau(0) = V_\tau(\gamma(\tau)).$$

Definition of Stationary Equilibrium

1. **Definition** The triple $\{P, F, \gamma\}$ is a *stationary equilibrium* if
2. $P(0) = \overline{P}$ and $P(x) = \underline{P}$ if $x \geq \gamma$
3. $F(x|\gamma)$ is a *stationary holdings distribution* the unique solution to

$$F(x|\gamma) = \int_0^\infty \left[1 - \Phi(\gamma|x') + \Phi(x|x')\right] F(dx'|\gamma)$$

4. Each consumer τ chooses an optimal car $z^*(\tau)$ given by equation (2) and $\forall x \in [0, \gamma]$ we have

$$F(x|\gamma) = \int_{\underline{\tau}}^{\overline{\tau}} I\left\{\tau | z^*(\tau) \leq x\right\} H(d\tau) \tag{3}$$

Intuitive Interpretation of Holdings Distribution $F(x|\gamma)$

1. Rust (1985) "Equilibrium Holding Distributions in Durable Asset Markets" *Transportation Research Part B: Methodological*
2. Using *renewal theory* Rust characterized $F(x|\gamma)$ as

$$F(x|\gamma) = \frac{\text{mean first passage time from 0 to } (x,\infty)}{\text{mean first passage time from 0 to } (\gamma,\infty)} \tag{4}$$

3. **Example:** if $\Phi(x'|x) = 1 - \exp\{-\lambda(x'-x)\}I\{x' \geq x\}$, then

$$\text{mean first passage time from 0 to } (x,\infty) = (1+\lambda x) \tag{5}$$

so

$$F(x|\gamma) = \frac{1+\lambda x}{1+\lambda\gamma} \tag{6}$$

Stationary Equilibrium: Homogeneous Consumers

1. If all consumers have same τ, they must be indifferent about which car to buy

$$M(x,\tau) + P(x) - \beta EP(x) = \text{constant}$$

2. This implies that P is the solution to the following *functional equation*

$$P(x) = \max\left[\underline{P}, \overline{P} - \beta EP(0) - M(0,\tau) - M(x,\tau) + \beta EP(x)\right]$$

3. **Theorem** $P(x) = \overline{P} - [V_\tau(x) - V_\tau(0)]$.

Stationary Equilibrium: Homogeneous Consumers

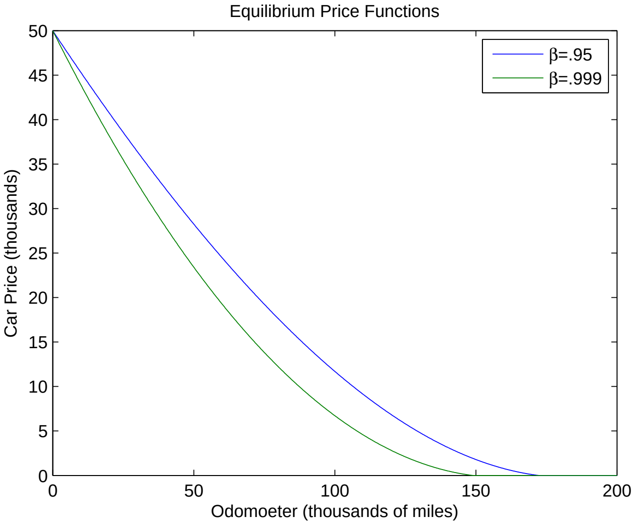
1. **Theorem** If $\Phi(y|x) = \max[0, 1 - \exp\{-\lambda(y - x)\}]$, then

$$
\begin{aligned}
P(x) &= \max\Bigg[\underline{P}, \underline{P} + \\
&\qquad \frac{1}{1-\beta}\int_x^\gamma M'_\tau(y)\,[1 - \beta\exp\{-\lambda(1-\beta)(y-x)\}]\,dy\Bigg] \\
F(x|\gamma) &= \frac{1+\lambda x}{1+\lambda\gamma} \quad x \in [0, \gamma]
\end{aligned}
$$

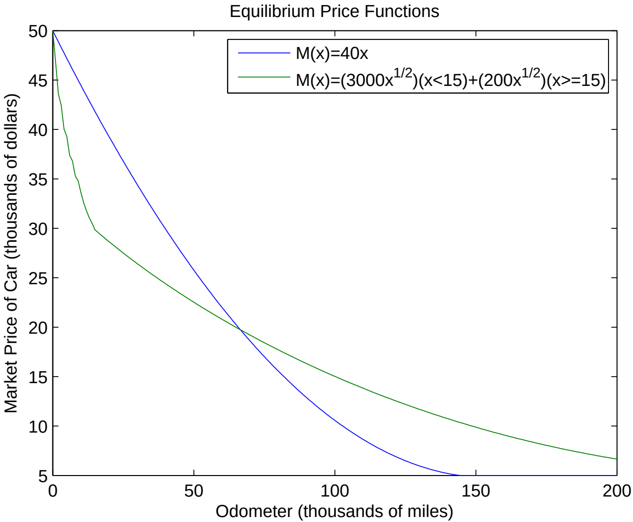
where γ is the unique solution to

$$
\overline{P} - \underline{P} = \frac{1}{1-\beta}\int_0^\gamma M'(y,\tau)\,[1 - \beta\exp\{-\lambda(1-\beta)(y-x)\}]\,dy
$$

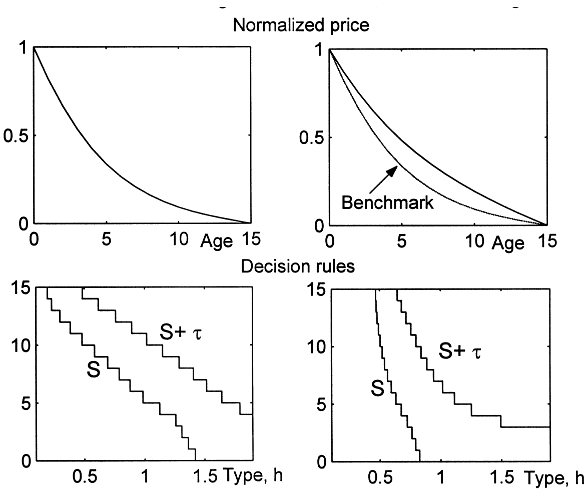
Equilibrium Price Functions, effect of discount factor β



Equilibrium Price Functions, effect of maintenance costs M



Stolyarov, *JPE* 2002



Esteban and Shum, *RAND* 2007

1. "We study the effects of durability and secondary markets on equilibrium firm behavior in the car market.
2. We construct a dynamic oligopoly model of a differentiated product market to incorporate the equilibrium production dynamics that arise from the durability of the goods and their active trade in secondary markets.
3. We derive an econometric model and estimate its parameters using data from the automobile industry over a 20-year period. Our estimates are used to provide a measure of the competitive importance of the secondary market."
4. For tractability, used a quality ladder model and assumed linear quadratic preferences and zero transactions costs
5. "Overall, we find that aggregate new-car production would increase by 12.08% for the 1987-1990 time frame were the secondary market to disappear temporarily."

Part II: Stationary equilibrium with transaction cost and consumer heterogeneity: theory

Modeling equilibrium trade in automobile markets

We extend Rust (1985) stationary flow equilibrium model

- Dynamic discrete choice equilibrium model
- Unit mass of (homogeneous) consumers
- Stationary flow equilibrium
- No transactions costs
- Continuous quality of durable good (odometer reading)

1. Add outside option
2. Add transactions cost
3. Add endogenous scrappage decisions at any car age
4. Allow for several types of consumer heterogeneity
5. Allow for multiple makes/models of cars of different vintages
6. Flexible utility specification with the possibility for driving and demand for gasoline in the applications

Baseline setup: idiosyncratic consumer heterogeneity

- Unit mass of consumers who live in the infinite horizon
- $i \in \{1, \ldots, J\}$ make/models of cars, age $a \in \{0, 1, \ldots, \bar{a}\}$
- Cars must be scrapped at age $\bar{a}$
- Car owners can:
  1. trade their current car (i, a) for a car (j, d)
  2. keep their current car, $(d = \kappa)$ (if $a < \bar{a}$)
  3. purge their current car (i, a) and remain without a car $i = \o$
- When existing car (i, a) is replaced or purged, there is additional *endogenous scrappage* choice:
  1. sell $s = 0_s$ the existing car (i, a) for the market price $P(i, a)$
  2. scrap $s = 1_s$ the existing car (i, a) at fixed buy-in price $\underline{P}$

Timing of events

1. Consumer enter the period with a car of make/model i and age a, or without a car $i = \emptyset$
2. Discrete trading/keeping $j \in \{\kappa, \emptyset, 1, \ldots, J\}$, $d \in \{0, 1, \ldots, \bar{a} - 1\}$, and scrappage $s \in \{0_s, 1_s\}$ choice immediately at the start of the period
3. The chosen car (j, d) is then utilized during the period, but can be involved in the accident with probability α
4. By the start of the next period:
   - with probability $1 - \alpha$ car has aged $a = d + 1$
   - with probability α car reach scrappage state $a = \bar{a}$

- Hence, it is impossible to start the period with a brand new car $\rightarrow$ the state variable a takes values from $a \in \{1, \ldots, \bar{a}\}$

Utility of car ownership and consumer heterogeneity

$$\text{Utility} = u(i,a) - \mu\big[\text{operating costs} + \text{trade and transaction costs}\big] + \epsilon$$

- Car utility $u(i,a)$ is a decreasing function of car age a that reflects
  - decreasing utility of car services
  - increasing cost of maintenance
- Marginal utility of money μ

Unit mass of idiosyncratically heterogeneous consumers

- **Extreme value** consumer types (taste shifters)
- GEV specification for $\epsilon \rightarrow$ nested choices to allow correlation between alternatives
- Logit choice probabilities and analytic expectations

Consumer's trading problem: no car or terminal age car

$$V(\emptyset, \epsilon) = \max \left[v(\emptyset, \emptyset) + \epsilon(\emptyset); \max_{\substack{j \in \{1,\dots,J\}, \\ d \in \{0,1,\dots,\bar{a}-1\}}} \left[v(\emptyset, j, d) + \epsilon(j, d) \right] \right]$$

$$V(i, \bar{a}, \epsilon) = \max \left[v(i, \bar{a}, \emptyset) + \epsilon(\emptyset); \max_{\substack{j \in \{1,\dots,J\}, \\ d \in \{0,1,\dots,\bar{a}-1\}}} \left[v(i, \bar{a}, j, d) + \epsilon(j, d) \right] \right]$$

where $v(\emptyset, j, d)$ and $v(i, \bar{a}, j, d)$ are values of trading to car of make/model j and age $d \in \{1, \dots, \bar{a} - 1\}$

Consumer's trading problem: car owners

$$V(i,a,\epsilon) = \max \left\{ \begin{array}{l} v(i,a,\kappa) + \epsilon(\kappa); \\[2ex] \max_{s\in\{1_s,0_s\}} \left[v(i,a,\emptyset,s) + \epsilon(\emptyset,s) \right]; \\[2ex] \max_{\substack{j\in\{1,\dots,J\},\\ d\in\{0,1,\dots,\bar{a}-1\},\\ s\in\{1_s,0_s\}}} \left[v(i,a,j,d,s) + \epsilon(j,d,s) \right] \end{array} \right\}$$

When existing car (i,a) is replaced, there is additional scrappage choice $s \in \{0_s, 1_s\}$: to sell or to scrap the replaced car.

Choice specific value functions for every state and choice

$$
\begin{aligned}
v(\emptyset,\emptyset) &= u(\emptyset) + \beta EV(\emptyset) \\
v(\emptyset,j,d) &= u(j,d) - \mu\big[P(j,d) + T_b(P,j,d)\big] + \\
&\quad + \beta\big(1-\alpha\big) EV(j,d+1) + \beta\alpha EV(j,\bar{a}) \\
v(i,\bar{a},\emptyset) &= u(\emptyset) + \mu\,\underline{P}(i) + \beta EV(\emptyset) \\
v(i,\bar{a},j,d) &= u(j,d) - \mu\big[P(j,d) - \underline{P}(i) + T_b(P,j,d)\big] + \\
&\quad + \beta\big(1-\alpha\big) EV(j,d+1) + \beta\alpha EV(j,\bar{a}) \\
v(i,a,\emptyset,1_s) &= u(\emptyset) + \mu\,\underline{P}(i) + \beta EV(\emptyset) \\
v(i,a,\emptyset,0_s) &= u(\emptyset) + \mu[P(i,a) - T_s(P,i,a)] + \beta EV(\emptyset) \\
v(i,a,\kappa) &= u(i,a) + \beta\big(1-\alpha\big) EV(i,a+1) + \beta\alpha EV(i,\bar{a}) \\
v(i,a,j,d,1_s) &= u(j,d) - \mu[P(j,d) - \underline{P}(i) + T_b(P,j,d)] + \\
&\quad + \beta\big(1-\alpha\big) EV(j,d+1) + \beta\alpha EV(j,\bar{a}) \\
v(i,a,j,d,0_s) &= u(j,d) - \mu[P(j,d) - P(i,a) + T_s(P,i,a) + T_b(P,j,d)] + \\
&\quad + \beta\big(1-\alpha\big) EV(j,d+1) + \beta\alpha EV(j,\bar{a})
\end{aligned}
$$

States Choices $\rightarrow$ Current period utility Future value

Expected value function

- Expected value function

$$EV(i,a) = \int_{\epsilon} V(i,a,\epsilon) f(\epsilon|i,a) d\epsilon$$

- Has analytic form under GEV: composition of logsum functions
- In the simplest case (GEV $\rightarrow$ EV1)

$$EV(i,a) = \sigma \log \left(\sum_{j,d,s} \exp \frac{v(i,a,j,d,s)}{\sigma} \right)$$

Bellman operator

After plugging in expressions for choice specific value function $v(i, a, j, d, s)$

$$EV(P) = \Gamma\big(EV(P), P\big)$$

- Rust engine replacement model on steroids!
- The EV vector is a fixed point to a contraction mapping
- Can be calculated by globally convergent method of successive approximations (VFI)
- Much better to use **Newton's method** in functional space $\rightarrow$ Newton-Kantorovich iterations
- *See NFXP manual and DSE2021 lecture #2*

Implied choice probabilities

1. Solution to DP problem $(EV(i,1),\ldots,EV(i,\bar{a})) \rightarrow$
2. Choice specific values $v(i,a,j,d,s)$ for all choices $j,d,s \rightarrow$
3. Analytic nested logit choice probabilities

- Simplest case:

$$\Pi(j,d,s|i,a) = \frac{\exp\left\{v(i,a,j,d,s)/\sigma\right\}}{\sum_{j',d',s'} \exp\left\{v(i,a,j',d',s')/\sigma\right\}}$$

- Choice probability for sell/scrap decision is separable under the additively separable transaction costs
- Implicitly depend on the market prices P

Ownership distribution

- Let $0 \le q_{i,a} \le 1$ denote the fraction of owners of make/model i cars aged a, $i \in \{1, \ldots, J\}$, $a \in \{1, \ldots, \bar{a}\}$
- Let $0 \le q_{\o} \le 1$ denote the fraction of consumers without a car

- **Ownership distribution**

$$q = \Big((q_{11}, \ldots, q_{1\bar{a}}), \ldots, (q_{J1}, \ldots, q_{J\bar{a}}), q_{\o} \Big) \in \mathbb{R}^{J\bar{a}+1}.$$

- q is measured in the beginning of the period before trading
- q changes from period to period due to trading, deterministic ageing of cars and stochastic accidents

Equilibrium

- Infinitely elastic supply of new cars at price $\overline{P}(j)$
- Infinitely elastic demand for scrap cars at price $\underline{P}(j)$
- $J(\bar{a}-1)$ tradable cars at prices $P = \big(P(1,1),\ldots,P(J,\bar{a}-1)\big)$

Definition (Stationary equilibrium)

Equilibrium is the pair (q, P) such that

1. Consumers maximize their expected discounted utility
2. Secondary market clears for all tradable cars
3. Ownership distribution is time invariant

Demand

Demand for cars of make/model j of age d:

$$D_{jd}(P,q) = \Pi(j,d|\emptyset,P)q_{\emptyset} + \sum_{i=1}^{J}\sum_{a=1}^{\bar{a}} \Pi(j,d|i,a,P)q_{ia}$$

- In equilibrium all intended choices take place $\rightarrow$
- Sum up choice probabilities from all owners
- Economy with continuum of consumers but finite number of tradable goods $\rightarrow$
- Choice probabilities = market shares

Supply

Supply of cars of make/model j of age d:

$$S_{jd}(P, q) = \big(1 - \Pi(\kappa | j, d, P)\big)\big(1 - \Pi(1_s | j, d, P)\big) q_{jd}$$

- Cars are supplied by owners who don't keep $1 - \Pi(\kappa | j, d, P) \rightarrow$
- And don't voluntarily scrap $1 - \Pi(1_s | j, d, P)$

Market clearing

$$D(P,q) = \Big((D_{11},\dots,D_{1,\bar{a}-1}),\dots,(D_{J1},\dots,D_{J,\bar{a}-1})\Big) \in \mathbb{R}^{J(\bar{a}-1)}$$
$$S(P,q) = \Big((S_{11},\dots,S_{1,\bar{a}-1}),\dots,(S_{J1},\dots,S_{J,\bar{a}-1})\Big) \in \mathbb{R}^{J(\bar{a}-1)}$$

Market clearing $\Leftrightarrow$ Zero excess demand condition:

$$ED(P,q) = D(P,q) - S(P,q) = 0$$

- System of $J(\bar{a}-1)$ equations: linear in q, non-linear in prices
- $J\bar{a}+1$ unknowns in the ownership distribution vector
- $J(\bar{a}-1)$ unknown prices

Trade transition probability matrix

$\Omega(P) =$ $J\bar{a}+1 \times J\bar{a}+1$ matrix

$$\begin{bmatrix} \Delta_{11}(P)+\Lambda_1(P) & \Delta_{12}(P) & \dots & \Delta_{1J}(P) & \Delta_{1\emptyset}(P) \\ \Delta_{21}(P) & \Delta_{22}(P)+\Lambda_2(P) & \dots & \Delta_{2J}(P) & \Delta_{2\emptyset}(P) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Delta_{J1}(P) & \Delta_{J2}(P) & \dots & \Delta_{JJ}(P)+\Lambda_J(P) & \Delta_{J\emptyset}(P) \\ \Delta_{\emptyset 1}(P) & \Delta_{\emptyset 2}(P) & \dots & \Delta_{\emptyset J}(P) & \Pi(\emptyset|\emptyset, P) \end{bmatrix}$$

Then $q \cdot \Omega(P)$ is distribution of cars after the trading phase

- $\Delta_{ij}(P)$ = trading components
- $\Lambda_{ij}(P)$ = keeping components

Matrix blocks for j-type car

$$\Delta_{ij}(P) = \begin{bmatrix} \Pi(j,1|i,1) & \dots & \Pi(j,\bar{a}-1|i,1) & \Pi(j,0|i,1) \\ \vdots & \ddots & \vdots & \vdots \\ \Pi(j,1|i,\bar{a}) & \dots & \Pi(j,\bar{a}-1|i,\bar{a}) & \Pi(j,0|i,\bar{a}) \end{bmatrix},$$

$$\Lambda_i(P) = \begin{bmatrix} \Pi(\kappa|i,1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \Pi(\kappa|i,\bar{a}-1) & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

- $\bar{a} \times \bar{a}$ matrices
- Last column pertains to the car of age $\bar{a}$, but collects choice probabilities for the new cars

Last column and bottom row of $\Omega(P)$

- Probabilities of buying (j, d)-cars by people without cars

$$\Delta_{\emptyset j}(P) = \Big[\Pi(j, 0|\emptyset), \ldots, \Pi(j, \bar{a} - 1|\emptyset), \Pi(j, 0|\emptyset)\Big] \in \mathbb{R}^{\bar{a}}$$

- Probabilities of transition to no car state by owners of (i, a)-cars

$$\Delta_{j\emptyset}(P) = \begin{bmatrix} \Pi(\emptyset|i, 1) \\ \vdots \\ \Pi(\emptyset|i, \bar{a}) \end{bmatrix} \in \mathbb{R}^{\bar{a}}$$

Physical transition probability matrix

$Q =$ $\qquad J\bar{a}+1 \times J\bar{a}+1$ matrix

$$\begin{bmatrix} Q_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & Q_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & Q_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & Q_J & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \qquad Q_j = \begin{bmatrix} 0 & 1-\alpha & \dots & 0 & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-\alpha & \alpha \\ 0 & 0 & \dots & 0 & 1 \\ 1-\alpha & 0 & \dots & 0 & \alpha \end{bmatrix}$$

- Ageing of cars with probability $1-\alpha$
- Total loss accidents with probability α
- Last column in each block is applies for the new cars, restores the order

$q \cdot \Omega(P)Q$ is ownership distribution in the next period

The stationary holdings distribution

$$\underbrace{q}_{t} \quad \rightarrow \quad \underbrace{q\Omega(P)}_{\text{after trading}} \quad \rightarrow \quad \underbrace{q\Omega(P)Q}_{t+1}$$

Condition for time invariance of the ownership distribution:

$$q = q\Omega(P)Q$$

Theorem (Uniqueness of stationary ownership distribution)

Let $\sigma > 0$. Then stationary ownership distribution is unique.

Proof.

With positive GEV scale parameter σ, the choice probabilities have full support, therefore transition matrix $\Omega(P)Q$ is irreducible and aperiodic. Uniqueness follows from the Fundamental theorem of Markov chains. $\square$

Existence of stationary equilibrium

Theorem (Equilibrium existence)

The stationary equilibrium for the automobile economy with the idiosyncratically heterogeneous consumers $(q^\star, P^\star)$ *exists, and in equilibrium it holds:*

$$\begin{aligned} q^\star &= q^\star \Omega(P^\star) Q, \\ 0 &= ED(P^\star, q^\star). \end{aligned}$$

- Only existence: $q^\star$ is unique, but unclear about $P^\star$
- However, have not seen any signs of multiplicity in computations

Equilibrium flow property

Theorem (Stationary flow equilibrium)

The stationary equilibrium $(q^\star, P^\star)$ in previous theorem has the flow property:

$$\underbrace{\sum_{a=1}^{\bar{a}-1} \Pi(1_s|j,a,P)\big(1-\Pi(\kappa|j,a,P)\big)q^\star_{ja} + q^\star_{j\bar{a}}}_{\text{all scrapped cars of make/model } j} = \underbrace{\sum_{i=1}^{J}\sum_{a=1}^{\bar{a}} \Pi(j,0|i,a,P)q^\star_{ia}}_{\text{all new cars of make/model } j}$$

- Follows from stationarity of ownership distribution $q^\star$, including it subvectors
- Also holds for the outside good (not owning the car), follows algebraically from the structure of $\Omega(P^\star)$ and Q
- So, we have stations flow equilibrium

How to compute stationary flow equilibrium?

Straightforward algorithm following the existence Theorem:

1. Equilibrium ownership distribution q is an implicit function of P from the stationarity condition $q = q\Omega(P)Q$
2. Express excess demand as a function of prices only $ED\big(P, q(P)\big)$
3. Numerically solve $ED\big(P, q(P)\big) = 0$ in prices
4. Finally, compute the corresponding stationary ownership distribution

The key to success:

- Use of the efficient gradient based solver (Newton method)
- Analytic derivatives $\rightarrow$ quick and precise
- Precise theoretically founded starting values $\rightarrow$ robust

Smoothness

To be able to apply Newton method we prove in a series of Lemmas that the following model object are continuously differentiable in prices P (*and* structural parameters + other fundamentals as α, $\overline{P}$, $\underline{P}$):

- Fixed point of Bellman equation EV
- Choice specific functions $v(i, a, j, d, s)$
- Choice probabilities $\Pi(j, d, s|i, a)$
- Invariant distribution of transition probability matrix $\Omega(P)Q$
- Excess demand $ED(P, q)$

Newton method is therefore applied:

1. When solving the DP problem (Newton-Kantorovich)
2. When solving for equilibrium prices
3. Later again when maximizing likelihood

Computational algorithm for $ED\big(P, q(P)\big)$

Use Newton method to solve the nonlinear system of $ED\big(P, q(P)\big) = 0$ with $J(\bar{a} - 1)$ equations and $J(\bar{a} - 1)$ unknown prices P.

Single computation of $ED\big(P, q(P)\big)$ for given prices P:

1. Solve the dynamic problem to find the fixed point of Bellman equation EV
2. Compute choice probabilities and form trade transition probability matrix $\Omega(P)$
3. Compute the invariant distribution q of $\Omega(P)Q$
4. Compute excess demand $ED(P, q)$

- Derivatives are computed by repeated application of chain rule
- Originate with derivatives of the utility function and trade costs
- Fit like "LEGO blocks" from the inner steps

Numerical performance

Good numerical performance already in Matlab

	$J = 1$	$J = 2$	$J = 5$	$J = 10$	$J = 25$
$\bar{a}$=20	0.081s	0.141s	0.674s	4.103s	1m 6.865s
$\bar{a}$=50	0.162s	0.720s	11.566s	1m 12.317s	
$\bar{a}$=100	0.680s	4.407s	1m 29.818s		

Possible to nest into iterative algorithms:

- Structural estimation
- Primary market modelling
- etc.

Persistent consumer heterogeneity

We extend the model to allow for:

1. Time-invariant consumer heterogeneity
2. Time variant consumer heterogeneity
3. The combination of the two

- Existence theorems
- Computational algorithm is linear in the number of types
- Allows for sorting of consumers into the ages and types of cars
  - Rich hold newer better cars, poor hold older worse cars
  - Gains from trade and longer surviving cars
  - High transaction costs suppress trade and lower scrappage age

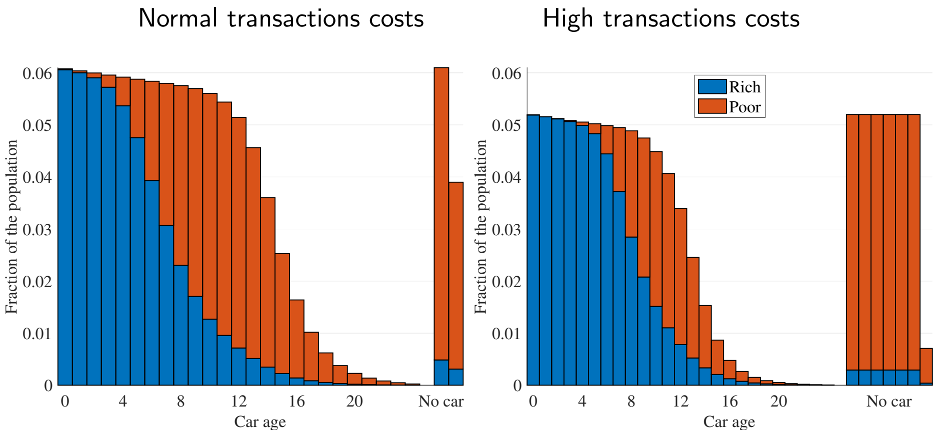
Updates in the heterogeneous consumer case

- Consumer types $\tau = 1, \ldots, N$
- Type fractions (time invariant types) or stationary distribution (time variant types) $f = (f_1, \ldots, f_N)$
- Consumer choice model has to be solved separately for each type τ
- Type-specific trading matrix $\Omega_\tau(P)$
- Type-specific ownership distribution q_τ
- Type-specific excesses demand $ED(P, q_\tau)$
- The equilibrium conditions become

$$\forall \tau \; q^\star_\tau = q^\star_\tau \Omega(P^\star) Q,$$
$$0 = \sum_{\tau=1}^{N} ED(P^\star, q^\star_\tau).$$

- Market clearing condition integrated over types

Illustrative example: ownership by two consumer types



- Sorting of consumers in each regime
- Heterogeneous effects of transaction costs

Double fixed point estimator

Full solution estimator for the equilibrium model

- Let θ denote the vector of structural parameters
- Data on car ownership and observed household characteristics x
- Likelihood function

$$L(\theta) = \sum_{\tau} \sum_{x'} \sum_{x} \log\big(\Pi(x'|x,\tau,\theta)\big) \underbrace{P(x'|x,\tau)q_\tau(x)f(\tau)}_{\sim\text{observed } x \to x'}$$

Three loops:

Outer Maximization of likelihood with respect to θ

Middle Equilibrium solver for price vector $P(\theta)$

Inner DP solver for fixed point of Bellman operator $EV(P,\theta)$

Summary of the theoretical part

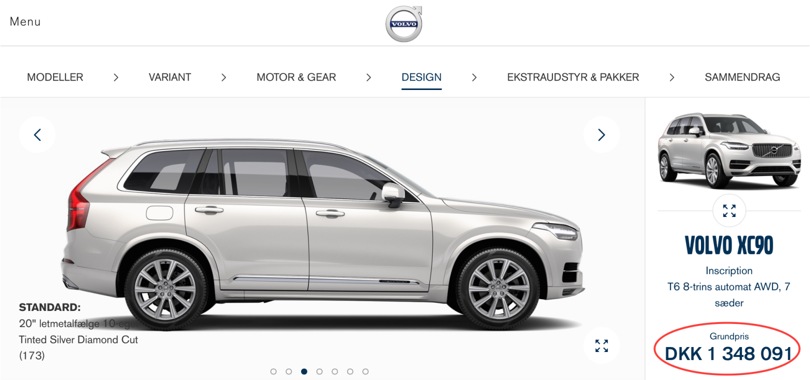
Empirical engine for analyzing equilibrium models of secondary markets of durable goods

- Computationally tractable in real empirical applications!
- Flexible specification of preferences and transaction costs $\rightarrow$
- Recover consumer heterogeneous preferences towards each type of car of every age
- Possible to use with no price data!
- Identity of car owners is not required: panel microdata on car ownership and trading can be represented by counts of transitions conditional on consumer types

Part III: Modelling Danish secondary market of automobiles

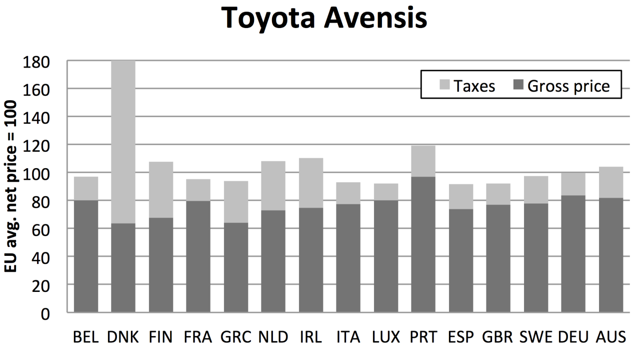
How much is a Volvo in Denmark?

204.816 US Dollars!



MSRP in US: $62.350

Danish car registration tax: 180%!
..... plus 25% VAT!!!



Car taxes in Denmark

Annual Revenue

- 30–50 billion DKK
- $\cong$ 2–3 pct. of GDP
- $\cong$ 4-7 pct. of total tax revenue
- Most revenue originate from taxation of ownership and registration of new cars.

It is also widely understood that transport externalities are rarely appropriately priced (e.g., Parry and Small (2005)).

- Underpricing congestion.
- Incorrect taxation of gasoline.

Simulating the effects of a hypothetical tax reform

Proposed Danish IRUC reform:

- lowers registration taxes, and
- raises usage taxes (road charging or gas tax).

Outcomes of interes

- Equilibrium dynamics of car ownership and type choice:
  - new car sales and trade in secondary markets
  - fleet age and scrappage,
  - value of the car stock.
- Driving, fuel demand, and emissions.
- Redistribution and welfare.
- Need to capture these effects simultaneously

To implement the counterfactual simulation:

1. Estimate the model using Danish register data
2. Cut the registration tax rates for new vehicles by half
3. Simultaneously increases the fuel tax rate such that revenue is unchanged
4. Compute economic/welfare/environmental implications

Utility specification

Consider a utility function of the form

$$u(a, x) = u_{\text{car}}(a) + u_{\text{drive}}(a, x) + \mu[\text{trade} + \text{transaction cost}]$$

Utility of ownership when $x = 0$

$$u_{\text{car}}(a) = \alpha_0 + \alpha_1 a + \alpha_2 a^2$$

Net utility from driving

$$u_{\text{drive}}(a, x) = (\gamma_0 + \gamma_1 a)x - \mu p x + \frac{\phi}{2} x^2$$

- x is vehicle kilometers driven each period
- p is the per kilometer cost of fuel plus any taxes,
- parameter vector $\theta = (\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \mu, \phi)$
- θ may be specific each consumer type/car type (for notational simplicity we have suppressed this dependence)

Demand for driving and gasoline

Assumption: *The probability of an accident and other physical deterioration in an automobile is independent of driving, x.*

This implies driving is a static subproblem of the overall DP problem that can be solved independently.

Structural driving equation implied by the direct utility function $u(x, a)$

$$x^*(p, a) = -\frac{1}{\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]. \tag{7}$$

Identification

We want to consider the identification of parameters $\theta = (\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \mu, \phi)$ from observations on:

1. consumer trading of automobiles
2. observed driving.

Reduced form driving equation

Structural driving equation implied by the direct utility function $u(x, a)$

$$x^*(p, a) = -\frac{1}{\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]. \tag{8}$$

Reduced form driving equation

$$x = d_0 + d_1 a + d_2 p. \tag{9}$$

where

$$\begin{aligned} d_0 &= -\frac{\gamma_0}{\phi} \\ d_1 &= -\frac{\gamma_1}{\phi} \\ d_2 &= \frac{\mu}{\phi} \end{aligned} \tag{10}$$

- reduced form parameters d_0, d_1 and d_2 are separately identified from data on driving, car age a and price per kilometer p of cars.
- Identifying variation in p across car types mainly due to different fuel efficiency of different types and different ages

Reduced form indirect utility of car ownership

Now plug the optimal driving into utility to obtain the indirect utility

$$
\begin{aligned}
u(a, x^*(p,a)) = v(a,p,\tau) &= \alpha_0 + \alpha_1 a + \alpha_2 a^2 - \frac{1}{2\phi}\left[\gamma_0 + \gamma_1 a - \mu p\right]^2 \\
&= u_0 + u_1 a + u_2 a^2
\end{aligned}
$$

where

$$
\begin{aligned}
u_0 &= \alpha_0 - \frac{1}{2\phi}\left[\gamma_0 - \mu p\right]^2 \\
u_1 &= \alpha_1 - \frac{\gamma_1}{\phi}\left[\gamma_0 - \mu p\right] \\
u_2 &= \alpha_2 - \frac{1}{2\phi}\left[\gamma_1^2\right].
\end{aligned}
$$

- (u_0, u_1, u_2) and μ are identified from an unrestricted or "reduced form" dynamic discrete choice model of car trading.
- Given μ, (u_0, u_1, u_2) and (d_0, d_1, d_2) are identified, then $(\alpha_0, \alpha_1, \alpha_2, \gamma_0, \gamma_1, \phi)$ can be separately identified

Estimation of parameters

Step 1: Least Squares Estimation of Reduced form driving parameters, (d_0, d_1, d_2)

Step 2: Maximum Likelihood Estimates of Equilibrium Model

- Reduced from utility parameters, (u_0, u_1, u_2)
- Marginal utility of money μ
- Purchase transaction cost
- Coefficients of binary logit model of accident rates: $\alpha(a) = \Lambda(\alpha_j + \alpha_j^a a)$
- Sales transaction cost

Step 3: Back out structural parameters

Table: Driving Regression Parameter Estimates

Dependent variable: thousands of kilometers driven per year			
γ_0	Intercept	19.07	(0.58)
$\hat{\gamma}_1^a/\phi_\tau$	Car age	-0.1325	(0.03)
$\hat{\gamma}_2^a/\phi_\tau$	Car age squared	-0.001975	(0.00)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Single, Rich	4.43	(0.70)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Couple, Poor	-3.752	(1.24)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, Low WD, Couple, Rich	-0.0325	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Poor	9.825	(0.79)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Single, Rich	12.33	(0.74)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Poor	6.436	(1.52)
$\hat{\gamma}_\tau/\phi_\tau$	Intercept, High WD, Couple, Rich	12.23	(1.27)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: light, green	-1.994	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, brown	4.345	(0.15)
$\hat{\gamma}_j/\phi_\tau$	Car dummy: heavy, green	3.606	(0.14)
μ/ϕ	Price (common)	-7.074	(0.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Single, Rich	-4.111	(1.02)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Couple, Poor	4.732	(1.84)
$\hat{\mu}_\tau/\phi_\tau$	Price, Low WD, Couple, Rich	0.2781	(1.16)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Poor	-6.41	(1.17)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Single, Rich	-9.892	(1.09)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Poor	-1.714	(2.29)
$\hat{\mu}_\tau/\phi_\tau$	Price, High WD, Couple, Rich	-9.007	(1.91)

Note: The first stage uses a rich dataset of driving and is a linear regression of driving on covariates. Standard errors in parentheses.

Table: Maximum Likelihood Estimates of Equilibrium Model

μ_τ: marginal utility of money		
Consumer type (τ)	Parameter est.	Standard error
Low WD, Single, Poor	0.1131	(0.1131)
Low WD, Single, Rich	0.1120	(0.1120)
Low WD, Couple, Poor	0.0939	(0.0939)
Low WD, Couple, Rich	0.1075	(0.1075)
High WD, Single, Poor	0.1036	(0.1036)
High WD, Single, Rich	0.1155	(0.1155)
High WD, Couple, Poor	0.0918	(0.0918)
High WD, Couple, Rich	0.1081	(0.1081)

Table: Maximum Likelihood Estimates of Equilibrium Model

$u_{\tau,j,0}$: intercept in quadratic indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Single, Poor	3.6486	3.1100	5.1437	4.7731
	(0.0179)	(0.0177)	(0.0282)	(0.0257)
Low WD, Single, Rich	4.0346	3.4641	5.7446	5.3490
	(0.0172)	(0.0172)	(0.0272)	(0.0248)
Low WD, Couple, Poor	2.4015	2.1452	3.5705	3.2012
	(0.0191)	(0.0177)	(0.0298)	(0.0273)
Low WD, Couple, Rich	3.2424	2.8169	4.6811	4.2764
	(0.0175)	(0.0172)	(0.0276)	(0.0252)
High WD, Single, Poor	3.8830	3.4334	5.3125	5.0547
	(0.0165)	(0.0163)	(0.0259)	(0.0236)
High WD, Single, Rich	4.7657	4.3197	6.5560	6.2390
	(0.0183)	(0.0181)	(0.0286)	(0.0261)
High WD, Couple, Poor	2.6648	2.4233	3.7449	3.5072
	(0.0206)	(0.0186)	(0.0324)	(0.0295)
High WD, Couple, Rich	3.5532	3.1710	4.9859	4.6954
	(0.0204)	(0.0193)	(0.0319)	(0.0291)

Table: Maximum Likelihood Estimates of Equilibrium Model

$u_{\tau,j,1}$: coefficient of age in quadratic indirect utility for car ownership				
	light, brown	light, green	heavy, brown	heavy, green
Low WD, Single, Poor	-0.1463	-0.0925	-0.2197	-0.1717
	(0.0009)	(0.0011)	(0.0014)	(0.0012)
Low WD, Single, Rich	-0.1591	-0.0988	-0.2415	-0.1955
	(0.0009)	(0.0011)	(0.0013)	(0.0012)
Low WD, Couple, Poor	-0.0986	-0.0616	-0.1599	-0.1128
	(0.0009)	(0.0010)	(0.0014)	(0.0012)
Low WD, Couple, Rich	-0.1311	-0.0843	-0.2056	-0.1543
	(0.0009)	(0.0010)	(0.0014)	(0.0012)
High WD, Single, Poor	-0.1394	-0.0828	-0.2136	-0.1697
	(0.0008)	(0.0010)	(0.0013)	(0.0011)
High WD, Single, Rich	-0.1646	-0.1072	-0.2552	-0.2074
	(0.0009)	(0.0011)	(0.0014)	(0.0012)
High WD, Couple, Poor	-0.1110	-0.0708	-0.1733	-0.1274
	(0.0010)	(0.0010)	(0.0016)	(0.0013)
High WD, Couple, Rich	-0.1496	-0.1010	-0.2210	-0.1735
	(0.0010)	(0.0011)	(0.0016)	(0.0013)

Table: Maximum Likelihood Estimates of Equilibrium Model

utility costs of transacting		
	common	no car
Low WD, Single, Poor	6.5800	1.7892
	(6.5800)	(1.7892)
Low WD, Single, Rich	6.4270	1.0740
	(6.4270)	(1.0740)
Low WD, Couple, Poor	6.5320	3.0808
	(6.5320)	(3.0808)
Low WD, Couple, Rich	6.5909	2.5745
	(6.5909)	(2.5745)
High WD, Single, Poor	6.2502	0.7775
	(6.2502)	(0.7775)
High WD, Single, Rich	6.4069	0.1682
	(6.4069)	(0.1682)
High WD, Couple, Poor	6.0172	2.3385
	(6.0172)	(2.3385)
High WD, Couple, Rich	6.2647	1.7889
	(6.2647)	(1.7889)

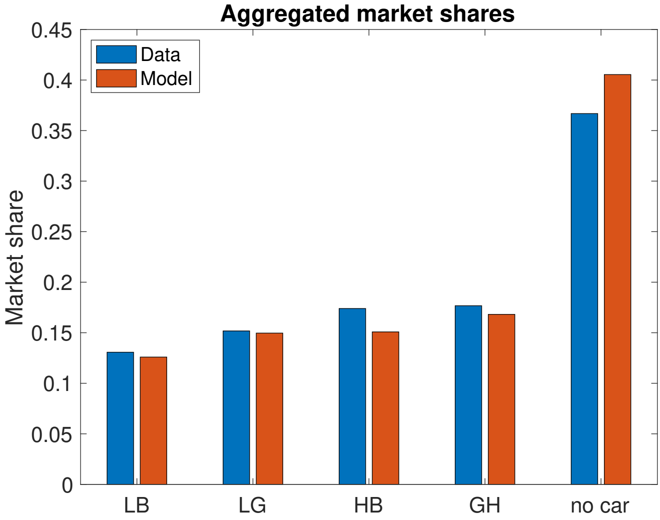
Table: Maximum Likelihood Estimates of Equilibrium Model

coefficients of binary logit model of accident rates: $\alpha(a) = \Lambda(\alpha_j + \alpha_j^a a)$				
	light, brown	light, green	heavy, brown	heavy, green
intercept	-5.6248	-6.0443	-5.6728	-5.7826
	(0.0119)	(0.0105)	(0.0095)	(0.0088)
age slope	0.1804	0.2216	0.2020	0.2048
	(0.0015)	(0.0011)	(0.0009)	(0.0009)

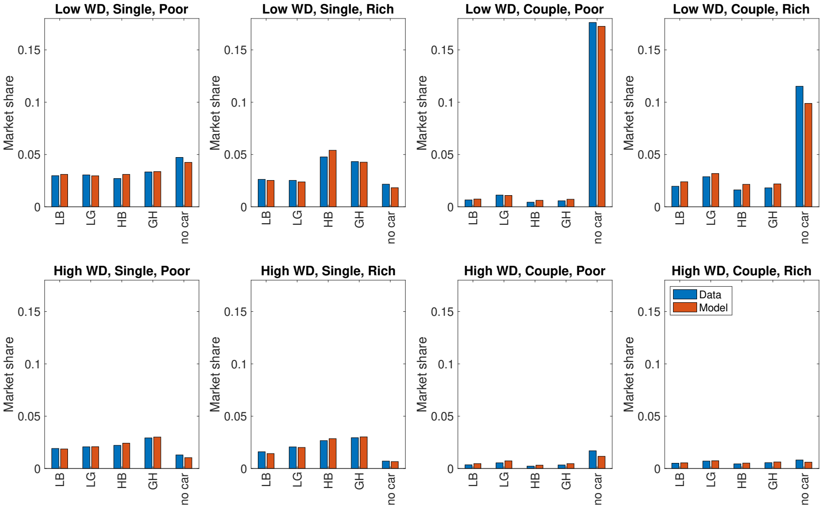
Table: Maximum Likelihood Estimates of Equilibrium Model

sales transactions costs (utility units)		
	Estimate	Standard Error
σ_s	0.3454	(0.3454)
sales transaction cost	0.9106	(0.9106)
sales transaction cost (inspection year)	-2.1929	(-2.1929)

Model fit: aggregate market shares



Model fit: market shares by consumer type

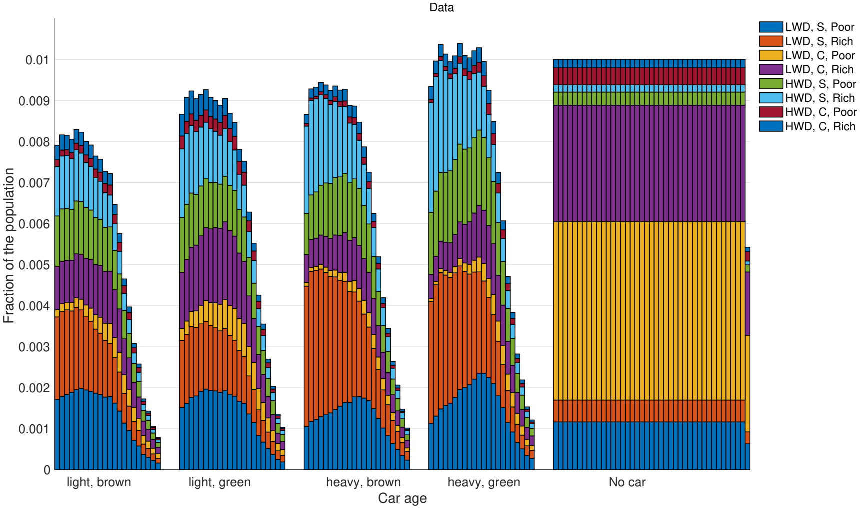


Gains from trade between rich and poor consumers

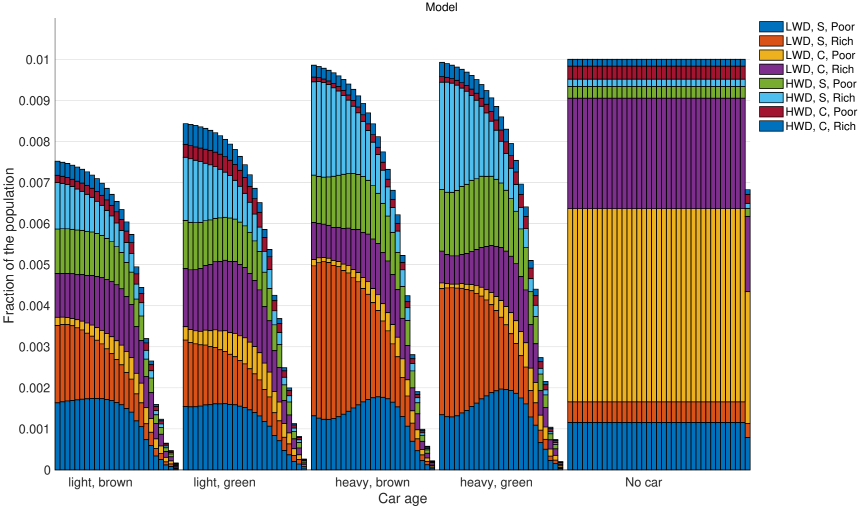
Rich mans Volvo

Poor mans Volvo

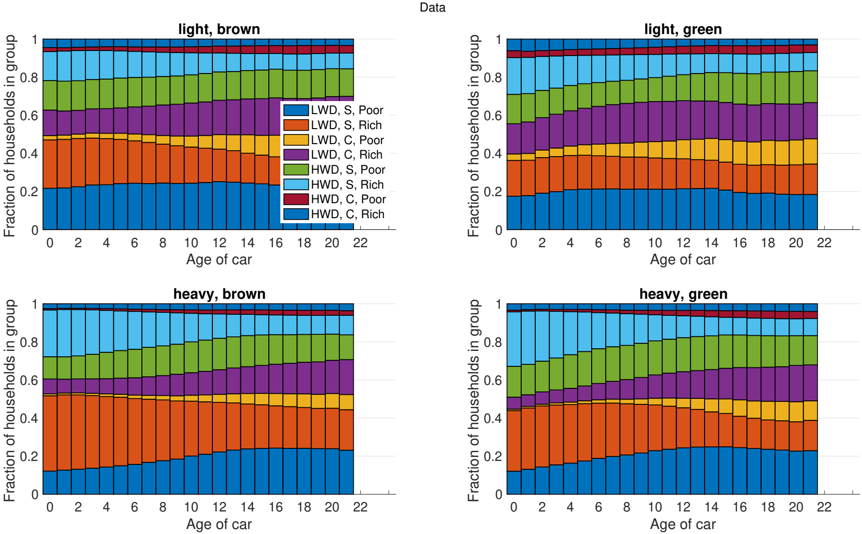
Model fit: observed ownership distribution



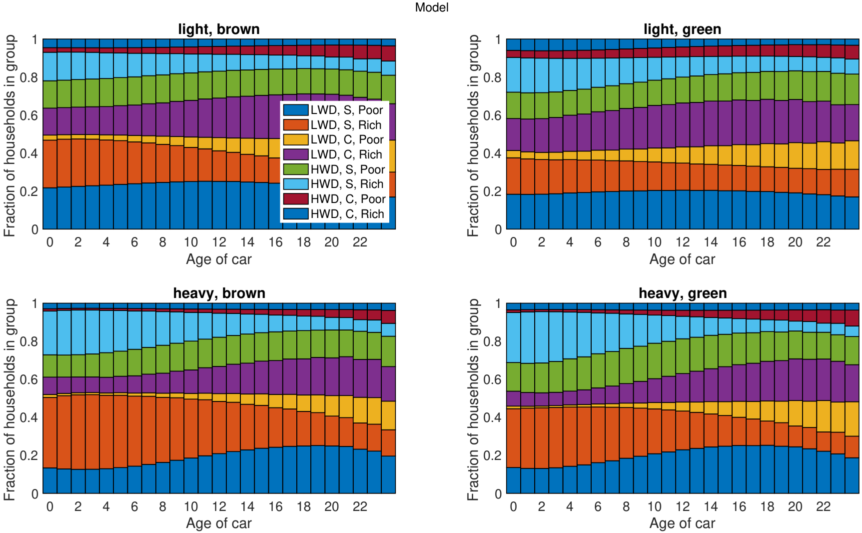
Model fit: predicted ownership distribution



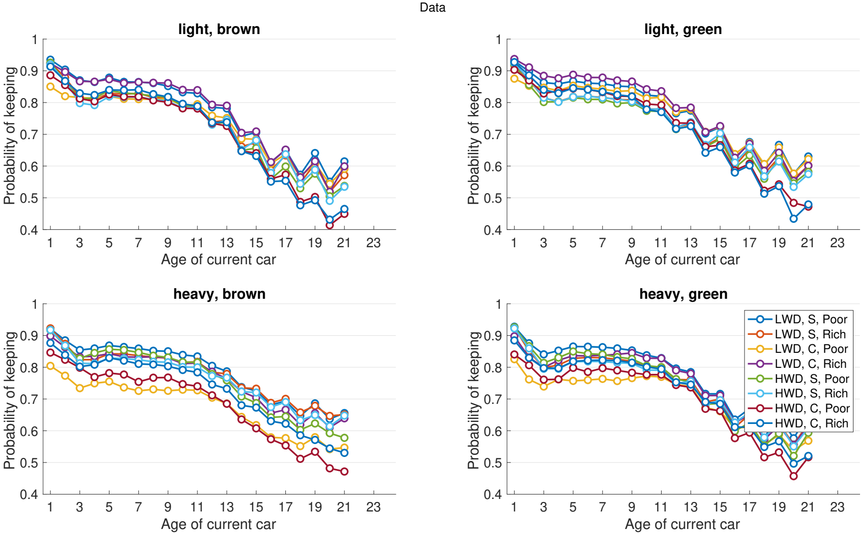
Model fit: observed sorting



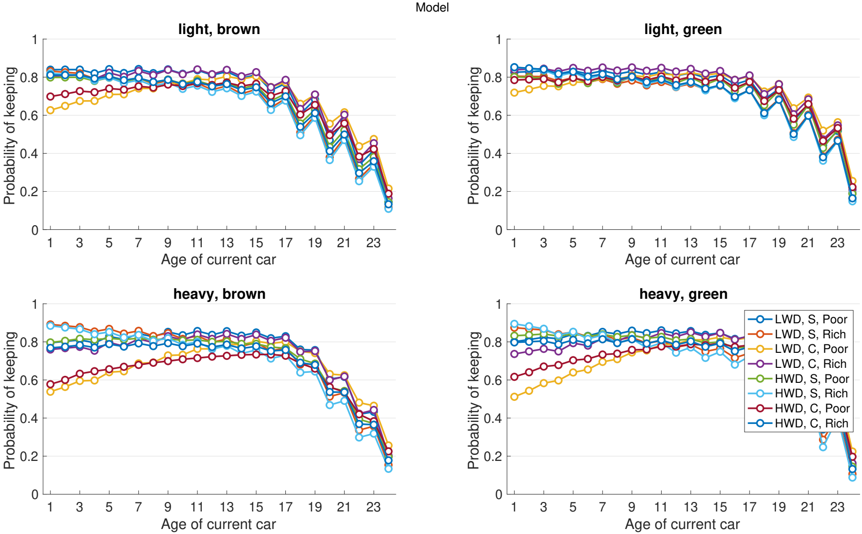
Model fit: predicted sorting



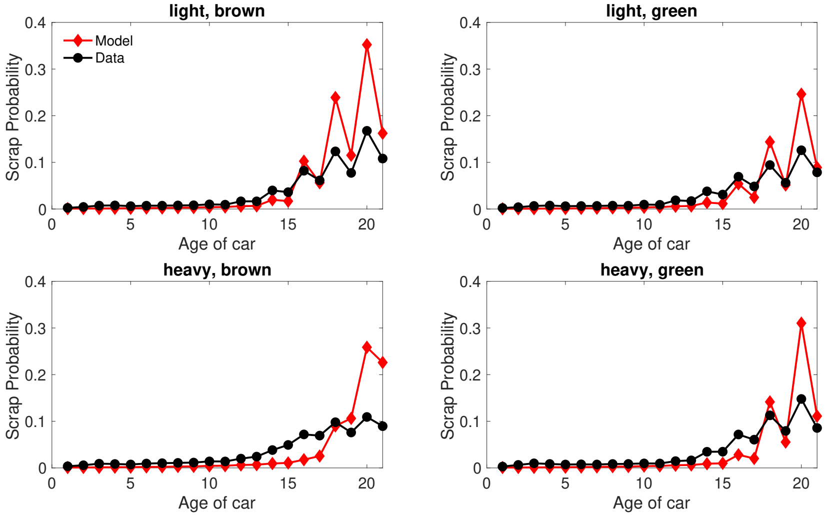
Model fit: observed keep probabilities



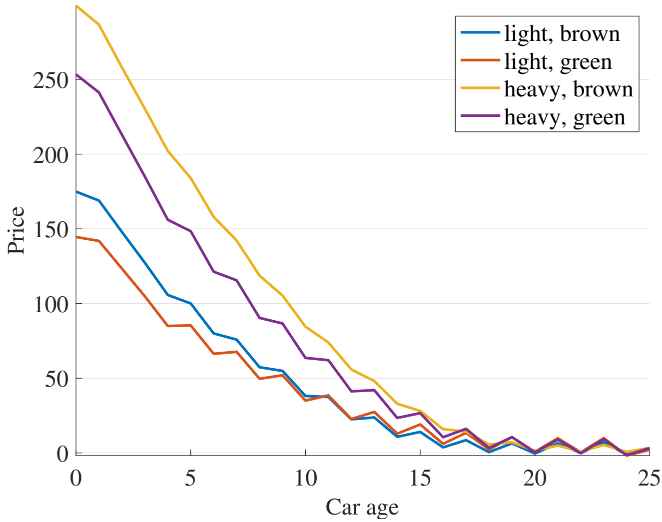
Model fit: predicted keep probabilities



Model fit: scrap probabilities



Equilibrium prices predicted by model



Counterfactual simulation

Halving registration tax corresponds to a drop in new car price by:
25.6% (new normal car) and 27.1% (new luxury car)

We consider the following four scenarios:

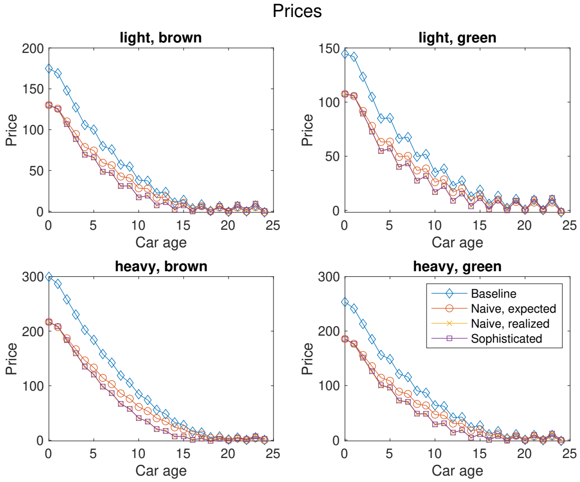
1. **Baseline:** Calibration under Danish tax rates from 2008.
2. **Naive, expected:** Non-equilibrium simulation: Both new and used car prices drops proportionally: between 25.6% (cheapest), 26.8% (most expensive car)
   Fuel taxes increase to keep total tax revenue neutral
   from 50% to 72.3% of the price at the pump
3. **Naive, expected:** Equilibrium As above + market equilibrium imposed. Not revenue neutral
4. **Sophisticated policy maker:** Equilibrium and revenue-neutral Fuel taxes increase only to 68.4%

Simulation of tax reform: Tax revenue

	Baseline	Naive expected	Naive realized	Sophisticated
Price, light, brown (1000 DKK)	174.902	130.110	130.110	130.110
Price, light, green (1000 DKK)	144.551	107.532	107.532	107.532
Price, heavy, brown (1000 DKK)	299.452	216.760	216.760	216.760
Price, heavy, green (1000 DKK)	253.397	185.508	185.508	185.508
VAT	0.250	0.250	0.250	0.250
Car tax (low)	1.050	0.525	0.525	0.525
Car tax (high)	1.800	0.900	0.900	0.900
Car tax (kink)	81.000	81.000	81.000	81.000
Fuel price (DKK/l)	8.322	15.006	15.006	13.177
Fuel tax (rel. to pump price)	0.500	0.723	0.723	0.684
Social surplus	8.362	10.350	7.378	9.175
Total tax revenue (1000 DKK)	8.758	8.758	6.702	8.758
Fuel tax revenue (1000 DKK)	3.737	4.653	4.432	5.711
Car tax revenue (1000 DKK)	5.021	4.105	2.271	3.047
Non-CO2 externalities (1000 DKK)	6.750	3.186	3.067	4.658
Externalities (1000 DKK)	7.373	3.483	3.350	5.097
Consumer surplus (1000 DKK)	6.977	5.076	4.026	5.514
CO2 (ton)	2.148	1.026	0.977	1.515
VKT (1000 km)	10.860	5.125	4.934	7.494
Fuel consumption (1000 liters)	0.898	0.429	0.409	0.633
E(car age)	6.507	2.883	4.146	5.382
Pr(no car)	0.367	0.556	0.557	0.425

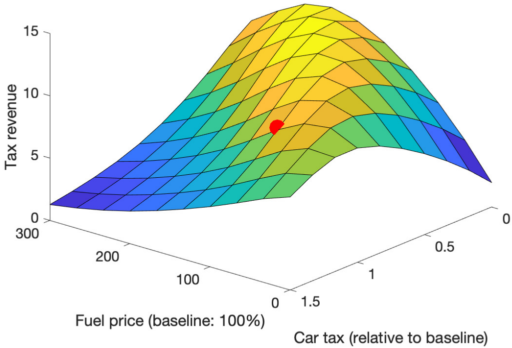
- Revenue neutral calculation without equilibrium effects leads to 23.5% lower revenue than expected!

Simulation of tax reform: Equilibrium prices



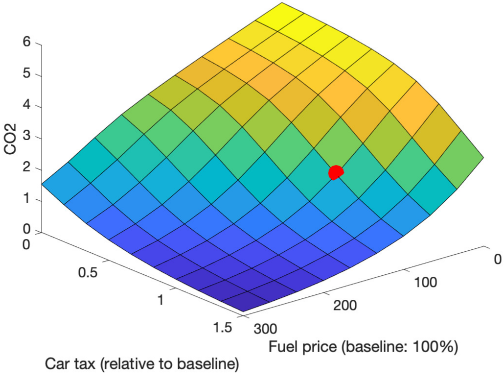
- Non-EQ based on proportional change in all prices
- But in equilibrium used car prices fall non-proportionally

Laffer curves for new car registration tax and fuel tax



The new car registration tax is changed with relative to the 100 baseline rates. The fuel tax is measured relative to the baseline level of 1. The total tax revenue comes from new car tax (from new car sales only) and fuel tax.

Co2 emissions for new car registration tax and fuel tax



The new car registration tax is changed with relative to the 100 baseline rates. The fuel tax is measured relative to the baseline level of 1. The total tax revenue comes from new car tax (from new car sales only) and fuel tax.

Driving, fuel, emissions and taxes

	Baseline	Naive expected	Naive realized	Sophisticated
Driving (1000 km)	10.860	5.125	4.934	7.494
Fuel price (DKK/l)	8.322	15.006	15.006	13.177
Fuel consumption (1000 liters)	0.898	0.429	0.409	0.633
Fuel tax (rel. to pump price)	0.500	0.723	0.723	0.684
Fuel tax revenue (1000 DKK)	3.737	4.653	4.432	5.711
CO2 (ton)	2.148	1.026	0.977	1.515
Social surplus	8.362	10.350	7.378	9.175
Consumer surplus (1000 DKK)	6.977	5.076	4.026	5.514
Non-CO2 externalities (1000 DKK)	6.750	3.186	3.067	4.658
Externalities (1000 DKK)	7.373	3.483	3.350	5.097

- Drop in consumer surplus smaller than environmental benefits
- Reform shifts taxation towards the harmful externality (driving)
- Reduction due to secondary market – substantial source of welfare for poor consumers who rely on cheaper older cars
- Welfare and revenue effects working through the secondary market equilibrium are first-order

Trade-off between Co2 emissions and welfare