

# Lecture 16: Solving and estimating directional dynamic games

2021 Econometric Society Summer School in  
Dynamic Structural Econometrics

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CRC/TR 224 and Econometric Society  
Summer School in Dynamic Structural Econometrics  
University of Bonn  
16-22 August, 2021

# Roadmap: dynamic games

- ➊ Collusion of Australian CPF (corrugated fiber packaging) producers  
[John Rust](#)
- ➋ Experiment with the model  
[John Rust, Fedor Iskhakov](#)
- ➌ State recursion algorithm
- ➍ Recursive lexicographical search (RLS) algorithm
- ➎ Full solution for the leapfrogging game  
[Fedor Iskhakov](#)
- ➏ Structural estimation of directional dynamic games  
with Nested RLS method

## Overview of results

- We extend the standard static textbook model of Bertrand price competition by allowing duopolists to undertake cost-reducing investments in discrete time
- Technological progress is exogenous and stochastic
- Each firm has a binary decision to acquire the state of the art production technology
- Even though this is a small extension of the classic static model of Bertrand price competition, surprisingly little is known about Bertrand competition in the presence of production cost uncertainty, especially in dynamic settings
- We show how to compute all equilibria of this game and show that this dynamic model of Bertrand price competition has surprisingly rich, complex, and counter-intuitive equilibrium outcomes.

## How do you find *all* Markov Perfect Equilibria?

The Markov Perfect Equilibrium (MPE) concept of Maskin and Tirole (1988) is now a widely used in *empirical IO*. However computing MPE remains a daunting computation problem

Quote (Hörner et. al. *Econometrica* 2011)

"Dynamic games are difficult to solve. In repeated games, finding some equilibrium is easy, as any repetition of a stage-game Nash equilibrium will do. This is not the case in stochastic games. The characterization of even the most elementary equilibria for such games, namely (stationary) Markov equilibria, in which continuation strategies depend on the current state only, turns out to be often challenging."

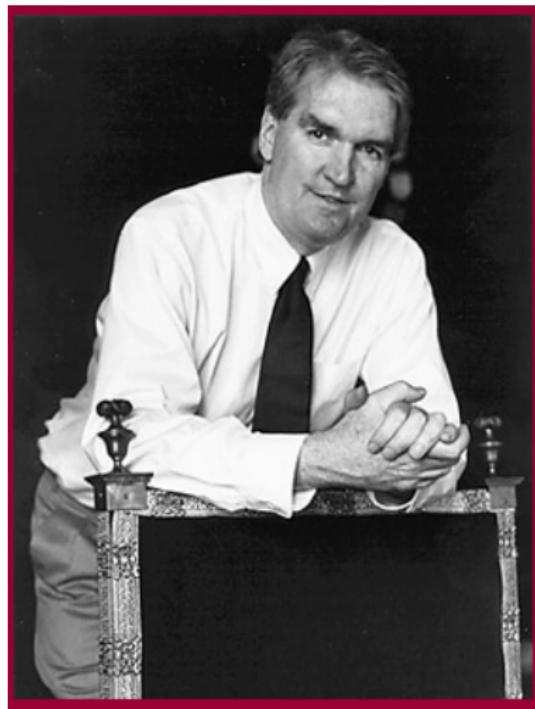
## Finding even a single MPE is challenging!

- How do people “find” MPEs?
- Theorists: Guess and Verify
- Applied people: Iterate on the player Bellman equations
- Pakes and McGuire (1994): some of the earliest work on computing MPE. Proposed a deterministic, iterative algorithm to compute MPE. Found a curse of dimensionality in trying to solve MPE model of firm dynamics with even moderate numbers of firms
- Pakes and McGuire (2001): Proposed a *stochastic algorithm* to approximate an MPE, in an attempt to break this curse of dimensionality

Eric Maskin: taught my game theory class at MIT



One of my classmates: Tim Kehoe



Another classmate: David Levine



Another classmate: Drew Fudenberg



## Another classmate: Jean Tirole



## Motivation: Collusion on the beach



## Peter Brown: Amcor Managing Director



## Harry Debney: Visy CEO



## Russell Jones: Chairman of Amcor



## Richard Pratt: Owner of Visy



## The Australian cardboard market

- The Australian market for *cardboard* (CFP) is essentially a duopoly
- Between 2000 and 2005 the two firms, *Visy* and *Amcor* colluded to raise the price of CFP
- I was hired to estimate the damage caused by the collusion, which requires predicting what CFP prices would have been in the absence of collusion
- My opinion was that the “but-for” CFP prices are those predicted by Bertrand price competition in the short run, with *leapfrogging investments* by the two firms over the longer run as they vie for low-cost leadership

# Amcor's New B9 Paper Mill

**Main Mill Site, Botany Bay Road, Botany Bay NSW**



*Source: Amcor*

## B9 is an example of leapfrogging

- Amcor's existing paper plant was over 50 years old
- "The B9 paper machine, so named as it is the ninth paper machine to operate at the company's Botany site, will produce more than 400,000 tonnes of paper annually when operating at full capacity and will deliver significant environmental benefits."
- Cost: \$500 million, the largest single investment in Amcor's 144 year history. "Largest and most innovative recycled paper machine of its kind in Australasia"
- "The machine is 330 metres long, and 22 metres high, and produces 1.6 km of paper per minute and reduces water consumption by 26%, energy usage by 34% and the amount of waste sent to landfill by 75%" (Nigel Garrard, Amcor CEO)

## But collusion caused B9 to be abandoned

- Amcor had planned B9 back in 1999, and at that time internal studies estimated huge rate of return for this investment because it would enable it to leapfrog Visy to become the low-cost producer of CFP in Australia.
- Amcor and Visy were locked in a price war that started in 1999, around the time the Amcor Board authorized the B9 investment.
- However when Visy and Amcor started to collude in 2000, the B9 project was curiously scrapped. B9 was not actually started until 2011, well after the end of the collusion in 2005. B9 only came online in February 2013.

## Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing Amcor and Visy before and after collusion
- but there was good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

## A cardboard corrugator



## Technological progress via cost-reducing investments

- in this industry, Amcor and Visy do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, Amcor and Visy purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

## Leapfrogging by Amcor lead to a price war

- the proximate cause of the collusion between Amcor and Visy was a price war in cardboard
- a key input to cardboard is *paper* and Amcor had a severe cost disadvantage relative to Visy due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- Visy on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however Amcor planned to invest in a new paper mill, B9, enabling it to produce CFP at substantially lower costs, thereby leapfrogging Visy to become the low cost leader in Australia

## Are price wars evidence of tacit collusion?

- The economic experts defending Amcor and Visy dismissed theory of Bertrand competition and leap frogging investments as naive and out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports a theory *tacit collusion* for repeatedly interacting duopolists
- In particular, they claimed that duopolists could achieve via tacit collusion the same discounted profits as they could via *explicit collusion*.
- This implies that the damage is *zero*.
- But if this is the case, and if tacit collusion is *legal*, why would Amcor and Visy have had any incentive to do illegal explicit collusion?

## Paucity of empirical support for tacit collusion

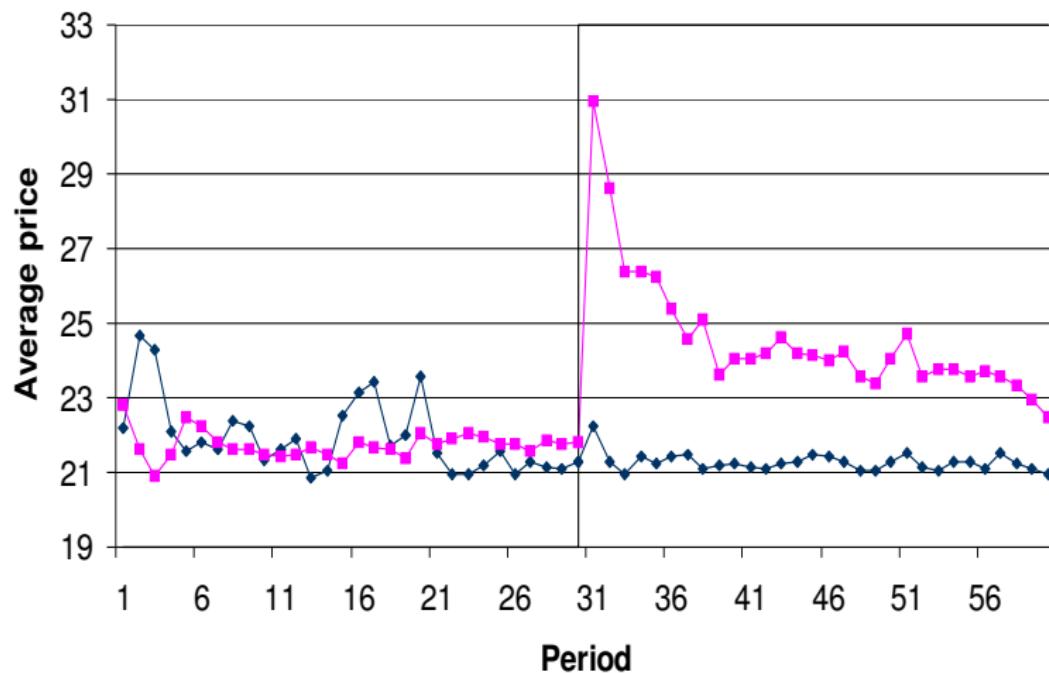
- Tacit collusion is hard to “observe” by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to “grow” tacit collusion in laboratory settings
- There are very few “field studies” that find evidence of tacit collusion outside of Bresnahan's (1987) JIE paper, “Competition and Collusion in the American Automobile Industry: the 1955 Price War”

## Conclusions of meta-study of over 500 experiments

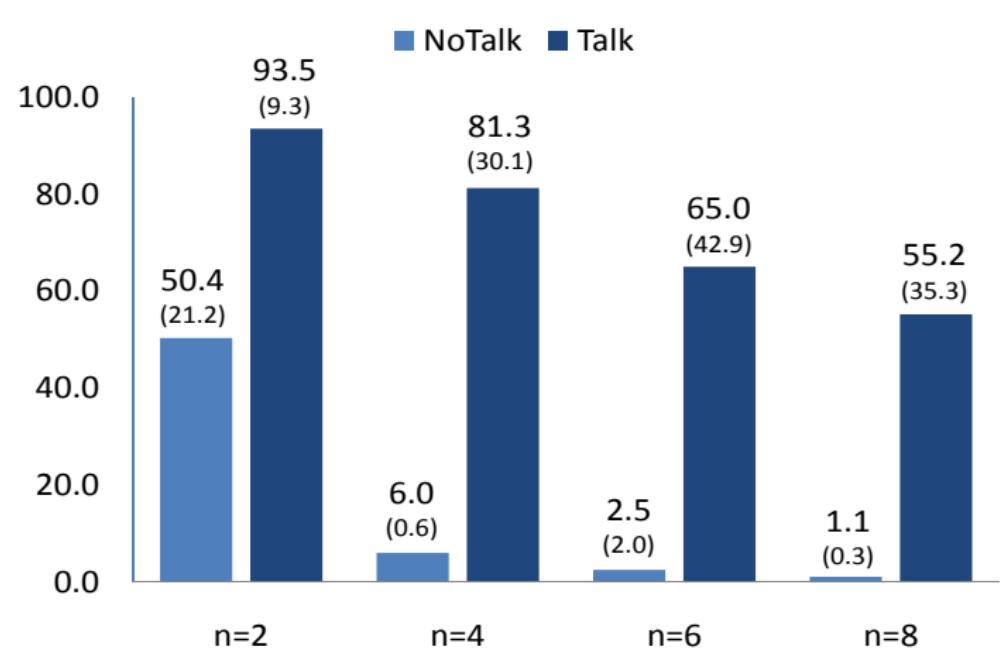
- Christoph Engel (2007) "Tacit Collusion The Neglected Experimental Evidence"
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) "Collusion through price ceilings? A search for a focal point effect"
- "Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude." (p. 2)

# Results of a laboratory duopoly

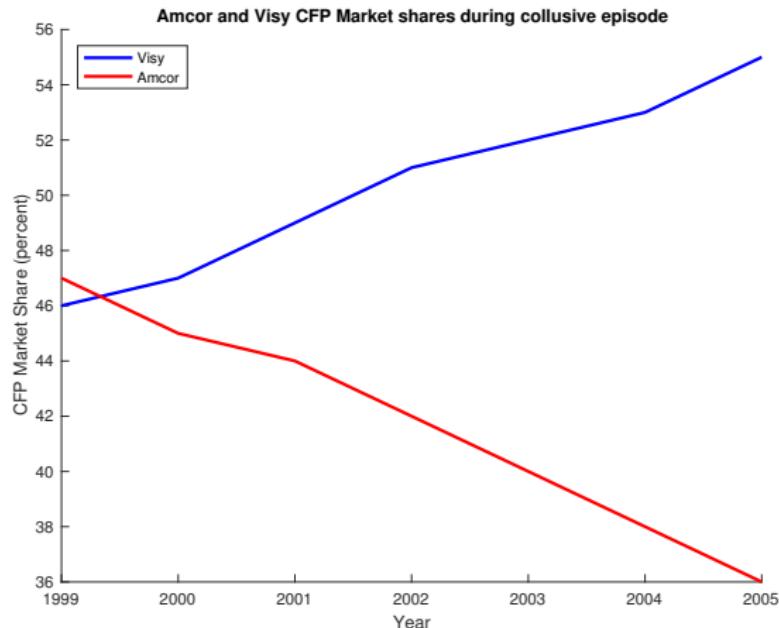
(Note: the Bertrand price is 21, the maximum cartel price is 48 and 28 is the price ceiling)



## Explicit communication is necessary for collusion



## Amcor and Visy collusive market shares



## David Rapson's reanalysis of Bresnahan 1987

- David Rapson (2009) "Tacit Collusion in the 1950s Automobile Industry? Revisiting Bresnahan (1987)"
- "This paper reexamines the competitive landscape in the 1950s U.S. automobile industry, and tests the robustness of the famous result from Bresnahan (1987) that firms were engaged in tacit collusion."
- Rapson uses a random coefficients logit model allows for more realistic demand behavior, including a broad set of possible substitution patterns in characteristic space.
- This enables firms to engage in a more realistic set of potential actions, including intrabrand or intrafirm, multiproduct strategic pricing.

## David Rapson's reanalysis of Bresnahan 1987

- "Relative to Bresnahan's framework, these improvements increase the likelihood that high price-cost markups will be attributed properly to either strategic oligopoly behavior or collusion." (p. 21).
- "*For no year can either of the forms of Bertrand competition be rejected in favor of tacit collusion.* This stands in contrast to Bresnahan's finding that firms were colluding in 1954 and 1956, with a price war in 1955."
- "These results accentuate the paucity of empirical evidence in favor of tacit collusion.
- Bresnahan's (1987) famous paper is one of the only studies that claim evidence of its occurrence."

## Nicolas de Roos' analysis of the US lysine cartel

- Theorists model collusion as an incentive-compatible, self-enforcing mechanism where price wars either do not exist in equilibrium, or if they exist, they are on the equilibrium path of a dynamic game of asymmetric information with cost or demand shocks.
- Papers such as “Cramton and Palfrey (1990) show that efficient collusion is attainable in which the lowest cost firm is allocated full production and the monopoly price is set.”
- However de Roos argues that these theories may be out of touch with reality “Both price wars contain elements of a bargaining or negotiation problem. Disagreements persisted about the appropriate market shares for the participants as well as the fundamental issue of exactly what form the cartel should take.”

## de Roos' analysis of the US lysine cartel, cont.

- “A second such issue relates to the existence of cheating in the lysine market. It appears that cheating occurred or was at least heavily suspected by cartel participants.” . . . “**where cheating is a problem for a cartel, this suggests the lack of an incentive compatible enforcement mechanism.**”
- A price war, prior to the cartel, was a result of ADM’s leap-frogging: “In 1988, ADM acquired a fermentation technique for lysine and, observed by its incumbent rivals, began production of the world’s largest lysine factory in 1989. ADM’s plant began production in February 1991, precipitating a severe price war. ”

# The Bertrand Investment Paradox

**Why should Bertrand competitors undertake cost-reducing investments?**

- Suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost  $c$
- Bertrand price competition following these investments will lead to a price of  $p = c$  and *zero profits for each firm*
- If each firm earns zero profits *ex post*, why would either have incentive to invest *ex ante*?

**The investment stage game is an anti-coordination game. Can the firms dynamically coordinate their investments in equilibrium, in order to avoid "bad" simultaneous investment outcomes?**

## The Riordan and Salant Conjecture

- In their 1994 *Journal of Industrial Economics* article, Riordan and Salant proved that in continuous time, if duopolists move alternately and technological progress is deterministic, then **investment preemption is the only possible equilibrium outcome**
- Further, they show this equilibrium is *completely inefficient* due to the excessively frequent investments of the preempting firm, a result they call **rent dissipation**
- They conjectured that their result does not depend on the alternating move assumption and that preemption (as opposed to leapfrogging) will be the generic equilibrium outcome in models of Bertrand price competition with cost-reducing investments.

# Solution to the Bertrand Investment Paradox

We show:

- Endogenous coordination is possible in equilibrium
  - leapfrogging (alternating investments) is possible
  - We show that the Riordan and Salant conjecture is wrong:  
leapfrogging, not preemption, is the generic outcome
- Price paths are piecewise flat and non-increasing
  - *Price wars* occur when the high cost firm leapfrogs its rival to become the new low-cost leader
  - These price wars lead to *permanent* price declines, unlike the conventional interpretation of price wars as punishments for periodic breakdowns in tacit collusion
- Equilibria are generally inefficient due to overinvestment
  - duplicative investments
  - excessively frequent investments

# Computing all equilibria

**Our findings are based on the computation of all Markov perfect equilibria of this dynamic game**

- New solution approach consisting of:
  - ① State recursion algorithm for finding stage equilibria
  - ② Recursive Lexicographic equilibrium Search (RLS) algorithm for finding all MPE paths
- Traditional solution approach (value function iterations, i.e. time recursion) fails in this model due to multiplicity of equilibria
  - Implementation of the Bellman operator induces an equilibrium selection rule
  - Not a contraction mapping, convergence is not guaranteed
- Danger of imposing symmetry
  - Most of MPE equilibria we find are asymmetric

## Part II: Dynamic Bertrand price competition (leapfrogging model)

# Dynamic Bertrand price competition

## Stochastic dynamic game

- Two Bertrand competitors,  $n = 2$ , no entry or exit
- Discrete time, infinite horizon ( $t = 1, 2, \dots, \infty$ )
- Each firm maximizes expected discounted profits, common discount factor  $\beta \in (0, 1)$
- Each firm has two choices in each period:
  - ① Price for the product
  - ② Whether or not to buy the state of the art technology

## Static Bertrand price competition in each period

- Continuum of consumers make static purchase decision
- No switching costs: buy from the lower price supplier

# Cost-reducing investments

## State-of-the-art production cost $c$ process

- Initial value  $c_0$ , lowest value 0:  $0 \leq c \leq c_0$
- Discretized with  $n$  points
- Follows exogenous Markov process and only improves
- Markov transition probability  $\pi(c_{t+1}|c_t)$   
 $\pi(c_{t+1}|c_t) = 0$  if  $c_{t+1} > c_t$

## Investment choice: dichotomous

- Investment cost of  $K(c)$  to obtain marginal cost  $c$
- One period construction time: production with technology obtained at  $t$  starts at  $t + 1$

# State space and information structure

## Common knowledge

- State of the game: cost structure  $(c_1, c_2, c)$
- State space is  $S = (c_1, c_2, c) \subset R^3: c_1 \geq c, c_2 \geq c$
- Actions are observable

## Private information

- In each period each firm incurs additive costs (benefits) from not investing and investing  $\eta \epsilon_{i,I}$  and  $\eta \epsilon_{i,N}$
- $\epsilon_{i,I}$  and  $\epsilon_{i,N}$  are extreme value distributed, independent across choice, time and firms
- $\eta \geq 0$  is a scaling parameter
- Investment choice probabilities have logit form for  $\eta > 0$

## Timing of moves

Pricing decisions are made simultaneously

Expected one period profit of firm  $i$  from Bertrand game ( $j \neq i$ )

$$r_i(c_1, c_2) = \begin{cases} 0 & \text{if } c_i \geq c_j \\ c_j - c_i & \text{if } c_i < c_j \end{cases}$$

Two versions regarding investment decisions

① Simultaneous moves:

- Investment decisions are made simultaneously

② Alternating moves:

- The “right to move” state variable  $m \in \{1, 2\}$ ,
- When  $m = i$ , only firm  $i$  can make a cost reducing investment
- $m$  follows an own Markov process  
(deterministic alternation as a special case).

# Actions and behavior strategies

## Two choices in each period

- $p_i(c_1, c_2, c) = \max(c_1, c_2)$  – Bertrand pricing decision
- $P_i^I(c_1, c_2, c)$  – probability of firm  $i$  to invest in state-of-the-art production technology  
 $P_i^N(c_1, c_2, c) = 1 - P_i^I(c_1, c_2, c)$  – probability not to invest

## Strategy profile

- $\sigma = (\sigma_1, \sigma_2)$  – pair of Markovian *behavior* strategies  
 $\sigma_i = (p_i(c_1, c_2, c), P_i^I(c_1, c_2, c)) \in \mathbb{R}_+ \times [0, 1]$
- Pure strategies included as special case

# Definition of Markov Perfect Equilibrium

## Definition (Markov perfect equilibrium (MPE))

MPE of Bertrand investment stochastic game is a pair of

- strategy profile  $\sigma^* = (\sigma_1^*, \sigma_2^*)$ , and
- pair of *value functions*  $V(s) = (V_1(s), V_2(s))$ ,  $V_i : S \rightarrow R$ ,

such that

- ❶ Bellman equations (below) are satisfied for each firm, and
- ❷ strategies  $\sigma_1^*$  and  $\sigma_2^*$  constitute mutual best responses, and assign positive probabilities only to the actions in the set of maximizers of the Bellman equations.

## Bellman equations, firm $i = 1$ , simultaneous moves

$$V_i(c_1, c_2, c) = \max [v_i^I(c_1, c_2, c) + \eta \epsilon_{i,I}, v_i^N(c_1, c_2, c) + \eta \epsilon_{i,N}]$$

$$v_i^N(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV_i(c_1, c_2, c|N)$$

$$v_i^I(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV_i(c_1, c_2, c|I)$$

With extreme value shocks, the investment probability is

$$P_i^I(c_1, c_2, c) = \frac{\exp\{v_i^I(c_1, c_2, c)/\eta\}}{\exp\{v_i^I(c_1, c_2, c)/\eta\} + \exp\{v_i^N(c_1, c_2, c)/\eta\}}$$

## Bellman equations, firm $i = 1$ , simultaneous moves

The expected values are given by

$$EV_i(c_1, c_2, c | \text{N}) = \int_0^c [P_j^I(c_1, c_2, c) H_i(c_1, c, c') + P_j^N(c_1, c_2, c) H_i(c, c, c')] \pi(dc' | c)$$

$$EV_i(c_1, c_2, c | I) = \int_0^c [P_j^I(c_1, c_2, c) H_i(c, c, c') + P_j^N(c_1, c_2, c) H_i(c, c, c')] \pi(dc' | c)$$

where

$$H_i(c_1, c_2, c) = \eta \log [\exp(v_i^N(c_1, c_2, c)/\eta) + \exp(v_i^I(c_1, c_2, c)/\eta)]$$

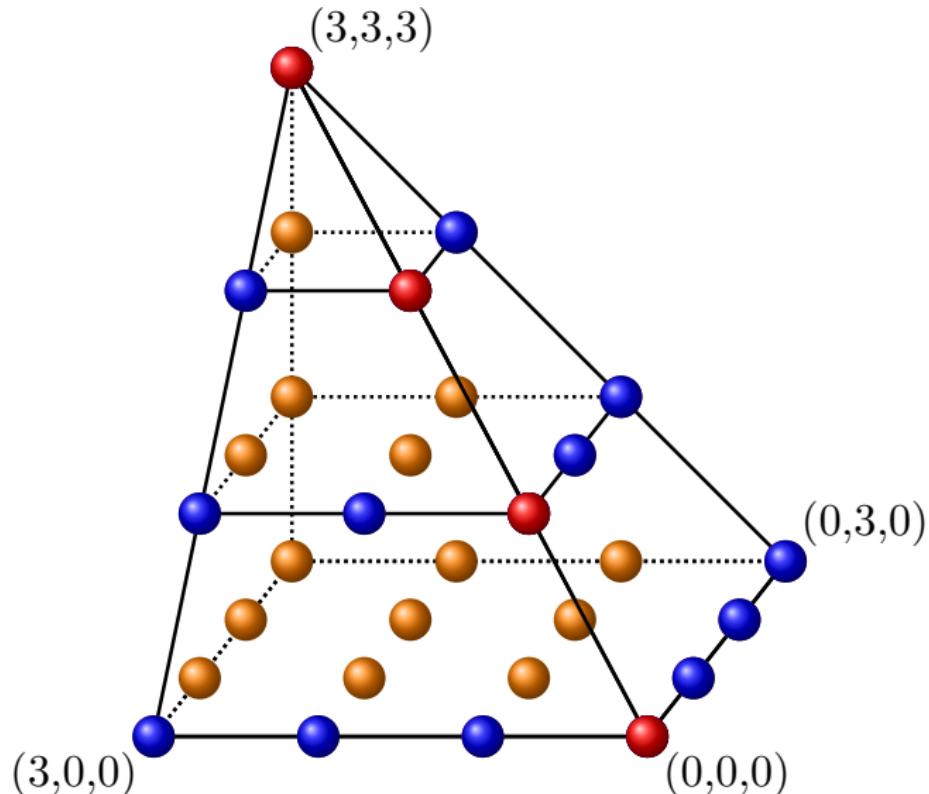
is the “smoothed max” or logsum function

# Leapfrogging Game



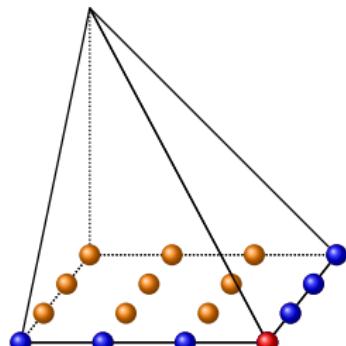
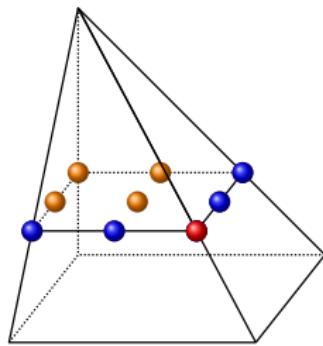
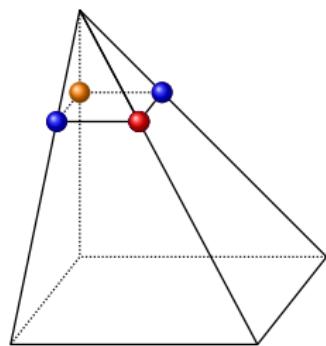
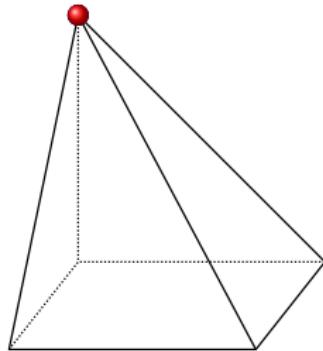
Discretized state space = a “quarter pyramid”

$$S = \{(c_1, c_2, c) | c_1 \geq c, c_2 \geq c, c \in [0, 3]\}, n = 4$$



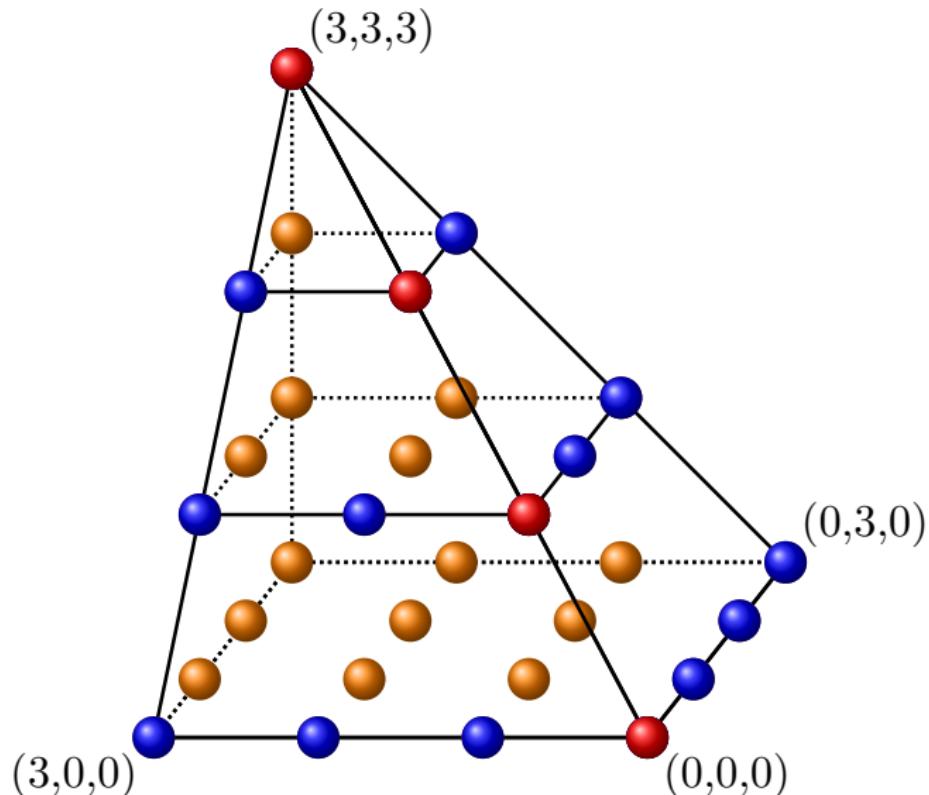
# Transitions due to technological progress

As  $c$  decreases, the game falls through the layers of the pyramid



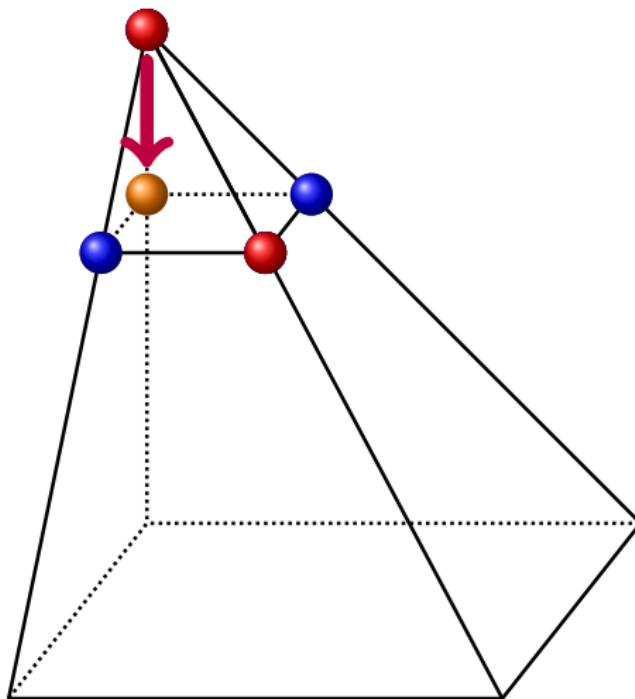
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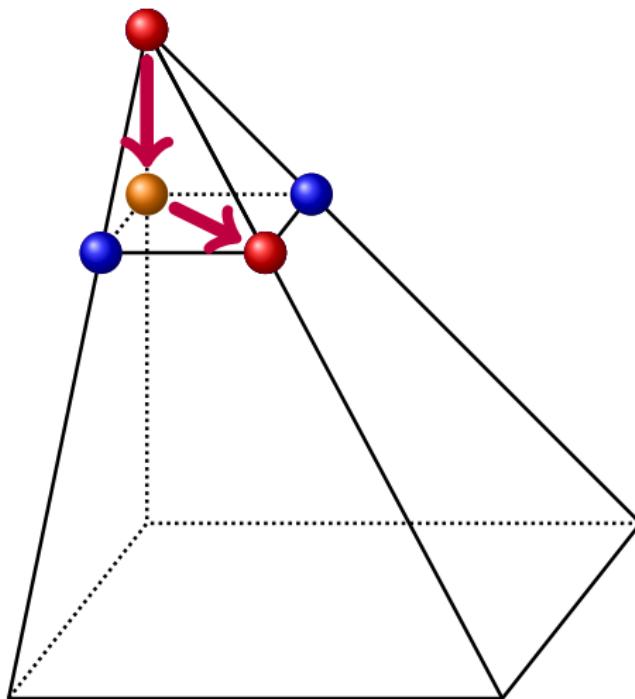
## Game dynamics: example

The game starts at the apex, as some point technology improves



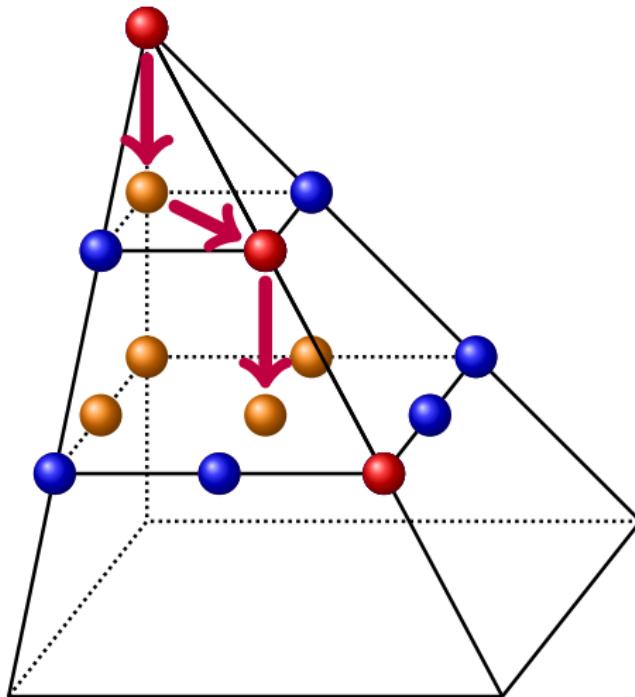
## Game dynamics: example

Both firms buy new technology  $c = 2 \rightsquigarrow (c_1, c_2, c) = (2, 2, 2)$



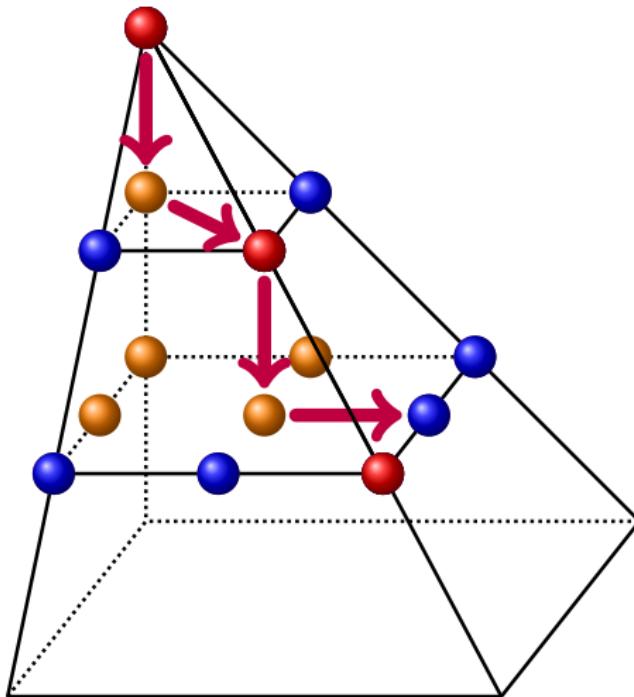
## Game dynamics: example

State-of-the-art technology becomes  $c = 1 \rightsquigarrow (c_1, c_2, c) = (2, 2, 1)$



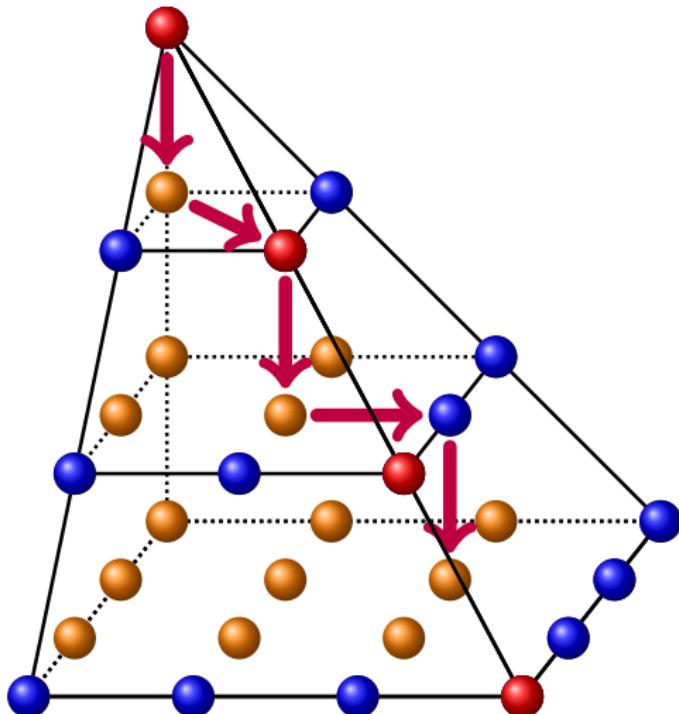
## Game dynamics: example

Firm 1 invests and becomes cost leader  $\rightsquigarrow (c_1, c_2, c) = (1, 2, 1)$



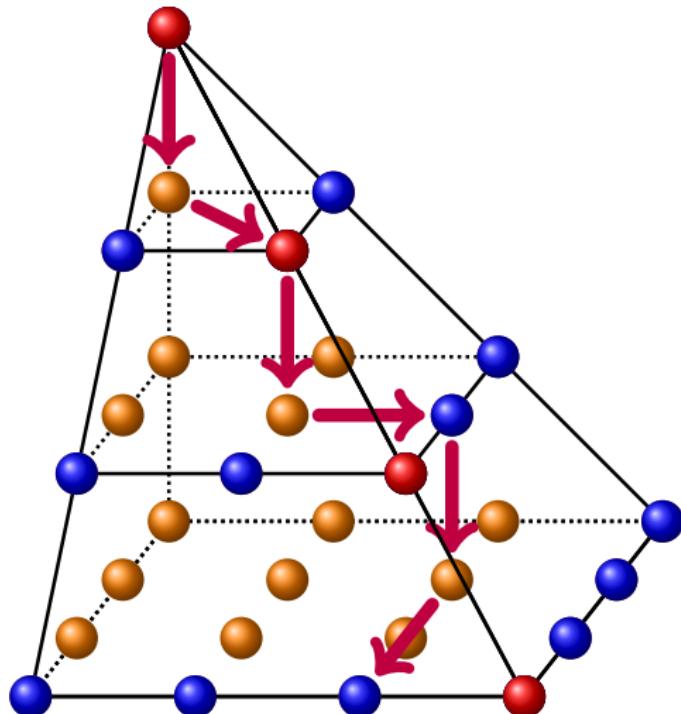
## Game dynamics: example

State-of-the-art technology becomes  $c = 0 \rightsquigarrow (c_1, c_2, c) = (1, 2, 0)$



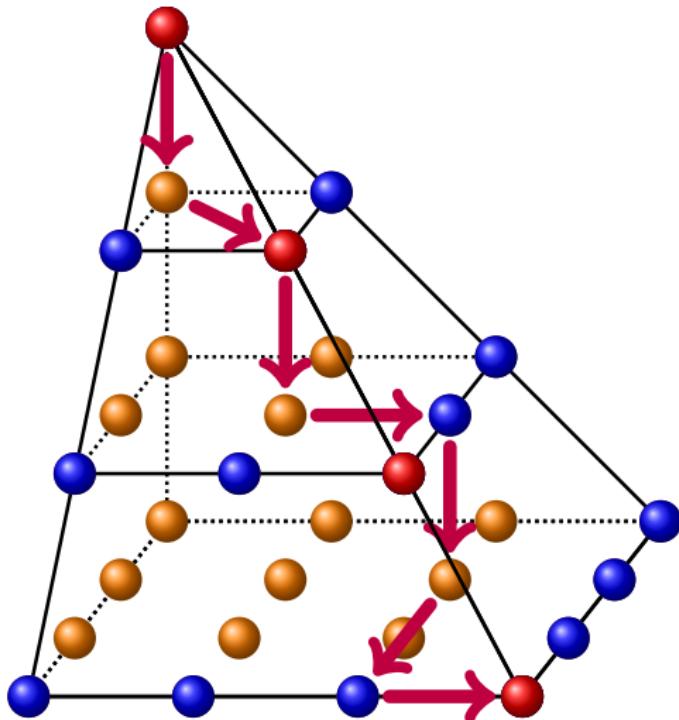
## Game dynamics: example

Firm 2 leapfrogs firm 1 to become new cost leader  $\rightsquigarrow (c_1, c_2, c) = (1, 0, 0)$



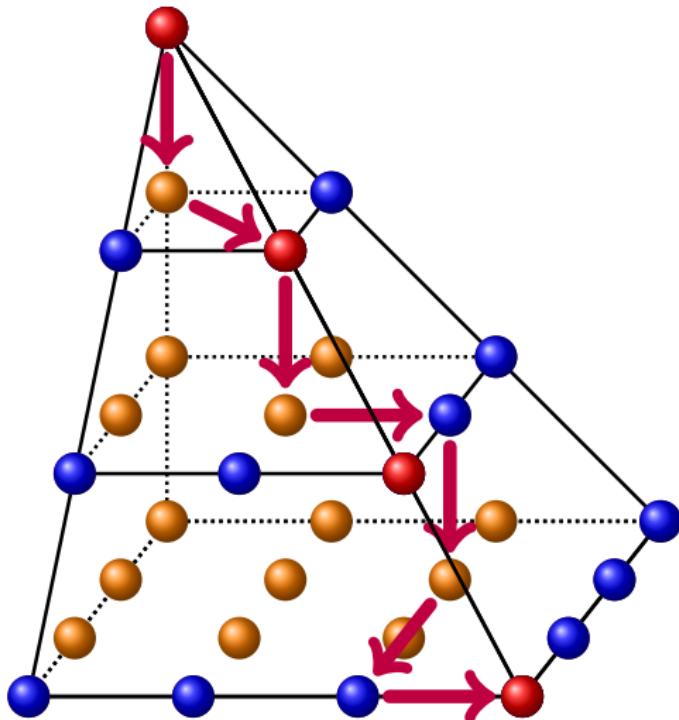
## Game dynamics: example

Firm 1 invests, and the game reaches terminal state  $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



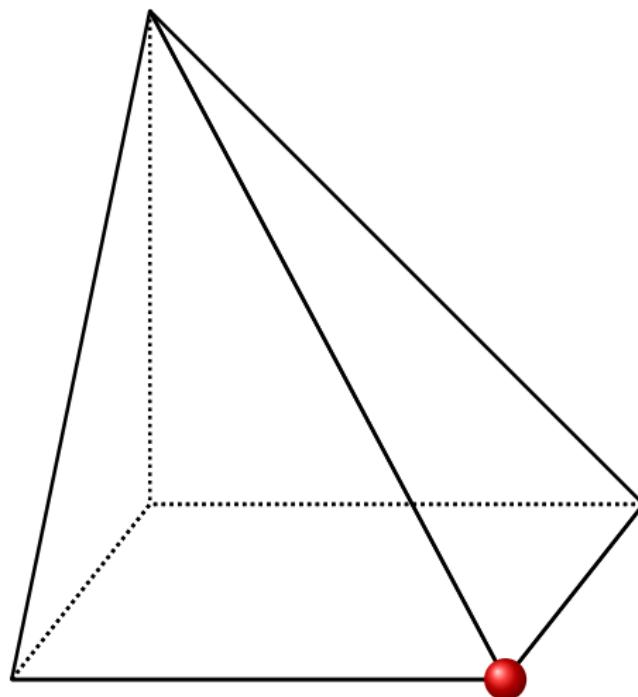
## Game dynamics: example

Firm 1 invests, and the game reaches terminal state  $\rightsquigarrow (c_1, c_2, c) = (0, 0, 0)$



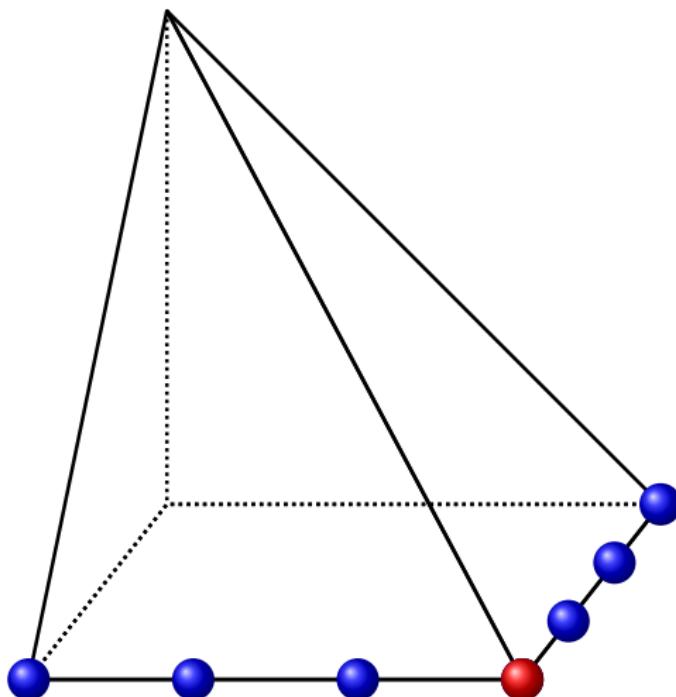
## State recursion

Solve terminal state  $(c_1, c_2, c) = (0, 0, 0)$



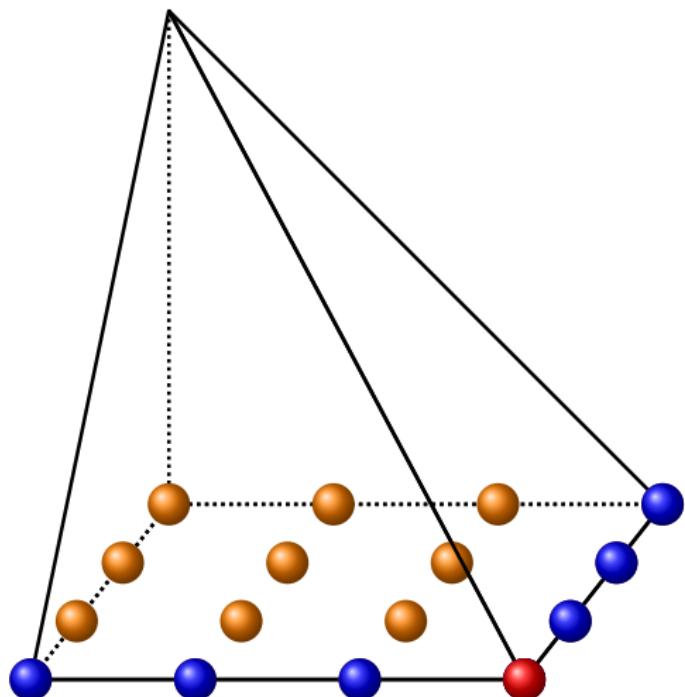
## State recursion

Solve bottom layer edges  $(c_1, c_2, c) = (0, c_2, 0)$  and  $(c_1, 0, 0)$



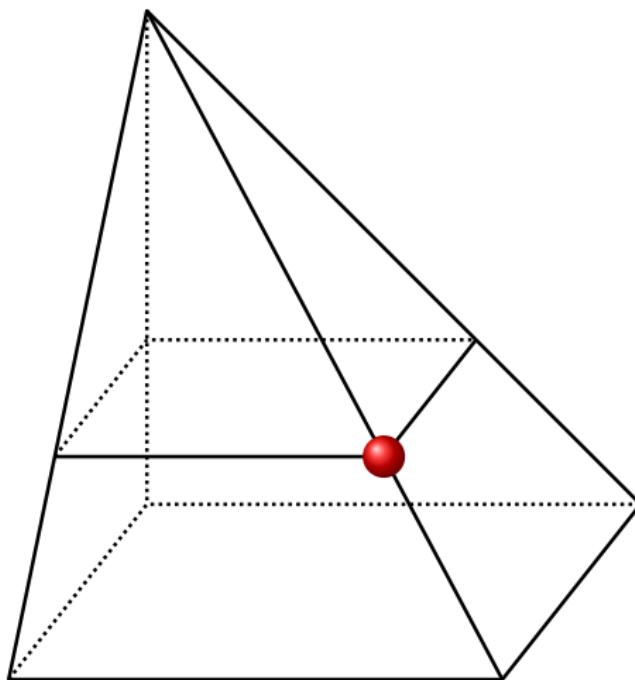
## State recursion

Solve bottom layer interior  $(c_1, c_2, c) = (c_1, c_2, 0)$



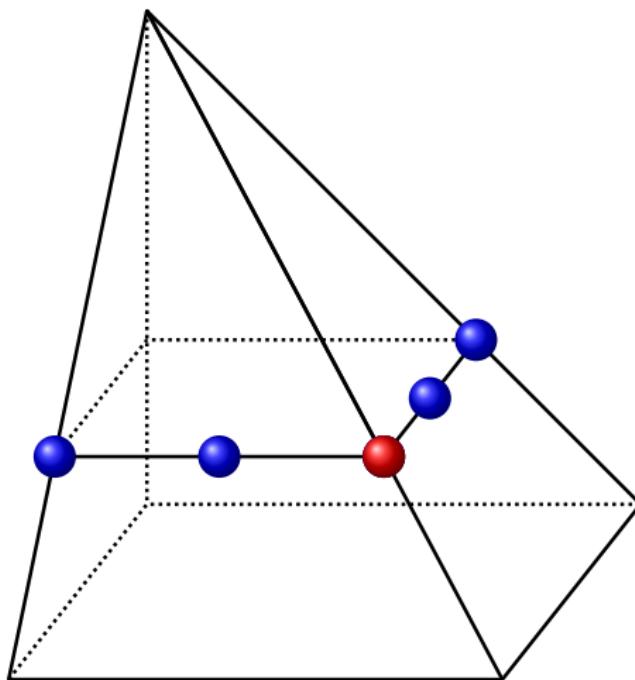
## State recursion

Solve the corner  $(c_1, c_2, c) = (1, 1, 1)$



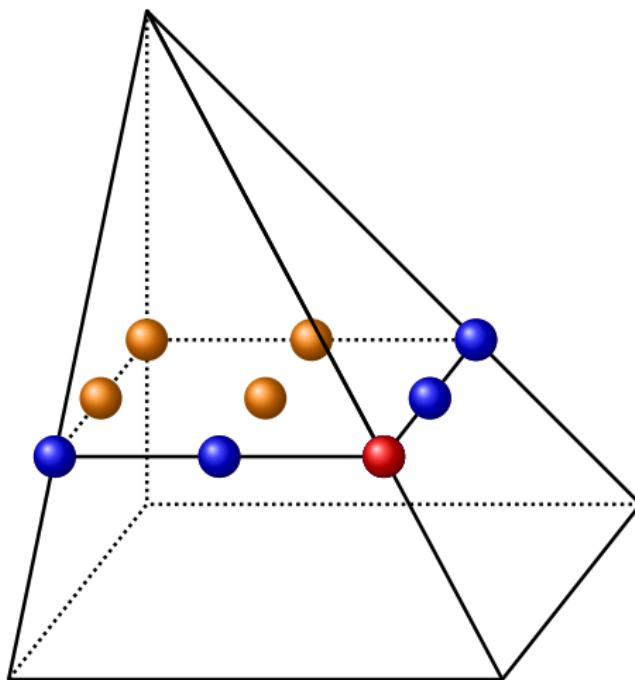
## State recursion

Solve edges  $(c_1, c_2, c) = (1, c_2, 1)$  and  $(c_1, 1, 1)$



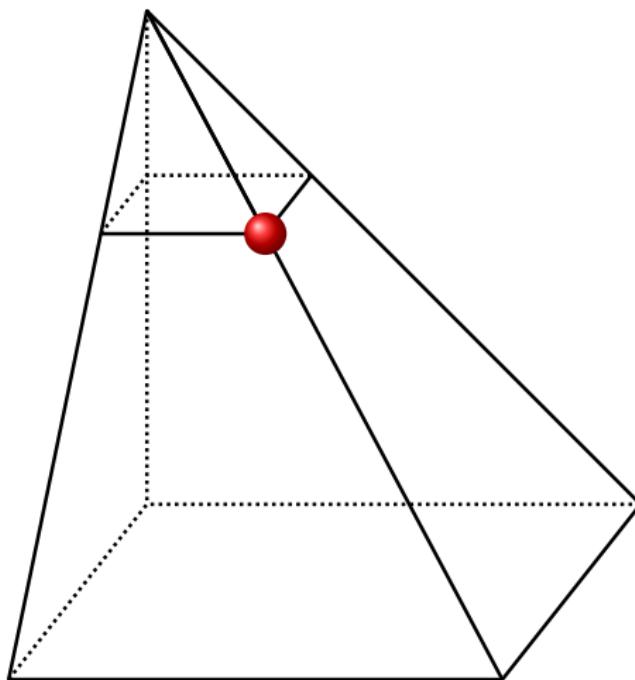
## State recursion

Solve interior  $(c_1, c_2, c) = (c_1, c_2, 1)$



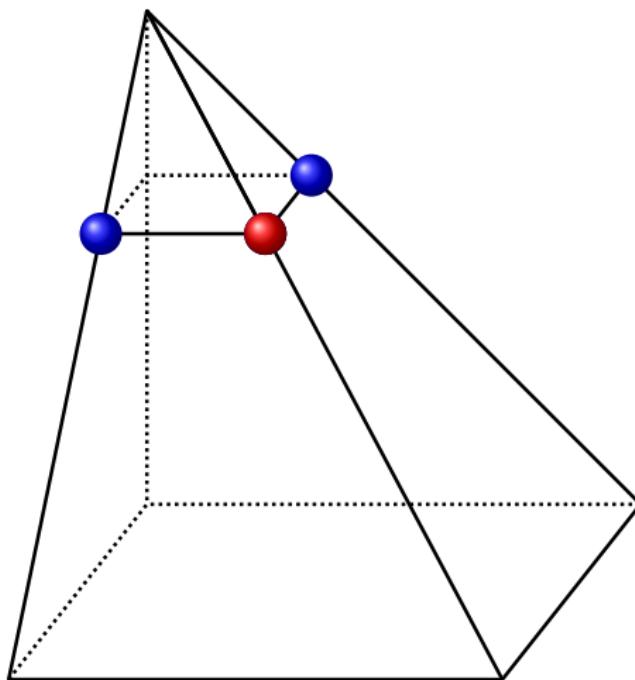
## State recursion

Solve the corner  $(c_1, c_2, c) = (2, 2, 2)$



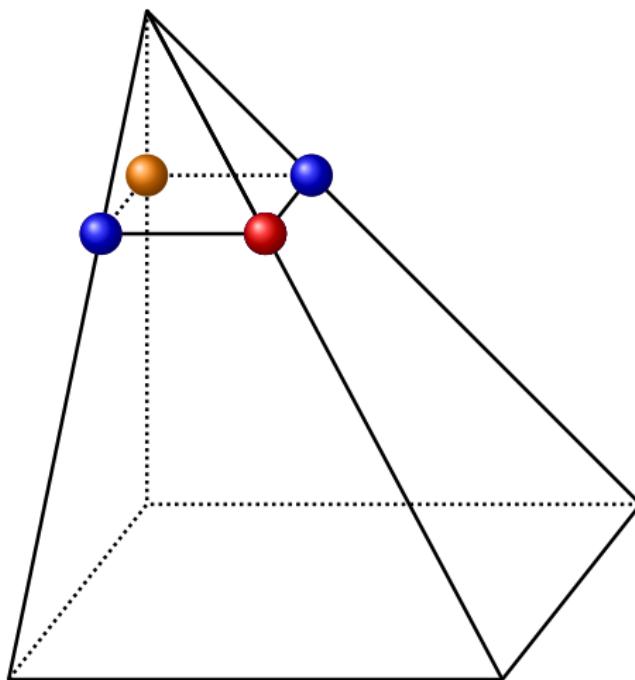
## State recursion

Solve edges  $(c_1, c_2, c) = (2, c_2, 2)$  and  $(c_1, 2, 2)$



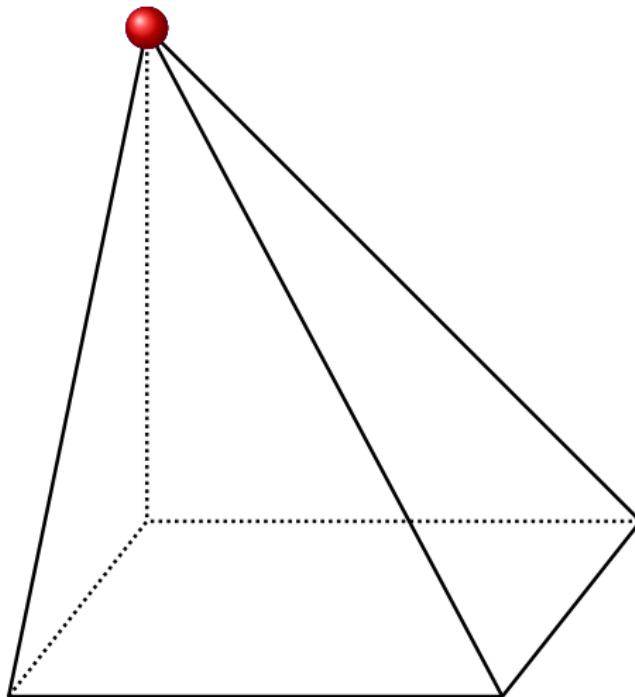
## State recursion

Solve interior  $(c_1, c_2, c) = (c_1, c_2, 2)$



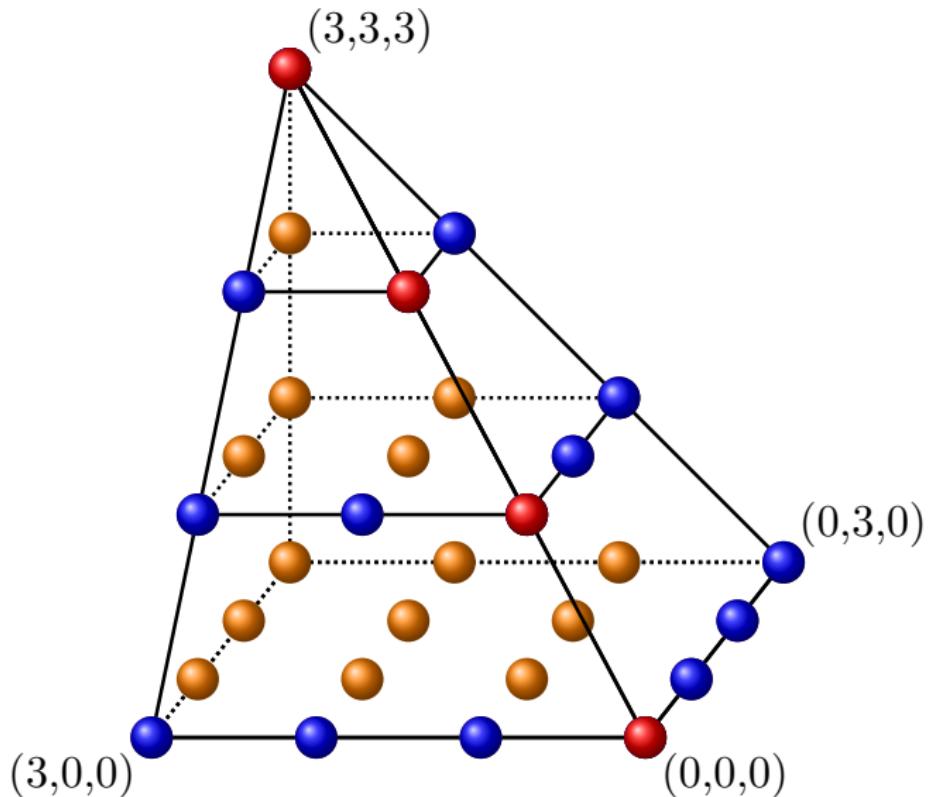
## State recursion

Finally solve the apex  $(c_1, c_2, c) = (3, 3, 3)$



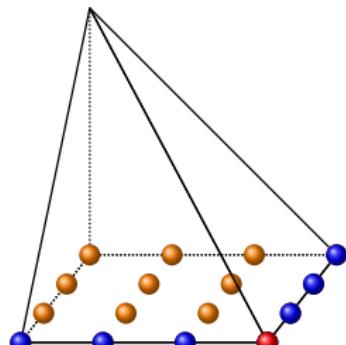
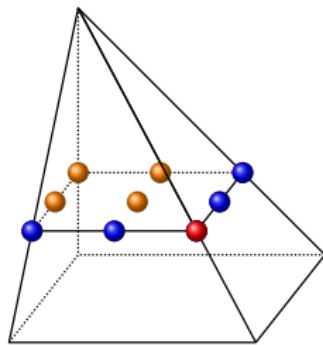
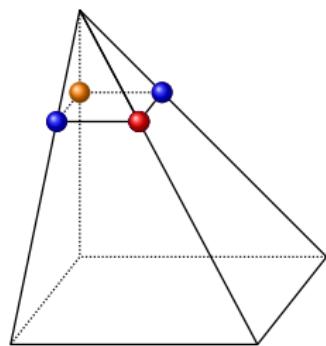
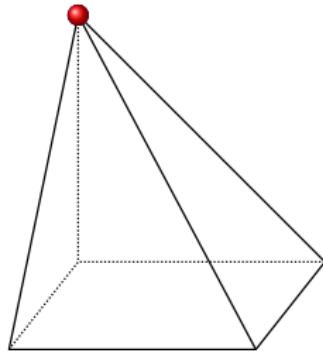
## State recursion

The whole game is solved!



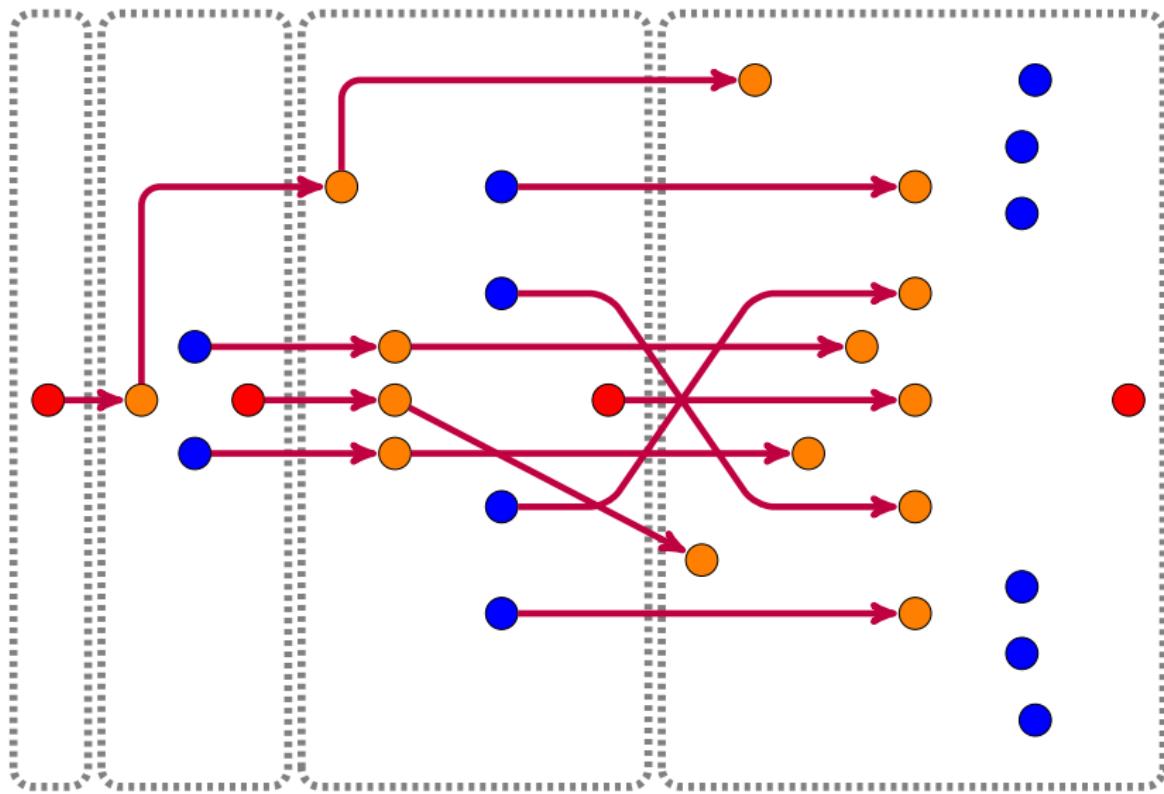
# Transitions due to technological progress

As  $c$  decreases, the game falls through the layers of the pyramid



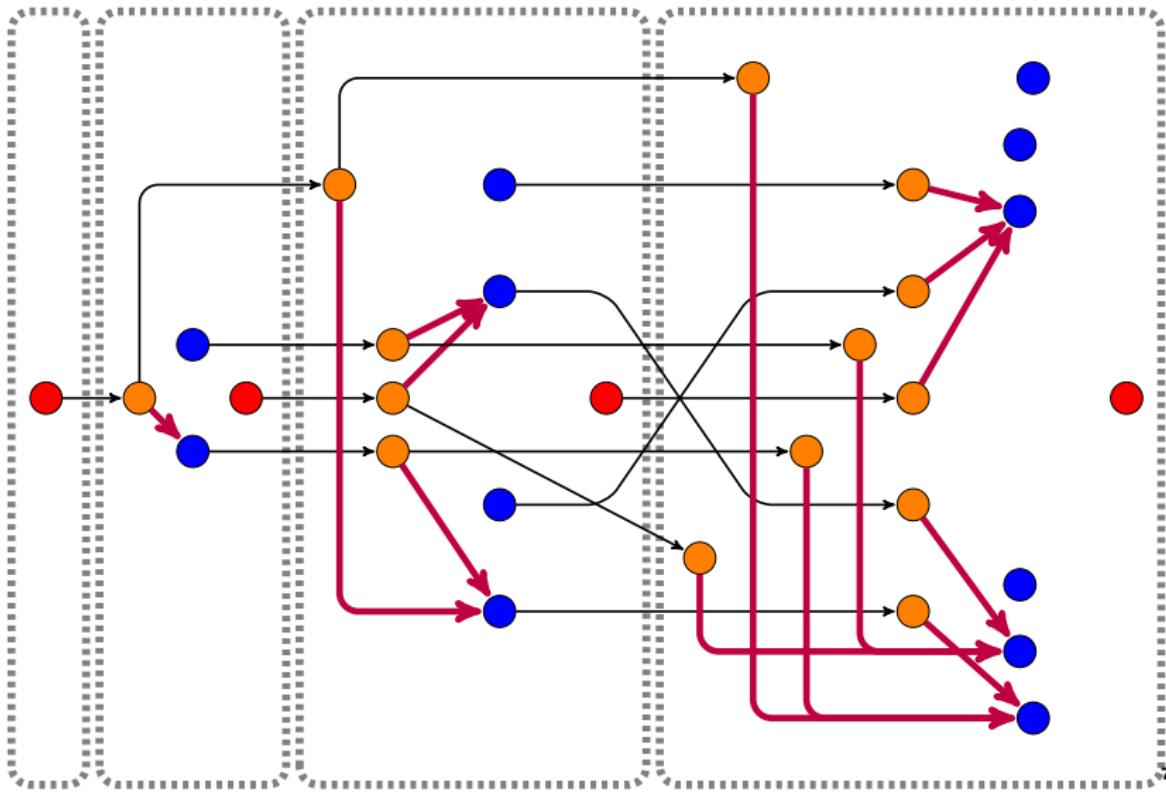
# Transitions due to technological progress

As  $c$  decreases, the game falls through the layers of the pyramid



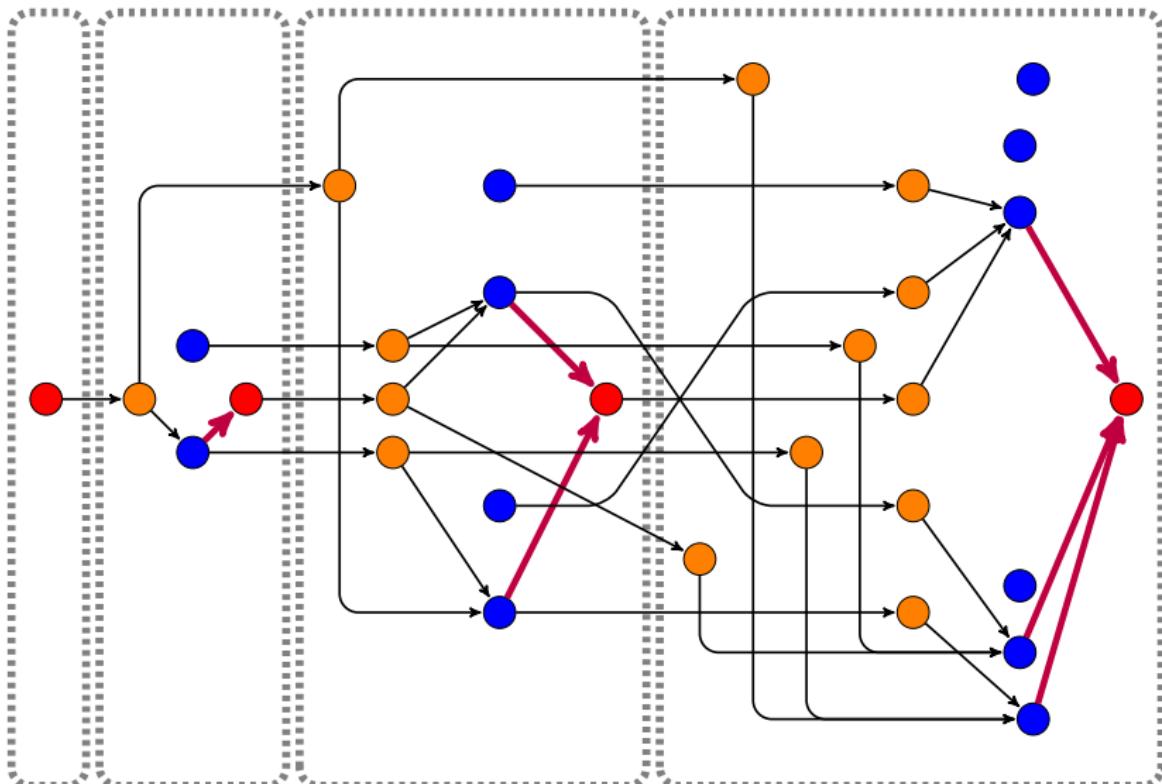
# Strategy-specific partial order on $S$

Strategy  $\sigma_1$  of firm 1: invest at all interior points



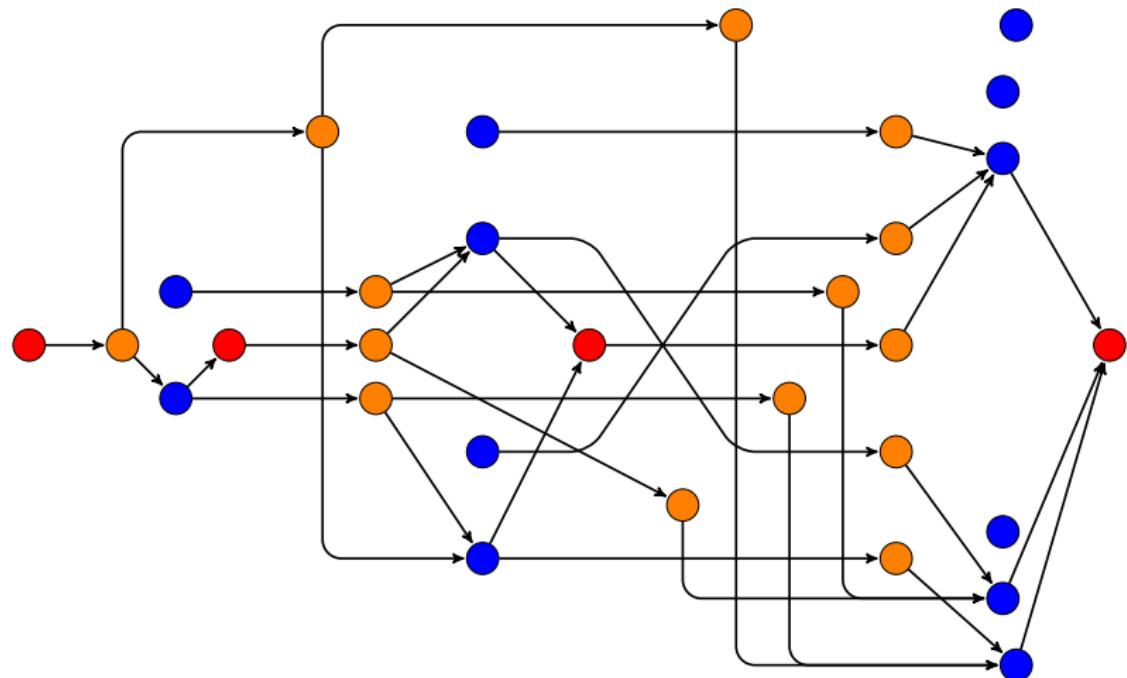
# Strategy-specific partial order on $S$

Strategy  $\sigma_2$  of firm 2: invest at all edge points



## Strategy-specific partial order on $S$

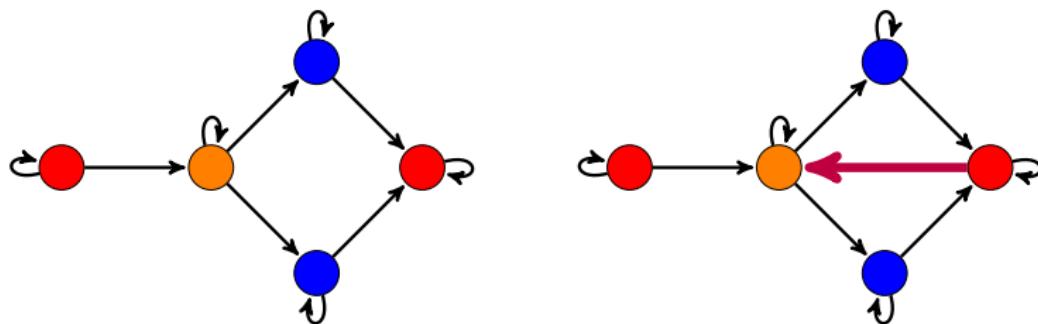
Strategy  $\sigma = (\sigma_1, \sigma_2)$  of both firms



## No loop (anti-cycling) condition

Hypothetical strategy profile inducing cycles

**Self-loops** appear when the game remains in the same state for two or more consecutive periods of time

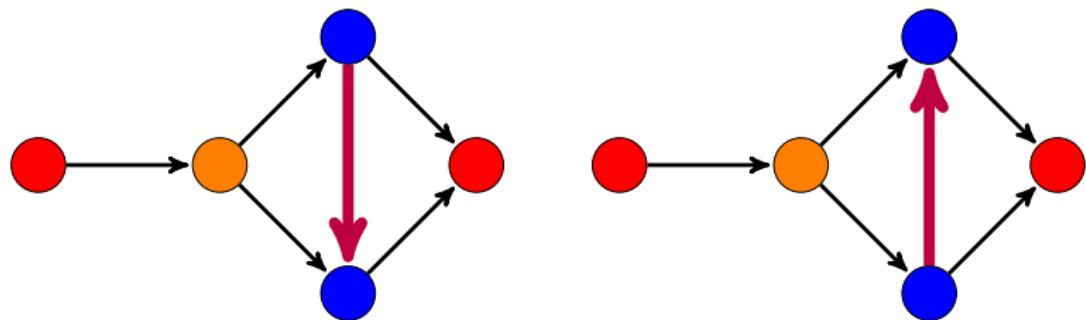


But loops between different states are not allowed

# Consistency of strategy specific partial orders

Two hypothetical inconsistent strategies

Two strategies that induce opposite transitions are **inconsistent**



Note that in both cases the no-loop condition is satisfied

# Definition of the Dynamic Directional Games

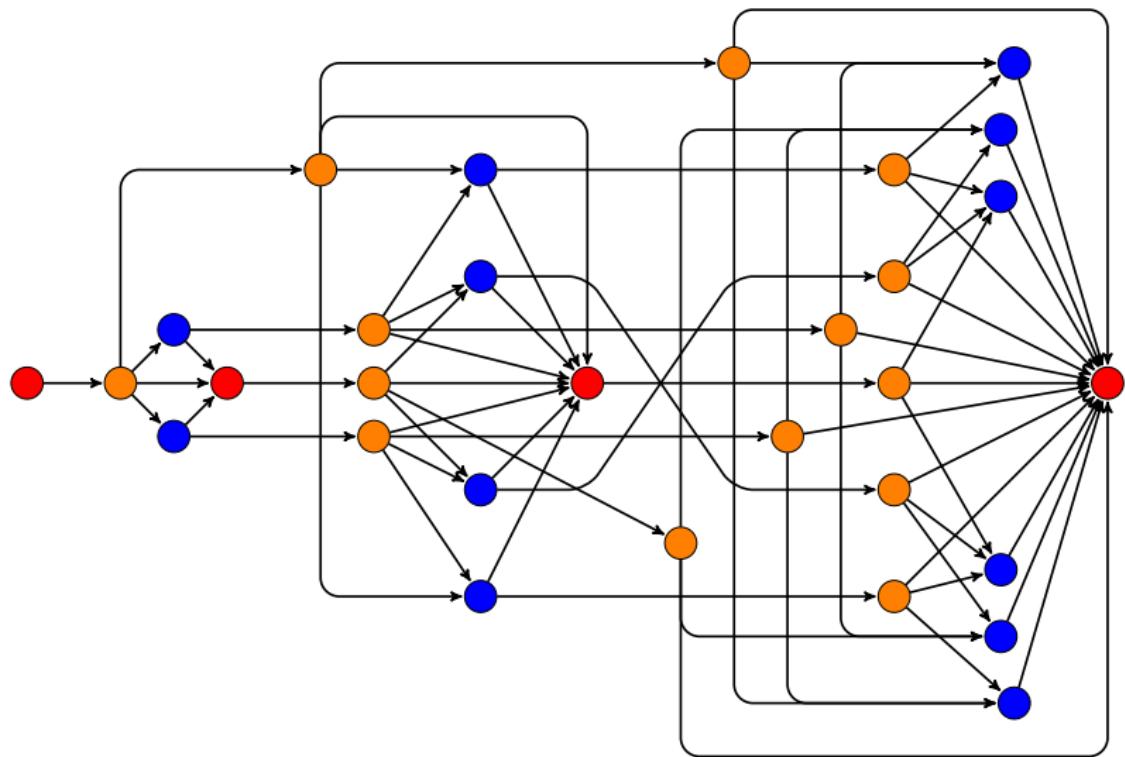
## Definition (Dynamic Directional Games, DDG)

Finite state Markovian stochastic game is a DDG if it holds:

- ① Every feasible Markovian strategy  $\sigma$  satisfies the no loop condition.
- ② Every pair of feasible Markovian strategies  $\sigma$  and  $\sigma'$  induce consistent partial orders on the state space.

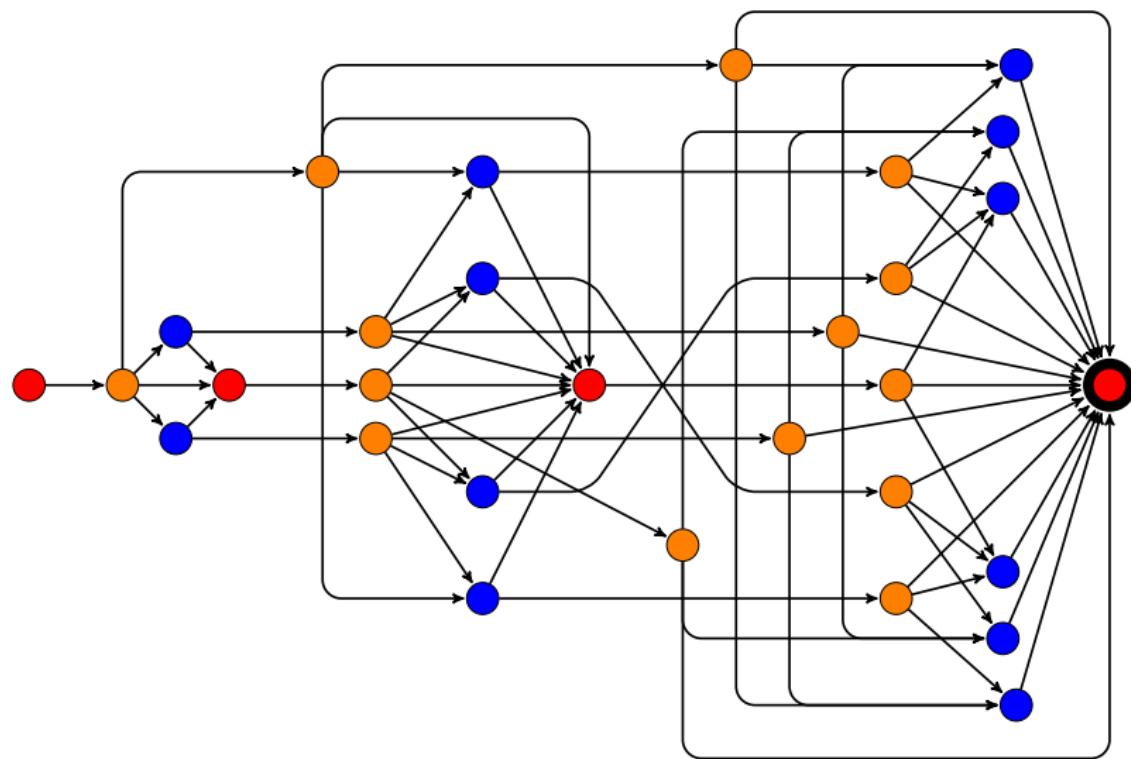
# Strategy independent partial order on $S$

Coarsest common refinement of partial orders induced by all strategies



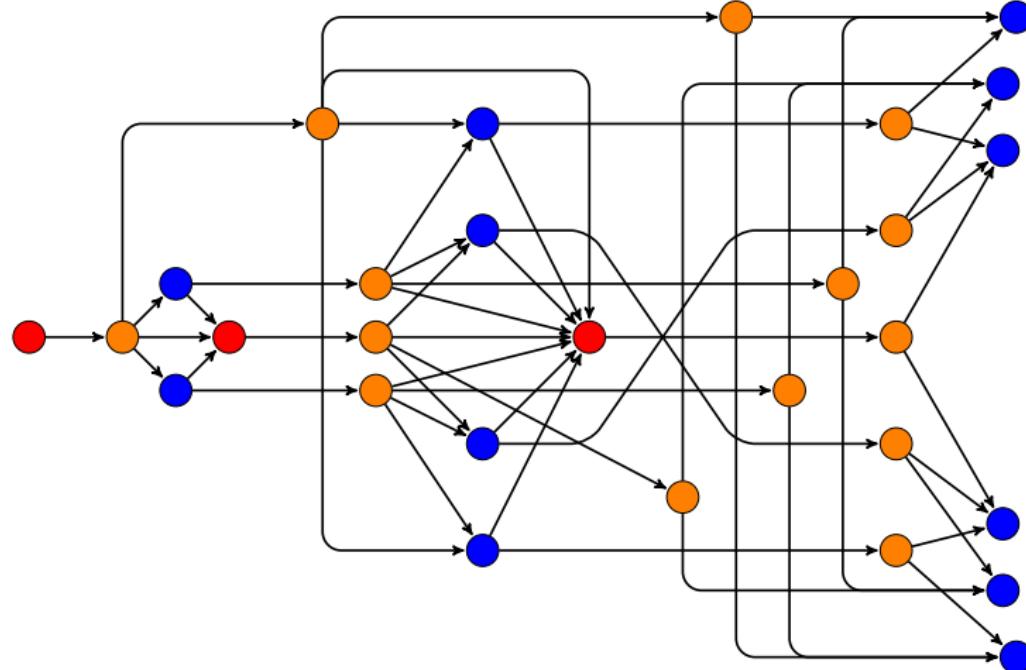
# DAG recursion to partition $S$ into stages

Identify terminal states



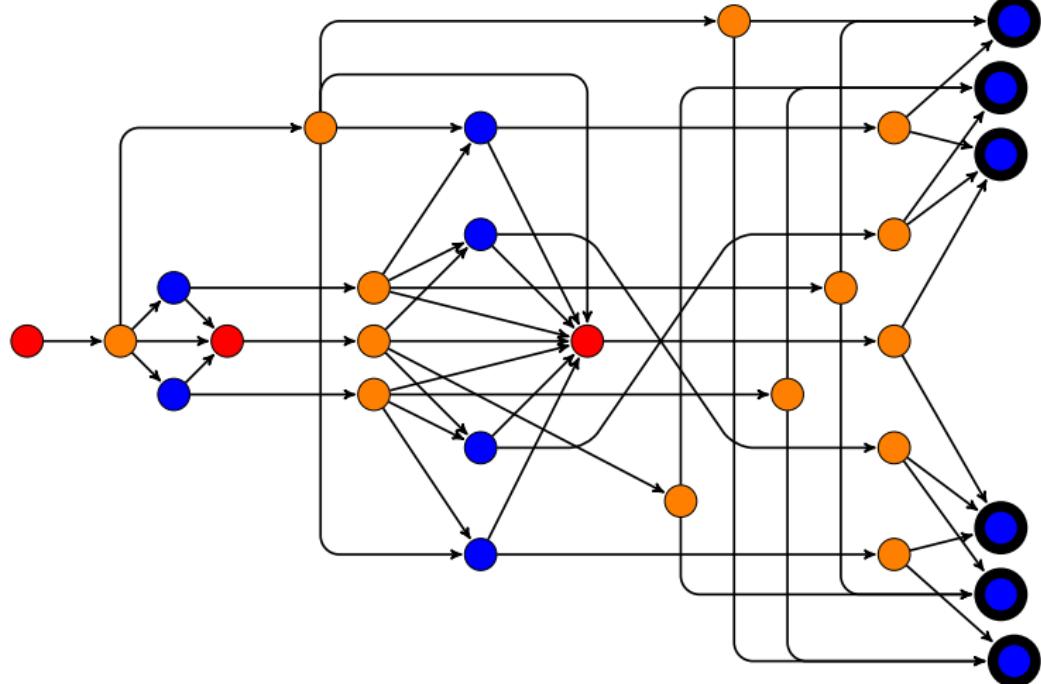
DAG recursion to partition  $S$  into stages

## Remove terminal states



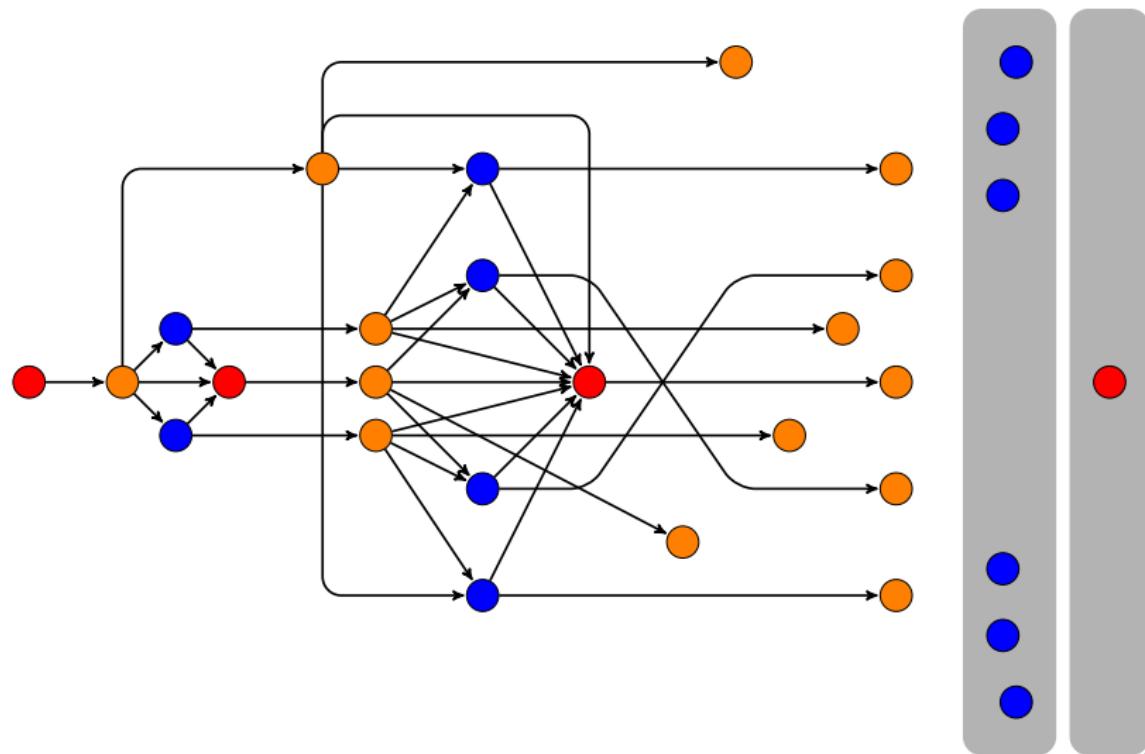
# DAG recursion to partition $S$ into stages

Identify terminal states



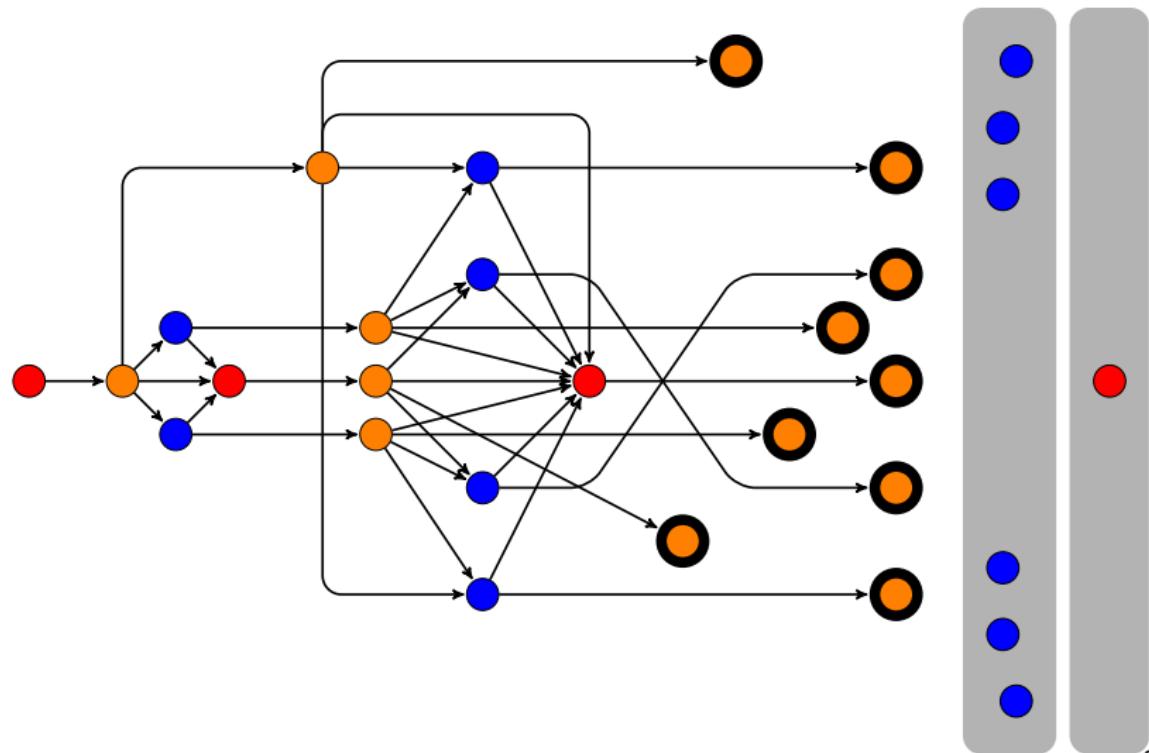
## DAG recursion to partition $S$ into stages

Remove terminal states



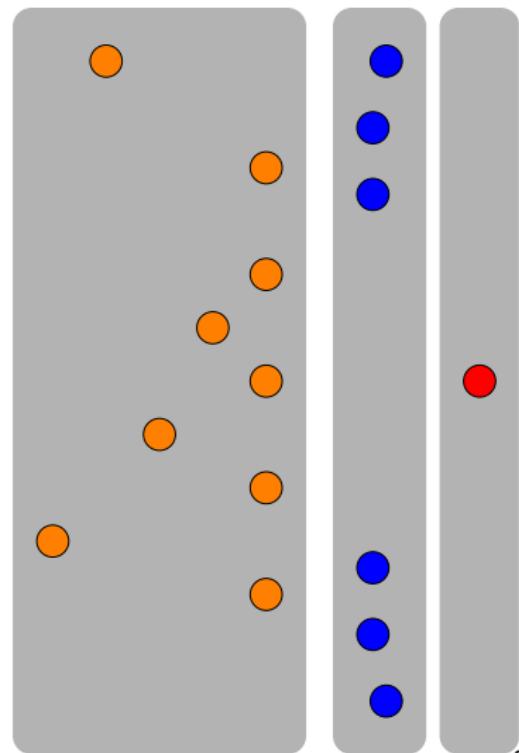
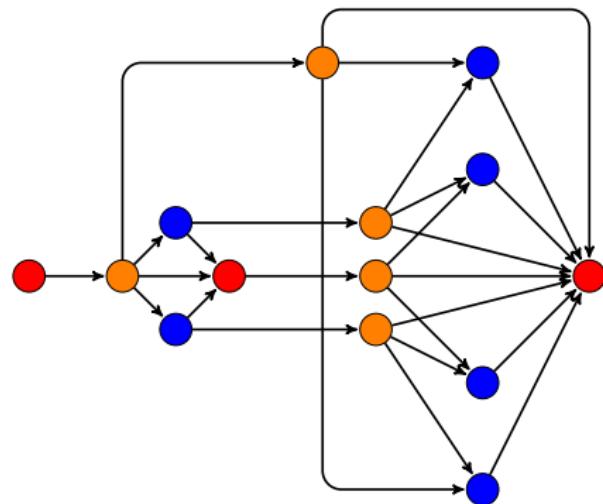
## DAG recursion to partition $S$ into stages

Identify terminal states



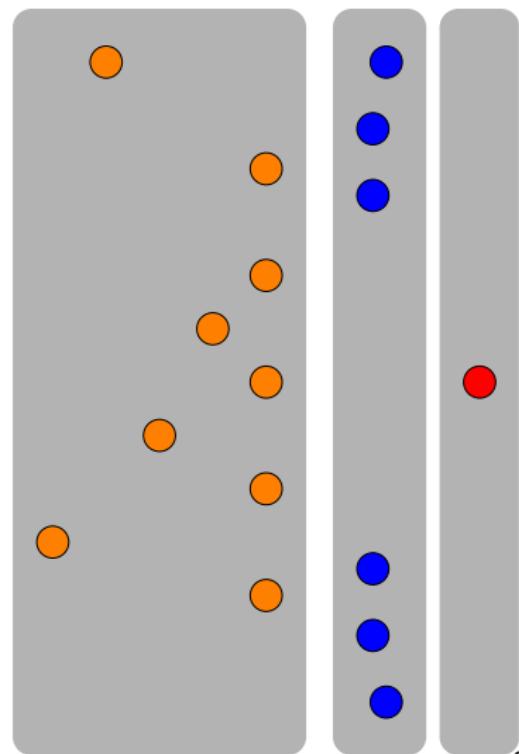
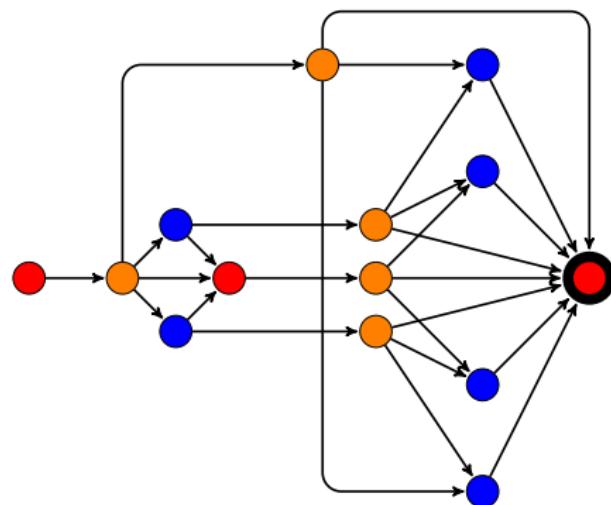
DAG recursion to partition  $S$  into stages

## Remove terminal states



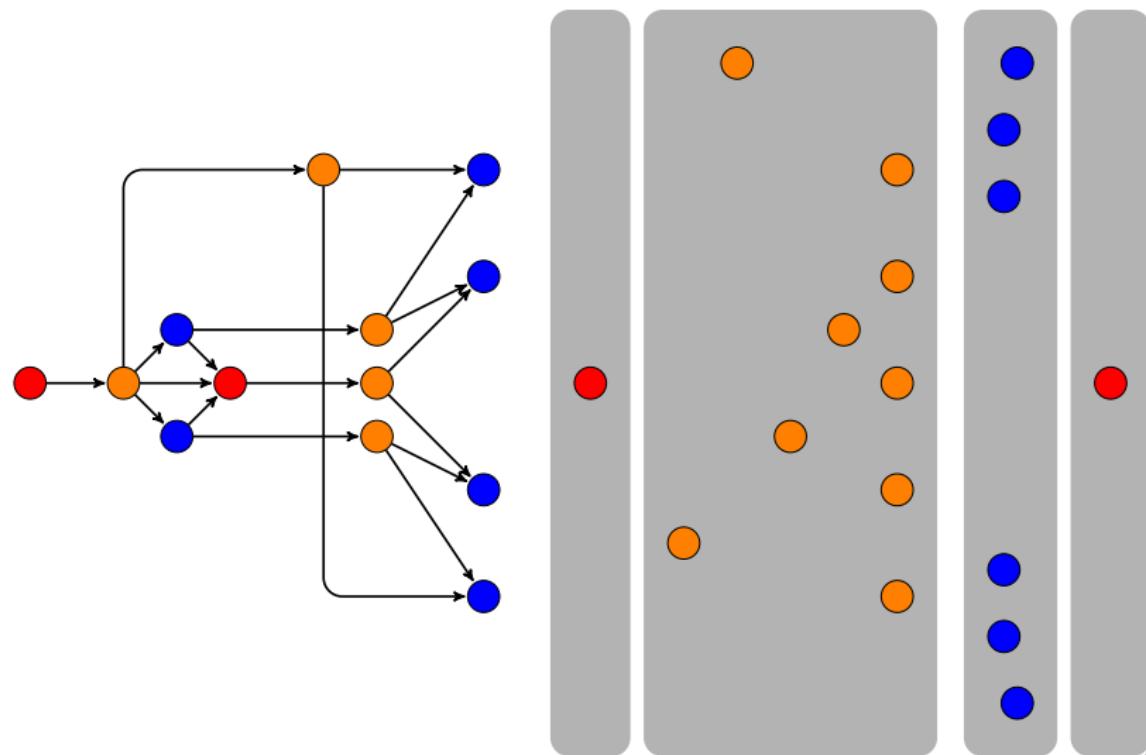
# DAG recursion to partition $S$ into stages

Identify terminal states



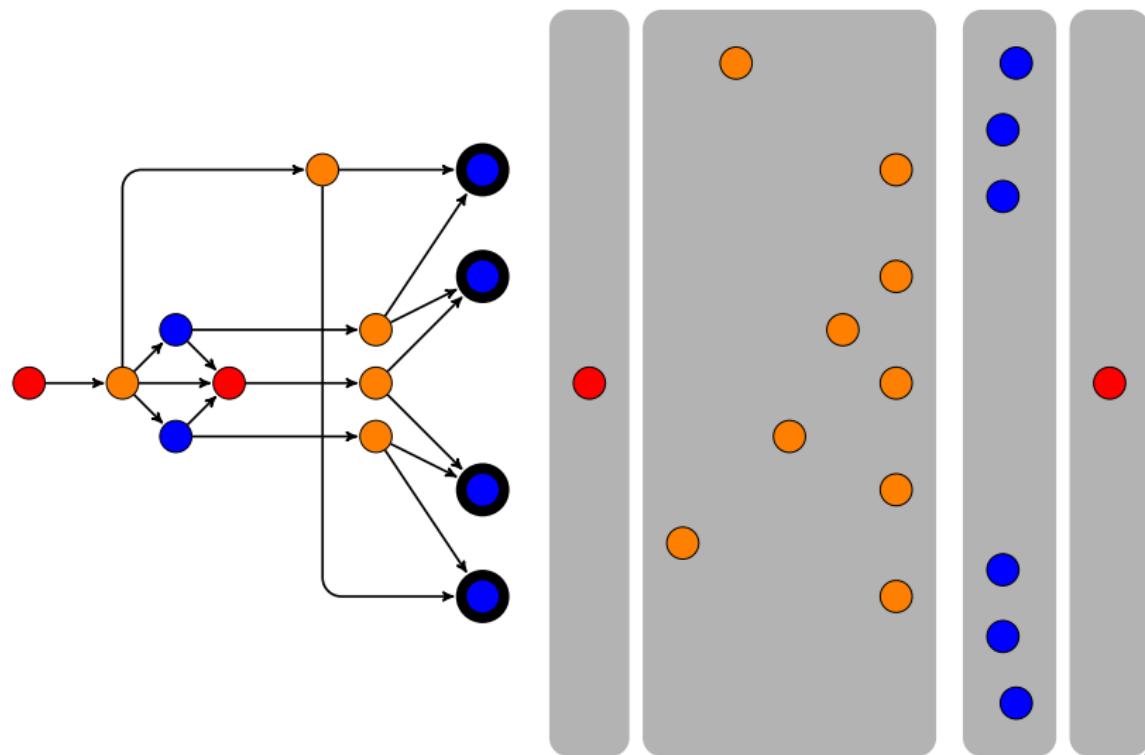
# DAG recursion to partition $S$ into stages

Remove terminal states



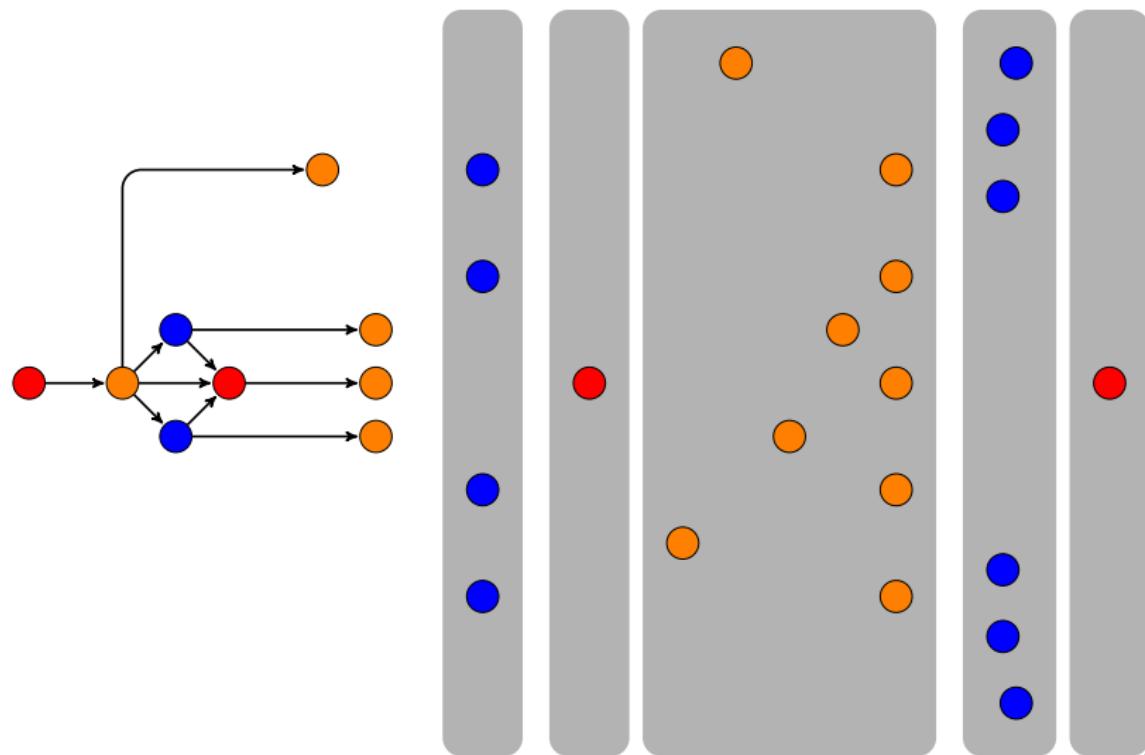
# DAG recursion to partition $S$ into stages

Identify terminal states



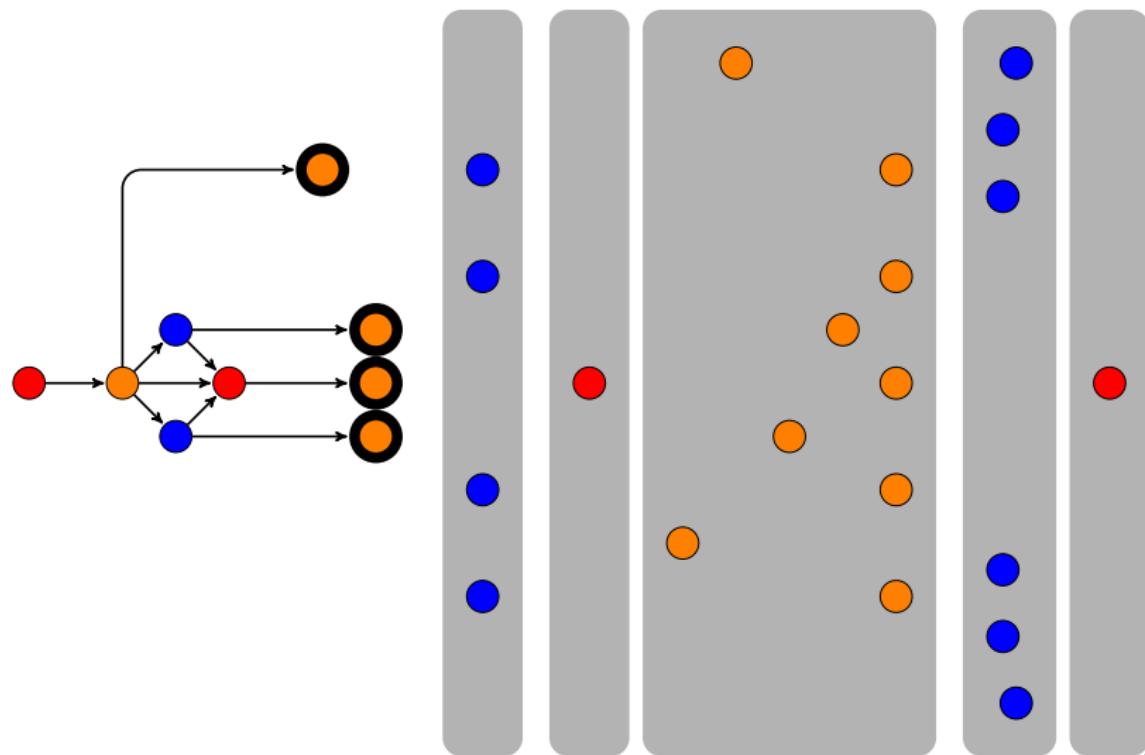
## DAG recursion to partition $S$ into stages

Remove terminal states



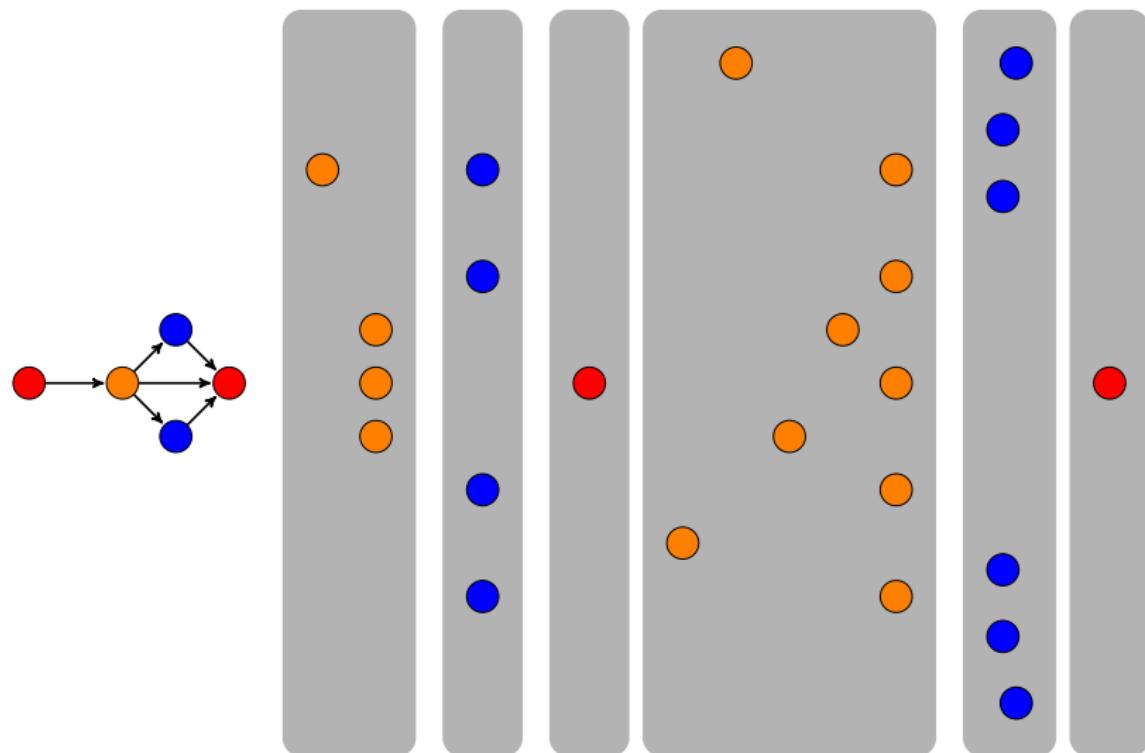
# DAG recursion to partition $S$ into stages

Identify terminal states



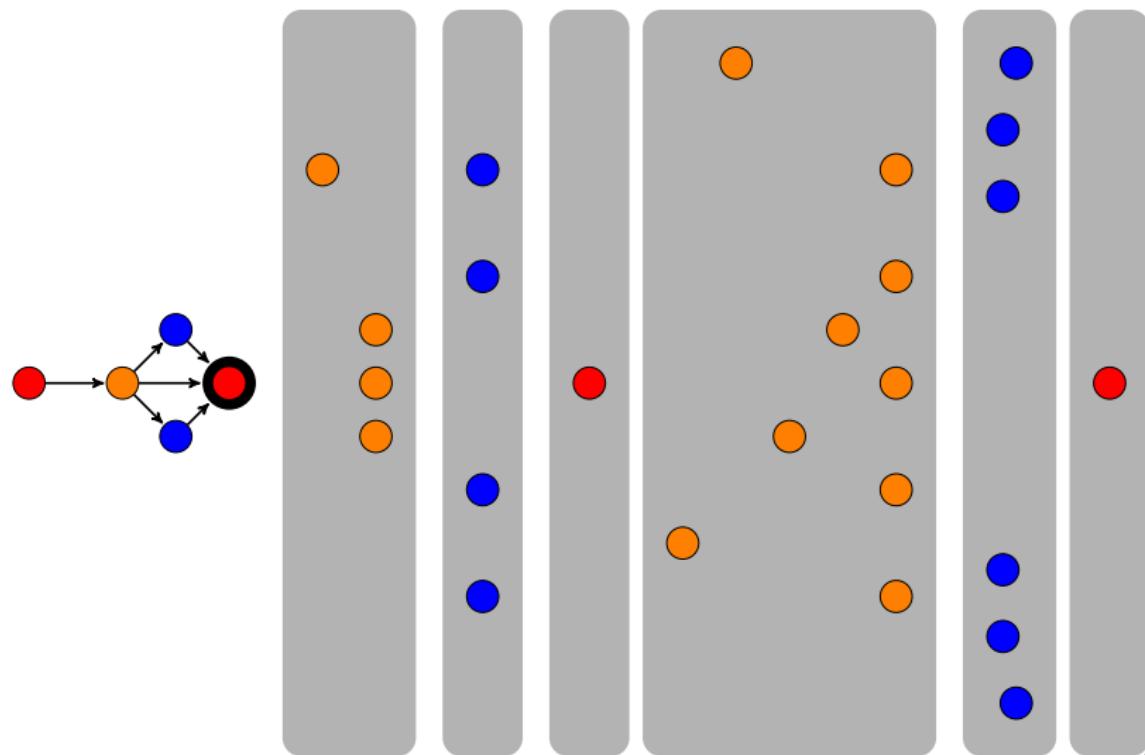
# DAG recursion to partition $S$ into stages

Remove terminal states



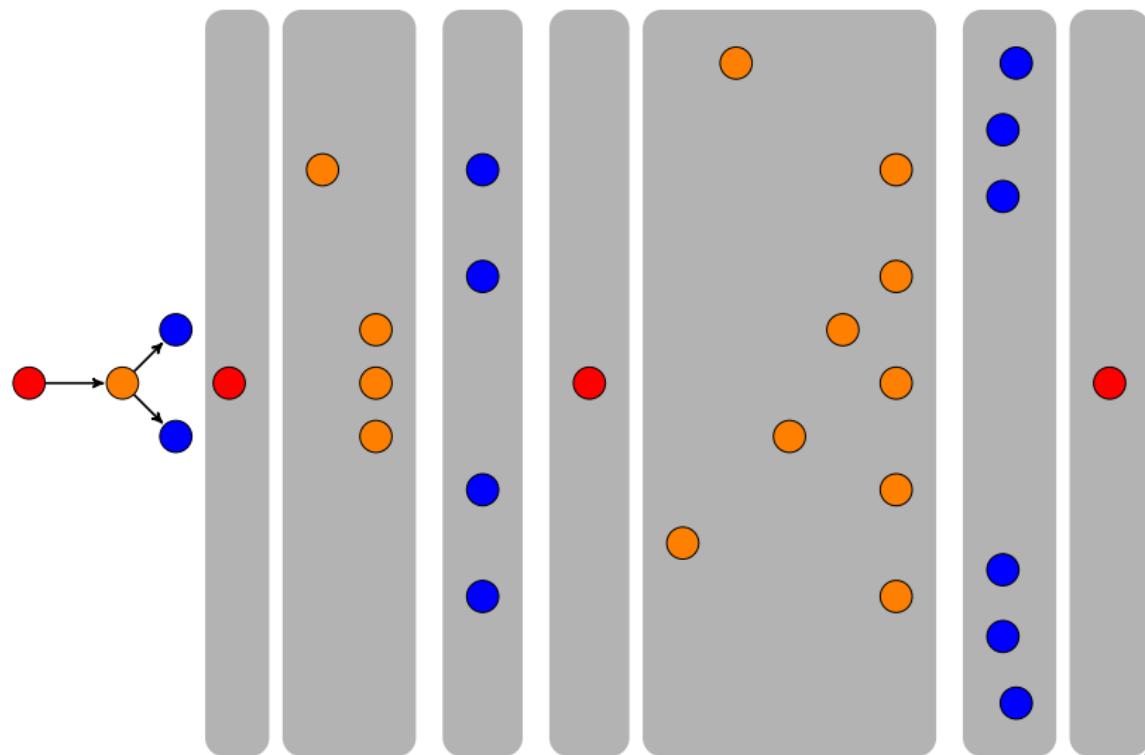
# DAG recursion to partition $S$ into stages

Identify terminal states



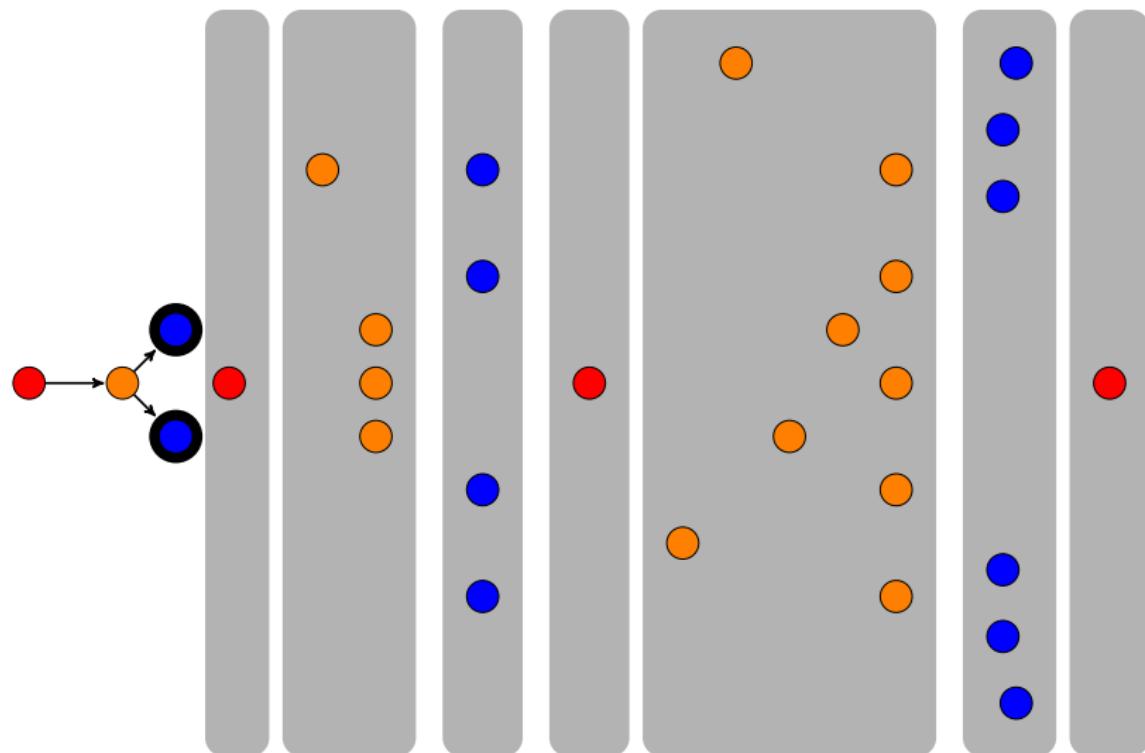
## DAG recursion to partition $S$ into stages

Remove terminal states



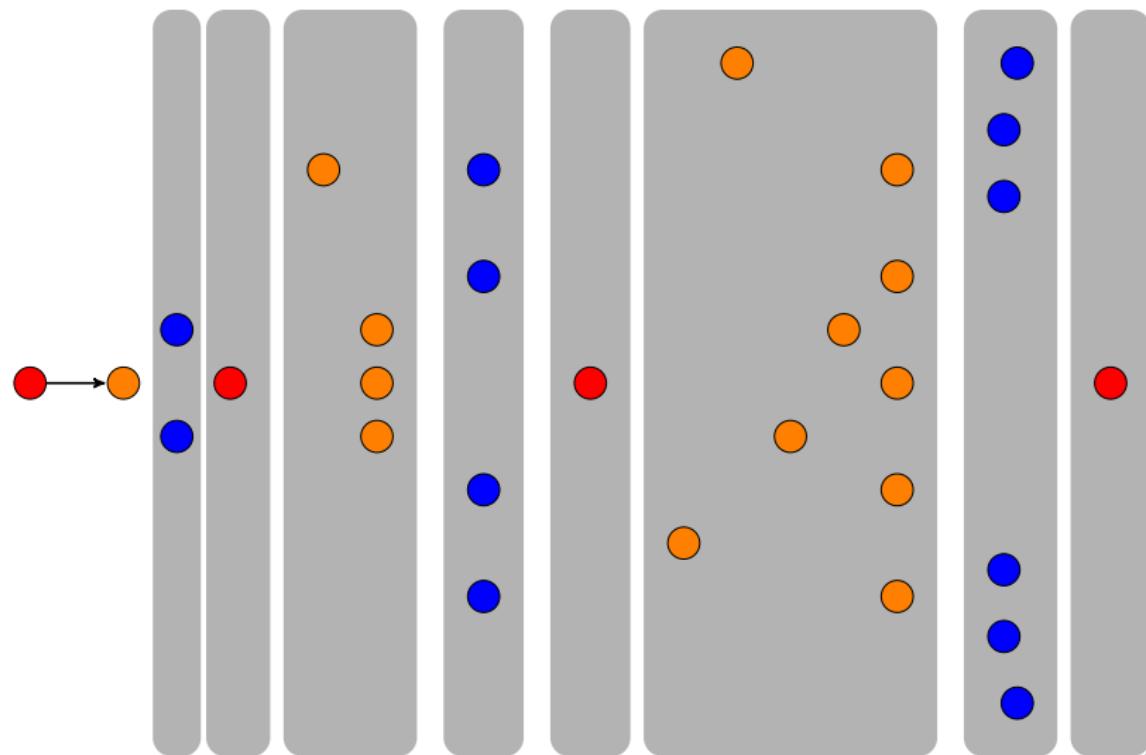
# DAG recursion to partition $S$ into stages

Identify terminal states



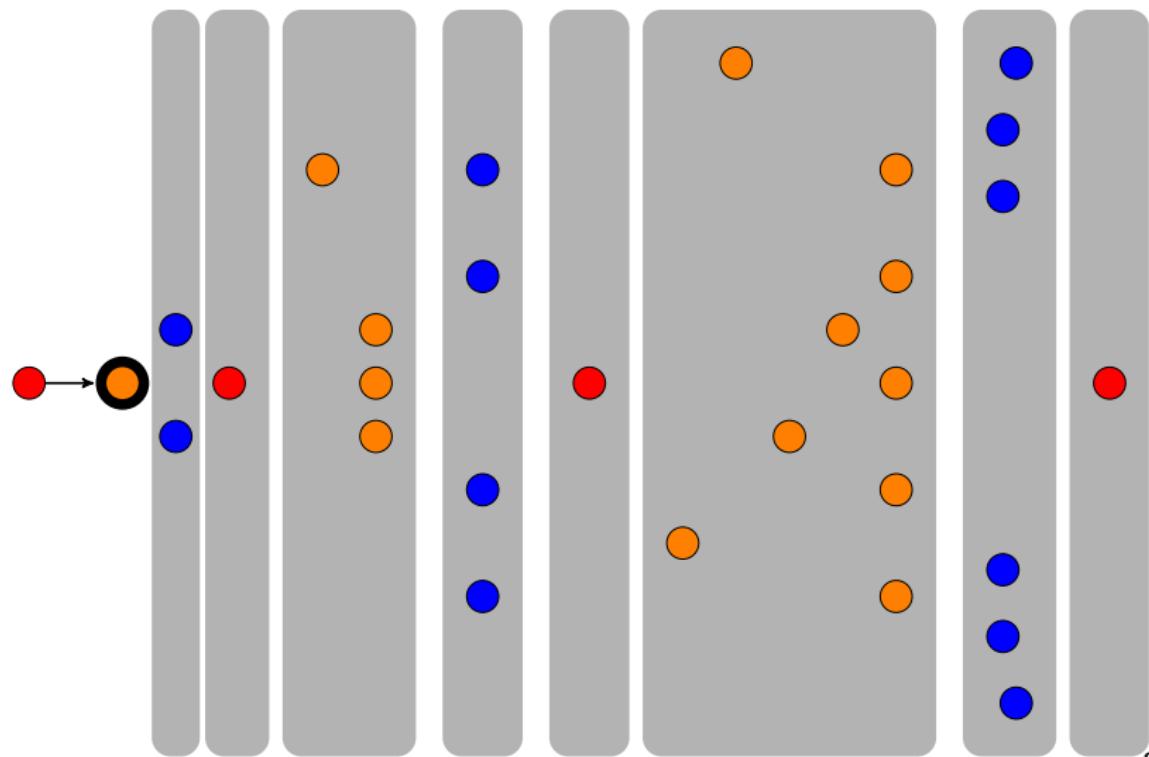
## DAG recursion to partition $S$ into stages

Remove terminal states



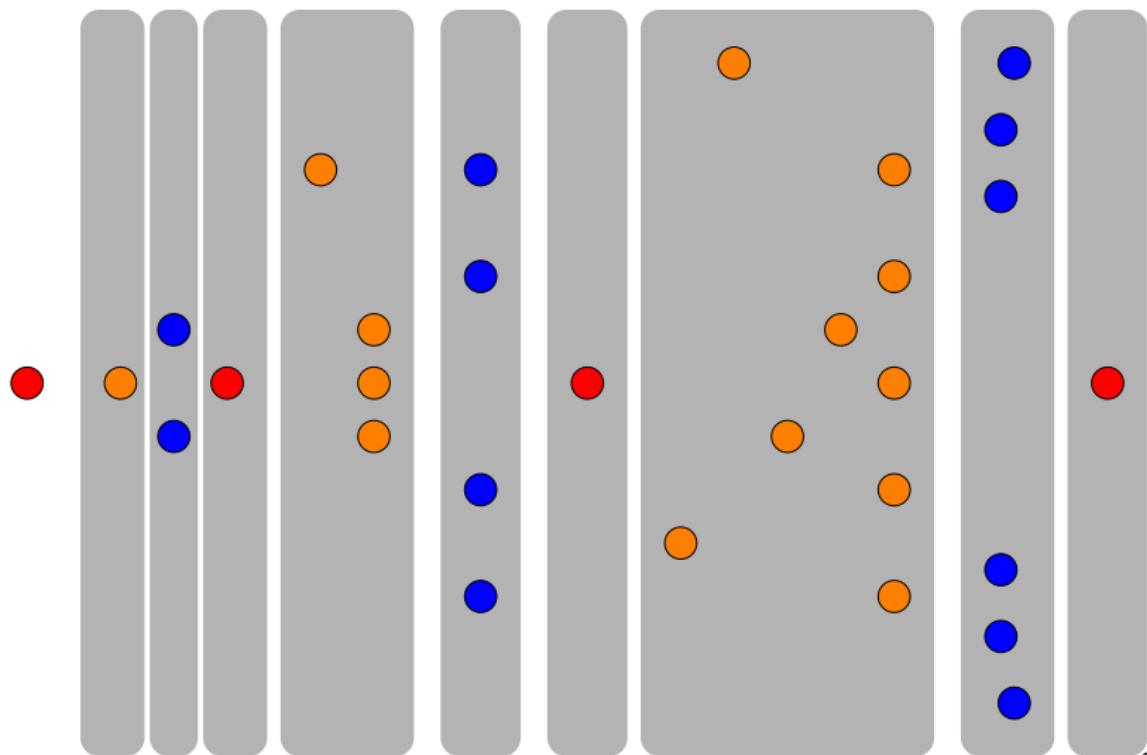
## Resulting partition of $S$ into stages

Identify terminal states



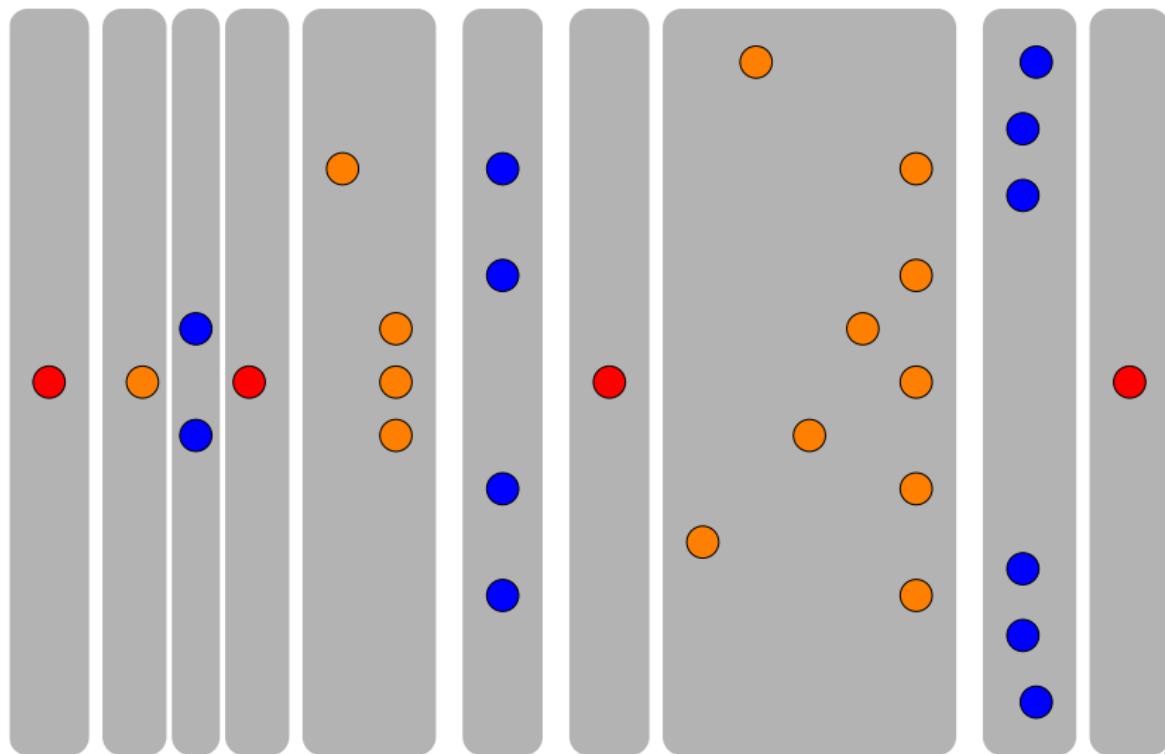
## Resulting partition of $S$ into stages

Remove terminal states



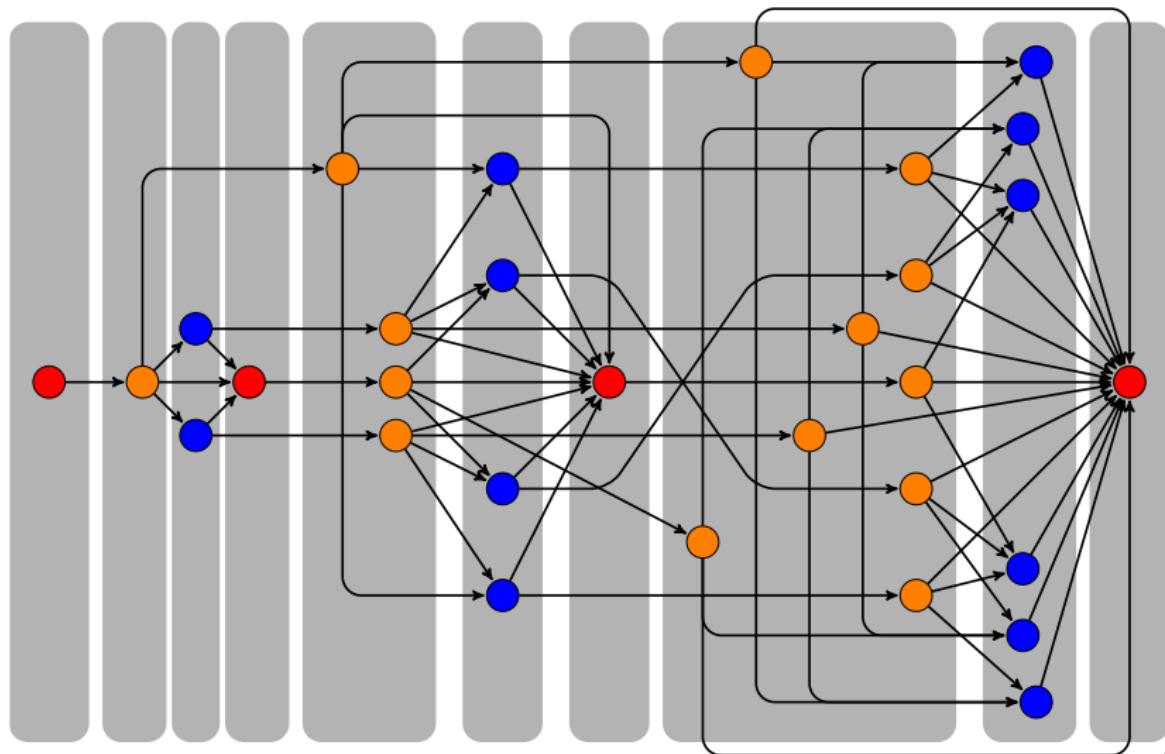
# Resulting partition of $S$ into stages

The stages of the game are totally ordered



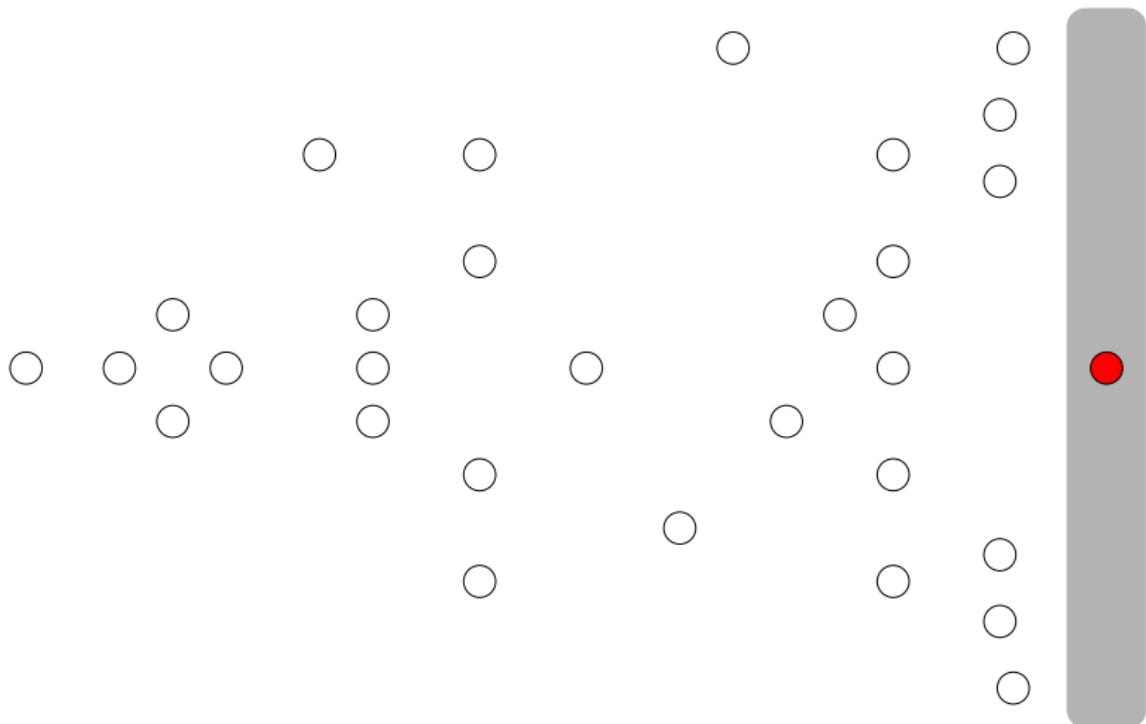
# Total order on the set of stages

Subgames of DDG follow the order of stages



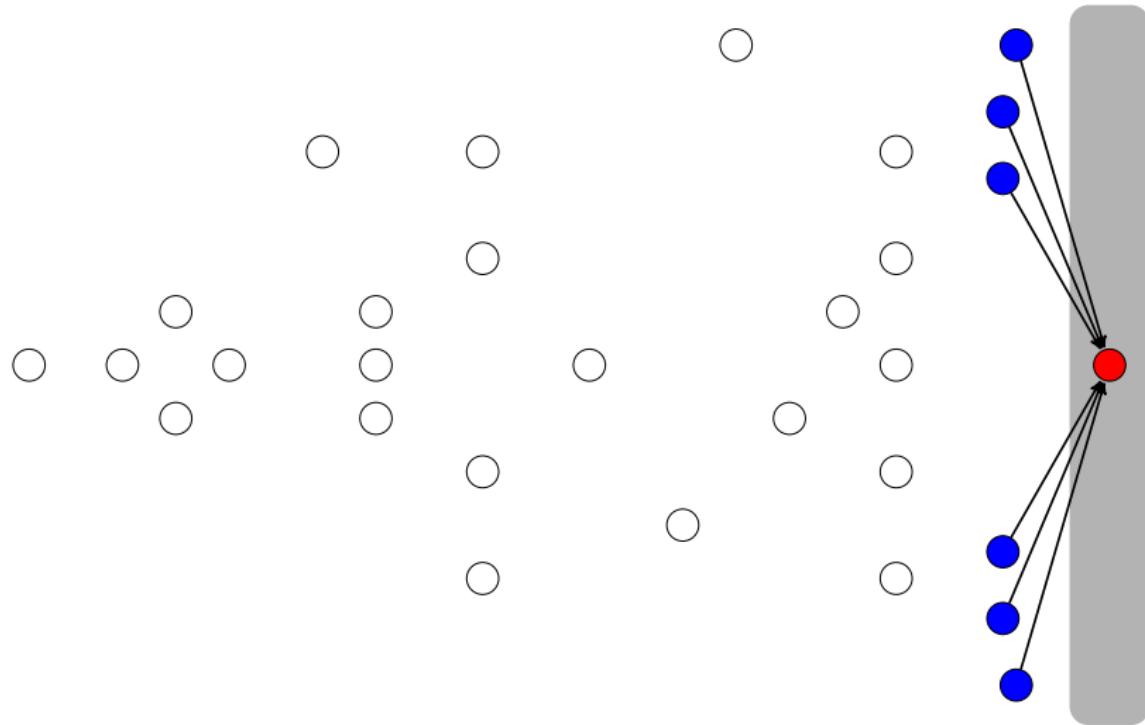
# State recursion algorithm

Backward induction on stages of DDG



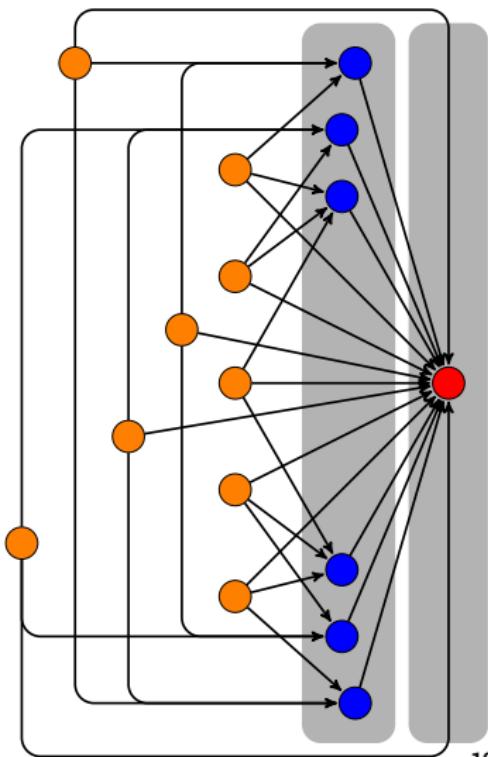
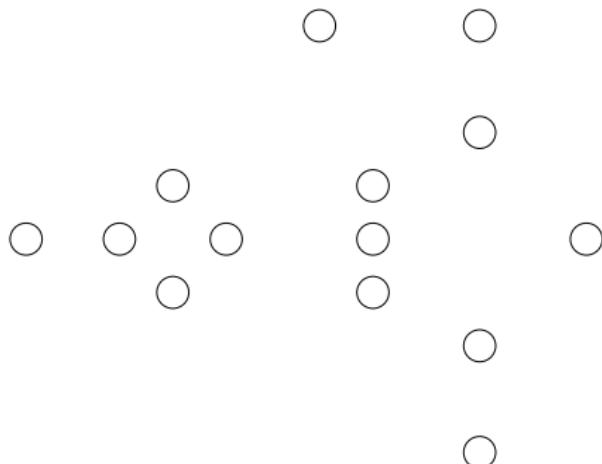
# State recursion algorithm

Backward induction on stages of DDG



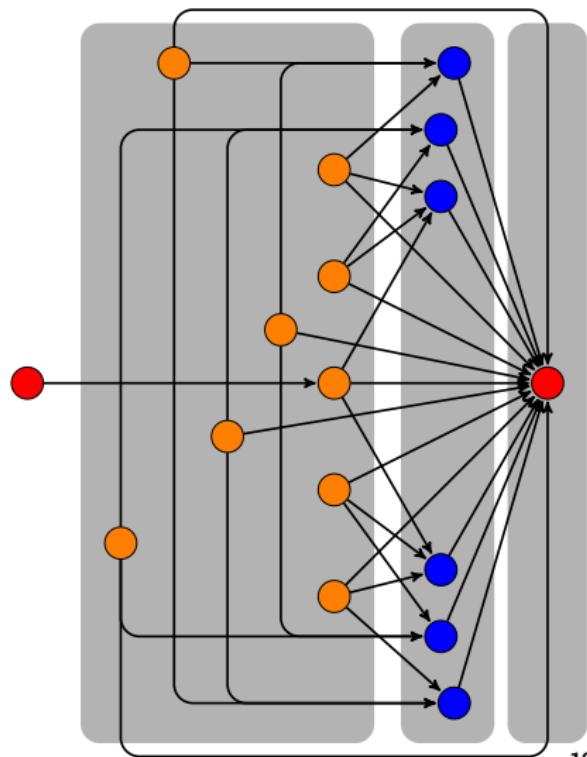
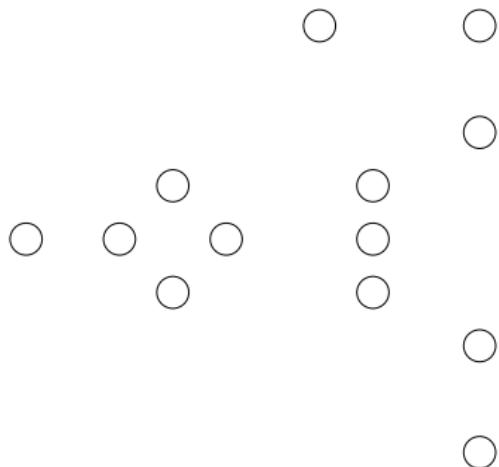
# State recursion algorithm

Backward induction on stages of DDG



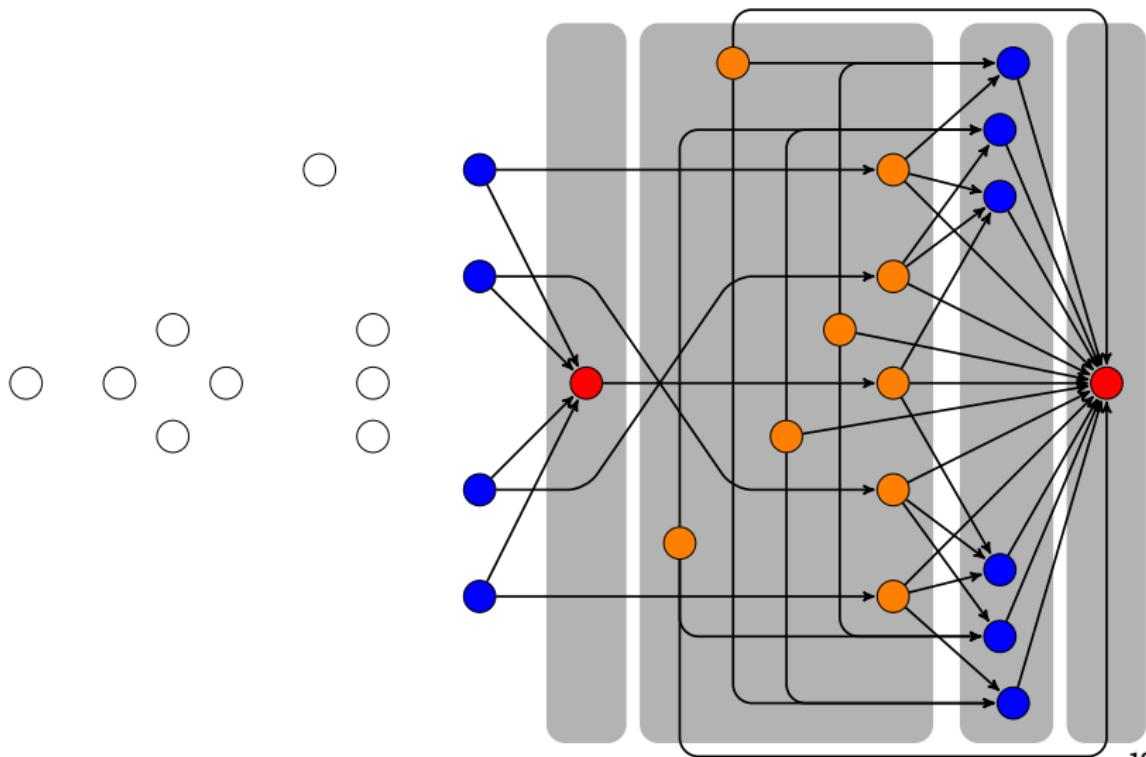
# State recursion algorithm

Backward induction on stages of DDG



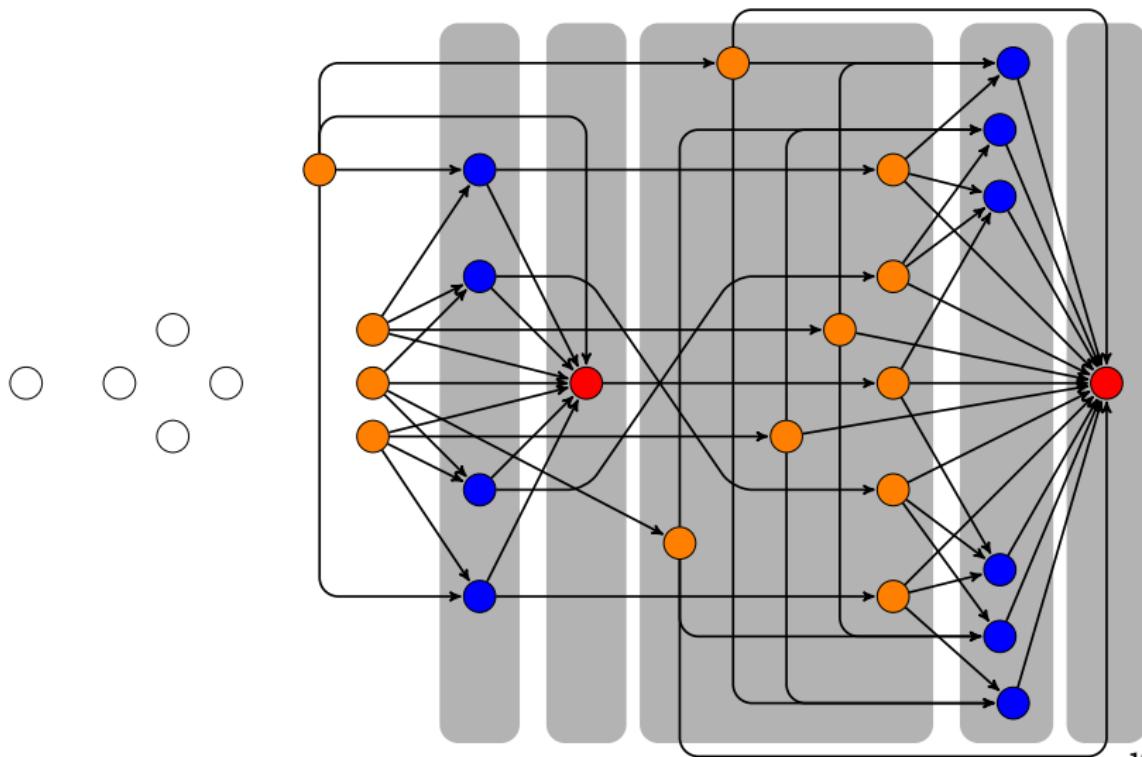
# State recursion algorithm

Backward induction on stages of DDG



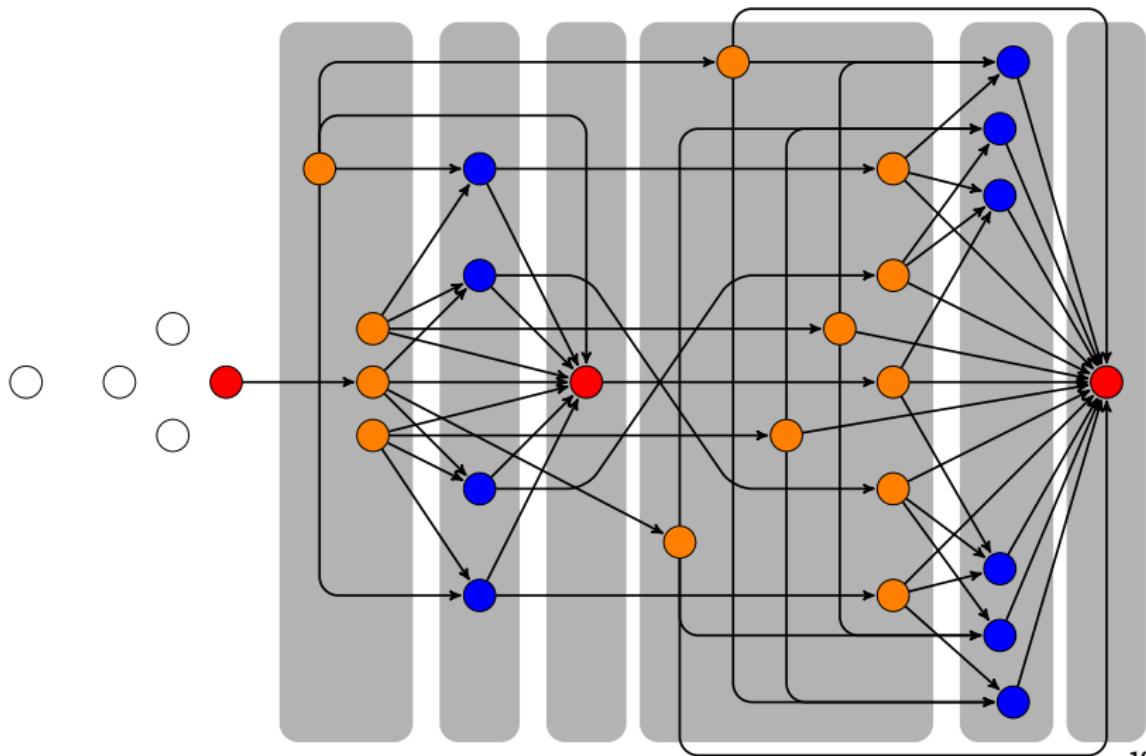
# State recursion algorithm

Backward induction on stages of DDG



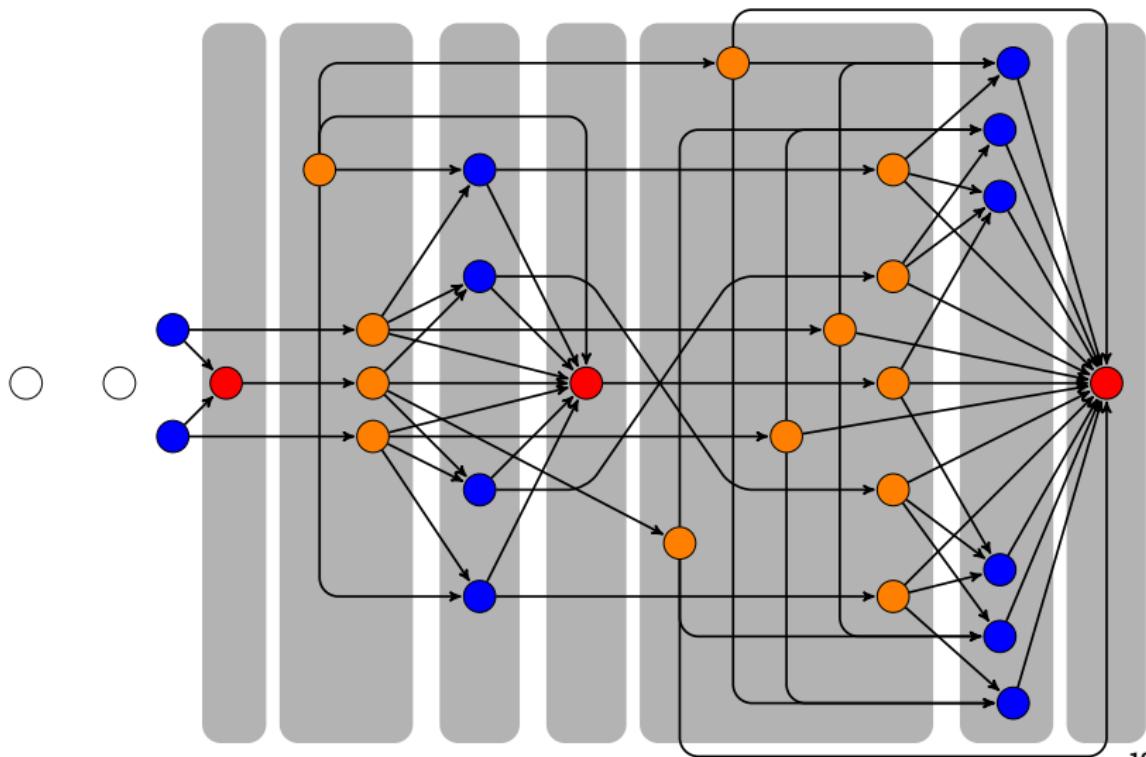
# State recursion algorithm

Backward induction on stages of DDG



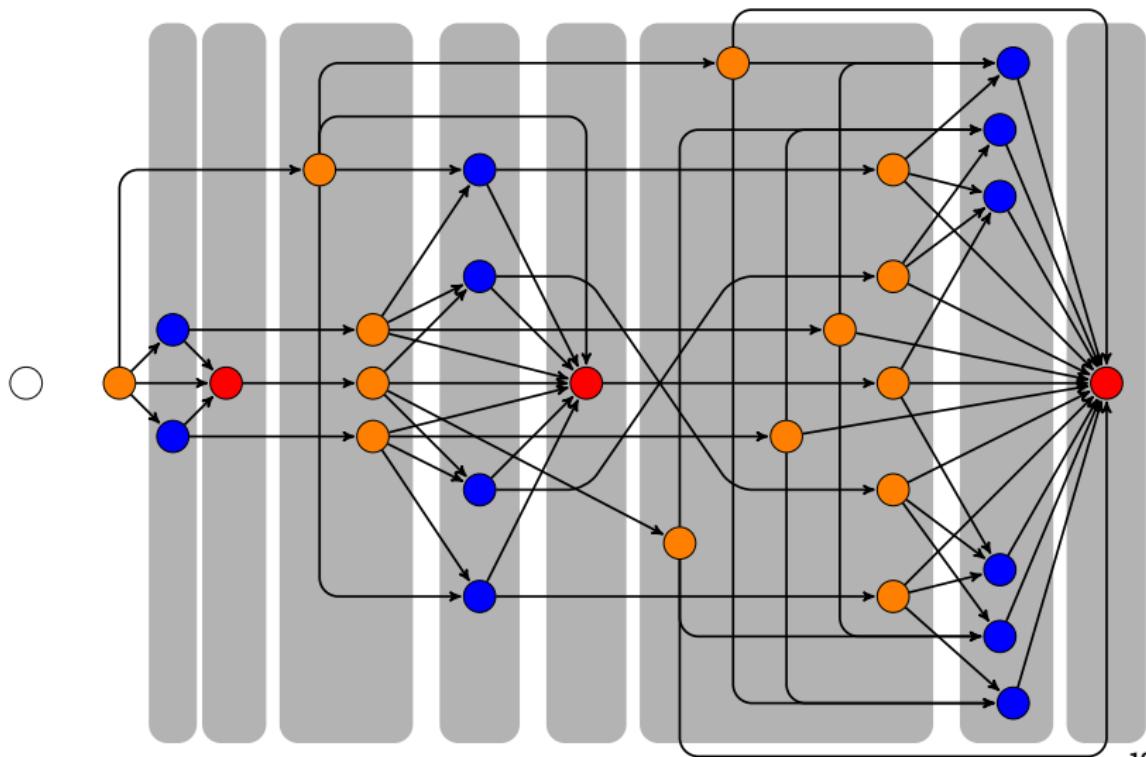
# State recursion algorithm

Backward induction on stages of DDG



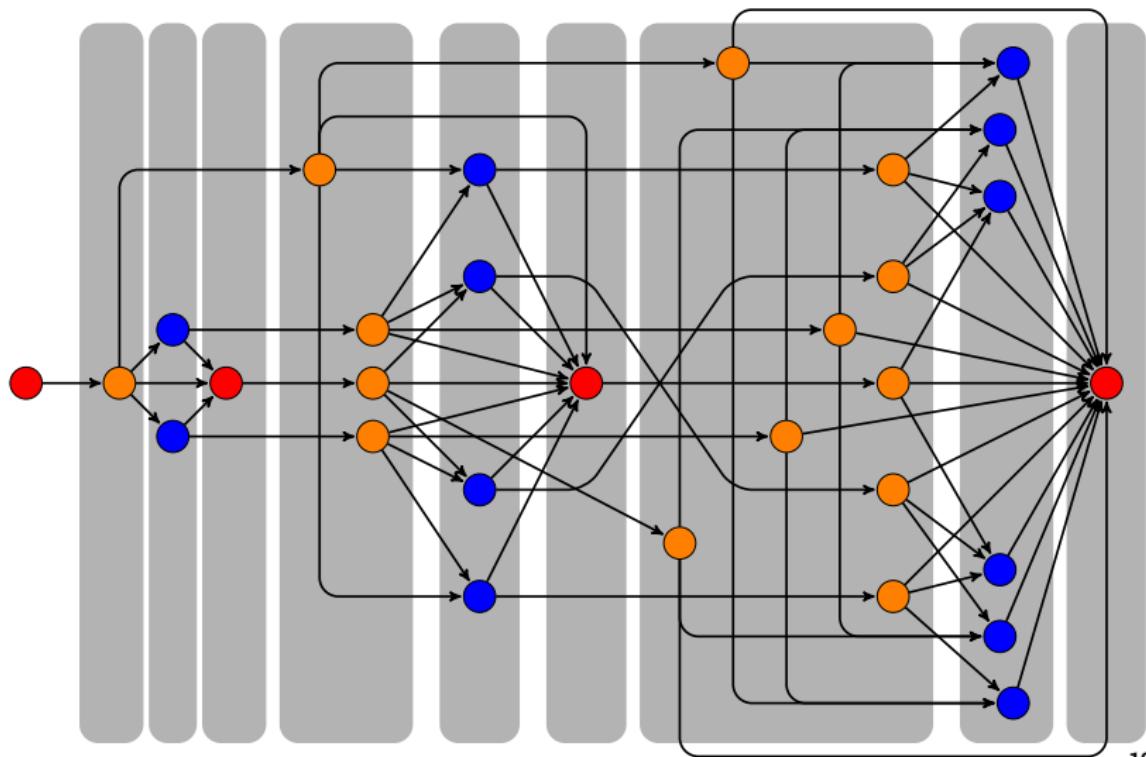
# State recursion algorithm

Backward induction on stages of DDG



# State recursion algorithm

Backward induction on stages of DDG

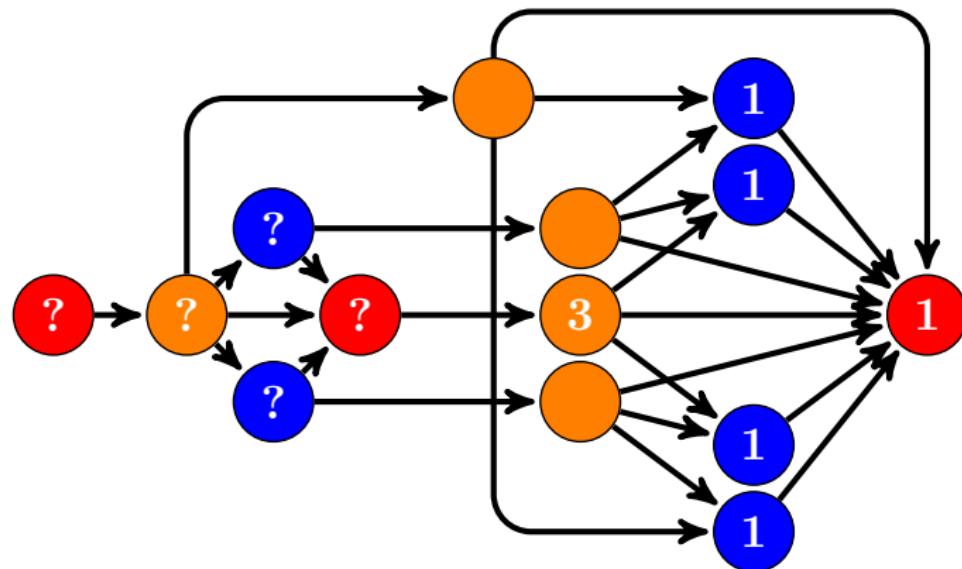


## State Recursion versus Backward Induction

- State recursion – generalization of backward induction
- Runs on state space instead of time periods
- Time ( $t$ ) evolves as  $t \rightarrow t + 1$  with probability 1
- For stages of state space ( $\tau$ ) transitions are stochastic and not necessarily sequential
- Yet, probability of going  $\tau \rightarrow \tau'$  is zero when  $\tau' < \tau$
- With multiplicity, state recursion is performed **conditional** of a particular **equilibrium selection rule (ESR)**

# Multiplicity of equilibria

Number of equilibria in the higher stages depends on the selected equilibria



# Recursive Lexicographic Search Algorithm

Building blocks of RLS algorithm:

- ➊ State recursion algorithm solves the game **conditional on** equilibrium selection rule (ESR)
- ➋ RLS algorithm efficiently cycles through **all feasible** ESRs

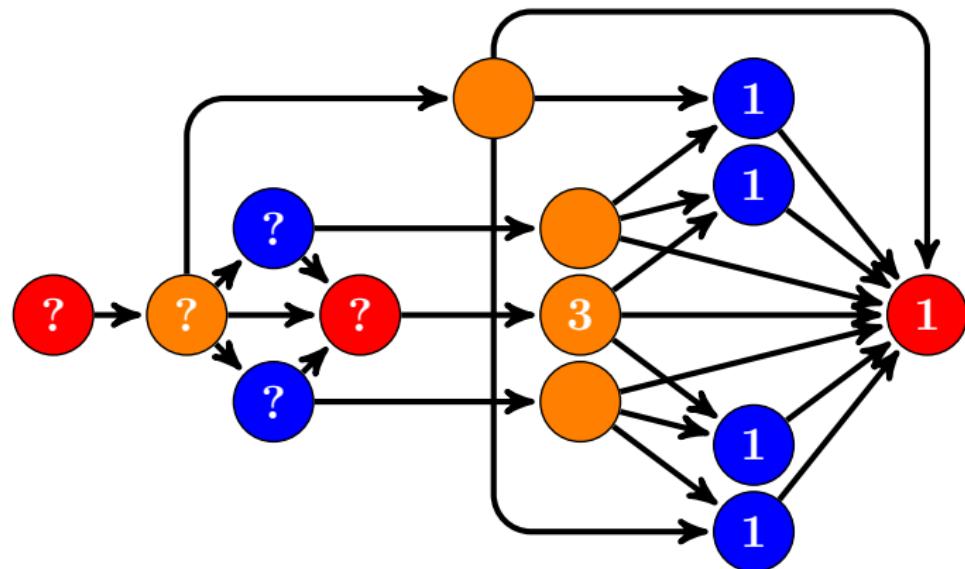
## Challenge:

- Choice of a particular MPE for any stage game at any stage
- may alter the **set** and even the **number** of stage equilibria at earlier stages

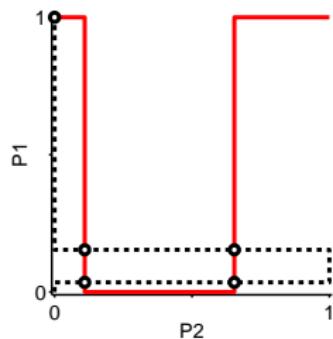
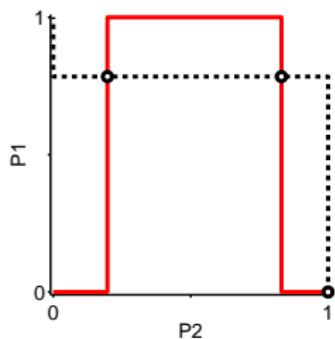
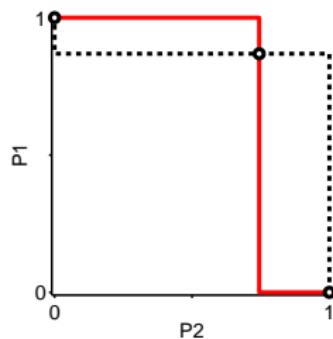
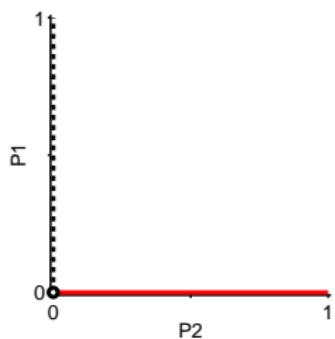
Need to find feasible ESRs

# Multiplicity of equilibria

Number of equilibria in the higher stages depends on the selected equilibria



## Best response correspondences of the two firms



# Represent ESR as string of digits

Use numbers in base- $K$  number system with digits  $0, 1, \dots, K - 1$

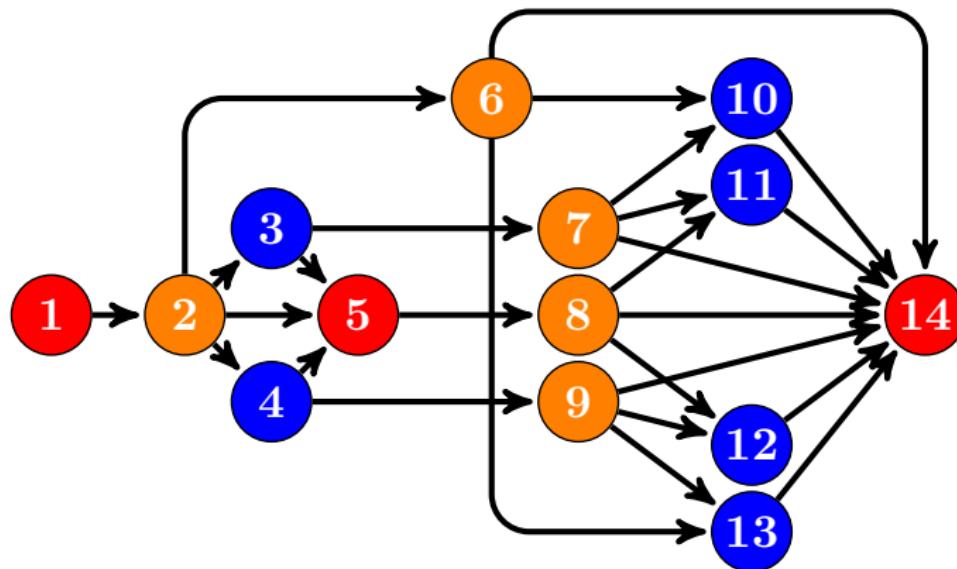
## Dependence preserving property:

Any point of the state space may depend on the points to the left (higher digits) and **not the points to the right** (lower digits)

ESR string	corner										edges					interior									
	c	e	e	e	e	i	i	i	i	c	e	e	i	c	c	c	c	c	c	c					
	14	13	12	11	10	9	8	7	6	5	4	3	2	1											
$c$	0	0	0	0	0	0	0	0	0	1	1	1	1	2											
$cl$	0	0	0	2	1	2	2	1	1	1	1	1	2	2	2										
$c2$	0	2	1	0	0	2	1	2	1	1	2	1	2	1	2										

# Digits of the ESR string mapped to the state space

Dependent preserving property



## All possible ESR in lexicographic order, $K = 3$

ESR string	c	e	e	e	e	i	i	i	i	c	e	e	i	c
Lexicograph	14	13	12	11	10	9	8	7	6	5	4	3	2	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	0	0	0	2
	0	0	0	0	0	0	0	0	0	0	0	0	0	10
	0	0	0	0	0	0	0	0	0	0	0	0	0	11
	0	0	0	0	0	0	0	0	0	0	0	0	0	12
	0	0	0	0	0	0	0	0	0	0	0	0	0	20
	0	0	0	0	0	0	0	0	0	0	0	0	0	21
	0	0	0	0	0	0	0	0	0	0	0	0	0	22
	0	0	0	0	0	0	0	0	0	0	0	0	10	0
	0	0	0	0	0	0	0	0	0	0	0	0	10	1
	...													
	2	2	2	2	2	2	2	2	2	2	2	2	2	0
	2	2	2	2	2	2	2	2	2	2	2	2	2	1
	2	2	2	2	2	2	2	2	2	2	2	2	2	2

4,782,969

## Variable base arithmetics

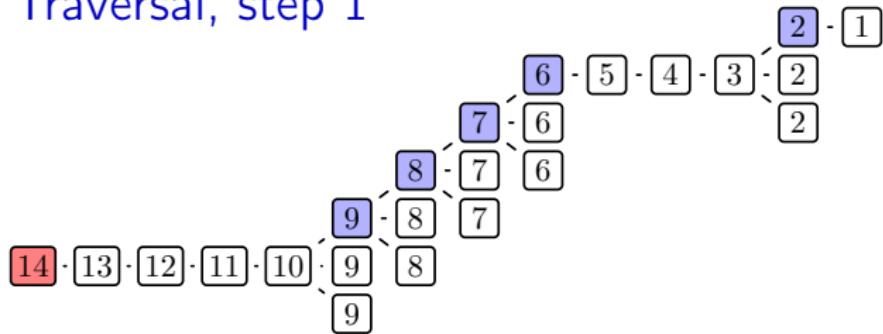
- Replace the base- $K$  ( $\text{mod}(K)$ ) arithmetics with variable base arithmetics
- Let  $(3 \ 1 \ 2)$  be bases  $\rightarrow$
- Allowed digits in the numbers are  $\{0, 1, 2\}$ ,  $\{0\}$  and  $\{0, 1\}$
- The 3-digit numbers in this system are:

$$\begin{array}{r r r} 0 & 0 & 0 \\ + & 1 & \\ \hline 0 & 0 & 1 \\ + & 1 & \\ \hline 1 & 0 & 0 \\ + & 1 & \\ \hline 1 & 0 & 1 \\ + & 1 & \\ \hline 2 & 0 & 0 \\ + & 1 & \\ \hline 2 & 0 & 1 \end{array} \rightarrow$$

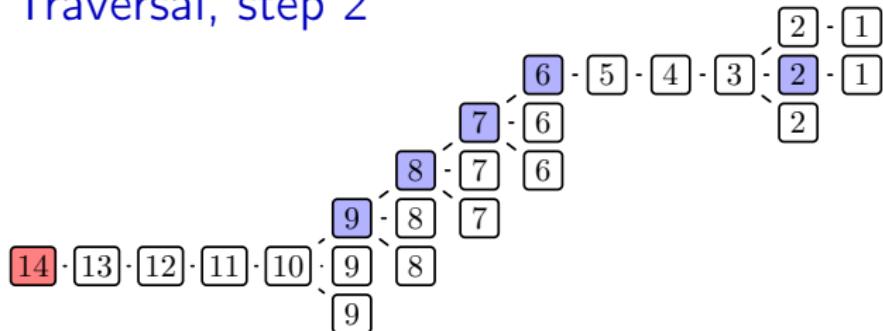
## RLS Algorithm: tree traversal

- ① Set ESR =  $(0, \dots, 0)$
- ② Run State Recursion using the current ESR
- ③ Save the number of equilibria in every stage game as  $ne(ESR)$
- ④ Add 1 to the ESR in bases  $ne(ESR)$  to obtain new feasible ESR
- ⑤ Stopping rule: successor function exceeds the maximum number with given number of digits
- ⑥ Return to step 2

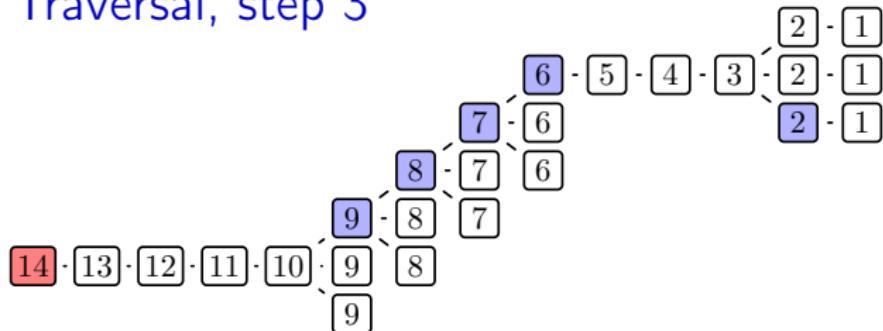
## RLS Tree Traversal, step 1



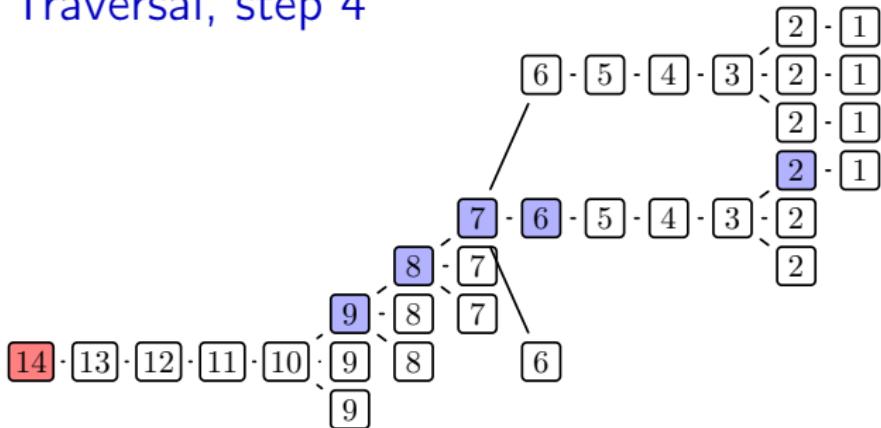
## RLS Tree Traversal, step 2



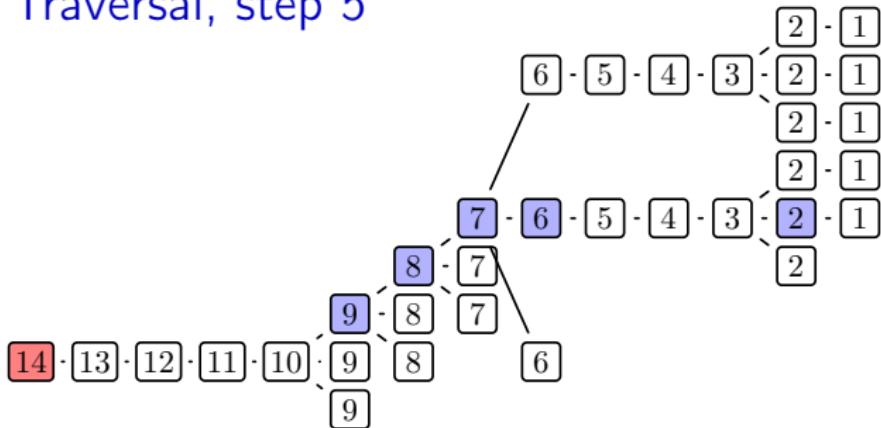
## RLS Tree Traversal, step 3



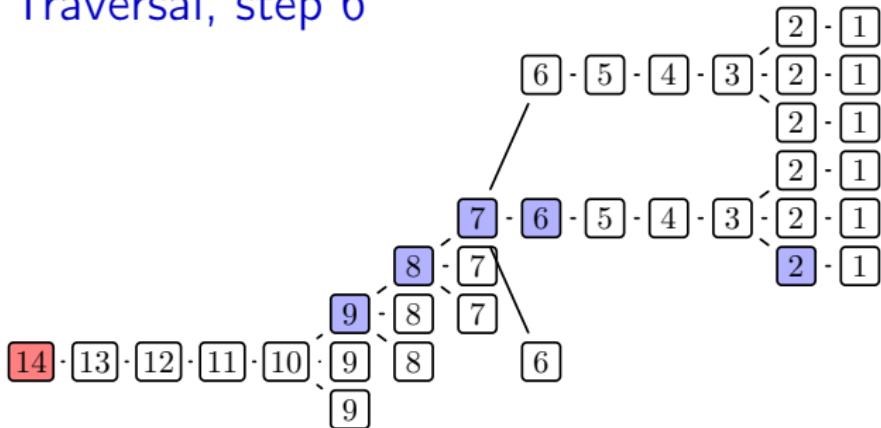
## RLS Tree Traversal, step 4



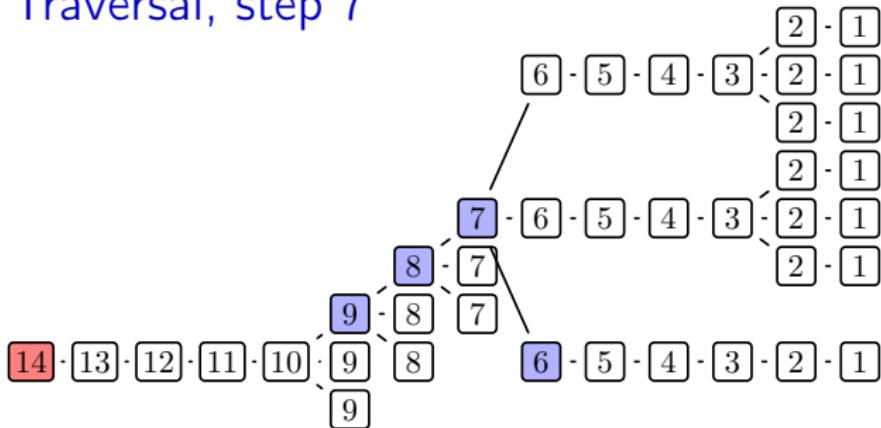
## RLS Tree Traversal, step 5



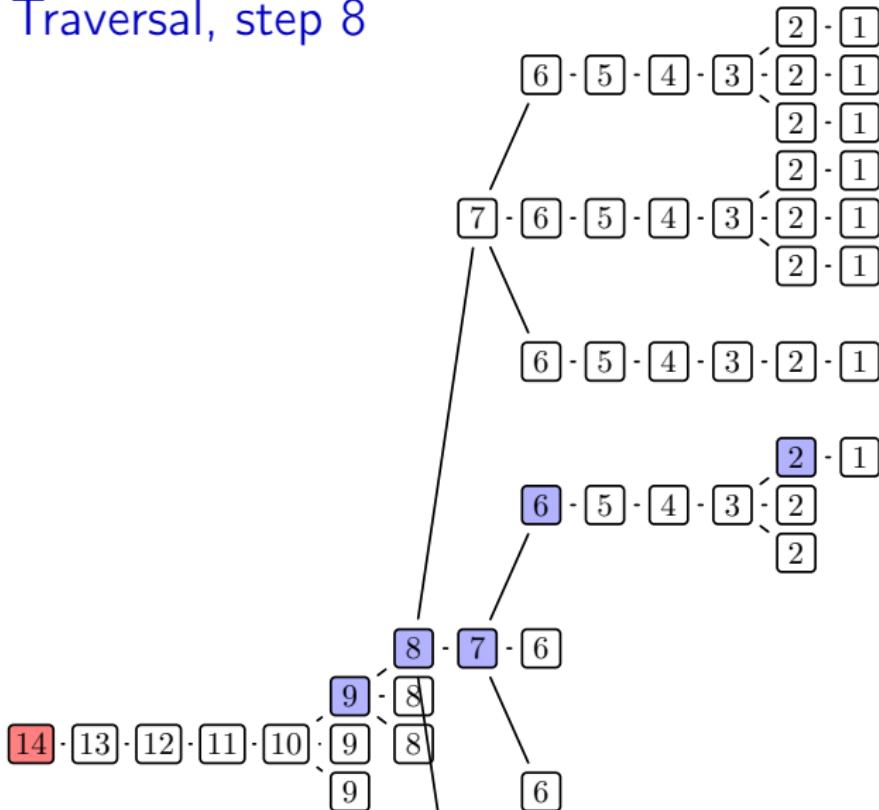
## RLS Tree Traversal, step 6



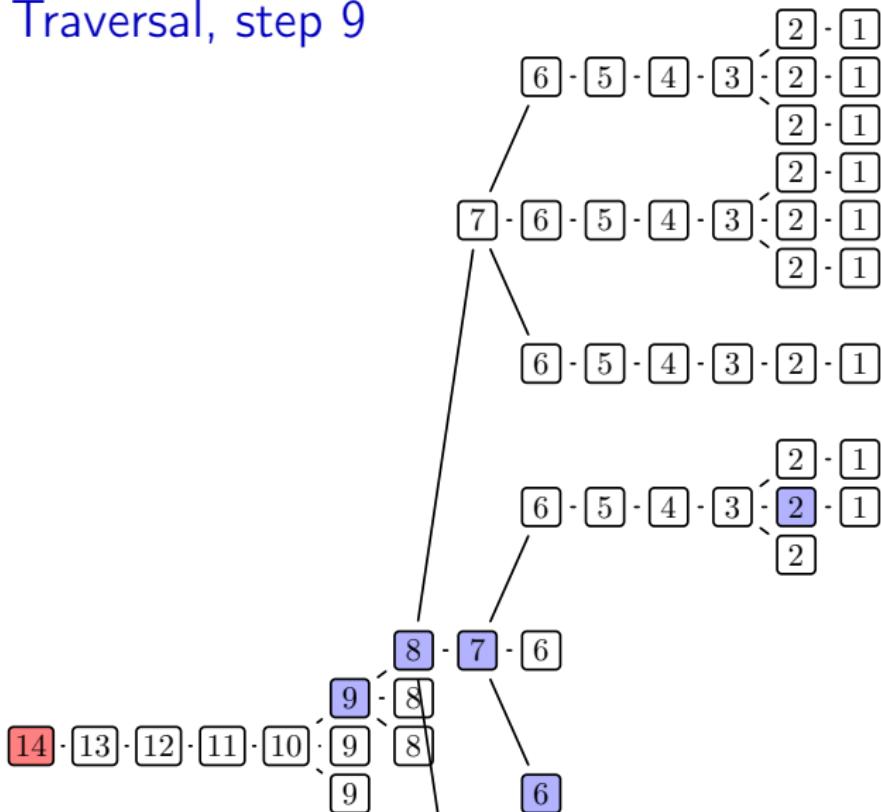
## RLS Tree Traversal, step 7



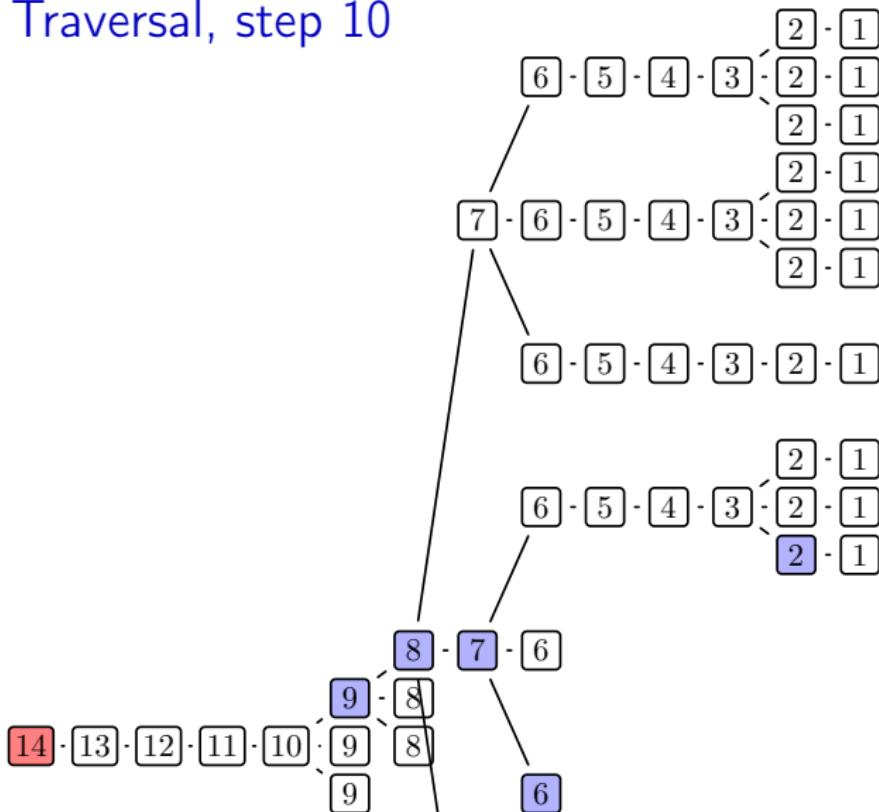
## RLS Tree Traversal, step 8



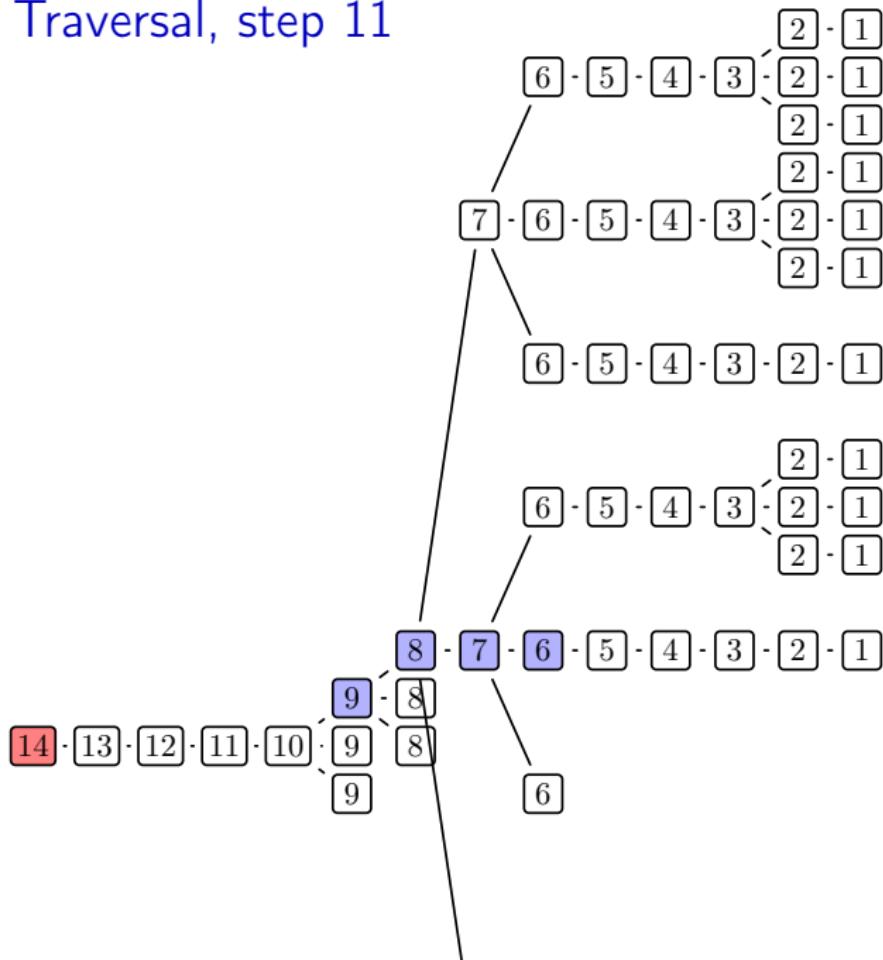
## RLS Tree Traversal, step 9



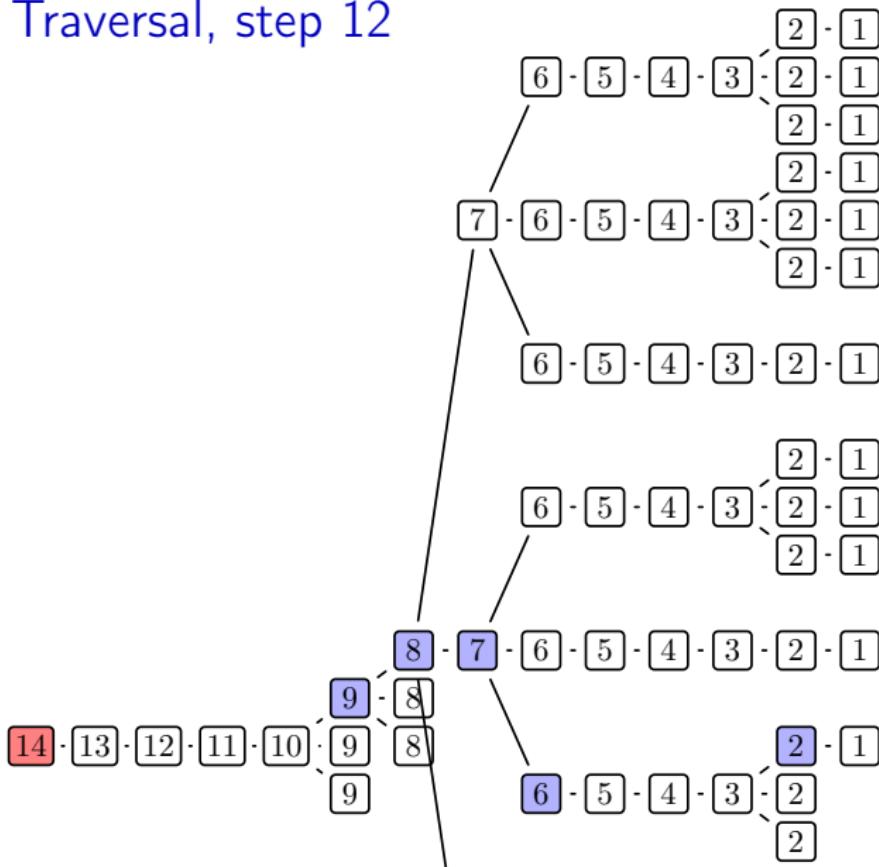
## RLS Tree Traversal, step 10



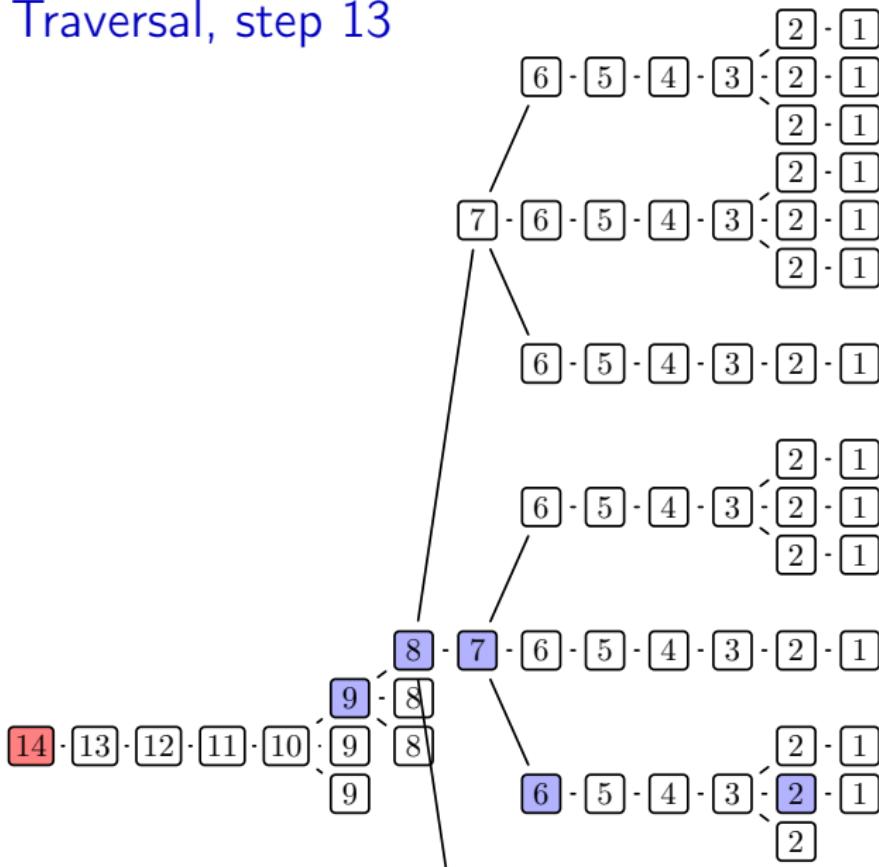
## RLS Tree Traversal, step 11



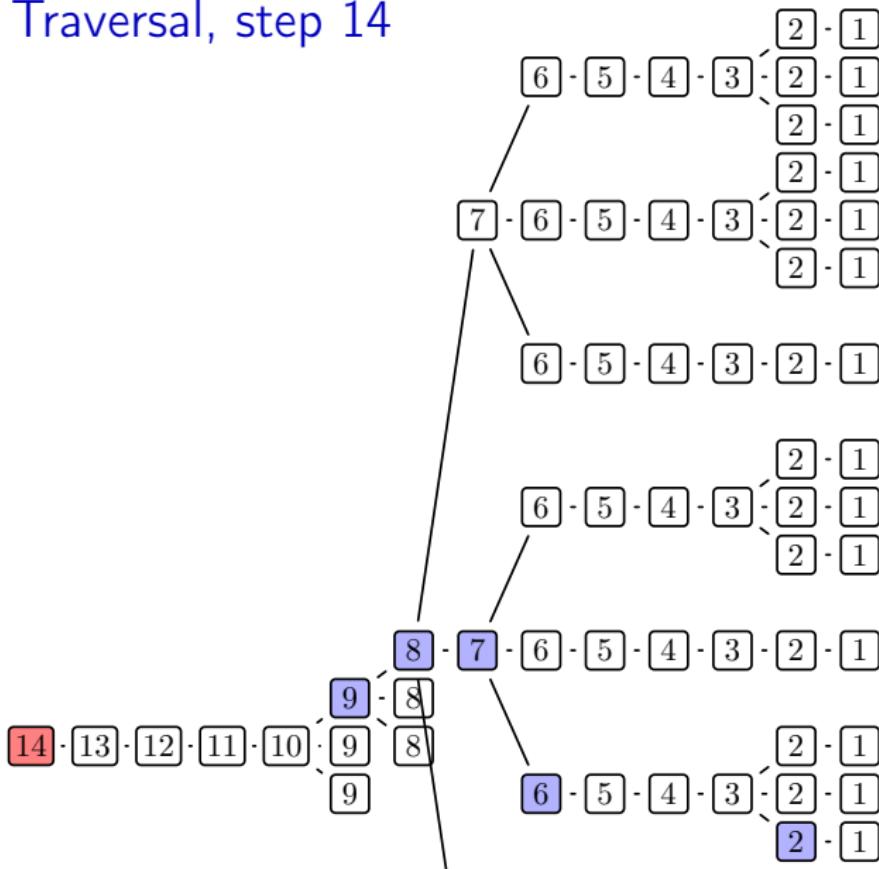
## RLS Tree Traversal, step 12



## RLS Tree Traversal, step 13



## RLS Tree Traversal, step 14



## Main result of the RLS Algorithm

Theorem (Decomposition theorem, strong)

*Assume there exists an algorithm that can find all MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.*

*Then the RLS algorithm finds all MPE of the DDG in a finite number of steps, which equals the total number of MPE.*



Iskhakov, Rust and Schjerning, 2016

## Main result of the RLS Algorithm

Theorem (Decomposition theorem, weak)

*Assume there exists an algorithm that can find at least one MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.*

*Then the RLS algorithm finds some (at least one) MPE of the DDG in a finite number of steps, which does not exceed the total number of MPE.*

## RLS algorithm: running times

$K = 3$

Simultaneous moves	$n = 3$	$n = 4$
Upper bound on number of MPE	4,782,969	3,948,865,611
Actual number of equilibria	127	46,707
Time used	0.008 sec.	0.334 sec.
Simultaneous moves	$n = 5$	
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728	
Actual number of equilibria		192,736,405
Time used		45 min.
Alternating moves	$n = 5$	
Upper bound on number of MPE	174,449,211,009,120,166,087,753,728	
Actual number of equilibria		1
Time used		0.006 sec.

## Solving for stage game MPE

Theorem (1, 3 or 5 MPE in simultaneous move stage games)

*When  $\eta = 0$  there exists an exact, non-iterative algorithm that finds all MPE of every d-stage game in the Bertrand pricing and investment game with simultaneous moves.*

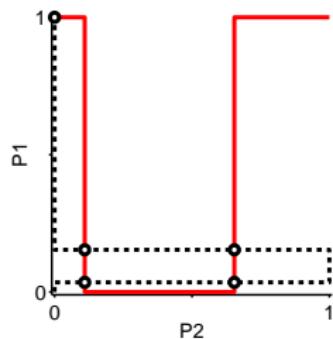
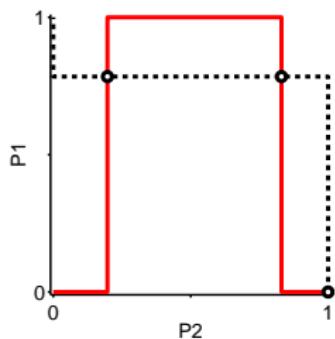
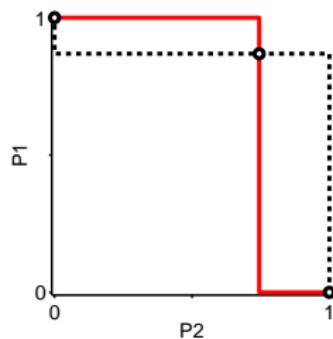
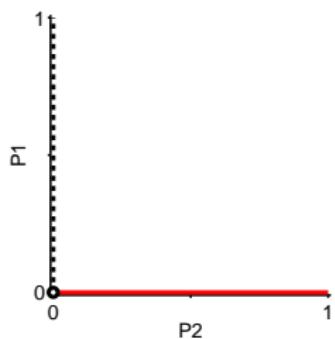
*The number of MPE in any d-stage game is either 1, 3 or 5.*

Theorem (1 or 3 MPE in alternating move stage games)

*When  $\eta = 0$  there exists an exact, non-iterative algorithm that finds all MPE of every d-stage game in the Bertrand pricing and investment game with alternating moves.*

*The number of MPE in any d-stage game is either 1 or 3.*

## Best response correspondences of the two firms



# Resolutions to Bertrand Investment Paradox

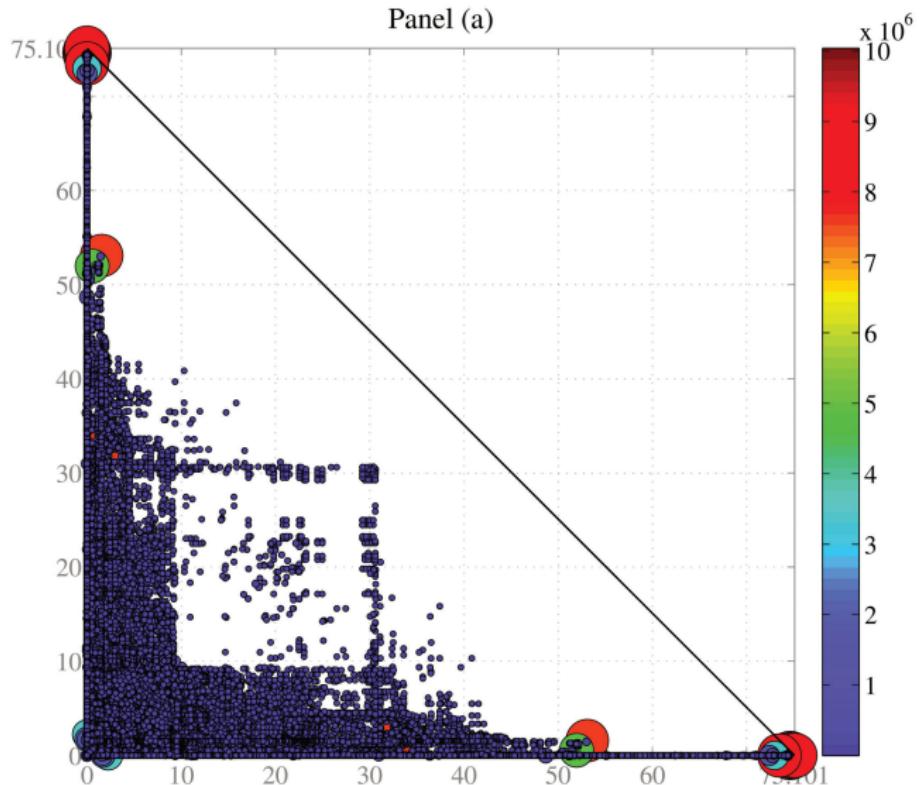
## Earlier work:

- Fudenberg et al. (1983 RIE), Reinganum (1985 QJE), Fudenberg and Tirole (1985 ReStud), ....
- Riordan and Salant (1994 JIE):  
Preemption and rent dissipation (unique equilibrium)

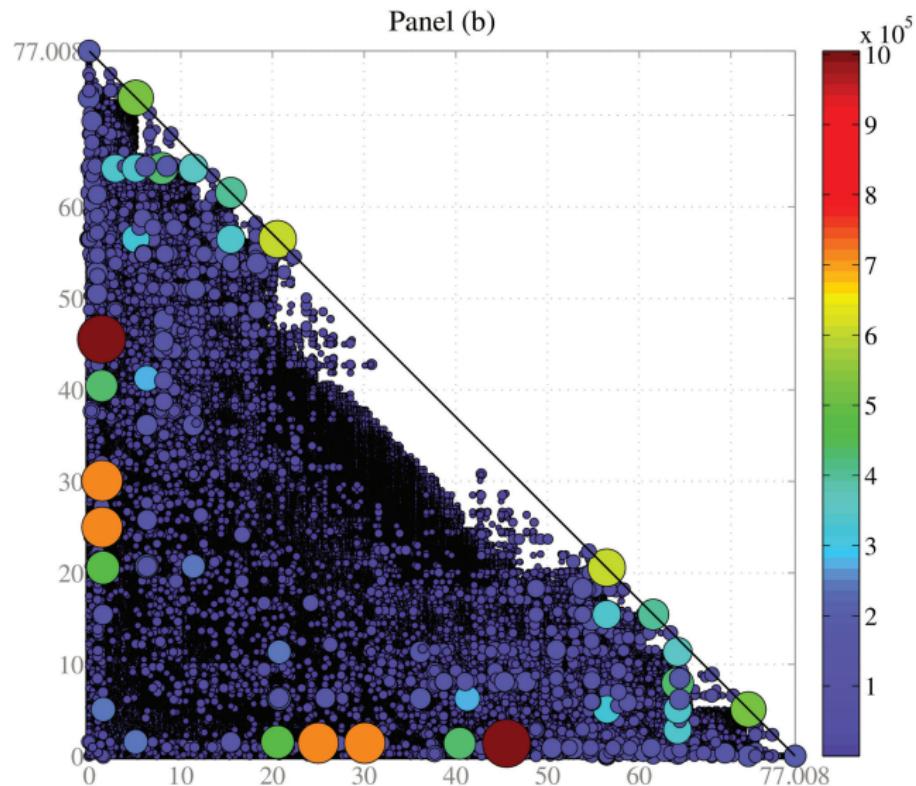
## We show:

- ① Many types of endog. coordination is possible in equilibrium
  - Leapfrogging (alternating investments)
  - Preemption (investment by cost leader)
  - Duplicative (simultaneous investments)
- ② The equilibria are generally inefficient due to over-investment
  - Duplicative or excessively frequent investments

## Pay-offs (deterministic tech progress)

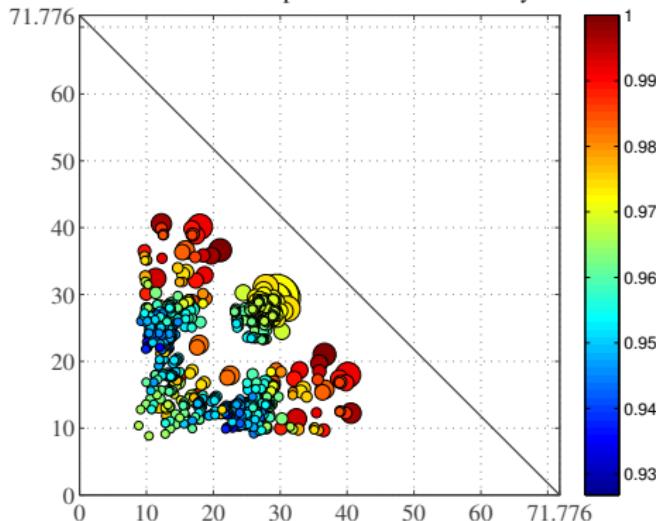


## Pay-offs (stochastic tech progress)

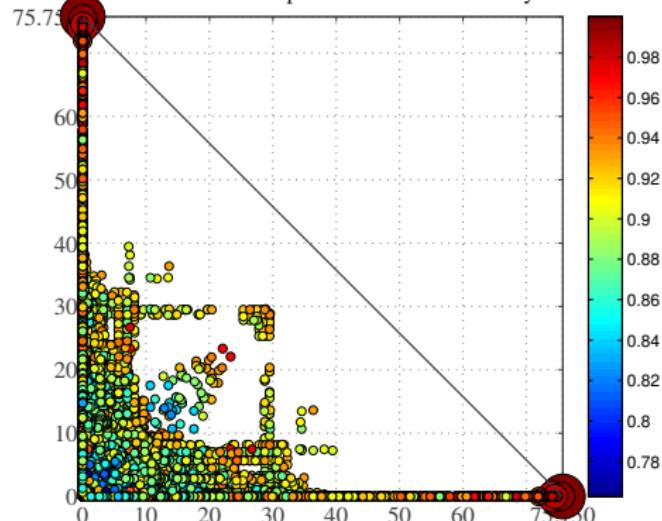


# Pay-offs: alternating vs simultaneous move games

Panel (a): Non-monotonic tech. progress  
17826 equilibria, 792 distinct pay-off points  
Size: number of repetitions Color: efficiency

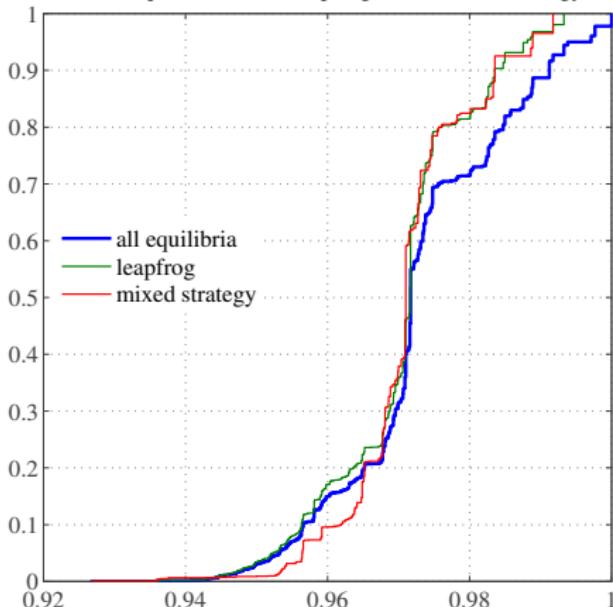


Panel (b): Simultaneous move  
28528484 equilibria, 16510 distinct pay-off points  
Size: number of repetitions Color: efficiency

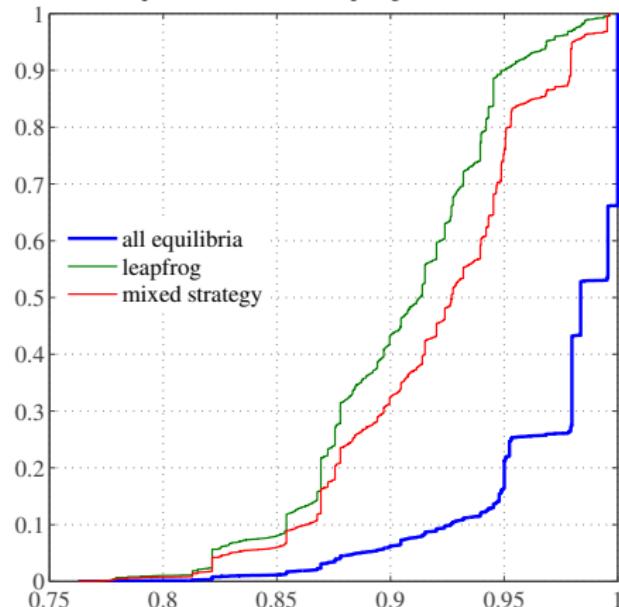


# Efficiency: alternating vs simultaneous move games

Panel (c): Non-monotonic tech. progress  
8913 equilibria, 7817 leapfrog, 2752 mixed strategy

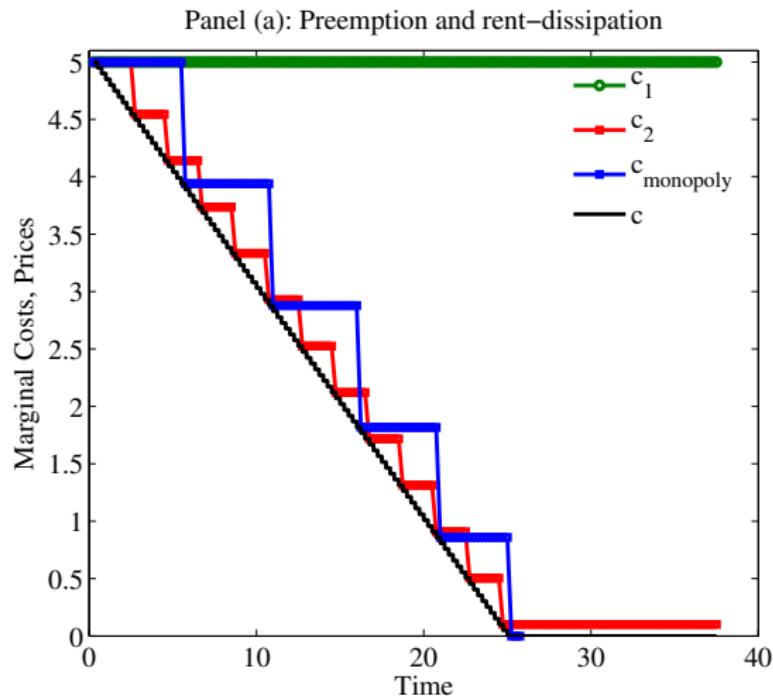


Panel (d): Simultaneous move  
14264242 equilibria, 2040238 leapfrog, 2730910 mixed strategy



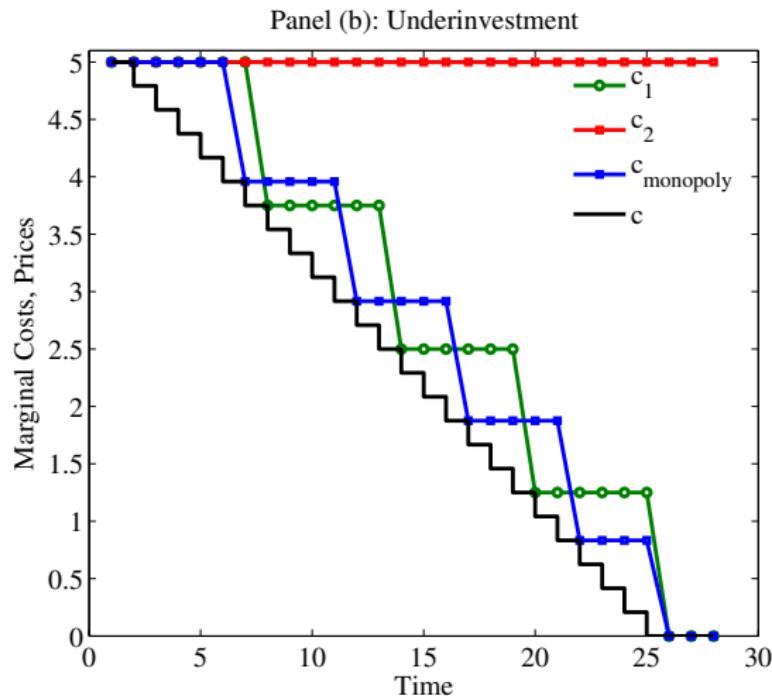
# Full preemption and rent dissipation

Confirm R&S the result with high K and small dt



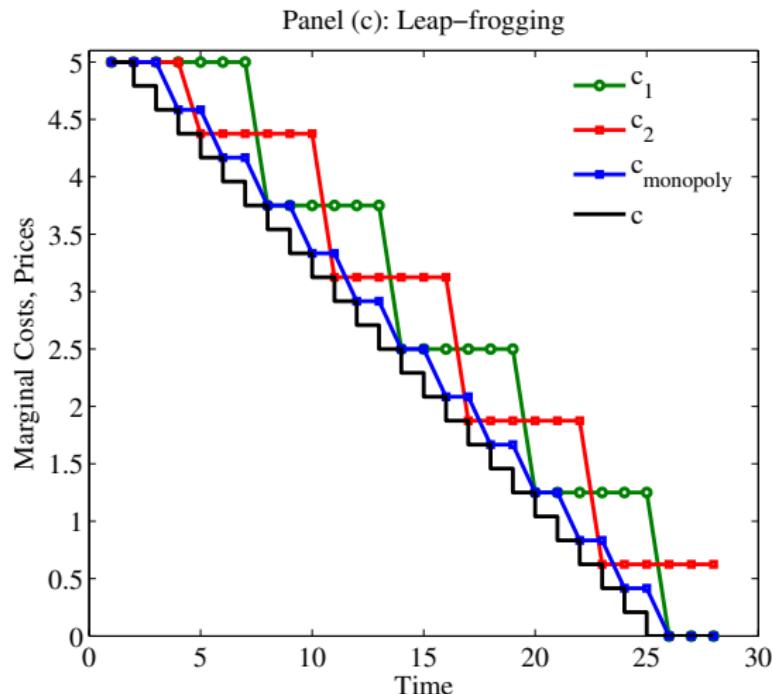
# Underinvestment

Rent-dissipation is not a general outcome - disappears when  $K$  is low relative  $dt$



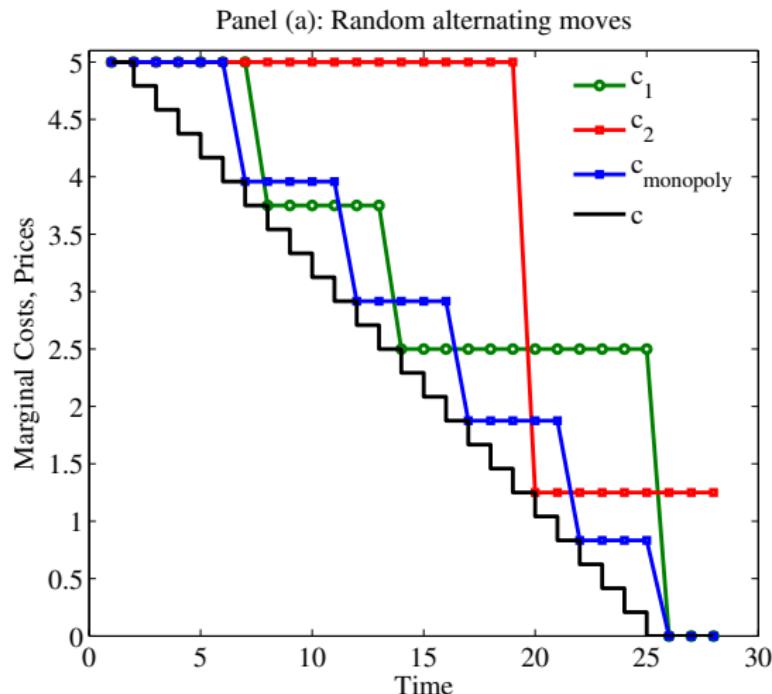
# Leap-frogging

Preemption is not the general outcome - disappears when K is even lower



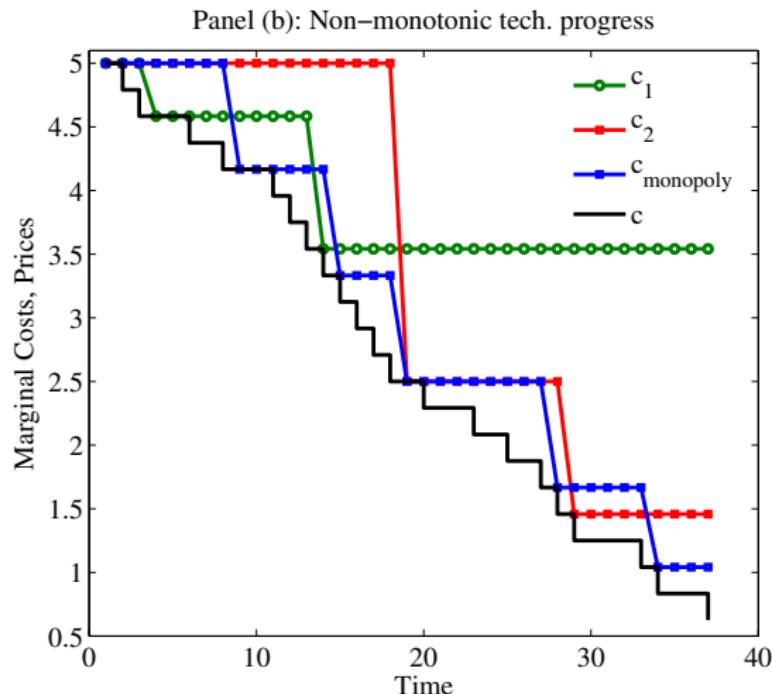
# Random alternation → Leapfrogging

Riordan and Salant's result is not robust



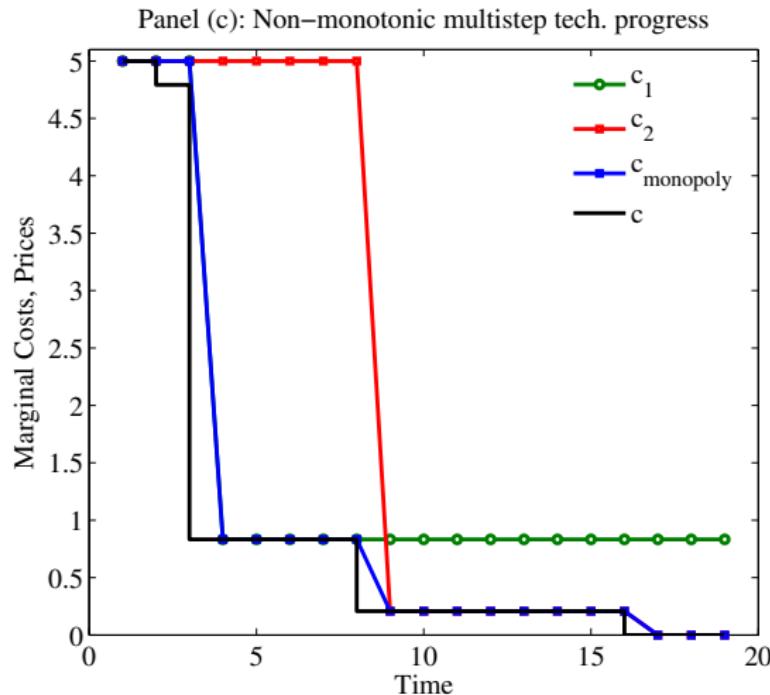
# Random onestep technology $\rightarrow$ Leapfrogging

Riordan and Salant's result is not robust



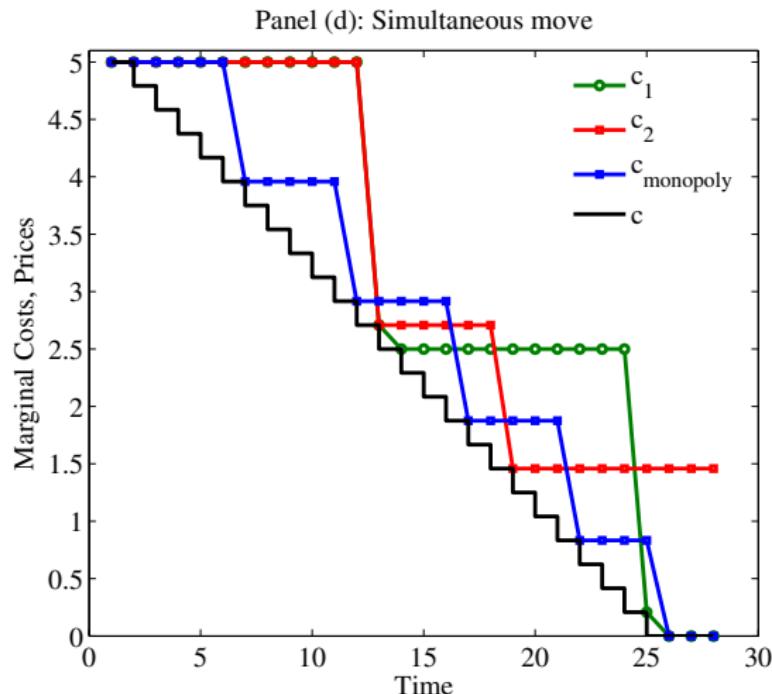
# Random multistep technology $\rightarrow$ Leapfrogging

Riordan and Salant's result is not robust

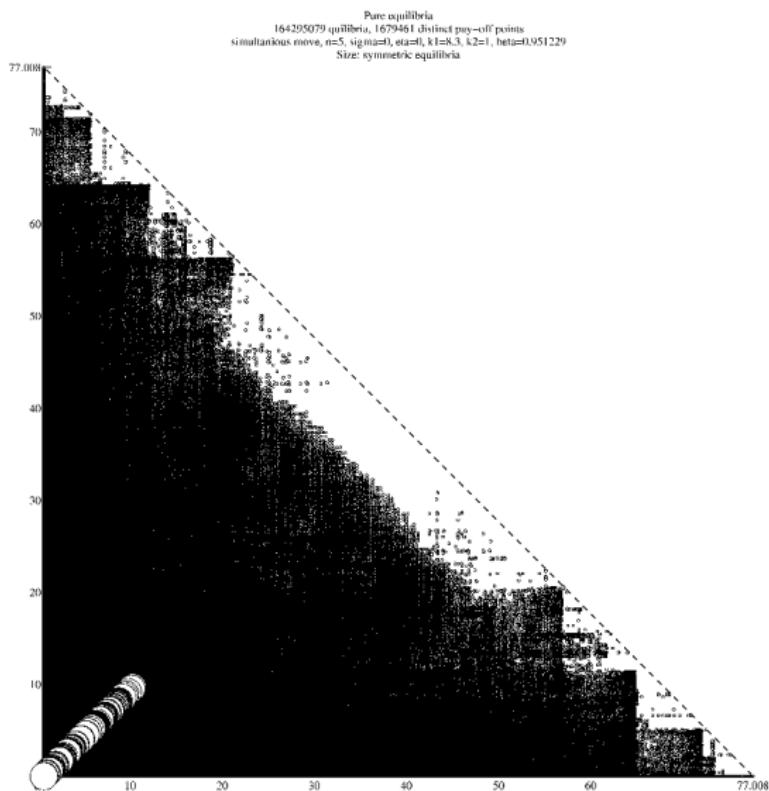


# Simultaneous moves: Leapfrogging

Riordan and Salant's conjecture is wrong



Symmetric equilibria:  $V_1(c_1, c_2, c) = V_2(c_2, c_1, c)$



# Failure of homotopy approach

## Homotopy parameter: $\eta$

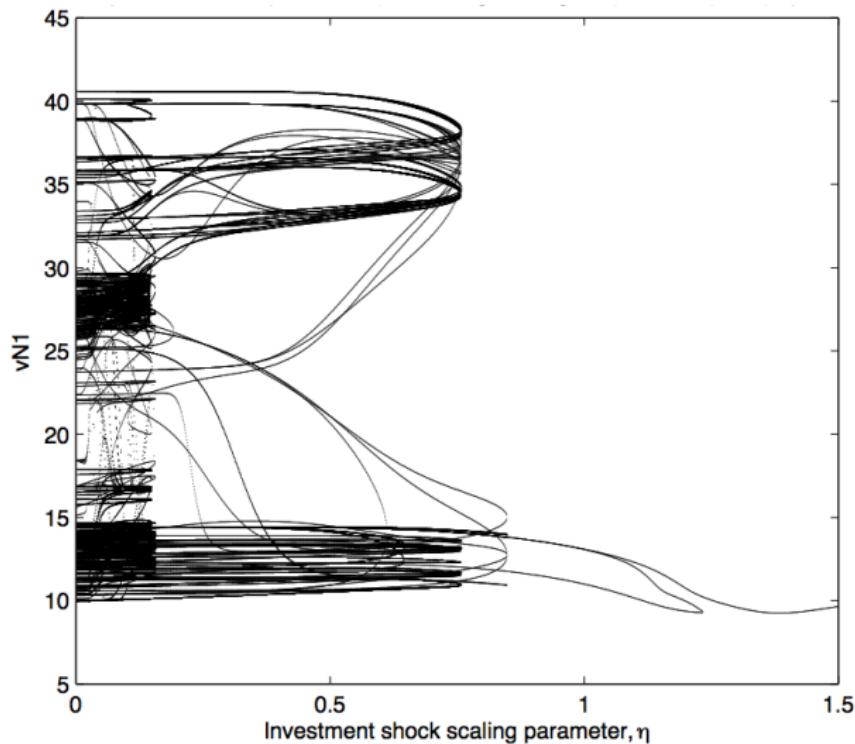
- In each period each firm incurs additive random costs/benefit from not investing and investing
- $\eta$  is a scaling parameter that index variance of idiosyncratic shocks to investment
- High  $\eta \rightarrow$  unique equilibrium  $\eta \rightarrow 0 \rightarrow$  multiple equilibria

## Problems:

- Multiplicity of equilibria  $\rightarrow$  too many bifurcations along the path
- Equilibrium correspondence is not lower hemi-continuous

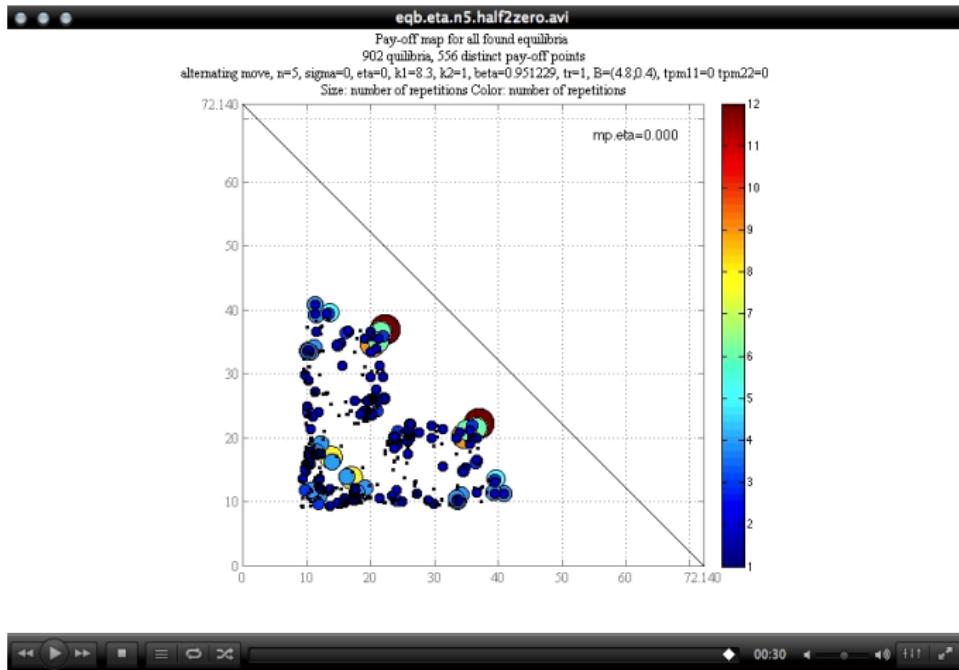
# Failure of homotopy approach

Equilibrium correspondance, alternating move game:  $V_{N,1}(c_0, c_0, c_0)$  vs.  $\eta$



# Failure of homotopy approach

Video: Set of equilibrium outcomes as variance of shocks decreases to zero



## Conclusions: Bertrand investments model

- Many types of endogenous coordination is possible in equilibrium
  - Leapfrogging (alternating investments)
  - Preemption (investment by cost leader)
  - Duplicative (simultaneous investments)
- Full rent dissipation and monopoly outcomes are supported as MPE.
- Numerous MPE equilibria and “Folk theorem”-like result
- The equilibria are generally **inefficient** due to **over-investment**
  - Duplicative or excessively frequent investments

## Conclusions: Solution of dynamic games

- When equilibrium is not unique the computation algorithm inadvertently acts as an **equilibrium selection mechanism**
- When directionality in the state space is present, state recursion algorithm is preferred to time iterations
- Plethora of Markov perfect equilibria poses new challenges:
  - How firms manage to coordinate on a particular equilibrium?
  - Increased difficulties for empirical applications.
  - Daunting perspectives for identification of equilibrium selection rule from the data.
- **Estimation of dynamic games with multiple equilibria**  
Nested Recursive Lexicographical Search (NRLS)

## Estimation of stochastic dynamic games

- ➊ Several decision makers (*players*)
  - ➋ Maximize discounted expected lifetime utility
  - ➌ Anticipate consequences of their current actions
  - ➍ Anticipate actions by other players in current and future periods (*strategic interaction*)
  - ➎ Operate in a stochastic environment (*state of the game*) whose evolution depend on the collective actions of the players
- 
- Estimate structural parameters of these models
  - Data on  $M$  independent markets over  $T$  periods
  - Multiplicity of equilibria

# Markov Perfect Equilibrium

- MPE is a pair of **strategy profile** and **value functions**:
- Bellman Optimality**

Each players solves their Bellman equation for values  $V$  taking other players choice probabilities  $P$  into account

- Bayes-Nash Equilibrium**

The choice probabilities  $P$  are determined by the values  $V$

- In compact notation

$$V = \Psi^V(V, P, \theta)$$

$$P = \Psi^P(V, P, \theta)$$

- Set of all Markov Perfect Equilibria

$$SOL(\Psi, \theta) = \left\{ (P, V) \mid \begin{array}{l} V = \Psi^V(V, P, \theta) \\ P = \Psi^P(V, P, \theta) \end{array} \right\}$$

## Maximum Likelihood

- Data from  $M$  independent markets from  $T$  periods

$$Z = \{\bar{a}^{mt}, \bar{x}^{mt}\}_{m \in \mathcal{M}, t \in \mathcal{T}}$$

Usually assume only one equilibrium is played in the data.

- For a given  $\theta$ , let

$(P^\ell(\theta), V^\ell(\theta)) \in SOL(\Psi, \theta)$  denote the  $\ell$ -the equilibrium

- Log-likelihood function is

$$\mathcal{L}(Z, \theta) = \max_{(P^\ell(\theta), V^\ell(\theta)) \in SOL(\Psi, \theta)} \frac{1}{M} \sum_{i=1}^N \sum_{m=1}^M \sum_{t=1}^T \log P_i^\ell(\bar{a}_i{}^{mt} | \bar{x}^{mt}; \theta)$$

- The ML estimator is  $\theta^{ML} = \arg \max_\theta \mathcal{L}(Z, \theta)$

# Estimation methods for stochastic games

## Maximum likelihood estimator

- Efficient, but expensive: need full solution method
- No problem with multiple equilibria

 Borkovsky, Doraszelsky and Kryukov (2010) All solution homotopy;  
Iskhakov, Rust and Schjerning (2016) RLS

## Two-step estimators

- Fast, but potentially large finite sample biases

 Bajari, Benkard, Levin (2007); Pakes, Ostrovsky, and Berry (2007);  
Pesendorfer and Schmidt-Dengler (2008)

$$\max_{\theta} \mathcal{L}(Z, \Psi^P(\Gamma(\theta, \hat{P}), \hat{P}, \theta))$$

# Estimation methods for stochastic games

## Nested psuedo-likelihood (recursive two-step)

- Bridges the gap between efficiency and tractability
- Unstable under multiplicity

 Aguirregabiria and Mira (2007); Pesendorfer and Schmidt-Dengler (2010); Kasahara and Shimotsu (2012)

## Math Programming with Equilibrium Constraints (MPEC)

- Reformulates ML problem as constrained optimization
- Should not be affected by multiplicity

 Su (2013); Egesdal, Lai and Su (2015)

$$\max_{(\theta, P, V)} \mathcal{L}(Z, P) \text{ subject to } V = \Psi^V(V, P, \theta), P = \Psi^P(V, P, \theta)$$

## Summary

- Propose robust and computationally feasible MLE estimator for **directional dynamic games (DDG)**, finite state stochastic games with particular transition structure
- Rely of full solution algorithm that provably computes all MPE under certain regularity conditions
- Employ smart discrete programming method to maximize likelihood function over the finite set of equilibria
- Provide Monte Carlo evidence of the performance
- Fully robust to multiplicity of MPE
- Relax single-equilibrium-in-data assumption
- Path to estimation of equilibrium selection rules