Unobserved Heterogeneity in Matching Games

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Agents in two-sided matching games vary in characteristics that are unobservable in typical data on matching markets. We investigate the identification of the distribution of unobserved characteristics using data on who matches with whom. In full generality, we consider many-to-many matching and matching with trades. The distribution of match-specific unobservables cannot be fully recovered without information on unmatched agents, but the distribution of a combination of unobservables, which we call unobserved complementarities, can be identified. Using data on unmatched agents restores identification.

I. Introduction

Matching games model the sorting of agents to each other. Men sort to women in marriage on the basis of characteristics such as income, schooling, personality, and physical appearance. Upstream firms sort to down-

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stream firms on the basis of the product qualities and capacities of each of the firms. This paper studies identification in transferable utility matching games. We study one-to-one matching, many-to-many matching, and matching with trading networks (e.g., Koopmans and Beckmann 1957; Gale 1960; Shapley and Shubik 1972; Becker 1973; Crawford and Knoer 1981; Kelso and Crawford 1982; Sotomayor 1992, 1999; Hatfield et al. 2013).

There has been recent interest in the structural estimation of transferable utility matching games (e.g., Dagsvik 2000; Choo and Siow 2006; Levine 2009; Siow 2009; Yang, Shi, and Goldfarb 2009; Graham 2011; Baccara et al. 2012; Chen and Song 2013; Galichon and Salanié 2015; Akkus, Cookson, and Hortaçsu 2016; Chiappori and Salanié 2016; Mindruta, Moeen, and Agarwal 2016; Chiappori, Salanié, and Weiss 2017; Ahlin 2018; Fox 2018). The papers we cite are unified in estimating some aspect of the preferences of agents in a matching game from data on who matches with whom as well as the observed characteristics of agents or of matches. The sorting patterns in the data combined with assumptions about equilibrium inform the researcher about the structural primitives in the market.

Some empirical papers structurally estimating matching games estimate how various structural or equilibrium objects, such as payoffs or preferences, are functions of the characteristics of agents or matches observed in the data while imposing strong assumptions about the distribution of the characteristics of agents or matches unobserved in the data. For example, Choo and Siow (2006) use a one-to-one transferable utility matching game to study the marriage market in the United States and estimate how the equilibrium payoffs of men for women vary by the ages of the man and the woman. Choo and Siow assume (if interpreted in terms of single-agent preferences rather than total match production) that men have identical preferences for women in the same measured demographic class (age 40–45 white females, say), ruling out men having preferences for characteristics of women such as personality and physical attractiveness that are not measured in the data. Fox (2018) uses a model of matching with trades to examine matching between automotive assemblers (downstream firms) and car parts suppliers (upstream firms) and asks how observed specialization measures in the portfolios of car parts sourced or supplied contribute to agent profit functions. In Fox's study, each trade's product quality is not directly measured and is only indirectly inferred.

Ackerberg and Botticini (2002) provide empirical evidence using instrumental variables that farmers and landlords sort on scalar unobservables such as risk aversion and monitoring ability, without formally estimating a matching game or the distribution of the scalar unobservables.

As the consistency of estimation procedures for matching games depends on assumptions on the unobservables, empirical conclusions might be more robust if the estimated matching games allow richly specified distributions of unobserved agent heterogeneity. This paper investigates what data on the sorting patterns between agents can tell us about the distributions of unobserved agent-specific and match-specific characteristics relevant for sorting.

We study the nonparametric identification of distributions of unobserved match heterogeneity and agent heterogeneity in two-sided, transferable utility matching games and the more general framework of matching with trading networks (Hatfield et al. 2013). We allow for this empirically relevant heterogeneity using data on only observed matches (who matches with whom), not data from, say, an online dating site on rejected profiles (Hitsch, Hortaçsu, and Ariely 2010) or on equilibrium transfers, such as wages in a labor market or prices in a hedonic equilibrium model of, say, housing (e.g., Brown and Rosen 1982; Ekeland, Heckman, and Nesheim 2004; Chiappori, McCann, and Nesheim 2010; Heckman, Matzkin, and Nesheim 2010; Eeckhout and Kircher 2011). Transfers are often confidential data in firm contracts and are rarely observed in marriage data (e.g., Choo and Siow 2006; Fox 2018). Our identification arguments use data on many markets with finite numbers of agents in each, following Fox (2010). This paper contributes to the small literature on the nonparametric (allowing unknown objects that are restricted to lie only in spaces of functions) identification of transferable utility matching games (Fox 2010, 2018; Graham 2011).

We first consider a baseline model, which is stripped down to focus on the key problem of identifying distributions of heterogeneity from sorting data. In our baseline transferable utility matching game, the primitive that governs sorting is the matrix that collects the production values for each potential match. The production level of each match is additively separable in observable and unobservable terms. The observable term is a match-specific characteristic. The unknown primitive is therefore the distribution (representing randomness across markets) of the matrix that collects the unobservable terms in the production of each match in a market. We call this distribution the distribution of match-specific unobservables. Match-specific unobservables nest many special cases, such as agent-specific unobservables.

We first show that the distribution of match-specific unobservables is not identified in a one-to-one matching game with data on who matches with whom but without data on unmatched or single agents. We provide two main theoretical results and many extensions. Our first main theoretical result states that the distribution of a change of variables of the unobservables, the distribution of what we call *unobserved complementari-*

ties, is identified. We precisely define unobserved complementarities below. Our identification proof works by tracing the joint (across possible matches in a market) cumulative distribution function of these unobserved complementarities using the match-specific observables. We also show that knowledge of the distribution of unobserved complementarities is sufficient for computing many counterfactual assignment probabilities. Our identification result for unobserved complementarities follows the emphasis on complementarities in observed characteristics in the literature on transferable utility matching games (e.g., Becker 1973). Our second main theoretical result says that the distribution of the primitively specified, match-specific unobservables is actually identified when unmatched agents are observed in the data.

Empirical researchers might be tempted to specify a parametric distribution of match-specific unobservables. Our theoretical results together suggest that estimating a matching model with a parametric distribution of match-specific unobservables will not necessarily lead to credible estimates without using data on unmatched agents, as a more general nonparametrically specified distribution is not identified. Also, we present an example of a multivariate normal distribution of match-specific characteristics whose parameters are not parametrically identified.

Our baseline model imposes additive separability between unobservables and observables in the production of a match. We show that we can simply condition on other observables not directly used in the previous identification arguments, relaxing additive separability in a setting with more observables. In another extension, we identify fixed-across-markets but heterogeneous-within-a-market coefficients on the match-specific characteristics used in the baseline model. This relaxes the assumption that the match-specific characteristics enter the production of each match in the same manner. Still another extension considers models in which key observables vary at the agent and not the match level and enter match production multiplicatively.

We discuss an extension to a model of matching with trades (Hatfield et al. 2013). In matching with trades, the same agent can make so-called trades as both a buyer and a seller and can have complicated preferences over the set of trades. An individual trade generalizes a match in that a trade can list other specifications, such as the number of board seats given to an investor (Uetake and Watanabe 2016).

As mentioned above, Choo and Siow assume (if interpreted in terms of single-agent preferences rather than total match production) that men have identical preferences for women in the same measured demographic class (and similarly for the preferences of women for men). Galichon and Salanié (2015) and Chiappori et al. (2017) call this preference restriction "separability." This preference restriction has also been used in Graham (2011) and Fox (2018). We show that assuming that men have identical preferences for women in the same measured demographic class has testable restrictions on the allowable distributions of unobserved complementarities.

There are many other modeling differences between our paper and the literature on transferable utility matching games following the approach of Choo and Siow (2006). We use data on many markets with finite numbers of players and different realizations of observables and unobservables in each market; the Choo and Siow approach has been applied to large markets, each market having a continuum of agents. We require at least one continuous, observable characteristic per match or per agent; the Choo and Siow literature in some cases uses only a finite number of observable characteristic values. The production functions corresponding to these finite observables are usually recoverable without further functional form assumptions; we require a particular match or agent characteristic to enter match production additively separably.

In simultaneous work, Agarwal (2015) and Agarwal and Diamond (2016) discuss the identification and estimation of distributions of scalar agent characteristics in nontransferable utility matching games with restrictions on preferences in order to ensure a unique stable match. Our work on transferable utility games allows match-specific characteristics without further restrictions on their joint distribution.

II. Baseline Identification Results

This section analyzes a two-sided, one-to-one matching game with transferable utility. This section imposes that all agents must be matched in order to focus purely on the identification coming from agent sorting and not from the decision to be single. We also begin the section with a simple space of explanatory variables. We change these assumptions later.

A. Baseline Model

We use the terms "agents" and "firms" interchangeably. Upstream firm u and downstream firm d can form a match $\langle u, d\rangle$. The monetary transfer from d to u is denoted as $t_{u,d}$; we will not require data on the transfers. The *production* or *total profit* from a match $\langle u, d\rangle$ is

$$z_{u,d} + e_{u,d}, \tag{1}$$

where $z_{u,d}$ is a scalar match-specific characteristic observed in the data and $e_{u,d}$ is a scalar match-specific characteristic unobserved in the data but observable to all firms in the matching game. One match-specific characteristic $z_{u,d}$ is the distance between the headquarters of firms u and d. Distance $z_{u,d}$ is always positive and likely enters match production with a negative sign; we can always construct a new regressor $\tilde{z}_{u,d} = -z_{u,d}$ that enters with a positive sign. The match-specific, unobserved characteristic $e_{u,d}$ generalizes special cases such as $e_{u,d} = e_u \cdot e_d$, where e_u and e_d are unobserved upstream and downstream firm characteristics, respectively. We allow a match-specific coefficient on each $z_{u,d}$ and, separately, use only agent-specific explanatory variables below.

We can more primitively model production for a match $\langle u, d\rangle$ as the sum of the profit of u and the profit of d, where the possibly negative transfer $t_{u,d}$ between d and u enters additively separably into both individual profits and therefore cancels in their sum. Utilizing some notation briefly to clarify, if the profit of u at some market outcome is $\pi^u_{u,d} + t_{u,d}$ and the profit of d is $\pi^d_{u,d} - t_{u,d}$, then the production of the match $\langle u, d\rangle$ is equal to $\pi^u_{u,d} + \pi^d_{u,d} = z_{u,d} + e_{u,d}$. Only production levels matter for the matches that form, and we will not attempt to identify upstream firm profits separately from downstream firm profits, except in the extension to matching with trades in Section VII.

There are N firms on each side of the market. The term N can also represent the set $\{1, \ldots, N\}$. The matrix

$$\begin{pmatrix} z_{1,1} + e_{1,1} & \cdots & z_{1,N} + e_{1,N} \\ \vdots & \ddots & \vdots \\ z_{N,1} + e_{N,1} & \cdots & z_{N,N} + e_{N,N} \end{pmatrix}$$

describes the production of all matches in a market, where the rows are upstream firms and the columns are downstream firms. Let

$$E = \begin{pmatrix} e_{1,1} & \cdots & e_{1,N} \\ \vdots & \ddots & \vdots \\ e_{N,1} & \cdots & e_{N,N} \end{pmatrix}, \quad Z = \begin{pmatrix} z_{1,1} & \cdots & z_{1,N} \\ \vdots & \ddots & \vdots \\ z_{N,1} & \cdots & z_{N,N} \end{pmatrix}$$

be the matrices of unobservables and observables, respectively, in a market. Because the scalar $z_{u,d}$ is an element of the matrix Z, we do not use uppercase and lowercase letters (or other notation) to distinguish random variables and their realizations. Whether we refer to a random variable or its realization should be clear from context.

A market is defined to be the pair (E, Z); agents in a market can match and agents in different markets cannot. A feasible one-to-one *assignment* A is a set of matches $A = \{\langle u_1, d_1\rangle, \ldots, \langle u_N, d_N\rangle\}$, where for this section each firm is matched exactly once. There are $N!$ feasible assignments. We refer to the following assumptions in the theoretical results.

ASSUMPTION.

A1. Each firm u or d is part of exactly one match in each feasible assignment A.

A2. The limiting data are on pairs (A, Z).
A3. The random matrix Z is independent of the random matrix E.
A4. The support of the random matrix Z is $\mathbb{R}^{N^2}$.
A5. Either the firm indices u and d have common meaning across markets or the random matrix E has a distribution that is exchangeable in agent indices.

We mentioned A1 just before. Assumption A2 says that a researcher observes the assignment A and the match-specific characteristics Z for many markets. The limiting data on (A, Z) from A2 directly identify $\Pr(A|Z)$, the probability of assignment A being the assignment in a competitive equilibrium (defined below) given the market-level match characteristics Z. Assumption A3 rules out omitted variable bias from having Z be dependent with E. We could in principle address the statistical dependence of E and Z with instrumental variables. We do not explore instrumenting. We should mention that the $e_{u,d}$ and $z_{u,d}$ for the realized matches in the observed assignment A will likely be statistically dependent because of the conditioning on the dependent variable A, part of the outcome to the matching game. The purpose of A4 is to use large explanatory variable values to identify the tails of distributions of heterogeneity. The prior example of the match characteristic distance does not vary over all of $\mathbb{R}^{N^2}$ because distance is computed using the agent-specific characteristics latitude and longitude. There are other examples of match-specific characteristics that conceptually can vary independently in N^2 dimensions: say the past experience of an upstream firm with the observed sector of a downstream firm. If all downstream firms in a matching market are in different sectors, then Z can conceptually vary in $\mathbb{R}^{N^2}$ after recentering experience. Assumption A5 clarifies that we consider two cases: (1) the same firms appear in each market, in which case u and d have specific meanings across markets; and (2) the labels u and d do not have specific meaning across markets, and so we assume that the random matrix E has a distribution that is *exchangeable in agent indices*. Define the random matrix E to be exchangeable in agent indices if the distribution of

$$E_{\pi_u,\pi_d} = \begin{pmatrix} e_{\pi_u(1),\pi_d(1)} & \cdots & e_{\pi_u(1),\pi_d(N)} \\ \vdots & \ddots & \vdots \\ e_{\pi_u(N),\pi_d(1)} & \cdots & e_{\pi_u(N),\pi_d(N)} \end{pmatrix}$$

is the same as E for all permutations π_u and π_d. If agent indices do not have common meaning across markets, then the distribution of the random matrix Z might also be exchangeable in agent indices, although we do not need to explicitly assume this.

We now discuss more details of the matching game. An *outcome* is a list of matches and transfers between matched agents:

$$\{\langle u_1, d_1, t_{u_1,d_1}\rangle, \ldots, \langle u_N, d_N, t_{u_N,d_N}\rangle\}.$$

We examine identification when researchers do not observe transfers, which are often part of confidential contracts. An outcome is *pairwise stable* if it is robust to deviations by pairs of two firms, as defined in references such as Roth and Sotomayor (1990, chap. 8). We omit standard definitions that can be easily found in the literature. An assignment A is called *pairwise stable* if there exists an underlying outcome (including transfers) that is pairwise stable. The assignment in a pairwise stable outcome is equivalent to the assignment in a *competitive equilibrium* in this model (Hatfield et al. 2013). To keep the same solution concept throughout the paper, we say that the paper uses the solution concept of competitive equilibrium.

The literature cited previously proves that the existence of a competitive equilibrium is guaranteed and that an assignment A is part of a competitive equilibrium if and only if it maximizes the *sum of production*

$$s(A; E, Z) = \sum_{\langle u,d\rangle \in A} (z_{u,d} + e_{u,d}).$$

If $z_{u,d}$ or $e_{u,d}$ has continuous support, $s(A; E, Z)$ has a unique maximizer with probability one and therefore the competitive equilibrium assignment is unique with probability one. The *sum of the unobserved production* of assignment A relative to the particular assignment $A_1 = \{\langle 1, 1\rangle, \ldots, \langle N, N\rangle\}$ is

$$\tilde{s}(A; E) = \sum_{\langle u,d\rangle \in A} e_{u,d} - \sum_{\langle u,d\rangle \in A_1} e_{u,d}. \tag{2}$$

Note that each match characteristic $z_{u,d}$ enters production (1) additively, the sign and coefficient on each $z_{u,d}$ in production is common across matches (normalized to be one), each $z_{u,d}$ has large support (A4), and Z is independent of E (A3). Similar large support explanatory variables have been used to prove point identification in the binary and multinomial choice literature (e.g., Manski 1988; Ichimura and Thompson 1998; Lewbel 1998, 2000; Matzkin 2007; Gautier and Kitamura 2013; Berry and Haile 2016; Fox and Gandhi 2016). In this literature, failure to have large support often results in identifying the cumulative distribution function (CDF) of the unobserved heterogeneity at a subset of points. Consider a binary choice model of buying a can of soda (or not) in which the large support regressor is the (negative) price of the soda, which varies across the data set. If we assume that consumers' willingness to pay for the can of soda is bounded by \$0 and \$10, we can identify the CDF of the willingness to pay for soda over all points if observable prices range between \$0 and \$10. If prices range only between \$0 and \$5, we can identify the fraction of consumers with values above \$5 by seeing the fraction who purchase at \$5. We cannot identify the fraction with values above \$6 or any value greater than \$5. If we do not restrict the support of the willingness to pay, we need prices to vary across all of $\mathbb{R}$ (including negative prices if consumers may have negative willingness to pay) to identify the CDF of the willingness to pay for soda at all points.

B. *Definition of Identification*

The unknown primitive whose identification we first explore is the CDF $G(E)$, which reflects how the match unobservables vary across matching markets. We do not restrict the support of E and we do not assume independence across the $e_{u,d}$'s within matching markets. This allows for many special cases, such as the case $e_{u,d} = e_u \cdot e_d$ mentioned earlier.

The probability of assignment A occurring given the match characteristics Z is

$$\Pr(A \mid Z; G) = \int_E 1[A \text{ competitive eq. assignment} \mid Z, E]\,\tilde{d}G(E), \qquad (3)$$

where $1[A \text{ competitive eq. assignment} \mid Z, E]$ is equal to one when A is a competitive equilibrium assignment for the market (E, Z). The symbol $\tilde{d}$ in $\tilde{d}G$ stands in for the differential symbol d from calculus, to distinguish the differential from our notation d for a downstream firm.

The distribution G is said to be *identified* whenever, for $G^1 \neq G^2$, $\Pr(A \mid Z; G^1) \neq \Pr(A \mid Z; G^2)$ for some pair (A, Z). The terms G^1 and G^2 give a different probability for at least one assignment A given Z. If G has continuous and full support so that all probabilities $\Pr(A|Z; G)$ are nonzero (for every (A, Z), $s(A; E, Z)$ will be maximized by a range of E) and continuous in the elements of Z, the existence of one such pair (A, Z) implies that a set of Z with positive measure satisfies $\Pr(A \mid Z; G^1) \neq \Pr(A \mid Z; G^2)$.

Our positive identification results will be constructive, in that we can trace a distribution such as $G(E)$ using variation in an object such as Z. Also, our identification arguments can be used to prove the consistency of a nonparametric mixtures estimator for a distribution G of heterogeneous unobservables E, as Fox, Kim, and Yang (2016) show for a particular, computationally simple mixtures estimator. The proof of consistency in Fox et al. for one estimator requires the heterogeneous unobservable (such as E) to have compact support, which is not required here for identification. A second estimator in Fox et al. allows the support of E to be $\mathbb{R}^{\dim(E)}$. For large markets, these estimators may have computational problems arising from the combinatorics underlying the set of matching game assignments. Fox (2018) uses a maximum score estimator to avoid these computational problems under a generalization of the setup in Choo and

Siow (2006) but does not estimate a distribution of unobservables. Our identification arguments do not address computational issues. Likewise, random variables such as E are of large dimension, and nonparametrically estimating a CDF such as $G(E)$ results in a rate of convergence that depends on the size of the random matrix E.

C. *Nonidentification of the Distribution of Match-Specific Characteristics*

As maximizing $s(A; E, Z)$ determines the assignment seen in the data, the ordering of $s(A; E, Z)$ across assignments A as a function of E and Z is a key input to identification. We can add a constant to the production of all matches involving the same upstream firm and the ordering of the production $s(A; E, Z)$ of all assignments will remain the same. This nonidentification result is unsurprising: the differential production of matches and hence assignments governs the identity of the competitive equilibrium assignment in any market.

We will show another nonidentification result. Consider the two realizations of matrices of unobservables

$$E_1 = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,N} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ e_{N,1} & e_{N,2} & \cdots & e_{N,N} \end{pmatrix},$$

$$E_2 = \begin{pmatrix} e_{1,1} & e_{1,2}+1 & \cdots & e_{1,N} \\ e_{2,1}-1 & e_{2,2}+1-1 & \cdots & e_{2,N}-1 \\ \vdots & \vdots & \ddots & \vdots \\ e_{N,1} & e_{N,2}+1 & \cdots & e_{N,N} \end{pmatrix}.$$

It is easy to verify that $s(A; E_1, Z) = s(A; E_2, Z)$ for all A, Z, which means that the competitive equilibrium assignment A is the same for E_1 and E_2, for any Z. Therefore, it is not possible to separately identify the relative frequencies of E_1 and E_2 in the data-generating process; the support of the random matrix E is too flexible.

We summarize the two counterexamples in the following nonidentification proposition. Note that nonidentification results always involve stating one set of conditions that are insufficient for identification.

Proposition 1. Let A1–A5 hold. Despite imposing these assumptions, the distribution $G(E)$ of market-level unobserved match characteristics is not identified in a matching game in which all agents must be matched.

Consider a simple example focusing on two upstream firms and two downstream firms. If we see the matches $\langle u_1, d_1\rangle$ and $\langle u_2, d_2\rangle$ in the data, we cannot know whether this assignment forms because $\langle u_1, d_1\rangle$ has high production, $\langle u_2, d_2\rangle$ has high production, $\langle u_1, d_2\rangle$ has low production, or $\langle u_2, d_1\rangle$ has low production.

D. *Unobserved Assignment Production*

The competitive equilibrium assignment A maximizes the function $s(A; E, Z) = \sum_{\langle u,d\rangle \in A}(z_{u,d} + e_{u,d})$. This looks like a fictitious single agent, the social planner, maximizing a utility function. Rough intuition from the multinomial choice literature, cited earlier, suggests that the distribution $H(\tilde{S})$ of

$$\begin{aligned}\tilde{S} &= (\tilde{s}(A_2; E), \ldots, \tilde{s}(A_{N!}; E)) \\ &= \left(\sum_{\langle u,d\rangle \in A_2} e_{u,d} - \sum_{\langle u,d\rangle \in A_1} e_{u,d}, \ldots, \sum_{\langle u,d\rangle \in A_{N!}} e_{u,d} - \sum_{\langle u,d\rangle \in A_1} e_{u,d}\right)\end{aligned}$$

might be identified, where the long vector $\tilde{S}$ collects the unobserved production of $N! - 1$ assignments relative to the reference assignment $A_1 = \{\langle 1, 1\rangle, \ldots, \langle N, N\rangle\}$. Directly citing the multinomial choice literature requires a vector of $N! - 1$ assignment-specific observables with support $\mathbb{R}^{N!-1}$, where a hypothetical assignment-specific observable would enter only $s(A; E, Z)$ for a particular A. Assignment-specific observables do not exist in our matching game. However, the distribution $H(\tilde{S})$ is identified using only the variation in match-specific characteristics Z assumed earlier.

LEMMA 1. Let A1–A5 hold. The distribution $H(\tilde{S})$ of unobserved production for all assignments is identified.

The proof, in the appendix, shows that large and product support on Z, A4, allows us to trace $H(\tilde{S})$. Failure of A4 means that the argument in the constructive proof for lemma 1 identifies $H(\tilde{S})$ at a subset of values of $\tilde{S}$.

EXAMPLE 1. Consider the case $N = 3$. The matrices of unobserved and observed match characteristics are

$$E = \begin{pmatrix} e_{1,1} & e_{1,2} & e_{1,3} \\ e_{2,1} & e_{2,2} & e_{2,3} \\ e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix}, \quad Z = \begin{pmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{pmatrix}.$$

There are six possible assignments,

$$\begin{aligned} A_1 &= \{\langle 1,1 \rangle, \langle 2,2 \rangle, \langle 3,3 \rangle\}, \\ A_2 &= \{\langle 1,2 \rangle, \langle 2,1 \rangle, \langle 3,3 \rangle\}, \\ A_3 &= \{\langle 1,3 \rangle, \langle 2,2 \rangle, \langle 3,1 \rangle\}, \\ A_4 &= \{\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,1 \rangle\}, \\ A_5 &= \{\langle 1,1 \rangle, \langle 2,3 \rangle, \langle 3,2 \rangle\}, \\ A_6 &= \{\langle 1,3 \rangle, \langle 2,1 \rangle, \langle 3,2 \rangle\}, \end{aligned} \tag{4}$$

and

$$\tilde{S} = \begin{pmatrix} \tilde{s}(A_2; E) \\ \tilde{s}(A_3; E) \\ \tilde{s}(A_4; E) \\ \tilde{s}(A_5; E) \\ \tilde{s}(A_6; E) \end{pmatrix} = \begin{pmatrix} e_{1,2} + e_{2,1} + e_{3,3} - (e_{1,1} + e_{2,2} + e_{3,3}) \\ e_{1,3} + e_{2,2} + e_{3,1} - (e_{1,1} + e_{2,2} + e_{3,3}) \\ e_{1,2} + e_{2,3} + e_{3,1} - (e_{1,1} + e_{2,2} + e_{3,3}) \\ e_{1,1} + e_{2,3} + e_{3,2} - (e_{1,1} + e_{2,2} + e_{3,3}) \\ e_{1,3} + e_{2,1} + e_{3,2} - (e_{1,1} + e_{2,2} + e_{3,3}) \end{pmatrix}. \tag{5}$$

Lemma 1 states that the distribution $H(\tilde{S})$ is identified.

E. Unobserved Complementarities

The random vector $\tilde{S}$ has $N! - 1$ elements. Estimating a joint distribution of $N! - 1$ elements is not practical in typical data sets. We now introduce the concept of unobserved complementarities as an intuitive, lower-dimensional random variable whose distribution is point identified if and only if $H(\tilde{S})$ is point identified. References such as Fox (2010) and Graham (2011) prove that complementarities in observed agent or match characteristics are identified using data on matches. The main identification theorem below states that the distribution of unobserved complementarities can be identified.

DEFINITION. The *unobserved complementarity* between matches $\langle u_1, d_1\rangle$ and $\langle u_2, d_2\rangle$ is

$$c_{u_1,d_1,u_2,d_2} = e_{u_1,d_1} + e_{u_2,d_2} - (e_{u_1,d_2} + e_{u_2,d_1}). \tag{6}$$

The unobserved complementarities capture the change in the unobserved production when two matched pairs $\langle u_1, d_1\rangle$ and $\langle u_2, d_2\rangle$ exchange partners and the matches $\langle u_1, d_2\rangle$ and $\langle u_2, d_1\rangle$ arise.

Fixing a realization of the unobserved match characteristics E, one can calculate the market-level array (of four dimensions) comprising all unobserved complementarities

$$C = (c_{u_1,d_1,u_2,d_2} \mid u_1, u_2, d_1, d_2 \in N). \tag{7}$$

We consider only values C formed from valid values of E.

There are N^4 values c_{u_1,d_1,u_2,d_2} in C given any realization E. Also, there are

$$\binom{N}{2}^2 = \frac{N^4}{4} - \frac{N^3}{2} + \frac{N^2}{4}$$

unobserved complementarities in C with a unique set of four firms: two upstream firms and two downstream firms; $\binom{N}{2}^2$ is on the order of N^4. However, all unobserved complementarities can be formed from a smaller set of other unobserved complementarities by addition and subtraction. Let

$$b_{u,d} = c_{1,1,u,d} = e_{1,1} + e_{u,d} - (e_{1,d} + e_{u,1}) \tag{8}$$

be an unobserved complementarity fixing the identities of the upstream firm u_1 and the downstream firm d_1 to both be one. Define the matrix

$$B = \begin{pmatrix} b_{2,2} & \cdots & b_{2,N} \\ \vdots & \ddots & \vdots \\ b_{N,2} & \cdots & b_{N,N} \end{pmatrix},$$

which contains all unique values of $b_{u,d}$ for a market. The matrix B contains $(N-1)^2$ elements. The following proposition shows we can restrict attention to B instead of C and hence focus on identifying the joint distribution $F(B)$ of the random matrix B.

PROPOSITION 2.

1. Every element of C is a linear combination of elements of B. The specific linear combination does not depend on the realizations of C or B.
2. For any CDF $F(B)$, there exists $G(E)$ generating $F(B)$ by the appropriate change of variables in (8).
3. If E is exchangeable in agent indices, then so is B.

This proposition does not require any of A1–A5. By the first part of the proposition, we can focus on identifying the distribution of the $(N-1)^2$ elements in B instead of all N^4 elements in C. The second statement in the proposition allows us to identify $F(B)$ without restrictions on the support of B or the dependence between the elements of B, as any $F(B)$ is compatible with some distribution $G(E)$ of the primitive matrix of match-specific unobservables E. The third statement in the proposition shows that in a typical empirical context in which the distribution of primitive unobservables is exchangeable in agent indices, the distribution of unobserved complementarities is also exchangeable in agent indices. We now present examples of some of the claims in the proposition.

EXAMPLE 1 ($N = 3$). There are $3! = 6$ assignments; $N^4 = 81$ and $\binom{N}{2}^2 = 9$. There are four unobserved complementarities in B:

$$\begin{aligned} B &= \begin{pmatrix} b_{2,2} & b_{2,3} \\ b_{3,2} & b_{3,3} \end{pmatrix} \\ &= \begin{pmatrix} e_{1,1} + e_{2,2} - (e_{1,2} + e_{2,1}) & e_{1,1} + e_{2,3} - (e_{1,3} + e_{2,1}) \\ e_{1,1} + e_{3,2} - (e_{1,2} + e_{3,1}) & e_{1,1} + e_{3,3} - (e_{1,3} + e_{3,1}) \end{pmatrix}. \end{aligned} \tag{9}$$

The first part of proposition 2 claims that the 81 elements in C can be constructed from the four elements in B. For one example,

$$c_{2,2,3,3} = e_{2,2} + e_{3,3} - (e_{2,3} + e_{3,2}) = b_{2,2} - b_{2,3} - b_{3,2} + b_{3,3}.$$

EXAMPLE 2. Let the distribution $G(E)$ be exchangeable in agent indices. Also let $G(E)$ be multivariate normal with zero means. The variance matrix of the distribution G is parameterized by the four parameters

$$\begin{aligned} \mathrm{Cov}(e_{u_1,d_1}, e_{u_2,d_2}) &= \psi_1 \quad \text{if } u_1 \neq u_2,\, d_1 \neq d_2, \\ \mathrm{Cov}(e_{u_1,d_1}, e_{u_2,d_1}) &= \psi_2 \quad \text{if } u_1 \neq u_2, \\ \mathrm{Cov}(e_{u_1,d_1}, e_{u_1,d_2}) &= \psi_3 \quad \text{if } d_1 \neq d_2, \\ \mathrm{Var}(e_{u_1,d_1}) &= \psi^2. \end{aligned}$$

One can use the properties of linear changes of variables for multivariate normal distributions to algebraically derive the distribution $F(B)$ of unobserved complementarities. The distribution $F(B)$ is itself exchangeable in agent indices (proposition 2.3) and is multivariate normal with a variance matrix with diagonal and off-diagonal terms

$$\begin{aligned} \mathrm{Cov}(b_{u_1,d_1}, b_{u_2,d_2}) &= \frac{1}{4}\nu^2 \quad \text{if } u_1 \neq u_2,\, d_1 \neq d_2, \\ \mathrm{Cov}(b_{u_1,d_1}, b_{u_2,d_1}) &= \frac{1}{2}\nu^2 \quad \text{if } u_1 \neq u_2, \\ \mathrm{Cov}(b_{u_1,d_1}, b_{u_1,d_2}) &= \frac{1}{2}\nu^2 \quad \text{if } d_1 \neq d_2, \\ \mathrm{Var}(b_{u_1,d_1}) &= \nu^2, \end{aligned}$$

where the new parameter $\nu^2 = 4(\psi^2 + \psi_1 - \psi_2 - \psi_3)$. This example shows the reduction of information from considering unobserved complementarities instead of unobserved match characteristics. In this example, $G(E)$ is parameterized by four parameters while the induced $F(B)$ has only one unknown parameter. An estimator for the parameters ψ_1, ψ_2, ψ_3, and ψ^2 will be inconsistent under A1–A5 as only a linear combination of those four parameters is identified.

F. *Identification of Unobserved Complementarities*

We have shown that $H(\tilde{S})$ is identified, where recall $\tilde{S} = (\tilde{s}(A_2, E), \ldots, \tilde{s}(A_{N!}, E))$. We now show that identification of $H(\tilde{S})$ gives the identification of $F(B)$, the distribution of unobserved complementarities.

Let

$$\tilde{r}(A; B) = \sum_{\langle u,d \rangle \in A} b_{u,d} - \sum_{\langle u,d \rangle \in A_1} b_{u,d}, \tag{10}$$

where for notational compactness we define $b_{u,1} = b_{1,d} = 0$ for all u and d. The term $\tilde{r}(A; B)$ gives the sum of the unobserved complementarities in B corresponding to the indices of the matches in A minus the same sum for $A_1 = \{\langle 1, 1\rangle, \ldots, \langle N, N\rangle\}$.

One of the main results of the paper is that the distribution $F(B)$ of unobserved complementarities is identified.

THEOREM 1.

1. $\tilde{s}(A; E) = \tilde{r}(A; B)$ for any A and where B is formed from E.
2. $\tilde{r}(A; B_1) = \tilde{r}(A; B_2)$ for all A if and only if $B_1 = B_2$.
3. If A1–A5 hold, the distribution $F(B)$ is identified.

The proof is in the appendix. The first part of the theorem states that the sum of unobserved match production for an assignment can be computed using the elements of B. Therefore, knowledge of B can be used to compute competitive equilibrium assignments, for example, for counterfactual analysis as we discuss in Section II.I. Likewise, knowledge of $F(B)$ lets one calculate assignment probabilities $\Pr(A|Z; F)$. The second part of the theorem states that there is a one-to-one mapping between the sums of unobserved assignment production for assignments and values of B. Therefore, the third part of the theorem states that as the distribution $H(\tilde{S})$ of the sums of unobserved match production for assignments is identified under A1–A5, so is the distribution $F(B)$ of unobserved match complementarities.

Example 1 ($N = 3$). By definition,

$$\begin{pmatrix} \tilde{r}(A_2; B) \\ \tilde{r}(A_3; B) \\ \tilde{r}(A_4; B) \\ \tilde{r}(A_5; B) \\ \tilde{r}(A_6; B) \end{pmatrix} = \begin{pmatrix} b_{1,2} + b_{2,1} + b_{3,3} - (b_{1,1} + b_{2,2} + b_{3,3}) \\ b_{1,3} + b_{2,2} + b_{3,1} - (b_{1,1} + b_{2,2} + b_{3,3}) \\ b_{1,2} + b_{2,3} + b_{3,1} - (b_{1,1} + b_{2,2} + b_{3,3}) \\ b_{1,1} + b_{2,3} + b_{3,2} - (b_{1,1} + b_{2,2} + b_{3,3}) \\ b_{1,3} + b_{2,1} + b_{3,2} - (b_{1,1} + b_{2,2} + b_{3,3}) \end{pmatrix}$$
$$= \begin{pmatrix} b_{3,3} - (b_{2,2} + b_{3,3}) \\ b_{2,2} - (b_{2,2} + b_{3,3}) \\ b_{2,3} - (b_{2,2} + b_{3,3}) \\ b_{2,3} + b_{3,2} - (b_{2,2} + b_{3,3}) \\ b_{3,2} - (b_{2,2} + b_{3,3}) \end{pmatrix} = \begin{pmatrix} -b_{2,2} \\ -b_{3,3} \\ b_{2,3} - (b_{2,2} + b_{3,3}) \\ b_{2,3} + b_{3,2} - (b_{2,2} + b_{3,3}) \\ b_{3,2} - (b_{2,2} + b_{3,3}) \end{pmatrix}, \tag{11}$$

where the second equality uses $b_{u,1} = b_{1,d} = 0$ for all u and d. Then using (9) for each of the four $b_{u,d}$'s and (5) for each of the five $\tilde{s}(A; E)$'s allows one to algebraically verify theorem 1.1 for $N = 3$. The interesting direction of theorem 1.2 for $N = 3$ states that $B_1 = B_2$ whenever $\tilde{r}(A; B_1) = \tilde{r}(A; B_2)$ for all A. This direction can be verified because $\tilde{r}(A_2; B)$ through $\tilde{r}(A_6; B)$ can be easily solved for the four elements of B. The less interesting direction of theorem 1.2 always holds by the definition of $\tilde{r}(A; B)$ to be a function of B. Given that we previously showed that $H(\tilde{S})$ is identified, $F(B)$ is also identified.

G. *Conditioning on Other Observed Variables X*

In addition to the match-specific characteristics Z, researchers often observe other match-specific and agent-specific characteristics, which we collect in the random variable X, which we think of as a long vector. Note that we number new assumptions in groups by letter with the number indicating in most cases the previous assumption that is superseded. We do not repeat unchanged assumptions.

Assumption. B2. The limiting data are on triples (A, Z, X).

Assumption B2 allows the direct identification of $\Pr(A|Z, X)$, the probability of assignment A being the competitive equilibrium assignment given the market-level match characteristics Z and X. Conditioning on X is straightforward, and the identification of unobserved complementarities conditional on X immediately follows from theorem 1.3, as that result does not use variation in X.

Corollary 1. Let A1 and B2 hold. Also let A3–A5 hold conditional on X. Then the distribution $F(B|X)$ is identified.

EXAMPLE 3. Let

$$X = (N, (x_u)_{u \in N}, (x_d)_{d \in N}, (x_{u,d})_{u,d \in N}),$$

where x_u is a vector of upstream firm characteristics, x_d is a vector of downstream firm characteristics, and $x_{u,d}$ is a vector of match-specific characteristics. Say match production is

$$(x_u \cdot x_d)' \beta_{u,d,1} + x'_{u,d} \beta_{u,d,2} + \mu_{u,d} + z_{u,d}, \tag{12}$$

where $\mu_{u,d}$ is a random intercept capturing unobserved characteristics of both u and d, $\beta_{u,d,1}$ and $\beta_{u,d,2}$ are random coefficient vectors specific to the match, and $x_u \cdot x_d$ is a vector of all interactions between upstream and downstream characteristics. Then define

$$e_{u,d} = (x_u \cdot x_d)' \beta_{u,d,1} + x'_{u,d} \beta_{u,d,2} + \mu_{u,d}$$

and let (6) define unobserved complementarities. Then we identify $F(B|X)$ by corollary 1. We could further attempt to unpack the identified $F(B|X)$ into the distribution of individual random coefficients and additive unobservables, such as the vectors $\beta_{u,d,1}$ and $\beta_{u,d,2}$ and the unobserved complementarities induced only by the scalar $\mu_{u,d}$ in the example production function (12). We would need to assume full independence between the primitive unobservables and the elements of X. Using (12), we can think of the definition of $b_{u,d}$, (8), as defining a system of $(N - 1)^2$ seemingly unrelated equations, relating $b_{u,d}$ to the elements of X, the random coefficients, and the additive unobservables. Masten (forthcoming) studies in part seemingly unrelated regressions with random coefficients and shows that the marginal distribution of each random coefficient or additive unobservable is identified but the joint distribution of the random coefficients and additive unobservables entering all equations is sometimes not identified.

H. *Heterogeneous Coefficients on Match Characteristics*

Define the production to a match $\langle u, d \rangle$ to be

$$e_{u,d} + \gamma_{u,d} \cdot z_{u,d}, \tag{13}$$

where $\gamma_{u,d}$ is a match-specific coefficient. We use the matrix $\Gamma = (\gamma_{u,d})_{u,d \in N}$. The matrix Γ is fixed across markets. Therefore, the $\gamma_{u,d}$ are fixed parameters to be identified and not random coefficients. Fixing coefficients across markets but not within markets makes sense in a context in which firm indices like u and d have a consistent meaning across markets. For example, the same set of upstream and downstream firms may participate in multiple matching markets, as in Fox (2018), where each market is a separate automotive component category. As we need the $z_{u,d}$'s to identify $F(B|X)$, we rule out the case in which any $\gamma_{u,d} = 0$. We also apply a scale normalization on production by setting $\gamma_{1,1} = \pm 1$.

ASSUMPTION.

B5. The firm indices u and d have common meaning across markets conditional on X.

B6. Every element $\gamma_{u,d}$ of Γ is nonzero and $\gamma_{1,1} \in \{-1, 1\}$.

THEOREM 2. Let A1, B2, B5, and B6 hold. Also let A3 and A4 hold conditional on X. Then the distribution $F(B|X)$ and the fixed matrix of parameters $\Gamma = (\gamma_{u,d})_{u,d\in N}$ are identified.

The proof is in the appendix. We could also study the production function (13) when each $\gamma_{u,d}$ is a random coefficient such that the random matrix $\Gamma = (\gamma_{u,d})_{u,d\in N}$ has some joint distribution $J(\Gamma)$ that describes how Γ varies across markets. An identification at infinity proof technique, where all but two $z_{u,d}$'s are set to $-\infty$, identifies the marginal distribution of each $\gamma_{u,d}$ but not the joint distribution $J(\Gamma)$ by reference to results on binary choice with random coefficients (Ichimura and Thompson 1998; Gautier and Kitamura 2013).

Consider a setup with observable types as in Choo and Siow (2006). Each agent u has a measured (in the data) agent type of w_u from a finite set of upstream firm types $\mathcal{W}^U$, and similarly, each agent d has a measured agent type of w_d from a finite set of downstream firm types $\mathcal{W}^D$. Here U and D are simply labels distinguishing terms for upstream and downstream firms. Let $W = ((w_u)_{u\in N}, (w_d)_{d\in N})$ be the random vector of type membership in a market. In marriage, observed types could refer to college and high school educated men and women. Let the production function be

$$e_{u,d} + \bar{\gamma}_{w_u,w_d} \cdot z_{u,d},$$

where

$$\bar{\Gamma} = (\bar{\gamma}_{w_u,w_d})_{w_u\in\mathcal{W}^U, w_d\in\mathcal{W}^D}$$

is a matrix of fixed parameters $\bar{\gamma}_{w_u,w_d}$ specific to a pair of agent types w_u and w_d. This suggested use of demographic classes is partially reminiscent of Chiappori et al. (2017), who use data over time on the US marriage market to estimate a different variance of the type I extreme value (logit) utility errors in a Choo and Siow (2006) style model for each male demographic class and for each female demographic class.

ASSUMPTION.

C2. The limiting data are on tuples (A, Z, X, W).

C6. Every element $\bar{\gamma}_{w_u,w_d}$ of $\bar{\Gamma}$ is nonzero and $\bar{\gamma}_{1,1} \in \{-1, 1\}$.

C7. The support of W conditional on X is such that all possible values of the pair of types w_u, w_d (given the finite sets $\mathcal{W}^U$ and $\mathcal{W}^D$) occur together with the pair 1, 1 in the support of W.

COROLLARY 2. Let A1, C2, C6, and C7 hold. Also let A3–A5 hold conditional on X and W. Then the distribution $F(B|X, W)$ and the fixed matrix of parameters $\bar{\Gamma} = (\bar{\gamma}_{w_u,w_d})_{w_u \in W^U, w_d \in W^D}$ are identified.

Proof. By C7, condition on a value of W that contains upstream firm type 1 and downstream firm type 1 (to use the scale normalization in C6) and a value of X. Even though u and d may lack common meaning across markets according to A5, we still wish to apply theorem 2, which uses B5 and hence some labels, say $\tilde{u}$ and $\tilde{d}$, with common meaning across markets. If all types w_u and w_d in a realization of W occur only once, then by conditioning on W we can interpret each firm type w_u or w_d in W as indexing a particular firm $\tilde{u}$ or $\tilde{d}$, as in B5. If W has two or more firms with the same type w_u or w_d, then the allocation of the two firms to $\tilde{u}$'s or $\tilde{d}$'s with constant meaning across markets can be done arbitrarily within each type w_u or w_d in each market, as long as $\tilde{u}$ and $\tilde{d}$ each correspond to the same type across markets for a given W. Then apply theorem 2, which is possible as W is fixed, A3 and A4 are the same once we condition on W and X, and C2 is similarly an analogue to B2. If there is a pair of types w_u, w_d that are not both in the particular W that has been conditioned on, simply condition on another value of W that contains that pair w_u, w_d (and the firm types in the scale normalization in C6) and repeat the use of theorem 2. QED

I. *Counterfactuals*

What counterfactuals are identified when $F(B|X)$ is identified?

EXAMPLE 4. Some counterfactuals require additional structure, which we now state in an example. Let

$$e_{u,d} = x'_{u,d}\beta + m_{u,d},$$

where $x_{u,d}$ is a vector of match-specific observables, $m_{u,d}$ is a match-specific unobservable, and β is a vector of homogeneous parameters. Let $X = (x_{u,d})_{u,d \in N}$. Define the unobserved complementarities in $m_{u,d}$ to be

$$o_{u_1,d_1,u_2,d_2} = m_{u_1,d_1} + m_{u_2,d_2} - (m_{u_1,d_2} + m_{u_2,d_1}).$$

The identification of $F(B|X)$ allows the identification of the parameter β under the additional restriction $E[m_{u,d}|X] = 0$ for all $u, d \in N$ and a linear independence condition on the vector

$$x_{u_1,d_1} + x_{u_2,d_2} - (x_{u_1,d_2} + x_{u_2,d_1}).$$

A full independence assumption between $m_{u,d}$ and X will also identify the CDF $I(O)$ of the random matrix $O = (o_{u,d})_{u,d\in N}$ from the identification of $F(B|X)$. After β and $I(O)$ are identified, the ability to compute assignment probabilities in theorem 1.1 identifies counterfactual assignment probabilities based on changes to X outside of its observed support or based on changes to β and I, as in many empirical matching papers (e.g., Choo and Siow 2006; Gordon and Knight 2009; Yang et al. 2009; Christakis et al. 2010; Baccara et al. 2012; Banerjee et al. 2013; Chen and Song 2013; Park 2013; Akkus et al. 2016; Pan 2017).

A smaller set of matching papers considers counterfactual experiments that restrict matching to a subset of observed agents (e.g., Uetake 2014; Yang and Goldfarb 2015; Uetake and Watanabe 2016). The distribution of the corresponding unobserved complementarities of a subset of agents is formed by marginalizing the identified $F(B|X)$, and the resulting assignment choice probabilities can be computed by theorem 1.1. Identifying $F(B|X)$ and not the primitive $G(E|X)$ does not permit the identification of all counterfactuals. For example, say a counterfactual doubles the standard deviation of match-specific unobservables, $e_{u,d}$. Then this change to $G(E|X)$ cannot be explored as $G(E|X)$ itself is not identified.

III. Testing

This section explores whether two special cases of the model can be tested and whether the general model as stated is overidentified.

A. *Testing That the Elements of E Are Identically and Independently Distributed*

We first explore testing the assumption that $F(E|X)$ is such that the elements $e_{u,d}$ of E are identically and independently distributed (i.i.d.) conditional on X, within a market. Can this assumption be tested on the basis of the identification result corollary 1? The answer is no.

PROPOSITION 3. Even if A1–A5 hold conditional on X, the assumption that the elements of E are i.i.d. conditional on X cannot always be tested.

Proof. The proof of nontestability is given by example. Example 2 shows an example of a multivariate normal distribution for E that is exchangeable in agent indices. By theorem 1.3, the parameter $\nu^2 = 4(\psi^2 + \psi_1 - \psi_2 - \psi_3)$ in example 2 is identified. The case of i.i.d. $e_{u,d}$ is that $\psi_1 = \psi_2 = \psi_3 = 0$. Two values of $(\psi^2, \psi_1, \psi_2, \psi_3)$ giving the same $\nu^2 = 4(\psi^2 + \psi_1 - \psi_2 - \psi_3)$ lead to the same assignment probabilities by theorem 1.1. Any identified value of v^2 can be explained solely with the parameter ψ^2, so the hypothesis that $\psi_1 = \psi_2 = \psi_3 = 0$ cannot be tested through the identification of v^2. QED

B. *Testing That $G(E|X)$ Satisfies the Separability Assumption from Choo and Siow (2006)*

We next consider testing a restriction on $G(E|X)$ used in Choo and Siow (2006) and labeled the *separability* assumption by Galichon and Salanié (2015) and Chiappori et al. (2017). The restriction applies when, following the setup of theorem 2, each upstream firm u has a measured (in the data) agent type of w_u in a typically finite set of upstream firm types $\mathcal{W}^U$ and, similarly, each downstream firm has a measured agent type of w_d in a set of types $\mathcal{W}^D$. Types are collected into the market-specific matrix W, as before.

Let $e_{u,d} = e^U_{u,d} + e^D_{u,d}$, where $e^U_{u,d}$ and $e^D_{u,d}$ are two other terms to be discussed now. The Choo and Siow (2006) separability assumption is that $e^D_{u,d} = e^D_{u',d}$ if u and u' satisfy $w_u = w_{u'}$, meaning that u and u' have the same types; and similarly, $e^U_{u,d} = e^U_{u,d'}$ if d and d' satisfy $w_d = w_{d'}$. The idea in marriage could be that each man u has identical preference shocks $e^U_{u,d}$ for women d in the same demographic group w_d. We prove the following testable necessary condition if the Choo and Siow restrictions hold. Recall the definition of unobserved complementarities in (6).

Proposition 4. Let the Choo and Siow separability restrictions hold. Then $c_{u_1,d_1,u_2,d_2} = 0$ when $w_{u_1} = w_{u_2}$ and $w_{d_1} = w_{d_2}$.

Proof. Using the definition of separability,

$$e_{u_1,d_1} = e^U_{u_1,d_1} + e^D_{u_1,d_1}, \quad e_{u_1,d_2} = e^U_{u_1,d_1} + e^D_{u_1,d_2},$$
$$e_{u_2,d_1} = e^U_{u_2,d_1} + e^D_{u_1,d_1}, \quad e_{u_2,d_2} = e^U_{u_2,d_1} + e^D_{u_1,d_2}.$$

It follows from algebra that

$$c_{u_1,d_1,u_2,d_2} = e_{u_1,d_1} + e_{u_2,d_2} - e_{u_1,d_2} - e_{u_2,d_1} = 0.$$

QED

One can identify $F(B|X)$ and then form the identified distribution of terms like $c_{u_1,d_1,u_2,d_2} = 0$ when $w_{u_1} = w_{u_2}$ and $w_{d_1} = w_{d_2}$. If the distribution does not put all mass on $c_{u_1,d_1,u_2,d_2} = 0$, the Choo and Siow separability assumption is rejected.

C. *Overidentification of $F(B|X)$*

Despite the model primitive $G(E|X)$ not being identified, the distribution $F(B|X)$ is overidentified.

Proposition 5. Let A1 hold and let A2–A5 hold conditional on X. Then the distribution $F(B|X)$ is identified only from $\Pr(A_1|Z, X)$, where $A_1 = \{\langle 1, 1\rangle, \dots, \langle N, N\rangle\}$. Therefore, $F(B|X)$ is overidentified.

Proof. The proof of lemma 1 works by setting $H(\tilde{S}|X) = \Pr(A_1|Z, X)$, where A_1 is the diagonal assignment $\{\langle 1, 1\rangle, \dots, \langle N, N\rangle\}$ and Z is a specific value chosen on the basis of the value of $\tilde{S}$. One can identify $H(\tilde{S}|X)$ and $F(B|X)$ if one observes only whether assignment A_1 occurs in a market or not; A_1 is just one of $N!$ assignments. Given the adding-up constraint that the sum of probabilities of assignments is always one, there are $N! - 2$ other probabilities $\Pr(A|Z, X)$ for each Z, X available to overidentify the model. QED

The necessity of using only one assignment probability in a proof of identification is analogous to identification arguments for the single-agent multinomial choice model, where only the probability of a single choice is necessary for identification (e.g., Thompson 1989; Lewbel 2000).

IV. Agent-Specific Characteristics

Match-specific z's with full support are not always available in data sets. For example, say that the induced match-specific characteristic $z_{u,d} = z_u \cdot z_d$, where z_u is an upstream firm characteristic and z_d is a downstream firm characteristic. Likewise, say $e_{u,d} = e_u \cdot e_d$, where e_u and e_d are unobserved agent-specific characteristics. The production function (1) becomes

$$e_u \cdot e_d + z_u \cdot z_d. \tag{14}$$

It is important for our identification argument that the unobservables have the same dimension as the observables and that observables and unobservables enter (11) using similar functional forms. All assignments would be part of a competitive equilibrium if the match production was instead $e_u + e_d + z_u + z_d$. Define the long vectors $\tilde{Z} = ((z_u)_{u\in N}, (z_d)_{d\in N})$ and $\tilde{E} = ((e_u)_{u\in N}, (e_d)_{d\in N})$.

Assumption.

- D2. The limiting data are on triples $(A, \tilde{Z}, X)$.
- D3. The random vector $\tilde{Z}$ is independent of the random vector $\tilde{E}$ conditional on X.
- D4. The support of the random vector $\tilde{Z}$ conditional on X is $\mathbb{R}^{2N}$.
- D5. Either the firm indices u and d have common meaning across markets conditional on X or the random vector $\tilde{E}$ has a distribution that is exchangeable in agent indices conditional on X.

Assumption D4 does not imply A4; the matrix Z of match characteristics in which $z_{u,d} = z_u \cdot z_d$ lacks support on $\mathbb{R}^{N^2}$ under D4.

Proposition 6. Let A1 and D2–D5 hold. The distribution $F(B|X)$ of unobserved complementarities is identified.

The short proof is in the appendix. One can alter the first lines of the proof of lemma 1, and then that lemma and hence the remainder of the identification machinery leading up to and including theorem 1 will apply to the agent-specific case.

V. Data on Unmatched Agents

We have considered matching games in which all agents have to be matched, A1. We infer $F(B|X)$ from sorting patterns in the data when only data on observed matches are available. For example, it may be unreasonable to assume that data on all potential entrants to a matching market exist. In other data sets, researchers can observe the identities of unmatched agents. Data are available, for example, on potential merger partners that do not end up undertaking mergers or on single people in a marriage market (e.g., Choo and Siow 2006; Uetake and Watanabe 2016). When data on unmatched agents do exist, we can identify the distribution of match-specific unobservables E.

Here, X can contain separate numbers of downstream firms N^D and upstream firms N^U. Let

$$E = \begin{pmatrix} e_{1,1} & \cdots & e_{1,N^D} \\ \vdots & \ddots & \vdots \\ e_{N^U,1} & \cdots & e_{N^U,N^D} \end{pmatrix}.$$

Use $\langle u, 0\rangle$ and $\langle 0, d\rangle$ to denote an upstream firm and a downstream firm that are not matched. An assignment A can be $\{\langle u_1, 0\rangle, \langle u_2, d_2\rangle, \langle 0, d_2\rangle\}$, allowing single firms.

ASSUMPTION. E1. Each firm u or d is part of exactly one match or is unmatched in each feasible assignment A.

We do not require match-specific characteristics $z_{u,0}$ and $z_{0,d}$ for unmatched firms; they can be included in X if present. The data-generating process is still (3). One difference is that a competitive equilibrium assignment needs to satisfy individual rationality: each nonsingleton realized match has production greater than zero.

THEOREM 3. Let E1 hold and let A2–A5 hold conditional on X. Then the distribution $G(E|X)$ is identified.

The proof shows that the distribution $G(E|X)$ can be traced using the probability that all agents are unmatched, conditional on Z. The individual rationality decision to be single identifies $G(E|X)$ while the sorting of matched firms to other matched firms identifies only $F(B|X)$. Using an individual rationality condition is more similar to the utility maximization assumptions used in the identification of single-agent discrete choice models and discrete Nash games (e.g., Lewbel 2000; Berry and Tamer 2007; Matzkin 2007; Berry and Haile 2016).

VI. Many-to-Many Matching

We now extend the previous results to many-to-many, two-sided matching. Consider a two-sided matching game in which upstream firm u can make a *quota* of q_u possible matches and downstream firm d can make

q_d possible matches. The researcher has data on the long vector $Q = ((q_u)_{u \in N}, (q_d)_{d \in N})$, and the quotas can vary across firms in the same market and across markets. The previous case of one-to-one matching is $q_u = q_d = 1$ for all firms. Leaving a quota slot unfilled gives production of zero for that slot. The number of upstream firms N^U may differ from the number of downstream firms N^D.

Let the production function for an individual match still be (1) and let the production of the matches of the upstream firm u with the pair of downstream firms d_1 and d_2 be equal to

$$z_{u,d_1} + e_{u,d_1} + z_{u,d_2} + e_{u,d_2}.$$

This implies *additive separability* in the production of multiple matches involving the same firm (Crawford and Knoer 1981; Sotomayor 1992, 1999). As in the one-to-one case, a competitive equilibrium assignment is proven to exist, to be efficient, and to be unique with probability one. Redefine the following objects to allow $N^U \neq N^D$:

$$E = \begin{pmatrix} e_{1,1} & \cdots & e_{1,N^D} \\ \vdots & \ddots & \vdots \\ e_{N^U,1} & \cdots & e_{N^U,N^D} \end{pmatrix},$$

$$Z = \begin{pmatrix} z_{1,1} & \cdots & z_{1,N^D} \\ \vdots & \ddots & \vdots \\ z_{N^U,1} & \cdots & z_{N^U,N^D} \end{pmatrix},$$

$$B = \begin{pmatrix} b_{2,2} & \cdots & b_{2,N^D} \\ \vdots & \ddots & \vdots \\ b_{N^U,2} & \cdots & b_{N^U,N^D} \end{pmatrix}.$$

Extending the model in Section II.H, let $\Gamma = (\gamma_{u,d})_{u \in N^U, d \in N^D}$ be the matrix of homogeneous parameters, if present.

Say first that the number of firms, the quotas, and the production functions are such that all firms make a number of matches equal to their quotas: there are no unused quota slots in equilibrium. Leaving no unused quota is feasible if

$$\sum_{u=1}^{N^U} q_u = \sum_{d=1}^{N^D} q_d.$$

In this case, every mathematical argument for the baseline model in Section II and many of the subsequent models extends to many-to-many matching. In particular, the distribution of unobserved complementarities $F(B|X)$ is identified using the sorting patterns in the data. Likewise, if unmatched firms are in the data and so quota slots can be left unused, the same analysis as in Section V applies.

ASSUMPTION.

- F1. $\sum_{u=1}^{N^U} q_u = \sum_{d=1}^{N^D} q_d$, and each firm u or d has a number of matches exactly equal to its quota, q_u or q_d, in each feasible assignment A.
- F2. The limiting data are on (A, Z, X, Q).
- F3. The random matrix Z is independent of the random matrix E conditional on X, Q.
- F4. The support of the random matrix Z conditional on X, Q is $\mathbb{R}^{N^U \cdot N^D}$.
- F5a. The firm indices u and d have common meaning across markets conditional on X, Q or the random matrix E has a distribution that is exchangeable in agent indices conditional on X, Q.
- F5b. The firm indices u and d have common meaning across markets conditional on X, Q.
- F6. Each firm u or d has a number of matches less than or equal to its quota, q_u or q_d, in each feasible assignment A. Firms can be unmatched.

COROLLARY 3. Consider the many-to-many matching model.

1. If F1–F4 and F5a hold, the distribution $F(B|X, Q)$ is identified in a model without the fixed matrix of parameters $\Gamma = (\gamma_{u,d})_{u \in N^U, d \in N^D}$.
2. If F1–F4, F5b, and B6 hold, the distribution $F(B|X, Q)$ and the fixed matrix of parameters $\Gamma = (\gamma_{u,d})_{u \in N^U, d \in N^D}$ are identified.
3. If F2–F4, F5a, and F6 hold, the distribution $G(E|X, Q)$ is identified in a model without the fixed matrix of parameters $\Gamma = (\gamma_{u,d})_{u \in N^U, d \in N^D}$.

The proof is omitted as it just checks previous mathematical arguments to see that properties unique to one-to-one matching are not used.

Corollary 3.3 can be extended to a simple version of matching with trades (Hatfield et al. 2013). Agents engage in trades ω from some finite set Ω. The production of a trade ω between buyer i and seller j is

$$z_\omega + e_\omega. \tag{15}$$

Define the vectors $\bar{Z} = (z_\omega)_{\omega \in \Omega}$ and $\bar{E} = (e_\omega)_{\omega \in \Omega}$. Let A be the set of trades that occur in a competitive equilibrium. A trade ω indexes the name of the buyer and the name of the seller and can specify other aspects, such as the quality or other specifications of the goods in question. In a labor market, trades could specify benefits such as health care plans and vacation time. In venture capital, a trade could specify the number of board seats a start-up gives a venture capitalist. Trades generalize our previous notion of a match. We require data on all aspects defining the trade ω; if quality is part of a trade, then the qualities for all trades in the set Ω must be measured. The price of trade ω is p_ω, although, as before, we study identification when prices are not observed in the data. Prices play the same role as transfers in the earlier matching models. Firms are not necessarily divided into buyers and sellers ex ante; a firm can be a buyer on some trades and a seller on other trades. In a model of mergers, a firm is not ex ante either a target or acquirer; these roles arise endogenously as part of a competitive equilibrium outcome. Two-sided, many-to-many matching is a strict special case of trading networks in which the profits of an upstream firm undertaking trades as a buyer are $-\infty$ and, likewise, the profits of a downstream making trades as a seller are $-\infty$.

Assumption.

- G1. All trades ω with positive production occur in an assignment A.
- G2. The limiting data are on triples $(A, \bar{Z}, X)$.
- G3. The random vector $\bar{Z}$ is independent of the random vector $\bar{E}$ conditional on X.
- G4. The support of the random vector $\bar{Z}$ conditional on X is $\mathbb{R}^{|\Omega|}$.
- G5. Either the trade index ω has common meaning across markets conditional on X or the random vector $\bar{E}$ has a distribution that is exchangeable in trade indices conditional on X.

Proposition 7. If G1–G5 hold, then $G(\bar{E}|X)$ is identified.

The proof is similar to the proof of theorem 3. Identification also holds if we introduce quotas to matching with trades, as in corollary 3.3.

VII. More General Matching with Trades

We now consider matching with trades in which firms have profit functions defined over portfolios of trades. Let $\Omega_i \subset \Omega$ be the set of trades in which i is either a buyer or a seller. The *individual profit* of a firm i undertaking the trades $\Psi_i \subseteq \Omega_i$ at prices p_ω for $\omega \in \Omega$ is

$$u(i, \Psi_i) + \sum_{\omega \in \Psi_{i\to}} p_\omega - \sum_{\omega \in \Psi_{\to i}} p_\omega, \tag{16}$$

where the set $\Psi_{i\to}$ is the trades in Ψ_i in which i is the seller and $\Psi_{\to i}$ is the trades in Ψ_i in which i is the buyer. Hatfield et al. (2013) prove that a competitive equilibrium assignment exists and is efficient (and therefore unique with probability one) under a condition on preferences called *substitutes*. A companion paper shows that the substitutes condition is equivalent to the indirect utility (profit) version of the direct utility (profit) in (16) being *submodular* for all vectors of prices, p_ω for $\omega \in \Omega$ (Hatfield et al. 2018, theorem 6). See the cited paper for a definition of submodularity. Submodularity of the indirect utility function is restrictive for many empirical applications. However, submodularity is a restriction only when the profit from a set of trades is not additively separable across the trades. Therefore, the underlying direct utility firm profits justifying proposition 7 imply that the corresponding indirect utility functions are submodular.

For all firms i and trades $\psi_i \subseteq \Omega_i$ let the *pretransfer profit* (or *valuation*) be

$$u(i, \Psi_i) = z_{i,\Psi_i} + e_{i,\Psi_i},$$

where z_{i,Ψ_i} is an observable specific to firm i and the set of trades Ψ_i and e_{i,Ψ_i} is an unobservable specific to firm i and the set of trades Ψ_i. Define $\hat{Z} = (z_{i,\Psi_i})_{i\in N,\Psi_i \subseteq \Omega_i}$ and $\hat{E} = (e_{i,\Psi_i})_{i\in N,\Psi_i \subseteq \Omega_i}$.

ASSUMPTION.

H1. Each firm i makes the trades Ψ_i, which can include not making any trades.
H2. The limiting data are on triples $(A, \hat{Z}, X)$.
H3. The random vector $\hat{Z}$ is independent of the random vector $\hat{E}$ conditional on X.
H4. The support of the random vector $-\hat{Z}$ conditional on X is a weak superset of the support of $\hat{E}$.
H5. The supports of $\hat{E}$ and $\hat{Z}$ imply that the corresponding indirect utility functions are submodular for all players for all realizations of $\hat{E}$ and $\hat{Z}$.
H6. Either the trade index ω and the agent index i have common meaning across markets conditional on X or the random vector $\hat{E}$ has a distribution that is exchangeable in agent and trade indices conditional on X.

Under H4, if the support of $\hat{E}$ is a product space, then the support of $-\hat{Z}$ must contain that product space. This is in principle a strong requirement because z_{i,Ψ_i} varies by the identity of the firm i and set of trades Ψ_i. If restrictions are placed on how e_{i,Ψ_i} varies across i and Ψ_i, then correspondingly less variation is needed in z_{i,Ψ_i}. On H5, we leave to other work the question of how to enforce submodularity in empirical applications.

Say that the profit from making no trades is zero and that the researcher observes data on firms that make no trades. Then the following identification result holds.

THEOREM 4. Let H1–H6 hold. The function $G(\hat{E} \mid X)$ is upper bounded by an identified function $\bar{G}(\hat{E} \mid X)$. The function $\bar{G}(\hat{E} \mid X) < 1$ if, for each X and $\hat{Z}$, there exists $\hat{E}$ with positive probability where trades occur.

We can identify a function $\bar{G}(\hat{E} \mid X)$ such that $G(\hat{E} \mid X) \le \bar{G}(\hat{E} \mid X)$ for all unobservables $\hat{E}$ and conditioning observables X. A distribution function reports a probability, so the trivial bound $\bar{G}(\hat{E} \mid X) = 1$ satisfies this property. If some assignment other than the assignment with no trades occurs with positive probability, then $\bar{G}(\hat{E} \mid X)$ is a tighter bound than the trivial bound of one. The bound is likely not sharp. Indeed, it is possible $\bar{G}(\hat{E} \mid X)$ is point identified and we do not know the proof.

The bound $\bar{G}(\hat{E} \mid X)$ in the proof of theorem 4 is actually $\Pr(A_0|Z, X)$ for some Z, where A_0 is the assignment in which no trades are made. The proof of theorem 4 extends the argument in the proofs of theorem 3 and proposition 7. In those proofs, an object like $G(\hat{E} \mid X)$ itself and not a bound equals $\Pr(A_0|Z, X)$. The reason is that the unobservables $e_{u,d}$ in theorem 3 and e_ω in proposition 7 correspond to the production of a match or trade, which is the sum of profits of the two firms for the match or trade. In theorem 4, the unobservables e_{i,Ψ_i} correspond to the profit of an individual firm i and not the production of all firms in the trades. The individual profit functions are not additively separable across individual trades, leaving no role for the concept of the production of a trade.

VIII. Conclusion

It has been an open question whether data on who matches with whom as well as match or agent characteristics are enough to identify distributions of unobservables in transferable utility matching games. Using data on only matched firms, one can identify distributions of what we call unobserved complementarities but not the underlying primitive distribution of match-specific (or agent-specific) unobservables. The distribution of complementarities is enough to compute differences in production levels across assignments and therefore many counterfactual assignment probabilities. We show that it is possible to identify heterogeneous-within-a-market coefficients on the large support match characteristics. The results extend naturally to two-sided, many-to-many matching.

If the data contain unmatched firms, the individual rationality decision to not be unmatched helps identify the distribution of primitively specified unobserved match characteristics, not just the distribution of unobserved complementarities. We partially extend this result to the fairly general case of matching with trades.

Appendix

Proofs

A. *Proof of Lemma 1*

Fix a realization E^* of the primitive unobservable, E. Using the elements of E^* and the large support on Z, set $z^*_{u,d} = -e^*_{u,d}$. Then $s(A; E^*, Z^*) = \Sigma_{\langle u,d \rangle \in A}(e^*_{u,d} + z^*_{u,d}) = 0$ for all assignments A.

The definition of the joint CDF $H(\tilde{S})$ at some vector of evaluation $\tilde{S}^*$ formed from E^* is

$$H(\tilde{S}^*) = \Pr_E(\tilde{s}(A;E) \le \tilde{s}(A;E^*), \forall\ A \ne A_1).$$

Here, each element of the vector $\tilde{S}^*$ is $\tilde{s}(A;E^*)$ for some $A \ne A_1$. Identification of H follows from

$$\begin{aligned} H(\tilde{S}^*) &= \Pr_E(\tilde{s}(A;E) \le \tilde{s}(A;E^*), \forall\ A \ne A_1) \\ &= \Pr_E(s(A;E,Z^*) - s(A_1;E,Z^*) \le s(A;E^*,Z^*) - s(A_1;E^*,Z^*), \forall\ A \ne A_1) \\ &= \Pr_E(s(A;E,Z^*) - s(A_1;E,Z^*) \le 0, \forall\ A \ne A_1) \\ &= \Pr_E(s(A;E,Z^*) \le s(A_1;E,Z^*), \forall\ A \ne A_1) \\ &= \Pr(A_1 \mid Z^*). \end{aligned}$$

Here the first line is the definition of the joint CDF, the second line adds the observed production of assignments A and A_1 to both sides of the inequality, the third line uses $s(A;E^*,Z^*) = 0$ for all A, the fourth line moves $s(A_1, E, Z^*)$ to the right side of the inequality for each A, and the fifth line uses the fact that assignment A_1 is a competitive equilibrium assignment whenever A_1 has a higher total production than all other assignments A.

B. *Proof of Proposition 2*

1. First Part of Proposition 2

For the first part of the proposition, we need to show that every element in C is a linear combination of elements in B. Note that any unobserved complementarity of the form c_{1,d_1,u,d_2} is equal to the difference of two elements of B:

$$\begin{aligned} c_{1,d_1,u,d_2} &= e_{1,d_1} + e_{u,d_2} - (e_{1,d_2} + e_{u,d_1}) \\ &= e_{1,1} + e_{u,d_2} - (e_{1,d_2} + e_{u,1}) - [e_{1,1} + e_{u,d_1} - (e_{1,d_1} + e_{u,1})] \\ &= b_{u,d_2} - b_{u,d_1}. \end{aligned}$$

Next, we represent an arbitrary unobserved complementarity c_{u_1,d_1,u_2,d_2} in terms of unobserved complementarities of the form c_{1,d_1,u,d_2}:

$$\begin{aligned} c_{u_1,d_1,u_2,d_2} &= e_{u_1,d_1} + e_{u_2,d_2} - (e_{u_1,d_2} + e_{u_2,d_1}) \\ &= e_{1,d_1} + e_{u_2,d_2} - (e_{1,d_2} + e_{u_2,d_1}) - [e_{1,d_1} + e_{u_1,d_2} - (e_{1,d_2} + e_{u_1,d_1})] \\ &= c_{1,d_1,u_2,d_2} - c_{1,d_1,u_1,d_2}. \end{aligned}$$

Because we have shown that any unobserved complementarity of the form c_{1,d_1,u,d_2} is a difference of two elements in B, c_{u_1,d_1,u_2,d_2} can be written as the sums and differences of elements in B.

2. Second Part of Proposition 2

For the second part of the proposition, we are given an $F(B)$ and need to find a $G(E)$ such that G generates F by the change of variables given by the definition of the unobserved complementarities in B, (8). Note that every element b_{u_2,d_2} of B contains a unique element e_{u_2,d_2} of E. Place a distribution G on E's such that $e_{1,d} = e_{u,1} = 0$ for all u, d and the other elements of each E are such that $e_{u,d} = b_{u,d}$ for some B in the support of $F(B)$. If each E of $G(E)$ has the same frequency as the paired B in $F(B)$, the distribution $G(E)$ generates $F(B)$.

3. Third Part of Proposition 2

Let π_u be a permutation of upstream firm indices and π_d a permutation of downstream firm indices. We wish to prove that B is exchangeable in agent indices if E is exchangeable in agent indices. The matrix B is formed from E by a linear transformation D, representing the formula (8) for each element of B. Dean and Verducci (1990, condition 2, theorem 4) provide a sufficient (and necessary) condition for a linear transformation to preserve exchangeability in all elements of a random vector. If we vectorize the matrices B and E, the argument in the first paragraph of the proof of theorem 4 of Dean and Verducci can be reproduced for our definition of exchangeability in agent indices. We skip this step of reproducing one direction of the proof of theorem 4 of Dean and Verducci for our different notion of exchangeability in agent indices for conciseness.

The sufficiency condition from Dean and Verducci that we need to verify is that for any permutation in agent indices of B, there exists a permutation of agent indices in E that gives B_{π_u,π_d} through the linear transformation D. This condition is satisfied for B and E. Given permutations of agent indices π_u and π_d themselves giving B_{π_u,π_d}, the same permutations of agent indices give E_{π_u,π_d}. It is clear that B_{π_u,π_d} is related to E_{π_u,π_d} through the linear transformation D by inspection of (8).

C. *Proof of Theorem 1*

1. First Part of Theorem 1

Using the definition of $\tilde{r}(A;B)$ gives

$$\begin{aligned}
\tilde{r}(A;B) &= \sum_{\langle u,d\rangle\in A} b_{u,d} - \sum_{\langle u,d\rangle\in A_1} b_{u,d} \\
&= \sum_{\langle u,d\rangle\in A} [e_{1,1} + e_{u,d} - (e_{1,d} + e_{u,1})] - \sum_{\langle u,d\rangle\in A_1} [e_{1,1} + e_{u,d} - (e_{1,d} + e_{u,1})] \\
&= \sum_{\langle u,d\rangle\in A} e_{u,d} - \sum_{\langle u,d\rangle\in A_1} e_{u,d} - \sum_{\langle u,d\rangle\in A} (e_{1,d} + e_{u,1}) + \sum_{\langle u,d\rangle\in A_1} (e_{1,d} + e_{u,1}) \\
&= \sum_{\langle u,d\rangle\in A} e_{u,d} - \sum_{\langle u,d\rangle\in A_1} e_{u,d} \\
&= \tilde{s}(A;E),
\end{aligned}$$

where the fourth equality uses the fact that each firm is matched the same number of times (in one-to-one matching, exactly once) in both the assignments A and A_1 and the last equality is just the definition of $\tilde{s}(A;E)$ in (2).

2. Second Part of Theorem 1

If $B_1 = B_2$, then $\tilde{r}(A;B_1) = \tilde{r}(A;B_2)$ simply because (10) is a definition of a function of B. For the other direction, assume $\tilde{r}(A;B_1) = \tilde{r}(A;B_2)$ for all A. Focus on a particular scalar unobserved complementarity $b_{u,d}$ in B. We will show that $b_{u,d}$ can be written as $\tilde{s}(A_2,E) - \tilde{s}(A_3,E)$ for particular assignments A_2 and A_3. As the first part of the theorem is that $\tilde{s}(A;E) = \tilde{r}(A;B)$ for any A and where B is formed from E, this implies that $b_{u,d}$ is the same in B_1 and B_2. Because $b_{u,d}$ was arbitrary, $B_1 = B_2$.

Let A_2 be an assignment that contains the matches $\langle u, d\rangle$ and $\langle 1, 1\rangle$. Let A_3 be the same assignment as A_2 except that A_3 includes the matches $\langle 1, d\rangle$ and $\langle u, 1\rangle$ and does not include $\langle u, d\rangle$ and $\langle 1, 1\rangle$. Then

$$\tilde{s}(A_2, E) - \tilde{s}(A_3, E) = e_{1,1} + e_{u,d} - (e_{u,1} + e_{1,d}) = b_{u,d}.$$

By the above argument and because the match $\langle u, d\rangle$ was arbitrary, $B_1 = B_2$.

3. Third Part of Theorem 1

The distribution $H(\tilde{S})$ is identified from lemma 1. The first part of theorem 1 shows that the change of variables from $\tilde{S}$ to the vector of all $\tilde{r}$ is one-to-one. The second part of theorem 1 shows that the change of variables from the vector of all $\tilde{r}$ to the matrix of unobserved complementarities B is one-to-one. Therefore, $F(B)$ is identified.

D. *Proof of Theorem 2*

If Γ is identified, then following a slightly modified version of the proof of lemma 1 and the same argument as the proof of theorem 1.3 demonstrate that F is also identified. So consider identifying Γ. Recall that the scale normalization is that $\gamma_{1,1} = \pm 1$. We can easily identify the sign of $\gamma_{1,1}$. Consider some assignment A that includes match $\langle 1, 1\rangle$. Then we can compare Z_1 and Z_2 that differ only in the value of $z_{1,1}$: $z^1_{1,1} > z^2_{1,1}$. If $\Pr(A \mid Z_1) > \Pr(A \mid Z_2)$, we conclude that $\gamma_{1,1} = +1$, and if $\Pr(A \mid Z_1) < \Pr(A \mid Z_2)$, we conclude that $\gamma_{1,1} = -1$. The main text rules out the case in which any $\gamma_{u,d} = 0$. In what follows, we focus on the case in which the sign of every $\gamma_{u,d}$ is identified and, in particular, $\gamma_{1,1} = +1$. The case of $\gamma_{1,1} = -1$ is symmetric.

We now show how to identify the arbitrary parameter $\gamma_{\tilde{u},\tilde{d}}$. Consider assignments $A^1 = \{\langle 1, 1\rangle, \langle \tilde{u}, \tilde{d}\rangle, \ldots\}$ and $A^2 = \{\langle 1, \tilde{d}\rangle, \langle \tilde{u}, 1\rangle, \ldots\}$ that are identical except for the explicitly listed matches. In a proof shortcut borrowing an idea from identification at infinity, let the matches not in $A^1 \cup A^2$ correspond to $z_{u,d}$'s where $\gamma_{u,d} z_{u,d} = -\infty$, so we consider only Z's in which the total production of any assignment other than A^1 and A^2 is $-\infty$ and hence $\Pr(A^1 \mid Z) + \Pr(A^2 \mid Z) = 1$. Set $z_{1,\tilde{d}} = 0$ and $z_{\tilde{u},1} = 0$. Then A^1 occurs whenever

$$z_{1,1} + e_{1,1} + \gamma_{\tilde{u},\tilde{d}} z_{\tilde{u},\tilde{d}} + e_{\tilde{u},\tilde{d}} \geq e_{1,\tilde{d}} + e_{\tilde{u},1},$$

or by (8), $z_{1,1} + \gamma_{\tilde{u},\tilde{d}} z_{\tilde{u},\tilde{d}} + b_{\tilde{u},\tilde{d}} \geq 0$. This decision rule is equivalent to a decision rule in a single-agent binary choice model. As $b_{\tilde{u},\tilde{d}}$ is fully independent from $z_{1,1}$ and $z_{\tilde{u},\tilde{d}}$, we can apply the results on binary choice from Manski (1988) under full independence and identify $\gamma_{\tilde{u},\tilde{d}}$.

E. *Proof of Proposition 6*

Condition all arguments on X. We first argue that the equivalent of lemma 1 holds. Fix a realization $((e^*_u)_{u\in N}, (e^*_d)_{d\in N})$ of the agent-specific unobservables $\tilde{E}$. Using the elements of $((e^*_u)_{u\in N}, (e^*_d)_{d\in N})$ and the large support on $((z_u)_{u\in N}, (z_d)_{d\in N})$, set $z^*_u = -e^*_u$ and $z^*_d = e^*_d$, the latter without a negative sign as the multiplication of two negatives is positive. Then $s(A; \tilde{E}^*, \tilde{Z}^*) = \sum_{\langle u,d \rangle \in A}(e_u^* \cdot e_d^* + z_u^* \cdot z_d^*) = 0$ for all assignments A. The rest of the proof is then identical to the corresponding portion of the proof of lemma 1, for the match-specific case.

The first two parts of theorem 1 do not refer to Z at all and do not impose any restrictions on the E matrix. So they automatically apply to the less general case in which $e_{u,d} = e_u \cdot e_d$. The main identification result, the third part of theorem 1, then follows for the agent-specific case from the slight modification to the proof of lemma 1 above and the first two parts of theorem 1.

F. *Proof of Theorem 3*

Condition on X. Let A_0 denote the assignment in which no agents are matched. By the normalization that being unmatched gives production zero, the sum of unobserved production for the assignment A_0 satisfies $\tilde{s}(A_0; E) = 0$ for all E. Let E^* be an arbitrary realization of the matrix of match-specific unobservables. Let $Z^* = (z_{u,d}^*)_{u,d \in N}$ be such that $z_{u,d}^* = -e_{u,d}^*$. Then $s(A; Z^*, E^*) = 0$ for all A and $S(A_0; Z^*, E) = 0$ for any E. Thus for all A and all $E \leq E^*$ elementwise, $S(A; Z, E) \leq 0 = S(A_0; Z^*, E)$. Further, if any element of E is greater than the corresponding element of E^*, assignment A_0 will not maximize $s(A, Z^*, E)$ and so A_0 will not be a competitive equilibrium assignment. Therefore, $G(E^*) = \Pr(E \leq E^*$ elementwise $\mid E^*) = \Pr(A_0 \mid Z^*)$.

G. *Proof of Proposition 7*

Condition on X. Let A_0 denote the assignment (of trades) in which no trades are made; then the sum of unobservables for this assignment is zero for all $\bar{E}$ and $\bar{Z}$. Let $\bar{E}^*$ be an arbitrary realization of $\bar{E}$, the vector of trade-specific unobservables. Let $\bar{Z}^*$ be such that $z_\omega^* = -e_\omega^*$ for all $\omega \in \Omega$. Define

$$s(A; \bar{E}, \bar{Z}) = \sum_{\omega \in A}(e_\omega + z_\omega)$$

to be the total production from an assignment. Then $s(A; \bar{Z}^*, \bar{E}^*) = 0$ for all A and $S(A_0; \bar{Z}^*, \bar{E}) = 0$ for any $\bar{E}$. Therefore, for all A and all $\bar{E} \leq \bar{E}^*$ elementwise, $S(A; \bar{Z}, \bar{E}) \leq 0 = S(A_0; \bar{Z}^*, \bar{E})$. Therefore, $G(\bar{E}^*) = \Pr(\bar{E} \leq \bar{E}^*$ elementwise $\mid \bar{E}^*) = \Pr(A_0 \mid Z^*)$.

H. *Proof of Theorem 4*

Condition on X. Let A_0 denote the assignment (of trades) in which no trades are made; then the sum of unobservables for this assignment is zero for all $\hat{E}$ and $\hat{Z}$. Let E^* be an arbitrary realization of the array of unobservables. Let $\hat{Z}^* = (z_{i,\Psi_i}^*)_{i \in N, \Psi_i \subseteq \Omega_i}$ be such that $z_{i,\Psi_i}^* = -e_{i,\Psi_i}^*$ for all $i \in N$, $\Psi_i \subseteq \Omega_i$. Define

$$s(A, \hat{E}, \hat{Z}) = \sum_{i \in N}(e_{i,\Psi_i^A} + z_{i,\Psi_i^A}),$$

where Ψ_i^A are the trades of i in A, to be the total profit or production from an assignment. Then $s(A; \hat{Z}^*, \hat{E}^*) = 0$ for all A and $S(A_0; \hat{Z}^*, \hat{E}) = 0$ for any $\hat{E}$. Therefore, for all A and all $\hat{E} \leq \hat{E}^*$ elementwise, $S(A; \hat{Z}, \hat{E}) \leq 0 = S(A_0; \hat{Z}^*, \hat{E})$.

Therefore, assignment A_0 will occur whenever $\hat{E} \leq \hat{E}^*$ elementwise. Can assignment A_0 occur for $\hat{E}$ not less than or equal (elementwise) to $\hat{E}^*$? For such an $\hat{E}$, there is at least one (i, Ψ_i) such that $e_{i,\Psi_i} > e^*_{i,\Psi_i}$. In this case, the valuation $u(i, \Psi_i)$ of i for Ψ_i at $\hat{Z}^*$, $e_{i,\Psi_i} + z^*_{i,\Psi} = e_{i,\Psi_i} - e^*_{i,\Psi_i}$ is positive. However, it could still be that at this $\hat{E}$ and $\hat{Z}^*$ a vector of prices for trades cannot be formed so that an assignment of trades other than A_0 is a competitive equilibrium assignment. So A_0 can occur at realizations of unobservables $\hat{E}$ not less than or equal (elementwise) to $\hat{E}^*$. Therefore, $G(\hat{E}^*) = \Pr_{\hat{E}}(\hat{E} \leq \hat{E}^*) \leq \Pr(A_0 \mid \hat{Z}^*)$. Define $\bar{G}(\hat{E}^* \mid X) = \Pr(A_0 \mid \hat{Z}^*, X)$.

References

Ackerberg, Daniel A., and Maristella Botticini. 2002. "Endogenous Matching and the Empirical Determinants of Contract Form." *J.P.E.* 110:564–91.

Agarwal, Nikhil. 2015. "An Empirical Model of the Medical Match." *A.E.R.* 105 (7): 1939–78.

Agarwal, Nikhil, and William Diamond. 2016. "Latent Indices in Assortative Matching Models." *Quantitative Econ.* 8 (3): 685–728.

Ahlin, Christian. 2018. "Group Lending, Matching Patterns, and the Mystery of Microcredit: Evidence from Thailand." Working paper, Michigan State Univ.

Akkus, Oktay, J. Anthony Cookson, and Ali Hortaçsu. 2016. "The Determinants of Bank Mergers: A Revealed Preference Analysis." *Management Sci.* 62 (8): 2241–58.

Baccara, Mariagiovanna, Ayse Imrohoroglu, Alistair Wilson, and Leeat Yariv. 2012. "A Field Study on Matching with Network Externalities." *A.E.R.* 102 (5): 1773–1804.

Banerjee, Abhijit, Esther Duflo, Maitreesh Ghatak, and Jeanne Lafortune. 2013. "Marry for What? Caste and Mate Selection in Modern India." *American Econ. J.: Microeconomics* 5 (2): 33–72.

Becker, Gary S. 1973. "A Theory of Marriage: Part I." *J.P.E.* 81 (July/August): 813–46.

Berry, Steven T., and Philip A. Haile. 2016. "Nonparametric Identification of Multinomial Choice Demand Models with Heterogeneous Consumers." Working paper, Yale Univ.

Berry, Steven T., and Elie Tamer. 2007. "Identification in Models of Oligopoly Entry." In *Advances in Economics and Econometrics: Theory and Applications; Ninth World Congress*, edited by Richard Blundell, Whitney Newey, and Torsten Persson. Cambridge: Cambridge Univ. Press.

Brown, James N., and Harvey S. Rosen. 1982. "On the Estimation of Structural Hedonic Price Models." *Econometrica* 50 (May): 765–68.

Chen, Jiawei, and Kejun Song. 2013. "Two-Sided Matching in the Loan Market." *Internat. J. Indus. Org.* 31:145–52.

Chiappori, Pierre-André, R. J. McCann, and Lars P. Nesheim. 2010. "Hedonic Price Equilibria, Stable Matching, and Optimal Transport: Equivalence, Topology, and Uniqueness." *Econ. Theory* 42 (2): 317–54.

Chiappori, Pierre-André, and Bernard Salanié. 2016. "The Econometrics of Matching Models." *J. Econ. Literature* 54 (3): 832–61.

Chiappori, Pierre-André, Bernard Salanié, and Yoram Weiss. 2017. "Partner Choice and the Marital College Premium." *A.E.R.* 107 (8): 2109–67.

Choo, Eugene, and Aloysius Siow. 2006. "Who Marries Whom and Why." *J.P.E.* 114 (1): 175–201.

Christakis, Nicholas, James Fowler, Guido W. Imbens, and Karthik Kalyanaraman. 2010. "An Empirical Model for Strategic Network Formation." Working Paper no. 16039, NBER, Cambridge, MA.
Crawford, Vincent P., and Elsie Marie Knoer. 1981. "Job Matching with Heterogeneous Firms and Workers." *Econometrica* 49 (2): 437–50.
Dagsvik, John K. 2000. "Aggregation in Matching Markets." *Internat. Econ. Rev.* 41 (February): 27–57.
Dean, A. M., and J. S. Verducci. 1990. "Linear Transformations That Preserve Majorization, Schur Concavity, and Exchangeability." *Linear Algebra and Its Applications* 127:121–38.
Eeckhout, Jan, and Philipp Kircher. 2011. "Identifying Sorting—in Theory." *Rev. Econ. Studies* 78:872–906.
Ekeland, Ivar, James J. Heckman, and Lars Nesheim. 2004. "Identification and Estimation of Hedonic Models." *J.P.E.* 112 (S1): S60–S109.
Fox, Jeremy T. 2010. "Identification in Matching Games." *Quantitative Econ.* 1 (2): 203–54.
———. 2018. "Estimating Matching Games with Transfers." *Quantitative Econ.* 9 (1): 1–38.
Fox, Jeremy T., and Amit Gandhi. 2016. "Nonparametric Identification and Estimation of Random Coefficients in Multinomial Choice Models." *RAND J. Econ.* 47 (1): 118–39.
Fox, Jeremy T., Kyoo il Kim, and Chenyu Yang. 2016. "A Simple Nonparametric Approach to Estimating the Distribution of Random Coefficients in Structural Models." *J. Econometrics* 195:236–54.
Gale, David. 1960. *The Theory of Linear Economic Models.* New York: McGraw-Hill.
Galichon, Alfred, and Bernard Salanié. 2015. "Cupid's Invisible Hand: Social Surplus and Identification in Matching Models." Working paper, New York Univ.
Gautier, E., and Y. Kitamura. 2013. "Nonparametric Estimation in Random Coefficients Binary Choice Models." *Econometrica* 81 (2): 581–607.
Gordon, Nora, and Brian Knight. 2009. "A Spatial Merger Estimator with an Application to School District Consolidation." *J. Public Econ.* 93:752–65.
Graham, Bryan S. 2011. "Econometric Methods for the Analysis of Assignment Problems in the Presence of Complementarity and Social Spillovers." In *Handbook of Social Economics*, edited by J. Benhabib, A. Bisin, and M. Jackson. Amsterdam: Elsevier.
Hatfield, John William, Scott Duke Kominers, Alexandru Nichifor, Michael Ostrovsky, and Alexander Westkamp. 2013. "Stability and Competitive Equilibrium in Trading Networks." *J.P.E.* 121 (5): 966–1005.
———. 2018. "Full Substitutability." Working paper, Univ. Texas.
Heckman, James J., Rosa L. Matzkin, and Lars Nesheim. 2010. "Nonparametric Identification and Estimation of Nonadditive Hedonic Models." *Econometrica* 78 (September): 1569–91.
Hitsch, Günter, Ali Hortaçsu, and Dan Ariely. 2010. "Matching and Sorting in Online Dating." *A.E.R.* 100 (1): 130–63.
Ichimura, H., and T. S. Thompson. 1998. "Maximum Likelihood Estimation of a Binary Choice Model with Random Coefficients of Unknown Distribution." *J. Econometrics* 86 (2): 269–95.
Kelso, Alexander S., and Vincent P. Crawford. 1982. "Job Matching, Coalition Formation, and Gross Substitutes." *Econometrica* 50 (November): 1483–1504.
Koopmans, Tjalling C., and Martin Beckmann. 1957. "Assignment Problems and the Location of Economic Activities." *Econometrica* 25 (January): 53–76.

Levine, Anna A. 2009. "Licensing and Scale Economies in the Biotechnology Pharmaceutical Industry." Working paper (April), Washington Univ. St. Louis.

Lewbel, Arthur. 1998. "Semiparametric Latent Variable Model Estimation with Endogenous or Mismeasured Regressors." *Econometrica* 66 (1): 105–21.

———. 2000. "Semiparametric Qualitative Response Model Estimation with Unknown Heteroscedasticity or Instrumental Variables." *J. Econometrics* 97 (1): 145–77.

Manski, Charles F. 1988. "Identification of Binary Response Models." *J. American Statis. Assoc.* 83 (September): 729–38.

Masten, Matthew. Forthcoming. "Random Coefficients on Endogenous Variables in Simultaneous Equations Models." *Rev. Econ. Studies.*

Matzkin, Rosa L. 2007. "Heterogeneous Choice." In *Advances in Economics and Econometrics: Theory and Applications; Ninth World Congress*, edited by Richard Blundell, Whitney Newey, and Torsten Persson. Cambridge: Cambridge Univ. Press.

Mindruta, Denisa, Mahka Moeen, and Rajshree Agarwal. 2016. "A Two-Sided Matching Approach for Partner Selection and Assessing Complementarities in Partners' Attributes in Inter-firm Alliances." *Strategic Management J.* 37 (1): 206–31.

Pan, Yihui. 2017. "The Determinants and Impact of Executive-Firm Matches." *Management Sci.* 63 (1): 185–200.

Park, Minjung. 2013. "Understanding Merger Incentives and Outcomes in the Mutual Fund Industry." *J. Banking and Finance* 37:4368–80.

Roth, Alvin E., and Marilda A. Oliveira Sotomayor. 1990. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Econometric Society Monographs. Cambridge: Cambridge Univ. Press.

Shapley, Lloyd S., and Martin Shubik. 1972. "The Assignment Game I: The Core." *Internat. J. Game Theory* 1:111–30.

Siow, Aloysius. 2009. "Testing Becker's Theory of Positive Assortative Matching." *J. Labor Econ.* 33 (2): 409–41.

Sotomayor, Marilda. 1992. "The Multiple Partners Game." In *Equilibrium and Dynamics: Essays in Honour of David Gale*, edited by Mukul Majumdar, 322–36. New York: Macmillan.

———. 1999. "The Lattice Structure of the Set of the Stable Outcomes of the Multiple Partners Assignment Game." *Internat. J. Game Theory* 28 (November): 567–83.

Thompson, T. Scott. 1989. "Identification of Semiparametric Discrete Choice Models." Discussion Paper no. 249, Univ. Minnesota.

Uetake, Kosuke. 2014. "Estimating a Model of Strategic Network Formation and Its Effects on Performance: An Application to the U.S. Venture Capital Markets." Working paper, Yale Univ.

Uetake, Kosuke, and Yasutora Watanabe. 2016. "Entry by Merger: Estimates from a Two-Sided Matching Model with Externalities." Working paper, Yale Univ.

Yang, Yupin, and Avi Goldfarb. 2015. "Banning Controversial Sponsors: Understanding Equilibrium Outcomes When Sports Sponsorships Are Viewed as Two-Sided Matches." *J. Marketing Res.* 52 (5): 593–615.

Yang, Yupin, Mengze Shi, and Avi Goldfarb. 2009. "Estimating the Value of Brand Alliances in Professional Team Sports." *Marketing Sci.* 28 (November–December): 1095–1111.