Exercise set 3: Zurcher Exercise

`Replicating John Rust (1987)`

Dynamic Programming Spring 2020,

by Patrick Kofod Mogensen and Maria Juul Hansen

Introduction

This week we will be looking at the first real world example of a structural estimation. Structural estimation has its roots in the Cowles Commision, and is an estimation method which bridges economic theory, and emperical economics. We saw last time that this is done by anchoring the estimation to a specific economic model. The validity and generality of results depend on the validity and generality of the model, but really this is just as true for a linear index logit, or any other parametric assumption.

This exercise set contains a lot of information. It is meant as a quick, rough introduction to the model and method. Great articles are John Rust (1987), John Rust (2000), John Rust's manuals on discrete Markov processes, and Aguirregabiria and Mira's (2010) survey on structural discrete choice models in Journal of Econometrics.

The model

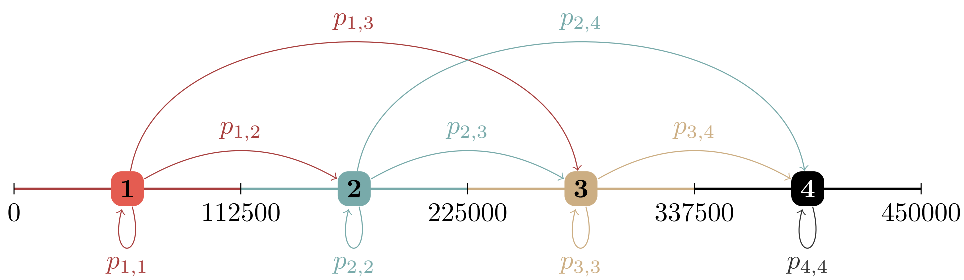
We are going to consider a model of optimal replacement. Madison Metropolitan Bus Company owns a fleet of buses, and their maintenance is managed by Harold Zurcher. Each month, he gets the buses in for maintenance. He registers their odometer readings and makes a binary decision: he can repair the engine of each bus or replace it. The replacement motor is expensive and built from scratch. However, the bus will run as a new one from the factory after a replacement. If he does not replace the engine, he has to do the usual maintainance (repair, oil changes, etc). If he does, he has to pay the replacement cost and do the usual maintenance of a new engine (oil, check everything is running fine, etc). As economists, we would model this using a cost function in two variables: odometer reading and the replacement decision

$$u(x_t, d_t, \theta_1) = \begin{cases} -c(x_t, \theta_c) & \text{if } d_t = 0 \\ -\left[\overline{P} - \underline{P} + c(0, \theta_c)\right] & \text{if } d_t = 1. \end{cases} \tag{1}$$

Here $RC \equiv \overline{P} - \underline{P}$ is the net replacement cost (scrap value minus engine cost). The binary choice of replacement is d_t, and the state (odometer reading) is x_t. The remaining part of the cost function u is the per period maintainance cost $c(x_t, \theta_c)$. Rust compares a few specifications, and we will use the linear version. The means that

$$c(x_t, \theta_c) = \theta_c \cdot 0.001 \cdot x_t.$$

To characterize the state space we define the set of possible values of x_t. We will assume that it is a bounded interval $[0, 450000]$ measured in miles. To handle the problem on a computer, we will discretize this state space into $N+1$ nodes, which results in N bins. This means that state 1 means that the odometer is in the interval $[0, \frac{450000}{N}]$, state 2 means that it is in $[\frac{450000}{N}, 2 \cdot \frac{450000}{N}]$, ..., and state N means$[(N-1) \cdot \frac{450000}{N}, N \cdot \frac{450000}{N}]$. Transitions mean moving from state n to m. If we abstract from the decision to replace the engine, the odometer reading can only increase. We model this uncontrolled transition **process** as a finite discrete Markov process. The idea can be seen in the figure below. We see that each state covers an interval of readings, not specific values.



Notice a few things. We have written this example such that if s is the current state $s+2$ is the upper bound on the possible state tomorrow, and state 4 is absorbing (i.e. the bus can't get away from this, unless the engine is replaced). The odometer readings are in principle unbounded and continuous, and these are features that we want to approximate. First of all, the largest odometer reading in the state space should be well larger than the largest observed mileage. Second of all, we need *enough* intervals to capture the difference in transition probabilities. If we keep the four intervals as above we would probably get $p_{j,j}$ very close to 1, and a finer grid would allow for a more flexible process.

Question

Take the above discrete Markov chain, and write out the transition matrix.

Controlled Markov Process

Now let us introduce the choice. The previous part was directional (x can only evolve in one direction: forward), and uncontrolled. However, remember that we have a control that enables us to reset x at a cost RC. The question is then: when should we replace the engine?

Details

There are a few details we need to be able to continue. For each control (keep/replace) there is an iid extreme value type 1 shock. This could represent different unobserved advantages and disadvantages of keeping or replacing. It has two great advantages. We can use the log-sum-trick, and it facilitates estimation. We could also have chosen another distribution, and in principle the theory would work. A normal distribution would lead to a probit likelihood function instead of a logit. The problem is that the solution method is slow due to numerical integration and an explicit maximization step.

The Bellman equation

In an infinite horizon problem with $\beta \in [0,1)$ we get a stationary solution to the dynamic optimization problem

$$V_\theta(x_t) = \sup_d E\left[\sum_{t=\tau}^{\infty} \beta^{t-\tau}\left(u(x_t, d_t, \theta_c) + \epsilon_t(d_t)\right) \;\middle|\; x_\tau, \epsilon_\tau, \theta_c, \theta_p\right].$$

Here θ_p is the parameter characterizing the transition probabilities. These are Markov probabilities in our case, so we simply get a matrix of parameters. The Bellman equation for this is *simply*

$$V_\theta(x,\epsilon) = \max_{d\in\{0,1\}}\left[u(x,d,\theta_c) + \epsilon(d) + \beta\int_y\int_\eta V_\theta(y,\eta)p(\mathrm{d}y,\mathrm{d}\eta|x,\eta,d,\theta_c,\theta_p)\right].$$

Here x and ϵ are the current period variables, and y and η are the corresponding future period variables. Using the conditional independence assumption, and the multivariate extreme value type 1 distributional assumption, we can rewrite

$$EV_\theta(x,i) = \int_y\left[\log\left\{\sum_{j\in\{0,1\}}\exp\left(u(x,j,\theta_c) + \beta EV_\theta(y,j)\right)\right\}\right]p(\mathrm{d}y|x,i,\theta_p). \tag{2}$$

The right-hand side is a contraction mapping in EV_θ, and is straight forward to calculate for parameters and EV_θ given. Notice that the discrete state formulation in combination with the Markov nature of the transitions allows us to simply replace the outer integral with a matrix multiplication. To simplify the transition process even further, we simplify the transition probabilities such that

$$\begin{aligned} p_{1,1} &= p_{2,2} = p_{3,3} = \ldots = p_0 \\ p_{1,2} &= p_{2,3} = p_{3,4} = \ldots = p_1 \\ p_{1,3} &= p_{2,4} = p_{3,5} = \ldots = p_{2}, \end{aligned}$$

and so on. So no matter if $x = 1$, $x = 2$ or something different, the probability of staying is the same, the possibility of increasing by one in the state space is the same, and so on. This simplifies the transition matrix and reduces the number of parameters dramatically. Remember each $p_{k,l}$ is a parameter in itself. We could have continuous distributions in stead, parameterized by a few parameters, but this would instead increase the difficulty of integrating over the transition process. Define the following

$$\begin{aligned} EV_\theta(0) &\equiv (EV_\theta(1,0), EV_\theta(2,0), \ldots, EV_\theta(N,0))' \\ EV_\theta(1) &\equiv (EV_\theta(1,0), EV_\theta(1,0), \ldots, EV_\theta(1,0))' \\ c(j,\theta) &\equiv (c(0,j,\theta), c(1,j,\theta), \ldots, c(N,j,\theta))'. \end{aligned}$$

Here EV_θ is written differently for $d = 1$ and $d = 0$. This is because if $d = 1$ we regenerate the state process to a *new* engine, and this means that we should consider the corresponding element of the expected value function. In other words, changing the motor today, means entering tomorrow with a brand new motor, which is the first element in the state space. The new vectors are complete versions of the functions on the state space.

$$EV_\theta(i) = P \times \left[\log \left\{ \sum_{j \in \{0,1\}} \exp\left(u(j,\theta_c) + \beta EV_\theta(j)\right) \right\} \right], \tag{3}$$

where the matrix multiplication is over an N-by-N matrix and a N-by-1 vector (log,Σ and exp are taken element-wise). It is super important that you understand this.

With the current model, we simply get logit choice probabilities of $d = k$

$$P(k|x,\theta) = \frac{\exp\left(u(x,k,\theta_c) + \beta EV_\theta(k)\right)}{\sum_{j\in\{0,1\}} \exp\left(u(x,j,\theta_c) + \beta EV_\theta(j)\right)}. \tag{4}$$

These define the likelihood function together with the transition probabilities (shown for a single bus)

$$l(\{x_\tau, d_\tau\}_{\tau=t}^T | x_\tau, d_\tau, \theta) = \prod_{t=\tau+1}^{T} P(d_t|x_t,\theta) p(x_t|x_{t-1}, d_{t-1}, \theta_p). \tag{5}$$

Including more observational units simply involve adding a subscript i and multiplying over them as well (since everything is iid conditional on the payoff-relevant states). We are going to take the natural logarithm, and calculate (and optimize) the log-likelihood function instead. This is then simply the sum of log choice probabilities and transition probabilities (at the observed data) over buses and periods. This gives us

$$\begin{aligned} \ell(\mathbf{x}, \mathbf{d}|x_\tau, d_\tau, \theta) &= \sum_t \sum_i \log\left(P(d_{i,t}|x_{i,t},\theta)\right) + \sum_t \sum_i \log\left(p(x_{i,t}|x_{i,t-1}, d_{i,t-1}, \theta_p)\right) && (6) \\ &= \ell^1 + \ell^2 && (7) \end{aligned}$$

Solving the model

So now we know how to solve the model. Solving the model means finding a fixed point in

$$EV_\theta = \Gamma(EV_\theta),$$

where $\Gamma : B \to B$ is a mapping defined by the right hand side of (3). Here B is the Banach space (a particular vector space) of bounded, measurable functions on $[0, \infty)$. Rust showed that this is a contraction mapping, which means that we can apply Γ successively, and as we proceed, our sequence of EV_θ-functions will approach the unique solution to the non-linear equation. We can also use a generalized version of Newton's method to find the zeros of an equivalent problem

$$(I - \Gamma)EV_\theta = 0,$$

where I is the identity operator on B . The identity operator is like a "1" in other spaces. It takes a function as its argument, and returns the same function. Equivalently the "0" on the right hand side is not a zero in the usual sense, but it is the zero element (or zero function) in B. If Γ was linear this would be a straightforward problem, but it is not. Since $(I - \Gamma)$ can be shown to have an invertible (Fréchet) derivative, we use Newton's method. Newton's method for solving non-linear equations is simply to create a Taylor expansion around a point (the point is our current function guess EV_θ^K)

$$0 = (I - \Gamma)EV_\theta^{K+1} \approx (I - \Gamma)EV_\theta^K + (I - \Gamma')(EV_\theta^{K+1} - EV_\theta^K),$$

where $(I - \Gamma')$ is the Fréchet derivative of $(I - \Gamma)$. To calculate Γ' we simply take (2) and differentiate with respect to EV. This turns out to be β times the transition matrix of the controlled process. What is the transition matrix of the controlled process? Well, we don't really develop the theory of Markov decision processes in this course, but it turns out that in this class of dynamic programming problems, the process (d, x) is also Markovian. That is, in our model we assume x followed a discrete Markov process unconditionally, but this then implies that even when we let the agent behave optimally, and therefore affect the evolution of x, we still get a Markov process. The transition matrix for this process in not too difficult to construct. Ask yourself: if $P(0|x_t)$ is the probability of not replacing given x_t, what is the probability of being in the same state tomorrow? It is just $P(0|x_t) \cdot p_0$. What is probability of increasing the state by 1? It is simply $P(0|x_t) \cdot p_1$. Continue this way, and you can fill out the entire controlled transition matrix (see p. 25 in the manual for further explanation). Since $(I - \Gamma')$ is linear and invertible, we get

$$\begin{aligned}
-(I - \Gamma)EV_\theta^K &= (I - \Gamma')(EV_\theta^{K+1} - EV_\theta^K) \\
-(I - \Gamma')^{-1}(I - \Gamma)EV_\theta^K &= EV_\theta^{K+1} - EV_\theta^K \\
EV_\theta^{K+1} &= EV_\theta^K - (I - \Gamma')^{-1}(I - \Gamma)EV_\theta^K. \qquad (8)
\end{aligned}$$

Using this procedure instead of contraction iterations can be dramatically faster than contraction iterations (also called value function iterations or succesive approximations). As mentioned before, we need to do this, because the solution enters into the likelihood function. Rust (2000) provides details on further improvements using Werner (1983)'s method.

Optimizing the likelihood

We now know how to solve the model and calculate the likelihood function. In principle we could simply try different values of the parameters, but this is extremely ineffective. Instead we use a quasi-Newton method called BHHH. As you hopefully know, this uses the outer product of the scores as the Hessian-updating formula. This means we do not have to calculate the Hessian, only the first order derivative. This means differentiating the likelihood contributions with respect to a vector of all the parameters: β,θ_C, and all elements of θ_p, and summing over buses and periods. This is a trivial task once we realize that we can use the implicit function theorem to calculate $\partial EV/\partial\theta$

$$\frac{\partial EV}{\partial\theta} = (I - \Gamma')\frac{\partial\Gamma(EV)}{\partial\theta}.$$

We normally assume β constant and leave it out of the vector. What about RC? Well, we simply take Γ evaluated at EV, and differentiate partially wrt. RC and θ_c. This is simply partial differentiation of (3) left-multiplied by $I - \Gamma'$.