Explaining Early Bidding in Informationally-Restricted Ascending-Bid Auctions

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Abstract

We introduce an empirical dynamic model of rationally inattentive bidding to explain early bidding in an online auction mechanism we call the Korean auction that was invented in 2002 by an executive of a car rental company in Korea. To thwart collusion, bidders are unable to see others' identities or bids and are only informed if they are the high bidder at each instant of the two minute auction. We suggest that modeling bidding as a perfect Bayesian equilibrium outcome cannot explain the early bidding behavior we observe in these auctions. We show that our model of rationally inattentive bidding under the concept of anonymous equilibrium predicts early bidding and final high bids, but the model fails quantitatively to predict the magnitude of first bids in these auctions, a phenonmenon we refer to as early overbidding. We argue that early overbidding is inconsistent with rational competitive bidding behavior as well as the hypothesis of collusive bidding in these auctions

Keywords: wholesale auctions, ascending bid, sealed bid, English, Japanese, open outcry and Korean auctions, bid sniping, jumping and creeping behaviors, collusion, informational restrictions, perfect Bayesian equilibrium, anonymous equilibrium, rational inattention, bidding frictions, early bidding and overbidding, informational free-riding, bounded rationality, dynamic programming, structural estimation, quasi maximum likelihood estimation, empirical mechanism design

JEL Classification Numbers: C57, C61, C72.

1 Introduction

We introduce a dynamic model of rationally inattentive bidding to explain early bidding in an online auction mechanism we call the *Korean auction* that was invented in 2002 by an executive of a car rental company in Korea.[1] The Korean auction is similar to a traditional ascending bid auction (often referred to as an *English auction*) except that it is *informationally restricted:* bidders are unable to see other bids or the number of other participating bidders. The only information provided to bidders is whether or not they are the current high bidder at each instant in the auction. The winner is the bidder who submitted the highest bid over the course of the two minute auction, and bidders are obligated to pay their highest bid if they win. Time priority is also enforced: if a bidder submits a bid equal to the current high bid they do not become the new current high bidder.

The rental company disposes of hundreds of rental cars in wholesale auctions each month. Before it invented the online Korean auction, the company sold its cars at different rental locations using *open outcry auctions* where bidders verbally called out bids under the guidance of an auctioneer.[2] Over 60 professional bidders are registered to participate in these auctions: most are car dealers seeking to buy used cars at wholesale prices for resale to consumers at a markup. Around year 2000 the rental executive suspected collusion by some of the bidders in these auctions. In 2002 he invented the informationally restricted Korean auction in order to defeat it, and he claims his design was successful. The new online auctions started in January 2003 and continued until 2007 when the executive decided the fixed costs of running them were too high, so he reverted to selling cars in open outcry auctions again, but this time at a wholesale auction house in Seoul.

[1] Due to a confidentiality agreement to obtain the data we are unable to reveal the specific company or the identities of bidders participating in these auctions.

[2] Open outcry auctions are also called English auctions, but a key difference is that submission of bids is endogenous and at the discretion of bidders. The English auctions that have been studied theoretically are actually *Japanese auctions* (also known as "thermometer" or "button" auctions, see Milgrom and Weber [1982]) where bidders observe an exogenously and continuously rising price and press buttons (or keep hands raised) to indicate their willingness to pay that price, with irreversible exit once they release the button or lower their hand. The winner is the last remaining bidder and the winning price is the value at which the penultimate bidder dropped out, an outcome that is strategically equivalent to a second price sealed bid auction. Open outcry auctions generally have far different outcomes with complex patterns of jump bidding and sniping, see, e.g. Avery [1998] and Isaac et al. [2007a]. Little is known theoretically about equilibrium bidding strategies in open outcry auctions, though Isaac et al. [2007b] computed equilibria numerically and showed that "jump bidding occurs due to strategic concerns and impatience." (p. 144).

We refer to the open outcry auctions held at the rental company as *Regime 0*, the online informationally restricted Korean auctions as *Regime 1*, and the open outcry auctions at the professional auction house as *Regime 2*. Cho et al. [2014] analyzed data from over 30,000 auctions from Regimes 1 and 2 and concluded that after controlling for car make, model and odometer that prices in regime 2 were nearly 10% *higher* than under regime 1. Thus, net of the auction house's 10 percent commission, the rental company received the same revenue per car sold, though it saved on the fixed costs of running its own online auctions. If collusion had resumed under regime 2, we should have observed *lower* prices compared to regime 1. Bidders can see each other and observe each others' bids in open outcry auctions. In the absence of collusion the *linkage principle* of Milgrom and Weber [1982] implies that prices in regime 2 should be higher than in the informationally restricted auctions in regime 1. However Cho et al. [2014] noted there are the higher prices might be explained by a larger number of bidders participating in auctions in regime 2 compared to regime 1.

The focus of this paper is understand bidding under the Korean auction (regime 1) assuming the informational restrictions were successful in defeating collusion. In section 2 we compare auction prices of specific makes and models at the end of regime 0 with auction prices just after the transition to regime 1. We find no systematic evidence of higher auction prices in regime 1 to back the executive's claims. However previous research finds that restricting information in dynamic auctions can inhibit collusion. Cramton and Schwartz [2000] recommended informational restrictions in FCC spectrum auctions to reduce suspected collusion. These include coarsening bids to 3 significant digits because bids in the billions "allowed for all kinds of signaling" in the less significant digits of the bids, and anonymizing bidder identities to reduce the possibility of retaliation for deviating from collusive agreements. Marshall and Marx [2009] also showed that suppressing bidder identities during and after an auction may successfully inhibit certain types of collusion. Bajari and Yeo [2009] provide empirical evidence that FCC auction design changes in response to these recommendations including "click box bidding" and suppressing bidder identities "limited firms' ability to tacitly collude" (p. 90) though "detecting collusion based solely on auction data can be difficult" (p. 100).

There are few previous structural empirical analyses of bidding in dynamic ascending bid auctions, with or without informational restrictions of the sort employed in the Korean auctions. Also, as we noted, there is very little theory characterizing equilibrium bidding behavior in open outcry auctions. The closest previous analysis to our's is Barkley et al. [2021] who analyze data from Texas auctions of certificate of deposit (CD) in 30 minute online auctions that are informationally restricted in the same way as the Korean rental car auctions: bidders cannot see each others' identities or each others' bids, but they are informed whenever their bid is the highest ("in the money"). They find that the informational restrictions as well as "bidding frictions" result in significant inefficiencies and "money left on the table" due to "submitting winning bids at rates well above the lowest bid needed to win" (p. 380) and "are costly both for revenue and allocative efficiency" (p. 376). They conclude that

> The choice of auction mechanism is puzzling given the estimated losses due to bidding frictions. Why should the auctioneer not run a sealed-bid uniform price auction to allocate funds? We suggest two reasons why the current mechanism may be preferred to such an alternative mechanism despite the losses due to frictions: collusion and corruption. Recent theoretical work has demonstrated the potential for collusive outcomes in similar games with more public information on player actions. Kamada and Kandori [2020] study a continuous-time game with a deadline and random arrival of opportunities to change one's action. In their model, all actions are perfectly observable, and players are able to coordinate to obtain results very near the full-collusion outcome. The coordinated equilibria of the type demonstrated in Kamada and Kandori [2020] may be one reason why the auction platform opted for providing banks with limited information.

However a static, anonymous first price sealed bid auction or second price auction reveals even less information than the informationally restricted ascending bid auctions, so this logic is not compelling as to why a dynamic auction mechanism should be preferred to a static one in terms of its ability to defeat collusion. It is known that repeated static auctions can also be subject to collusion via bidding rings that employ self-enforcing side payment mechanisms to reduce the bids in auctions and lower winning prices (see, e.g. Graham and Marshall [1987], McAfee and McMillan [1992], and Mailath and Zemsky [1991]). In any event, we also agree with the view that it is very difficult to detect collusion from bid data alone. If there is collusion that lowers bids, our structural analysis will reflect this via estimated valuations that are lower than they would be absent collusion, but the auctions might otherwise appear to be competitive at the lower inferred valuations. Thus, our analysis proceeds on the assumption there is no collusion, though we return to this important question at the end of the paper since our empirical analysis does not uncover any obvious signs of collusion such as artificially low "fake bids" designed to create an impression of active, competitive bidding by members of a bidding ring.

Our main goal and the key contribution of this paper is to improve our understanding of bidding behavior in dynamic auctions, and conduct a limited type of *empirical mechanism design* to assess whether, in the absence of collusion, the rental car company might have raised more revenue by removing the informational restrictions or adopting simpler static auction mechanisms such as first price sealed bid auctions or second price auctions. A natural first approach to analyzing the Korean starts with the hypothesis that in the absence of any significant bidding frictions, the informational restrictions in the Korean auctions do not constitute a binding restriction on bidders compared to a traditional open outcry auction. By a process of *bid creeping* that we illustrate in section 2, bidders can learn the high bid at any moment of the auction and avoid early overbidding that Barkley et al. [2021] found to be so prevalent in the Texas CD auctions. This suggests that in a frictionless world the Korean auction should be strategically equivalent to the Japanese clock/thermometer auction: i.e. bidders should keep bidding up to their valuations and sequentially drop out when the curent high bid exceeds their valuations, leaving the winner as the bidder with the highest valuation, but paying an amount equal to the valuation of the second highest bidder.

However an alternative way to view bidding behavior in the Korean auction is as a realization of a *perfect Bayesian equilibrium* (PBE), where all bidders employ dynamic Nash equilibrium strategies and Bayesian updating to determine when and how much to bid during the auction. We show that when the auction has a *hard close* (i.e. all bidders can submit bids at the last instant of the auction and be guaranteed that their bid will be accepted), there is always an *uninformative PBE* that is strategically equivalent to the outcome of a first price sealed bid auction (though it is a "Bayesian" version of this equilibrium since bidders only have a prior distribution of the number of bidders who might enter into any given auction, see McAfee and McMillan [1987a]). Another way to say this is that in the uninformative PBE all bidders *snipe* — they make no early bids and only bid at the last possible instant.

Our auction data strongly reject the hypothesis that bidding behavior is consistent with an uninformative PBE of the Korean auction, and so is the bidding behavior in the Texas CD auctions in Barkley et al. [2021]. Could there exist other *informative PBE* that are consistent with the early bidding we observe? In section 3 we provide an example of a two bidder, two period version of the Korean auction where the *only* PBE is the uninformative PBE that is strategically equivalent to a first price sealed bid auction. Whether there exist informative PBEs in more general environments is an open question, but we agree with Barkley et al. [2021] that it is infeasible to compute or even characterize them:

> During the auction, in equilibrium a bidder recognizes her past bidding in this auction has possibly influenced her rivals' bids, and this in turn affects the terms of trade she now faces. Moreover, the manner in which her rivals respond to her earlier bids depend on their own valuations. Therefore each bidder keeps track of the information she accumulates during the auction along the equilibrium path to form more precise beliefs about the valuations of her rivals. Defining a perfect equilibrium entails her forming beliefs based on her own past deviations from equilibrium bidding behavior as well. In such models solving all the equilibria, including those with pooling or mixed strategies, is a daunting if not impossible task with current computational technology.

In section 4 we introduce a new, computationally feasible approach to modeling bidding behavior in the Korean auction. We assume that bidders have rational but non-Bayesian beliefs and are subject to bidding frictions that include a form of rational inattention inspired by work by Sims [2003] and Matějka and McKay [2015]. A bidder's beliefs about the conditional probability distribution of the high bid at any point during the auction constitutes a "sufficient statistic" for calculating their optimal bidding strategy. We assume that rational, experienced bidders know this family of conditional probability distributions, and thus we bypass the intractable and extremely high dimensional Bayesian updating problem involved in calculating a PBE. This allows us to recast the problem from one of computing a PBE to the much simpler problem of computing a Nash equilibrium to an *anonymous game* where we solve for each bidder's equilibrium strategy as a *single agent dynamic programming problem (DP)* subject to the constraint that bidders have rational beliefs about the stochastic process governing the high bid in the auction.[3] Since the bidders in our data set are experienced professional bidders who have participated in thousands of auctions, the rational expectations assumption seems plausible, and we can estimate bidders' beliefs non-parametrically via a two step estimation process using our extensive data on bids in large numbers of auctions for homogenous types of cars.

However to explain the early bidding behavior we observe in these auctions, we hypothesize certain bidding frictions, including the assumption that due to various distractions during the auction, bidders are not always paying attention to the computer screen and continuously updating their information during the auction. This inattention is *rational* in the sense that when solving the DP problem to calculate an optimal bidding strategy, bidders account for the probability that they might be distracted later during the auction. Accounting for this and other bidding frictions causes rationally inattentive bidders to *bid earlier and higher* than they would in a frictionless environment where there is no issue of inattention or other impediments to bidding.

We solve for optimal bidding strategies by discretizing time into 121 bidding instants t during the auction running from $t = 0$ to the final possible bid at $t = 120$. We use numerical DP to compute bidding strategies that maximize bidders' expected payoff from participating in the auction. Our model involves 4 unknown parameters (v, c, p, σ) where v is the bidder's valuation of the car being auctioned, c is the psychological cost (or benefit if negative) of submitting a bid at any instant, p is the probability that a bidder is distracted and not able to bid at any instant t, and σ is a scale parameter of an extreme value distribution representing idiosyncratic costs/benefits of bidding any any instant t.

We use a semi-parametric two step fixed-effects quasi-maximum likelihood (QML) estimation approach to estimate these 4 parameters for 4029 auction/bidder pairs in who participated in 533 auctions of a homogeneous Korean car known as Avante XD. Our approach differs from that of Barkley et al. [2021] in that we use numerical DP to solve for the the bidding strategies of each bidder in each auction under the assumption that each bidder has rational beliefs, so outcomes are realizations of anonymous equilibria of

[3] Computing Nash equilibria of anonymous games is tractable and in some cases can be done in polynomial time, see Daskalakis and Papadimitriou [2015] and Cheng et al. [2017].

the Korean auction. The QML estimator finds the 4 parameters (or a total of more than 16,000 parameters in total for the 4029 auction/bidder pairs) that best fits the observed sequence of bids of each bidder in each auction. Our approach allows us to conduct detailed simulations of bidding behavior under various counterfactuals, including predicting the effect of eliminating the informational restriction on bidding behavior. In contrast, the estimation approach in Barkley et al. [2021] "only impose a subset of the conditions that define a best reply. The premise for our analysis is that bidders do not play dominated strategies, and do not bid above their valuation." (p. 377). Their approach does not identify bidder/auction-specific valuations and friction parameters, and instead focuses on estimating the distribution of bidder valuations, and is limited in its ability to evaluate counterfactual predictions of changes in auction mechanisms since as they note, "In principle the set of equilibria could be computed using the structural estimates but there are two practical barriers to this approach. The first is computational: the number of equilibria is not determined and each equilibrium is a complicated non-stationary function in continuous time." (p. 392).

Our key empirical findings are summarized in section 5. We find that our model can provide a qualitative explanation of the early bidding behavior we observe in these auctions, and the model provides good quantitative predictions of the *final high bid* submitted in the auctions. However we find that even allowing for rational inattention and bidding frictions, our model typically substantially underpredicts the initial bids submitted in these auctions, a phenomenon we refer to as *early overbidding.* We interpret the failure of our model to predict all bids well as a rejection of our assumption of bidder rationality, and an indication of *bounded rationality* or *animal spirits* among the bidders that causes them to bid up prices faster and earlier in the auction compared to what rationally inattentive bidders would do. Via a counterfactual simulation, we show that our estimated rationally inattentive bidders outperform the human bidders in these auctions, earning higher expected profits by submitting lower bids than the human bidders typically submit. Thus, our finding is inconsistent with the hypothesis of sophisticated rational bidders who are colluding to lower bids in these auctions.

We conduct other counterfactual simulations that involve changing the auction rules, but maintaining the assumption of no collusion among the bidders. We find that if all bidders behaved according to our model of rational bidding but with rational inattention and bidding frictions, the rental company would earn higher expected revenue in a static first price auction, or even more in a static second price auction. The reason for this is that the (non-Bayesian) learning by bidders during the dynamic online auction enables them to acquire more information that allows the winner to pay less for the item than in a comparable first price or second price sealed bid auction. However due to the element of irrationality/animal spirits among most bidders, we find that expected revenues earned from the Korean auction exceed the expected revenues the company would earn if it were to use static first price sealed bid auctions, but are not as high as the revenues predicted from a static second price sealed bid auction.

The latter also implies that the Korean auction is not strategically equivalent to a Japanese auction or a static second price sealed bid auction. Even if there are no bidding frictions, the Korean auction is not strategically equivalent to a Japanese or static second sealed bid auction because it is generally not optimal for bidders to continue to bid up to their valuations in the Korean auction as it is in the Japanese auction. Finally, we predict the impact of dropping the informational restriction in the Korean auction, so that it becomes an electronic version of an open outcry ascending bid auction, but one where bidders are anonymous. We prove that in the absence of bidding frictions, doing this leads to "informational free-riding" that eliminates the incentive of bidders to bid before the final instant in the auction. This causes all bidders to adopt bid-sniping strategies, making the equilibrium in this dynamic auction strategically equivalent to the equilibrium in a static first price sealed bid auction.

2 Auction Data

A large rental car company provided us with detailed data on auctions of all cars sold under its new online auction system between 2003 and 2007, the regime 1 period before it switched back to open outcry auctions but ones run at a professional auction house in Seoul. We have data on 11,259 individual car auctions during regime 1. Due to confidentiality restrictions we cannot identify the company, its location, or individual bidders.

The company used the informationally restricted online auction mechanism it designed to sell hundreds of it cars each month in a sequence of back to back two minute auctions, where a single car was sold in each auction. Bidders were given advance notice of the auctions so they could physically inspect the cars prior to the auction. Bidders would typically not undertake detailed mechnical inspections, but rather a brief "walk around" to inspect the interior and exterior condition of the car. They can request a copy of the vehicle's maintenance history, including the total amount spent on maintenance, dates of maintenance, records of accidents, and so forth. Our data includes the same maintenance and accident records that were available to the bidders, but we do not have the information gained from the physical inspection of the vehicles.

Our data includes time stamps and the amounts of each bid and the identities of each bidder in each auction. Some time stamps are a few milliseconds past the two minute auction closing time, but in general, we excluded auctions took longer than 121 seconds as many of these auctions reflected special circumstances (such as technical problems with the auction server) that required extra time to complete the auction.[4] In all instances, these slightly late bids were allowed as valid bids in the auction. There is no reservation price so in virtually every auction, the company sold the car to the highest bidder regardless of the value of the winning bid. The only exceptions that we are aware of are a small number of auctions where a bidder made a data entry error, keying in a bid that is far in excess of any reasonable value for the car. In such cases, the company cancelled the auction outcome as invalid and re-auctioned the car at a later date.

2.1 Effect of informational restrictions on potential collusion

We start our empirical analysis by comparing auction prices for specific makes and models of cars under Regime 0 (the open outcry auctions held onsite which the rental manager suspected were subject to collusion) and Regime 1 (the informationally restricted online

[4]Some auctions lasted 10 minutes but we do not know if there was some technical problem during these auctions that cause the company to extend the usual 2 minute duration, or whether there is some other reason for having a longer than usual duration. Overall we excluded 198 auctions whose total duration exceeded 121 seconds.

Table 1: Two sample t-tests for the effect of auction mechanism on collusion

Model	**Regime 0** Oct 1, 2002 to Dec 31, 2002	**Regime 1** Jan 1, 2003 to Mar 31, 2003	**Regime 1** Jan 1, 2003 to Jun 30, 2003	**P-values for two sample t-tests for equal means**
EF Sonata 1.8	5279 (1048),n=81	5148 (746),n=72	4919 (700),n=89	.815,.995 .975,.976
EF Sonata 2.0	5867 (1594),n=137	6043 (1359),n=46	7161 (1432),n=17	.235,.001 .019,.005
Dynasty 3.0	11633 (2496),n=25	13043 (2458),n=23	12934 (1757),n=13	.027,.035 .016,.560
Grandeur XG 2.0	11295 (1399),n=18	11081 (1055),n=14	11123 (978),n=15	.687,.659 .692,.456
Grandeur XG 2.5	12626 (2150),n=67	11504 (1974),n=50	11827 (1356),n=78	.998,.995 .999,.157
Galloper 7	7109 (1480),n=45	7477 (1473),n=61	7776 (1263),n=53	.103,.010 .025,.123
Magnus 2.0	7614 (1170),n=11	6665 (1576),n=16	6503 (506),n=6	.957,.992 .980,.640

auctions that the manager was convinced had defeated collusion). We have only limited data on auctions of 568 cars in Regime 0, over the period October through December 2002. We have far more data from Regime 1 with 580 cars sold via the company's informationally restricted online platform from January to end of March 2003, and and additional 633 cars auctioned from April through June, 2003. However the rental company owns a diverse set of different makes and models of cars and to assess the "causal effect" of the auction mechanism on possible collusion, it is desirable to use a matching estimator that conditions on individual makes and models of cars, as well as age and odometer value.

Unfortunately, when we try to do this conditioning to assess the causal effects, we find there are far fewer observations on the most popular makes and models, reducing the power of a test for the effect of auction mechanism on possible collusion. Table 1 reports the mean and standard deviations (and the number of observations n) for the most popular individual makes/models of cars auctioned by the rental company, for three periods: 1) Regime 0 from October through December, 2002, 2) Regime 1 from January through

March 2003 (period 1), and 3) Regime 1 from April though June 2003 (period 2). We split up the first 6 months of auctions under Regime 1 to assess the natural variability in auction prices over time under Regime 1 and to assess any potential "transition effects" after the new auction mechanism was adopted. The final column of the table reports the P-values of two-sample t-test for equality of means, against the alternative hypothesis that mean prices under Regime 1 are higher than under Regime 0. There are four P-values reported for the following four tests: 1) equality of Regime 0 mean versus the mean under Regime 1 period 1, 2) Regime 0 mean versus Regime 1, period 2, 3) Regime 0 mean versus pooled data for Regime 1 periods 1 and 2, and 4) Regime 1 period 1 versus Regime 1 period 2. The first two P values are reported on the top line and the next line reports the bottom two P-values in the order given above.

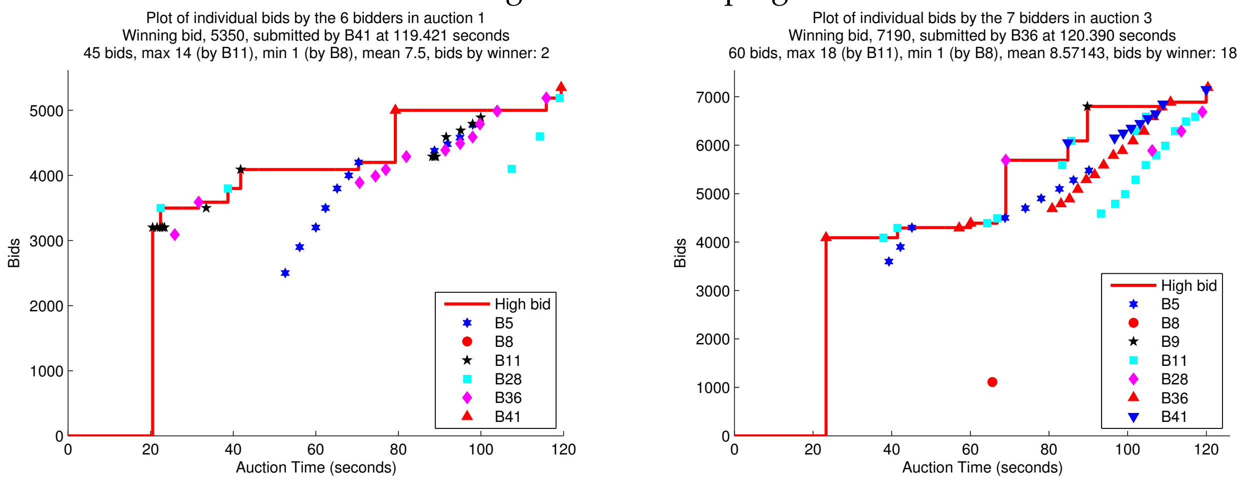
We can see that due to limited numbers of observations and high standard deviations of the auction prices, we do not find strong evidence supporting the manager's claim that the auction mechanism he invented had successfully defeated collusion. Prices are not typically statistically significantly higher in either period of Regime 1 compared to Regime 0. Only for three particular models: EF Sonata 2.0, Dynasty 3.0, and Galloper 7 is there moderate evidence of significantly higher prices under Regime 1. Mean auction prices for Magnus 2.0 and Grandeur XG 2.5 and EF. Sonata 1.8 models are actually lower under regime 1, though the difference is not satistically significant.

In the remainder of this section we will focus on analyzing the bidding behavior under the Korean auction regime under the maintained assumption that this mechanism was successful in defeating collusion. We speculate that the manager's firm belief that it did may be based on information that we do not have access to.

2.2 Detailed analysis of bidding strategies in individual auctions

The left panel of figure 1 plots bids for an auction on January 26, 2005 which we refer to as Auction 1 since it was the first auction that was held on that day. A four door mid-sized sedan was sold in this auction. We have more precise details on the exact make and model but for purposes of explaining how the auction works it suffices to mention that

Figure 1: Bid creeping

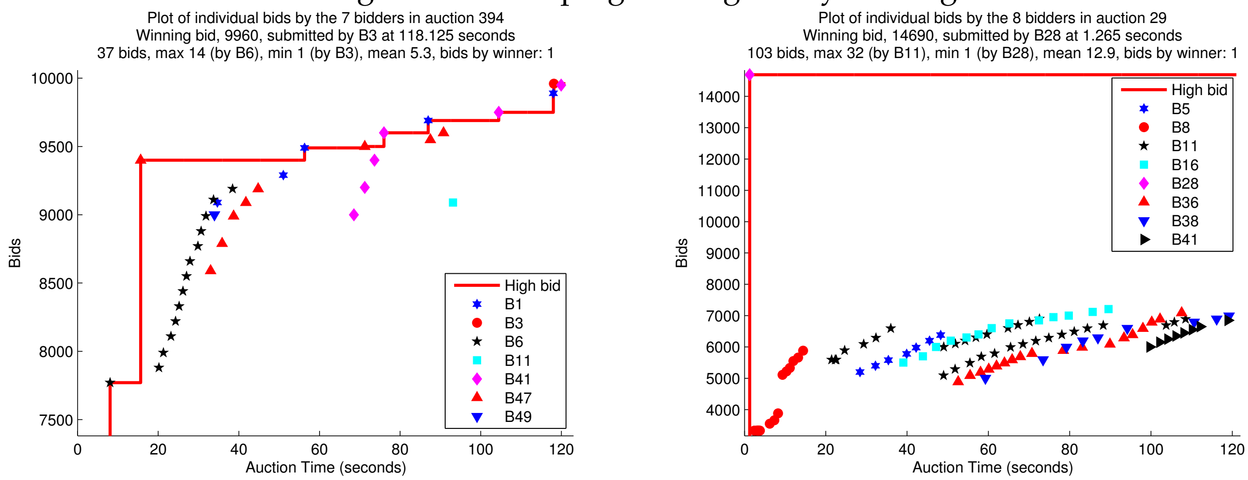


the car was about two years old with approximately 40,000 miles on its odometer at the time of sale. As we can see from the figure, there were six different bidders participating in this auction. We also see that the bids are generally montonically increasing, though not all bidders are active at every possible instant. The winning bidder in this auction, B41, delayed submitting its bid until there were approximately 50 seconds remaining in this two minute auction, and it made only one further revision to its first bid, raising it from \$5000 to approximately \$5400 at the very last instant of the two minute auction.

We observe a variety of other bidding behaviors including bidders who post bids earlier in the auction and make frequent changes to their bids. These bidders, such as B5, appear to be "probing" or "testing" the market to find the smallest bid they need to submit to become the highest bidder. We refer to such bidding behavior as *bid creeping.* However B5 never succeeded in placing a highest bid, and so never learned what the current high bid was during the auction. B5's final bid was just over \$4500 submitted with less than 30 seconds remaining in the auction. At that point B5 might have reached his "reservation value" though whether this equals his valuation for the car is question we will return to. But one theory is that the final high bid is indeed just equal to the bidder's true valuation of the car being auctioned.

The right panel of Figure 1 plots another auction where there are 7 participating bidders and a different firm won the auction, B36. This bidder behaved differently than the winning bidder in auction 1 (B41) by virtue of being the first bidder to place a bid in the

Figure 2: Bid sniping and high early bidding



auction, with a bid of $4000 just seconds after the start of the auction, and then consistently increasing its bid in a series of small steps over nearly the entire duration of the auction until it placed the winning bid of approximately $7100 in the final instant of the auction. B36 and B41 appear to be "dueling" with each other to maintain the highest bid. B41 delayed his first bid until approximately the last 30 seconds of the auction, and his first bid was higher, $6000. But in this auction, unlike in auction 1, B41 did increase its subsequent bids in small increments while he appeared to be engaged in a duel with B36 to maintain the high bid. Notice that both B41 and B36 placed bids in the very last instant of the auction, though B36 succeeded in bidding just slightly higher, winning the auction.

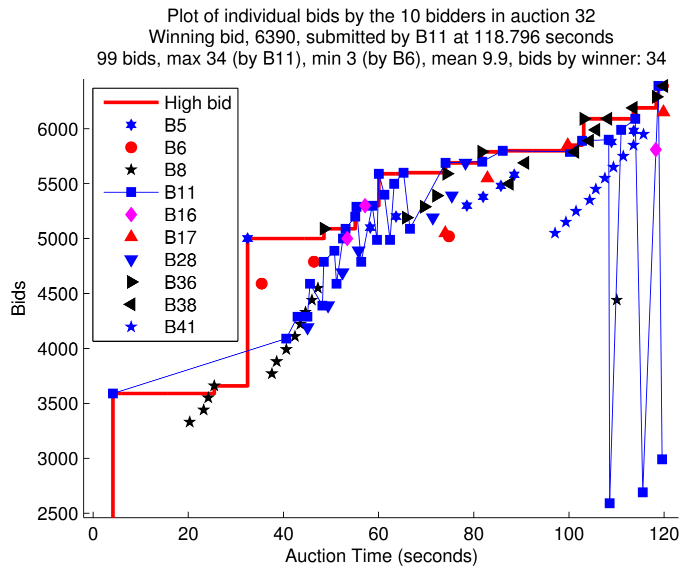
In addition to bid creeping strategies, we also see bidders who use *jump bidding* and *bid sniping* strategies. The latter are bidders who place a single large bid at the very end of the auction. The left panel of figure 2 below plots the bids placed in auction 394. This auction was won by B3 just by a hair, with a bid by B3 of $9960 at 118.125 seconds that exceeded the final bid by B41 of $9950 at 119.953 seconds. Note that B3 won by placing only a single bid 1.875 seconds before the end of the auction, whereas B41 started bidding with an initial bid of $9000 at 68 seconds into the auction and steadily increased its bid in five subsequent revisions until placing its final bid of $9950 less than one tenth of a second before the end of the auction. As we show shortly, bid sniping is a relatively infrequently used strategy in these auto auctions.

The right panel of Figure 2 illustrates an auction with an exteme example of *early* *overbidding.* We will subsequently show that less extreme forms of early overbidding is quite common in these auctions. In this case B28 places a high bid of $14,690 only 1.265 seconds after the auction started and remains the high bidder for the duration of the auction. This bid is so high that the attempts by all of the other bidders to increase their bids to become the highest bidder in the remainder of this auction are completely futile. The highest of all of these other bids was $7210 by B16, less than half as much as B28 bid. Thus, it appears, at least from a simple *ex post* analysis of this auction, that B28 could have purchased this car much more cheaply by starting to bid low and gradually increasing its bid over the course of the auction, instead of precommitting to a very high bid at the start of the auction. While it is easy to make these judgements in hindsight, having access to data that no single bidder possesses individually, it does appear hard to rationalize a strategy of making a high initial bid on purely *a priori* grounds. Bidding a high amount early in these auctions comes with the risk of overpaying, whereas bidding a low amount early in the auction entails a risk of being outbid later in the auction and of revealing potentially valuable information to other bidders.

Auction 32, shown in figure 3 below, exhibits the non-monotonic bidding behavior that we observe by some bidders in some auctions. We see that the winning bidder in this auction, B11, frequently *reduced* its bid including reducing its bid to very low values, $3000 and below, (lower than any other bids, including its own bids, earlier in the auction), but then dramatically increased its bid to win in the final milliseconds of the auction. It does not seem reasonable to attribute the frequent reductions in bids to keyboard errors or "trembles" on the part of B11. Instead, the bids seems to be intentional, perhaps out of boredom or an effort to confuse other bidders. However the auction rules guarantee that other bidders would not observe and thus be completely unaware of the bid reductions by B11. Thus, reducing a bid serves no informational or strategic purpose, so it is hard to rationalize B11's bid reductions in this auction. Most of the other bidders placed monotonically increasing bids, though one other bidder, B8, also reduced its bid to just below $4500 from a previous bid of $6000 before raising its bid again to about $6200 in the final seconds of the auction.

In auction 32, the winner was B11 though there was a subsequent bid by B38 just 47

Figure 3: Non-monotonic bidding



milliseconds before the end of the auction equal to the $6390 placed by B11. The auction software uses time priority as a tie breaker in case multiple bidders have the same highest bid, so this is why B11 won. Notice that after submitting its winning bid of $6390 1.204 seconds before the end of the auction, B11 submitted another bid of $2990 438 milliseconds before the end of the auction. Could this reduction in its bid have affected the other high bidder, B38? The answer is no since bidders are committed to paying the highest bid they submitted during the auction even if they subsequently reduce their bids. Further no other bidders will observe a bidder who reduces their bid (since the software only tells each bidder whether or not their highest bid is also the highest bid in the auction so far). Thus, B11's high bid of $6390 was still "on the table" even though he reduced it to $2990 438 milliseconds before the end of the auction. If B38's last bid at 47 millesconds before the end of the auction had been slightly higher, it is doubtful that B11 would have realized it had been outbid in time to submit a slightly higher bid. Thus, we can see no strategic advantage to adopting B11's non-monotonic bidding strategy in this auction.

These auction results raise some immediate questions: if the auction rules allow bidders to reduce their bids, could it ever be an equilibrium response for the bidders to use *non-monotonic bidding strategies* — i.e. placing high bids early on in the auction to try to learn where the high bid might be, but lowering their bid later in the auction at B11 did in auction 32 shown above? If the answer is no, then these examples suggest that some sort of model of boundedly rational bidding behavior may be more appropriate for analyzing these auctions. In our model of bidding, we have treated bid reductions as simply a sign of boredom on the part of a bidder, and thus we assume it has no strategic value and do not attempt to explain this behavior. In our structural analysis in section 5, we have suppressed all bid reductions, and only attempt to explain "real bids" — i.e. bids that improve on a bidder's previous high bid.

We also see that there is some randomness in the time the auction actually ends. Though the auctions are supposed to last 120 seconds, we see a number of auctions where the last bid is time stamped at over 120 seconds, in some cases as late as 121 seconds. The auction software accepts these late bids perhaps to allow for some communication delays. However it makes the determination of the actual *termination rule* somewhat complicated. How do bidders know when it will be too late to place a bid? This probabilistic nature of when some of these auctions end complicates the use of bid sniping strategies since it is not a practical possibility to submit a single bid in the very "last instant" of the auction. To our knowledge the rental company was not intentionally using a *soft close* similar to the rules used by Amazon and eBay discussed in Roth and Ockenfels [2002].

It is not immediately clear, given the limited information conveyed in the early stages of the auction, what value bidders get from participating early in the auction as opposed to the strategy of waiting to the last minute and placing a single clinching bid. If all bidders adopted the latter strategy, this auction would start to resemble a one shot first price sealed bid auction, and the auction outcomes could probably be well approximated using this standard and commonly used auction model.

What we cannot say from this simplistic, descriptive look at the auction data is what effect the option to place early bids has on a bidder's subsequent bidding decisions? From a single bidder's standpoint, it appears that having the option to bid early has value as it enables a bidder to safely "test the waters" by placing low bids and gradually increase them, hoping that they can obtain a good deal. On the other hand, given the coarse nature of information revealed over the auction (with each bidder being unable to see competing bids) it is not immediately clear that bidders will do any better under this auction format than they could in a standard one shot first price sealed bid auction, a mechanism that precludes any early bidding and within-auction learning.

However given the large number of auctions we observe and the large number of bids submitted by each bidder in each auction, we think these data present both an interesting theoretical and empirical challenge. We observe a variety of bidding behaviors in these auctions, with a combination of early bidders and late bidders, bid creepers, jump bidders, and bid-nipers who submit a single bid in the very last few seconds of the auction. In particular, the number of bids submitted per second increases dramatically in the final second of the auction as bidders jockey frenetically to submit the winning bid.

2.3 Auction Statistics

We now present a few statistics summarizing the auctions and bidding behavior to help quantify the frequency of different types of qualitative bidding strategies, including bid sniping and pre-emptive high bidding. We are studying a population of experienced professional bidders: the universe of 67 registered bidders who participated in these auctions represent broker/dealers and they bid in an average of 1281 auctions. The bidder with the most experience was B11 who bid in 7167 auctions, though 20 of the 67 bidders were infrequent participants who bid in fewer than 100 auctions.

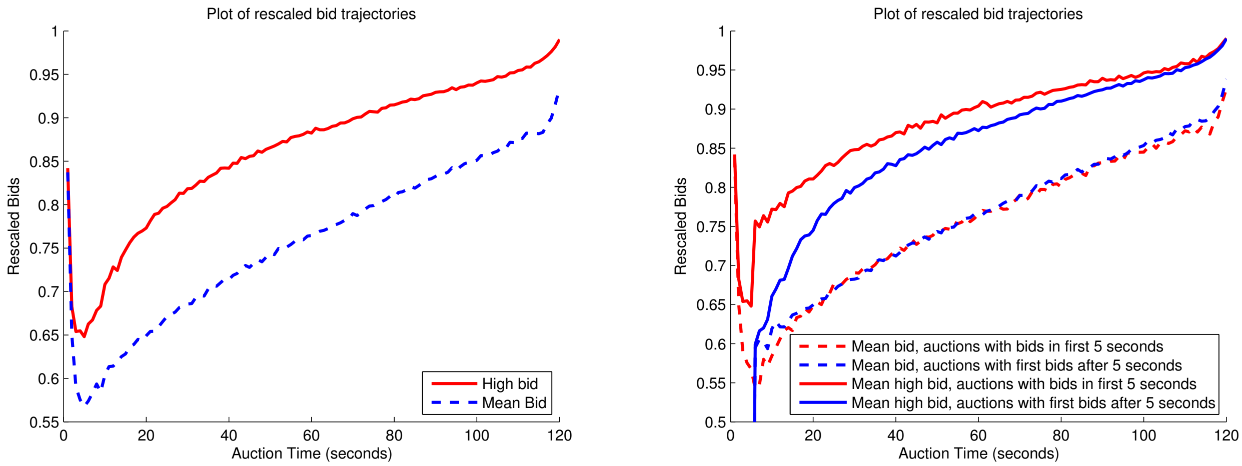
We observe substantial variation in the "win rate" (i.e. the fraction of auctions that a bidder participates in and submits the winning bid), ranging from a low of 0% for bidder B45 to a high of 39% for bidder B72. The average win rate among the 67 bidders is 13%. There is no obvious correlation in the number of auctions bidders participate in and the win rate: some of the most frequent bidders have below average win rates, such as B11 (the bidder that participated in the largest number of auctions) or B49, who participated in 5385 auctions but has a win rate of just 4.4%. The bidder with the highest win rate, 39%, participated in only 211 auctions. Of course, having a high win rate is not necessarily evidence of a "successful" bidding strategy since it is possible to increase the win rate by bidding more aggressively, but at the potential cost of overpaying for the vehicles the bidder wins. Some differences in the win rate may also be due to bidders who specialize in buying certain types of cars that may be in higher demand than other types of less desirable vehicles.

Of the 11,259 vehicles sold in the auctions, the average selling price was \$9621, though there is substantial heterogeneity: the highest sale price was \$100,000 and the smallest was \$100. We generally have the information on the make and model of the vehicles that were sold at auction, however we do not have data on the odometer and age of sale for all 11,281 of these vehicles. We know the age of the vehicle for a subsample of 8722 auctions, and the average age of these cars was 1063 days (or about 2.9 years old), with the oldest vehicle auctioned was just over 6 years old, and the youngest was only 56 days old (from date of acquisition as a new car). In some cases relatively new cars are sold due to accidents, or due to other special circumstances. We have information on the odometer value of the car in 7344 auctions, and the average value is 73,719.[5]

To better understand statistical regularities in the two minute bidding procees, figure 4 plots the mean values of the *rescaled bids* in the auctions as a function of the elapsed time in the auction. We rescaled the bids by dividing the bids by the winning (highest) bid in the auction, and thus all of the rescaled bids are in the $[0,1]$ interval. Since this is an ascending price auction, the bids naturally increase as a function of elapsed time in the auction. The left panel of figure 4 compares the mean high bid and the mean of all bids received as a function of elapsed time t. We see the anomalous result that the expected bid trajectory actually tends to *decrease* during the first five seconds of the auction before they turn around and start increasing during the remaining 115 seconds of the auction. This anomalous pattern reflects a compositional effect, a reflection of the prevalence of *early winning bids* such as was illustrated in the case of auction 29 in the right panel of figure 2. There are 1046 auctions out of the universe of 11,259 auctions where bids were submitted within the first second of the start of the auction. In 255 (24%) of these auctions the winning bid was also submitted in the first second of the auction, and in the remaining

[5]The rental company we study sells its rental cars at an average age and odometer value that are significantly larger than the average age and odometer values that typical rental car companies in the U.S. sell their vehicles, even though there is a recent trend toward holding vehicles longer even by U.S. rental car companies. Cho and Rust [2010] analyzed the replacement decisions of the rental car company who provided our data and found that this firm could increase its profits significantly (by up to 40%) by selling its rental vehicles at about 150,000km and 5 years of age.

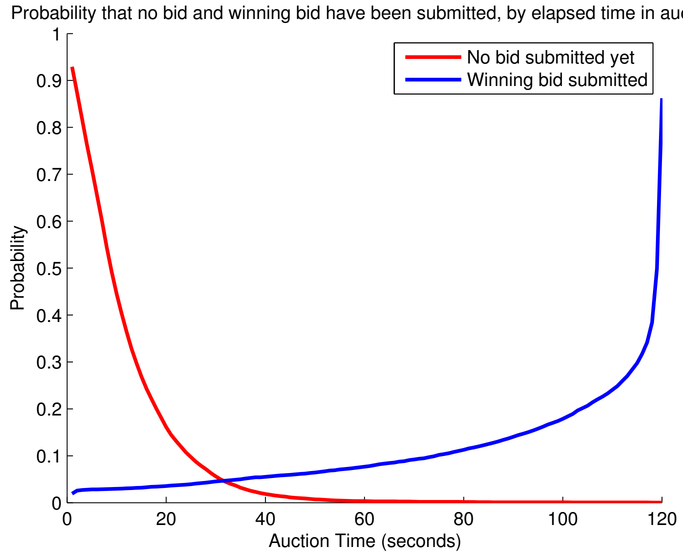
Figure 4: Rescaled Bid Trajectories



791 auctions where bids were submitted in the first second but the winning bid was made after the first second of the auction, the mean rescaled high bid was 79%. This is still higher than the mean rescaled value of the first bid in a typical auction, which is 64.5% and was submitted 14 seconds into the auction. So we conclude that there appears to be some heterogeneity in bidding in these auctions: some bidders place relatively high early bids very early into the auction for certain cars.

The right hand panel of figure 4 sheds further light on this. It compares the mean bid and mean high bid for bids received at different times during the auction for two different subsamples of auctions: a) 8242 auctions where there were no bids in first 5 seconds, and b) 3548 auctions where there were bids in first 5 seconds of the auction. We see that the average of *all bids* received are essentially identical for these two subsamples after the first five seconds of the auction, whereas the average of the *high bids* received in subsample b) is significantly higher. This leads us to conclude that the difference in trajectories are due to a subset of "overly eager bidders" who place very high bids right away in the auction for certain vehicles, perhaps for vehicles that they have especially high valuations for, or which they may have overbid for. Additional evidence in favor of this is that the mean sale price for the 1046 vehicles where bids were placed in the first second of the start of the auction is $10610, over $1000 more than the average selling price in all 11,259 auctions. If we further condition on the 255 auctions where the winning bid was submitted in the first second of the auction, the mean selling price is almost $1000 higher, $11324. The standard

Figure 5: Cumulative probabilities of first bid and winning bid by elapsed time in auction

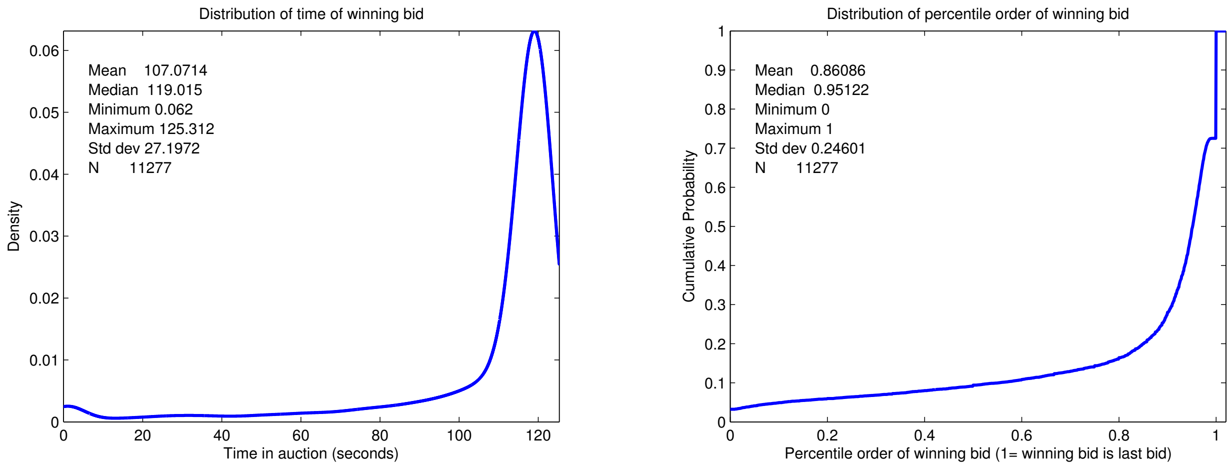


deviation of this mean sales price in these 255 auctions is $436, so the price difference is statistically significant.

It is not clear that bidding a large amount for a vehicle in the opening instant of an auction is an especially wise strategy. Recalling our discussion of the outcome of auction 29 in figure 2, these high early bids could reflect a naive bidding startegy that cause these bidders to significantly overpay relative to a more patient bidding strategy used by most of the bidders, which is to start bidding at a low price and gradually raise the bid over the course of the auction in an attempt to discover what the highiest bids of the other bidders will be. We found that 41 of the 67 bidders placed these pre-emptive early bids in the first second of the auction. However of these, four bidders — B28, B11, B47 and B10 — made such pre-emptive early bids in an average of 17 auctions each.

Figures 5 and 6 confirm that the dominant pattern is for the bidders to *delay* the submission of bids and for the winning bid to be submitted quite late in the two minute auction. The red curve in figure shows the cumulative probability that no bid has been submitted in the auction as a function of time in the auction. Thus, a first bid has been submitted in virtually all of the auctions by the 60 second point, the median time of submission of a first bid is about 10 seconds, and the mean time at which a first bid is submitted is 14 seconds into the auction.

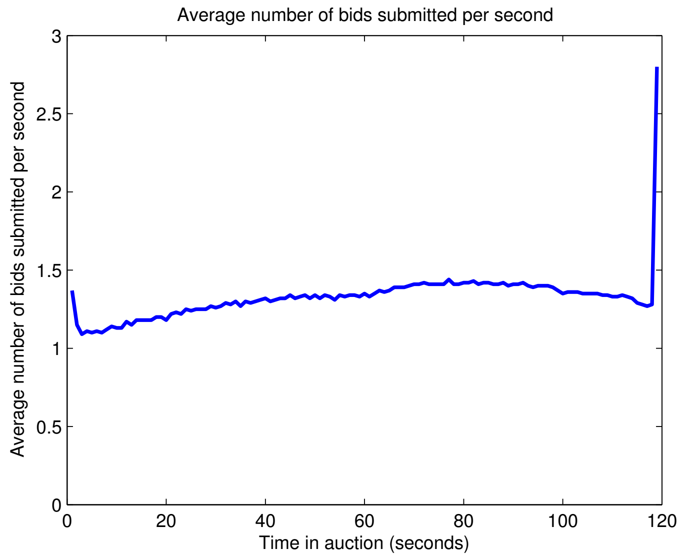
Figure 6: Distribution of submission time and order of the winning bid in the auction



The blue curve plots the cumulative distribution of the time of submission of the winning bid. This distribution is clearly skewed towards the end of the auction: the median time at which the winning bid is submitted is at 119 seconds, just 1 second before the end of the auction and the mean time is 107 seconds. The left hand panel of figure 6 also displays the probabilitiy density of the time at which the winning bid is submitted and we do indeed see a small bump in this density around 0 that reflects a small fraction of auctions where high pre-emptive submitted just after the auction starts. The right hand panel of figure 6 displays the order in which the winning bid is received, i.e. it displays the cumulative distribution of the winning bid *percentile ordering*. That is, if the winning bid is the first bid that was placed in the auction, its percentile ordering is 0 since there are no bids that were placed before it. On the other hand if the winning bid was the last bid submitted in the auction, its percentile is 1 since all of the bids in the auction where placed before it. We find that the winning bid was the first bid in only 3.2% of the auctions, whereas it as the last bid submitted in 27% of the auctions. The mean percentile is 86%, which means in a typical auction 86% of the bids were submitted before the winning bid, and 14% of the bids will are submitted after the winning bid was submitted. The median value is even higher: 95%, which indicates that in 50% of the auctions only 5% of the bids in the auctions are submitted after the winning bid.

Figure 7 provides additional evidence that the rate of bidding activity is the highest

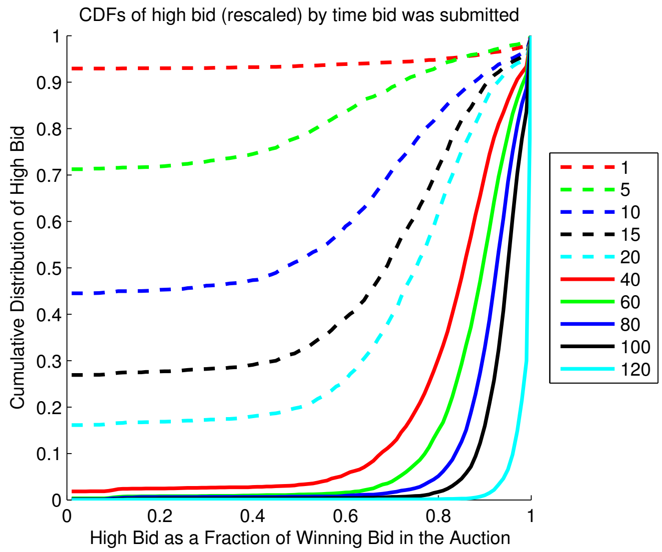
Figure 7: Bids submitted per second in the auction



in the final second of the auction. It shows the average number of bids submitted in one second intervals in the auction at various times in the auction. On average one new bid arrives each second in the auction, but in the final second (i.e. all bids arriving after 119 seconds into the auction) an average of 2.8 bids are received. Since there is some randomness in the exact ending time of the auctions (37% of the auctions received a last bid after 120 seconds, and the mean time of the last bid for these auctions was 120.5 seconds), we would expect a somewhat higher value of the mean bids per second given that our definition of the "last second" of the auction is an interval that actually lasts an average of 1.2 seconds when we account for late bids that are allowed by the auction software. However the value of 2.8 bids in the last second cannot be accounted by the fact that the "last second" actually last 1.2 seconds when we account for late arriving bids. The increased bidding frequency in the last second is also accompanied by an acceleration in the bidding increments, as reflected by the convex shape of the the mean value high bid in the final second of the auction as shown in left hand panel of figure 4.

The overall conclusion that we draw from this is that a) winning bids are submitted very late in the auction, and b) as a consequence, they do not remain on the the table for very long: the mean duration of the winning bid is 11.8 seconds, but the median duration is 0.875 seconds, reflecting the fact that the fact that in 50% of the auctions the winning bid

Figure 8: Cumulative distributions of recsaled bids by elapsed time in auction



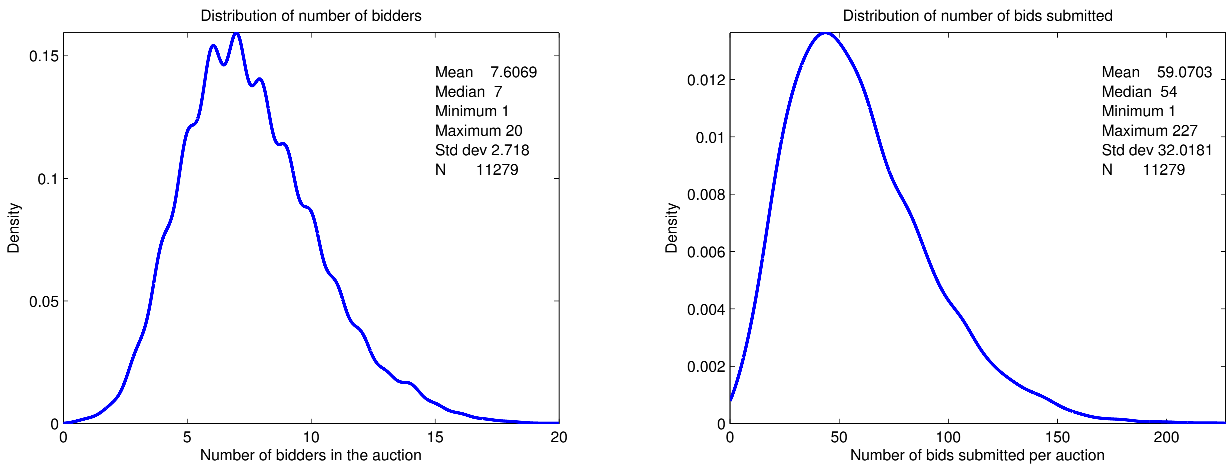
was submitted after 119 seconds into the auction, with less than a second of remaining time before the auction ends. There is a notable acceleration in both the rate of submission of bids and in the rate of increase in both the high bid and the average bid in the closing second of the auction.

Figure 8 plots the cumulative distributions of the rescaled *high bids* as a function of the time in the auction. Consistent with the ascending bid nature of the auction, we see a natural pattern of stochastic dominance in these distributions. That is, if $F_t(b)$ is the CDF of the high bid received by second t into the auction, we have $F_s \succ F_t$ in the sense of first order stochastic dominance if $s > t$. So we see that in the first second of the auction, the CDF $F_1(b)$ indicates that the high bid is 0 in over 90% of the auctions — that is, no bid has been submitted by the end of the first second in over 90% of all auctions. The fraction of auctions where no bid has been submitted steadily falls as t increases, so that by 10 seconds into the auction the blue dotted line in figure 8 indicates that a positive high bid is on the table in more than 50% of the auctions in our data set, and the median value of the high bid is about 50% of the eventual winning bid submitted in the auction. The distribution of bids collapses about a unit mass as a rescaled bid of 1 (i.e. the high bid in the auction) as $t \to 120$. However the convergence to a unit mass is not perfect, and we see that $F_{120}(1) < 1$ due to the fact that the auction software allows for short delays in submission of final bids and final bids were received shortly after 120 seconds in over 36% of all auctions in our data set. Thus, there is some residual uncertainty that a high bid could be bettered even for high bids that are submitted slightly after 120 seconds into the auction.

Figure 9 shows the distribution of the number of bidders participating in an auction and the total number of bids submitted per auction (by all bidders). We see that there is are an average of 7.6 bidders participating in an auction and an average of 59 bids are submitted, so each bidder submits on average 7.8 bids per auction. Interestingly, there were 22 auctions where only a single bidder was bidding in the auction. Of course, reasonable prices are possible in this situation because of the fact that bidders are not aware of the number of other bidders who are participating in any auction: they only see whether their current bid is the highest or not. While a bidder who is the only bidder in the auction will see that they have the highest bid in every instant (after they have submitted their bid), the bidder may conclude that they have overbid, or that there are other bidders who have not yet submitted a bid and, as a result, the bidder may actually be tricked into increasing their bid in the auction even though they are actually facing no competition at all! Indeed, the average number of bids submitted in the 22 auctions where only a single bidder had entered the auction was 7.7, virtually the same as the average number of bids per bidder in auctions when there are multiple bidders participating in the auction.

The fact that we found that most winning bids are placed very close to the end of the auction and that a high proportion of winning bids are actually the last bid submitted in the auction leads to the question as to whether *bid sniping* is a frequently used bidding strategy in these auctions. We define an auction as being won by a *bid sniper* if the winning bidder submits a single bid in the final two seconds of the auction. Using this definition, we find that 520 of the 11,259 auctions we analyzed (4.6%) were won by bid snipers. A total 37 of the 67 bidders in these auctions used bid sniping strategies, though it was far from the exclusive strategy/behavior that they exhibited. The most frequent bid snipers

Figure 9: Distribution of number of bidders and bids submitted in an auction



were B58 (who sniped in 134 of the 1520 auctions they bid in, or 8.8% of the time), follows by B65 who sniped in 3.9% of the auctions they participated in, B23 who sniped 2.6% of the time and B1 who sniped 2.3% of the time. However these are the frequencies of *successful sniping*, i.e. where the bidder was able to win. We should put the adjective "successful" in quotation marks since it is not evident *a priori* that sniping is an effective bidding strategy. Similar to high pre-emptive early bidding, snipers are not engaging in the opportunity to learn the current high bid earlier in the auction and thus could be winning by significantly overpaying for the cars they bought.

We can consider a weaker notion of bid sniping by defining a sniper to be any bidder who submits a single bid in the last two seconds of the auction, regardless of whether they win the auction. A bid sniper was present in 2502 of the 11,259 auctions (or 22% of all auctions). In most of these auctions only a single sniper was present, but there were multiple snipers in 293 auctions. A total of 54 of the 67 bidders in our data set engaged in bid sniping in some auction, though their proclivity to snipe varied significantly, from less than 1% to a high of 26% by the bidder who sniped most frequently, B28. The win rate for bid snipers is about the same as the overall win rate: about 13%, however a number of bidders who frequently snipe have a significantly higher win rate from the auctions where they sniped compared to ones where they did not snipe. For example B58 has a win rate of 34% on the auctions where it sniped, which is higher than its overall win rate of 29%. B1, who sniped in 9% of the auctions it participated in, had a 26% win rate for these auctions compared to an overall win rate of 14%. For the auctions where a bidder sniped but lost the auction, their (losing) bid was on average 83% of the winning bid. The bidders who sniped more frequently and had higher win rates in the auctions they bid in also bid a higher fraction of the winning bid in the auctions they lost. For example, the most frequent sniper, B58, submitted bids that were an average of 90% of the winning bid in the auctions where B58 sniped but lost the auction. B1, another frequent sniper whose win rate in auctions where it sniped is significantly higher than its overall win rate, bid 93% of the winning bid in the auctions where it sniped but lost.

Besides pre-emptive early bidding and bid sniping, we observe a fair amount of what we might describe as irrational or *uninformative bidding* in the auctions. This occurs whenever a bidder submits a bid that is at the same value or lower than an previous bid that the bidder had already submitted. Given the rules of the auction where only the highest submitted bid is recorded, there appears to be no rationale for submitting these types of bids, and they can't even have a signalling value to other bidders since the auction software (by virtue of only showing whether or not a bidder's current bid is the current highest bid in the auction) prevents other bidders from being aware that a bidder has lowered their lower bid. In some cases we might expect a submission of a bid that is lower than a previously submitted bid would be simply a mistake, a typing error on the keyboard for example. However for most bidders who do this, it happens too frequently to chalk this up to a mistake. For example the bids placed by B11 in auction 32 in figure 3 show a systematic zig-zag pattern that can only be ascribed to an intentional pattern of bidding. However it is not clear what the objective of this is, other than perhaps being a sign of boredom or capriciousness by the person placing the bids. Out of 85798 bidding histories we analyzed (the total number of bid histories by all bidders who participated in the 11,259 auctions in our data set), we observed bidders repeating the same bid at one or more instances in the auction in 12502 cases (14.6% of the histories) and one or more decreases in their bid in 6556 cases (7.6% of the histories).

Overall, while we do observe both pre-emptive early bidding and sniping behavior and a high incidence of uninformative bidding in these auctions, by far the most commonly used bidding behavior that we observe is *bid creeping* where a bidder makes a succession of increasing bids closely spaced in time in an attempt to find out what the current high bid is. Examples of bid creeping are the bids by B5, B11, B28 and B36 in auction 1 in figure 1. Bid creeping seems to be reasonable strategy for *learning* what the current high bid in the auction is, since it avoids the risk of overbidding that might be implied by a *bid jumping* strategy which is similar to bid creeping but involves bid sequences that are spaced further apart and jump up in higher increments compared to bid creeping. Examples of bid jumping strategies include the sequences of bids by B41 in auction 1 in figure 1, and B1 in auction 394 in figure 2. Of course the dividing line between bid creeping and bid jumping is a fuzzy one: the two behaviors are both consistent with a desire to learn what the current high bid in the auction is, but in bid creeping the bidder is willing to make a much larger number of bids in rapid succession, each one only slightly higher than the previous one, whereas in bid jumping the bidder seems to have a higher psychic cost of placing bids and tends to make fewer bids at more widely spaced intervals of time in the auction, and the increments over the previous bids are larger. Thus, bid jumpers seem to behave as if they had a higher cost of submitting bids and/or are more willing to take the risk of overbidding to become the current higher bidder relative to what we observe for bid creepers.

In summary, we have identified a number of different bidding behaviors in these auctions: 1) pre-emptive early bidding, 2) bid sniping, 3) non-monotonic bidding, 4) bid jumping, and 5) bid creeping. We analyzed the 11,259 auctions in our database with regard to the type of strategies employed by the *winning bidder* and we found that bid creeping was the predominant strategy employed by the winning bidders, in 52% of all auctions. We found that bid jumping and behaviors that involve a mix of creeping and sniping were the next most common behavior, used by the winning bidder in 20% of the auctions. We observed bid sniping in nearly 5% of all auctions (where the winner submitted a single bid in the remaining 2 seconds of the auction), and pre-emptive early bidding in nearly 3% of all auctions (where the winner submitted a single bid but in the first 2 seconds of the auction).

When we analyzed the types of bidding behaviors on a bidder-by-bidder basis, we find a distribution of behaviors for each of the bidders — i.e. no bidder always bids exclusively using one type of "strategy" (e.g. bid sniping) in all of the auctions they participate in. We tabulated the distribution of various types of bidding behaviors for the 67 bidders in the auction and the most common behavior for virtually all of the bidders is bid creeping, and the next most common behavior was bid jumping (or a mix of creeping and jumping behaviors).

In the next section we analyze this auction from a theoretical perspective to see what we can say about the properties of rational, equilibrium bidding strategies, and whether the actual bidding behavior we observe in these auctions at all resembles the equilibrium bidding behavior predicted by auction theories.

3 Can game-theoretic models explain early bidding?

In this section we consider dynamic, equilibrium models of bidding in the rental car auctions, and whether the behavior we observe could be consistent with a Perfect Bayesian Equilibrium (PBE) of the auction, formulated as a dynamic game of incomplete information. Due to the severe restrictions on information provided to bidders in these auctions, the amount they can learn about each other over the course of the auction is limited. As we noted in the previous section, there were 22 auctions where only a single bidder participated and the average number of bids submitted in these auctions was essentially the same as in auctions where multiple bidders are present. Apparently, it is even difficult for bidder to learn even the most basic fact, such as whether other bidders are present!

In the introduction we presented an intuitive argument that under frictionless bidding, all bidders should be able to bid frequently enough to learn the high bid at each point in the auction, and thus bidding outcomes in the Korean auction should approximate the outcome of a Japanese auction – i.e. bidders should continue to bid in small increments until they reach their valuation for the cars and then stop further bidding. This bidding strategy is sometimes called *straigthforward bidding* and implies that except in the case with only a single bidder the outcome of the Korean auction should be strategically equivalent to a Japanese auction, which is in turn strategically equivalent to a static second price sealed bid auction.

However this casual intuition is not a substitute for developing a rigorous game-theoretic model of bidding, as has been recognized by Isaac et al. [2007a] who showed that in open outcry auctions "straightforward bidding is not even typically part of a Nash equilibrium in the non-clock ascending auction, much less a dominant strategy." (p. 145). Things are even more complicated in the Korean auction, where the informational restrictions imply that the natural equilibrium concept is Perfect Bayesian Equilibrium.

In this section we sketch the elements of a PBE model of bidding in the Korea auction and show that when the auction has a *hard close* (i.e. all bidders who wait to submit their bids at the last instant $T = 120$ in the two minute auction are guanranteed that their bids will be recorded), then there is always an *uninformative PBE* that is strategically equivalent to a first price sealed bid auction (though modified as in McAfee and McMillan [1987a] to account for a common knowledge prior over the unknown number of bidders participating in the auction). As we noted in the introduction, this equilibrium implies that all bidders use bid sniping strategies, which is manifestly inconsistent with the actual bidding behavior we observe in these auctions as we showed in the previous section.

The question is whether there exist other *informative PBE* that involve early bidding consistent with what we observe in the data, as well as the other features such as jump bidding, bid creeping, and the occasional bid sniping that we documented in the previous section. We show, via a simple two period two bidder example, that there is no guarantee that informative PBE generally exist. Further, as we noted in the introduction, even if informative PBE might exist in more complex settings than the simple 2x2 case we consider in this section, computing informative PBE is an intractable computational problem. Thus, the main message of this section is that we need to consider an alternative approach to explain the early bidding behavior that we observe in the Korean auctions.

For this reason, below we only sketch the elements of a symmetric PBE model of bidding, without attempting to formulate a version maximum generality due to the computational and theoretical difficulties in computing or characterizing PBEs. We adopt the "conditional independent private values" framework introduced by Li et al. [2000] where the valuations of all bidders are *IID* draws from a density of valuations $f(v|\mu)$ that depends on parameter μ that is public knowledge amongst all potential bidders in the auction. We assume the distribution $f(v|\mu)$ is also common knowledge amongst all possible bidders, as well as a discrete probability density $g(n)$ for the number of bidders who arrive stochastically to bid in any given auction.

We assume that μ represents a combination of information that can be obtained from observable characteristics of the vehicle (e.g. the make, model and vintage, number of kilometers on the odometer, number of accdients, cumulative maintanence costs, etc) and other characterics that can be observed from a physical inspection of the vehicle, such as the presence of dents, or how clean/dirty the car's interior is, and so forth. One example of how μ might be modeled is via a lognormal hedonic regression

$$\mu = \exp\{X\gamma + \epsilon\}, \tag{1}$$

where X is a vector of characteristics that all bidders *and* the econometrician can observe (e.g. make, model, vintage, odometer, etc) and ϵ represents unobserved characteristics of the car that the bidders can observe from a phyiscal inspection that we as econometricians do not observe. We can think of μ as akin to a "blue book value" — i.e. the value of the car to the average bidder.

An example of a distribution of valuations $f(v|\mu)$ might be a lognormal distribution where we write

$$\tilde{v} = \exp(X\gamma + \epsilon + \nu\} = \mu \exp\{\nu\}. \tag{2}$$

where ν represents a component for bidder's *idiosyncratic private valuation* of the car that could be higher or lower than the appraised value μ, where we impose the restriction that $E\{\exp\{\nu\}\} = 1$. If ν and ϵ are normally distributed, and $\mathrm{cov}(\nu, \epsilon) = 0$, then if the appraised value μ is common knowledge among the bidders but the idiosyncratic components ν are private information, then the auction has the structure of independent private values, but one that is conditional on the component μ that itself may not be directly observable by the econometrician.

Definition: A *Symmetric Perfect Bayesian Equilibrium* of the Korean auction is a strategy $b_t = \gamma(X_t)$ that specifies the optimal bid b_t for any bidder participating in the auction as a function of the bidder's *state* X_t which is the 4-tuple $X_t = (v, \rho_t, b_t, h_t)$ where:

1. v is the bidder's private valuation of the car being auctioned,

2. ρ_t is a Bayesian posterior distribution over the product space of the number of bidders participating in the auction and the valuations of the other competing bidders in this auction based on all information observed by the bidder up to time t and making use of the common knowledge about the density $g(n)$ over number of bidders, the density $f(v|\mu)$ for competing bidders' valuations, and the strategy γ,

3. b_t is the bidder's highest submitted bid as of instant t in the auction and

4. h_t is a $\{0, 1\}$ indicator of whether b_t is the highest bid at instant t per the rules of the Korean auction.

Bids b_t are restricted to be non-negative integers with $b_t = 0$ being interpreted as a decision not to bid at instant t. The strategy γ must constitute a best response in every state X_t and time $t \in [0, T]$, that is, $b_T = \gamma(X_T)$ must maximize the expected terminal payoff $W_T(X_T, b) = E\{(v - b)|X_T\}$ and for every $t < T$, and $b_t = \gamma(X_t)$ maximizes $E\{W_t(X_t, b)|X_t\}$ where $W_t(X_t, b) = E\{W_T(X_T, \gamma(X_T))|X_t, b\}$ is defined recursively as the conditional expectation of the terminal payoff at time t in the state X_t given that the bidder bids b, assuming, recursively, optimal bidding behavior for all $s \in (t, T]$ and that the bidder updates their beliefs ρ_t according to Bayes Rule with probability 1 for all possible histories $\{X_t\}$ and all $t \in [0, T]$.

Here, we have assumed that a symmetric PBE exists and can be found within the class of *pure strategies* $b_t = \gamma(X_t)$, but we can generalize the definition to allow for the possibility of mixed strategies by interpreting $\gamma(X_t)$ as a conditional probability distribution over the set of possible bids (nonegative integers).

The complexity of solving for a PBE is mainly due to the high dimensionality of the family of posterior beliefs $\{\rho_t\}$ which are conditional probability measures that depend on the current state X_t that have support over the number of bidders n and the joint distribution $(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ over the valuations of all possible competing bidders other than the bidder in question, bidder i. This posterior distribution must be updated continuously, even in the event of no new information (such as if bidder i has not yet submitted a bid in the auction), since bidder i's priors and their beliefs about the equilibrium bidding strategy γ enables them to form updated beliefs about these quantities purely as a function of elapsed time in the auction. Thus, it should be clear that computing even a since PBE is a very duanting undertaking.

However there is a symmetric PBE that is easy to compute: the *uninformative PBE.*

Lemma 1 *Suppose the Korean auction has a hard close: i.e. any bidder can wait to the final instant T and be guarnteed that the bid they submit at that last instant will be recorded. Let $b_T = \gamma(v)$ be the symmetric Bayesian equilibrium bidding strategy to a first price sealed bid auction when the number of bidders is unknown but all bidders have common knowledge of the distribution $g(n)$ of the number of bidders participating in the auction and of $f(v|\mu)$ the density for valuations of other competing bidders that is described in McAfee and McMillan [1987a]. Then γ constitutes a symmetric, uninformative PBE of the Korean auction in which all bidders wait until the final instant T and submit their bids $b_T = \gamma(v)$.*

The proof of Lemma 1 is quite elementary. First, by construction, there is no deviation bid by any of the bidders at time T that can improve their expected payoff under this candidate uninformative equilibrium. So we only need to check if there is any profitable unilateral deviation by any of the bidders prior to T. However it is easy to see that there isn't because if it is common knowledge that all competing bidders will submit their bids at the last instant T, there is nothing a deviating bidder can learn by submitting their bid prior to T. This early bid also cannot have any impact on the bids that will be submitted by the other bidders, so we conclude there is no profitable deviation from this candidate PBE. Given that there are no bids submitted prior to T, the only relevant information for submitting a bid at time T is just the bidder's valuation v, so each bidder uses the symmetric Bayesian Nash equilibrium bidding strategy $b_T = \gamma(v)$ calculated by McAfee and McMillan [1987a] and all the criteria for a PBE given in our definition are satisfied.

Given that there is always a trivial uninformative PBE, the next question is whether there exist other informative PBE. It is much harder to determine whether this is the case since it involves finding an informative PBE and verifying that early bidding will occur with positive probability along the equilibrium path. As we noted above, this is an intractable computation and theoretical problem that we are unaware anyone has solved either in this context or in related contexts (e.g. continuous double auctions). However we now consider a simple example where we can show that *no informative PBE exists.*

We consider the simplest possible case, where time is discrete and there are only two bidding opportunities, at $t = 1$ and $t = 2$. We also assume that only two bidders participate in the auction, and this is common knowledge i.e. $g(2) = 1$. Now consider a candidate informative symmetric equilibrium. This means that in period $t = 1$ the two bidders use the same bidding strategy $\gamma_1(v)$ to place bids given their valuation v in time period $t = 1$ of the auction, and then at period $t = 2$ they place updated bids from two bid functions $\{\gamma_2(v, b, \rho_0, 0), \gamma_2(v, b, \rho_1, 1)\}$ that condition on their bid b at $t = 1$ and whether they were revealed as the high bidder or not at $t = 1$, including their posterior belief ρ_0, and ρ_1, respectively. If $h_1 = 1$, the bidder was informed they had the high bid in period $t = 1$ so they update their posterior belief about their opponent's valuation from $F(v|\mu)$ to the conditional probability distribution ρ_1 given below, and they use the bid function $b_2 = \gamma_2(v, b, \rho_1, 1)$ to compute their bid in period 2. However if they were not revealed to be the high bidder at $t = 1$, then their bid function is given by $\gamma_2(v, b, \rho_0, 0)$ where ρ_0 is the bidder's posterior belief about their opponent's valuation if they learn they were not the high bidder at $t = 1$, which we also derive below.

Since both bidders are using the same bid function in period $t = 1$, if this bid function is strictly monotonic (as seems natural given we are considering an informative PBE), the information on whether they submitted the high bid is equivalent to learning whether or not their opponent's valuation for the car being auctioned is higher or lower than their own known valuation. Applying Bayes Rule to update beliefs based on the information learned at $t = 1$, if bidder 1 has valuation v_1 and learned they submitted the high bid at $t = 1$, their posterior belief about the valuation of their opponent, bidder 2, at the start of period $t = 2$ is given by the conditional distribution (CDF)

$$\rho_1(v|\mu, v_1) \equiv F(v|\mu, v \leq v_1) = \frac{F(v|\mu)}{F(v_1|\mu)}, \quad v \leq v_1, \text{ 0 otherwise} \tag{3}$$

where $F(v|\mu)$ is their prior belief of the CDF of their opponents' valuation which can potentially depend on the public signal μ both bidders received about the quality of the car being sold prior to the start of the auction. Similarly if bidder 2 learns they are the low bidder at stage $t = 1$ their posterior belief about their opponent's valuation is given by

$$\rho_0(v|\mu, v_2) = F(v|\mu, v \geq v_2) = \frac{F(v|\mu) - F(v_2|\mu)}{1 - F(v_2|\mu)} \quad v \geq v_2, \text{ 0 otherwise.} \tag{4}$$

Now consider the equations determining the equilibrium bidding strategies, where we assume a symmetric Bayesian Nash equilibrium of the two period game. The strategies are *ex ante symmetric* in the sense that both biddders use the same single bidding strategy $(\gamma_1(v), \gamma_2(v, b, \rho_1, 1), \gamma_2(v, b, \rho_0, 0))$ to determine their in both periods of the auction, even though this symmetric equilibrium does reflect an *endogenous information asymmetry* that arises in period $t = 2$ when the bidders submit their stage $t = 2$ bids. Of course, this asymmetry results from the information the bidders receive about whether they submitted the high bid in stage $t = 1$. Even though this is dynamic game of incomplete information, the players start off using the same bidding strategy $\gamma_1(v)$ to determine their initial bids at $t = 1$, because there has been no information released at this stage to cause them to bid differently. But at period $t = 2$ the bidders have learned which of them submitted the high bid and this information causes them to use potentially different bidding strategies at $t = 1$ given by $\gamma_2(v, b, \rho_1, 1)$ and $\gamma_2(v, b, \rho_0, 0)$, respectively.

We solve the bidding game by backward induction starting at stage $t = 2$. The bidding strategies $\gamma_2(v, b, \rho_1, 1)$ and $\gamma_2(v, b, \rho_0, 0)$ must be mutual best responses, so they must satisfy the following equations

$$\begin{aligned}
\gamma_2(v, b, \rho_1, 1) &= \underset{b' \geq b}{\operatorname{argmax}}(v - b') \int_0^v I\{\gamma_2(v', \gamma_1(v'), \rho_0, 0) \leq b'\} f(v'|\mu) dv' / F(v|\mu) \\
\gamma_2(v, b, \rho_0, 0) &= \underset{b' \geq b}{\operatorname{argmax}}(v - b') \int_v^\infty I\{\gamma_2(v', \gamma_1(v'), \rho_1, 1) \leq b'\} f(v'|\mu) dv' / [1 - F(v|\mu)],
\end{aligned}$$

where

$$\gamma_1(v) = \underset{b \geq 0}{\operatorname{argmax}} W_1(v, b), \tag{5}$$

where $W_1(v, b)$ is the bidder's expected payoff from bidding b at period $t = 1$ given by

$$\begin{aligned}
W_1(v, b) = {} & (v - \gamma_2(v, b, \rho_1, 1)) \left[\int_0^v I\{\gamma_2(v', \gamma_1(v'), \rho_0, 0) \leq \gamma_2(v, b, \rho_1, 1)\} f(v'|\mu) dv' \right] + \\
& (v - \gamma_2(v, b, \rho_0, 0)) \left[\int_v^\infty I\{\gamma_2(v', \gamma_1(v'), \rho_1, 1) \leq \gamma_2(v, b, \rho_0, 0)\} f(v'|\mu) dv' \right].
\end{aligned}$$

This is a system of functional equations whose solution gives the symmetric PBE of the Korean auction, assuming a solution exists. Notice that the period $t = 1$ bidding function $\gamma_1(v)$ affects the period $t = 2$ bid functions $(\gamma_2(v, b, \rho_1, 1), \gamma_2(v, b, \rho_0, 0))$, and conversely the period $t = 2$ bid functions determine the period $t = 1$ bid function γ_1. Note also from equation (6) that γ_1 must also be a best response to itself.

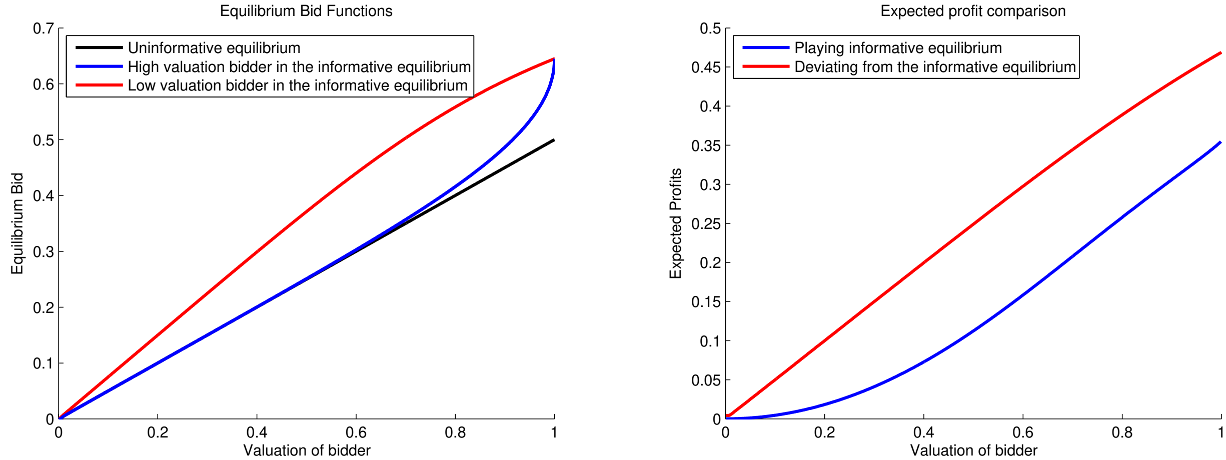
The amount the winner pays in period $t = 2$ is $\max(b', b)$, where b' is the amount bid in stage $t = 2$ and b is the amount bid in stage $t = 1$. This is a consequence of the auction rules already discussed and is reflected in equation (5). Thus, even if either bidder lowered their bid in stage $t = 2$, they would still be obligated to pay the amount they bid in stage $t = 1$, b, if this turned out to be the high bid in the auction.

If an informative PBE exists, it will exhibit an *endogenous asymmetry* — i.e. the period $t = 2$ bid functions $\gamma_2(v, \gamma_1(v), \rho_1, 1)$ and $\gamma_2(v, \gamma_1(v), \rho_0, 0)$ will not be the same. More specifically, the candidate equilibrium solution in period 2 results in more agressive bidding by the low valuation bidder, i.e. the one who submitited the low bid in the first stage. We now show that $\gamma_2(v, \gamma_1(v), \rho_0, 0) > \gamma_2(v, \gamma_1(v), \rho_1, 1)$. If the equilibrium bid functions are continuous, there will be pairs v_1 and v_2 such that $v_1 > v_2$ but $\gamma_2(v_2, \gamma_1(v_2), \rho_0, 0) > \gamma_2(v_1, \gamma_1(v_1), \rho_1, 1)$. That is, a symmetric equilibrium of the dynamic auction will exhibit *ex post inefficiency* even though the symmetric equilibrium of the single shot first price sealed bid auction is *ex post* efficient. This implies that the expected revenue in the static sealed bid auction will not necessarily equal the expected revenue from the dynamic auction. Thus, the endogenous asymmetries that arise in the dynamic auction imply that it is not generally revenue equivalent to the static sealed bid first price auction.

To say more, we make a further assumption that $f(v|\mu)$ is a uniform distribution, so that both of the bidders' valuations are *IID* $U(0, 1)$ random variables. Further we conjecture, and subsequently verify, that the period $t = 2$ bid functions do not depend on the bids submitted in period $t = 1$, so we can write them as $\gamma_2(v, \rho_0, 0)$ and $\gamma_2(v, \rho_1, 1)$, respectively. The only additional restriction we need to verify is that the stage $t = 1$ equilibrium bid function $\gamma_1(v)$ is strictly monontonic and positive for $v > 0$.

Now, assume there is a unique asymmetric Bayesian equilibrium to the period $t = 2$ bidding game defined by the solution to the system of ordinary differential equations (ODE) for the inverse bid functions $\{\gamma_2^{-1}(b, \rho_1, 1), \gamma_2^{-1}(b, \rho_0, 0)\}$ (where the inverse is in the first argument, i.e. $\gamma_2^{-1}(b, \rho_i, i)$ is the valuation that results in a bid equal to b for $i \in \{0, 1\}$), with the boundary conditions $\gamma_2^{-1}(0, \rho_1, 1) = 0$ and $\gamma_2^{-1}(0, \rho_0, 0) = 0$ and $\gamma_2^{-1}(\overline{b}, \rho_1, 1) = \gamma_2^{-1}(\overline{b}, \rho_0, 0)$ where $\overline{b} = \gamma_2(1, \rho_1, 1) = \gamma_2(1, \rho_0, 0)$ is the maximum bid that either bidder would submit in the second period of the auction for any possible bid b in period 1 of the auction. The system of ODEs for $\{\gamma_2^{-1}(b, \rho_0, 0), \gamma_2^{-1}(b, \rho_1, 1)\}$ can be derived from first order conditions to each of the bidder's optimal bidding strategies in

Figure 10: Equilibrium bids and deviation payoffs in a candidate informative PBE



the second stage of the game, from equation (5) above. This system is given by

$$\begin{aligned}\frac{\partial\gamma_2^{-1}}{\partial b}(b,\rho_0,0) &= \frac{\gamma_2^{-1}(b,\rho_0,0)}{\gamma_2^{-1}(b,\rho_1,1)-b}\\ \frac{\partial\gamma_2^{-1}}{\partial b}(b,\rho_1,1) &= \frac{\gamma_2^{-1}(b,\rho_1,1)-\gamma_2^{-1}(b,\rho_0,0)}{\gamma_2^{-1}(b,\rho_0,0)-b}.\end{aligned} \tag{6}$$

We solved system (6) as a free boundary value problem since the end point boundary condition $\gamma_2^{-1}(\bar{b},\rho_0,0)=\gamma_2^{-1}(\bar{b},\rho_1,1)=1$ involves the unknown maximum bid $\bar{b}$.

For comparison, the left hand panel of figure 10 plots the equilibrium bid functions for the uninformative PBE in period $t=2$ where $\gamma_1(v)=0$ and the period $t=2$ bid function is the unique symmetric equilibrium to a first price sealed bid auction, $\gamma_2(v) = v/2$. There is no endogenous asymmetry in the stage two bid functions in this case, of course, because there are no bids placed in period $t=1$ by either bidder, and thus, neither bidder learns anything from period $t=1$ of the game. It follows that stage $t=2$ is equivalent to the BNE of a single stage first price sealed bid auction with uniform valuations. By Lemma 1, this is also a PBE of the overall 2 period Korean auction.

Now consider the case of an informative PBE and assume one such equilibrium exists. Then, γ_1 is strictly monotonic and strictly positive for $v>0$ and the players observe $I\{\gamma_1(v_1)>\gamma_1(v_2)\}$, and this allows each of them to deduce whether they have the high valuation for the item or not which creates the endogenous asymmetry in period $t=2$. Now the red and blue bidding functions in the left hand panel of figure 10 show the unique equilibrium for the period $t = 2$ bid functions $\gamma_2(v, \rho_0, 0)$ and $\gamma_2(v, \rho_1, 1)$ and several things are immediately apparent. First, we see that $\gamma_2(v, \rho_0, 0) \geq \gamma_2(v, \rho_1, 1) \geq v/2$, with strict inequality for sufficiently large values of $v \in (0, 1]$. This implies that the bidder who learns they are the low valuation bidder will bid more aggressively to win the auction in stage $t = 2$ than the bidder who learns they have the high valuation for the item.

However we also see that under the endogenously asymmetric period $t = 2$ equilibrium of the auction, *both types of bidders bid strictly more than the bidders would bid in period* $t = 2$ *under the uninformative equilibrium.* This suggests that it is not in the interest of the bidders to reveal their hand by submitting informative bids in stage $t = 1$ of the auction. The right hand panel of figure 10 verifies that this is the case. It demonstrates that in fact *a symmetric informative PBE cannot exist in this case.*

The blue line in the right hand panel of figure 10 plots the the conditional expected payoff to a bidder at the start of the game, as a function of their valuation v. This bidder reasons that if they place an informative bid at this stage, with probability v they will turn out to be the high valuation bidder and so in stage $t = 2$ they will receive an expected payoff of $(v - \gamma_2(v, \rho_1, 1))\gamma_2^{-1}(\gamma_2(v, \rho_1, 1), \rho_1, 1)/v$. However with probability $1 - v$ they will learn they are the low valuation and then they will receive an expected payoff of $(v - \gamma_2(v, \rho_0, 0))(\gamma_2^{-1}(v, \rho_0, 0) - v)/(1 - v)$. The red line in figure 10 is simply the weighted average of these two expected payoffs at period $t = 2$ using weights v and $1 - v$, respecctively.

Now consider the red line in the right hand panel of figure 10. It plots the *deviation payoff* as a function of the bidder's valuation v from submitting a stage $t = 1$ bid of 0 rather than the equilibrium bid $\gamma_1(v)$. Assume that the other bidder is playing the informative PBE, then if the conjectured informative PBE indeed an equilibrium, it should not pay for the bidder to deviate and submit a bid of 0 in the first stage. However we see that in fact, it does pay to deviate. The deviation payoff is given by $\max_b (v - b)\gamma_2^{-1}(b, \rho_1, 1)/v$, which is the payoff a bidder expects from submitting a bid of 0, which leads the other bidder to conclude with probability 1 that they are the high valuation bidder and thus uses the less aggressive bidding stategy $\gamma_2(v, \rho_1, 1)$ in stage $t = 2$. It is better for a bidder to be certain of bidding against the less aggressive bidder than having some probability of facing a more aggressive bidder in stage $t = 2$ and thus the bidder concludes that there is no advantage to him to submitting a serious bid in stage $t = 1$ and thereby revealing information about his valuation for the item.

Lemma 2 *In the two period, two bidder version of the Korean auction, if bidders have independent uniformly distributed valuations, there is no symmetric informative PBE.*

Proof Though the solutions above were calculated numerically, we can solve for the bidding strategies numerically using the results in Kaplan and Zamir [2012] which confirms that the numerical calculations and our conclusions are indeed correct.

We do not know if there may be informative PBE in other versions of the Korean auction with more time periods and bidders, or if there is an informative *asymmetric* PBE even in the 2×2 case above. However we think it could be quite challenging to demonstrate the existence of non-trivial informative PBE to this bidding game. However to the extent that our result on the non-existence of informative PBE extends to other versions of the Korean auction, it casts doubt on the relevance of the PBE solution concept, and may suggest that bidders are incapable of behaving according to the extremely demanding standard of rationality implicit in the PBE solution concept.

Indeed, our reading of the experimental literature on auctions shows there is already fairly pervasive evidence that subjects typically do *not* behave according to the predictions of Nash equilibrium. For example, Isaac et al. [2012] conducted an experimental study of first price and second price static sealed bid auctions where the number of bidders participating in the auctions were unknown to the bidders. They concluded that "We observe systematic deviations from risk neutral bidding in FP auctions and show theoretically that these deviations are consistent with risk averse preferences." (p. 516). Other studies such as Dyer et al. [1989] also find that bidders do not bid according to the Nash equilibrium theory with risk-neutral payoffs.[6]

[6] Isaac et al. [2012] argue that the predictions of the Nash equilibrium theory provide a better approximation to actual bidding behavior in their laboratory experiments if subjects are modeled as *risk-averse* expected utility maximizers instead of as risk-neutral expected payoff maximizers. Other theories including *loss aversion*, theories of decision making under ambiguity such as theories involving *probability weighting* have also been invoked to try to explain the discrepancy between the predictions of Nash equilibrium models of bidding in auctions and laboratory evidence.

Finding models that better approximate the actual behavior of bidders is important, since many of the results of standard auction theory no longer hold if bidders are not rational risk neutral expected utility maximizers. For example, Levin and Ozdenoren [2004] study first price sealed bid auctions where the number of bidders is unknown. They model ambiguity aversion about the number of bidders using the maxmin expected utility framework of Gilboa and Schmeidler [1989] and find that revenue equivalence between the first price and second price sealed bid auctions no longer holds: sellers strictly prefer a first price auction because of overbidding by ambiguity-averse bidders.

Less is known about experimentally how individuals learn and update their bidding strategies in dynamic environments. An experimental study of sequential Bayesian updating in the "centipede game" by El-Gamal et al. [1993] found evidence in favor of PBE and sequential rationality. Experiments in simpler non-auction environments such as El-Gamal and Grether [1995] have tested the hypothesis that human subjects use Bayesian learning (i.e. Bayes rule) to make decisions and inferences. They concluded that "even though the answer to 'are experimental subjects Bayesian?' is 'no,' the answer to 'what is the most likely rule that people use?' is 'Bayes's rule.' " (p. 1144).

Thus, it remains unclear whether it is reasonable to model bidders as rational, risk-neutral expected payoff maximizers who use PBE bidding strategies. However the biggest argument against this approach is our inability to solve for an informative PBE. In the next section we propose to model bidding behavior in Korean auction using a slightly relaxed equilibrium concept, *anonymous equilibrium,* that drops the assumption that bidders learn via Bayesian updating but maintains the assumption of bidder rationality including the assumption that bidders have correct beliefs about the stochastic process for the high bid in the Korean auction, while relaxing the assumption of continuous attention and frictionless bidding by allowing for the possibility of *rationally inattentive bidding.*

4 A dynamic model of rationally inattentive bidding

In this section, we develop an alternative model of dynamic bidding behavior at the Korean auction that is both computationally tractable and capable of explaining the early bidding we observe, including the heterogeneous bidding strategies we documented in section 2. Our new approach is similar in many respects to the traditional game-theoretic approach that leads to the definition of a PBE presented in the previous section. In particular, we assume bidders are rational optimizers who adopt bidding strategies that maximize their expected payoffs from bidding in the auction. Instead of the PBE concept, we use an alternative equilibrium concept of Nash equilibrium that has been recently developed for *anonymous games*. This concept is similar to the notion of a rational expectations equilibrium (or self-confirming equilibrium) in market games, but is extended to environments where agents playing these games are nonatomic and can, thus, individually have measurable influence overall outcomes; see the paper by Simone Cerreia-Vioglio, Fabio Maccheroni, and Schmeidler [2022]. Two key differences exist between our notion of anonymous equilibrium in the Korean auction model and a PBE:

1. Instead of continually updating beliefs concerning the number of other bidders and their valuations using Bayes' Rule in a PBE, our equilibrium only requires bidders to have beliefs about the stochastic process of the high bid at the auction and these beliefs are fixed and thus not carried as state variables in bidders' dynamic programming problems.
2. As in Barkley et al. [2021], we admit several bidding frictions, including time-varying psychological benefits to and cost of submitting and updating bids, as well allowing for ationally inattentive bidding.

The anonymous equilibrium concept requires bidders to use dynamically optimal strategies, but maintains that after their experience at bidding in thousands of individual rental-car auctions bidder learning leads them to converge to fixed, rational beliefs concerning the stochastic process of the high bid at the auction, which we shall demonstrate constitutes the relevant "sufficient statistic" for successful bidding. Unlike in a PBE, where Bayes' Rule provides an operational procedure for belief formation and updating, our equilibrium concept is agnostic on how bidders learn and converge to rational beliefs concerning the stochastic process of the high bid in the auction. Indeed, it may seem to be an unrealistic requirement given that the informational restrictions at these auction only admit *endogenous sampling* of the high bid over the course of each auction. Research by George A. Hall and Rust [2021] has demonstrated that the endogenous sampling problem can be overcome, so it may not be unrealistic to assume that experienced bidders converge to accurate beliefs concerning the stochastic process of the high bid at these auctions. Furthermore, the computational savings from this assumption are enormous because it implies that we no longer have to carry around high-dimensional posterior beliefs ρ_t as a state variable and perform the subtle updating of beliefs using knowledge of the equilibrium bidding strategies $\{\gamma_t\}$ in order to solve a bidder's dynamic programming problem.[7]

The intuition for why we can generate early informative bidding in an anonymous equilibrium is due to the exogeneity of bidders' beliefs. Even though bidders realize that their own bidding behavior enables them to have some effect on the winning price in auctions where they are the high bidder, in an anonymous equilibrium their beliefs concerning the stochastic process of the high bid are fixed and, therefore, unaffected by their bidding strategies. This converts the problem of finding an equilibrium bidding strategy into a single-agent dynamic-programming problem, or "game against nature" where bidders' beliefs about the law of motion for the high bid constitutes the law of motion for nature. In contrast, the PBE solution concept involves more complicated and subtle reasoning on the part of bidders that enables them to realize how their strategies affect their beliefs concerning the stochastic process of the high bid at the auction. The example we provided in the previous section illustrates how the more subtle reasoning involved in constructing a PBE leads bidders to conclude that attempts to gain information early on in the auction work to their collective disadvantage, ruling out existence of informative PBE. In contrast, we shall demonstrate that in an anonymous equilibrium, early bidding helps bidders learn what the high bid at the auction is, enabling them to win the auction by paying less, on average, than they would pay if all the bidders were to bid sniping strategies and submit a single bid at the last instant of the auction.

[7] Our equilibrium concept can extended to allow for ϵ-equilibrium versions of anonymous equilibrium, including the ϵ-estimated equilibrium concept of Cerreia-Vioglio et al. [2022] that involves two key requirements: "1) Every player best-responds to his beliefs (optimality). 2) The belief of every player is consistent with what he can observe (ϵ-discrepancy)" (p. 111). See Daskalakis and Papadimitriou [2015] and Cheng et al. [2017] for recent results demonstrating that anonymous equilibria are much easier to compute compared than standard (nonanonymous) Nash equilibrium or PBE.

4.1 Definition of Anonymous Equilibrium in the Korean Auction

Before going into the details of our model of rationally-inattentive bidding, we first provide a high-level description of a successive approximations algorithm for computing an ϵ-anonymous equilibrium in the Korean auction, following Cerreia-Vioglio et al. [2022], where ϵ is a tolerance defining how closely we require bidders' beliefs to be correct or self-confirming. Let $\mathcal{B}_0$ be any initial guess for bidders' beliefs concerning the stochastic process governing the high bid in a set of Korean auctions for a homogenous type of used rental car. We shall describe precisely what the notation $\mathcal{B}_0$ means in more detail below, but at this point assume that given $\mathcal{B}_0$ we can solve bidders' DP problems to determine their optimal bidding strategies which we can write as functions $\gamma(\mathcal{B}_0, \boldsymbol{\tau})$ that depend on the bidder's *type* $\boldsymbol{\tau}$ and their (common) belief $\mathcal{B}_0$ concerning the stochastic process of the high bid at the auction. Via repeated IID stochastic simulations of a given set of n bidders with types $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ using the calculated strategies $(\gamma(\mathcal{B}_0, \boldsymbol{\tau}_1), \ldots, \gamma(\mathcal{B}_0, \boldsymbol{\tau}_n))$, we can construct updated beliefs $\mathcal{B}_1$ that constitute the stochastic process for the high bid track implied by the initial guess for bidders' beliefs, $\mathcal{B}_0$. We can write the composition of these two operations (for example, solving for the optimal strategies followed by simulating auctions using them to generate and realized high bid tracks), as the mapping $\mathcal{B}_1 = \Lambda(\mathcal{B}_0)$, which can be viewed as the updated beliefs implied from the initial guess $\mathcal{B}_0$. We can repeat this sequence of solution and simulations repeatedly, so that at the generic iteration t of this updating process we have the following successive approximations iteration or *best-response belief mapping*:

$$\mathcal{B}_{t+1} = \Lambda(\mathcal{B}_t). \tag{7}$$

Definition: Anonymous Equilibrium *If the best-response belief mapping (7) has a fixed point $\mathcal{B}_\infty$, then the corresponding collection of optimal bidding strategies $\{\gamma(\mathcal{B}_\infty, \boldsymbol{\tau})\}$ for an appropriately defined collection of bidder types $\boldsymbol{\tau}$ constitutes an anonymous equilibrium of the Korean auction.*

In practice, we do not find an exact fixed point via the successive approximations procedure, but only an approximate fixed point after t iterations where a convergence critierion $\|\mathcal{B}_{t+1} - \Lambda(\mathcal{B}_t)\| \leq \epsilon$ is satisfied, where $\|\cdot\|$ is an appropriate distance function over beliefs that we define below.

Definition: ϵ-Anonymous Equilibrium *An ϵ-anonymous equilibrium is any belief $\mathcal{B}$ for the stochastic process of the high bid at Korean auctions and the corresponding collection of optimal bidding strategies $\{\gamma(\mathcal{B}, \boldsymbol{\tau})\}$ for an appropriately defined collection of bidder types $\boldsymbol{\tau}$ satisfying $\|\mathcal{B} - \Lambda(\mathcal{B})\| \leq \epsilon$.*

With this overview of our concept of equilibrium, we now provide additional details concerning the underlying assumptions on which these definitions are based, and then we introduce the specifics of our model of rationally-inattentive bidding at Korean auctions. First, our analysis conditions on auctions of a specific make and model of commonly-auctioned rental cars: the Avante XD. This is a fairly generic passenger car in Korea and one of the most frequently-purchased car used by rental companies. We assume that not only are these cars generic, but the stochastic properties of the auctions in which they are sold are also generic as well and, hence, the data from these auctions can be aggregated for purposes of statistical analysis and estimating bidders' beliefs. In particular, we rule out the possibility of unique items being auctioned that would invalidate our assumption that the stochastic process of the high bid are IID realizations from an underlying anonymous equilibrium for Korean auctions of generic Avante XD rental cars. We make this operational via the following assumptions:

Assumption 1 (Conditionally-independent private values): *If there are n bidders at an auction, their valuations of the rental car on sale are IID draws from a conditional density $f(v|\mu)$ where we refer to μ as public value of the car. It is a random variable that is a function of a vector of characteristics $\mathbf{x}$ of a specific car being auctioned and additional information ϵ that bidders can observe from a physical inspection of the car prior to the auction but which we as the econometrician cannot observe. The public variable μ is common knowledge to the bidders, as is the distribution $f(v|\mu)$. If there are K cars being auctioned, we also assume that their public values $\{\mu_1, \ldots, \mu_K\}$ are IID draws from some distribution $H(\mu)$, which is also common knowledge amount the bidders.*

Assumption 2 (Random arrival of bidders): *The number of bidders n arriving to participate in a auction of a car with public value μ is a realization from a discrete probability distribution $g(n|\mu)$ which is common knowledge among the bidders.*

Assumption 3 (Time discretization): *Bidding at the Korean auction occurs during $T = 121$ discrete bidding instants at seconds $t = 0, 1, \ldots, 120$ during the auction. Any bid that is recorded in the continuous-time interval $[t, t + 1)$ is treated as having been submitted at bidding instant t. Bidders are updated about the high bids received in these intervals at bidding instants $t = 1, 2, \ldots, 120, 121$, so that any bids submitted at instant $t = 120$ have a one-second grace period for final bids to be recorded and bidders are informed of the auction outcome (and the winning bidder and bid) at final bidding instant at second $t = 121$.*

Assumption 4 (Bidder types): *Consider an auction of car with public value μ. The type of a bidder is a vector $\boldsymbol{\tau} = [v, c, p, \sigma]^\top$ where v is the bidder's valuation for the car being auctioned, c and σ are location/scale parameters of an extreme-value distribution governing the bidder's psychological cost of updating bids during the auction, and p is the probability that the bidder is distracted and unable to bid at a given bidding instant during the auction. he psychological bidding costs are IID extreme-value draws, and bidder distraction are* IID *Bernoulli draws at each of the $T = 121$ bidding instants during the auction. Conditional on a bidder's valuation v, the remaining components of the bidder's type (c, p, σ) IID draws from a conditional distribution $Q(c, p, \sigma|v)$ which is common knowledge among the bidders.*

Assumption 5 (Bidder beliefs): *Bidders individually regard auctions as having generic IID outcomes and share a common belief $\mathcal{B}$ concerning the probability measure governing the stochastic process for the high bid in each auction, which we denote by $\{\overline{b}_t\}$, where $\overline{b}_t$ is the highest submitted bid up to bidding instant t in the auction.*

Assumption 6 (Bidder optimality): *Bidders adopt bidding strategies that maximize their expected payoff from participating in the auction, formulated as a single agent game against nature where "nature" is the combination of the rules of the Korean auction and the bidder's beliefs $\mathcal{B}$ about the probability law of $\{\overline{b}_t\}$, the stochastic process for the high bid in the auction. Bidders are rationally-inattentive: they take into account the possibility that they will be inattentive in later instants of the auction when calculating their optimal bids in earlier instants, and the also account for the impact of their psychological bidding costs that constitute additional bidding frictions' when determining their optimal bidding strategies.*

Assumptions 1 to 6 imply that individual auctions are "generic" and have generic realized "high bid tracks" $\{\overline{b}_t\}$ where $\overline{b}_t$ denotes the high bid at time t in the auction. If there are n bidders partcipating in the auction with types $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$, the assumptions imply that the bidders use optimal bidding strategies in response to a common belief $\mathcal{B}$ concernig the stochastic process of $\{\overline{b}_t\}$ that take the the form $(\gamma(\mathcal{B}, \boldsymbol{\tau}_1), \ldots, \gamma(\mathcal{B}, \boldsymbol{\tau}_n))$. By Assumption 3 the IID structure of psychological bidding costs and bidder attention, implies these strategies will appear stochastic from the standpoint of outsiders who do not observe these costs or whether the bidder is inattentive at any given instant. Together with the rules of the Korean auction, this implies that the high bid track $\{\overline{b}_t\}$ is a well defined stochastic process, and the high bid tracks for different auctions will IID stochastic processes.

Lemma 3 *Assumptions 1 to 6 imply that the high bids at each instant of the Korean auctions $\{\overline{b}_t\}$ are well defined stochastic processes that are IID across different auctions.*

4.2 Dynamic programming solution for the optimal bidding strategy

In this section, we demonstrate that a bidder's belief $\mathcal{B}$ concernig the stochastic process of the high bid in the Korean auction constitutes a "sufficient statistic" that enables the bidder to calculate their optimal bidding strategy $\gamma(\mathcal{B}, \boldsymbol{\tau})$ by dynamic programming. We begin by describing a bidder's beliefs about the high bid in the auction, $\{\overline{b}_t\}$, in a more convenient form for carrying out the DP recursions. Let $\lambda_{t+1}(b|b_t, h_t)$ denote the conditional CDF for the high bid b at the auction submitted at second t given that the bidder's current high bid at second t in the auction was b_t. The only information communicated to the bidders during the auction is the high bidder indicator h_t, a variable that takes the value $h_t = 1$ if the bidder's bid b_t was the highest bid submitted at the auction up to to second t and $h_t = 0$ otherwise. At $t = 0$, none of the bidders has submitted a bid yet, so by definition $b_0 = 0$ and $h_0 = 0$ with probability one, and $\lambda_1(b|b_0, h_0)$ is the CDF for the high bid submitted at $t = 0$. We use the subscript $t = 1$ because the outcome of any bids submitted at second $t = 0$ are only revealed to bidders at the start of second $t = 1$. Thus, b_1 denotes the bid submitted by the bidder at $t = 0$, and the bidder conditions on this bid and the information h_1 to form expectations of the high bid at $t = 1$ via the CDF $\lambda_2(b|b_1, h_1)$. If the bidder chose not to bid at $t = 0$, then we set $b_1 = 0$. In this case, $h_1 = 0$ even if no other bidders submitted a bid at $t = 0$. So h_t will only equal one the first period t when a positive bid has been tendered and the bidder in question had submitted the highest bid so far at the auction.

Assumption 7: *A bidder's beliefs concernig the stochastic process of the high bid at the auction is encoded by the family of conditional CDFs given by*

$$\mathcal{B} \equiv \{\lambda_{t+1}(b|b_t, h_t)|t = 0, 1, \ldots, 120\}. \tag{8}$$

For technical reasons we assume for each t that $\lambda_{t+1}(b|b_t, h_t)$ is continuously differentiable with respect to b for $b > b_t$, and also differentiable with respect to b_t for each fixed $b > b_t$, for $t \in \{0, 1, \ldots, T\}$.

We can now to write the Bellman equation recursions to solve for the bidder's optimal bidding strategy at the Korean auction. The solution will depend on the bidder's type $\boldsymbol{\tau} = [v, c, p, \sigma]^\top$, his beliefs $\mathcal{B}$, and their *state* (b_t, h_t), for $t = 0, \ldots, 121$. To economize on notation we will drop the non-time-varying variables $\boldsymbol{\tau}$ and $\mathcal{B}$ from expressions for the bidder's value functions, which we denote by W_t and their optimal bidding decision rule, which we denote by γ_t.

The backward induction starts in the terminal period, $T+1 = 121$ where the bidder has potentially submitted a final bid at second $T = 120$, which via our notation is denoted as b_{121}. The auction software then transmits the information h_{121} which equals zero if the bidder had submitted the highest bid at the last period (and, thus, won the auction), or zero otherwise. The terminal value function is $W_{T+1}(b_{T+1}, h_{T+1})$ given by

$$W_{T+1}(b_{T+1}, h_{T+1}) = (v - b_{T+1})I\{1 = h_{T+1}\}. \tag{9}$$

W_{T+1} is a post-decision value function, since it specifies the *ex post* payoff at the end of the auction. Let $w_T(b, b_T, h_T)$ denote the bid-specific value function. This is the *ex ante* expected value to the bidder at $T = 120$ from submitting a bid of b given their state at T is (b_T, h_T). The bid-specific value function is given by

$$w_T(b, b_T, h_T) = \mathbb{E}[W_T(b_{T+1}, h_{T+1})|b, b_T, h_T] = (v - b)\lambda_{T+1}(b|b_T, h_T). \tag{10}$$

Note that this is the same expected payoff function as a first-price seald-bid auction when all bidders choose to snipe, so $b_T = 0$ for all bidders in the auction. In this sense, our model includes a theory of bidding at a static first-price, sealed-bid auction as a special case.[8] Assume that the Korean auction has a final warning signal so the bidder is not distracted and decides to submits a bid at time T. The final optimal bid is given by

$$\gamma_T(b_T, h_T) = \underset{b \geq b_T}{\operatorname{argmax}} \left[w_T(b, b_T, h_T) \right]. \tag{11}$$

Generally, it will be optimal to improve the current bid, so $\gamma_T(b_T, h_T) > b_T$. As we saw in section 2, however, bidders frequently choose not to improve their bids, perhaps because they are distracted at certain instants in the auction, but we also want to allow the possibility that a bidding friction—a psychological cost a bidder incurs from calculating an improved bid—might deter the bidder from submitting a strictly improved bid even if the bidder was not distracted. We allow for this via another state variable $\epsilon_T = (\epsilon_T(0), \epsilon_T(1))$,

[8]It will, however, only be an anonymous equilibrium of the larger dynamic Korean auction if all bidders' beliefs assign zero probability to early bidding in the auction, that is, only if $\lambda_{t+1}(0|0, 0) = 1$ for $t < T = 120$.

where $\epsilon_T(0)$ is the monetary equivalent of a psychological cost (if negative) or benefit (if positive) associated with not improving the current bid b_T and $\epsilon_T(1)$ is the corresponding benefit or cost associated with submitting the optimal bid $\gamma_T(b_T, h_T)$. Thus, we define the value function $W_T(b_T, h_T, \epsilon_T)$ by

$$W_T(b_T, h_T, \epsilon_T) = \max \left[w_T(b_T, b_T, h_T) + \epsilon_T(0), \max_{b \geq b_T} [-c + w_T(b, b_T, h_T) + \epsilon_T(1)] \right]. \quad (12)$$

By Assumption 4, the shocks have a Type-1 extreme-value distribution with having mean normalized to zero and scale parameter σ. This implies that the probability of submitting a bid $\gamma_T(b_T, h_T)$ at the last instant $T = 120$ is $P_T(\gamma_T(b_T, h_T)|b_T, h_T)$ given by a binomial logit formula

$$P_T(\gamma_T(b_T, h_T)|b_T, h_T) = \frac{\exp\{[w_T(\gamma_T(b_T, h_T), b_T, h_T) - c]/\sigma\}}{\exp\{w_T(b_T, b_T, h_T)/\sigma\} + \exp\{[w_T(\gamma_T(b_T, h_T), b_T, h_T) - c]/\sigma\}}. \quad (13)$$

Define the expected value function $EW_T(b_T, h_T)$ as the expectation of $W_T(b_T, h_T, \epsilon_T)$ with respect to ϵ_T but conditioning on (b_T, h_T). By the well-known property of Type 1 extreme-value distributions, we have the following closed-form solution for this expectation

$$\begin{aligned} EW_T(b_T, h_T) &= \int_{\epsilon_T} W_T(b_T, h_T) q(\epsilon_T) d\epsilon_T \quad (14) \\ &= \sigma \log \Big(\exp\{w_T(b_T, b_T, h_T)/\sigma\} + \exp\{w_T(\gamma_T(b_T, h_T), b_T, h_T)/\sigma\} \Big), \end{aligned}$$

where $q(\epsilon_T)$ is the probabilty density function of a bivariate Type-1 extreme-value random variable having mean zero and scale parameter σ. $EW_T(b_T, h_T)$ is relevant only if the bidder was not distracted at instant T and so observed ϵ_T and made a choice of whether to improve his bid. If the bidder was distracted, then no improved bid would be submitted and the value in this case is just $w_T(b_T, b_T, h_T)$, the same value as a conscious decision not to bid. Therefore, the bid-specific value function at bidding instant $T-1$ is $w_{T-1}(b, b_{T-1}, h_{T-1})$ given by

$$\begin{aligned} w_{T-1}(b, b_{T-1}, h_{T-1}) = \quad & [p w_T(b, b, 1) + (1-p) EW_T(b, 1)] \, \lambda_T(b|b_{T-1}, h_{T-1}) + \quad (15) \\ & [p w_T(b, b, 0) + (1-p) EW_T(b, 0)] \, [1 - \lambda_T(b|b_{T-1}, h_{T-1})]. \end{aligned}$$

Equation (16) shows the rational inattention aspect of our model: at instant $T-1$ the bidder by definition is not distracted, since he is considering the value of making alternative bids $b \geq b_{T-1}$ at that point. Yet the bidder is self-aware that at instant T, so there is a probability p that he will be distracted, and thus unable to make any further adjustments to his bid in that case. If the bidder tenders b at $T-1$, he will be the high bidder at instant T with probability $\lambda_T(b|b_{T-1}, h_{T-1})$ so $h_T = 1$. In equation (16), the first line shows the expected payoff for this outcome. The second line covers the case where the b is not the high bid, so $h_T = 0$. In both cases, if the bidder is distracted, no further bid is made and this has value $w_T(b, b, 1)$ or $w_T(b, b, 0)$ depending on whether b is the high bid or not, and if the bidder is not distracted, the expected values are $EW_T(b, 1)$ and $EW_T(b, 0)$, respectively.

Using $w_{T-1}(b, b_{T-1}, h_{T-1})$ we can define the optimal bid function $\gamma_{T-1}(b_{T-1}, h_{T-1})$ in the same way as we did for period T in equation (11), and a probability of bidding $P_{T-1}(\gamma_{T-1}(b_{T-1}, h_{T-1})|b_{T-1}, h_{T-1})$ defined similarly as in equation (13), and finally an expected value function $EW_{T-1}(b_{T-1}, h_{T-1})$ defined similarly as $EW_T(b_T, h_T)$ is defined in equation (15). The backward induction then proceeds via the same formulas for all other bidding instants for $t = T-2, \ldots, 1, 0$. We will denote by $\gamma(\mathcal{B}, \boldsymbol{\tau})$ the full sequence of optimal bid functions

$$\gamma(\mathcal{B}, \boldsymbol{\tau}) = (\gamma_0, \gamma_1, \ldots, \gamma_T), \tag{16}$$

where we have suppressed the dependence on the non-time-varying arguments $\mathcal{B} = \{\lambda_{t+1}(b|b_t, h_t)|t = 0, \ldots, T\}$ and $\boldsymbol{\tau} = [v, c, p, \sigma]^\top$ to simplify notation.

Note that the actual bidding strategy $\gamma(\mathcal{B}, \boldsymbol{\tau})$ is stochastic due the effect of the *IID* bidding cost state variables $\{\epsilon_t\}$ and the Bernoulli process for bidder attention. Let a_t be a Bernoulli random variable that equals 1 if the bidder is paying attention at bidding instant t and 0 otherwise. Also let $u_t(b_t, h_t, \epsilon_t)$ be another Bernoulli random variable that equals 1 if it is optimal to bid in state (b_t, h_t, ϵ_t) when $a_t = 1$, i.e.

$$u_t(b_t, h_t, \epsilon_t) = \begin{cases} 1 & \text{if } w_t(b_t, b_t, h_t) + \epsilon_t(0) \geq w_t(\gamma_t(b_t, h_t), b_t, h_t) - c + \epsilon_t(1) \\ 0 & \text{if } w_t(b_t, b_t, h_t) + \epsilon_t(0) < w_t(\gamma_t(b_t, h_t), b_t, h_t) - c + \epsilon_t(1). \end{cases} \tag{17}$$

Then we can write the full bidding strategy as the function $\gamma_t(b_t, h_t, \epsilon_t, a_t)$ given by

$$\gamma_t(b_t, h_t, \epsilon_t, a_t) = \begin{cases} b_t & \text{if } a_t = 0 \\ b_t & \text{if } a_t = 1 \text{ and } u_t(b_t, h_t, \epsilon_t) = 0 \\ \gamma_t(b_t, h_t) & \text{if } a_t = 1 \text{ and } u_t(b_t, h_t, \epsilon_t) = 1. \end{cases} \tag{18}$$

The extended bidding strategy given in equation (18) above is what we will use both to simulate outcomes in the Korean auction and as a basis for inference. Intuitively, the bidder will not improve their current bid b_t at bidding instant $t+1$ if either 1) they are inattentive, $a_t = 0$, or 2) they are attentive, but due to the realized psychological bidding shocks $(\epsilon_t(0), \epsilon_t(1))$ they conclude it is not optimal for them to improve their bid. However a key restriction of this model is that in the third case where the bidder is paying attention and the shocks $(\epsilon_t(0), \epsilon_t(1))$ are such that it is optimal for them to place a bid, the amount of the bid, $\gamma_t(b_t, h_t)$, is a deterministic function of (b_t, h_t). This implies that our model of bidding in the Korean auction is *statistically degenerate* — that is, the probability of observing any other bid $b \neq \gamma_t(b_t, h_t)$ is zero under our rationally inattentive model of bidding behavior and this rules out the direct use of maximum likelihood for inference. For this reason in the next section we introduce a *quasi-maximum likelihood estimator* (QML) for conducting inference in this model. The QML estimator is tolerant of observed bids b that are not equal to the predicted optimal bid $\gamma_t(b_t, h_t)$ and will enable us to find values for the bidder's type $\boldsymbol{\tau} = (v, c, p, \sigma)$ that "best fit" the bidder's behavior in individual auctions. Using the QML estimates of bidder types, we will show in section 5 that the estimated model is capable of explaining the observed early bidding behavior and heterogeneous bidding strategies that we documented in section 2.

4.3 Effect of informational restrictions on optimal bidding

Before we describe the QML estimator, we consider how the bidder's DP problem can be modified to solve for the optimal bidding strategy when a key informational restriction in the Korean auction is dropped. Specifically, suppose that the auction software is modified to display the highest submitted bid at each instant during the auction, although the identity of the bidder holding the highest bid is still suppressed. The high bid indicator

h_t still show each bidder whether they are currently the high bidder or not, but now every bidder observes the high bid in the auction even if $h_t = 0$. We refer to this auction mechanism and an *anonymous open outcry auction.* This change in rules makes the anonymous open outcry auction a slightly informationally restricted version of a standard open outcry auction, the difference being that in most open outcry auctions all bidders can see the identities of competing bidders and not just the high bid at any given moment. As the name suggests, the anonymous open outcry auction can be formulated as an anonymous game, and an anonymous equilibrium can be computed using the same approach we outlined above for the Korean auction.

How are bidding strategies affected by dropping the informational restriction that each bidder only knows the high bid when they are the high bidder? It is easy to see that their beliefs about the CDF of the high bid in the auction no longer depends on the binary indicator h_t. Instead, we reinterpret the state variable b_t in the DP problem to be the *current high bid in the auction* as opposed to the bidder's own high bid, our intrerpretation of b_t under the Korean auction's informational restrictions. Without this restriction, bidders' beliefs can be fully described by a family of conditional CDFs $\{\lambda_{t+1}(b|b_t)|t = 0, 1, \ldots, T\}$, where $\lambda_{t+1}(b|b_t)$ is the conditional probability that the high bid submitted at bidding instant t is less than or equal to b given that the high bid submitted up through time $t-1$ is b_t. From this it is easy to show that the Bellman equations and optimal bidding strategies *do not depend on the high bid indicator h_t.* The intuitive implication of relaxing the informational restriction in the Korean auction is that bidders in an anonymous open outcry auction have no incentive to bid early in the auction to learn the high bid since they are provided this information for free. This implies that relaxing the informational resriction leads to *informational free-riding* and we show below that under certain assumptions, all bidders prefer to "remain in the background" and only submit a single bid at the last instant, T. That is, bid-sniping is the only anonymous equilibrium of this game, so it is strategically equivalent to the equilibrium of an anonymous static sealed bid auction. In order to establish this result we need to introduce a mild assumption about bidders' beliefs about the distribution of the high bid, namely that $\lambda_{t+1}(b|b_t)$ is *stochastically increasing in the current high bid b_t.* This assumption seems reasonable in an ascending bid auction. When this holds, we can show that there is no strategic reason for early bidding to occur if there are no bidding frictions. However we do allow for rational inattention, which plays a role similar to that of a "soft-close" in online auctions on Amazon discussed by Roth and Ockenfels [2002].

Definition: We say the family $\{\lambda_{t+1}(b|b_t)|t = 0, 1, \ldots, T\}$ is *stochastically increasing in b_t* if for any $b'_t \geq b_t$ we have:

$$\lambda_{t+1}(b|b'_t) \leq \lambda_{t+1}(b|b_t), \tag{19}$$

for all $b \geq b_t$ and all $t \in \{0, 1, \ldots, T\}$.

Assumption 8 The family of beliefs about the high bid $\{\lambda_{t+1}(b|b_t)|t = 0, 1, \ldots, T\}$ is stochastically increasing in b_t and satisfies the derivative condition

$$\nabla_{b_t} \lambda_{t+1}(b|b_t) \leq 0, \tag{20}$$

where ∇_{b_t} denotes the derivative of $\lambda_{t+1}(b|b_t)$ with respect to its conditioning argument, b_t, for all $b \geq b_t$ and $t \in \{0, 1, \ldots, T\}$ Further, $\lambda'_{t+1}(b|b_t) > 0$ for $b > b_t$ where $\lambda'_{t+1}(b|b_t)$ is the derivative of the CDF $\lambda_{t+1}(b|b_t)$ with respect to b, which we assume is a continuous function of b for any $b > b_t$ and all $t \in \{0, 1, \ldots, T\}$.

Theorem 1 *If Assumptions 1 to 8 hold, optimal bidding strategies and value functions in the anonymous open outcry auction are independent of h_t, the indicator for whether the bidder holds the current high bid. If $c = \sigma = 0$ (i.e. no bidding frictions), the optimal bidding strategies involve bidding zero until the final instant T of the auction, so all bidders adopt optimal bid-sniping strategies and the anonymous equilibrium of the anonymous open outcry auction is strategically equivalent to the anonymous equilibrium of an anonymous static first price sealed bid auction.*

The proof of Theorem 1 is provided in Appendix A. The theorem encapsulates the intuition that the key informational restriction in the Korean auction that suppresses the current high bid unless a bidder was the first to submit a bid that equals the current high bid is essential for explaining the early bidding we observe, at least when there are no bidding frictions except for rationally inattentive bidding.

4.4 Two-step quasi maximum-likelihood fixed-effects estimator

We employ a two-step estimation approach, where the first step involves nonparametric estimation of bidders' beliefs $\mathcal{B} = \{\lambda_{t+1}(b|b_t, h_t)|t = 0, \ldots, 120\}$,while the second step involves estimating bidder/auction-specific types $\boldsymbol{\tau} = (v, p, c, \sigma)$ for each bidder at each auction in our auction data set. The latter is done using a structural nested DP quasi-maximum likelihood estimator (QMLE) where we repeatedly solve for a bidder's optimal bidding strategy for different candidate values of $\boldsymbol{\tau}$ using the first stage nonparametric estimates of beliefs about the high bid, $\hat{\mathcal{B}}$. Since we nonparametrically estimate the conditional distribution of the high bid, $\lambda_{t+1}(b|b_t, h_t)$ in the first stage, our estimation approach imposes rational expectations on the part of all bidders. We recognize, however, that estimation noise exists in our first-stage estimates of beliefs, so we subsequently check whether the weaker condition of ϵ-anonymous equilibrium holds by re-simulating data using the estimated bidding strategies for all bidders at all auctions and calculating the difference $\epsilon \equiv \|\hat{\mathcal{B}} - \Lambda(\hat{\mathcal{B}})\|$. If this difference is sufficiently small, then we can conclude that our estimated structural model of rationally-inattentive bidding constitutes an ϵ-anonymous equilibrium of the Korean auction.

We adopt a *fixed effects* approach to estimation of the unknown parameters of our model of rationally-inattentive bidding in Korean auctions. The values v are clearly both bidder and auction-specific by Assumption 1. To allow for maximum heterogeneity, we estimate the other three parameters (p, c, σ) separately for each bidder/auction pair in our data set as well. It is easiest to begin by explaining how to estimate $\boldsymbol{\tau}$ via full maximum-likelihood, and after deriving the full likelihood it will help motivate why we chose to estimate $\boldsymbol{\tau}$ using a QMLE instead.

To derive the likelihood, we consider an extension of the model developed in the previous section that is *statistically nondegenerate*;that is, it assigns positive probability to any possible observed sequence of bids by a given bidder in the auction. The data we have for an individual bidder in a specific auction are $\{(b_t, h_t)|t = 1, \ldots, T\}$ where $T = 121$, where we cleaned the data to remove nonmonotonic bids submitted by any bidder. Recall our timing convention: b_1 denotes the bid the bidder submitted at $t = 0$ (or zero if no bid was submitted) and $h_1 = 1$ if that was the highest bid among all bids submitted in the interval $[0, 1)$. Continuing, b_t is the bid submitted at second $t - 1$, that is, in the time interval $[t - 1, t)$ and $h_t = 1$ if that bid was the highest bid submitted in the auction up to time t or 0 otherwise. Finally b_{121} denotes the final bid submitted at the last second of the auction $T = 120$, that is, in the time interval $[120, 121)$ where as noted in section 2 the auction software accepts bids that arrive slightly after the two-minute mark.

Suppose our extended, nonstatistically degenerate model results in a *bid transition* *probability* of the form $P(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$. Then the likelihood for a bidder in a particular auction is given by

$$L(\boldsymbol{\tau}) = \prod_{t=0}^{120} f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) \tag{21}$$

where f is a conditional probability given by

$$f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) = \begin{cases} p + (1-p)[1 - P(b_{t+1}|b_t, h_t, \boldsymbol{\tau})] & \text{if } b_{t+1} = b_t \\ (1-p)P(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) & \text{if } b_{t+1} > b_t \end{cases} \tag{22}$$

and subject to the initial condition $(b_0, h_0) = (0, 0)$. We explain the formula for the density $f(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ in equation (22) as follows: If the bidder fails to improve his bid at instant t, ($b_{t+1} = b_t$), the probability this can happen equals the sum of the probability p that the bidder was not paying attention at t plus the probability the bidder was paying attention but chose not to improve the bid. This is reflected by the second term in the first line of equation (22) where $P(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ is a multinomial logit formula for the probability of submitting a bid of b_{t+1} at instant t that we derive below. If the bidder does improve his bid ($b_{t+1} > b_t$), then the probability of this occurring is in the second line of equation (22) and it requires that the bidder not be inattentive and to choose a bid higher than their best previous bid b_t.

The MLE uses a nested solution approach where an outer hill climbing algorithm searches for a 4×1 vector $\hat{\boldsymbol{\tau}}$ that maximizes $L(\boldsymbol{\tau})$, and an inner DP algorithm is called to solve the bidder's DP problem each time the likelihood is evaluated for a given value of $\boldsymbol{\tau}$. Note that bidder beliefs are fixed at the nonparametric estimates $\hat{\mathcal{B}}$ from the first step throughout all second step structural estimations for all bidder/auction pairs.

We can modify the DP model in the previous section to be statistically nondegenerate by 1) assuming bids are discrete, say restricted to the positive integer values (which is the case in our data), and 2) introducing a separate Type-1 extreme *bidding shock* for every possible integer bid b. When we do the latter, we need to modify the formula for the value function in equation (12), and then generic time t step of the backward induction the value function is $W_t(b_t, h_t, \epsilon_t)$ given by

$$W_t(b_t, h_t, \epsilon_t) = \max\left[w_t(b_t, b_t, h_t) + \epsilon_t(0), \max_{b > b_t}[-c + w_t(b, b_t, h_t) + \epsilon_t(b)]\right]. \tag{23}$$

This version of the model has enough bid-specific shocks to generate a positive probability of observing any possible bid b, since it implies the following multinomial logit conditional choice probability for $P(b|b_t,h_t)$

$$P_t(b|b_t,h_t) = \begin{cases} \frac{\exp\{[w_t(b,b_t,h_t)-c]/\sigma\}}{\exp\{w_t(b_T,b_T,h_T)/\sigma\}+\sum_{b\geq b_t}\exp\{[w_T(b,b_t,h_t)-c]/\sigma\}} & \text{if } b > b_t \\ \frac{\exp\{w_t(b,b_t,h_t)/\sigma\}}{\exp\{w_t(b_T,b_T,h_T)/\sigma\}+\sum_{b\geq b_t}\exp\{[w_T(b,b_t,h_t)-c]/\sigma\}} & \text{if } b = b_t \end{cases} \tag{24}$$

Unfortunately, the full likelihood approach involves a substantial computational burden because it requires exhaustive evaluation of the value functions at all possible integer bids b greater than or equal to b_t. The model we presented in the previous section only involves two shocks, $[\epsilon_t(0),\epsilon_t(1)]$, where $\epsilon_t(0)$ represents unobserved costs/benefits of not improving the current bid, and $\epsilon_t(1)$ are the unobserved costs/benefits corresponding to submitting the optimal bid $\gamma_t(b_t,h_t)$. We use more efficient continuous maximization algorithms to find $\gamma_t(b_t,h_t)$ that require relatively few evaluations of an interpolated version of the bid-specific value functions $w_t(b,b_t,h_t)$, which is far faster than a "brute-force" exhaustive evaluation that is required by the full likelihood approach.

Another attractive feature of the model we presented in the previous section is that its tight prediction of the optimal bid $\gamma_t(b_t,h_t)$ is useful assessing how well our model actually fits the data. We can think of $\gamma_t(b_t,h_t)$ as akin to a nonlinear regression function that constitutes the model's predicted optimal bid, so we can directly evaluate the "residuals" $e_t(\hat{\boldsymbol{\tau}}) = b_{t+1} - \gamma_t(b_t,h_t,\hat{\boldsymbol{\tau}})$ at the estimated value of each bidder's type $\hat{\boldsymbol{\tau}}$. In the next section, we demonstrate the ability to compare actual bids to those predicted by our model, which will prove to be extremely informative not only about model fit, but also about the behavior of the bidders. The logit model in equation (24) has such a rich specification of unobservables that it can "rationalize" any observed bidding behavior, though even the fully "saturated" specification does still impose testable restrictions that can be assesses by a variety of specification tests. That said, we chose to begin with the QMLE estimator.

To implement the QMLE estimator, we need to define a "quasi-likelihood" of observing a bid of b that differs from the optimal bid $\gamma_t(b_t,h_t,\boldsymbol{\tau})$ predicted by the model when the current parameter is $\boldsymbol{\tau}$. For this purpose we use the following binary logit probability

$\Pi(b|b_t, h_t, \boldsymbol{\tau})$ given by

$$\Pi(b|b_t, h_t, \boldsymbol{\tau}) = \frac{\exp\{w_t(b, b_t, h_t)/\omega\}}{\exp\{w_t(b, b_t, h_t)/\omega\} + \exp\{[w_t(\gamma_t(b_t, h_t) - c]/\omega\}}, \tag{25}$$

where $\omega \geq 0$ is a *penalization parameter* that controls how hard the QMLE tries to fit observed bids b_{t+1} via the predicted optimal bids from the model $\gamma_t(b_t, h_t, \boldsymbol{\tau})$. When ω is set to a small value, actual bids b_{t+1} that are far from the optimal bid $\gamma_t(b_t, h_t, \boldsymbol{\tau})$ result in very small values for $\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ and, thus, low values for the QMLE objective function as we will see below. Indeed, for any bid b we have $w_t(b, b_t, h_t) \leq w_t(\gamma_t(b_t, h_t), b_t, h_t)$, so it follows that

$$\Pi(b|b_t, h_t, \boldsymbol{\tau}) \leq \frac{1}{1 + \exp\{-c/\sigma\}}. \tag{26}$$

and $\Pi(b|b_t, h_t, \boldsymbol{\tau})$ is maximized at $b = \gamma_t(b_t, h_t)$. When ω is small there is high penalization and $\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ will be close to zero for observed bids that are far fom the optimal bid predicted by the model, $\gamma_t(b_t, h_t, \boldsymbol{\tau})$.

The QMLE is defined similarly to the MLE using the following formula for the quasi-likelihood for the observed bidding data

$$QL(\boldsymbol{\tau}) = \prod_{t=0}^{120} f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) \tag{27}$$

where f is a conditional probability given by

$$f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) = \begin{cases} p + (1-p)[1 - P(\gamma_t(b_t, h_t)|b_t, h_t, \boldsymbol{\tau})] & \text{if } b_{t+1} = b_t \\ (1-p)P(\gamma_t(b_t, h_t)|b_t, h_t, \boldsymbol{\tau})\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) & \text{if } b_{t+1} > b_t \end{cases} \tag{28}$$

Comparing equations (22) and (28), we see that the key difference between the likelihood $L(\boldsymbol{\tau})$ and the quasi-likelihood $QL(\boldsymbol{\tau})$ is the presence of the binary logit probability $\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ that constitutes a penalty term for observed bids b_{t+1} that differ too much from the predicted optimal bid $\gamma_t(b_t, h_t, \boldsymbol{\tau})$.[9]

[9]The likelihood $L(\boldsymbol{\tau})$ and quasi-likelihood $QL(\boldsymbol{\tau})$ are smooth functions of $\boldsymbol{\tau}$, but calculating their gradients requires recursive evaluation of the gradients of the bid-specific value functions $\{\nabla_{\boldsymbol{\tau}} w_t(b, b_t, h_t, \boldsymbol{\tau})\}$. This is done in tandem with the recursive calculation of $\{w_t(b, b_t, h_t, \boldsymbol{\tau})\}$ itself, using piecewise polynomial interpolation over a grid of bid values. MATLAB code to solve for optimal bidding strategies and estimate this model via the structural two-step QMLE is available on request.

5 Results and Counterfactuals

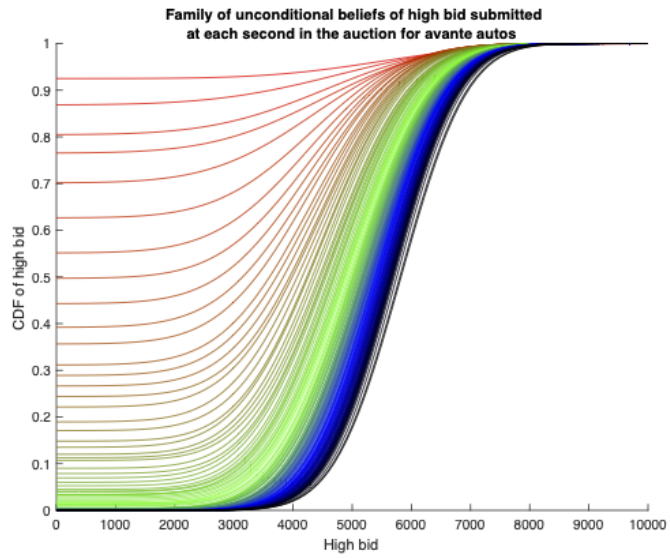
We analyze data from 533 auctions of Hyundai Avante Elanta XD cars with 1.6L engines. These are generic passenger sedans without any unique features that would suggest that we should treat different units as "unique items" in violation of Assumption 1 of section 4. There are a total of 4029 bidder/auction observations in these 533 auctions, which we used to estimate an equivalent number of 4×1 type vectors τ using the QMLE defined in section 5. Thus, we should consider a single "observation" as the complete record of bids for a given bidder in a given auction.

However before we could do these estimations we had to perform the first step our two step estimation procedure and use the bid data from the 4029 bidding histories to non-parametrically estimate bidders' beliefs $\hat{\mathcal{B}} = \{\hat{\lambda}_{t+1}(b_{t+1}|b_t, h_t)|t = 0, \ldots, 120\}$. We also estimated a parametric models of beliefs using a truncated normal specification and found these estimates closely match the non-parametric estimates but are substantially smoother and less noisy. Therefore in our actual estimation results, we opted to use the parametrically estimated family of beliefs. We start by describing these results before presenting our structural estimation results.

5.1 Estimated beliefs

The bid data from the 4029 bidder/auction observations provide up to 121 indidivual "bid-second" level observations of the high bid at different seconds in the auction. Figure 11 plots our estimates of the family of "unconditional beliefs" $\{\hat{\lambda}_{t+1}(b_{t+1}|0,0)\}$ at bid instants $t = 0, 1, \ldots, 120$. This family constitutes the *ex ante* beliefs about the distribution of high bids at each second of the auction for bidders who have not yet submitted bids in the auction. The unscaled CDFs are qualitatively similar (in terms of shape) to the rescaled CDFs we plotted for a much larger sample of vehicles in figure 8 in section 2. In particular, we see the same pattern of strict stochastic dominance in this family of CDFs over successive seconds in the auction consistent with the fact that these are ascending bid auctions. The probability that the high bid equals 0 (i.e. no bids were submitted) starts at over 90% in the first second of these auctions and rapidly drops to zero as t approaches

Figure 11: Unconditional CDFs for high bids for Avante cars by elapsed time in auction

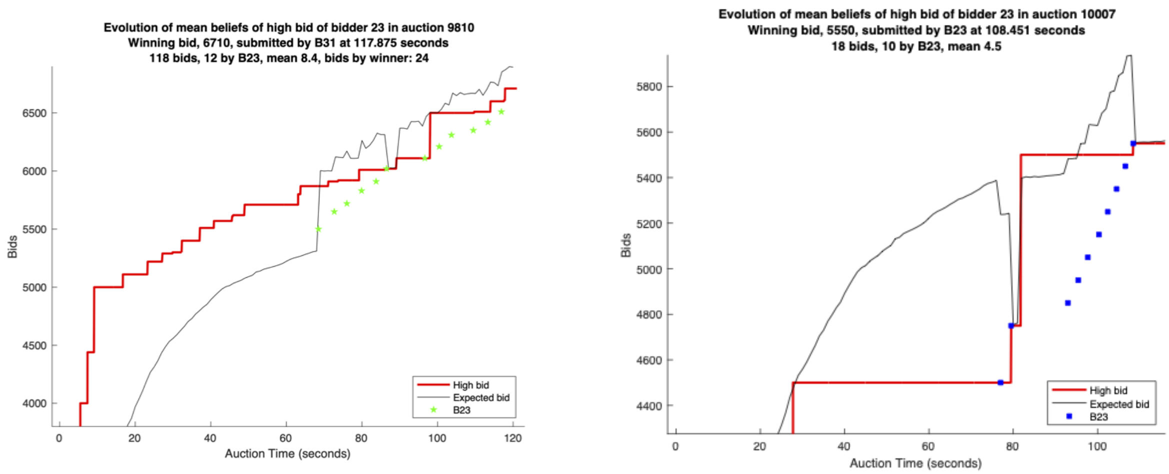


$T = 120$. At the last bidding instant, the median high bid is approximately \$6000.

We also estimated a separate family of conditional CDFs $\{\hat{\lambda}_{t+1}(b_{t+1}|b_t, 1)\}$ that conditions on the previous high bid b_t in the preceding second of the auction (and hence the second conditioning argument is $h_t = 1$), and a third family of CDFs $\{\hat{\lambda}_{t+1}(b_{t+1}|b_t, 0)\}$ that conditions on bids b_t that are lower than the current high bid $\overline{b}_t$ in the preceding section of the auction. These families were estimated via truncated regression methods (Tobit models).

Figure 12 illustrates how bidder beliefs evolve during an auction by plotting two example paths for expected high bids, $E\{\overline{b}_{t+1}|b_t, h_t\} = \int_{b_t}^{\infty} b\lambda_{t+1}(db|b_t, h_t)$ of bidder ID 23 (B23) in two different auctions. The left panel of the figure shows the evolution of the expected high bid in auction 9810. B23 did not submit a bid until approximately 70 seconds into the auction, and the first bid submitted did not turn out to be the high bid. This is reflected in the concave shaped black line in the figure that roughly parallels the red high bid track until the instant B23 submits his bid, at which point B23's expectation of the high bid jumps to a value above the high bid track. B23 continues to "bid creep" but does not succeed in capturing the high bid until about second $t = 90$, when we see B23's expectation of the high bid decreases to a value virtually equal to b_{90}. That is, once a bidder becomes a high bidder at some point in the auction, they have high confidence that they

Figure 12: Expected high bids for specific examples



will remain the high bider for the next few seconds. However a few seconds later, B23's high bid is leapfrogged by a higher bid of some other bidder in the auction and then B23's belief of the expected high bid jumps back up above the high bid track and remains there until the end of the auction since B23's subsequent attempt at bid creeping did not enable B23 to regain the high bid and win this auction.

The right hand panel of figure 12 plots the evolution of B23's mean beliefs of the high bid in auction ID 10007. Similar to the previous case, the *ex ante* expectation of the high bid (i.e. *ex ante* in the sense that B23 has not submitted any bids to gather information on the high bid in the auction) follows a concave shape, but in this case it lies substantially above the red realized high bid track. At $t = 78$ the bidder submits a bid equal to the current high bid $\overline{b}_{78}$ but due to the time-priority rules in the Korean auction, B23 is not informed that his bid equals the high bid, so B23's information is $h_{78} = 0$, i.e. he does not hold the high bid at that point. Nevertheless, B23's expectation of the high bid falls in response to this information, so $E\{\overline{b}_{79}|b_{28}, 0\} < E\{\overline{b}_{79}|0, 0\}$. Then at $t = 80$ B23 submits a higher bid that is the high bid at that point, $b_{80} = \overline{b}_{80}$, so B23 receives the information $h_{80} = 1$ and this decreases B23's expectation of the high bid to a value only slightly higher than b_{80}. But then a few seconds later some other bidder submits a substantially higher bid that increases the high bid to about \$5500, and this causes B23's beliefs about the expected high bid to jump up too, but to a value of approximately \$5400.

In the remainng 30 seconds of auction 10007 B23 executes a sequence of "bid creeps" that are initially unsuccessful in capturing the high bid status until second $t = 108$ of the auction where B23 becomes the high bidder with a bid of $b_{108} = 5550$. B23 opts not to make any further improvements in bid for the remainder of the auction, retaining his high bid status for the rest of the auction and winning it with his high bid of $b_{108} = 5550$. Thus, we can see how even with our model of "fixed beliefs" our model captures "learning" in these auctions. Bidders are not "learning" in a Bayesian sense, but instead they are "learning" about the high bid in the auction through the process of bidding during the auction. We now show how bidders' rational beliefs about the evolution of the high bid in the auction is reflected in the *magnitudes* of the bids — i.e. we will wish to assess how close their actual bids $\{b_t\}$ are to the optimal bids $\{\gamma_t(b_t|b_{t-1}, h_{t-1})\}$ predicted by the structural estimates of our model of rationally inattentive bidding.

5.2 Estimated types

We estimated 4029 4×1 type vectors $\hat{\tau}$ using our structural nested DP QML estimator by repeatedly solving for optimal bidding strategies in the Korean auction to find ones that best fit the actual bidding behavior of the 4029 bidder/auction observations in our data set. In all of these solutions we keep the beliefs of all bidders fixed at the values we estimated in the first stage, $\hat{\mathcal{B}}$ described above. Though there are at most 121 bidding instants in a single auction, we find that there is sufficient information to point-identify the 4029 type vectors for all bidder/auction pairs in our data set. We acknowledge potential econometric issues with our QML estimation approach, including the standard "incidental parameters problem" that can affect the consistency of fixed effect estimators, as well as the issue of how to think about asymptotics if the data we use to estimate the type of each bidder is limited to the observed bids submitted during each 120 second auction. For now, we will not to worry about these econometric issues and treat our QMLE as akin to a "calibration" that gives us substantial flexibility to best fit observed bidding behavior, as well as capturing the substantial heterogeneity in bidding strategies that we documented in section 2.

Figure 13: Estimated valuations v and costs of bidding c

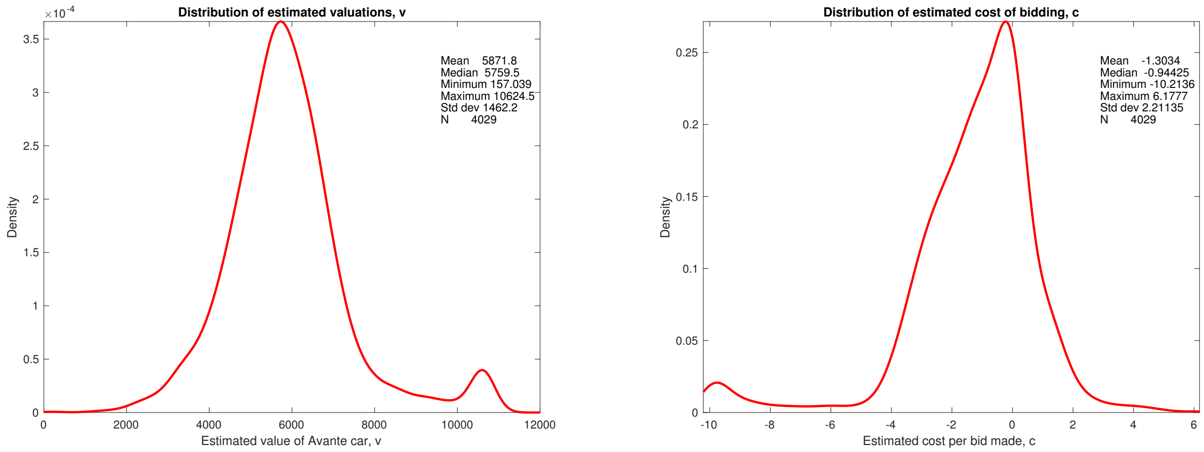
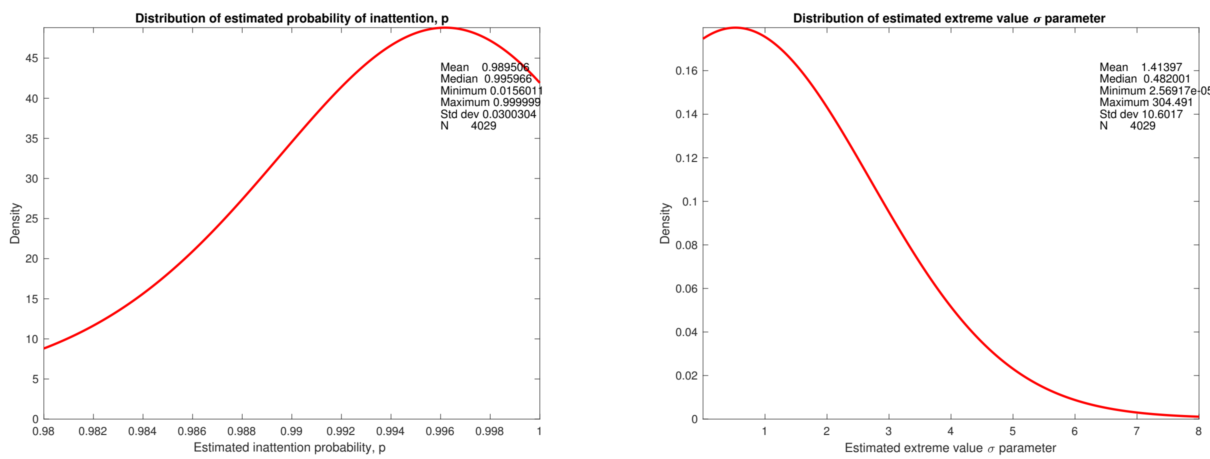


Figure 14: Estimated inattention probabilities p and scale parameters σ



With estimates of over 16,000 parameters, it is obviously impractical to display them in a table (though we can provide a data set containing the estimated parameters on request). Instead we display the results graphically via univariate kernel density plots of each of the four components of bidders' types, $\tau = (v, c, p, \sigma)$ in figures 13 and 14.

The left panel of figure 13 plots the distribution of estimated valuations, v. The mean valuation for the 4029 bidders participating in the 533 Avante auctions was \$5872. However we see substantial heterogeneity, and though the distribution seems to be approximately normally distributed, there is an interesting "bump" in the upper tail of valuations. We need to do further analysis to see if there is something unique to these valuations, such as the potential that we mis-identified a subset of Avante cars that have additional features (e.g. luxury interiors) that make them more valuable to bidders than the generic Avante XD 1.6L cars that we identified. Since unique features for a subset of vehicles would violate our Assumption 1, it would be appropriate to remove these cases from our estimation results. However for now we retain them.

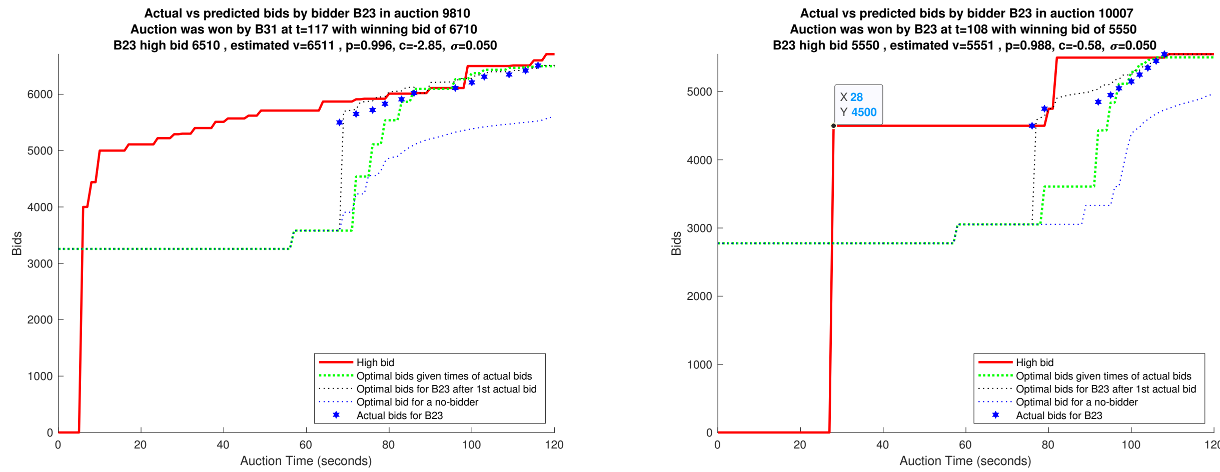
The right panel of figure 13 plots the distribution of the location term c for the distribution of idiosyncratic psychological costs or benefits a bidder perceives from updating the current bid. The mean value of c is -1.3, indicating that the average bidder perceives a small *benefit* to updating their bid, above and beyond the benefit inherent from learning more about the current high bid. However the mode of the distribution is positive, so about 40% of all bidders perceive a psychological cost that constitutes a slight deterrent to frequently updating their bid, i.e. a *bidding friction.*

The left panel of figure 14 plots the distribution of inattention probabilities, p. Surprisingly, the mean probability of being inattentive at any point during the auction is nearly 99%. The reader may wonder, if bidders are not paying attention to the auction 99% of the time, how is it possible that they ever find an opportunity where they are paying attention in order to place even a single bid in the auction? Recall that our implicit assumption is that inattention is an *IID* Bernoulli process with parameter p and there are 121 bidding instants during the auction. The probability is $1 - (.99)^{121} = .7036$ that the bidder will be paying attention at one or more instants, and therefore able to submit one or more bids during an auction. However this does point out a shortcoming of our specification: it is hard for our model to replicate "bid creeping" when inattention is an *IID* Bernoulli process with such large values of p: this makes it unlikely to observe a sequence of several bids occuring only seconds apart from each other.[10]

The final, right hand panel of figure 14 plots the distribution of σ scale parameters for the extreme value distributed idiosyncratic component of psychological bidding costs.

[10]Of course, bidders who bid creep have smaller estimated values of p, since the QMLE estimator reduces p to increase the likelihood of observing sequences of such sequences of bids. However we can also estimate a two state Markov model of rational inattention in future work. In a Markovian model if a bidder is attentive at instant t they may be more likely to be attentive at $t + 1$, and conversely for bidders who are inattentive. A Markov model requires estimation of two probabilities for bidders who are currently attentive and inattentive, respectively.

Figure 15: Actual versus predicted optimal bids by B23 in auctions 9810 and 10007



The values of σ are on average fairly large, meaning that there is a fair degree of idiosyncratic variation in these costs that determine whether a bidder will update their bid, conditional on not being inattentive at a given instant in the auction. The overall message is that the estimated model predicts that there is *a fair degree of randomness in the times at which different bidders submit their bids in Korean auctions.* We will explore this further below, but to foreshadow, our model of rationally inattentive bidding is much better at predicting *how much bidders bid* but it is much harder to predict *the total number of bids and the specific times at which bidders submit their bids.*

5.3 Evaluation of model fit at level of individual bidders and auctions

We now present evidence on the ability of our model to fit bidder behavior, particularly to assess how closely the model predicts a bidder's actual sequence of bids in an auction, conditioning on the times when the bidder submitted those bids. Figure 15 compares the actual bids of B23 with the bids that the estimated model of rationally inattentive bidding would be optimal in the two auctions we used previously to illustrate the evolution of B23's beliefs about the high bid (see figure 12).

In each panel the blue stars plot the actual bids submitted by B23 in auctions 9810 and 10007, respectively. The red line is the high bid track and the dotted lines are counterfactual optimal bids predicted by the estimated model under three scenarios:

1. The blue dotted line plots the optimals bid for a bidder who has not submitted a bid previously in the auction, that is, it plots the points $\{\gamma_t(0,0)|t = 0,\ldots,120\}$. This line is the path of the *optimal first bid, for all possible instants when this first bid could be submitted in the auction.*

2. The black dotted line plots the optimal bids for a bidder *conditioning on the times and bids the bidder actually submitted in the auction.* This line is the path of bids $\{\gamma_t(b_t,h_t^c)|t = 0,\ldots,120\}$ where $\{(b_t,h_t^c)\}$ is the actual sequence of realized bids and h_t^c is a counterfactual high bid indicator that may differ from the actual high bid indicator h_t to the extent that the predicted optimal bid from the model may be above or below the high bid track (which is treated as fixed except in cases where the counterfactual optimal bid becomes the high bid in the auction).

3. The green dotted line plots the optimal bids for a bidder that *condition on the times in the auction that the bidder submitted their bids but not the values of their actual bids.* This line is the recursively calculated path of bids $\{\gamma_t(b_t^*,h_t^*)|t = 0,\ldots,120\}$ where $\{b_t^*,h_t^*)\}$ are the model's prediction of the optimal bid that the bidder should have submitted in the previous period (as opposed to the bidder's actual bid and the actual high bid indicator, (b_t,h_t)). As in case 2 above, h_t^* is a counterfactual high bid indicator that equals 1 if the previous period optimal predicted bid was the highest, or 0 otherwise, where we take the high bid track in the auction as fixed except in periods where the bidder in question has submitted the highest counterfactual bid.

To better understand these three different counterfactual optimal bid paths, consider the left hand panel of figure 15. B23 did not submit a first bid in auction 9810 until $t = 70$. The model's prediction of the first bid is about 3500, but the actual first bid submitted by B23 was nearly 5200, an "overbid" of nearly 700. If we constrain the model to bid only at the times B23 actually bid in this auction, the predicted counterfactual bid path for B23 is the green dotted line. The black dotted line can be viewed as an *off-the-equilibrium bid path.* That is, the model takes the initial overbid as a given, and then calculates what bids would be subsequently optimal given that initial "deviation". The model's prediction for the optimal second bid taking the initial overbid as given is actually slightly higher than the actual second bid by B23. Then taking B23's second bid as another deviation off the equilibrium path, we see the optimal third bid, which is also higher than B23's actual third bid.

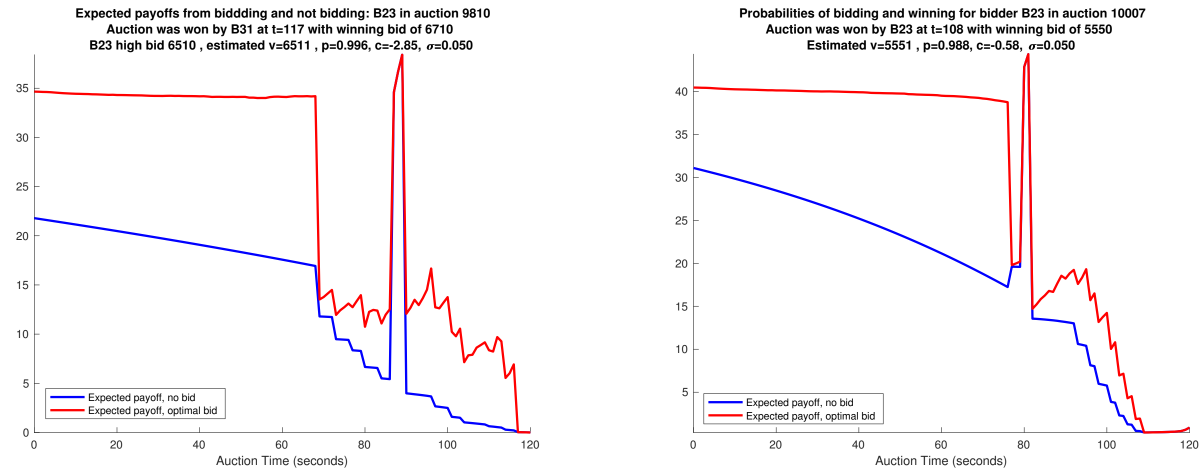
Thus, except for the initial "overbid" at $t = 70$, the model predicts the subsequent path of actual bids well. Notice also how the unrestricted optimal bid path (green dotted line) rapidly increases and catches up to nearly equal the actual bids after $t = 80$. As a result, the model almost perfectly predicts the actual final bid of 6510 by B23 at $t = 117$. The blue dotted line can be used to see what bid B23 should have optimally submitted had B23 followed a "bid sniping" strategy and waited until $T = 120$ to submit a single bid. The model predicts that the optimal snipe bid is the last point on the blue dotted line at $T = 120$, or approximately a bid of 5500. The bidder would have lost if it had followed the model's advice, but B23 also lost with the actual bids that he submitted, so one may wonder if the "overbid" at $t = 70$ has any real significance.

Now consider the right hand panel of figure 15, which compares actual and counterfactual bids for auction 10007. Here again we observe a substantial "overbid" in the first bid B23 submits $t = 80$, but the optimal bid path (green dotted line) rapidly catches up to the actual bid path so that the model nearly perfectly predicts B23's final bid of 5550 at $t = 108$. In this case B23 won the auction, and B23 would have also won this auction if he had followed the predicted optimal bidding strategy. However if B23 would have adopted a bid sniping strategy and waited until $T = 120$ to submit the optimal snipe bid of 4950, B23 would have lost auction 10007.

There is one other feature of figure 15 that is worth pointing out: *in both auctions, B23's final bid was just \$1 less than his valuation for the two cars being auctioned.* B23's actual bidding behavior (involving both frequent early bidding and initial overbidding) lead to B23 leaving these auctions with very little surplus. B23 lost auction 9810, so he received zero profit. B23 won auction 10007, but his final bit of 5550 was only \$1 less than B23's estimated valuation of $v = 5551$. This leads to the question: can it be optimal for the winner of an auction to bid so aggressively that they nearly exhaust all the surplus from participating in the auction? The bidding behavior of B23 is suggestive of optimal bidding behavior in a Japanese auction: keep bidding until the high bid reaches your valuation and then drop out.

We will return to this question shortly, but we first seek to answer a previously raised question: how important is the *timing* and *frequency* of bidding to the overall success in the auction, separately from the *level of bidding* that we already considered above? Also, how well can our model predict the total number and timing of bids? Figure 16 provides some insight into these questions by plotting the expected payoffs from the decision not

Figure 16: Payoffs to bidding and not bidding for B23 in auctions 9810 and 10007

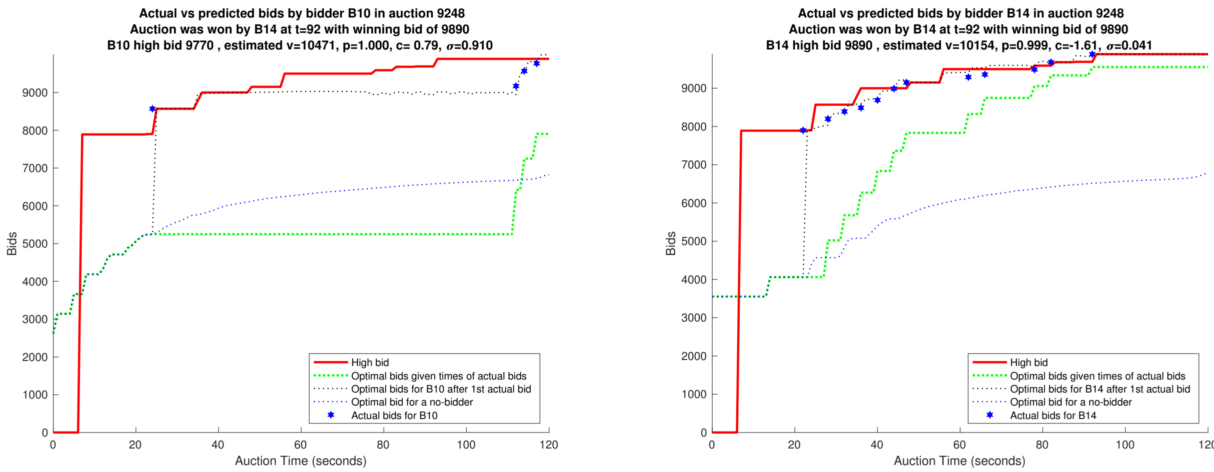


to bid, $w_t(b_t, b_t, h_t)$ and submitting an optimal bid, $w_t(\gamma_t(b_t, h_t), b_t, h_t)$ at each second in the auction. The panels plot two lines (where the value of submitting the optimal bid is the red line, and the value of not bidding is the blue line) for B23 in auctions 9810 and 10007, respectively. We see that conditional on being attentive, there is a net gain of approximately $10 from submitting a bid starting from the very first second of the auction, and this grows the longer the bidder has not yet bid in the auction.

However once the bidder has submitted a bid, the net gain in the value of updating the bid falls to nearly zero for a few periods (as illustrated by the times the red and blue lines in figure 16, around the times B23 starts "bid creeping"). This is an indication that the bidding frequency for B23 is higher than what the estimated model would predict, since the estimated model predicts after B23 submits a bid the immediate gain from further improving their bid largely disappears. The model predicts that B23 should wait for more time to elapse and the gap between the value of bidding and not bidding to increase before submitting more bids. In this sense, it is challenging for the model to explain "bid creeping" even in the absence of rational inattention.

Now we return to the question raised above: does our model of rationally inattentive bidding predict that bidders should behave as is optimal in a Japanese auction and keep bidding until they reach their valuations and then drop out? In the introduction we raised the question of whether this strategy would emerge as an optimal bidding strategy in a

Figure 17: Actual versus predicted optimal bids by B10 and B14 in auction 9248

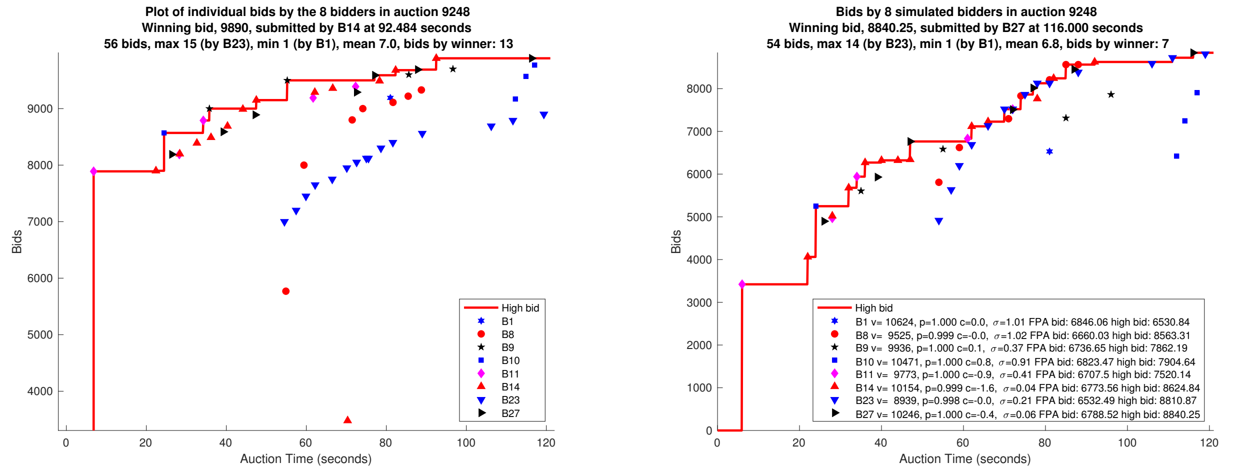


frictionless environment that would make the Korean auction strategically equivalent to both a Japanese auction and a static second price sealed bid auction.

Figure 17 illustrates that this intuition is incorrect, at least in the presence of rational inattention and bidding frictions. It plots the actual and predicted optimal bids for B10 (a losing bidder) and B14 (the winner) of auction 9248. In both cases the final bids of both bidders are well below their estimated valuations. In this auction we estimate the valuation of B10 to be $v = 10471$ whereas the valuation of the winner, B14, is estimated to be $v = 10154$. Another bidder (not shown) is B1 with an estimated valuation of $v = 10624$. If this auction was strategically equivalent to a Japanese auction, then B1 should have won the auction and paid a price of $1 more than B10's valuation, i.e. 10472. We also see that B10's final bid was 9770, well below their valuation. B14's final bid is also well below its valuation and the estimated valuations of all of the other bidders in auction 9248. Thus, we conclude that our model of bidding behavior does not predict that outcomes of Korean auctions are strategically equivalent to Japanese auctions. Part of the reason may be the bidding frictions in our model, but we will show that strategic equivalence fails to hold even when there are no bidding frictions or rational inattention among the bidders.

Comparing figures 15 and 17, we see evidence of endogenous asymmetries in the optimal bid strategies in our model. Our model predicts that bidders with low valuations are *weak bidders* who tend to bid more aggressively and in some cases nearly up to their

Figure 18: Actual versus simulated bids in auction 9248 (conditioning on actual bid times)

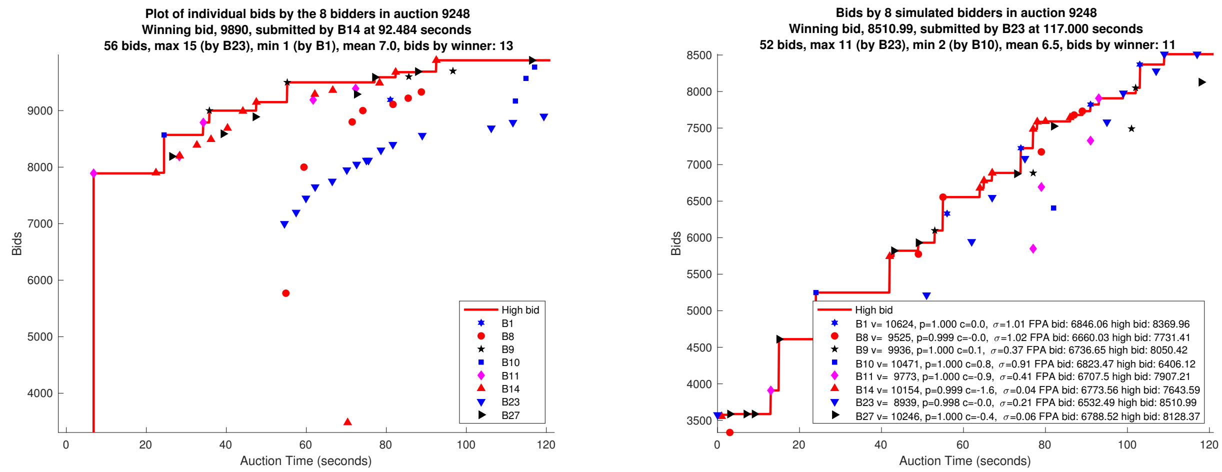


valuations for the car before stopping (as we illustrated for B23 above), but bidders with relatively high valuations take advantage of being *strong bidders* and bid less aggressively. In auction 9248 the final bids of high valuation bidders are significantly lower than their estimated valuations, so when a bidder such as B14 wins, they earn a significantly higher profit conditional on winning.

The other key point to notice from figure 17 is that we continue to find a pattern of *early overbidding* on the part of both B10 and B14. In particular, in both cases their first bids are significantly higher than what our estimated model predicts is optimal. In fact, we can use auction 9248 to show that early overbidding can have significant welfare consequences. To see this, consider figure 18 that compares the actual outcome of auction 9248 (left panel) with the predicted outcome if all bidders would have bid at exactly the same times that they actually did, but at the values that are optimal according to the estimated models for each of the 8 participating bidders.

In particular, while the actual winning bid in auction 9248 was 9890 by B14 at $t = 92$, in the counterfactual simulation of auction 9248 the winning bid is more than 1000 lower: 8840 by B27 at $t = 116$. Though B27 is only the second highest valuation bidder, this bidder wins a substantially larger profit, $10246 - 8840 = 1406$ than the predicted profit of the actual winner B14, $10154 - 9890 = 264$. These calculations and a comparison of the high bid tracks in the two panels of figure 18 suggest that the high early bidding pushes

Figure 19: Actual versus simulated bids in auction 9248 (simulated bid times)



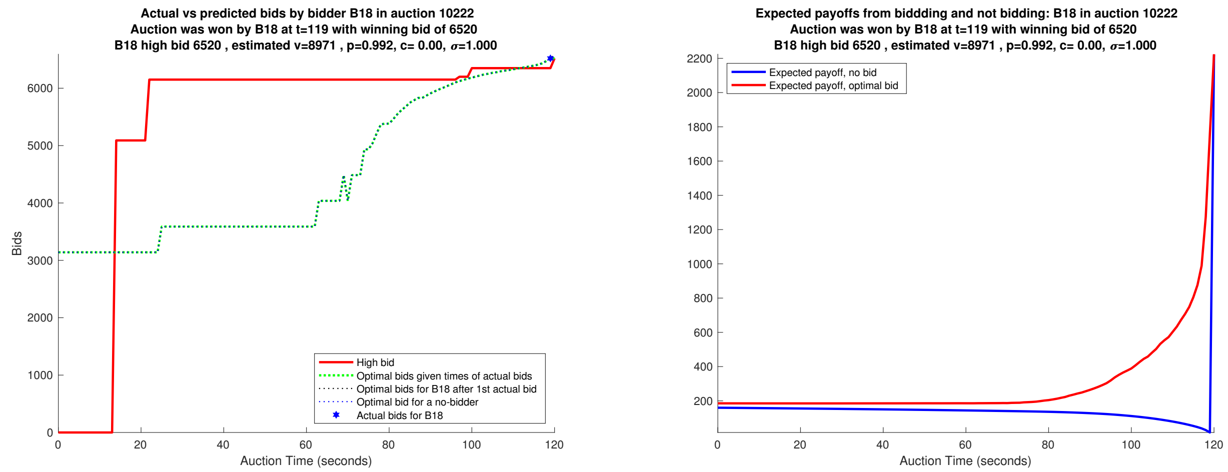
up bids over the entire course of the auction and causes the winning bidder to end up paying significantly more than the counterfactual winning bid predicted by our model.

Note that the counterfactual simulation in figure 18 is a *conditional simulation*. That is we conditioned on the times the actual bidders in auction 9248 submitted their bids and calculated what would have happened instead if the bids were chosen by our estimated bidding strategies.[11] Figure 19 presents a *full stochastic simulation*. That is, we condition only the identities of the 8 actual participating bidders, but otherwise we simulate the number of bids, the timing of these bids, and the value of the bids for all 8 bidders over the course of the the auction using the optimal bidding strategies implied by the QMLE estimates of the types τ of the 8 actual bidders who bid in this auction.

The full simulation matches several features of the actual auction, including the total number of bids and the early bidding we observe in the actual auction. However in the full simulation the early bidding starts out at significantly lower prices (nearly 8000 first bid in the actual auction vs just over 3500 in the full simulation of auction 9248), and though prices steadily rise through the remainder of the auction, the final winning price in the full simulation of auction 9248 is 9890-8511=1379 lower than the winning bid of 9890 in the actual auction. Further, from the right hand panel of figure 19 we can see for

[11]The right panel of figure 18 shows 54 bids placed in the conditional simulation of action 9248 whereas the actual number of bids submitted in this auction was 56. This is because in two instances the optimal bidding strategy chose not to improve the bid whereas the actual bidders did improve their bids at the two corresponding bid times.

Figure 20: Actual versus simulated bids for B18 in auction 10222

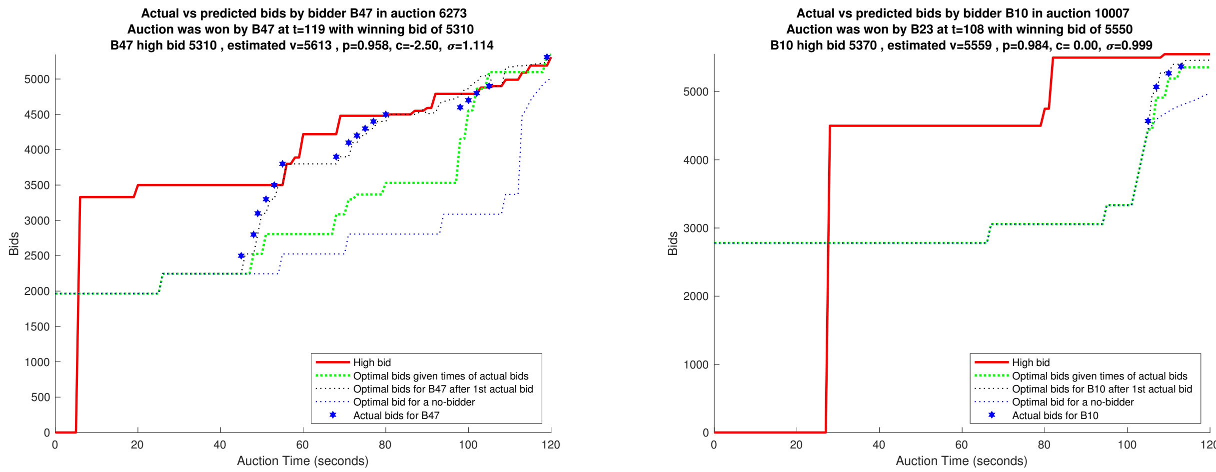


all 8 bidders, their final high bid in the full simulation is significantly lower than their estimated valuations. This is further evidence that the bidding behavior predicted by our anonymous equilibrium model of the Korean auction is not strategically equivalent to bidding in a Japanese auction where all losing bidders only exit the auction once the current high bid exceeds their valuations.

Thus, we can conclude already that our model of rationally inattentive bidding in the Korean auction can *qualitatively* explain the early bidding behavior we observe the actual auctions. However, the results so far suggest that *quantitatively* our estimated model is unable to match the magnitudes of the first bids in these auctions, a phenomenon we refer to as *early overbidding*.

Of course we can show many examples where our model does provide very accurate predictions of all bids submitted during the auction. This is particularly true for *bid snipers* since the model only needs to adjust its estimate of the valuation to match the single bid submitted by the bidder in the final few seconds of the auction. We illustrate this in figure 20. The left panel plots the predicted optimal bid trajectory and shows that it correctly predicts the actual bid submitted by B18 at $t = 119$. The right hand panel plots the expected value of submitting an optimal bid (red line) versus the value of not bidding (blue line). The two lines are close to each other for the first 80 seconds of the auction, indicating that there is no huge gain to B18 from submitting a bid early in the

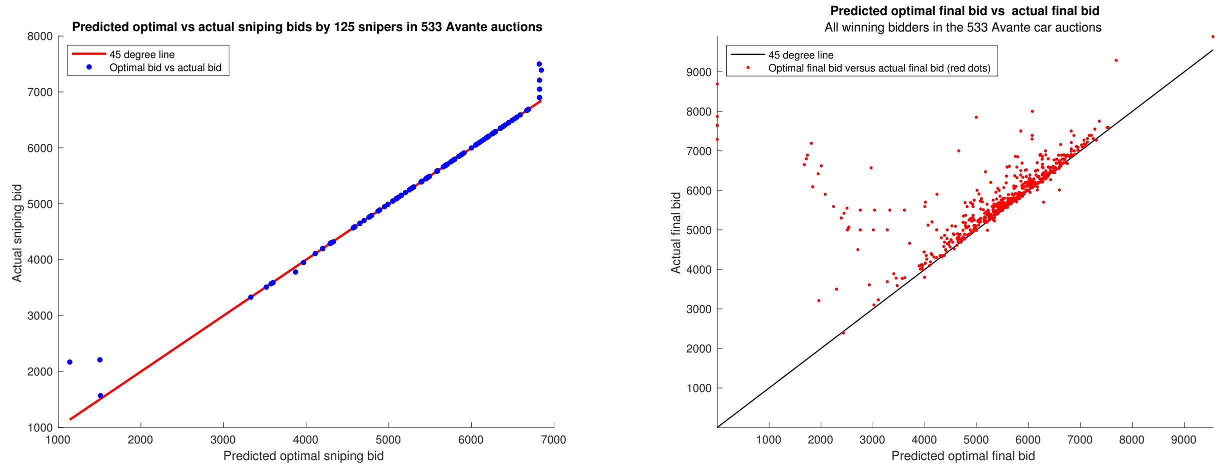
Figure 21: Actual versus simulated bids for B47 in auction 6273 and B10 in auction 10007



auction. The value of submitting a bid starts to rise rapidly in the remaining 40 seconds of the auction whereas the value of not bidding converges to zero at t approaches $T = 120$. Note that once B18 submits their optimal bid of 6520 at $t = 119$, the gain from further bidding falls to zero. In this case B18 won the auction, and it is another example where the optimal bidding strategy implies a final high bid that is significantly below B18's estimated valuation of $v = 8971$. This case is not atypical: we will show shortly that our model provide accurate predictions of bids by bid snipers, whereas it tends to underpredict the final high bid of non-snipers (early bidders).

Figure 21 shows that our model can provide accurate predictions of the final high bid even for bidders who submit multiple bids during the auction. The left panel plots the actual bids for B47 in auction 6273 and we see that the optimal final bid (green and black dotted lines) are very close to the actual final bid of 5310 submitted at $t = 119$. The right hand panel plots actual and predicted optimal bids for B10 in auction 10007. Here again, the model not only accurately predicts the final high bid submitted by B10, but the three previous bids as well. Note that in both of these cases, the final high bids by B47 and B10 are 303 and 189 below the estimated valuations, respectively. This is further evidence that our model generally does not predict that bidders should bid up to their valuation before stopping, and their actual bidding behavior can be well approximated by the predicted optimal bidding behavior from our model of rationally inattentive bidding.

Figure 22: Actual versus predicted final bids 125 bid snipers and all 533 winning bidders



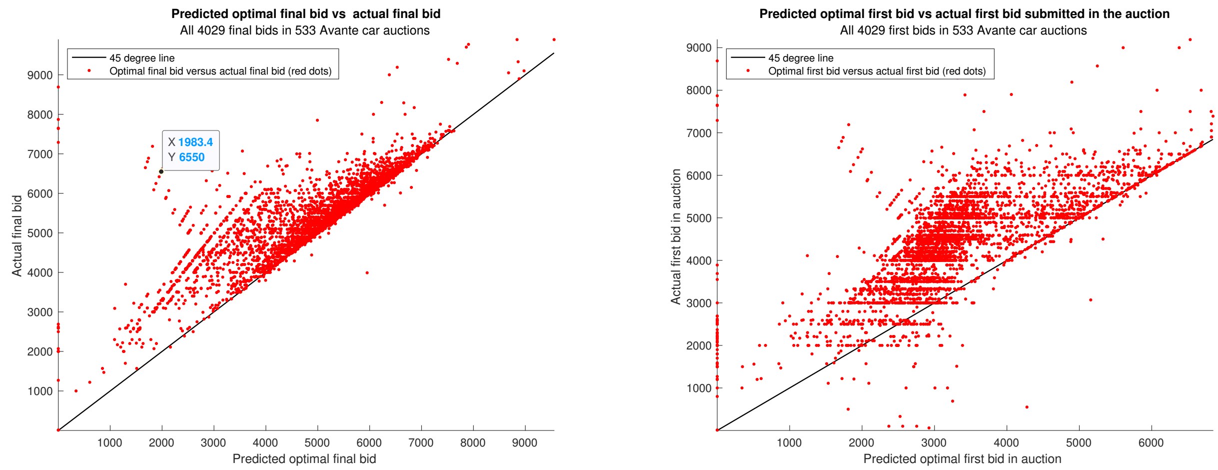
5.4 Evaluation of model over all Avante auctions

So far we have evaluated model fit at the level of individual bidders and auctions, and showed that our estimated model of rationally inattentive bidding does exhibit the early bidding behavior we observed in our data set. It also captures the heterogeneity in bidding strategies that we identified in section 2 including bid creeping and bid sniping. However we have shown examples of bids that our model is unable to predict, particularly the phenomenon of early overbidding — the tendency for the model to underpredict the magnitude of the first bid submitted during the auction.

The caveat to this is that our model is able to predict the bids submitted by snipers very well. We illustrated that via a single example in figure 20, but the left panel of figure 22 shows this is true in general by plotting predicted vs actual bids for 125 bid-snipers that we identified among the 4029 bidder/auction pairs in our data set. We classified a bid to be a sniped bid if the bidder submitted only a single bid in the auction and this bid was submitted after second $t = 118$, i.e. within the last 2 seconds of the auction. We see that except for a few cases our model nearly perfectly predicts their bids.

The right hand panel of figure 22 plots predicted vs actual final bids for all winning bidders in the 533 Avante auctions we analyzed. These dots are also highly concentrated along the 45 degree line, indicating that our model also does a very good job of predicting the final bids of non-bid-snipers. However there is a very interesting general pattern in

Figure 23: Actual versus predicted final and first bids for all 4029 bidder/auction pairs



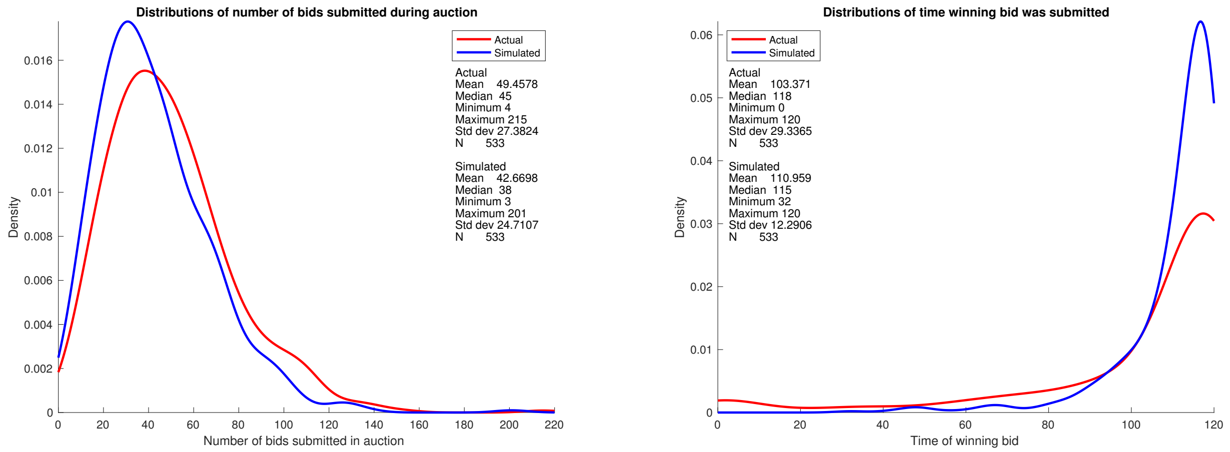
the prediction errors in final bids: the red dots lie mostly above the 45 degree line, indicating that *our model generally underpredicts the final high bids submitted by winning bidders.* In a few cases our model dramatically underpredicts the actual winning bid submitted by the winning bidder.

The pattern of underprediction emerges even more clearly in figure 23. The left panel plots predicted optimal final bid against the actual final bid for all 4029 bidder/auction pairs in our data set, while the right panel plots predicted vs optimal *first bids* for these same pairs. It is evident that the dominant pattern is one of underprediction of bids, but much nore so for first bids than final bids. This reflects the pattern we have already seen in our analysis of individual bidder data: our model substantially underpredicts the first bids but not only modestly underpredicts final bids. What is the reason for the systematic downward bias in predicted first and final bids? Shouldn't a good econometric model result in unbiased predictions? We will return to this question, but before doing so we provide further insight on our model's ability to fit the data by comparing actual vs simulated distributions of other outcomes in the 533 Avante auctions.

5.5 Comparing actual and simulated auctions of Avante cars

We conducted simulations of the 533 Avante car auctions using our estimated model of rationally inattentive bidding by conditioning on the identities of the bidders participat-

Figure 24: Actual versus simulated number of bids and time of winning bid

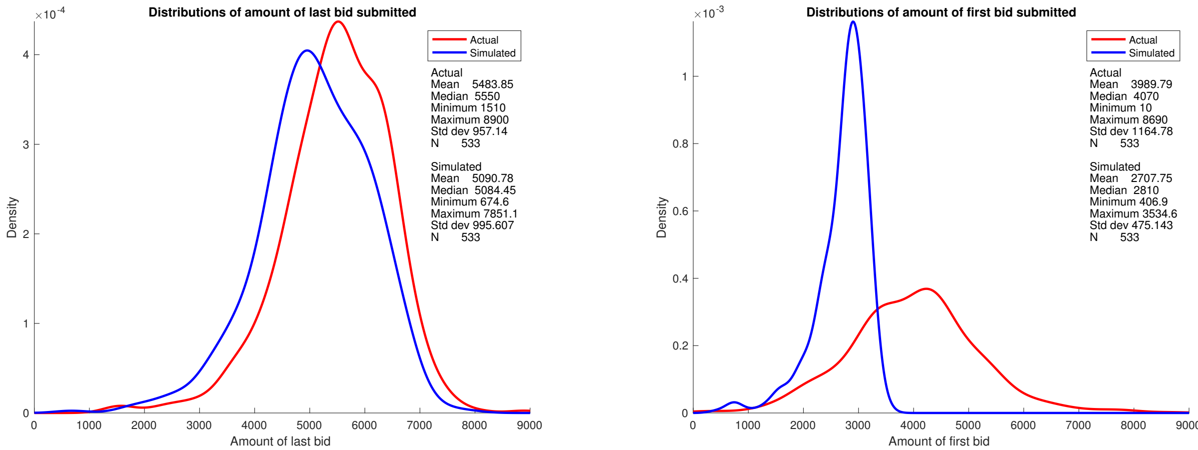


ing in each of the auctions, but otherwise all other aspects of their behavior including the number and timing of bids and the amount of bids, are the values that were dynamically simulated using the estimated bidding strategies for each of the actual bidders who bid in each auction. Overall, we used all 4029 estimated bidding strategies to simulate auction outcomes in the 533 Avante car auctions.

Our simulation strategy guarantees that the distribution of number of bidders participating in individual auctions (similar to the distribution plotted in the left panel of figure 9 but restricted to the 533 Avante auctions) in our simulations is identical to the actual distribution by construction. However by comparing simulated vs actual distributions of other outcome variables, we gain more insight into features of the data that our model may not explain so well.

The left panel of figure 24 plots simulated vs actual number of bids submitted in an auction. Our model underpredicts an average of 7 fewer bids compared to what we see in the data. The right panel compares simulated and actual distributions of the time the winning bid was submitted in the auction. Our model captures the general pattern that bidders wait until the last seconds in the auction to submit their final high bids, but the simulated bidders wait longer on average, submitting the winning bid at $t = 111$ seconds into the auction compared to $t = 103$ in the data. Thus, our model cannot fully capture the phenomenon of "high early bidding" where certain bidders submit seemingly irrationally

Figure 25: Actual versus simulated winning bids and first bids

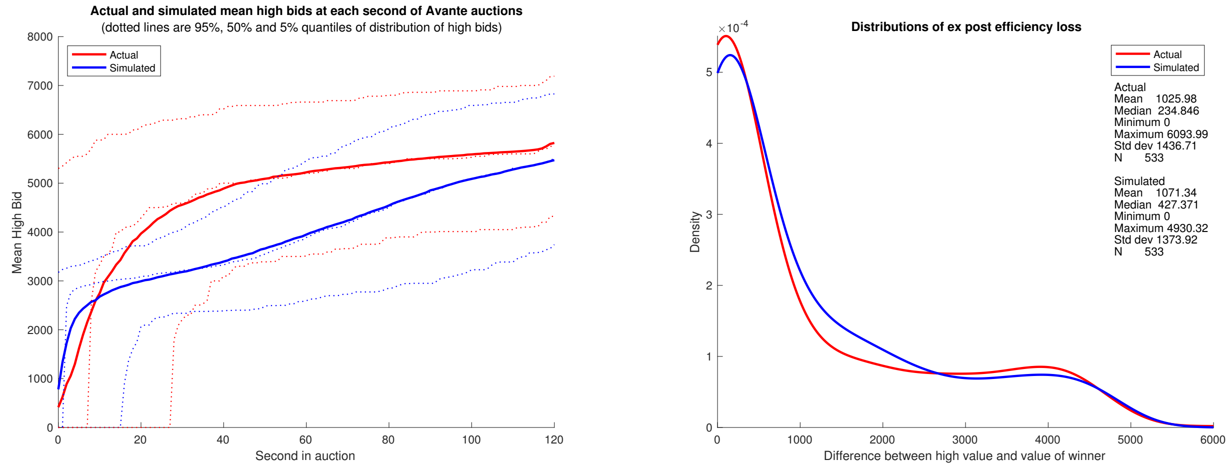


high early winning bids in the auction as we illustrated in the right panel of figure 2 in section 2.

Figure 25 plots simulated vs actual distributions of first bids and final bids in the 533 Avante auctions. We see that our model underpredicts both quantities, but the problem is much more severe for first bids than final bids. The average underprediction in final bids is $5550 - 5091 = 459$ whereas the average underprediction in first bids is $3990 - 2708 = 1282$, almost three times larger. This confirms the early overbidding phenomemon that we have already illustrated above is one of the problematic aspects of our model. However the terminology "overbidding" suggests that the problem is with the bidders: is this overbidding evidence of some form of "bidder irrationality" or a symptom of a specification error in our model? We will return to this question below.

We conclude our analysis of model fit with figure 26 which plots simulated vs actual distributions of high bid trajectories and *ex post* efficiency losses in the 533 Avante auctions. The right panel shows that the model is able to capture the distribution of *ex post* efficiency losses in these auctions, where we define the *ex post* efficiency loss to be the difference between the highest valuation among the bidders participating in the auction less the valuation of the winner. Auction mechanisms such as the Japanese auction or the second price sealed bid auction are predicted to be fully efficient, so the distribution of efficiency losses in those auctions would be predicted to be a point mass at 0. In the

Figure 26: Actual versus simulated bid trajectories and distributions of efficiency losses



Korean auctions, in contrast, we see large *ex post* efficiency losses averaging over \$1000 per auction. Considering that the average actual winning bid in these auctions is \$5824, these losses are indeed substantial.

The left panel of figure 26 plots the mean value of the high bid trajectories as well as the median trajectory and the 5% and 95% quantiles of the "coordinate distributions" of high bid paths $\{\overline{b}_t\}$ (i.e. the marginal distributions of $\overline{b}_t$ for $t = 0, 1, \ldots, 120$). Recall our definition of ϵ-anonymous equilibrium, definition 4.1 in section 4. For the model to be in an ϵ-equilibrium, bidders' beliefs about the distribution of the high bid must be sufficiently close to the actual distribution of high bids resulting from those beliefs and the implied optimal bidding behavior. That is, beliefs should be approximately self-confirming in the sense that the difference $\|\mathcal{B} - \Lambda(\mathcal{B})\|$ should be small, where Λ is the operator that maps bidders' beliefs into actual auction outcomes, i.e. the composition of the DP solution operator to compute optimal bidding strategies given beliefs, and a simulation operator that generates the implied distribution of high bid paths implied by bidders' optimal bidding strategies.

Looking at the left panel of figure 26 we conclude that regardless how we might define the distance metric $\|\cdot\|$ to measure the difference between "rational beliefs" (i.e. the ones implied by our data, which result in our first stage estimates of the distribution $\hat{\mathcal{B}}$ of high bids in these auctions) and the distribution of high bids implied by our model of rationally inattentive bidding, the difference is too large to argue that our two step estimation approach has indeed been able to explain observed bidding behavior as an ϵ-anonymous equilibrium of our model of rationally inattentive bidding. That is, we treat the discrepancy shown in the left hand panel of figure 26 as further evidence that our model is not able to explain observed bidding behavior in the Korean auctions.

5.6 Counterfactual experiments

So far we have shown that our dynamic model of rationally inattentive bidding is capable of providing a *qualitative* explanation of the early bidding behavior we observe in the Korean auctions, but it fails to provide a sufficiently accurate *quantitative* predictions of this behavior. In particular, the key feature that our model fails to capture is *early overbidding.* In this section we run several counterfactual simulations with our model to try to get further insight into the nature of the nature of early overbidding and whether it is symptomatic of a problem for bidders by leading them to bid more than necessarily to win cars in the Korean auctions.

A key question whether the model's failure to predict the early overbidding a symptom of specification error in our model, or evidence of "irrationality" on the part of bidders in these auctions? Our position is that the most likely explanation for our model's inability to fit the data is that there is some sort of *bounded rationality,* impatience, or *animal spirits* on the part of many bidders explaining their proclivity to submit a high first bid early in these auctions, or the objective of some of the bidders may not necessarily be to maximize expected profit.[12]

To convince the reader that early overbidding is problematic for bidders we conduct a first counterfactual experiment where we calculate counterfactual profits for each of the bidders in the auction under the assumption that instead of using the bidding strategies they actually employed, that their bids were those predicted by our estimated model

[12] We do not take a stand as to whether early overbidding is a result of an inability to perfectly calculate optimal bids consistent with the bounded rationality notion of Simon [1957] or due to the effect of emotions or "animal spirits" as emphasized by Akerlof and Shiller [2009] or other reasons (risk aversion or desire to win an auction for the sake of winning), though exploring these different possible explanations is an interesting direction for future research.

of rationally inattentive bidding. In the counterfactual we loop through all 4029 bidder/auction pairs and for each bidder in each of these cases we calculate the profit the bidder would have earned from a *unilateral deviation* in their bidding strategy. The unilateral deviation is determined by the estimated optimal bidding strategy for the bidder in each auction the bidder partcipates in, but we also *condition on the times each bidder actually bid* just as we did in our evaluations of actual bids vs model predictions in section 5.3. The bids of all other bidders are treated as fixed in these counterfactual simulations, except that if counterfactual bid path submitted by the bidder in question becomes the high bid at any point (or conversely if counterfactual bid is lower and no longer the high bid at that point), then we adjust the high bid track appropriately. This implies that some of the other bidders (whose bids are treated as fixed in this counterfactual simulation) may emerge as winners of the auction in cases where the bidder in question won the auction but the counterfactual deviation bid path is lower than the actual one causing the bidder in question to lose the auction. We credit the new (human) winner of the counterfactual auction with additional profits in such cases.

We can think of the profits earned by bidders under this counterfactual as the predicted profits they would have earned if they used the strategy predicted by our estimated model to calculate the magnitude of their bids at the times they chose to submit them. In what follows we distinguish actual outcomes when the bidding is done entirely by the *human bidder* versus the counterfactual outcomes when the values of bids are calculated by the algorithmic *deviation bidder.* Table 2 summarizes the outcomes of the 4029 counterfactual simulations. We see the most common outcome is the both the human bidder and deviation bidder lost the auction, and the next most common outcome is that both the human bidder and deviation bidder won the auction. Note the asymmetry in the remaining two outcomes: it is much more frequent for the human bidder to win and the deviation bidder to lose the same auction than vice versa.

Table 3 displays the effects of the counterfactual simulation on the profits and final bids of selected bidders in these auctions (standard errors for the mean of each quantity in the table is presented below in parentheses). The column "Win rates" compares the fraction of auctions that each bidder actually won (Act) vs the fraction won under the

Table 2: Summary of counterfactual simulation 1

1. The human bidder won the actual auction and the deviation bidder won the counterfactual auction. *297 counterfactual simulated auctions, or 7.4% of the 4029 counterfactual simulations, had this outcome.*

2. The human bidder won the actual auction but the deviation bidder lost the counterfactual auction. *236 counterfactual simulated auctions, or 5.9%, had this outcome.*

3. The human bidder lost the actual auction, but the deviation bidder won the the counterfactual auction. *16 counterfactual simulated auctions, or 0.4%, had this outcome.*

4. The human bidder lost the actual auction and the deviation bidder lost the counterfactual auction. *3480 counterfactual simulated auctions, or 86.4%, had this outcome.*

counterfactual simulation (CF). It is interesting to notice the high degree in heterogeneity across different bidders in terms of the number of auctions they bid in, and the variation in their win rates, ranging from a low of 5.4% for B47 to a high of 37.1% for B3. We also see big variation in the estimated expected profits (see column Mean Profits, All Auctions, Act). Expected profits range from a low of \$5 for B15 to a high of \$567 for B58.[13]

The bidders generally win fewer auctions under the counterfactual bidding strategy. In a number of cases the win rates from the counterfactual deviation strategy are only half as high as the actual win rates. However note that in all cases the deviation bidders earn higher profit conditional on winning than the actual bidders (see the columns All Auctions, Auctions won). We also see that in all cases the deviation bidders bid less on average both in the auctions they won and in the auctions they lost. In many cases the deviation bidders bid signficantly less than their human counterparts.

Under an optimal bidding strategy a bidder chooses a bid to solve the trade-off between the probability of winning and the expected profits conditional on winning. Table 3 suggests that for most of the bidders, they have erred by choosing final bids that are too high, sacrificing higher profits conditional on winning in favor of a higher win rate. There are several bidders (B1, B3, B6, B10, B14 and B28) whose actual expected profits are higher

[13]In separate analysis not shown here in the interest of space, we do not find any relationship between "bidder experience" (as proxied by the number of auctions a bidder participated) and expected profits.

Table 3: Counterfactual 1: actual vs simulated auction outcomes for selected bidders

Bidder	**Win rates**		**Mean profits**				**High bids**			
Number of			All auctions		Auctions won		All auctions		Auctions won	
Auctions	Act	CF	Act	CF	Act	CF	Act	CF	Act	CF
1	20.2	11.6	207	126	1023	1086	5671	5311	5821	5575
163			(4)	(3)	(34)	(53)	(6)	(7)	(29)	(56)
3	37.1	11.4	282	205	760	1797	6182	5542	6330	5390
35			(15)	(19)	(49)	(297)	(21)	(30)	(50)	(507)
5	5.5	4.8	16.0	17.4	294	363	5146	4695	5955	6027
146			(0.5)	(0.6)	(23)	(31)	(7)	(9)	(64)	(82)
6	13.5	6.1	33	24	246	390	5708	5288	5925	6083
163			(0.7)	(0.7)	(10)	(21)	(5)	(6)	(30)	(54)
8	10.8	3.8	23	43	212	1126	4935	4123	5168	4476
315			(0.4)	(0.9)	(10)	(86)	(3)	(4)	(32)	(116)
9	9.9	5.0	144	153	1449	3086	5448	4799	5528	4968
323			(2)	(2)	(55)	(78)	(3)	(3)	(25)	(80)
10	12.3	5.7	68	66	550	1161	4879	4282	5566	5096
227			(2)	(2)	(33)	(112)	(5)	(6)	(30)	(110)
11	9.7	5.5	29	58	301	1049	5344	4695	5707	5124
361			(0.3)	(1.1)	(8)	(69)	(3)	(4)	(20)	(64)
14	15.6	7.2	46	33	292	454	4690	4199	5270	5412
167			(1)	(1)	(10)	(21)	(7)	(8)	(46)	(52)
15	6.8	6.8	4.9	216	72	3165	4664	3986	6370	3277
44			(.5)	(27)	(14)	(1380)	(21)	(28)	(517)	(950)
16	15.2	10.1	95	161	624	1589	5428	5040	5368	4610
158			(2)	(5)	(29)	(120)	(6)	(7)	(39)	(97)
17	14.9	7.7	59	66	398	857	5528	5162	5781	5597
181			(1)	(2)	(13)	(68)	(5)	(6)	(27)	(53)
23	21.6	14.9	103	132	479	891	5383	5106	5504	5290
148			(3)	(4)	(30)	(63)	(5)	(7)	(24)	(41)
28	15.6	10.7	122	117	783	1094	5641	5230	6158	5593
225			(2)	(2)	(23)	(49)	(4)	(5)	(22)	(51)
32	33.7	19.8	48	230	141	1166	5828	5499	5982	5269
86			(2)	(12)	(9)	(128)	(9)	(14)	(26)	(115)
36	6.0	3.8	15	76	242	2011	5675	5087	6506	4964
132			(0.6)	(6)	(29)	(752)	(6)	(9)	(139)	(572)
47	5.4	5.9	9.2	12.8	170	216	5661	5363	5955	5863
203			(0.4)	(0.3)	(25)	(15)	(4)	(5)	(95)	(79)
58	28.9	26.7	567	720	1961	2700	5854	5475	6500	5580
45			(27)	(32)	(118)	(135)	(26)	(33)	(80)	(164)

than what their deviation bidder earns.[14] However for most of the bidders expected profits are significantly higher from following the counterfactual deviation bidding strategy. Over all 4029 bidder/auction pairs, mean actual profits were \$79 per bidder per auction, and \$601 per auction won. However in the counterfactual simulation, average profits were \$97 per bidder per auction, or \$1247 per auction won. The counterfactual strategies achieved higher expected profits by reducing bids. This lowered the win rate to 7.8% over all bidders, compared to 13.2% on average under their actual bidding strategies.

We take this finding as strong evidence in favor of the hypothesis that the reason our model fails to explain the early overbidding by the bidders in these auctions is that most of the bidders are bidding suboptimally. Specifically they are overbidding relative to what our model predicts is optimal. The first counterfactual demonstrates that most bidders in out sample are not using "best response" bidding strategies — in the process of trying to model their bidding behavior we have constructed alternative bidding strategies that enables them to earn significantly higher expected profits. This should not be possible if the bidders were using equilibrium bidding strategies.

However the reader may feel a bit queasy about the nature of the "counterfactual" outcome in the first counterfactual simulation since we have treated the bids of all other human bidders other than the deviation bidder as fixed (subject only to the minor caveats discussed above). To allay these concerns, we conducted a second counterfactual experiment where all bidders in each of the 533 auctions use our optimal estimated bidding strategies, $\{\gamma_t(b_t, h_t, \hat{\tau}_{j,a}) | t = 0, \ldots, 120\}$ where $\hat{\tau}_{j,a}$ is our estimate of the type of bidder j who participated in auction a. To compare with counterfactual 1 we condition on the actual times each of the human bidders submitted bids for each of the 4029 bidder/auction pairs. However other than this, *the bids of all bidders are those chosen by the model in Counterfactual 2.*

[14]Though the counterfactual prediction is supposed to be "optimal" the optimality is relative to our Assumption 1 that all bidders have common beliefs $\mathcal{B}$ and participate in "generic" auctions for Avante rental cars. However if certain bidders self-select into participation for certain types of Avante cars (e.g. screening on the μ parameter in our conditional independent private values Assumption 1 where bidder valuations are drawn according to the conditional density $f(v|\mu)$), it is possible that such bidders have "superior beliefs" for the subset of auctions they actually participate in relative to the "average beliefs" $\hat{\mathcal{B}}$ for all 533 Avante auctions we have estimated. If so, these bidders would have an informational advantage that enables them to earn higher profits than optimal bidding strategies solved with average beliefs $\hat{\mathcal{B}}$.

Table 4: Counterfactual 2: actual vs simulated auction outcomes for selected bidders

Bidder	**Win rates**		**Mean profits**				**High bids**			
Number of			All auctions		Auctions won		All auctions		Auctions won	
Auctions	Act	CF	Act	CF	Act	CF	Act	CF	Act	CF
1	20.2	23.9	207	340	1023	1422	5671	5253	5821	5609
163			(4)	(5)	(34)	(34)	(6)	(7)	(29)	(27)
3	37.1	20.0	282	222	760	1112	6182	5439	6330	5461
35			(15)	(17)	(49)	(128)	(21)	(29)	(50)	(222)
5	5.5	3.4	16.1	13.8	294	402	5146	4586	5955	5642
146			(0.5)	(0.5)	(23)	(29)	(7)	(9)	(64)	(101)
6	13.5	19.0	33	83	246	435	5708	5205	5925	6011
163			(0.7)	(1.4)	(10)	(11)	(5)	(6)	(30)	(25)
8	10.8	10.8	23	52	212	478	4935	4048	5168	4766
315			(0.4)	(0.6)	(10)	(8)	(3)	(4)	(32)	(24)
9	9.9	8.0	144	220	1449	2732	5448	4754	5528	5474
323			(2)	(3)	(55)	(62)	(3)	(3)	(25)	(23)
10	12.3	11.9	68	80	550	673	4879	4199	5566	5279
227			(2)	(1)	(33)	(27)	(5)	(6)	(30)	(27)
11	9.7	8.9	29	49	301	554	5344	4595	5707	5404
361			(0.3)	(0.5)	(8)	(12)	(3)	(4)	(20)	(29)
14	15.6	16.2	46	112	292	694	4690	4038	5270	4312
167			(1)	(2)	(10)	(13)	(7)	(8)	(46)	(35)
15	6.8	4.5	4.9	44.5	72	981	4664	3921	6370	4184
44			(0.5)	(5.1)	(14)	(295)	(21)	(29)	(517)	(697)
16	15.2	17.7	95	165	624	931	5428	4930	5368	5036
158			(2)	(3)	(29)	(21)	(6)	(7)	(39)	(33)
17	14.9	16.0	59	128	398	796	5528	5097	5781	5611
181			(1)	(2)	(13)	(30)	(5)	(6)	(27)	(31)
23	21.6	29.0	103	224	479	772	5383	4956	5504	5124
148			(3)	(4)	(30)	(19)	(5)	(7)	(24)	(20)
28	15.6	13.3	122	100	783	752	5641	5143	6158	5707
225			(2)	(1)	(23)	(15)	(4)	(5)	(22)	(24)
32	33.7	26.7	48	135	141	506	5828	5408	5982	5748
86			(2)	(4)	(9)	(23)	(9)	(13)	(26)	(35)
36	6.0	7.6	15	83	242	1094	5675	4998	6506	5948
132			(0.6)	(3)	(29)	(133)	(6)	(9)	(139)	(66)
47	5.4	7.9	9.2	25.5	170	323	5661	5271	5955	5711
203			(0.4)	(0.5)	(25)	(13)	(4)	(5)	(95)	(45)
58	28.9	26.7	567	597	1961	2240	5854	5432	6500	6056
45			(27)	(27)	(118)	(119)	(26)	(33)	(80)	(87)

Table 4 displays the results of this second counterfactual comparison for selected bidders. We now see that in many cases counterfactual win rates are as high or higher than the actual win rates, and significantly higher than under the first counterfactual. With only a few exceptions expected profits conditional on winning the auction are higher under this counterfactual compared to actual. As a result, we also find significant increases in expected profits over all auctions bidders participated in relative to the actual expected profits. There are only 3 bidders (B3, B5 and B28) for whom counterfactual expected profits are lower than we estimated these bidders actually earned. Of course, this is due to the fact that all bidders typically submit lower bids under counterfactual 2, whereas in counterfactual 1 only a single "deviation bidder" lowered their bids while the bids of all other bidders in the auction were held fixed at their actual values.

Over all 4029 bidder/auction pairs mean actual profits were \$79 per bidder per auction, and \$601 per auction won. However in the second counterfactual simulation, average profits were \$123 per bidder per auction, or \$932 per auction won. The counterfactual strategies achieved higher expected profits by reducing bids. However the overall average win rate is by definition the same as the actual average win rate equal to $533/4029 = .132$ under the second counterfactual simulation. We conclude that most of the bidders in these auctions would be better off if they were to use our estimated bidding strategies to serve as their "algorithmic traders" to bid on their behalf in these auctions. Our conclusions are robust to relaxing the assumption that bidders only bid at the times their human "master" chose to bid in these auctions. In another counterfactual where we let all of the algorithmic bidders choose the number, timing and amounts of their bids, we reach similar conclusions as we have in counterfactual 2 above. In particular, average counterfactual profits in this case are \$115 per bidder per auction, and \$875 per auction won. Average winning bids in the 533 auctions are about the same (\$5398) as in counterfactual 2 (\$5345), and both are significantly lower than the actual value (\$5824).

We conclude this section with a final counterfactual simulation where all of the bidders in the 533 Avante auction were to use *frictionless optimal trading strategies* to serve as their agents and bid on their behalf in all auctions. This will also enable us to answer the conjecture raised in the introduction, namely, is the frictionless bidding outcome in the

Korean outcome strategically equivalent to the Japanese auction? That is, will the highest valuation bidder win each auction but at a price just $1 above the valuation of the second highest bidder?

To implement counterfactual 3 we re-solved all 4029 bidder/auction-specific bidding strategies, using the estimate of each bidder's valuation for the car in each auction they participated in, but setting the other inattention and bid friction parameters to 0. Thus, the types of the bidders in this simulation are of the form $\hat{\tau}_{j,a} = (\hat{v}_{j,a}, 0, 0, 0)$ where $\hat{v}_{j,a}$ is the QMLE estimate of the valuation of bidder j in auction a.

Results from the counterfactual 3 simulations are presented in table 5 for the same selected set of bidders as we displayed in the corresponding tables for counterfactuals 1 and 2. We see that in this case, expected profits (over all auctions each bidder practicipated in) are strictly higher under the counterfactual compared to actual. This is due to two effects: 1) expected profits earned by the frictionless bidders conditional on winning are significantly higher than counterfactual 2 where bidders were subject to inattention and bidding frictions, and 2) the win rates are generally as high or higher compared to win rates of bidders subject to inattention and bidding frictions.

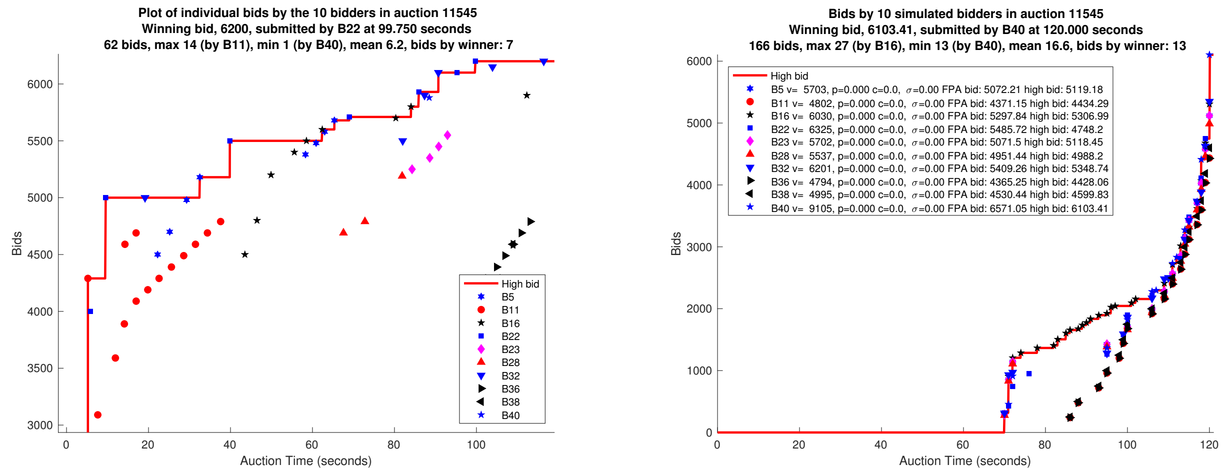
What about the conjecture that bidding by frictionless bidders in the Korean auction should be strategically equivalent to a Japanese auction? We show this conjecture does not hold using a specific case in figure 27 as a counterexample. The left panel plots the actual data on bids in auction 11545 whereas the right panel shows the result of counterfactual simulation outcome where all of 8 human bidders are replaced by their frictionless "bidding agents". Paradoxically, there is no early bidding in the frictionless bidder simulation. We can also see that the bidder with the highest valuation, B40 $v = 9105$ wins the auction when all bidders use frictionless bidding strategies, whereas the actual auction was won by B22 who is estimated to have the second highest valuation of $v = 6325$. Thus, while it is true that we happened to get an *ex post* efficient outcome for auction 11545, the winning price of 6103 is lower than the second highest valuation, 6325.

The left panel of figure 28 compares the actual high bid trajectories to those that arise under frictionless bidding. Paradoxically, we see there is no early bidding in the frictionless bidding scenario. While a typical bidder submits a much larger number of bids when

Table 5: Counterfactual 3: actual vs simulated auction outcomes for frictionless bidders

Bidder	Win rates		Mean profits				High bids			
Number of			All auctions		Auctions won		All auctions		Auctions won	
Auctions	Act	CF	Act	CF	Act	CF	Act	CF	Act	CF
1	20.2	22.0	207	432	1023	1957	5671	5169	5821	5387
163			(4)	(7)	(34)	(41)	(6)	(5)	(29)	(21)
3	37.1	31.4	282	582	760	1850	6182	5326	6330	5611
35			(15)	(33)	(49)	(123)	(21)	(18)	(50)	(55)
5	5.5	4.1	16.1	42.9	294	1044	5146	4718	5955	5109
146			(0.5)	(1.9)	(23)	(171)	(7)	(5)	(64)	(97)
6	13.5	8.6	33	78	246	912	5708	5098	5925	5206
163			(0.7)	(1.7)	(10)	(29)	(5)	(4)	(30)	(54)
8	10.8	7.6	23	71	212	930	4935	4523	5168	4837
315			(0.4)	(1.0)	(10)	(31)	(3)	(2)	(32)	(30)
9	9.9	32.2	144	783	1449	2433	5448	5057	5528	5544
323			(2)	(4)	(55)	(16)	(3)	(2)	(25)	(7)
10	12.3	13.2	68	137	550	1034	4879	4536	5566	5076
227			(2)	(2)	(33)	(19)	(5)	(4)	(30)	(18)
11	9.7	9.4	29	125	301	1332	5344	4852	5707	5158
361			(0.3)	(1.4)	(8)	(37)	(3)	(2)	(20)	(22)
14	15.6	15.0	46	118	292	787	4690	4300	5270	4875
167			(1)	(2)	(10)	(24)	(7)	(5)	(46)	(25)
15	6.8	6.8	4.9	216	72	3165	4664	3986	6370	3277
44			(.5)	(27)	(14)	(1380)	(21)	(28)	(517)	(950)
16	15.2	10.1	95	161	624	1589	5428	5040	5368	4610
158			(2)	(5)	(29)	(120)	(6)	(7)	(39)	(97)
17	14.9	7.7	59	66	398	857	5528	5162	5781	5597
181			(1)	(2)	(13)	(68)	(5)	(6)	(27)	(53)
23	21.6	14.9	103	132	479	891	5383	5106	5504	5290
148			(3)	(4)	(30)	(63)	(5)	(7)	(24)	(41)
28	15.6	10.7	122	117	783	1094	5641	5230	6158	5593
225			(2)	(2)	(23)	(49)	(4)	(5)	(22)	(51)
32	33.7	19.8	48	230	141	1166	5828	5499	5982	5269
86			(2)	(12)	(9)	(128)	(9)	(14)	(26)	(115)
36	6.0	3.8	15	76	242	2011	5675	5087	6506	4964
132			(0.6)	(6)	(29)	(752)	(6)	(9)	(139)	(572)
47	5.4	5.9	9.2	12.8	170	216	5661	5363	5955	5863
203			(0.4)	(0.3)	(25)	(15)	(4)	(5)	(95)	(79)
58	28.9	26.7	567	720	1961	2700	5854	5475	6500	5580
45			(27)	(32)	(118)	(135)	(26)	(33)	(80)	(164)

Figure 27: Actual vs counterfactual outcomes with frictionless bidding in auction 11545

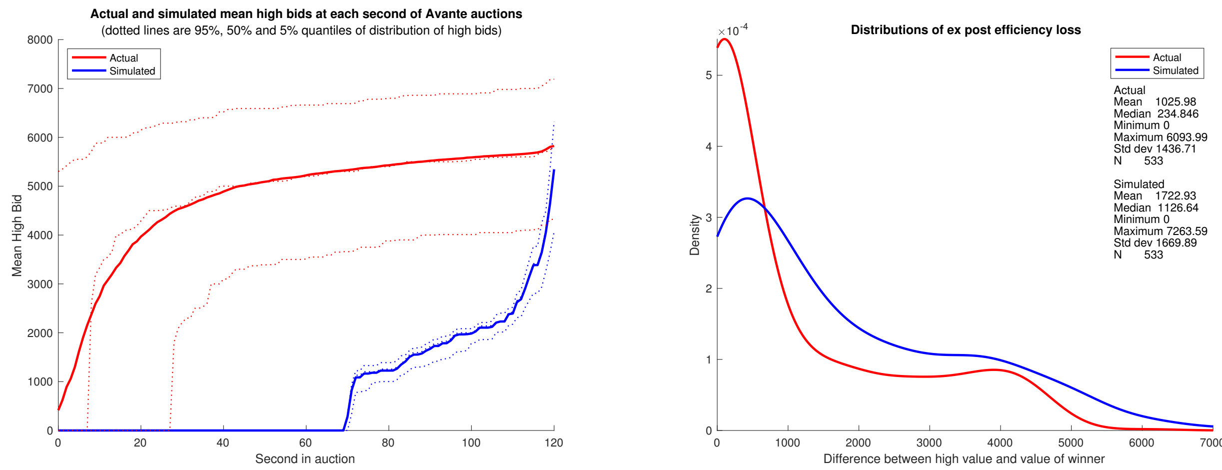


bidding is frictionless, the bids are concentrated in the last half of the auction, and rise most rapidly in the final 5 seconds of the auction, giving rise to the convex shape of the mean high bid trajectory in the frictionless case, which contrasts with the concave shaped mean bid trajectory under actual bidding in these auctions.

The right panel of figure 28 compares the distribution of *ex post* efficiency losses under frictionless bidding to the losses that actually occur, where we presume bidding frictions are fairly significant. In particular, the mean efficiency loss in the frictionless bidding counterfactual, 1723, is nearly 70% than the mean efficiency loss that we estimate occurs under actual bidding. This is also a paradoxical finding since we would expect that reducing bidding frictions should enable bidders to approximate the outcome of a Japanese auction, resulting in the highest valuation bidder winning the auction but paying $1 over the valuation of the second highest bidder and no *ex post* efficiency losses. As we see, this conjectured outcome is not what actually happens under frictionless bidding.

We conclude this section with table 6 which compares expected revenues to the seller (the rental car company) under several different scenarios. The first column presents the actual mean winning price in the 533 Avante auctions. The second column presents the expected winning price under the assumption that all bidders behave according to our estimated model of rationally inattentive bidding. The expected winning price is significantly less in this case, which both indicates that our model is unable to fit the data,

Figure 28: Bid trajectories and efficiency losses for frictionless bidders



but it also illustrates the issue of overbidding in the actual auctions we have discussed above. Our explanation for the rejection of our model is that bidders in the Korean auction overbid, and thus we are pointing to a rejection of the assumption of *rationality* as the primary explanation for the rejection of our model.

The third column of table 6 presents the expected winning price under the assumption of frictionless bidding. It is lower than the expected winning price in the presence of bidding frictions, which seems intuitive. With frictionless bidding the bidders are better able to learn what the high bid is and avoid early overbidding, enabling the winners to pay less to win the auction.

The fourth and fifth columns are the expected winning bids if the auctions were organized as static first price and second price sealed bid auctions, respectively. We see that if the bidders behaved according to our model, the rental company would be better off switching to a first price sealed bid auction compared to the more complicated dynamic Korean auction mechanism. However due to the early overbidding that occurs in the Korean auctions, the rental company actually earns more money from the latter auctions.

The final column shows the expected winning price if the rental company adopted a static second price sealed bid auction mechanism. This results in the highest expected revenue of the options considered here. If bidders bid rationally in this auction, the outcome will also be *ex post* efficient, and it appears that the seller can capture most of the efficiency losses that occur under the Korean auction in the form of significantly higher auction revenue. The seminal paper by Myerson [1981] proves that with in a world of independent private values and no collusion, the optimal auction mechanism is a second-price sealed bid auction with an appropriate reserve price. Our empirical findings are in line with Myerson's theoretical predictions, though we have not attempted to calculate an optimal reserve price and predict how that would affect the expected winning bid, since the optimal reserve price depends on the distribution of valuations, $f(v|\mu)$ which in turn depends on the individual characteristics of each car being sold, μ, which in turn depends on variables we do not observe.

Table 6: Expected winning bids under different scenario and auction mechanisms

Korean auction Actual	Korean auction with frictions	Korean auction without frictions	First price sealed bid	Second price sealed bid
5824 (2)	5511 (2)	5345 (1)	5789 (2)	6276 (3)

Of course our conclusions in table 6 depend on the assumption of no collusion occuring. However if there is collusion it is not clear that the informational restrictions in the dynamic Korean auction are more effective in curbing collusion than the even stronger informational restrictions inherent in anonymized versions of a static first price or second price sealed bid auction, especially with an appropriately calculated reserve price. These auctions are much simpler to implement and bid in, and convey even less information to bidders than the Korean auction does. The main argument for adopting the more complex Korean auction mechanism seems to be to exploit the "irrational exuberance" of the bidders. However there is also extensive laboratory evidence of irrational bidding in second price auctions that leads to overbidding in that format as well.[15] Thus, it is not clear that the Korean auction is the best choice of auction mechanism, even considering the possibility of collusion and irrationality on the part of bidders.

[15] See, for example, Kagel et al. [1987] who find laboratory subjects overbid by 11% relative to their valuations in second price sealed bid auctions. They conclude that "Although we observe persistent overbidding in second-price auctions, clear economic forces are at work limiting the size of the overbid." (p. 1302).

6 Conclusion

In this paper we have analyzed data from a new informationally restricted online ascending bid auction mechanism that we call the Korean auction, since it was designed by the executive of a Korean car rental company to defeat suspected collusion by professional bidders participating in the previous auction mechanism it employed, a standard physical "open outcry" auction. In the new online auction system bidders cannot observe the identities of other bidders or the bids they submit: the only information they receive is a binary indicator of whether they are the current high bidder or not.

A natural way to model bidding behavior in such auctions is to employ the rational, game-theoretic approach that has been so successfully applied in the literature on auctions. In our context, this would argue for modeling bidding behavior as a perfect Bayesian equilibrium (PBE) outcome. We showed that if the Korean auction has a "hard close" there will always exist an *uninformative PBE* where all bidders adopt bid sniping strategies that involve submitting a single bid at the last second of the two minute auction. This PBE would then be strategically equivalent to the equilibrium of a static first-price sealed bid auction.

However we argued that it is unclear whether *informative PBE* exist with early bidding reflecting the incentive for "price discovery" prior to the last instant in the auction. We provided a simple two period, two bidder example where the *only* PBE is the uninformative PBE with no early bidding. Whether informative PBE can be shown to exist in other versions of the Korean auction with more bidders and time periods is an open question, but we argued that given present technology, it is computationally infeasible to try to solve for an informative PBE (assuming one exists) and, if so, determine if the bidding behavior it implies is consistent with the prevalent early bidding we observe in the Korean auctions.

Instead of giving up on a structural analysis of our rich dataset on bidding in these auctions, we have introduced a computationally tractable model of rationally inattentive bidding under a relaxed definition of equilibrium known as an ϵ-anonymous equilibrium. We have shown that this model can explain the early bidding behavior we observe, at least in a qualitative sense. We assume that in any given instant of the auction, there is a probability p that the bidder is distracted and not paying attention to the auction. The inattention is rational in the sense that bidders are aware of their inattention and compensate by bidding earlier and higher in the instants where they are not distracted and able to pay attention to the auction. Thus, rational inattention has effects akin to introducing a "soft close" in the Korean auction, which is known to create an incentive for earlier bidding compared to bid sniping as Roth and Ockenfels [2002] have shown.

We developed a new fixed effects quasi-maximum likelihood estimation algorithm and estimated the 4 dimensional type parameters $\tau = (v, c, p, \sigma)$ of our structural model of bidding behavior using a nested numerical solution approach where we explicitly calculate optimal bidding strategies for each bidder using numerical dynamic programming. In all, we estimated 4029 bidder/auction-specific types for the set of all bidders who participated in 533 auctions for a generic type of rental car, Avante XD 1.6L. The key component of the type for each bidder is v, their private valuation of the car being auctioned. The other 3 components, (c, p, σ) are parameters governing *bidding frictions* that include the probability of being inattentive p and the psychological cost or benefit of submitting a bid c, and a scale parameter σ for an Extreme value distribution governing the idiosyncratic component of bidding costs.

Even in the presence of frictions, our model is a fundamentally rational model of bidding behavior, where we use rational inattention and modest bidding frictions in an attempt to explain the bidding behavior we observe. We find that introducing these modest frictions into the model takes us a great distance toward explaining observed bidding behavior, but we conclude that not even our model is capable of explaining the *early overbidding* that is prevalent in our data set. That is, though our econometric model is able to do a reasonable job of predicting the final bids submitted in these auctions, it systematically underpredicts the size of the initial bid submitted by most bidders in these auctions.

We argue that this early overbidding is symptomatic of boundedly rational behavior on the part of the professional bidders in this auction and constitute *bidding mistakes* that end up pushing prices higher later in the auction, causing the human bidders to pay a higher price than our estimated model predicts if all bidders behaved according to our rationally inattentive model of bidding. We demonstrated this via a series of counterfactual experiments where we used our estimated bidding strategies to serve as algorithmic "bidding agents" that can recommend bids to the professional human bidders, or take over completely for them and bid on their behalf in the auctions they participated in.

We showed that our optimal bidding algorithms consititute successful deviation strategies that increase the expected profits from participating in the auction. The gain in profits occurs when we consider only unilteratal deviations by a single bidder in each auction who follows the advice of our optimal bidding algorithm, but is even larger when all bidders adopt the bidding algorithms since the algorithms end up pushing down the winning prices in the auctions by a greater amount compared to the case where only unilateral deviations by a single bidder in each auction are considered. We "compromised" our algorithmic bidders by subjecting them to the same inattention and bidding friction parameters that we estimated for the human bidders, and despite this, we found that prices in these auctions would have been lower and bidders would have been better off if they had been employed our algorithmic bidding strategies to bid on their behalf.

We also conducted a counterfactual simulation with completely frictionless bidding, which we achieve by simulating auction outcomes when the bidding friction parameters in the bidder type vectors are set to 0, i.e. we simulated the auctions where all bidders have types of the form $\tau = (v, 0, 0, 0)$. One might conjecture that with completely frictionless bidding the Korean auction would be strategically equivalent to a Japanese auction where bidders observe a common exogenously rising price clock and have a dominant strategy to remain in the auction until this price clock exceeds their valuation for the object being auctioned (known as "straightforward bidding"). We showed that the equilibrium outcome with frictionless bidding in the Korean auction is not strategically equivalent to a Japanese auction. Instead it results in lower expected prices compared to equivalent auctions with bidding frictions, but the frictionless outcome is subject to *ex post* inefficiency, unlike the Japanese auction outcome where the highest valuation bidder always wins the auction at a price \$1 higher than the valuation of the second highest valuation bidder.

We conclude that winning auction prices would be significantly lower and expected profits earned by the bidders would be significantly higher if they were all to use our frictionless optimal bidding strategies as their bidding agents in the Korean auctions. We see this as not only providing convincing evidence against the hypothesis of bidder rationality, but also against the hypothesis of bidder collusion. Indeed, if there was a collusive bidding ring operating during the Korean auction, one of their feasible collusive strategies would be to use bidding algorithms to serve as their bidding agents in these auctions. Our results show that doing this would have resulted in lower prices and higher expected profits than what we actually observed.

Thus, even though a direct comparison of auction prices before and after the rental executive switched from the company's traditional open outcry auctions to the new informationally restricted online auctions did not reveal a significant increase in prices in the new regime consistent with his belief that the Korean auction thwarted collusion, our structural estimation results provide no reason to suspect the executive was wrong in his belief either.

However it remains unclear to us whether the more complex dynamic informationally restricted Korean auction mechanism is superior to much simpler auction mechanisms such as a simple static second price auction with an appropriately determined reservation price. If the latter auction was run anonymously, it reveals even less information to bidders than the Korean auction does and thus seems no worse in terms of its susceptibility to collusion. We have shown that a static second price sealed bid auction without a reservation price generates higher expected revenue than the Korean auction. It would be interesting try to convince the rental car company to run an experiment using the second price auction mechanism to see how it performs in practice. However as we noted in the introduction, the company switched back to open outcry auctions in 2007, but ones run at a professional wholesale auction house in Seoul. Even after paying a 10% commission to the auction house, Cho et al. [2014] found that net prices of cars auctioned at the auction house are as high as those sold online via the Korean auction mechanism.

A caveat is that we have adopted the conditionally independent private values framework for our analysis (see Assumption 1 of section 4). We believe this is an appropriate framework for used car auctions, but it may not be approriate for oil lease auctions, for example, where there is a significant "common value" component. In such cases the linkage principle of Milgrom and Weber [1982] provides a rationale for information revelation during the auction. The Korean auction may be a way to balance the desire for gradual information relevation to improve collective learning and "price discovery" against the need to anonymize bidders to thwart collusion. We discussed recommendatons of this nature that have been employed with apparent success in FCC spectrum auctions, where "activity rules" discourage informational free-riding and encourage early bidding in the auction to promote price discovery, see Wilson [2002]. The Korean auction does not need activity rules due to the incentives the informational restrictions create to bid early to learn the high bid.

Another interesting agenda for future research to try to extend our model and relax some of our assumptions to see if a behavioral version of our model might be able to explain the early overbidding we observe. For example we hypothesize that bidders are not just maximizing expected profit from participating in the auction, but they may also have to fulfill a "quota." This puts an independent value on simply winning an auction, even if means "overpaying". Since the auctions are run in a back-to-back format over successive days, there might be ways to test such hypotheses by conditioning on the number of cars of a given type already won by the bidder in previous auctions on the same day or previous day. If there is a quota-effect, we should observe a greater tendency for overbidding in auctions later in the sequence compared to earlier ones where the bidder has higher option value of meeting their quota if they fail to win in earlier auctions.

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Appendix A: Proof of Theorem 1

This appendix provides the proof of the Theorem in section 4.3. Our proof is by induction. We will show inductively starting in the final period T that a) the value functions and optimal bid functions are independent of the high bid indicator, h_t, $t = 0, 1, \ldots, T$, and b) for all $t < T$ the optimal bid function $\gamma_t(b_t) = 0$, whereas in the last period, the optimal bid function $\gamma_T(b_T)$ is the anonymous equilibrium bid function for an anonymous static first price sealed bid auction, where beliefs are given by $\lambda_{T+1}(b|0)$, i.e. bidders' *ex ante* anonymous equilibrium beliefs about the high bid in an anonymous version of a static sealed bid auction.

Consider the optimal bid function $\gamma_T(b_T)$ in the last period T where we assume the bidder is paying attention and observes the high bid so far in the auction, b_T and considers what final bid $\gamma_T(b_T)$ to submit. It is easy to see that if $b_T > v$, the optimal bid is zero, $\gamma_T(b_T) = 0$. Otherwise, the optimal final bid is defined by

$$\gamma_T(b_T) = \underset{b \geq b_T}{\operatorname{argmax}} (v - b)\lambda_{T+1}(b|b_T). \tag{29}$$

In general the optimal bid function will involve a strict improvement over the current high bid, i.e. $\gamma_T(b_T) > b_T$ and involves "shading" i.e. bidding below the bidder's true valuation v

$$\gamma_T(b_T) = v - \frac{\lambda_{T+1}(\gamma_T(b_T)|b_T)}{\lambda'_{T+1}(\gamma_T(b_T)|b_T)}, \tag{30}$$

where $\lambda'_{T+1}(b|b_T)$ is the derivative of the CDF $\lambda_{T+1}(b|b_T)$ with respect to its first argument, b. But it is possible that the constraint $b \geq b_T$ is binding, so the optimal bid equals b_T. But since time priority is binding in the auction, matching the current high bid will result in zero probability of winning the auction. A necessary condition for $\gamma_T(b_T) = b_T$ is that $\frac{\partial}{\partial b} w_T(b_T, b_T) < 0$, where $w_T(b, b_T) = (v - b)\lambda_{T+1}(b|b_T)$ is the terminal period bid-specific value function. It follows that $\gamma_T(b_T)$ is given by

$$\gamma_T(b_T) = \begin{cases} 0 & \text{if } b_T > v \\ b_T & \text{if } b_T \leq v \text{ and } \gamma_T(b_T) = b_T \text{ for } \gamma_T(b_T) \text{ in (29)} \\ \gamma_T(b_T) & \text{if } b_T \leq v \text{ and } \gamma_T(b_T) > b_T \text{ for } \gamma_T(b_T) \text{ in (30)} \end{cases} \tag{31}$$

Substituting the optimal decision rule $\gamma_T(b_T)$ into the decision-specific value function $w_T(b, b_T) = (v-b)\lambda_{T+1}(b|b_T)$ we obtain the *ex ante* value function $W_T(b_T)$ given by

$$W_T(b_T) = \begin{cases} 0 & \text{if } v < b_T \\ 0 & \text{if } b_T \le v \text{ and } \gamma_T(b_T) = b_T \text{ for } \gamma_T(b_T) \text{ in (29)} \\ w_T(\gamma_T(b_T), b_T) & \text{if } b_T \le v \text{ and } \gamma_T(b_T) > b_T \text{ for } \gamma_T(b_T) \text{ in (30).} \end{cases} \tag{32}$$

Note that $w_T(b_T, b_T) = 0$ due to time priority rules: just matching an existing high bid of b_T will not succeed in winning the auction, so $\lambda_{T+1}(b_T|b_T) = 0$.

The next step is to show that $W_T'(b_T) \le 0$ for any b_T, where $W_T'(b_T)$ is the derivative of W_T at b_T. Using formula (30) it is easy to see that the result holds when $v < b_T$ since $W_T(b_T) = 0$ in this region. If $b_T \le v$ and $\gamma_T(b_T) = b_T$ (the binding constraint case), $W_T'(b_T) = 0$ for the same reason. Now consider the final case in the last equation of (32) where $\gamma_T(b_T) > b_T$ and hence we have an interior optimum. In this region we have

$$W_T'(b_T) = \frac{\partial}{\partial b} w_T(\gamma_T(b_T), b_T) + \frac{\partial}{\partial b_T} w_T(\gamma_T(b_T), b_T) \tag{33}$$

$$= 0 + (v - b_T)\nabla_{b_T}\lambda_{T+1}(b_T|b_T). \tag{34}$$

In the last equation of (34) we appealed to the Envelope Theorem, so the first term equals zero when $b = \gamma_T(b_T) > b_T$. By Assumption 8, $\lambda_{T+1}(b|b_T)$ is stochastically increasing in b_T, so we have $\nabla_{b_T}\lambda_{T+1}(b_T|b_T) \le 0$. It follows that $W_T'(b_T) \le 0$ in this region as well.

Now we define the bid-specific value function at period $T-1$, $w_{T-1}(b, b_{T-1})$, which is the expected payoff at time $t = T-1$ from submitting a bid of b given that the current high bid at the start of $T-1$ is b_{T-1}. Let $p \in [0,1]$ be the probability that the bidder is inattentive (and therefore unable to bid) at time T. But as of the perspective at time $T-1$, our definition of $w_{T-1}(b, b_{T-1})$ presumes that the bidder is attentive at $T-1$ and thus able to calculate an optimal bid. Let $\gamma_{T-1}(b_{T-1})$ be the optimal bid function at period $T-1$. We want to show that either $\gamma_{T-1}(b_{T-1}) = 0$ or $\gamma_{T-1}(b_{T-1}) = b_{T-1}$, so in either case either the bidder does not bid at $T-1$ or does not submit an improved bid at $T-1$, and with time priority for the high bid, either case can be viewed as equivalent to not bidding. Clearly if $v < b_{T-1}$ we have $\gamma_{T-1}(b_{T-1}) = 0$. Now suppose that $v \ge b_{T-1}$ so there is a potential for profiting by submitting an improved bid, $b > b_{T-1}$.

At period $T-1$ when $v \geq b_{T-1}$ the bid-specific value function $w_{T-1}(b, b_{T-1})$ takes the form

$$
\begin{aligned}
w_{T-1}(b, b_{T-1}) &= \left[p w_T(b,b) + (1-p) W_T(b)\right] \lambda_T(b|b_{T-1}) \\
&\quad + \int_b^\infty \left[p w_T(b',b') + (1-p) W_T(b')\right] \lambda_T'(b'|b_{T-1}) db'
\end{aligned} \tag{35}
$$

The term in the first line of equation (36) is the expected utility in the case where the bid the bidder submitted, b, is the high bid submitted at $T-1$. With probability p the bidder will be inattentive at T and thus unable to make a final bid, and the value in this case is just $w_T(b,b)$. But with probability $(1-p)$ the bidder is attentive and able to calculate and submit an optimal bid $\gamma_T(b)$, and this has value $W_T(b)$ as calculated above in equation (30). The second line of equation (36) covers the case where the high bid submitted at $T-1$ exceeds the bidder's bid b.

Taking the *partial derivative* of $w_{T-1}(b, b_{T-1})$ with respect to b we obtain

$$
\begin{aligned}
\frac{\partial}{\partial b} w_{T-1}(b, b_{T-1}) &= \left[p \nabla_b w_T(b,b) + (1-p) W_T'(b)\right] \lambda_T(b|b_{T-1}) \\
&\quad + \left[p w_T(b,b) + (1-p) W_T(b)\right] \lambda_T'(b|b_{T-1}) \\
&\quad - \left[p w_T(b,b) + (1-p) W_T(b)\right] \lambda_T'(b|b_{T-1}) \\
&= \left[p \nabla_b w_T(b,b) + (1-p) W_T'(b)\right] \lambda_T(b|b_{T-1}).
\end{aligned} \tag{36}
$$

However it is clear that the last line of the expression for $\frac{\partial}{\partial b} w_{T-1}(b, b_{T-1})$ is non-positive since we have shown above that $\nabla_b w_T(b,b) \leq 0$ and $W_T'(b) \leq 0$. It follows that $w_{T-1}(b, b_{T-1})$ is maximized at $b = b_{T-1}$, so $\gamma_{T-1}(b_{T-1}) = b_{T-1}$. We summarize this as

$$
\gamma_{T-1}(b_{T-1}) = \begin{cases} 0 & \text{if } v < b_{T-1} \\ b_{T-1} & \text{if } v \geq b_{T-1} \end{cases}, \tag{37}
$$

and the corresponding value function $W_{T-1}(b_{T-1})$ is given by

$$
W_{T-1}(b_{T-1}) \equiv w_{T-1}(\gamma_{T-1}(b_{T-1}), b_{T-1}) = \begin{cases} 0 & \text{if } v < b_{T-1} \\ w_{T-1}(b_{T-1}, b_{T-1}) & \text{if } v \geq b_{T-1} \end{cases}. \tag{38}
$$

Now we calculate $W_{T-1}'(b_{T-1})$ and show that $W_{T-1}'(b_{T-1}) \leq 0$. This obviously holds when $v < b_{T-1}$ where $W_{T-1}(b_{T-1}) = 0$. Now consider the case where $v \geq v_{T-1}$. In this case $W'_{T-1}(b_{T-1}) = \nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$ where $\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$ is the *total derivative* of $w_{T-1}(b_{T-1}, b_{T-1})$ defined in equation (36). Calculating this, we have

$$\begin{aligned}
\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1}) &= \nabla_{b_{T-1}} \Big([p w_T(b_{T-1}, b_{T-1}) + (1-p) W_T(b_{T-1})]\, \lambda_T(b_{T-1}|b_{T-1}) \\
&\quad + \int_{b_{T-1}}^{\infty} \left[p w_T(b', b') + (1-p) W_T(b')\right] \lambda'_T(b'|b_{T-1}) \Big) \\
&= \left[p \nabla_{b_{T-1}} w_T(b_{T-1}, b_{T-1}) + (1-p) W'_T(b_{T-1})\right] \lambda_T(b_{T-1}|b_{T-1}) \\
&\quad + [p w_T(b_{T-1}, b_{T-1}) + (1-p) W_T(b_{T-1})]\, \lambda'_T(b_{T-1}|b_{T-1}) \\
&\quad + [p w_T(b_{T-1}, b_{T-1}) + (1-p) W_T(b_{T-1})]\, \nabla_{b_{T-1}} \lambda_T(b_{T-1}|b_{T-1}) \\
&\quad - [p w_T(b_{T-1}, b_{T-1}) + (1-p) W_T(b_{T-1})]\, \lambda'_T(b_{T-1}|b_{T-1}) \\
&\quad + \nabla_{b_{T-1}} \left[\int_{b_{T-1}}^{\infty} \left[p w_T(b', b') + (1-p) W_T(b')\right] \lambda'_T(b'|b_{T-1}) db' \right]. \\
&= \left[p \nabla_{b_{T-1}} w_T(b_{T-1}, b_{T-1}) + (1-p) W'_T(b_{T-1})\right] \lambda_T(b_{T-1}|b_{T-1}) \\
&\quad + [p w_T(b_{T-1}, b_{T-1}) + (1-p) W_T(b_{T-1})]\, \nabla_{b_{T-1}} \lambda_T(b_{T-1}|b_{T-1}) \\
&\quad + \nabla_{b_{T-1}} \left[\int_{b_{T-1}}^{\infty} \left[p w_T(b', b') + (1-p) W_T(b')\right] \lambda'_T(b'|b_{T-1}) db' \right]. \\
&\leq 0.
\end{aligned} \tag{39}$$

The final inequality in (39) follows from the third equation for $\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$ and noting the first term of that third equation is non-positive since we have already shown that $\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1}) \leq 0$ and $W_T(b_{T-1}) \leq 0$. The second term of the last equation in (39) follows from Assumption 8 and the property that $\lambda_{T-1}(b_{T-1}|b_{T-1})$ is stochastically monotone in the conditioning argument of the CDF $\lambda_{T-1}(b|b_{T-1})$ for any $b \geq b_{T-1}$. The final term in the third equation for $\nabla_{b_{T-}} w_{T-1}(b_{T-1}, b_{T-1})$ in (39) also follows from Assumption 8, since the stochastic monotonicity of $\lambda_{T-1}(b|b_{T-1})$ in b_{T-1} implies that the conditional expectation of any non-increasing function of b will be non-increasing in the conditioning argument, b_{T-1}.

We have now completed a full induction step. We proved that if $w_T(b, b)$ and $W_T(b)$ are non-increasing functions of b then $\gamma_{T-1}(b_{T-1})$ equals 0 or b_{T-1} depending on whether $v < b_{T-1}$ or $b \geq b_{T-1}$. Then we showed that $w_{T-1}(b, b)$ and $W_{T-1}(b)$ are also non-increasing functions of b. A similar argument as used above to derive $\gamma_{T-1}(b_{T-1})$ implies that $\gamma_{T-2}(b_{T-2})$ either equals 0 or b_{T-2}, depending on whether $v < b_{T-2}$ or $v \geq b_{T-2}$.

It follows by induction that these properties hold for all time periods $t = 0, 1, \ldots, T-1$. However at the start of the auction the high bid is by definition equal to 0, i.e. $b_0 = 0$. This implies that $\gamma_t(b_t) = 0$ for $t = 0, 1, \ldots, T-1$. The only positive bid that is made is at the last period T where bidders, using the same inductive logic, will realize no other bidder will place a positive bid before T, so the distribution of the high bid will be $\lambda_{T+1}(b|0)$. Thus, all bidders will bid-snipe and the anonymous equilibrium of the ascending bid auction will be strategically identical to the anonymous equilibrium of a static first price sealed bid auction as claimed in Theorem 1.

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