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Structural Econometrics of Auctions: A Review

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Abstract

We review the literature concerned with the structural econometrics of observational data from auctions, discussing the problems that have been solved and highlighting those that remain unsolved as well as suggesting areas for future research. Where appropriate, we discuss different modeling choices as well as the fragility or robustness of different methods.

Keywords: auctions; structural econometrics.

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1

Introduction and Motivation

One of the great success stories in economics during the latter half of the twentieth century was the systematic theoretical investigation of incomplete information in various economic environments—for example, moral hazard in insurance markets and adverse selection in such market institutions as auctions, to name just two. Developed in tandem with the incredible advances in game theory since the Second World War, an important by-product of this research program was the formulation of many incomplete information problems as the design of optimal mechanisms.¹

From an econometrician’s perspective, theoretical models of incomplete information are particularly compelling because they provide richer explanations of the observed heterogeneity in data than ones relying on measurement error or productivity shocks. For example, when individual agents use private information strategically to make decisions, the equilibrium interactions of those decisions with the decisions of others can generate new explanations of observed phenomena, particularly in the field of industrial organization.

¹The book by Tilman Börgers [2015], with chapter by Daniel Krähmer and Roland Strausz, provides a useful introduction to the topic.

As is common in economics, however, econometric and empirical developments involving models of incomplete information lagged behind those in theory. For even though using theoretical models to put structure on economic data was advocated by the late Nobel laureates Tryge Haavelmo [1944] and Tjalling C. Koopmans [1947, 1949, 1953] as well as associates of Koopmans at the Cowles Foundation, Yale University, in the 1950s (for example, the late Nobel laureate Leonid Hurwicz [1950]), the approach did not become generally accepted, let alone widespread, until the last quarter of the twentieth century, particularly in response to the devastating evaluation of reduced-form methods by the Nobel laureate Robert E. Lucas, Jr. [1976], which is now commonly referred to as the Lucas critique.

In this review, we summarize a subset of this literature—that devoted to the structural econometric analysis of observational data from auctions, as opposed to data from either laboratory or field experiments, although we mention some studies involving the latter two in passing.

A classic, canonical problem in economic theory involves a seller of an object who faces several potential buyers. Although the seller probably knows the object’s value to himself, he typically has little or no information concerning what value any one of the potential buyers places on the object. Similarly, each potential buyer may have a good notion of the object’s value to himself, but probably knows little about the seller’s valuation or the valuations of the other potential buyers. In short, the seller as well as the potential buyers face incomplete, asymmetric information.

How should the seller dispose of the object? One commonly used method of sale involves announcing a take-it-or-leave-it price and then selling the object to the first who accepts that price. Another involves the seller’s engaging in pair-wise negotiations with individual potential buyers, either sequentially or simultaneously. Yet a third involves selling the object at auction. In short, a set of different selling mechanisms exists, from which the seller must choose, guided by some objective—an auction being one of those potential choices. The choice of mechanism by the seller typically depends on many factors, for example, the ob-

jective of the seller and transaction costs to name just two.

Auctions are ubiquitous in market economies. They are also ancient, their durability suggesting that the institution serves an important allocational role. Perhaps the most important feature defining environments within which auctions are used is the existence of incomplete information of some sort—the asymmetry of information between the seller and the potential buyers. Because the potential buyers have no incentive to tell the seller or any of their competitors anything about those valuations, the auction plays an important allocative role by inducing the potential buyers to reveal to the seller information concerning their valuations of the object.

The way potential buyers form their valuations remains an open question in economics. In fact, in auction theory, researchers are unusually vague about what generates the demand structure, unlike in standard demand theory where considerable care is taken to specify the structure of preferences. Suffice it to say, however, when economic theorists come to modeling the asymmetry in information as well as the heterogeneity in valuations across agents, they employ random variables. Typically, each potential buyer is assumed to demand at most one unit of the object in question. In the simplest model, for each potential buyer, the marginal utility of the one unit is assumed an independent realization of a continuous random variable. By and large, the budget constraint and issues of substitution are ignored.

With such an austere model, it is indeed somewhat surprising that economists can provide any practical insights concerning how to bid at auctions, let alone how to structure the institution with any purpose. In the workhorse model of auction theory, which was first developed by the late Nobel laureate William S. Vickrey [1961], each of a known number N of potential buyers draws an individual-specific random valuation independently from the same differentiable cumulative distribution function (cdf) $F_V(v)$ that has corresponding probability density function (pdf) $f_V(v) = dF_V(v)/dv$. In Vickrey's model, he assumed that V is distributed uniformly on the unit interval: the specific value of his draw is that potential buyer's private information; it represents the monetary value of the object to him. Economic theorists refer to

this as the symmetric independent private-values (IPV) model because the draws are independent and the valuations are bidder specific. Also, because the valuations are drawn from the same distribution (urn so to speak), the bidders are *ex ante* symmetric.

Different auction formats (open-outcry versus sealed-bid) and different pricing rules (pay-your-bid versus second-price) provide potential buyers with different incentives concerning how to bid. For example, under the pay-your-bid rule, a bidder's action (his bid) determines what he pays should he win, whereas under the second-price rule, the action (bid) of his nearest rival determines what the winner pays.

In equilibrium, different functions map the private information of potential buyers (their values) into their actions (their bids). For example, open-outcry (sometimes referred to as oral) auctions can be conducted in two different ways. In the first, the price is set very low, perhaps at zero, and then allowed to rise more or less continuously until only one participant remains active in the auction. That remaining active bidder is the winner, and he pays what the last other active bidder was willing to pay—sometimes plus a small increment. (Later, in the context of electronic auctions held on the Internet, the relative magnitude of that increment is shown to be important.)

Economic theorists have typically chosen to model oral auctions as clocks, where the price rises continuously with the movement of a clock hand. In this case, the winner of the auction is the participant with the highest valuation and he pays what his nearest rival (that participant with the second highest value) was willing to pay.

Thus, the oral, ascending-price auction guarantees the efficient allocation of the object: the participant with highest valuation wins the auction. Such an auction is sometimes referred to as a second-price auction because in the absence of bid increments the winning bid is the second-highest bid, which happens to be the second-highest valuation as well.

In economics, this outcome has special meaning: the second-highest valuation represents the opportunity cost of the object for sale—its value in the next-best alternative. Thus, one can see why economists are naturally attracted to mechanisms that have this property.

As a technical aside, within the IPV model, the equilibrium at an oral, ascending-price auction (sometimes referred to as an English auction) has a special structure. It is a weakly dominant-strategy equilibrium: each participant has an incentive to reveal his private information, that is, to tell the truth concerning his value by continuing to bid up to his value, regardless of what his rivals do.

The second way to conduct an oral auction involves initially setting the price very high, and then allowing it to fall continuously; the winner is the first participant to cry out a bid, and he pays his bid. In practice, these oral auctions are typically implemented using a clock, where the hand (or a digital panel) lists the current price. Participants affirm their willingness to pay the current price by pushing a button which stops the clock at that price. Such auctions are often referred to as Dutch auctions, perhaps because they are frequently used in the Netherlands to sell fish and flowers.

Because the price that the winner pays is related to his action (crying out or pushing the button), he has an incentive to shave his bid—to wait longer before pushing the button to stop the clock. In theoretical models of Dutch auctions within the symmetric IPV model, the equilibrium is not a dominant-strategy equilibrium, but rather a Bayes-Nash equilibrium, which is a much stronger notion of equilibrium. Although the Bayes-Nash equilibrium bid function is an increasing function of a bidder's value, it has a slope that is less than one: each bidder is deceptive when bidding; he does not tell the truth, but rather bids less than his value. Again, however, the winner is the participant with the highest valuation, so objects are allocated efficiently at Dutch auctions.

In general, economic theorists have found that the seller's expected revenue depends first and foremost on the information available to potential buyers and then on the auction format and pricing rules employed, the amount of competition, and the attitudes of potential buyers toward risk. Within the symmetric IPV model, however, under the assumption that potential buyers are risk neutral with respect to winning the object for sale, a remarkable result obtains: revenue equivalence (in expectation). That is, if the same object were sold under the two different institutions, then the average winning bid at the English

auction would equal the average winning bid at the Dutch auction.

To most people, this revenue equivalence result is at first somewhat surprising because at English auctions considerable information is revealed during the course of bidding, whereas at Dutch auctions no information is revealed until the winner has been determined. Within the symmetric IPV model, however, information plays no extra role in determining the average winning price since each bidder's private information (his value) is, by assumption, statistically independent of the private information of his rivals (their values): knowing something about the values of his rivals provides no extra information to a bidder concerning his own valuation, or his likelihood of winning the auction. Therefore, no bidder at an English auction can learn anything more about his valuation from the actions (bids) of his rivals. Once one realizes this fact, the equivalence of average winning bids is clear: at a Dutch auction, assuming he wins because he has the highest value, a representative participant forms his bid so that he will, on average, just beat his nearest rival, the bidder with the second-highest valuation.

Similar analyses have been performed for the sealed format under different pricing rules. In fact, theorists have shown that sealed auctions at which the highest bidder wins the auction and pays what he bid are strategically equivalent to Dutch auctions. Consequently, the Bayes-Nash equilibrium bid function at a sealed, pay-your-bid auction is identical to that at a Dutch auction. It has also been shown that sealed auctions at which the highest bidder wins the auction, but pays the bid of his closest rival, are strategically equivalent to English auctions, so it is a dominant strategy at these auctions for bids to tell the truth, too.

Under the assumption of risk-neutral potential buyers, expected revenue equivalence follows. That is, if the same object were sold under the two different institutions, then the average selling price at a sealed, pay-your-bid auction would equal the average selling price at a sealed, second-price (also known as Vickrey) auction. This result, which was first outlined by Vickrey [1961] and then proved in general by the Nobel laureate Roger B. Myerson [1981] as well as John G. Riley and William F. Samuelson [1981], is the celebrated Revenue Equivalence Theorem

(RET), perhaps the best known result in auction theory.

In its full generality, the RET states that any combination of auction format and pricing rule that has the same probability of assigning a winning bidder generates the same expected revenue to the seller. In particular, the RET predicts that the expected revenues garnered by the seller at sealed auction formats will be the same as those earned at oral auction formats under either pay-your-bid or second-price rules, at least for one-shot, single-object auctions when the distribution from which the values are drawn is the same for all potential buyers, who are also risk neutral.

From a policymaker's perspective, an important issue involves choosing the selling mechanism that garners the most revenue for the seller, on average. To a large extent, the structure of the optimal selling mechanism depends on the informational environment. Within the symmetric IPV model, given the RET, one question arises naturally: Can one still improve on the structure of the four combinations of auction format and pricing rule? Myerson as well as Riley and Samuelson showed that devising a selling mechanism that maximizes the seller's expected gain involves choosing the reserve price r , the minimum price that must be bid, optimally. In the previous notation, the optimal reserve price r^* solves the following equation:

$$r^* = v_0 + \frac{[1 - F_V(r^*)]}{f_V(r^*)}, \quad (1.1)$$

where v_0 denotes the seller's valuation of the object at auction.

The presence of a binding reserve price means that some fraction of the time the object at auction goes unsold. In other words, an inefficiency is introduced because $r^* > v_0$, meaning some bidders might value the object more than the seller, but the object goes unsold. This inefficiency highlights the tension between expected revenue maximization on the part of the seller in particular and allocative efficiency in the economy in general.

Historically, the literature concerned with mechanism design was sometimes criticized as lacking practical value because the optimal selling mechanism (in this case, the optimal reserve price r^*) typically depends on a primitive like $F_V(\cdot)$, the distribution of the valuations,

which is often unknown to the designer. In the past, because the distribution of valuations has been unknown, calculating the optimal reserve price, the optimal selling mechanism, for a real-world auction seemed impossible.

From an econometrician's perspective, auctions are particularly attractive because the rules of an auction govern how the potential buyers must behave during the selling process—specifically, how bids must be tendered, who wins the auction, what the winner pays, and so forth. These rules place incredible structure on the data generating process, unlike in some other economic applications. In particular, under certain conditions, the twin hypotheses of optimization and equilibrium allow the econometrician to identify the unobserved distribution of valuations from the observed distribution of bids. In other words, part of the structural econometric approach to auctions is an identification strategy. Another part involves reverse-engineering an estimate of the distribution of latent types (for example, valuations) from the observed distribution of actions (the bids). Yet a third part is referred to as comparative institutional design—using the estimate of the distribution of latent valuations to improve on auction design.

For example, at auctions within the IPV model, the equilibrium bidding strategies of potential buyers are increasing functions of their valuations. At English auctions, under the clock model, for instance, the dominant strategy of bidders who lose the auction is to bid their valuations. Thus, in principle, it is possible to estimate the underlying probability law of valuations using the empirical distribution of bids from a cross-section of auctions. Because a researcher can recover the primitives of the economic model, the Lucas critique is circumvented; the researcher can then also entertain comparative institutional design, for instance, comparing outcomes under alternative market institutions not observed in the data.

Despite this success, several potential problems remain. For example, at English auctions, as shown later, the winning bid does not reveal complete information concerning the winner's actual valuation of the object for sale. Next, in the presence of a binding reserve price, the empirical distribution of observed bids represents a truncated sample

of data: only those potential buyers whose valuations exceeded the reserve price chose to bid. Finally, in the presence of a binding reserve price, the joint distribution of bidding and nonparticipation depends on the number of potential buyers, but finding a measure of potential competition is often impossible; when it can be done, the specific proxy is often inaccurate. In short, for the past three decades or so, econometricians have had much to occupy themselves, at least when it comes to analyzing observational data from auctions.

At the polar extreme of the IPV model is the pure common-value (CV) model. Within the pure CV model, an alternative device is used to describe the motivation of potential buyers—in particular, a continuous random variable that represents individual-specific signals concerning the object’s true, but unknown, value. This true, but unknown, value will be revealed only after the auction has ended, when the winner has been determined and the transaction price paid. Regardless of the winner, however, the value of the object is the same to all.

In the baseline model, the conceptual experiment involves each potential buyer’s receiving a draw from a signal distribution. Conditional on his draw, a bidder is then assumed to act purposefully, using the information in his signal along with Bayes’ rule to maximize either the expected profit or the expected utility of profit from winning the auction. Another frequently-made assumption is that the signal draws of potential buyers are independent and that those potential buyers are *ex ante* symmetric—their draws coming from the same distribution of signals.

Under these assumptions, a researcher can then focus on a representative agent’s decision rule (policy function) when characterizing equilibrium behavior. Robert B. Wilson [1977] invented this model to demonstrate that the winner’s curse could not obtain in equilibrium among rational bidders.

The winner’s curse is perhaps the second-best known result in auction theory. First conjectured to be a problem in oil exploration by Edward C. Capen, Robert V. Clapp, and William M. Campbell [1971], the prospect of the phenomenon has captured the imaginations of many. For a readable history of the topic, see the book by the Nobel laureate

Richard H. Thaler [1992]; the book by John H. Kagel and Dan Levin [2002] provides additional information—both theoretical and experimental.

In oil exploration, for instance, the idea is that within the CV model, even if signals are unbiased estimates of the true value of the as yet undiscovered mineral resource, the maximum signal is an overestimate of the oil's true value. Thus, potential buyers who do not take this fact into account and who then bid their signals will systematically overbid and lose money: the winner is cursed.

One of Wilson's contributions was to point out that rational bidders in equilibrium will anticipate this overestimation problem, and adjust accordingly. That is, among rational bidders the winner's curse cannot obtain as an equilibrium.

More importantly, however, Wilson demonstrated that when the number of potential buyers is large (tends to infinity) the winning bid at sealed, pay-your-bid auctions converges almost surely to the true, but unknown, value of the object. In other words, the auction format and pricing rule play an important role in aggregating the disparate, individual pieces of information held by the bidders. In short, auctions play a critical role in the process referred to as price discovery.

Paul R. Milgrom [1979] subsequently provided a precise characterization of the structure the signal distribution must possess in order for this convergence property to hold; Wolfgang Pesendorfer and Jeroen M. Swinkels [1997] have referred to this as full information aggregation.

In summary, within both the IPV and the CV models, results depend on five important assumptions: (1) noncooperative behavior among participants; (2) only one object is for sale, so these are one-shot, single-object auction games; (3) the valuation or signal draws are independent; (4) potential buyers draw their values from the same distribution; (5) potential buyers are risk neutral. To understand the effects of relaxing each of these five assumptions individually, we examine here only the IPV model, the most commonly used informational paradigm in auction theory.

When potential buyers are symmetrically risk-averse (that is, each

has the same von Neumann-Morgenstern utility function), participants at sealed, pay-your-bid and oral, descending-price auctions bid more aggressively than they would had the same object been sold at oral, ascending-price or sealed, second-price auctions. Under each pricing rule, both auction formats yield efficient allocations: the highest-valuation bidder wins the auctions. Expected winning bids are, however, higher at pay-your-bid auctions (either Dutch or sealed) than at second-price auctions (either English or Vickrey); see Charles A. Holt, Jr. [1980], Riley and Samuelson [1981] as well as Steven A. Matthews [1983]. In other words, the RET breaks down.

When potential buyers are risk neutral with respect to winning the object, but their valuations are drawn independently from different distributions (different urns so to speak), some important economic issues arise. If potential buyers are *ex ante* asymmetric, often referred to as the asymmetric IPV model, then at English and Vickrey auctions it remains a weakly dominant strategy for bidders to reveal their private information—that is, to tell the truth and to bid up to their private values. In short, under the clock model, the highest-valuation bidder wins the auction and pays what his nearest rival was willing to pay, so the outcome is efficient. The winning bid is the second-highest valuation, which represents the opportunity cost of the object for sale.

On the other hand, within the asymmetric IPV model, at pay-your-bid auctions, Bayes-Nash equilibrium behavior is much more complicated. Characterizing this equilibrium behavior requires solving systems of nonlinear differential equations for which a mathematical property known as the Lipschitz condition does not hold. Consequently, only numerical methods are in general feasible and few clean results exist in particular.

Here is what is known: Consider the case of just two bidders, one of whom is strong and one who is weak. In this case, weak means that the cdf of valuations of the weak bidder is everywhere to the left of that of the strong bidder; see the papers by the Nobel laureate Eric S. Maskin and Riley [2000a, 2000b]. Under these assumptions, for the same valuation, the strong bidder will bid less than the weak bidder: Vijay Krishna [2010] has described this as “weakness breeds aggres-

sion.” Suffice it to say that the winning bid need not bear any relation to the opportunity cost, either in realization or in expectation. Most important, disturbingly, it is possible for the highest-valuation bidder to lose the auction. In other words, the outcomes at sealed, pay-your-bid and oral, descending-price auctions can be inefficient. In general, however, except by using the numerical methods described in Timothy P. Hubbard and Paarsch [2014], few predictions concerning the expected-revenue ranking of the different auction formats and pricing rules exist, which makes structural econometric empirical work especially important.

When bidders’ valuations are symmetric, but dependent, the revelation of private information through bidding can be important to the equilibrium outcome. Thus, the winning bids at English auctions are more informative than those at sealed or Dutch auctions because considerably more information is revealed during the course of bidding at English auctions; see, for example, the work of Pesendorfer and Swinkels [2000]; Han Hong and Matthew Shum [2004] as well as Hong, Paarsch, and Pai Xu [2013].

In order to construct equilibria to auction games under dependence, economic theorists have been forced to impose a specific structure on the form of the dependence. Mathematicians, following the path blazed by Samuel Karlin [1968], refer to this structure as multivariate total positivity of order two, or MTP_2 for short, whereas in an influential and classic paper, Milgrom and Robert J. Weber [1982] coined the term affiliation to describe this form of dependence.

Under symmetric affiliation, Milgrom and Weber derived a powerful result and coined the term linkage principle to describe it. In single-object auction models, where the signals of the risk-neutral potential buyers are symmetrically affiliated, the linkage principle states that a seller can expect to increase revenues by providing more information to bidders, both before and during the auction. An implication of the linkage principle is that English auctions will, on average, earn more revenue for the seller than sealed auctions, under which no information is released, or similar auction formats that reveal less information to potential buyers. Again, the RET breaks down. According to Motty

Perry and Philip J. Reny [1999], “the linkage principle has come to be considered one of *the* fundamental lessons provided by auction theory.”

Thus, the presence of some degree of dependence (a common-value component so to speak) in the signals of potential buyers is critical to the validity of the linkage principle. The affiliated-values (AV) model is a generalization of the pure CV model, and the symmetric IPV model is nested within the AV model. Within the symmetric AV model, the conditional expectation of any monotonic function of the signals of all bidders is an increasing function of any individual bidder’s own signal. When the signals of bidders are dependent in this manner, information released by the seller or information the seller provides concerning the bids made by other participants (by virtue of the seller’s choice of auction format) helps bidders refine their beliefs concerning the true value of the object for sale, which in turn induces them to bid more aggressively than they would in the absence of such information.

Of course, when bidders coöperate with one another (for example, collude), knowing the exact nature of such behavior is critical to any analysis. As a case in point, collusion is easier to sustain in environments that are rich in information: more information is released at English auctions than at sealed ones, or other less open auction formats. At the risk of belaboring the obvious, investigating collusion is difficult because collusive arrangements often differ substantially across economic environments; a deep understanding of economic institutions is required.

John Asker [2010] produced an excellent example of what the structural approach can deliver when the workings of a cartel are known. Asker focused on a cartel of stamp dealers in the 1990s. His data contained records of an entire year’s worth of activities involving the bidding ring members—including side payments, and behavior in both “knockout” and “target” auctions. The structural approach allowed him to assess damages imposed by the ring, to consider the extent to which market efficiency was compromised, and to evaluate how much the ring benefited from collusion.

Like the work of Robert H. Porter and J. Douglas Zona [1993, 1999] as well as Martin Pesendorfer [2000], Asker’s research documented and

investigated a collusive scheme *ex post*, although economists are making progress on testing for collusion too; see, for example, the research of Vadim Marmer, Artyom Shneyerov, and Uma Kaplan [2017].

Admitting several objects complicates matters considerably as the research of Weber [1983], for example, has shown. In fact, economic theorists distinguish between multi-object and multi-unit auctions. At multi-unit auctions, it matters not which unit a bidder wins, but rather the aggregate number of units he wins, while at multi-object auctions it matters which specific object(s) a bidder wins. An example of a multi-object auction would involve the sale of an apple and an orange, whereas one of a multi-unit auction would involve the sale of two identical apples.

When several units of the same object are sold sequentially, the analysis depends on whether potential buyers demand just one unit (referred to by Milgrom [2004] as singleton demand) or several units. Within the symmetric IPV model, when potential buyers have singleton demand, Weber [1983] demonstrated that the equilibrium price path under the four combinations of auction formats and pricing rules follows a martingale: the expectation of the price of the next unit at auction equals the price of the last unit sold.

Characterizing equilibrium in a private-values model of a sequential, multi-unit auction when multi-unit demand is admitted involves solving an asymmetric-auction game for each unit. In a special case, when potential buyers have multi-unit demand that follows a Poisson process, Stephen G. Donald, Paarsch, and Jacques Robert [2006] demonstrated that the equilibrium price path follows a submartingale: on average, the equilibrium price rises over consecutive auctions. As one can see, small changes can have large effects on the equilibrium predictions, even within the simplest of informational paradigms, the IPV model.

To our knowledge, only Perry and Reny [1999] have investigated the effect of affiliation in multi-unit auctions. In fact, they provided a counterexample that demonstrates the Milgrom-Weber ranking breaks down in multi-unit auctions with affiliation.

When several, say K , units of a good are simultaneously for sale, at least two important questions arise: Who will be the winning bid-

ders? What price(s) will those winners pay? For example, Milgrom [1981] developed a natural generalization of the Wilson [1977] model. In Milgrom's model, each bidder submits a price and the auctioneer then aggregates these demands, allocating the units to those bidders with the highest K submitted bids. The winners then pay a uniform price—specifically, the highest rejected bid.

Pesendorfer and Swinkels [1997] built on this research by investigating a sequence of auctions at which both the number of potential buyers N_t and the number of units K_t increase. They demonstrated that a necessary and sufficient condition for full information aggregation is that $K_t \rightarrow \infty$ and $(N_t - K_t) \rightarrow \infty$, a condition they referred to as double largeness. Under this condition, non-negligible supply can be a substitute for the strong signal structure required in Wilson [1977] as well as Milgrom [1979, 1981]. Ilan Kremer [2002] has investigated this further.

Even though it is heartening to know there are conditions under which transaction prices will converge in probability to the true, but unknown, values of objects for sale, the rate at which these prices converge is probably of more practical relevance and value. In particular, Hong and Shum [2004] asked the following question: How large must N be to be large enough? They then investigated the rates of information aggregation in common-value environments. Knowing the conditions under which transaction prices provide potentially useful estimates of the unknown values of objects is important to understanding the price discovery process because in practice neither the number of bidders nor the number of units for sale at an auction ever really gets to infinity.

Of course, the pricing rule investigated in Wilson [1977] and Milgrom [1979, 1981] as well as Pesendorfer and Swinkels [1997, 2000] is not the only pricing rule that could be used under a sealed format. For example, another pricing rule would involve allocating the K units to those bidders who tendered the highest K bids, but each winner would then pay what he bid for the unit(s) he won. In general, at multi-unit auctions, different auction formats and different pricing rules induce different equilibrium behavior and, thus, translate into different transaction prices as well as potentially different expected revenues for

sellers. Hence, as Matthew O. Jackson and Kremer [2004, 2006] have emphasized, understanding the effects of auction formats and pricing rules has important practical relevance. Even small changes can have effects, as has been illustrated by Claudio Mezzetti and Ilia Tsetlin [2008, 2009].

At the risk of sounding banal, when potential buyers demand more than one object, investigating multi-object auctions is even more complicated than multi-unit auctions. One mechanism that has been investigated extensively is the multi-object extension of the Vickrey auction, referred to in the economics literature as the Vickrey-Clarke-Groves (VCG) mechanism—in honor of the contributions of the late Edward H. Clarke [1971] and Theodore Groves [1973] as well as Vickrey. Although the VCG mechanism has many attractive features, if say K objects exist, then in order to know how to bid a potential buyer must evaluate all of the $K!$ combinations of the objects. For large numbers of objects, searching through $K!$ different combinations is a computationally intractable problem, belonging to the complexity class known as NP-hard. In short, for this reason (and others) the VCG mechanism is impractical for many real-world applications; Michael H. Rothkopf, Thomas J. Teisberg, and Edward P. Kahn [1990]; Lawrence M. Ausubel and Milgrom [2006] as well as Rothkopf [2007] have all provided helpful discussions.

An alternative to the VCG mechanism is the so-called generalized second-price (GSP) auction, which scales well for large numbers of objects. The GSP auction has generated literally hundreds of billions in advertising revenues for the Internet search company Google; for more details on this auction, see the paper by Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz [2007].

That noted, for relatively small K , the VCG mechanism has been implemented successfully on the Internet, too. For instance, Google actually switched from the GSP auction to the VCG mechanism for its auctions of contextual advertisements in 2012; Google’s chief economist, Hal R. Varian, and Christopher Harris [2014] have documented some of the reasons behind this change. In addition, we have learned from industry experts that other Internet firms have used the

VCG mechanism at online advertising auctions.

In summary, the two most important considerations in auction theory are allocative efficiency and price discovery. Within the symmetric IPV model, allocative efficiency is guaranteed; under risk neutrality, revenue equivalence obtains. Despite these positive results, the seller can increase his expected revenues by introducing a binding reserve price. An optimally-chosen, binding reserve price increases expected revenues, but introduces inefficiencies, too: some fraction of the time the object goes unsold. Within the pure CV model, price discovery is guaranteed under a variety of assumptions. Symmetric affiliation permits a ranking of auction formats and pricing rules: those formats and rules that release more information will yield higher average revenues. In the presence of asymmetries, allocative efficiency and price discovery can breakdown.

Having whetted the reader's appetite with an *amuse bouche* of economic theory as it pertains to auctions, in the remaining sections of this review we employ these results to put structure on the data generating processes of observational data from auctions. To this end, in the next section, we develop several different theoretical models of equilibrium bidding under different auction formats and pricing rules within the three commonly-used informational environments; these models provide the foundations on which the econometric models developed later in the review rest. Following that, in section 3 we illustrate how to construct the mapping from the theoretical models to the observable data that can be used to estimate empirical specifications. In section 4, we examine the thorny issue of identification, while in section 5 we outline two basic approaches to doing empirical work using auction data—the reduced-form and the structural. We devote section 6 to describing several successful strategies used when estimating empirical specifications. In the next three sections, we then document the nitty gritty of dealing with asymmetries and dependence as well as multi-unit and multi-object auctions—computationally intensive work. We summarize and conclude as well as provide suggestions for future research in section 10, and collect in appendices any results too cumbersome for inclusion in the text.

Before proceeding any further, however, we should also note that this review is intended to augment those reviews and surveys of the econometrics of auctions that came before it—specifically, those by Kenneth Hendricks and Paarsch [1995]; Isabelle Perrigne and Quang H. Vuong [1999]; Paarsch and Hong [2006]; Susan C. Athey and Philip A. Haile [2007]; Hendricks and Porter [2007] as well as Brent R. Hickman, Hubbard, and Yiğit Sağlam [2012]. It is impossible for us to acknowledge fully the influence that these papers have had on our work. Suffice it to say, we owe a debt to those who came before us.

1.1 Recommended, Additional Reading concerning Auctions

The classic reference concerning auctions is the book by Ralph Casady, Jr. [1967]. That said, even though this book has many interesting anecdotes and facts, it is somewhat out of date, and provides no guidance in theoretical modeling. At the risk of shameless self-promotion, for those with little experience in economics, we recommend the elementary book by Hubbard and Paarsch [2015], which is concerned only with auctions. For those with a bit more training in economics, we recommend the beautifully written book by the late John McMillan [2002], which is concerned with markets in general, but deals with auctions as well. At the next level, even though each is over thirty years old, we recommend the surveys by Milgrom [1987] as well as R. Preston McAfee and McMillan [1987b]—definitely worth reading. For a light technical treatment of market design, the book by Guillaume Haeringer [2017] is a wonderful read. For graduate students and professional economists, the books by Paul D. Klemperer [2004], Milgrom [2004], and Krishna [2010] are essential reading. In fact, in our opinion, no one interested in conducting research in the structural econometrics of auctions should begin without having read Krishna’s textbook first.

2

Theoretical Models

As mentioned in section 1, the workhorse of auction theory is the IPV model, which was invented by Vickrey [1961] and then investigated further by Holt [1980] and Myerson [1981] as well as Riley and Samuelson [1981]. At the polar extreme of the IPV model is the pure CV model, which was invented by Wilson [1977]. Milgrom and Weber [1982] then invented the AV model, which is more general than either the IPV or the CV models. In this section, we characterize the equilibrium bid functions within these theoretical models under different auction formats (oral versus sealed) and pricing rules (pay-your-bid versus second-price); these equilibrium bid functions are the foundations of the econometric models developed later in the review.

2.1 IPV Model

Because the IPV model is the simplest theoretical model within which to work, we begin there, but first we develop some notation, which we subsequently augment as needed in later parts of the review.

2.1.1 Initial Notation

Suppose that potential buyers at the auction are members of a set $\mathcal{N} = \{1, 2, \dots, N\}$ where the letter n indexes the members. Because the main focus in auction theory is asymmetric information, which economic theorists have chosen to represent as random variables, the bulk of our notation centers around a consistent way to describe random variables. Typically, we denote random variables by uppercase Roman letters—for example, V . Realizations of random variables are then denoted by lowercase Roman letters; for example, v is a realization of V .

We denote the pdf and cdf of the random variable V by f_V and F_V , respectively. When there are different distributions (urns so to speak), we again use the subscript to refer to a given bidder's distribution, but use the set \mathcal{N} numbering. Hence, f_1 and F_N for specific bidders, but f_n and F_n in general. If necessary, a vector (V_1, V_2, \dots, V_N) of random variables is denoted \mathbf{V} , while a realization without potential buyer n is denoted \mathbf{v}_{-n} .

The lowercase Greek letters β and σ are used to denote equilibrium bid functions—for example, σ for the equilibrium bid function at a sealed, pay-your-bid auction where the choice variable is s , whereas β denotes the equilibrium bid function at a sealed, second-price auction where the choice variable is b . Again, if necessary, we use $\boldsymbol{\sigma}$ to collect all strategies, and \mathbf{s} to collect the choice variables, while $\boldsymbol{\sigma}_{-n}$ is used to collect all the strategies except that of bidder n and \mathbf{s}_{-n} collects all the choices except that of bidder n . We use φ to denote the inverse-bid function and $\boldsymbol{\varphi}_{-n}$ to collect all of the inverse-bid functions, excluding that of bidder n . Similarly, we use $\boldsymbol{\beta}$ to collect all strategies and \mathbf{b} to collect the choice variables under the second-price rule, where the $-n$ subscript collects all except that of bidder n .

That said, because we draw from several different fields of economics, inevitably the same symbol is used more than once, and can mean different things in different contexts. For example, in this section concerned with theoretical models, X_n denotes a signal of potential buyer n , while in the section concerned with estimation \mathbf{x}_t denotes a vector of covariates at auction t . The reader should be aware of this, and adjust accordingly.

2.1.2 Dominant Strategy Equilibria

Within the symmetric IPV model, the simplest combination of auction format and pricing rule to examine is the Vickrey auction: the sealed, second-price auction. At Vickrey auctions, potential buyers tender their bids in secret in sealed envelopes; the potential buyer who submits the highest bid wins the object, but he pays the bid of his nearest opponent, the second-highest bid tendered. If the winner has no opponents, then he pays the reserve price, the minimum price which must be bid at the auction, if one has been declared; when no reserve price has been set, the solo bidder gets the object for free.

How should potential buyer n bid at Vickrey auctions? Well, to bid less than his valuation v_n would be to reduce his chances of winning without changing the price he would pay if he wins the auction. If he does not win the auction by bidding less than his valuation, then the price paid by the winning bidder could change. This would occur if potential buyer n 's bid were lower than some other bid which in turn was less than v_n . To bid more than v_n could change the outcome when some other bidder has submitted a bid higher than n 's valuation but lower than his new bid. Raising the bid above v_n may allow n to win, but he would then risk paying more than the object is worth to him, which is clearly suboptimal. Hence, the weakly dominant strategy of the n^{th} potential buyer is to bid his valuation v_n . The equilibrium decision rule (policy function) is

$$B_n = \beta(V_n) = V_n. \quad (2.1)$$

Introducing the notation $(k : N)$ as a subscript to denote the k^{th} highest order statistic from a sample of size N , the winner is the participant who tendered the highest bid $B_{(1:N)}$ or $V_{(1:N)}$, but he pays what his next nearest opponent was willing to pay $B_{(2:N)}$ or $V_{(2:N)}$, so the price paid is $V_{(2:N)}$. The Vickrey auction ensures the economically-efficient allocation of the object and the winning bid represents the object's opportunity cost—its value in the next-best alternative use.

Even though constructing the dominant-strategy equilibrium at the Vickrey auction is simple and the argument elegant, it is sometimes difficult to convince others that, within the IPV model, equation (2.1)

constitutes the equilibrium at an English auction, at least for nonwinners. Many of the difficulties arise because reality gets in the way. A variety of different rules exists at oral, ascending-price auctions. At some auctions, the bidding behavior of participants is essentially secret or difficult to observe. At others, discrete bid increments are imposed. To isolate the important features, Milgrom and Weber [1982] constructed a theoretical model of an English auction that is referred to as the clock model. Both simple and elegant, the clock model implies that equation (2.1) applies to all nonwinners at English auctions.

The mechanics of the Milgrom-Weber clock model are as follows: First, the clock is set initially at some minimum (reserve) price; in the absence of a reserve price let this minimum price be zero. Subsequently, the price is allowed to rise continuously. As the price rises, a bidder must decide whether to continue or to exit. When described in this fashion, it is clear what potential buyer n , who is a nonwinner of the auction, should do: continue until the price reaches his valuation v_n . Thus, when all but one of the bidders have dropped out, according to the Milgrom-Weber clock model, the auction ends.

What does this imply about the winning bid? For N potential buyers, having ordered valuations

$$v_{(N:N)} < v_{(N-1:N)} < \dots < v_{(3:N)} < v_{(2:N)} < v_{(1:N)},$$

the bidder having valuation $v_{(N:N)}$ will drop out first, followed by the bidder having valuation $v_{(N-1:N)}$. This will continue with the bidder having $v_{(3:N)}$ dropping out second to last and the bidder having $v_{(2:N)}$ dropping out last. When all but one of the bidders have dropped out, the remaining bidder is the winner and the price he pays is the last bid his last opponent was willing to pay, the second order statistic $v_{(2:N)}$.

In summary, the optimal bid function for bidder n at an English auction with $N(\geq 2)$ bidders is to stay in the auction until either the price reaches n 's valuation v_n and then drop out or to stop bidding when all others have dropped out. Put formally, for the n^{th} nonwinner the dominant strategy is to bid his valuation v_n , so equation (2.1) constitutes the equilibrium policy function within the clock model of an English auction. The participant with the highest valuation should continue to bid until the potential buyer with the second-highest valua-

tion drops out, that is, the winning bid W will be $V_{(2:N)}$, which is why English auctions are sometimes referred to as second-price auctions: the winning bid is the second-highest order statistic from a sample of N valuations.

Like its cousin the Vickrey auction, the English auction ensures the efficient allocation of the object as the potential buyer with the highest valuation wins the object. Note, too, that the winning bid represents the opportunity cost of the object as this would be the value of the object in its next-best alternative use. Thus, winning bids realized at English auctions can be useful in estimating the opportunity costs of the objects for sale—information that is very important to economists and policymakers alike.

2.1.3 Derivation of Symmetric Bayes-Nash Equilibrium

Consider now a seller who invites sealed tenders from N potential buyers. After the close of tenders, the bids are opened more or less simultaneously and the object is awarded to the highest bidder; the winner then pays the seller what he bid.

The bid function that characterizes rational, equilibrium behavior at the sealed, pay-your-bid auction is not a dominant-strategy equilibrium bid function. Instead, the equilibrium bid function is referred to as a Bayes-Nash equilibrium bid function, involving an equilibrium concept that is an extension of Nash's original one.

To demonstrate this claim, consider the symmetric IPV model where each potential buyer has a private value for the object for sale and knows his private value, but not those of his competitors. As before, assume that V_n , the value of potential buyer n , is an independent draw from the cdf $F_V(v)$, which is continuous, having an associated pdf $f_V(v)$, but now impose some additional structure—namely, that $f_V(v)$ is strictly positive on the compact interval $[\underline{v}, \bar{v}]$ in which the valuations live, where $\underline{v} \geq 0$. In addition, assume that the number of potential buyers N as well as the cdf $F_V(v)$ and the support $[\underline{v}, \bar{v}]$ are common knowledge. Initially, assume potential buyers are risk neutral, too.

When potential buyer n , who has valuation v_n , tenders bid s_n , he

receives the following payoff:

$$\text{Payoff}(v_n, s_n) = \begin{cases} v_n - s_n & \text{if } s_n > s_m \text{ for all } n \neq m \\ 0 & \text{otherwise.} \end{cases}$$

Assume that potential buyer n chooses s_n to maximize his expected payoff, in this case his expected profit,

$$U_n(v_n, s_n) = (v_n - s_n) \Pr(\text{win}|s_n). \quad (2.2)$$

What is the structure of $\Pr(\text{win}|s_n)$? Within this model, the identities of the potential buyers (their subscript n) are irrelevant because all potential buyers are *ex ante* identical. Therefore, without a loss of generality, we can focus on the problem faced by potential buyer n . Suppose the opponents of bidder n use a bid strategy that is a strictly increasing, continuous, and differentiable function $\sigma(v)$. Bidder n will win the auction with tender s_n when all of his opponents bid less than him because their valuations of the object are less than his. In absolutely gory detail,

$$\begin{aligned} \Pr(\text{win}|s_n) &= \Pr(S_1 < s_n, S_2 < s_n, \dots, S_{n-1} < s_n, S_{n+1} < s_n, \dots, S_N < s_n) \\ &= \Pr[(S_1 < s_n) \cap (S_2 < s_n) \cap \dots \cap (S_{n-1} < s_n) \cap \\ &\quad (S_{n+1} < s_n) \cap \dots \cap (S_N < s_n)] \\ &= \prod_{m \neq n} \Pr(S_m < s_n) \\ &= \prod_{m \neq n} \Pr[\sigma(V_m) < s_n] \\ &= \prod_{m \neq n} \Pr[V_m < \sigma^{-1}(s_n)] \\ &= F_V [\sigma^{-1}(s_n)]^{N-1} \\ &\equiv F_V [\varphi(s_n)]^{N-1}, \end{aligned}$$

which means equation (2.2) can be written as

$$U_n(v_n, s_n) = (v_n - s_n) \Pr(\text{win}|s_n) = (v_n - s_n) F_V [\varphi(s_n)]^{N-1} \quad (2.3)$$

where $\varphi(\cdot)$ is the inverse-bid function. Differentiating equation (2.3)

with respect to s_n yields the following first-order condition:

$$\begin{aligned} \frac{dU_n(v_n, s_n)}{ds_n} &= F_V[\varphi(s_n)]^{N-1} + \\ & (v_n - s_n)(N-1)F_V[\varphi(s_n)]^{N-2} f_V[\varphi(s_n)] \frac{d\varphi(s_n)}{ds_n} = 0. \end{aligned} \quad (2.4)$$

In a Bayes-Nash equilibrium, $\varphi(s)$ equals v . Also, under monotonicity, we know from the inverse function theorem that $d\sigma(v)/dv$ equals $ds/d\varphi(s)$, so dropping the n subscript yields

$$\frac{d\sigma(v)}{dv} + \sigma(v) \frac{(N-1)f_V(v)}{F_V(v)} = \frac{(N-1)v f_V(v)}{F_V(v)}. \quad (2.5)$$

In summary, within the symmetric IPV model, equilibrium behavior is characterized by a first-order ordinary differential equation (ODE); that is, the differential equation involves only the valuation v , the bid function $\sigma(v)$, and the first derivative of the bid function $d\sigma(v)/dv$, which we often denote in a short-hand by $\sigma'(v)$. Although the valuation v enters nonlinearly through the functions $f_V(v)$ and $F_V(v)$, the differential equation is considered linear because $\sigma'(v)$ can be expressed as a linear function of $\sigma(v)$. These features make the solution to this differential equation tractable, but as we will see in the section 2.4.6, they only hold within the symmetric IPV model.

Equation (2.5) is among the few differential equations that have a closed-form solution. Using a notation that will be familiar to students of introductory calculus, we note that if differential equations are of the following form:

$$y' + p(x)y = q(x),$$

then following the text by William E. Boyce and Richard C. DiPrima [1977], there exists a function $\nu(x)$ such that

$$\nu(x)[y' + p(x)y] = [\nu(x)y]' = \nu(x)y' + \nu'(x)y.$$

Thus,

$$\nu(x)p(x)y = \nu'(x)y.$$

When ν is positive, as it will be in the auction case because it is the ratio of two positive functions multiplied by a positive integer,

$$\frac{\nu'(x)}{\nu(x)} = p(x),$$

so

$$\log[\nu(x)] = \int_{x_0}^x p(u) \, du,$$

whence

$$\nu(x) = \exp \left[\int_{x_0}^x p(u) \, du \right].$$

Therefore,

$$\nu(x)y = \int_{x_0}^x \nu(u)q(u) \, du + k,$$

for some constant k , or

$$y = \frac{1}{\nu(x)} \left[\int_{x_0}^x \nu(u)q(u) \, du + k \right],$$

where k is chosen to satisfy an initial condition $y(x_0)$ equals y_0 .

To solve equation (2.5) in a closed-form, a condition relating v and s must be known. Fortunately, economic theory provides this known relationship at one critical point. In the absence of a reserve price, the minimum price that must be bid, $\sigma(\underline{v})$ equals \underline{v} ; that is, a potential buyer having the lowest value \underline{v} will bid his value. In the presence of a reserve price r , $\sigma(r)$ equals r .

The appropriate initial condition, together with the differential equation, constitute an initial-value problem, which has the following unique solution:

$$\sigma(v) = v - \frac{\int_{\underline{v}}^v F_V(u)^{N-1} \, du}{F_V(v)^{N-1}}. \quad (2.6)$$

This is the symmetric Bayes-Nash equilibrium bid function of the n^{th} bidder; it was characterized by Holt [1980] as well as Riley and Samuelson [1981].

Basically, when potential buyer n bids, he shaves his valuation v_n by the term to the right of the minus sign in equation (2.6). How much potential buyer n shaves his bid depends on the number of potential buyers N , how high the reserve price r is (if one exists), how high his valuation v_n is, and the shape of the valuation distribution as captured by $F_V(\cdot)$.

Potential buyer n shaves less when N is larger than when N is small: competition can matter. In fact, as N gets very large, this model of an auction converges to the perfectly competitive model of sale. Potential

buyer n also shaves less when r is large than when r is small. Note, too, that potential buyer n shaves more when v_n is large than when v_n is small, but he also bids more when v_n is large than when v_n is small. That is how the auction induces the highest-valuation potential buyer to reveal his willingness to pay the most for the object.

Note, too, however, that when V is defined on the interval $[\underline{v}, \bar{v}]$, S is defined on the interval $[\sigma(\underline{v}), \sigma(\bar{v})] \equiv [\underline{v}, \bar{s}(N)]$, where

$$\bar{s}(N) \equiv \sigma(\bar{v}) = \bar{v} - \frac{\int_{\underline{v}}^{\bar{v}} F_V(u)^{N-1} du}{F_V(\bar{v})^{N-1}} = \bar{v} - \int_{\underline{v}}^{\bar{v}} F_V(u)^{N-1} du < \bar{v}.$$

In other words, the support of S is different from that of V . Moreover, the support of S depends in a highly nonlinear way on $F_V(v)$ as well as N , not just \underline{v} and \bar{v} (or r and \bar{v} when a reserve price is imposed).

Suffice it to say that the relationship between σ and V is quite complicated. Not surprisingly the relationship between the winning bid at sealed, pay-your-bid auctions and the opportunity cost $v_{(2:N)}$ is unclear.

What about Dutch auctions? Under the conditions of the symmetric IPV model, the oral, descending-price and the sealed, pay-your-bid auctions are strategically equivalent. To see this, consider the decision problem faced by a potential buyer at a Dutch auction, where the price starts high and then falls continuously until someone stops it. Depending on the bidder's valuation for the object, he must decide at what point to stop the auction by signaling his willingness to pay the existing price. But this situation is strategically identical to that faced by a bidder at a sealed, pay-your-bid auction, who must decide how high to bid for the object. Thus, the optimal bid function that characterizes rational, equilibrium behavior at the oral, descending-price auction is the same as at the sealed, pay-your-bid auction.

2.2 CV Model

Within the pure CV model, the release of information is critical. In order to simplify the theoretical development of equilibrium bidding within the pure CV model, we first characterize the equilibrium at sealed, pay-your-bid auctions, and then describe the complications that

arise under different assumptions concerning information release, but first some notation.

Within the pure CV model, the true, but unknown, value is denoted by v , whereas the estimator of (the signal concerning) v is denoted by X . The pdf and cdf of X conditional on v are denoted by $f_{X|V}(x|v)$ and $F_{X|V}(x|v)$, respectively, while the prior density of V is denoted by $g_V(v)$.

2.2.1 Pay-Your-Bid Auctions

To construct an equilibrium to the auction game, we focus on the Bayes-Nash notion: Suppose that the $(N - 1)$ opponents of potential buyer n are using a common bidding rule $\sigma(x)$ that is increasing and differentiable in their estimate x . Since estimators are modeled as independent draws from a common distribution, conditional on the true value v , the probability of player n winning with bid s_n is

$$\Pr(\text{win}|s_n) = F_{X|V}[\sigma^{-1}(s_n)|v]^{N-1} \equiv \pi[\sigma^{-1}(s_n)|v].$$

Conditional on his estimate x_n , the expected pay-off from bid s_n is

$$\int_{-\infty}^{\infty} (v - s_n) \pi[\sigma^{-1}(s_n)|v] \ell(v|x_n) \, dv,$$

where by Bayes' rule

$$\ell(v|x_n) = \frac{f_{X|V}(x_n|v)g_V(v)}{\int_{-\infty}^{\infty} f_{X|V}(x_n|v)g_V(v) \, dv},$$

is proportional to

$$\int_{-\infty}^{\infty} (v - s_n) \pi[\sigma^{-1}(s_n)|v] f_{X|V}(x_n|v) g_V(v) \, dv.$$

Maximizing behavior implies that the optimal bid solves the following first-order condition:

$$\begin{aligned} & - \int_{-\infty}^{\infty} \pi[\sigma^{-1}(s_n)|v] f_{X|V}(x_n|v) g_V(v) \, dv + \\ & \int_{-\infty}^{\infty} (v - s_n) \pi'[\sigma^{-1}(s_n)|v] \frac{d\sigma^{-1}(s_n)}{ds_n} f_{X|V}(x_n|v) g_V(v) \, dv = 0. \end{aligned} \quad (2.7)$$

Symmetry among bidders implies

$$s_n = \sigma(x_n). \quad (2.8)$$

Substituting equation (2.8) into equation (2.7), and requiring equation (2.8) to hold for all feasible x , yields the following differential equation for σ :

$$\begin{aligned} \sigma'(x) \int_{-\infty}^{\infty} \pi(x|v) f_{X|V}(x|v) g_V(v) \, dv + \\ \sigma(x) \int_{-\infty}^{\infty} \pi'(x|v) f_{X|V}(x|v) g_V(v) \, dv = \\ \int_{-\infty}^{\infty} v \pi'(x|v) f_{X|V}(x|v) g_V(v) \, dv. \end{aligned} \quad (2.9)$$

Equation (2.9), too, is a first-order, linear ODE of the following form:

$$\sigma'(x) + p(x)\sigma(x) = q(x),$$

which has solution

$$\sigma(x) = \frac{1}{\nu(x)} \left[\int_{\underline{x}}^x \nu(u) q(u) \, du + k \right],$$

where

$$\nu(x) = \exp \left[\int_{\underline{x}}^x p(u) \, du \right],$$

with the constant k chosen to satisfy a boundary condition of the form $\sigma(x_0) = s_0$. The winner is the participant with the highest estimate $x_{(1:N)}$ and he pays $\sigma[x_{(1:N)}]$.

At this point, economists are wont to conduct comparative statics, with the number of potential buyers N being the natural candidate. Two effects accompany increased competition: For low numbers of potential buyers, the competitive effect prevails and the bid function increases in N . Eventually, however, the adverse selection effect takes over. A bidder rationally anticipates that if he wins, then it is because his estimate is unusually high relative to those of his opponents, so he shaves his bid. Because the winning bid is a monotonic function of $X_{(1:N)}$, its distribution is related to that of the highest order statistic for a sample of size N from the distribution of estimators.

The reader will have surely noticed that the pure CV model is computationally much more complicated than the IPV model, if only because solutions to equation (2.9) typically do not have closed forms. What can one do to ameliorate the computational burdens encountered when calculating the equilibrium bid functions? Implementing the CV model requires putting additional structure on $p(x)$ and $\nu(x)q(x)$, so these functions can be easily integrated, either analytically or numerically. This clearly restricts the classes of functions for $f_{X|V}(x|v)$ and $g_V(v)$ that a researcher can entertain. Researchers working on sealed, pay-your-bid auctions have basically adopted two different strategies.

2.2.2 Bid Functions that Are Proportional to Estimates

Michael H. Rothkopf [1969, 1971] and Albert K. Smiley [1979] chose to focus on equilibria in which the bid function is proportional to the estimate x , so

$$\sigma(x) = \rho x$$

where the constant of proportionality ρ (< 1) depends on the parameters of the distribution and the number of potential buyers. Smiley provided the following conditions upon $F_{X|V}(x|v)$ and $g_V(v)$ that are sufficient to obtain these results:

$$F_{X|V}(kx|kv) = F_{X|V}(x|v) \quad \text{for } k > 0$$

and

$$g_V(v) \propto \frac{1}{v^2}.$$

The former condition is a homogeneity assumption that restricts the way in which the cdf of the estimator X shifts in response to a shift in v . Smiley noted that the latter condition concerning $g_V(v)$ implies the following: once a bidder has made an unbiased estimate of v , the posterior expected value of v is just the estimate itself. This means that the prior is neutral in the sense that the bidder's prior expectations concerning v do not shift the posterior expected value of v away from the estimate. Smiley demonstrated that the Gumbel, Log-Normal, and Weibull distributions satisfy these two conditions, so the equilibrium bid functions are proportional to the signal.

However compelling the above approach may appear, Richard Engelbrecht-Wiggans and Weber [1979] demonstrated that multiplicative strategies do not constitute a Bayes-Nash equilibrium in general. In short, one must implement the model of Wilson [1977] in its gory detail. One approach that accomplishes this in a tractable way involves making stark assumptions concerning the conditional distribution of signals.

2.2.3 Bid Functions for Normally Distributed Estimates

Building on the research of Rothkopf [1980], Stuart E. Thiel [1988] proposed an alternative approach, which was to consider distributions of X (or distributions of a transformation of X) that live within the location-scale family. In such cases, one can standardize X (by subtracting out the mean v and by scaling by the standard deviation ω) to obtain

$$Z = \frac{X - v}{\omega}.$$

If potential buyers have identical and diffuse prior distributions concerning v , and the estimation errors are normal and independent of the true value, then Dan Levin and James L. Smith [1991], in a comment concerning Thiel [1988], demonstrated that the following characterizes the family of symmetric Bayes-Nash equilibria:

$$\sigma(x) = x - \delta_N \omega + \psi \exp \left[-x \zeta_{(1:N)} / \omega \right], \quad (2.10)$$

where ψ (≤ 0) is a parameter and

$$\delta_N = \frac{\int_{-\infty}^{\infty} N z^2 \Phi(z)^{N-1} \phi(z) \, dz}{\int_{-\infty}^{\infty} N z \Phi(z)^{N-1} \phi(z) \, dz} \equiv \frac{\int_{-\infty}^{\infty} N z^2 \Phi(z)^{N-1} \phi(z) \, dz}{\zeta_{(1:N)}}.$$

Here $\phi(\cdot)$ and $\Phi(\cdot)$ denote respectively the pdf and the cdf of a standard normal random variable, while $\zeta_{(1:N)}$ denotes the expectation of the standardized estimator $Z_{(1:N)}$ from a sample of size N . Note that equation (2.10) implies a continuous family of equilibrium bid functions, one of which, $\psi = 0$, is linear in the estimate.

2.2.4 Second-Price Auctions

In order to illustrate how one might proceed in the case of the English auction, adopt a model of closed exit; that is, assume that potential buyers do not observe when their opponents drop out of the auction. Initially, assume no reservation price exists, so that all potential buyers attend the auction. In the absence of impatient bidders, the above model eliminates any dynamic considerations that the clock model of an English auction would introduce.

Under these assumptions, the English auction is strategically equivalent to a sealed, second-price auction; see the paper by Sushil Bikhchandani and Riley [1991] as well as the survey by Wilson [1992]. A rational bidder will bid so that if he has the highest estimate he bids, on average, what his nearest neighbor would value the object at, given that information. This turns out to be the conditional expectation of v , given that the first and second order statistics for the estimates are equal. That is, the equilibrium bid function is

$$\begin{aligned}\beta(x) &= \mathbb{E} \left[V \mid X_{(1:N)} = x, X_{(2:N)} = x \right] \\ &= \frac{\int_{-\infty}^{\infty} v F_{X|V}(x|v)^{N-2} f_{X|V}(x|v)^2 g_V(v) \, dv}{\int_{-\infty}^{\infty} F_{X|V}(x|v)^{N-2} f_{X|V}(x|v)^2 g_V(v) \, dv}.\end{aligned}\tag{2.11}$$

In general, equation (2.11) does not have a closed-form solution, but it can be calculated numerically if distributions for $F_{X|V}(x|v)$ and $g_V(v)$ have been specified.

Theory predicts that, for a fixed x , $\beta(x)$ may rise with competition, but will eventually fall. Again, two effects are at work: With more competition bidders must be more aggressive, but with a higher N bidders must recognize the adverse selection effect (namely, the winner will be the bidder with the most optimistic estimate of the object's value), and thus bids must be shaved. This latter effect eventually prevails, often at relatively low numbers of potential buyers, for instance just three. Such predictions are often important when ruling out particular theoretical models of auctions.

Of course, another (in some circumstances more realistic) situation is where the exit of opponents is observed. In such circumstances, however, knowing exactly at what prices those bidders exited the auction

is critical. In the past, prior to the advent of Big Data, collecting such information was prohibitively costly, but today in certain cases it may not be.

2.3 AV Model

Milgrom and Weber [1982] invented the AV model in order to incorporate dependence in signals/valuations among potential buyers. The AV model is more general than either the IPV or the CV models: the IPV and the CV models are special cases of the AV model. The key technical problem faced by Milgrom and Weber was to impose enough structure on the form of dependence in the signals/valuations to ensure the existence of a Bayes-Nash equilibrium. Whether that equilibrium is unique is sometimes in question; uniqueness of equilibrium is particularly important within the structural econometric approach.

2.3.1 Different Forms of Dependence

Several different types of dependence exist. Most people are familiar with Galton's coefficient of linear correlation, which depends on the covariance between two random variables V_1 and V_2 collected in the vector \mathbf{V} :

$$\sigma_{12} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [v_1 - \mathbb{E}(V_1)][v_2 - \mathbb{E}(V_2)] f_{\mathbf{V}}(v_1, v_2) dv_1 dv_2.$$

Other forms of dependence exist as well. For example, Samuel Karlin [1968] introduced the term multivariate total positivity of order two, or MTP_2 for short. In economics, Milgrom and Weber [1982] coined the term affiliation when referring to MTP_2 . Affiliation is a condition concerning the joint distribution of \mathbf{V} : using the example from Milgrom and Weber [1982], two random variables are affiliated if high (low) values of each are more likely to occur than high and low or low and high values. This seems like a relatively innocuous condition. Affiliation is frequently assumed in models of incomplete information, such as the adverse selection models employed in auction theory.

Following Karlin [1968], in the case of continuous random variables, suppose (V_1, V_2, \dots, V_N) have joint pdf $f_{\mathbf{V}}(\mathbf{v})$. Consider \mathbf{v}' and \mathbf{v}'' . The

random variables \mathbf{V} are said to be affiliated if

$$f_{\mathbf{V}}(\mathbf{v}' \vee \mathbf{v}'') f_{\mathbf{V}}(\mathbf{v}' \wedge \mathbf{v}'') \geq f_{\mathbf{V}}(\mathbf{v}') f_{\mathbf{V}}(\mathbf{v}'') \quad (2.12)$$

where

$$(\mathbf{v}' \vee \mathbf{v}'') = [\max(v'_1, v''_1), \max(v'_2, v''_2), \dots, \max(v'_N, v''_N)]$$

denotes the component-wise maxima of \mathbf{v}' and \mathbf{v}'' , sometimes referred to as the join, while

$$(\mathbf{v}' \wedge \mathbf{v}'') = [\min(v'_1, v''_1), \min(v'_2, v''_2), \dots, \min(v'_N, v''_N)]$$

denotes the component-wise minima, sometimes referred to as the meet.

Essentially, under affiliation, with continuous random variables, the off-diagonal elements of the Hessian of the logarithm of the joint pdf are all non-negative; that is, the joint pdf is log-supermodular, so

$$\frac{\partial^2 \log f_{\mathbf{V}}(\mathbf{v})}{\partial v_i \partial v_j} \geq 0.$$

Under joint normality of random variables, affiliation requires that all the pair-wise covariances be weakly positive.

How is affiliation related to other forms of dependence? Consider two continuous random variables V_1 and V_2 , having joint pdf $f_{\mathbf{V}}(v_1, v_2)$ as well as conditional pdf $f_{V_2|V_1}(v_2|v_1)$ and conditional cdf $F_{V_2|V_1}(v_2|v_1)$. Introduce $g(\cdot)$ and $h(\cdot)$, functions that are nondecreasing in their arguments. Luciano I. de Castro [2007] noted that affiliation implies that

- a) $[F_{V_2|V_1}(v_2|v_1)/f_{V_2|V_1}(v_2|v_1)]$ is decreasing in V_1 , often referred to as a decreasing inverse hazard rate, which implies that
- b) $\Pr(V_2 \leq v_2|V_1 = v_1)$ is nonincreasing in V_1 , also referred to as positive regression dependence, which implies that
- c) $\Pr(V_2 \leq v_2|V_1 \leq v_1)$ is nonincreasing in V_1 , also referred to as left-tail decreasing in V_1 , which implies that
- d) $\text{cov}[g(V_1), h(V_2)]$ is positive, which implies that
- e) $\text{cov}[g(V_1), h(V_2)]$ is positive, which implies that

f) $\text{cov}(V_1, V_2)$ is positive.

The key point is that affiliation is a much stronger form of dependence than positive covariance. Having explored weaker forms of dependence, de Castro noted that affiliation is a stronger condition than is necessary to guarantee the existence of a pure strategy equilibrium, which means that affiliation can in principle be tested.

Using a graph, derived from an example first presented by Milgrom and Weber, and then developed further by de Castro and Paarsch [2010], we can depict the space of affiliated distributions using two random variables: four possible outcomes for two random variables each having just two values as depicted in Figure 2.1. The $(1, 1)$ and $(2, 2)$ points are more likely than the $(2, 1)$ or $(1, 2)$ points. Letting θ_{ij} denote the probability of (i, j) , affiliation in this example is then reduced to total positivity of order two (or TP_2 for short), namely,

$$\theta_{11}\theta_{22} \geq \theta_{12}\theta_{21}.$$

Put another way, the determinant of the matrix

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix}$$

must be weakly positive. Independence in valuations obviously satisfies the lower bound on this determinantal inequality. Note, too, that affiliation restricts distributions to a part of the simplex depicted in Figure 2.2. In that figure, it is the region on the simplex that appears to be a semicircle rising from the line where $\theta_{11} + \theta_{22}$ equals one. In order to draw this figure, symmetry was imposed, so θ_{12} and θ_{21} are equal; thus, the intercept for θ_{12} is one half.

Another important point to note is that affiliation is a global restriction. It is well known that a function is MTP_2 (affiliated), if and only if, it is TP_2 in all collections of four points. To see the importance of this fact, introduce another value for each random variable, 3; five additional points then appear, as is depicted in Figure 2.3. Affiliation requires that the probabilities at all collections of four points satisfy TP_2 ; that is, the following additional six inequalities must hold:

$$\theta_{12}\theta_{23} \geq \theta_{13}\theta_{22}, \quad \theta_{22}\theta_{33} \geq \theta_{23}\theta_{32}, \quad \theta_{21}\theta_{32} \geq \theta_{22}\theta_{31},$$

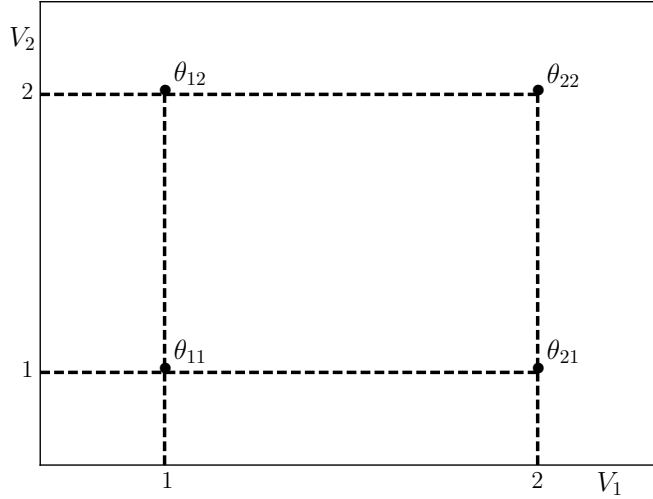


Figure 2.1: Affiliation—Two Values and Two Random Variables

$$\theta_{11}\theta_{33} \geq \theta_{13}\theta_{31}, \quad \theta_{12}\theta_{33} \geq \theta_{13}\theta_{32}, \quad \theta_{21}\theta_{33} \geq \theta_{23}\theta_{31}.$$

Of course, symmetry would imply that θ_{ij} equal θ_{ji} for all i and j , so the joint probability mass function (pmf) of the two random variables and three values under symmetric affiliation can be written as the following matrix:

$$\Theta = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{bmatrix} = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

where the determinants of all (2×2) submatrices must be positive, which reduces to the following inequalities:

$$ab \geq d^2, \quad df \geq be, \quad bc \geq f^2, \quad \text{and} \quad ac \geq e^2.$$

And all the points must also live on the simplex, so

$$0 \leq a, b, c, d, e, f < 1 \text{ and } a + b + c + 2d + 2e + 2f = 1.$$

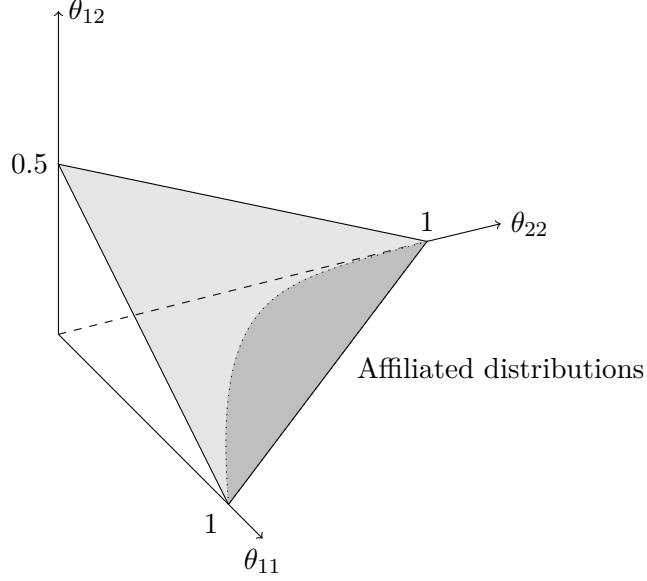


Figure 2.2: Probability Set under Affiliation and Alternative

Adding values to the random variables expands the number of determinantal restrictions required to satisfy TP_2 , thus restricting the space of distributions that can be entertained. Likewise, adding random variables, particularly if the random variables are assumed symmetric, also restricts the space of distributions that can be entertained. For example, suppose that a third random variable is added. The pmf for triplets of values (v_1, v_2, v_3) , where $v_n = 1, 2, 3$ and $n = 1, 2, 3$, can be represented as an array whose slices can then be represented by the following three matrices for players 1 and 2, indexed by the values of player 3:

$$\Theta_1 = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}, \quad \Theta_2 = \begin{bmatrix} d & b & f \\ b & h & g \\ f & g & i \end{bmatrix}, \quad \text{and} \quad \Theta_3 = \begin{bmatrix} e & f & c \\ f & g & i \\ c & i & j \end{bmatrix}.$$

In general, affiliation puts perhaps incredible structure on the relationships among signals/valuations, which in turn delivers very sharp theoretical predictions—for instance, the linkage principle of

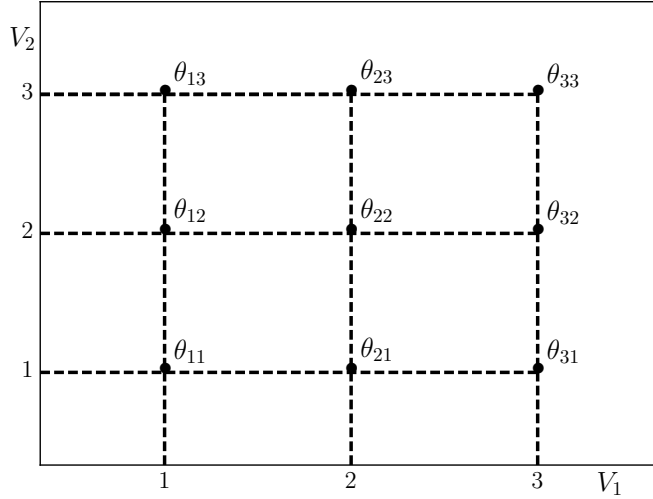


Figure 2.3: Affiliation—Three Values and Two Random Variables

Milgrom and Weber. Whether the assumption of affiliation is supported by data remains an open question, which is presumably why researchers continue to investigate it—for instance, see the paper by de Castro and Paarsch [2010].

2.3.2 Some Additional Notation

Within the AV model, the N potential buyers each have a valuation V_n for the object, and receive a private signal X_n concerning V_n . Potential buyer n only observes X_n prior to the beginning of the auction; he does not observe any of the valuations, V_n for all $n \in \mathcal{N}$, or any other bidder's signal, X_m , for $m \neq n$. Assume that the valuations and signals are jointly distributed according to the distribution function $F_{\mathbf{V}\mathbf{X}}(v_1, \dots, v_N, x_1, \dots, x_N)$.

Note that in a private-values model $V_n = X_n$ for $n \in \mathcal{N}$ (that is, each bidder knows his true valuation for the object), while in a

common-value model $V_n = V_0$ for $n \in \mathcal{N}$ (that is, the value of the object is the same to all potential buyers, but none of them knows the true value of the object). In this case, the X_n s are noisy signals of the, true but unknown, V . Typically, however, one would expect that both common- and private-valued components are present in the valuation that a potential buyer has for the object on sale. In general, V_n is a function of all the signals of the potential buyers as well as other information variables that may not be observed by any of them.

When the joint distribution $F_{\mathbf{V}\mathbf{X}}(\cdot)$ is symmetric with respect to $n = 1, \dots, N$, the model is symmetric; otherwise it is asymmetric. Under independence of valuations, the joint distribution $F_{\mathbf{V}\mathbf{X}}(\cdot)$ can be factored into marginal distributions of V_n, X_n , $n \in \mathcal{N}$, namely,

$$F_{\mathbf{V}\mathbf{X}}(v_1 \dots, v_N, x_1 \dots, x_N) = \prod_{n=1}^N F_n(v_n, x_n).$$

Different combinations of common-value and private-values assumptions, symmetric and asymmetric distribution assumptions, and independence and dependence assumptions create a host of possible options for each auction format and pricing rule, which is why the paper by Milgrom and Weber [1982] is such an important one in economic theory.

In general, potential buyer n 's equilibrium bid is a function of all the private signals: the form of the bid function depends on the auction format and the pricing rule as well as the joint distribution of signals and valuations. For example, at sealed auctions, the bid function depends only on X_n .

2.3.3 Sealed, Pay-Your-Bid Auctions

The sealed, pay-your-bid auction proceeds as follows: Having observed $X_n = x$, potential buyer n chooses a bid s_n to maximize his expected payoff, given his opponent's equilibrium behavior. In symbols,

$$\sigma(x) = \operatorname{argmax}_s \mathbb{E} \left\{ (V_n - s) \mathbf{1} \left[X_m \leq \sigma_m^{-1}(s), m \neq n | X_n = x \right] \right\},$$

where $\mathbf{1}(\mathbf{A})$ is an indicator function of the event \mathbf{A} and $\sigma_m(\cdot)$ denotes the equilibrium bid function for $m = 1, \dots, N$. Using the inverse-bid

notation, $\varphi_m(\cdot) \equiv \sigma_m^{-1}(\cdot)$, which are collected in the vector $\boldsymbol{\varphi}(\cdot)$, with $\boldsymbol{\varphi}_{-n}(\cdot)$ denoting that vector with $\varphi_n(\cdot)$ excluded, the first-order condition of this maximization problem is:

$$\begin{aligned} & \sum_{m \neq n} \frac{\partial G_n [\boldsymbol{\varphi}_{-n}(s) | X_n = x]}{\partial x_m} \frac{1}{\sigma'_m [\varphi_m(s)]} \\ & \mathbb{E} [V_n | X_n = x, X_m = \varphi_m(s), X_\ell \leq \varphi_\ell(s), \ell \neq m, n] - \\ & G_n [\boldsymbol{\varphi}_{-n}(s) | X_n = x] - \\ & \sigma_n(x) \sum_{m \neq n} \frac{\partial G_n [\boldsymbol{\varphi}_{-n}(s) | X_n = x]}{\partial x_m} \frac{1}{\sigma'_m [\varphi_m(s)]} = 0 \end{aligned} \quad (2.13)$$

where $G_n(\boldsymbol{x}_{-n} | X_n = x)$ denotes the conditional distribution of \boldsymbol{X}_{-n} , the $(N - 1)$ subvector of the signals excluding X_n , given $X_n = x$.

The system of N first-order conditions defined by equation (2.13) implicitly defines the collection of N equilibrium bid functions $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$. Under certain assumptions, these equations simplify. For example, in the symmetric IPV model $V_n = X_n$ for $n \in \mathcal{N}$, so the equilibrium bid functions are defined by the first-order ODE given in equation (2.5), which has the solution given in equation (2.6).

Even though the AV model is quite general, as will be noted later in section 4, it is unidentified, except in some of the special cases described in that section. In other words, the elegance and power of the AV model does not translate fully into empirical specifications that can be implemented.

2.4 Complications

Perhaps the most glaring complication introduced by reality is a binding reserve price, which is why we investigate it first. A second theoretical complication, which cannot necessarily be observed, is risk aversion. We investigate this next because it is a relatively straightforward adjustment to the model, which delivers additional flexibility. As a bridge between a binding reserve price and a full model of endogenous participation, which can arise for several different reasons, we next investigate an exogenous, but stochastic, number of opponents. Finally, we investigate both asymmetric bidders and bid increments, which are theoret-

ically and practically relevant complications. Because the IPV model is especially tractable, we illustrate the complications using it.

2.4.1 Binding Reserve Price

In the presence of a binding reserve price, the number of participants at an auction M is random—endogenous. To see this, consider \mathcal{N} , the set of potential buyers who have values that are collected in the vector (V_1, V_2, \dots, V_N) . For a pre-announced reserve price r , denote attendance at the auction by the Bernoulli random variable A_n $n = 1, 2, \dots, N$, where

$$A_n = \begin{cases} 1 & \text{when } V_n \geq r, \\ 0 & \text{otherwise.} \end{cases}$$

In words, potential buyer n attends the auction (A_n is one) when his value is above or equal to the reserve price, otherwise he does not. Now, $\Pr(A_n = 1)$ is the probability that V_n is in the right tail of the distribution of values, above r . We know, however, that $\Pr(V_n \geq r)$ is $[1 - \Pr(V_n < r)]$, but this is just $[1 - F_V(r)]$. The number of participants at an auction M is the sum of the $\{A_n\}_{n=1}^N$, that is,

$$M = \sum_{n=1}^N A_n.$$

In statistics, the random variable M is referred to as a binomial random variable and its pmf of a realization m is

$$p_M(m|F_V, N, r) = \binom{N}{m} [1 - F_V(r)]^m F_V(r)^{N-m}. \quad (2.14)$$

When the reserve price is low, as in figure 2.4, the probability of no one attending the auction, the left-most \bullet in the graph above 0, is quite small, around 0.00032. On the other hand, when the reserve price is relatively high, as in figure 2.5, the probability of no one attending the auction, the left-most \bullet in the graph above 0, is around 7.78 percent, still small, but 243 times larger than in the first case: a 200 percent increase in the reserve price results in 2,330 percent increase in the probability that no one attends the auction, an elasticity of over 11.

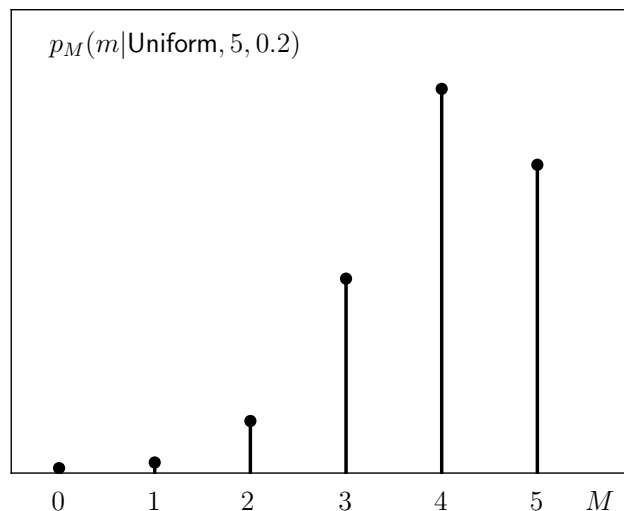


Figure 2.4: pmf of Number of Participants—Low Reserve Price

Thus, in practice, it seems likely that participation will be fairly responsive to changes in the reserve price. In other words, the reserve price matters.

Note that the probability of just one potential buyer's participating in the auction is

$$p_M(1|F_V, N, r) = N[1 - F_V(r)]F_V(r)^{N-1}.$$

In the event that only one potential buyer attends the auction, that participant gets the object on sale for the reserve price at both English and Vickrey auctions, so knowing this probability is important when deciding on the optimal reserve price.

From the perspective of an applied econometrician, the presence of a binding reserve price means that the number of participants at an auction is endogenous. The practical implication of this fact is that the assumption of exogeneity maintained in regression methods is not satisfied if all the bids or just the winning bid are regressed on different features (including the number of bidders at the auction).

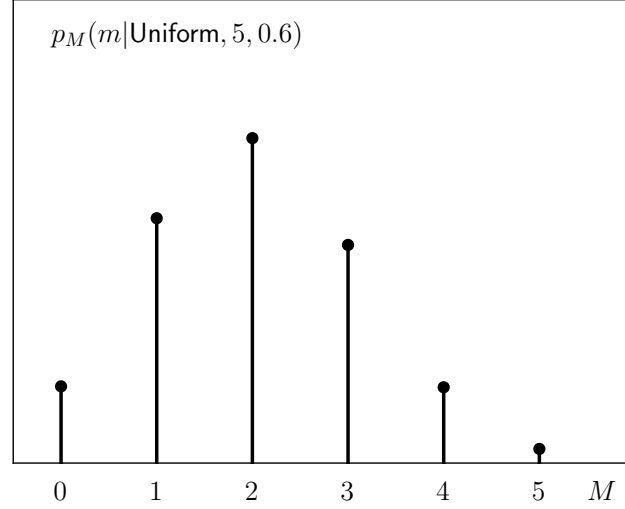


Figure 2.5: pmf of Number of Participants—High Reserve Price

2.4.2 Risk Aversion

The RET does not hold when potential buyers are risk averse. Following Riley and Samuelson [1981] as well as Matthews [1983], we demonstrate that expected revenues at pay-your-bid auctions, either oral or sealed, exceed those at second-price auctions, either oral or sealed.

Let $\hat{\sigma}(\cdot)$ denote the bid function at pay-your-bid auctions with risk-neutral potential buyers and $\tilde{\sigma}(\cdot)$ denote the bid function when potential buyers are risk averse. Given a von Neumann-Morgenstern utility function $u(\cdot)$, the decision problem faced by a potential buyer having valuation v under risk aversion is to report an x to maximize

$$u[v - \tilde{\sigma}(x)]F_V(x)^{N-1}.$$

The necessary first-order condition corresponding to this decision problem is

$$\begin{aligned} & -u'[v - \tilde{\sigma}(x^*)]\tilde{\sigma}'(x^*)F_V(x^*)^{N-1} + \\ & u[v - \tilde{\sigma}(x^*)](N-1)F_V(x^*)^{N-2}f_V(x^*) = 0. \end{aligned}$$

Equilibrium behavior requires that x^* equal v . This can then be written as

$$\tilde{\sigma}'(v) = \frac{u[v - \tilde{\sigma}(v)]}{u'[v - \tilde{\sigma}(v)]} \frac{(N-1)f_V(v)}{F_V(v)}.$$

With risk neutrality, where $u(Y)$ equals Y and $u'(Y)$ is one, the corresponding first-order condition for the bid function is:

$$\hat{\sigma}'(v) = [v - \hat{\sigma}(v)] \frac{(N-1)f_V(v)}{F_V(v)}.$$

When $u(\cdot)$ is strictly concave, because $u(0)$ can be normalized to zero without loss of generality, it is clear that, for Y exceeding zero,

$$\frac{u(Y)}{u'(Y)} > Y.$$

If $\tilde{\sigma}(v)$ weakly exceeds $\hat{\sigma}(v)$ for some v , then we have the following:

$$\tilde{\sigma}'(v) \geq [v - \hat{\sigma}(v)] \frac{(N-1)f_V(v)}{F_V(v)} = \hat{\sigma}'(v).$$

This implies that $\tilde{\sigma}(v')$ weakly exceeds $\hat{\sigma}(v')$ for all v' . At the boundary, where \underline{v} is assumed to be zero for simplicity, $\hat{\sigma}(0)$ and $\tilde{\sigma}(0)$ both equal zero, so it follows that

$$\tilde{\sigma}(v) \geq \hat{\sigma}(v)$$

for all positive v . In words, under the pay-your-bid rule, participants bid more aggressively when they are risk averse than when they are risk neutral. Since the expected revenues are identical under both the pay-your-bid and the second-price rules with risk neutrality, when bidders are risk averse, expected revenues will be higher at pay-your-bid auctions than at second-price auctions.

One particularly tractable specification of the von Neumann-Morgenstern utility function is the so-called hyperbolic absolute risk aversion (HARA) family, where

$$u(Y) = \eta Y^{1/\eta} \quad \eta \geq 1.$$

Under this specification, the equilibrium bid function at pay-your-bid auctions, either oral or sealed, is then

$$\sigma(v) = v - \frac{\int_{\underline{v}}^v F_V(u)^{\eta(N-1)} du}{F_V(v)^{\eta(N-1)}}.$$

See Holt [1980].

2.4.3 Stochastic Numbers of Opponents

As one might expect, introducing a stochastic number of opponents can be done in a number of ways. We outline one—perhaps the most tractable at sealed, pay-your-bid auctions within the symmetric IPV model when potential buyers are risk neutral. McAfee and McMillan [1987a] conducted a detailed analysis of this case.

Suppose that the number of opponents O follows the Poisson law, having pmf

$$p_O(o) = \frac{\lambda^o \exp(-\lambda)}{o!} \quad \lambda > 0, \quad o = 0, 1, \dots$$

Because potential buyers do not know how many opponents they will face, the expected profit function is then

$$\begin{aligned} U_n(v_n, s_n) &= \sum_{o=0}^{\infty} (v_n - s_n) F_V \left[\sigma^{-1}(s_n) \right]^o p_O(o) \\ &= \sum_{o=0}^{\infty} (v_n - s_n) F_V \left[\sigma^{-1}(s_n) \right]^o \frac{\lambda^o \exp(-\lambda)}{o!} \\ &= (v_n - s_n) \exp \left[-\lambda \left(1 - F_V \left[\sigma^{-1}(s_n) \right] \right) \right] \\ &\quad \sum_{o=0}^{\infty} \frac{(F_V \left[\sigma^{-1}(s_n) \right] \lambda)^o \exp(-\lambda F_V \left[\sigma^{-1}(s_n) \right])}{o!} \\ &= (v_n - s_n) \exp \left[-\lambda \left(1 - F_V \left[\sigma^{-1}(s_n) \right] \right) \right], \end{aligned}$$

whence derives the following first-order condition:

$$\begin{aligned} 0 &= -\exp \left[-\lambda \left(1 - F_V \left[\sigma^{-1}(s_n) \right] \right) \right] + \\ &\quad (v_n - s_n) \exp \left[-\lambda \left(1 - F_V \left[\sigma^{-1}(s_n) \right] \right) \right] \lambda f_V \left[\sigma^{-1}(s_n) \right] \frac{d\sigma^{-1}(s_n)}{ds_n}, \end{aligned}$$

which yields the following ODE:

$$\sigma'(v) = [v - \sigma(v)] \lambda f_V(v).$$

The Bayes-Nash equilibrium bid function is

$$\sigma(v) = v - \frac{\int_v^v \exp[\lambda F_V(u)] \, du}{\exp[\lambda F_V(v)]}.$$

How does this bid function compare to that when N is known?

Throughout this review, we will sometimes illustrate various results using a solved example, often graphically. Specifically, we assume that V belongs to the Weibull family on the interval $[\underline{v}, \bar{v}]$, so its pdf is

$$f_V(v|\alpha_1, \alpha_2) = \alpha_1 \alpha_2 v^{\alpha_2-1} \exp(-\alpha_1 v^{\alpha_2}) \quad (2.15)$$

$$\alpha_1 > 0, \alpha_2 > 0, \underline{v} \leq v \leq \bar{v},$$

where we assume that $\alpha_1 = 1$, $\alpha_2 = 2$, $\underline{v} = 0.01$, $\bar{v} = 4$, and $N = 5$. In Figure 2.6, we depict the difference between equilibrium bid functions under the second-price rule $\beta(v)$ as well as the two under the pay-your-bid rule—one calculated with a known $N = 5$, while the other is calculated assuming a Poisson-distributed number of opponents where $\lambda = 4$.

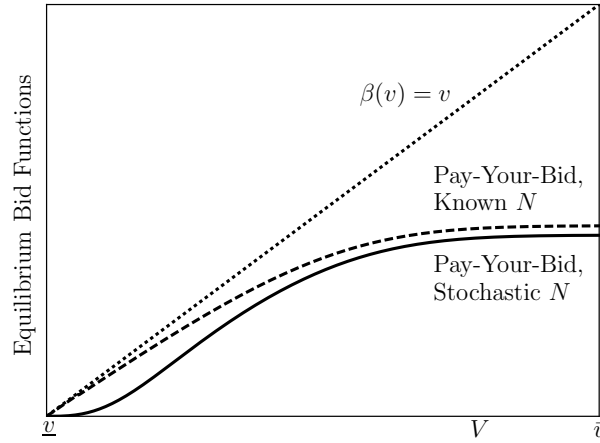


Figure 2.6: Known and Stochastic Numbers of Bidders

In this example, the equilibrium bid function under a Poisson-distributed, stochastic number of buyers is everywhere below that when the number of buyers is known. Why? Under the pay-your-bid pricing rule, in equilibrium, rational potential buyers shave proportionately more when N is small than when it is large. In other words, the profit

margins are proportionately higher when N is small than when it is large. For instance, in the uniform example, the profit margin is 50 percent when $N = 2$; 33.3 percent when $N = 3$; 25 percent when $N = 4$; and 20 percent when $N = 5$, but only 10 percent when $N = 10$. In this Poisson example, more of the mass is below the mean λ than above it. Thus, in equilibrium, a rational bidder tenders less at each level of v because he gambles that fewer than four opponents will typically attend even though on average four do attend.

In this example, the equilibrium bid function under the second-price rule remains $\beta(v) = v$, which is also depicted in the figure. This would seem to violate the RET, but under a Vickrey auction the winning bid would be zero $\exp(-\lambda)$ fraction of time (that is, when no other opponent participates in the auction), while the expectation of the winning bid for O greater than zero is indexed by $(o + 1)$, so would be smaller than the expectation of $V_{(2:N)}$ when $N < O + 1$, where the bulk of the mass is. In short, revenue equivalence is maintained.

2.4.4 Endogenous Participation

An especially troublesome complication is endogenous participation: What happens if the auctioneer's policy choices affect the set of bidders who participate? This is a practically important question because in many real-world auction markets relatively small fractions of eligible bidders actually participate. For instance, Hendricks, Joris Pinkse, and Porter [2003] reported that, on average, only 25 percent of eligible firms participated in U.S. Minerals Management Service "wildcat auctions" held between 1954 to 1970. Tong Li and Xiaoyong Zheng [2009] found that only about 28 percent of planholders in auctions for Texas Department of Transportation mowing contracts actually submitted bids. Similar results have been reported for timber auctions; see, for example, Athey, Jonathan Levin and Enrique Seira [2011], Li and Bingyu Zhang [2015]; online auctions by Patrick Bajari and Ali Hortaçsu [2003]; and highway procurement auctions by Elena Krasnokutskaya and Katja Seim [2011], among others. To rationalize such observations, researchers have turned to models in which auction participation is costly and potential buyers endogenously choose whether

to undertake these costs. As we will see, however, such models may suggest substantially different policy conclusions than our baseline model of exogenous participation.

The analysis of endogenous participation has a long history in the auction literature; several different theoretical models have been proposed. In practice, most empirical work can be viewed within a framework which, in the spirit of Vivek Bhattacharya and Andrew Sweeting [2015], we refer to as a standard auction with simultaneous entry. In this model, entry and bidding proceed in two stages: In the first stage, potential buyers make simultaneous entry decisions, with entry unrestricted subject to the payment of a (potentially private) entry cost. In the second stage, all bidders who chose to enter compete at a standard auction; that is, an auction where, if made, the award is to the highest bidder—provided the highest bid exceeds the reserve price.

Broadly speaking, studies of endogenous participation within standard auctions with simultaneous entry differ along two main dimensions: The first pertains to the baseline auction environment—for example, common-value versus private-values, bidding rules, asymmetric bidders, and so forth. The second pertains to the structure of the entry process specifically—for example, whether bidders have private information prior to entry and whether this information pertains to valuations or entry costs. Differences in both dimensions turn out to have substantial implications for auction design and performance.

Given the previous discussion devoted to modeling the post-entry auction, we focus here on models of the entry process. Specifically, within the symmetric IPV model with simultaneous entry, we propose a novel framework that unifies four key models of endogenous participation: (1) the mixed-strategy entry model of Levin and Smith [1994]; (2) the perfectly-selective entry model of Samuelson [1985]; (3) the private-costs entry model of Jingfeng Lu [2008, 2010], Diego Moreno and John Wooders [2011] as well as Li and Zheng [2009] among others; and (4) the arbitrarily-selective or affiliated-signal model of Lixin Ye [2007]; Marmer, Shneyerov, and Xu [2013]; James W. Roberts and Sweeting [2013]; Matthew L. Gentry and Li [2014]; and Gentry, Li, and Lu [2017]. These four entry models provide the foundations of the vast

majority of empirical studies concerned with endogenous participation.

One exception is online auctions, where the number of potential buyers is very large—effectively infinite. With infinite numbers of potential buyers, these environments obviously require somewhat different modeling techniques. Even then, however, insights from settings with finite numbers of potential buyers are frequently useful. For instance, Bajari and Hortag su [2003] investigated a model of online auctions in which the number of buyers follows a Poisson distribution with an equilibrium entry rate determined by a zero-profit condition analogous to the mixed-strategy entry model of Levin and Smith [1994]. Similarly, Hickman, Hubbard, and Paarsch [2017] assumed that potential buyers first choose whether to participate in an online market based on their known values, with participating buyers being matched randomly to selling counterparts according to a Poisson process. This yields a participation model similar in spirit to the perfectly selective model of Samuelson [1985].

In what follows, we first illustrate features of equilibrium common to all four models, and then discuss the potential policy implications of endogenous entry.

Unifying Framework

Suppose $N \geq 2$ symmetric potential buyers compete for a single object at a standard auction with simultaneous entry. Entry and bidding proceeds in the following stages: In the first, each potential buyer $n \in \mathcal{N}$ observes the realization t_n of a private pre-entry signal T_n potentially correlated with his *ex ante* unknown private valuation V_n . Next, all potential buyers simultaneously decide whether to enter, where potential buyer n with signal t_n must pay cost $c(t_n)$ to enter. In the third stage, $M \leq N$ bidders choose to enter, and then learn the realizations of their private valuations. Finally, the M entrants compete at a standard auction with reserve price $r \geq \underline{v}$. For concreteness, suppose that the number of potential buyers N is common knowledge prior to entry, but the number of bidders M is revealed only after the auction concludes.

The alternative assumption that bidders observe M prior to bidding

is also frequently employed. In this case, equilibrium bidding strategies would, of course, be conditioned on the realization of M , whereas in the setting considered here they are not. By integrating over realizations of M , one can show (at least in the symmetric IPV case) that, prior to entry, bidders expect participation to yield the same average profit when M is observed as when M is unobserved. Hence, at least in settings with independent, private values equilibrium entry strategies are invariant to whether M is observed, or not.

Let $F_{VT}(v, t)$ denote the joint distribution of bidder n 's value-signal pair (V_n, T_n) , $F_{V|T}(v|t)$ denote the cdf of V_n conditional on T_n , and $F_V(v)$ denote the marginal cdf of V_n . For simplicity, we assume that private information is independent across bidders: $(V_n, T_n) \perp (V_m, T_m)$ for $n \neq m$. In addition, we assume that higher pre-entry signals are good news in the sense of yielding both weakly lower entry costs and stochastically higher distributions of valuations: In symbols, for any t_n, t'_n such that $t'_n \geq t_n$, we have both $F_{V|T}(v|t'_n) \leq F_{V|T}(v|t_n)$ and $c(t'_n) \leq c(t_n)$. For simplicity, we normalize pre-entry signals so that T_n has a marginal uniform distribution on $[0, 1]$ for each $n \in \mathcal{N}$. Provided pre-entry signals are continuously distributed, this normalization is without loss of generality because monotone transformations of signals preserve information.¹

Within this framework, the Levin and Smith [1994] model of mixed-strategy entry obtains when we set $c(t_n) = c$ and $T_n \perp V_n$, whereas the Samuelson [1985] model of perfectly-selective entry obtains when we set $c(t_n) = c$ and $t_n = F_V(v_n)$. The heterogeneous-cost models of Lu [2008, 2010], Moreno and Wooders [2011] as well as Li and Zheng [2009] follow if $c(t_n)$ depends on t_n , but $T_n \perp V_n$. Finally, the affiliated-signal or arbitrarily selective models of Ye [2007]; Marmer et al. [2013]; Roberts and Sweeting [2013]; Gentry and Li [2014]; as well as Gentry et al. [2017] obtain when $c(t_n) = c$, but V_n and T_n depend on one another. To our knowledge, this is the first analysis explicitly

¹More precisely, given any initial signal \tilde{T}_n following a continuous marginal distribution $F_{\tilde{T}}$, we can define an equivalent normalized signal T_n by $T_n = F_{\tilde{T}}(\tilde{T}_n)$. Then T_n and \tilde{T}_n are economically equivalent in the sense that the conditional distributions $F_{V|T}(\cdot|T_n)$ and $F_{V|\tilde{T}}(v|\tilde{T}_n)$ are identical, but T_n has a marginal distribution that is uniform by construction.

nesting all four entry models.

Equilibrium with Endogenous Participation

Focus on a Bayes-Nash equilibrium in symmetric, monotone strategies. By definition, entry strategies in any such equilibrium will involve a symmetric signal threshold t^* such that potential buyer n enters if and only if $T_n \geq t^*$. Our main goal here is to characterize this equilibrium entry threshold t^* .

Toward this end, first consider an arbitrary entry threshold \bar{t} . Let $F^*(v|\bar{t})$ denote the (potentially selected) distribution of valuations among bidders electing to enter at threshold \bar{t} —that is, any bidder drawing $T_n \geq \bar{t}$. Recall that T_n is marginally distributed uniformly, so we can express this distribution $F^*(v|\bar{t})$ as:

$$F^*(v|\bar{t}) = \frac{1}{(1 - \bar{t})} \int_{\bar{t}}^1 F_{V|T}(v|t) dt = \frac{[F_V(v) - F_{VT}(v, \bar{t})]}{(1 - \bar{t})}.$$

Next, suppose that on entry bidder n discovers value realization v_n . What is the probability that this bidder wins against any given potential rival m ? Clearly, if m elects not to enter, then n will win against m by default; otherwise, n will win against m if $V_m \leq v_n$. Therefore, the overall probability that entrant n with valuation v_n wins against potential rival m at threshold \bar{t} is

$$\Psi(v_n|\bar{t}) = \bar{t} + (1 - \bar{t})F^*(v_n|\bar{t}) = \bar{t} + [F_V(v_n) - F_{VT}(v_n, \bar{t})].$$

Assume the post-entry auction is standard in the sense defined above. Provided the highest bid weakly exceeds the reserve price, the auctioneer then awards the object to the highest bidder. Let $\pi_r(v_n|\bar{t}, N)$ denote the expected profit of an entrant with valuation $v_n \geq r$ competing against $(N - 1)$ potential opponents who enter according to \bar{t} . Levering the revenue equivalence insights of Riley and Samuelson [1981] and Myerson [1981], $\pi_r(v_n|\bar{t}, N)$ must equal the expected difference between v_n and the maximum between the reserve price and the highest valuation among entering rivals, in the event that v_n is the highest valuation among entrants. Noting that the cdf of the highest valuation among entering rivals is given by $\Psi(v_n|\bar{t})^{N-1}$, we may express this expectation

as

$$\pi_r(v_n|\bar{t}, N) = v_n \Psi(v_n|\bar{t})^{N-1} - r \Psi(r|\bar{t})^{N-1} - \int_r^{v_n} y \, d\Psi(y|\bar{t})^{N-1},$$

where the first term reflects n 's valuation times the probability n wins, the second term corresponds to n 's payment when no rival entrant has a valuation that exceeds the reserve price, and the third term corresponds to n 's payment when at least one rival entrant has a valuation above the reserve. Applying integration by parts to the final integral, we may rewrite this expression somewhat more concisely as

$$\pi_r(v_n|\bar{t}, N) = \int_r^{v_n} \Psi(y|\bar{t})^{N-1} \, dy. \quad (2.16)$$

Consider now potential buyer n choosing whether to enter based on his pre-entry signal $t_n \in [0, 1]$. Assuming that all opponents enter according to threshold \bar{t} , this potential buyer expects post-entry profit

$$\begin{aligned} \Pi_r(t_n|\bar{t}, N) &= \int_r^{\bar{v}} \pi_N(v|\bar{t}, r) \, dF_{V|T}(v|t_n) \\ &= \int_r^{\bar{v}} \int_r^v \Psi(y|\bar{t})^{N-1} \, dy \, dF_{V|T}(v|t_n). \end{aligned}$$

Exchanging the order of integration in the final expression, this expected profit function can be expressed in the following slightly more convenient form:

$$\Pi_r(t_n|\bar{t}, N) = \int_r^{\bar{v}} [1 - F_{V|T}(v|t_n)] \Psi(v|\bar{t})^{N-1} \, dv. \quad (2.17)$$

Finally, consider the marginal equilibrium entrant—namely, the bidder drawing signal $T_n = t^*$. Assuming that $t^* \in (0, 1)$ (so the entry decision is nontrivial), this entrant must be just indifferent to entry when all rivals also enter according to threshold $\bar{t} = t^*$. In other words, the equilibrium entry threshold t^* must satisfy the break-even condition

$$\Pi_r(t^*|t^*, N) = c(t^*). \quad (2.18)$$

Given the assumptions concerning $F_{V|T}(\cdot|t_n)$, it can be easily shown that $\Pi_r(t_n|\bar{t}, N)$ is decreasing in r , increasing in t_n , strictly increasing in \bar{t} , and decreasing in N . Hence, the right-hand side of equation (2.18)

is strictly increasing in t^* . Because $c(t)$ is assumed decreasing in t , it follows that the equilibrium entry threshold t^* will be unique. Furthermore, because $\Pi_r(t_n|\bar{t}, N)$ is decreasing in both r and N , t^* will be increasing in both r and N .

Taking equations (2.16) and (2.18) together, we then obtain a complete characterization of the unique, symmetric, monotone equilibrium in any standard auction with simultaneous entry.

Implications of Endogenous Participation

How does accounting for endogenous participation change our understanding of the underlying economic environment? The critical difference is that both M and valuations will now respond to the auctioneer's policy choices. For instance, setting a binding reserve price $r \geq \underline{v}$ will not only extract surplus from entrants, but also cause some marginal entrants to exit; the latter channel is entirely absent in settings with fixed N . Therefore, not surprisingly, by accounting for endogenous participation, one typically finds much smaller net benefits from binding reserve prices. In fact, in the mixed-strategy model with $V_n \perp T_n$ and $c(t_n) = c$, Levin and Smith showed that the seller optimally sets a nonbinding reserve price.² Therefore, when endogenous participation is relevant, accounting for it can have substantial practical implications.

Endogenous participation also opens the door to an entirely new set of policy questions concerning efficient regulation of entry. Increased actual competition is obviously of enormous importance for the seller: Jeremy Bulow and Klemperer [1996] showed that the seller is always better off attracting an additional bidder than setting an optimal reserve price.³ Remarkably, however, it does not follow from

²More generally, in settings with heterogeneous pre-entry information (that is, either $c(t_n) \neq c$ as analyzed by Lu and Ye as well as Moreno and Wooders, or $V_n \not\perp T_n$ as analyzed by Gentry et al.), a revenue-maximizing seller will typically have an interior optimum $r \geq \underline{v}$. But the revenue gains from setting this optimal reserve will typically be much smaller than would be predicted in models with exogenous participation.

³Notably, this is true even when M does not respond to r . As noted above, such an endogenous response substantially reduces the auctioneer's gains from setting an optimal reserve.

this that more potential buyers are always good! In fact, as shown by Li and Zheng [2009], increasing N , by discouraging equilibrium entry, can strictly decrease the seller's expected revenues. In other words, from the seller's perspective, simultaneous entry can be a strikingly ineffective vehicle for harnessing potential competition N .

This raises a natural follow-up question: Can the seller generate higher revenues by regulating and/or restricting entry? As shown in a series of papers by Ye [2007]; Roberts and Sweeting [2013]; Bhattacharya, Roberts, and Sweeting [2014] as well as Bhattacharya and Sweeting [2015], the answer to this question is unequivocally yes. In fact, several mechanisms involving sequential entry, sequential bidding, and bidding for entry rights frequently yield higher revenue, and sometimes also higher bidder surplus, than do auctions with simultaneous entry. We do not discuss these here, but the interested reader is directed to the excellent survey by Bhattacharya and Sweeting [2015] for further details. We do note, however, that endogenous entry introduces a fascinating set of policy questions that are at present incompletely understood. Thus, this area provides potentially important research opportunities for those entering this area.

2.4.5 Bid Increments

At many auctions, bid increments are part of the rules. Initially, these bid increments were ignored, whether their magnitude was large, or not. Subsequently, researchers have learned that even small bid increments can have important implications in the structural econometrics of auctions. Thus, in this section, we outline the implications of bid increments under the common auction formats and pricing rules. We also investigate the effect of bid increments on a new auction format and pricing rule used on the Internet—electronic auctions.

Second-Price Auctions

Suppose potential buyers can only tender bids in discrete increments, where $\mathcal{K} = \{k_0, k_1, k_2, \dots, k_K, k_{K+1}\}$ denotes the set of acceptable

bids. Here k_{i+1} equals $(k_i + \Delta_i)$ for positive increments Δ_i and all $i = 1, \dots, K$. The lowest admissible bid, the reserve price, is denoted by k_1 , while the highest observed bid, but not necessarily the highest admissible one, by k_K . In principle, a potential buyer could submit $k_{K+1} = (k_K + \Delta_K)$. For nonparticipants, the null bid is denoted by k_0 which, without loss of generality, can be normalized to be zero. If the increments are the same, then one can suppress the subscript and just refer to the increment as Δ . In the event of ties, the object is allocated at random to one of the tied bidders. Specifically, if ℓ bidders each submit the same highest bid k_j , then one is chosen at random to be the winner with probability $(1/\ell)$.

At Vickrey auctions, a potential buyer will tender a bid that is the highest k_j less than his valuation v_j . In short, the link between observed bids and latent valuations has been broken, which will have important implications that we discuss later in the review.

Consider what will happen within a Milgrom-Weber digital clock model of an English auction: as the clock ticks upward in discrete increments, the dominant strategy of a potential buyer is to drop out in the interval $[k_i, k_{i+1})$ if his valuation lies within that interval; otherwise, he should remain active in the auction. Eventually, one of two things can happen: (1) only one bidder remains and he pays the current price, which is above the second-highest valuation, but below the highest valuation; (2) two (or more) bidders remain at one increment, but none remain at the next, in which case the object is awarded at random to one of the remaining ℓ bidders, who pays the second-to-last price, which could be below the second-highest valuation, below the third-highest valuation, or even lower—depending on how many bidders remained at the penultimate price. Again, the link between observed bids and latent valuations is broken.

Sealed, Pay-Your-Bid Auctions

Paarsch and Jacques Robert [2003] developed a model to investigate the implications of bid increments at sealed, pay-your-bid auctions. As above, potential buyers are restricted to tendering bids in discrete increments contained in the set \mathcal{K} . Paarsch and Robert worked within

the symmetric IPV model, but developed their theoretical model in terms of observables—specifically, the number of potential buyers N as well as the distributions of nonparticipation and observed bids, denoting the probability of observing bid k_i by π_i , where π_0 denoted the probability of not participating at the auction. Collect the π_i s in the vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots, \pi_K)$. Introducing the cdf $\Pi_i = \sum_{j=0}^i \pi_j$, the probability that a potential buyer bids k_i or less; collect all of these in the vector $\boldsymbol{\Pi}$ which equals $(\pi_0, \Pi_1, \dots, \Pi_{K-1}, 1)$.

A representative potential buyer of type v seeks to maximize the expected profit from winning the auction, that is,

$$\max_{k_j \in \mathcal{K}} (v - k_j) \Pr(\text{win} | k_j).$$

Denote by Γ_i the equilibrium probability of winning if k_i is submitted; collect these probabilities in $\boldsymbol{\Gamma} = (\Gamma_0, \Gamma_1, \dots, \Gamma_K, \Gamma_{K+1})$.

What is the relationship between $\boldsymbol{\pi}$ and $\boldsymbol{\Gamma}$? If someone does not participate at the auction, then the probability of his winning is zero, so $\Gamma_0 = 0$. Also, if someone bids more than the observed maximum, then he is sure to win, so $\Gamma_{K+1} = 1$. For the remaining Γ_i s, one can verify directly that the following constitutes a symmetric, Bayes-Nash equilibrium:

$$\Gamma_i = \frac{(\Pi_i)^N - (\Pi_{i-1})^N}{N(\Pi_i - \Pi_{i-1})} \quad \forall i = 1, 2, \dots, K. \quad (2.19)$$

The numerator of equation (2.19) is the probability that the highest bid is exactly equal to k_i , while the denominator is the expected number of potential buyers submitting bid k_i .

For valuation v , when confronted by $\boldsymbol{\Gamma}$, a representative bidder's best response involves bidding k_i when the following inequalities hold:

$$(v - k_i)\Gamma_i \geq (v - k_j)\Gamma_j \quad \forall j \neq i. \quad (2.20)$$

In words, the expected profit from bid k_i weakly exceeds that for any alternative bid k_j . Of course, this is just the definition of k_i 's being optimal. This set of inequalities is the discrete analog to the equilibrium first-order condition for expected-profit maximization in the continuous-valuation model—namely, equation (2.5).

Electronic Auctions

During the past two decades, electronic auctions (EAs) have become important market mechanisms—reducing frictions so that large numbers of buyers and sellers can be brought together to trade an incredibly diverse range of goods. Bajari and Hortacısu [2003] were among the first to investigate these auctions using a structural approach; their survey, Bajari and Hortacısu [2004], influenced greatly the direction the literature took.

EAs are dynamic auctions that have inherited many of the features of traditional auction formats and pricing rules, but other features that derive from their online implementations exist, too. One especially important difference for some EAs (for example, eBay) is proxy bidding: a bidder reports a number to the server (which is kept private until the bidder is outbid) that represents the maximal amount he authorizes the server to bid; in turn, the server acts on the bidder's behalf to increase his standing offer up to his reported maximum.

Early reduced-form research by the Nobel laureate Alvin E. Roth and Axel Ockenfels [2002] sought to explain why proxy bidding was not used more often than it is. Instead, bidders sometimes use a strategy referred to as sniping—waiting until the last minute to tender bids. Kenneth Steiglitz [2007] wrote a very accessible introduction to this phenomenon (and others present at EAs), while Denis Nekipelov [2007, 2008] developed and estimated structural econometric models of the phenomenon.

That noted, because of proxy bidding, researchers have typically modeled EAs as some variant of a second-price auction because the winner (usually) pays a price linked to the second-highest proxy bid for the object on sale. In implementations of proxy bidding, when the EA software overtakes one bidder's maximum bid on behalf of another, it forces the new lead bidder, whenever possible, to surpass the former lead bidder by a commonly known, discrete amount Δ , referred to as the bid increment.

Because the EA software forces the lead bidder to surpass the second-highest bid by Δ , a necessary exception occurs when the top two proxy bids are within Δ of one another, because a jump in the full

amount Δ would then surpass the high bidder's maximum authorized bid. In this eventuality, the price is set at the value of the high bid. In other words, the winner pays his bid as under the pay-your-bid rule.

Hickman [2010] derived the theoretical implications of the EA pricing rule and found that dominant strategies ceased to exist in the symmetric IPV model. In equilibrium, participants engage in bid shaving, similar to that at pay-your-bid auctions because the winner's bid determines the transaction price with positive probability.

To simplify matters, Hickman focused on the case involving a constant Δ , but his logic carries over to the case of nonconstant Δ (multiple Δ_i s) as shown by Hickman et al. [2017]. Under symmetry, Hickman was able to simplify calculations by focusing on Z , the highest private value of a representative bidder's $(N - 1)$ opponents, which has cdf

$$F_Z(z) = F_V(z)^{N-1}$$

and pdf

$$f_Z(z) = (N - 1)F_V(z)^{N-2}f_V(z).$$

Denote by $\gamma(v)$ the symmetric bid function at an EA with bid increment Δ . Two different cases determine the sale price: (1) the highest losing bid is within Δ of the winner's bid, so he pays his own proxy bid; (2) the top two bids are farther apart, and the winner pays the highest losing bid plus Δ . Note that the bid increment Δ only enters the pricing equation directly when the second-price rule is triggered. Therefore, at the margin, when choosing his bid, a participant only considers how the bid increment controls the threshold at which the pay-your-bid rule is triggered. Define this threshold function $\tau(c)$ as follows:

$$\tau(c) = \begin{cases} \underline{v}, & c \leq \underline{v} + \Delta, \\ c - \Delta, & \underline{v} + \Delta \leq c. \end{cases} \quad (2.21)$$

What does $\tau(c)$ mean? Suppose c is the highest bid. Whenever c is less than $\underline{v} + \Delta$ the pay-your-bid rule is triggered with certainty; otherwise, this only occurs when the maximum competing bid is within Δ of c .

Denote by \mathbf{C} the vector of all bids and let C_{-n}^{\max} denote the maximum bid among player n 's opponents. Bidder n 's optimization problem

can be expressed as

$$\max_{c \in \mathbb{R}^+} \left\{ (v_n - c) \Pr [\tau(c) < C_{-n}^{\max} \leq c] + \right. \\ \left. [v_n - \mathbb{E}(C_{-n}^{\max} | C_{-n}^{\max} \leq \tau(c)) - \Delta] \Pr [C_{-n}^{\max} \leq \tau(c)] \right\}. \quad (2.22)$$

The first term in the sum corresponds to winning the EA under the pay-your-bid rule; the second term corresponds to winning the EA under a second-price rule.

Hickman showed that the probability of winning under the pay-your-bid rule given equilibrium bid function $\gamma(v)$ with inverse $\gamma^{-1}(c)$ is

$$\Pr [\tau(c) < C_{-n}^{\max} \leq c] = \Pr (\gamma^{-1} [\tau(c)] < Z \leq \gamma^{-1}(c)) \\ = F_Z (\gamma^{-1}(c)) - F_Z (\gamma^{-1} [\tau(c)]),$$

while under the second-price rule, it is

$$\Pr [C_{-n}^{\max} \leq \tau(c)] = \Pr (Z \leq \gamma^{-1} [\tau(c)]) = F_Z (\gamma^{-1} [\tau(c)]).$$

The conditional expectation in the second term of expression (2.22) is

$$\mathbb{E} [C_{-n}^{\max} | C_{-n}^{\max} \leq \tau(c)] = \frac{\int_{\underline{v}}^{\gamma^{-1}[\tau(c)]} \gamma(u) f_Z(u) du}{F_Z (\gamma^{-1} [\tau(c)])}.$$

Maximizing a bidder's expected surplus yields a boundary-value problem, that defines the function $\gamma(v)$:

$$\gamma'(v) = \frac{[v - \gamma(v)] f_Z(v)}{F_Z(v) - F_Z [\gamma^{-1} (\tau [\gamma(v)])]}, \quad (2.23)$$

where $\gamma(\underline{v}) = \underline{v}$. In short, EA equilibrium behavior solves this differential equation, collapsing to the pay-your-bid strategy over the interval $[\underline{v}, \gamma^{-1}(\underline{v} + \Delta))$ since the pay-your-bid rule is triggered with certainty over this region.

Using the continuing solved example from section 2.4.3, in Figure 2.7, we depict the difference between equilibrium bid functions under the second-price rule $\beta(v)$, the EA rule $\gamma(v)$, and the pay-your-bid rule $\sigma(v)$, when Δ is five percent of the maximum valuation \bar{v} . Depending on the increment and the distribution as well as the number of potential buyers, $\gamma(v)$ will be closer to (or farther from) $\beta(v)$.

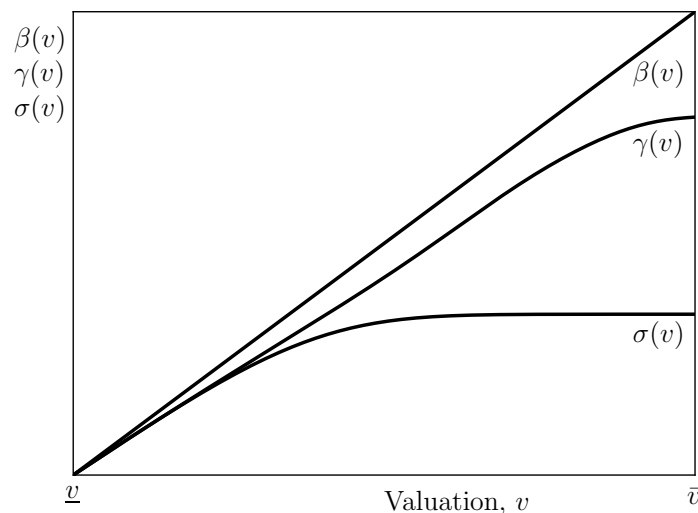


Figure 2.7: Bid Functions—Second-Price, EA, Pay-Your-Bid Pricing Rules

2.4.6 Asymmetries

When potential buyers draw their valuations independently from different distributions (different urns), this is often referred to as the asymmetric IPV model. Since the research of Vickrey [1961], economists have known that such asymmetries matter.

At English and Vickrey auctions, it remains a dominant strategy to reveal one's private information—that is, to tell the truth and to bid up to one's value. In short, under the clock model, the highest-valuation bidder wins the auction and pays what his nearest rival was willing to pay, so the outcome is efficient. The winning bid is the second-highest valuation, which represents the opportunity cost of the good.

The situation at pay-your-bid auctions is quite different. In order to investigate the essence of the technical problems that arise within the asymmetric IPV model, consider just two potential buyers in the absence of a reserve price, assuming risk neutrality. (Heterogeneous risk

attitudes across bidders would also generate an asymmetric model.) The two-bidder case allows us to highlight the challenges asymmetries present relative to the symmetric case considered in the previous sections.

To begin, suppose that bidder 1 gets an independent draw from urn 1, denoted $F_1(v_1)$, while bidder 2 gets an independent draw from urn 2, denoted $F_2(v_2)$. Assume that the two valuation distributions have the same support $[\underline{v}, \bar{v}]$. The largest of the two bids wins the auction, and the winner pays what he bid.

Now, $U_1(v_1, s_1)$, the expected profit of bid s_1 to player 1, can be written as

$$U_1(v_1, s_1) = (v_1 - s_1) \Pr(\text{win}|s_1),$$

while $U_2(v_2, s_2)$, the expected profit of bid s_2 to player 2, can be written as

$$U_2(v_2, s_2) = (v_2 - s_2) \Pr(\text{win}|s_2).$$

Assuming each potential buyer n is using a bid s_n equal to $\sigma_n(v_n)$ that is monotonically increasing in his value v_n , we can write the probability of winning the auction as

$$\begin{aligned} \Pr(\text{win}|s_n) &= \Pr(S_m < s_n) \\ &= \Pr[\sigma_m(V_m) < s_n] \\ &= \Pr[V_m < \sigma_m^{-1}(s_n)] \\ &= \Pr[V_m < \varphi_m(s_n)] \\ &= F_m[\varphi_m(s_n)]. \end{aligned}$$

Thus, the expected profit function for bidder 1 is

$$U_1(v_1, s_1|\varphi_2, F_2) = (v_1 - s_1)F_2[\varphi_2(s_1)],$$

while the expected profit function for bidder 2 is

$$U_2(v_2, s_2|\varphi_1, F_1) = (v_2 - s_2)F_1[\varphi_1(s_2)].$$

The presence of bidder m 's inverse-bid function in bidder n 's objective makes clear the trade-off bidder n faces: by submitting a lower bid, he increases the profit he receives when he wins the auction, but he decreases his probability of winning the auction.

To construct the Bayes-Nash equilibrium-bid function, first maximize each expected profit function with respect to its argument. The necessary, first-order conditions of these maximization problems are the following:

$$\begin{aligned}\frac{\partial U_1(v_1, s_1 | \varphi_2, F_2)}{\partial s_1} &= -F_2[\varphi_2(s_1)] + (v_1 - s_1)f_2[\varphi_2(s_1)]\frac{d\varphi_2(s_1)}{ds_1} = 0 \\ \frac{\partial U_2(v_2, s_2 | \varphi_1, F_1)}{\partial s_2} &= -F_1[\varphi_1(s_2)] + (v_2 - s_2)f_1[\varphi_1(s_2)]\frac{d\varphi_1(s_2)}{ds_2} = 0.\end{aligned}$$

A Bayes-Nash equilibrium is characterized by the following pair of differential equations:

$$\begin{aligned}\varphi_2'(s_1) &= \frac{F_2[\varphi_2(s_1)]}{[\varphi_1(s_1) - s_1]f_2[\varphi_2(s_1)]} \\ \varphi_1'(s_2) &= \frac{F_1[\varphi_1(s_2)]}{[\varphi_2(s_2) - s_2]f_1[\varphi_1(s_2)]}.\end{aligned}\tag{2.24}$$

These differential equations allow us to describe some essential features of the problem. First, as within the symmetric IPV model, each individual equation constitutes a first-order differential equation as the highest derivative term in each equation is the first derivative of the function of interest. Unlike within the symmetric IPV model, however, the functions we seek are the inverse-bid functions $\varphi_n(\cdot)$, not the bid functions $\sigma_n(\cdot)$. Even though we would like to solve for the bid functions, it is typically impossible within the asymmetric IPV model. In both cases, the first-order conditions that obtain from bidders maximizing their expected profit involve the inverse-bid function and its derivative. Within the symmetric IPV model, however, we were concerned with an equilibrium in which all (homogenous) bidders adopted the same bidding strategy $\sigma(v)$. This, together with monotonicity of the bid function, allowed us to map the first-order condition from a differential equation characterizing the inverse-bid function $\varphi(s)$ to a differential equation characterizing the bid function $\sigma(v)$.⁴ In the asymmetric environment, this is typically impossible to do because, in general,

$$\varphi_1(s) \neq \varphi_2(s).$$

⁴Essentially, this is just an application of the implicit function theorem.

The inverse-bid functions $\varphi_n(s)$ are helpful because they allow us to express the probability of winning the auction for any choice s : bidder n considers the probability that the other bidder will draw a valuation that will induce him to submit a lower bid in equilibrium than the bid player n submits. Because the bidders draw valuations from different urns, they do not use the same bidding strategy, so the type for which it is optimal to submit a bid s is, in general, different for the two bidders. Furthermore, because we can no longer translate either differential equation into one that characterizes the bid function directly, neither differential equation is linear. Finally, note that each differential equation involves a bid s , the derivative of the inverse-bid function for one of the players $\varphi'_n(s)$, and the inverse-bid functions of each of the bidders $\varphi_1(s)$ as well as $\varphi_2(s)$: mathematicians would refer to this system of ODEs as nonautonomous because the system involves the bid s explicitly.⁵ This last fact highlights the interdependence among players that is common to game-theoretic models. Thus, in terms of deriving the equilibrium inverse-bid functions within the asymmetric IPV model, one must solve a nonlinear system of first-order ODEs.

Theorists have determined how equilibrium (inverse-) bid functions should behave at the boundaries of the valuation (bid) support. Specifically, the inverse-bid functions must satisfy $\varphi_n(\underline{v}) = \underline{v}$ and $\varphi_n(\bar{s}) = \bar{v}$ for all $n \in \mathcal{N}$. Because \bar{s} is not known, but rather must be solved for as part of the equilibrium, the asymmetric auction problem is considered a free boundary-value problem. On top of all these challenges, the principal technical difficulty associated with solving models of asymmetric auctions is that the system of ODEs does not satisfy the Lipschitz condition at the left endpoint \underline{v} because, at that point,

$$\frac{f_V(v)}{F_V(v)} = \frac{f_V(v)}{\int_{\underline{v}}^v f_V(u) \, du}$$

is unbounded, even in the symmetric model as our notation suggests. This means the differential equations are not well posed. As we have noted, unlike in the symmetric setting, an analytic solution is not possible and so numerical methods are required. Hubbard and Paarsch

⁵A nonautonomous ODE is sometimes referred to as time-dependent. Here, bid-dependent is a better description.

[2014] provided an extensive discussion of the challenges involved with solving asymmetric auctions numerically as well as the techniques researchers employ in these settings.

In Figure 2.8, we depict the equilibrium bid functions in a model of a sealed, pay-your-bid auction where bidder 1 has valuations from the pdf given in equation (2.15), with $\alpha_1 = \frac{2}{3}$ and $\alpha_2 = 2$, while bidder 2 has valuations from the same pdf, but with $\alpha_1 = 0.5$ and $\alpha_2 = 2$. In this case, bidder 1 is the weaker of the two bidders, as the cdf of his valuations is everywhere to the left of those of bidder 2; see Figure 2.9. Except at the endpoints, the bid function of the weak bidder (the dashed line) is everywhere above that of the strong bidder the solid line: weakness breeds aggression. Consequently, it is possible for bidder 1 to win the auction, even though his valuation is less than that of bidder 2: inefficiencies can obtain.

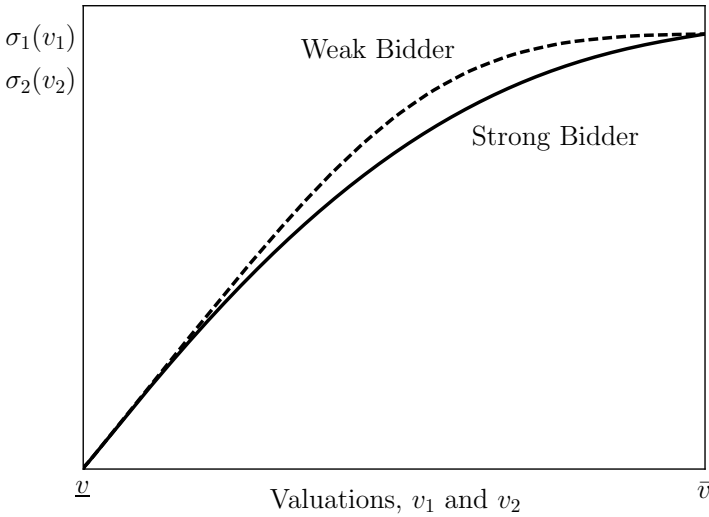


Figure 2.8: Sealed, Pay-Your-Bid Equilibrium—Asymmetric IPV Model

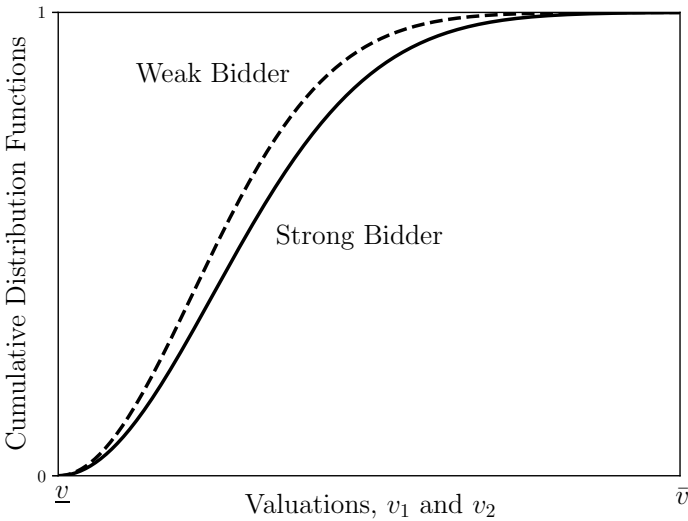


Figure 2.9: Weak and Strong Bidders

3

Mapping Theory to Observables

In this section, we illustrate how to map from theory to observables. Even though we employ the IPV model as the working example (again because of its tractability), the strategy demonstrated applies to CV and AV models as well.

To fix ideas, consider the following: In the most relevant application of auction theory to policymaking—mechanism design—the key object of interest is the distribution of private values, but those random variables are unobserved, latent: only the actions of potential buyers—their participation decisions and the bids tendered—are observed. Without knowing the cdf of valuations $F_V(v)$, however, calculating the optimal reserve price defined in equation (1.1) is impossible. What to do?

Undertaking empirical work when the object of interest is a latent, unobserved random variable is not new to economists. For example, in the literature concerning labor supply, a theoretical object known as the reservation wage is unobserved: whether an agent works, and if so, at what wage, are the observables. In short, researchers of labor supply use economic theory to put structure on the relationship between the unobserved reservation wage and the observed earned wage. Thus, it would seem natural then to pursue the same strategy when examining

data from auctions.

3.1 Deriving the pdfs of Bids, and Winning Bids

Researchers conducting empirical analyses of auction data are indeed fortunate because the link between the theory and the observables is particularly tight. Below, we illustrate this fact by examining the links between the distributions of bids under both pay-your-bid and second-price rules (as well as the distributions of winning bids under those rules as well as different auction formats) and the distribution of valuations.

3.1.1 Vickrey Auctions

As with the theory, we begin with Vickrey auctions, where the relationship between the cdf of bids at sealed, second-price auctions $F_B(b)$ and the cdf of valuations $F_V(v)$ is trivial because $B = \beta(V) = V$. That is, because the equilibrium bid function is the identity function at Vickrey auctions, the distribution of bids is the distribution of valuations. Therefore, $f_V(v)$, the pdf of valuations, is $f_B(b)$, the pdf of bids, while the joint density of all the bids collected in the vector \mathbf{B} and evaluated at realization \mathbf{b} is

$$f_{\mathbf{B}}(\mathbf{b}) = \prod_{n=1}^N f_B(b_n) = \prod_{n=1}^N f_V(b_n).$$

3.1.2 Sealed, Pay-Your-Bid Auctions

At sealed, pay-your-bid auctions within the IPV model, the relationship is more complicated than in the Vickrey case. Recall that the equilibrium bid function $\sigma(V)$ is a monotonic function of V .¹ In Figure 3.1, the area (mass) under the $f_V(v)$ curve between v_0 and $(v_0 + dv)$,

¹To verify this, note that $\sigma'(v)$, the derivative of $\sigma(v)$, is strictly positive on the interval $(\underline{v}, \bar{v}]$ because

$$\sigma'(v) = [v - \sigma(v)] \frac{(N-1)f_V(v)}{F_V(v)} > 0.$$

denoted A, must equal the area (mass) under the $f_S(s)$ curve between s_0 and $(s_0 + ds)$, denoted B. This can be written in symbols as

$$f_S(s) \, ds = f_V(v) \, dv.$$

When $\sigma(v)$ is monotonic, there exists a unique function $\sigma^{-1}(\cdot)$ such that

$$\sigma^{-1}(s) = \sigma^{-1}[\sigma(v)] = v,$$

where

$$\frac{d\sigma^{-1}(s)}{ds} = \left[\frac{d\sigma(v)}{dv} \right]^{-1} = \frac{dv}{ds}.$$

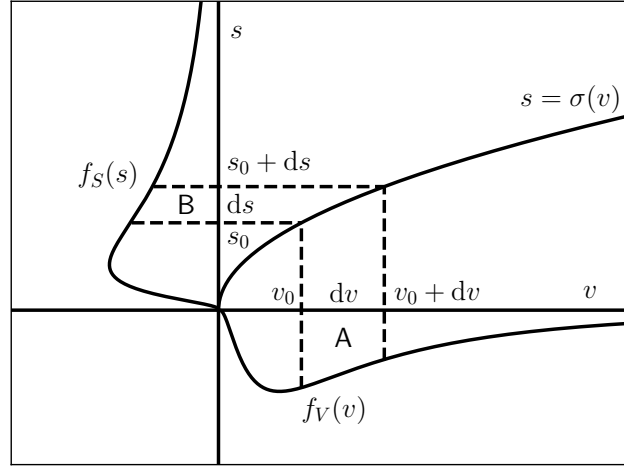


Figure 3.1: Nonlinear Transformation of Random Variable

As mentioned in section 2.1.3, when V is defined on the interval $[\underline{v}, \bar{v}]$, S is defined on the interval $[\underline{v}, \sigma(\bar{v})]$, where

$$\bar{s} \equiv \sigma(\bar{v}) = \bar{v} - \frac{\int_{\underline{v}}^{\bar{v}} F_V(u)^{N-1} \, du}{F_V(\bar{v})^{N-1}} = \bar{v} - \int_{\underline{v}}^{\bar{v}} F_V(u)^{N-1} \, du < \bar{v}.$$

In words, the support of S is different from that of V . That noted, the cdf of S , which is the probability of V 's being less than some value v , is

$$F_S(s) = \Pr(S \leq s) = \Pr[\sigma(V) \leq s] = \Pr[V \leq \sigma^{-1}(s)] = F_V[\sigma^{-1}(s)].$$

Differentiating $F_S(s)$ or $F_V[\sigma^{-1}(s)]$ with respect to s yields the pdf of S

$$\begin{aligned} f_S(s) &= \frac{dF_S(s)}{ds} \\ &= \frac{d}{ds} F_V[\sigma^{-1}(s)] \\ &= f_V[\sigma^{-1}(s)] \frac{d\sigma^{-1}(s)}{ds} \\ &= f_V[\sigma^{-1}(s)] \frac{dv}{ds}. \end{aligned}$$

Therefore,

$$f_S(s|F_V, N) = \frac{f_V[\sigma^{-1}(s)]}{\sigma'[\sigma^{-1}(s)]} \quad s \in [\underline{v}, \bar{s}(N)],$$

where we have stressed the dependence of $f_S(\cdot)$ on both $F_V(\cdot)$ and N by conditioning on them. We have also stressed that \bar{s} , the support of S , depends on the number of potential buyers N , not the number of actual bidders M .

The derivative of the equilibrium bid function given in equation (2.6) can be found by brute-force calculation

$$\begin{aligned} \sigma'(v) &= 1 - \frac{F_V(v)^{N-1} F_V(v)^{N-1} - (N-1) F_V(v)^{N-2} f_V(v) \int_{\underline{v}}^v F_V(u)^{N-1} du}{[F_V(v)^{N-1}]^2} \\ &= \frac{(N-1) f_V(v) \int_{\underline{v}}^v F_V(u)^{N-1} du}{F_V(v)^N} \\ &= [v - \sigma(v)] \frac{(N-1) f_V(v)}{F_V(v)}. \end{aligned}$$

Direct substitution yields

$$f_S(s|F_V, N) = \frac{F_V[\sigma^{-1}(s)]^N}{(N-1) \int_{\underline{v}}^{\sigma^{-1}(s)} F_V(u)^{N-1} du},$$

whence one can derive in a straightforward manner the joint density of all the bids \mathbf{S} . Clearly, being able to calculate $\sigma^{-1}(s)$ is key to calculating the pdf of S .

3.1.3 Dutch Auctions

At Dutch auctions, only the winning bid W is observed. What then? In what follows, the distributions of order statistics will be required, for instance, the $\max(V_1, V_2, \dots, V_N)$. Instead of deriving the distribution of each order statistic individually, we derive the pdf of the k^{th} highest order statistic from a sample of N independent, identically-distributed draws from $F_V(\cdot)$. From that distribution, one can obtain the distribution of any order statistic required.

Let X denote $V_{(k:N)}$. For X to be the k^{th} highest order statistic and fall within the interval $[x, x + \Delta x)$, there must be $(N - k)$ below x and $(k - 1)$ draws above $[x + \Delta x)$. The probability of this event is

$$\Pr \{X \in [x, x + \Delta x)\} = \frac{N!}{(N - k)!(1 - 1)!(k - 1)!} F_V(x)^{N-k} [F_V(x + \Delta x) - F_V(x)] [1 - F_V(x + \Delta x)]^{k-1}.$$

Here, the expression

$$\frac{N!}{(N - k)!(1 - 1)!(k - 1)!}$$

is the multinomial combinatoric formula for the number of ways that one can get $(N - k)$ draws below x , $(k - 1)$ draws above $(x + \Delta x)$, and one between x and $(x + \Delta x)$, in N draws.

Deriving $f_X(x)$ simply involves calculating the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Pr \{X \in [x, x + \Delta x)\}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{[F_V(x + \Delta x) - F_V(x)]}{\Delta x}.$$

Recalling that $(1 - 1)! = 0! = 1$, this implies

$$f_X(x|F_V, N, k) = \frac{N!}{(N - k)!(k - 1)!} F_V(x)^{N-k} [1 - F_V(x)]^{k-1} f_V(x). \quad (3.1)$$

One convenient way of summarizing the relationship between $F_X(x|F_V, N, k)$, cdf of X , and $F_V(v)$ as well as N and k is

$$F_X(x|F_V, N, k) = \frac{N!}{(N-k)!(k-1)!} \int_0^{F_V(x)} u^{N-k}(1-u)^{k-1} du. \quad (3.2)$$

Differentiating equation (3.2) with respect to x yields the pdf defined by equation (3.1), which is a comforting check.

Note that $F_X(\cdot)$ is a monotonic transformation of $F_V(\cdot)$, where (conditional on knowing N and k) the monotonic function has a known form—the incomplete Beta function multiplied by a known constant. In other words, knowing $F_X(\cdot)$ and N is tantamount to knowing $F_V(\cdot)$, which will be an important result to know in section 4, the section concerned with identification.

We can now easily find the pdf of $Z = \max(V_1, V_2, \dots, V_N) = V_{(1:N)}$, which is

$$f_Z(z|F_V, N) = f_X(z|F_V, N, 1) = NF_V(z)^{N-1}f_V(z),$$

where

$$F_Z(z|F_V, N) = F_X(z|F_V, N, 1) = F_V(z)^N.$$

The winning bid is the equilibrium bid function $\sigma(\cdot)$, listed in equation (2.6), evaluated at Z , so the pdf of W , which is defined on the interval $[\underline{v}, \bar{s}]$, is the following:

$$\begin{aligned} f_W(w|F_V, N) &= \frac{f_Z[\sigma^{-1}(w)|F_V, N]}{\sigma'[\sigma^{-1}(w)]} \\ &= \frac{NF_V[\sigma^{-1}(w)]^{N-1}f_V[\sigma^{-1}(w)]^{N-1}}{\sigma'[\sigma^{-1}(w)]} \\ &= \frac{NF_V[\sigma^{-1}(w)]^{2N-1}}{(N-1) \int_{\underline{v}}^{\sigma^{-1}(w)} F_V(u)^{N-1} du}. \end{aligned}$$

As with the distribution of any one bid at a sealed, pay-your-bid auction, this pdf is a nontrivial function of both covariates (such as the number of potential buyers N , not the actual number who showed up) as well as $F_V(\cdot)$, the distribution of the latent valuations. Moreover, the support of this random variable depends on these as well—facts that will prove troublesome and useful when it comes to estimation.

3.1.4 English Auctions

In the absence of a binding reserve price, under the clock model, when the dropout bids of nonwinners are observed, the distribution of dropout bids simply follows the distribution of valuations. Unlike at Vickrey auctions, however, a complication exists: the winner does not reveal his valuation. All one knows about the winner's valuation is that it is above the observed winning bid—in the right tail of the distribution.

When the dropout prices of nonwinners cannot be observed, one must focus on the winning bid. The pdf of the winning bid at an English auction is the distribution of the second-highest order statistic $Y = V_{(2:N)}$. Under the clock model, with at least two bidders present, the pdf of the winning bid at an English auction is then

$$\begin{aligned} f_Y(y|F_V, N) &= f_X(y|F_V, N, 2) \\ &= N(N-1)F_V(y)^{N-2}[1 - F_V(y)]f_V(y). \end{aligned}$$

In Figure 3.2, using the dotted line and the example from section 2.4.5, we depict the pdf of V ; using the dashed line with $N = 5$, we depict the pdf of Y

$$\begin{aligned} f_Y(y|\alpha_1, \alpha_2, N) &= N(N-1)\alpha_1\alpha_2y^{\alpha_2-1} [1 - \exp(-\alpha_1y^{\alpha_2})]^{N-2} \\ &\quad \exp(-2\alpha_1y^{\alpha_2}), \end{aligned}$$

and, for contrast, using the solid line, we depict the pdf of Z

$$f_Z(z|\alpha_1, \alpha_2, N) = N\alpha_1\alpha_2z^{\alpha_2-1} [1 - \exp(-\alpha_1z^{\alpha_2})]^{N-1} \exp(-\alpha_1z^{\alpha_2}).$$

3.1.5 Binding Reserve Price

In the presence of a binding reserve price, deriving the pdf of bids, or just the winning bid, can be complicated for English auctions, even under the clock model. Here, we focus on the winning bid, but our logic applies to all of the bids as well.

When the reserve price binds, the number of participants at an auction is a random variable. In the case of the English auction, if a participant faces no opponents, then he should simply tender the

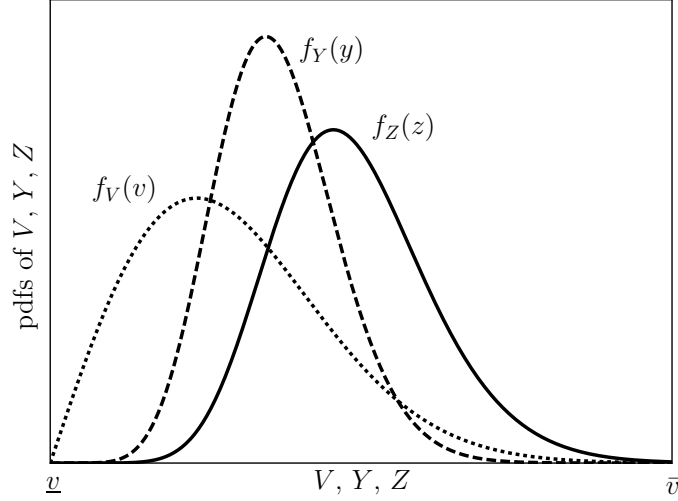


Figure 3.2: $f_V(v)$, $f_Y(y)$, and $f_Z(z)$, Weibull(1,2) Family, $N = 5$

reserve price, regardless of his valuation. When no one attends the auction, the object goes unsold. In short, the pdf of bids at an English auction can have point masses; that is, it is a mixture of continuous and discrete elements.

With this information, one can now specify the pmf/pdf of the winning bid at an English auction with a binding reserve price, assuming the clock model. A fraction of the time, no potential buyers attend the auction, so the good goes unsold, which one can define as $W = 0$. The probability of this event is

$$p_M(0|F_V, N, r) = F_V(r)^N.$$

Another fraction of the time, only one potential buyer attends the auction, in which case the winning bid is the reserve price r . The probability of this event is

$$p_M(1|F_V, N, r) = NF_V(r)^{N-1} [1 - F_V(r)].$$

Finally, when two or more potential buyers attend the auction, the winning bid is determined by the second-highest order statistic from a sample of size N . The discrete/continuous “density” is then

$$f_W(w|F_V, N, r) = \left\{ F_V(r)^N \right\}^{\mathbf{1}(W=0)} \times \\ \left\{ N F_V(r)^{N-1} [1 - F_V(r)] \right\}^{\mathbf{1}(W=r)} \times \\ \left\{ N(N-1) F_V(w)^{N-2} [1 - F_V(w)] f_V(w) \right\}^{\mathbf{1}(W>r)},$$

where $\mathbf{1}(W = 0)$ is an indicator function of a winning bid of zero, $\mathbf{1}(W = r)$ is an indicator function of a winning bid of the reserve price r exactly, and $\mathbf{1}(W > r)$ is an indicator function of a winning bid greater than the reserve price. In Figure 3.3, we depict this discrete/continuous density function using the Weibull example from above as well as a reserve price of $r = 0.74$.

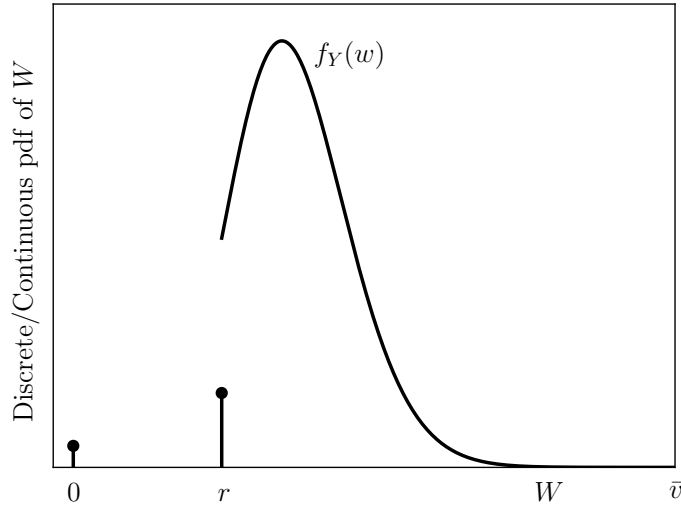


Figure 3.3: pdf of Winning Bid at English Auction with Binding Reserve Price

3.2 Empirical Specifications

For economists, the natural way in which to decompose observed bids (or winning bids) is to find the mean of the random variable in question, and then to write the bid (or the winning bid) as the sum of that mean and a mean-zero error term. For example, in the case of the Vickrey auction,

$$\mathbb{E}(B_n) = \int_{\underline{v}}^{\bar{v}} v f_V(v) \, dv = \mathbb{E}(V) \equiv \mu_V,$$

so

$$B_n \equiv \mu_V + \varepsilon_n^V,$$

where ε^V is a mean-zero random variable that follows the distribution $F_V(v - \mu_V)$.

In the case of sealed, pay-your-bid auctions, a similar decomposition can be constructed, namely,

$$\mathbb{E}(S_n) = \int_{\underline{s}}^{\bar{s}} s f_S(s|F_V, N) \, ds \equiv \mu_S(N),$$

so

$$S_n = \mu_S(N) + \varepsilon_n^S.$$

Unfortunately, the relationship between μ_V and $\mu_S(N)$ is a complicated one. In addition, even though, by construction ε_n^S has zero mean, its variance depends on N , which can complicate matters of inference in data sets where the numbers of potential buyers vary substantially across auctions.

Under the symmetric IPV model, however, with risk-neutral potential buyers, the RET informs one that the winning bid W has expectation $\mathbb{E}[V_{2:N}] \equiv \mu_Y(N)$. In symbols, at auction t ,

$$W_t = \mu_Y(N_t) + \varepsilon_t^{SW},$$

whereas at English auctions, in the absence of a binding reserve price, one obtains

$$W_t = \mu_Y(N_t) + \varepsilon_t^{EW}.$$

The distributions of ε_t^{SW} and ε_t^{EW} are, however, quite different because the pdfs of the winning bids at pay-your-bid and second-price auctions are quite different.

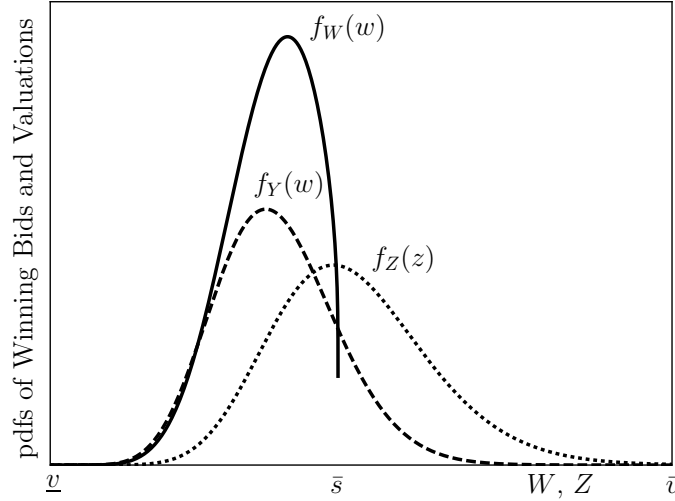


Figure 3.4: pdfs of Winning Bids and Valuations, No Reserve Price

To see this, consider Figure 3.4 in which are depicted the pdfs solved for the Weibull example employed previously—in the absence of a binding reserve. In this figure, $f_W(w)$, the pdf of the winning bids at pay-your-bid auctions, is depicted by the solid line, which is strictly positive at the upper bound of support \bar{s} , a theoretical property of the model. The pdf of the winning bid at second-price auctions $f_Y(w)$ is depicted by the dashed line, where for contrast $f_Z(z)$, the pdf of the largest order statistic, is depicted by the dotted line: equilibrium behavior greatly changes the distribution of bids that obtains from a distribution of latent valuations.

Do the two pdfs really have equal means? Because $f_W(w)$'s upper bound of support is quite inside that of the upper bound of support of $f_Y(w)$, the naked eye has trouble discerning that the two winning bids have the same expected value. But, yes, as a matter of fact, the two random variables do have the same mean. Using both numerical integration and computer simulation, we have verified that in this example

the mean of the winning bid under the pay-your-bid rule and the mean selling price under a second-price rule are around 1.09. In fact, the RET tells us these means will be the same. Obviously, the variance associated with $f_W(w)$ is less than that associated with $f_Y(w)$, a fact that may be important to the seller—for example, if he were risk-averse.

In Figure 3.5, using the same example and the solid line, we depict the discrete/continuous density function of the winning bid at a pay-your-bid auction in the presence of a binding reserve price, contrasting its features with the dashed line of the English auction. Whereas the English auction has a point mass at r , the pdf of the winning bid at the pay-your-bid auction simply has a positive density. Both have a point mass at 0—when the object goes unsold because no one participates at the auction because all N have valuations less than the reserve price r .

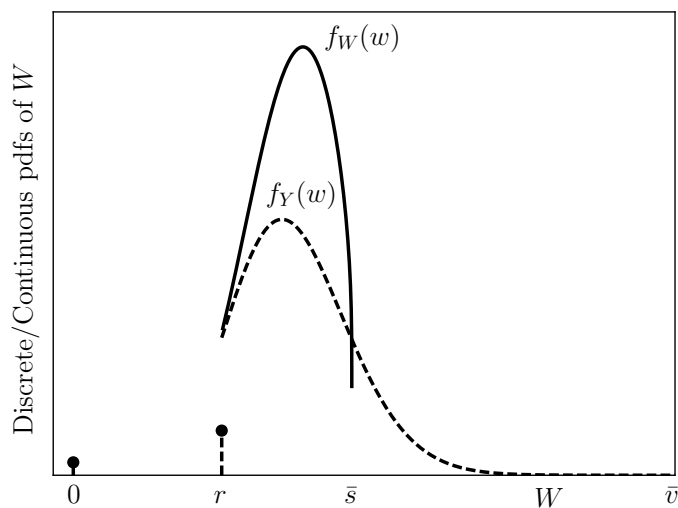


Figure 3.5: pdf of Winning Bid at Pay-Your-Bid Auction, Binding Reserve

3.3 Complications

Perhaps the greatest complication encountered when implementing the empirical specifications described above is one of measurement: a key covariate of interest, the number of potential buyers N , is often difficult to acquire. Researchers then often use the number of participants M as a proxy for N . Unfortunately, in the presence of a binding reserve price, the number of participants M is an endogenous random variable; that is, it is measured with error, and that error is correlated with the ε s above. If the method of least squares (regression) is used to estimate the model, then one of the important maintained assumptions (namely, that the covariate and the error have zero covariation) is violated, which in turn leads to biased and inconsistent parameter estimates, which in turn makes inference impossible.

Another, related complication is that it is often unknown whether certain auctions were not held because no one attended. In the presence of a binding reserve price, however, such important conditioning information is central to specifying either the conditional regression function or the appropriate likelihood function.

Yet a third complication involves the behavior of the seller: if the seller has preferences over auction formats or pricing rules, then these preferences can manifest themselves in the choice of institution employed to sell the object. Although such choices may not matter within the symmetric IPV model, within other models it can then mean that the data observed across auctions are not randomly sampled—sample selection is the term used by the Nobel laureate James J. Heckman [1978, 1979].

4

Identification

Whether one can learn what one would like from data depends crucially on a condition referred to by econometricians as identification. Broadly speaking, the question of identification revolves around determining whether a researcher who has been provided the true population distribution (imagine an omniscient oracle) could calculate the object of interest. That is, under the most optimistic conditions possible, could a researcher solve the problem at hand?

In many fields, particularly those involving controlled experiments, identification exists: well-designed experiments ensure that researchers can learn the objects of interest. In other fields, particularly in the social sciences, data are often not generated by controlled experiments. Instead, data are simply gathered under a variety of different sampling schemes, which we have referred to as observational data previously. When observational data are used, in order to obtain identification, researchers are sometimes forced to make identifying assumptions, which cannot be tested. Why? Well, without the identifying assumptions, the empirical specification could not be estimated in the first place.

4.1 Notation and Definitions

We first review some notation describing unknown model primitives: Prior to bidding, each potential buyer n observes a private signal X_n of his potentially unknown valuation V_n . Letting the superscript 0 signify the truth, denote the true joint distribution of $(V_1, \dots, V_N, X_1, \dots, X_N)$ by $F_{\mathbf{V}\mathbf{X}}^0$ and the true joint distribution of (V_1, \dots, V_N) by $F_{\mathbf{V}}^0$. In potentially asymmetric settings, denote the true marginal distribution of V_n by F_n^0 , with F_V^0 denoting the true common marginal distribution of (V_1, \dots, V_N) when symmetry is assumed.

Next, introduce a notation describing auction-specific observables: Denote the random vector (B_1, \dots, B_N) of all bids tendered at second-price auctions by \mathbf{B} and the random vector (S_1, \dots, S_N) of all bids tendered at pay-your-bid auctions by \mathbf{S} . In some instances, the econometrician may observe only a subset of these. Depending on the setting, denote the joint distribution of bids by $F_{\mathbf{B}}^0$ or $F_{\mathbf{S}}^0$, and the marginal distribution of bids by F_B^0 or F_S^0 . When bidder-specific distributions are required, we introduce G^0 to denote bid distributions; for example, at an asymmetric pay-your-bid auction, the bid distribution of bidder n is denoted by G_n^0 .

At Dutch and English auctions, order statistics of bids and valuations play a critical role in the analysis. For $k = 1, \dots, N$, denote the k^{th} highest order statistic of the random variable by $Y_{(k:N)}$. Let \mathbf{B}_{-n} or \mathbf{S}_{-n} denote bids submitted by all bidders except n , and $G_{(1:-n)}^0(\cdot | b_n)$ or $G_{(1:-n)}^0(\cdot | s_n)$ denote the distribution of the maximum rival bid, that is, $\max \mathbf{B}_{-n}$ or $\max \mathbf{S}_{-n}$ conditional on the event $B_n = b_n$ or $S_n = s_n$.

All arguments below also extend immediately to an arbitrary vector of observables \mathbf{z} characterizing the auction environment. In general, when such observables \mathbf{z} are relevant, one would simply condition all observed bid distributions on realizations of \mathbf{Z} . For clarity, however, we explicitly indicate this conditioning only when auction-specific observables play an important role in the identification argument.

Denote the true distribution of $V_{(k:N)}$ by $F_{(k:N)}^0$. Recall two important properties of the order statistics $[V_{(1:N)}, \dots, V_{(N:N)}]$. First, as seen in equation 3.2 above, if (V_1, \dots, V_N) are independently drawn from a

symmetric distribution F_V^0 , then

$$F_V^0(v) = Q_{(k:N)}[F_{(k:N)}^0(v)],$$

where $Q_{(k:N)}$ denotes the quantile function of the k^{th} highest of N independent and identical uniform draws—a known, strictly monotone function defined implicitly by the identity

$$F = \frac{N!}{(N-k)!(k-1)!} \int_0^{Q_{k:N}(F)} u^{N-k}(1-u)^{k-1} du. \quad (4.1)$$

Second, even when latent values (V_1, \dots, V_N) are independent, the order statistics $[V_{(1:N)}, \dots, V_{(N:N)}]$ are not. Therefore, when exploring identification and inference based on order statistics, we must bear in mind this lack of independence.

Finally, introduce some definitions relevant in settings with affiliated values. Let $\omega_n(x_n, \mathbf{x}_{-n}) \equiv \mathbb{E}(V_n | x_n, \mathbf{x}_{-n})$ denote bidder n 's full-information valuation function; that is, the expectation of V_n which bidder n would form after having observed both his own signal realization $X_n = x_n$ and all rival signal realizations $\mathbf{X}_{-n} = \mathbf{x}_{-n}$. As before, bidders have private values when full-information valuations do not depend on rival signals, so $\omega_n(x_n, \mathbf{x}_{-n}) = \mathbb{E}(V_n | x_n)$. More generally, in settings where $\omega_n(x_n, \mathbf{x}_{-n})$ depends nontrivially on rival signals, we say that bidders have nonprivate values. In settings with private values, we adopt the convention that $X_n = V_n = \mathbb{E}(V_n | X_n)$, while with potential nonprivate values, we normalize X_n to have a marginal uniform distribution. In either case, the affiliated values hypothesis implies that $\omega_n(x_n, \mathbf{x}_{-n})$ is strictly increasing in bidder n 's own signal x_n , and at least weakly increasing in each element of rival signals \mathbf{x}_{-n} (strictly when valuations are strictly nonprivate). Because monotone transformations of signals preserve information, both normalizations are without loss of generality.

Of course, in practice, a bidder observes only his signal X_n , not the signals of his rivals \mathbf{X}_{-n} , which in turn leads to the winner's curse introduced in section 1. Because bidder n wins only when his signal is relatively greater than all opponents, bidder n 's expected valuation conditional on X_n and winning is typically less than his expected valuation conditional on X_n alone. Denote bidder n 's pivotal valuation

at signal $X_n = x_n$ by $w_n(x_n)$; that is, bidder n 's expectation of V_n conditional on both $X_n = x_n$ and just outbidding his highest rival $m \neq n$. Letting $(\sigma_1, \dots, \sigma_N)$ denote an equilibrium strategy profile, define $w_n(x_n)$ as follows:

$$w_n(x_n) \equiv \mathbb{E} \left[V_n | X_n = x_n, \max_{m \neq n} \sigma_m(X_m) = \sigma_n(x_n) \right]. \quad (4.2)$$

Under the hypothesis that observed bids (S_1, \dots, S_N) embody equilibrium strategies $(\sigma_1, \dots, \sigma_N)$ —that is, $S_n = \sigma_n(X_n)$ for all n —we can equivalently characterize $w_n(x_n)$ as follows:

$$w_n(x_n) = \mathbb{E}(V_n | S_n = S_n, \max_{m \neq n} S_m = S_n).$$

From an identification perspective, this latter characterization is often useful because bids are directly observed—especially at pay-your-bid auctions.

4.2 Vickrey Auctions

To begin, we consider identification in the simplest possible context, the Vickrey auction. Because they are rarely encountered in practice, pure Vickrey auctions are in some sense a theoretical ideal. Nevertheless, Vickrey auctions represent a particularly convenient setting within which to introduce the main ideas of identification because equilibrium strategies are very simple at such auctions.

First, suppose that bidders have private values, so under the normalizations above, $V_n = X_n$ for all $n \in \mathcal{N}$. In this case, the unique dominant-strategy equilibrium of potential buyers is to bid their values: $B_n = V_n$ for all $n \in \mathcal{N}$. Identification is trivial because the joint distribution of values (V_1, \dots, V_N) simply equals the observed joint distribution of bids (B_1, \dots, B_N) . Thus, we deduce the latent structural primitive F_V^0 directly from the identity $F_V^0 = F_B^0$.

Suppose, instead, that potential common values exist in the environment. In other words, bidder n 's full-information valuation $\mathbb{E}(V_n | x_n, \mathbf{x}_{-n})$ can depend on rival signals \mathbf{x}_{-n} . In this case, it is optimal for bidder n having signal realization $X_n = x_n$ to submit bid $b_n = w_n(x_n)$, where $w_n(x_n)$ is the pivotal valuation defined in (4.2)

above. Assuming that observed bids arise from play of such an equilibrium,

$$(B_1, \dots, B_N) = [w_1(X_1), \dots, w_N(X_N)] \quad (4.3)$$

In words, (B_1, \dots, B_N) is now not equivalent to observation of (V_1, \dots, V_N) , but to observation of $[w_1(X_1), \dots, w_N(X_N)]$.

What can one learn from knowledge of $[w_1(X_1), \dots, w_N(X_N)]$? Unfortunately, without additional information, not all that much. In fact, one cannot even reject the possibility that $w_n(X_n) = \mathbb{E}(V_n | X_n, \mathbf{X}_{-n}) = V_n$ for all $n \in \mathcal{N}$; that is, the underlying environment is one of private values. This, in turn, leads to our first negative identification result, which is due initially to Jean-Jacques Laffont and Vuong [1996]: Without additional information, every AV model is observationally equivalent to some APV model.

Of course, here the critical caveat is “without additional information.” This raises an obvious question: What additional information would be sufficient to establish identification? Subsequent research has provided a number of positive answers to this question: within the CV model, Hendricks and Porter [1988] as well as Li, Perrigne, and Vuong [2000]; with binding reserve prices, Hendricks et al. [2003] as well as Jonathan B. Hill and Shneyerov [2013]; with exogenous variation in competition, Jorge Balat, Haile, Hong, and Shum [2016] as well as Paulo J. Somaini [2015] all established identification. We return to these potential solutions in detail in section 4.4.2.

4.3 English Auctions

Relative to identification in models of Vickrey auctions, identification in models of English auctions involves three practical challenges. First, even assuming that bidders drop out exactly at their values, not all valuations are revealed because the auction ends when the second-to-last bidder exits—the high bidder’s valuation never having been revealed. Second, bids may not reveal valuations—in practice, jump bidding frequently occurs at English auctions, which appears inconsistent with the clock model. Finally, data on real-world open outcry English auctions are often incomplete—the auctioneer may record, for instance, the win-

ning bidder and the final sale price, but not the identities of the losing bidders or the bids they submitted. This can be a significant obstacle to empirical analysis, particularly when the set of auction participants is otherwise unknown.

In this section, we focus on approaches to identification in models of English auctions which resolve the first two practical challenges noted above, assuming that the econometrician observes the set of participating bidders. We return to the question of incomplete data on the number of bidders in section 4.6 below.

4.3.1 English Auctions within the IPV Model

We begin with the ideal case: English auctions within the IPV model when the set of participants is known. In this case, we can lever independence to compensate for lack of data concerning high valuations, which turns out to extend naturally to settings with potential jump bids. We first consider the simple Milgrom and Weber [1982] clock model, in which the only challenge to identification is unobserved high bids. We then proceed to discuss the alternative incomplete approach of Haile and Elie Tamer [2003], which can accommodate potential jump bids.

Clock Model

Within the IPV clock model, each nonwinning bidder $n \in \mathcal{N}$ will remain active until the price equals his valuation v_n , at which point he will drop out. The auction concludes when only one bidder remains—that winner paying the second-highest bidder’s dropout price. Therefore, the dropout history reveals the first $(N - 1)$ order statistics of (V_1, \dots, V_N) , or in the notation above $[V_{(N:N)}, \dots, V_{(2:N)}]$. Under independence, the identification problem is to recover F_n^0 , the individual marginal distribution of values for each bidder n , from observation of the order statistics $[V_{(N:N)}, \dots, V_{(2:N)}]$.

Under symmetry, demonstrating identification is straightforward: If (V_1, \dots, V_N) are drawn independently from F_V^0 , then for any $k = 1, \dots, N$, the identity

$$F_V^0(v) = Q_{(k:N)}[F_{(k:N)}^0(v)]$$

defines $F_{(k:N)}^0$, the cdf of the k^{th} order statistic $V_{(k:N)}$. Here, $Q_{(k:N)}$ is a known strictly monotone function defined implicitly by equation (4.1). Observing any single order statistic $V_{(k:N)}$ is sufficient to identify the latent primitive distribution F_V^0 . Provided the number of potential buyers N and the transaction price $V_{(2:N)}$ are observed, the symmetric IPV clock auction model is identified. Additional bids provide overidentifying restrictions, which may help either to reduce sampling variability in estimation or to examine model specification.

What if bidders are asymmetric? In this case, the objective is to recover the marginal distribution F_n^0 for each bidder n , accounting for the fact that in general $F_m^0 \neq F_n^0$. As before, the challenge is that the winner's valuation is never observed: It is only known that this valuation lies above the transaction price $V_{(2:N)}$.

Suppose the econometrician observes the set of active bidders, the identity of the winning bidder, and the transaction price $V_{(2:N)}$. Given these data, Athey and Haile [2002] as well as Tatiana Komarova [2013b] have established that under the clock model the asymmetric IPV model is identified.

Athey and Haile [2002] noted that the asymmetric clock model is isomorphic to a statistical competing risks model, for which Isaac Meilijson [1981] had previously explored identification. Komarova formalized this result, providing a rigorous proof of identification in generalized competing risks models which resolved several important gaps in Meilijson's prior analysis—most notably, providing sufficient conditions under which a model rationalizing the data exists, and under which this solution is guaranteed to be unique. We do not reproduce these results in detail here, referring interested readers to the original papers for those details. For current purposes, however, the key conclusion is that under the clock model the asymmetric IPV model is identified under essentially the same data requirements as the symmetric IPV clock—requiring the identity of the winning bidder as well.

Unfortunately, behavior at real-world English auctions frequently diverges substantially from that predicted by the clock model. In particular, jump bidding (increasing the standing bid by a discrete amount) has no analog within the clock model. Absent the structure of the clock

model, it is unclear how observed bids should be interpreted. How to proceed?

Haile and Tamer [2003] proposed an elegant solution to this problem based on the following incomplete model of bidding at English auctions. Assume the symmetric IPV model, where $F_n^0 = F_V^0$ for all $n \in \mathcal{N}$ and $V_m \perp V_n$ for $m \neq n$. In place of the full Milgrom and Weber [1982] clock model, Haile and Tamer maintained the following two natural hypotheses on bidder behavior: (1) no bidder tenders more than he is willing to pay and (2) no bidder allows a rival to win at a price he is willing to beat. Because these hypotheses flow directly from the basic premise of economic rationality, as opposed to any particular structure on the bidding game, the economic appeal of these restrictions is obvious.

Under these hypotheses, analysis is robust in the sense that the two accommodate a wide variety of observed behaviors that are inconsistent with the Milgrom-Weber clock model. The practical cost is that under the two hypotheses alone the bidding model is incomplete in the sense that many potential bid distributions F_B could be consistent with a given latent distribution of values F_V^0 . Haile and Tamer showed, however, that the two hypotheses are sufficient to permit substantial progress on important economic questions.

Consider the first hypothesis: No bidder tenders more than he is willing to pay. This statement is equivalent to the statement that $B_n \leq V_n$ for all bidders $n \in \mathcal{N}$, whence follows $B_{(k:N)} \leq V_{(k:N)}$ for all $k = 1, \dots, N$. Letting $F_{(k:N)}^0$ and $G_{(k:N)}$ denote the distributions of $V_{(k:N)}$ and $B_{(k:N)}$, respectively, this in turn implies the following first-order stochastic dominance relationship:

$$F_{(k:N)}^0(y) \leq G_{(k:N)}(y)$$

for all $k = 2, \dots, N$. Applying the monotone transformation $Q_{(k:N)}$ defined in Equation (4.1) to both sides of the inequality above, and recalling that $F_V^0(v) = Q_{(k:N)}[F_{(k:N)}^0(v)]$, we obtain

$$F_V^0(y) = Q_{(k:N)}[F_{(k:N)}^0(y)] \leq Q_{(k:N)}[G_{(k:N)}(y)] \quad (4.4)$$

for all $k = 2, \dots, N$. Therefore,

$$F_V^0(y) \leq \min_{k=2, \dots, N} Q_{(k:N)}[G_{(k:N)}(y)]. \quad (4.5)$$

The unknown primitive distribution F_V^0 is identified pointwise at the upper bound.

Consider now the second hypothesis: No bidder allows a rival to win at a price he is willing to beat. Translated into order statistics, this can hold only if $V_{(2:N)} \leq B_{(1:N)}$, which of course implies $V_{(k:N)} \leq B_{(1:N)}$ for all $k > 2$. In terms of the relevant distributions $F_{(2:N)}^0$ and $G_{(1:N)}$, this in turn implies the stochastic dominance relationship

$$F_{(2:N)}^0(y) \geq G_{(1:N)}(y).$$

Applying the strictly monotone transformation $Q_{(2:N)}$ to both sides of this inequality, Haile and Tamer obtained a pointwise lower bound on the unknown primitive distribution F_V^0 :

$$F_V^0(y) = Q_{(2:N)}[F_{(2:N)}^0(y)] \geq Q_{(2:N)}[G_{(1:N)}(y)]. \quad (4.6)$$

If the observed bids are generated under the clock model (so losers bid exactly up to their values), then both $G_{(k:N)}(y) = F_{(k:N)}^0(y)$ for all $k = 2, \dots, N$ and $G_{(1:N)}(y) = F_{(2:N)}^0(y)$. In words, both the pointwise upper bound (4.5) and the pointwise lower bound (4.6) collapse to F_V^0 , yielding point identification of the model.

Haile and Tamer then proceeded to translate these bounds on F_V^0 into robust bounds on counterfactual objects of interest, such as seller revenue and the seller's optimal reserve price. Using an application to United States Forest Service timber auctions, they demonstrated that bounds on revenue can be quite informative, although bounds on the optimal reserve price may remain wide. The latter observation raises interesting follow-up questions regarding how reserve prices should be set given only bounds on the optimal reserve; see, for example, Gaurab Aryal and Dong-Hyuk Kim [2013] for one analysis of this issue. Given the minimal structure imposed on equilibrium bid strategies, however, the bounds analysis of Haile and Tamer remain a remarkable empirical accomplishment.

That said, the bounds analysis of Haile and Tamer also raised at least two open questions: First, can one construct analogous bounds in asymmetric and/or affiliated private-values environments? Andrés Aradillas-Lopez, Amit Gandhi, Daniel Quint [2013] as well as Komarova [2013a] explored both issues, with some positive progress. Second, to what extent can the bounds be sharpened? Recently, Andrew Chesher and Adam M. Rosen [2017] demonstrated that the Haile-Tamer bounds are not sharp; they also characterized the sharply identified set. Estimation based on this characterization remains, however, an open problem.

4.3.2 English Auctions within the APV Model

What if, instead, the environment is affiliated private-values? Unfortunately, without further structure, results in this case are quite negative: even in the best case, assuming symmetric bidders, Athey and Haile [2002] demonstrated that, without independence, the clock model is unidentified. The intuition is simple: In the absence of independence, identification requires recovering the joint distribution of the N valuations (V_1, \dots, V_N) , but equilibrium bidding reveals (at most) the lowest $(N - 1)$ order statistics $[V_{(2:N)}, \dots, V_{(N:N)}]$. More unknowns exist than observables—an immediate failure of identification.

Even though identifying the joint distribution F_V^0 is, in general, infeasible without independence, Aradillas-Lopez et al. [2013] as well as Komarova [2013a] have demonstrated that one may nevertheless make some progress. Within a general asymmetric APV model, under a variety of data assumptions, Komarova constructed identified bounds on the marginal distributions (F_1^0, \dots, F_N^0) ; in many cases, these bounds are quite tight. In symmetric, ascending-price auctions with correlated private-values, Aradillas-Lopez et al. demonstrated that, given exogenous variation in the number of potential buyers N , one can construct informative counterfactual bounds on buyer and seller surplus.¹ In turn, these counterfactual bounds support informative policy analysis—even

¹Aradillas-Lopez et al. invented the correlated private-values information structure, which is a generalization of the APV model. For expositional reasons, we focus on the APV model here.

in the absence of identification of F_V^0 .

4.3.3 English Auctions within the AV Model

Given that one cannot demonstrate identification within the APV model, it should come as no surprise that the more general AV model is also unidentified. Indeed, even under the clock model, a simple interpretation of bids is impossible: The dropout price of bidder n now reflects the expectation of bidder n 's valuation conditional on his own signal, the hypothesis that his valuation is currently pivotal, and dropout points by all prior bidders. In applications, jump bidding substantially complicates the analysis further. To date, relatively little progress on these fronts has been made.

4.4 Sealed, Pay-Your-Bid Auctions

Relative to models of English or Vickrey auctions, the main challenge for identification in models of sealed, pay-your-bid auctions is that bidding is strategic: Bids no longer correspond directly to values. Because the bidding strategies in these models are, however, reasonably well understood, identification turns out to be relatively straightforward to establish in many cases of interest.

4.4.1 Private-Values Models

We begin with the simplest case: sealed, pay-your bid auctions, where potential buyers have independent private values. For the moment, we restrict attention to risk-neutral potential buyers; we return to the important question of risk aversion in section 4.6.4. Suppose that observed bids are generated from play of an equilibrium strategy profile $(\sigma_1, \dots, \sigma_N)$, so $S_n = \sigma_n(V_n)$ for each $n \in \mathcal{N}$. Within this environment, the pioneering, fundamental insight concerning identification, which was suggested by Laffont and Vuong [1996] and formalized by Emmanuel Guerre, Perrigne, and Vuong [2000], is that for each bidder n the inverse-bid strategy σ_n^{-1} is identified directly from observed equilibrium bids. Thus, even though bidding is strategic, one can recover

values from bids through the identified mapping $V_n = \sigma_n^{-1}(S_n)$: The twin hypotheses of optimization and equilibrium deliver identification.

This profound, powerful insight almost singularly revolutionized the literature. The underlying intuition, however, is remarkably simple. Consider a simple pay-your-bid pricing rule, where N potentially asymmetric bidders have independent, private values. Denote by G_n^0 the true equilibrium distribution of bids S_n submitted by bidder $n = 1, \dots, N$. Let $G_{(1:-n)}^0(s_n) = \prod_{m \neq n} G_m^0(s_m)$ denote the true distribution of the maximum bid submitted by any rival $m \neq n$. Under the equilibrium hypothesis, the expected profit of bidder n having private value v_n and submitting bid s_n is then

$$U_n(v_n, s_n) = (v_n - s_n)G_{(1:-n)}^0(s_n).$$

In equilibrium, the bid s_n actually submitted by bidder n must maximize $U_n(v_n, s_n)$ taking rival behavior (embodied in the cdf $G_{(1:-n)}^0$) as given. This implies that s_n must satisfy the following necessary first-order condition:

$$(v_n - s_n)g_{(1:-n)}^0(s_n) = G_{(1:-n)}^0(s_n).$$

Re-arranging this first-order condition, one ultimately obtains the identified inverse-bid function of Guerre et al. [2000]:

$$v_n = s_n + \frac{G_{(1:-n)}^0(s_n)}{g_{(1:-n)}^0(s_n)} \equiv \sigma_n^{-1}(s_n), \quad (4.7)$$

which is often referred to as the GPV transformation in honor of the contribution of Guerre, Perrigne, and Vuong. Note that the only unknown on the right-hand side is the true equilibrium distribution of bids submitted by n 's opponents. By hypothesis the true equilibrium bid distribution is known, and the bids are observed, so the inverse-bid function $\sigma_n^{-1}(s_n)$ is identified.

What if, instead, the environment satisfies affiliated private values? In this case, Li, Perrigne, and Vuong [2002] showed that the relevant inverse-bid function is

$$v_n = s_n + \frac{G_{(1:-n)}^0(s_n | s_n)}{g_{(1:-n)}^0(s_n | s_n)} = \sigma_n^{-1}(s_n), \quad (4.8)$$

where $G_{(1:-n)}^0(\cdot|s_n)$ now denotes the true distribution of the maximum bid among n 's opponents, conditional on the realization $S_n = s_n$. Given that the full joint distribution of bids is observed, this conditional distribution is also identified, so $\sigma_n^{-1}(s_n)$ is identified.

Note that, if private values (v_1, \dots, v_N) are independent, then equilibrium bids (s_1, \dots, s_N) will also be independent. In this case, we will have $G_{(1:-n)}^0(s_n|s_n) = G_{(1:-n)}^0(s_n)$ and $g_{(1:-n)}^0(s_n|s_n) = g_{(1:-n)}^0(s_n)$, so that (4.8) reduces to (4.7). In other words, the IPV inverse-bid function (4.7) is a special case of the APV inverse-bid function (4.8). Since the IPV model is a special case of the APV model, this should come as no surprise.

This analysis also sheds light on the following important question: When could observed bidding behavior have arisen from equilibrium play in some IPV or APV model? Recall from section 2 that, under either the IPV or the APV model, the equilibrium bidding strategy $\sigma_n(v_n)$ must be strictly increasing in v_n . Therefore, the inverse-bid function $\sigma_n^{-1}(s_n)$ must also be strictly increasing in s_n . Recalling that $\sigma_n^{-1}(s_n)$ is linked to observed bid distributions through either (4.7) or (4.8) above, this turns out to be the key restriction imposed by equilibrium bidding. Specifically, if bids (s_1, \dots, s_N) are affiliated and the right-hand side of (4.8) is strictly increasing in s_n , then there exists an APV model rationalizing observed bidding behavior. Similarly, if (s_1, \dots, s_N) are independent and (4.7) is strictly increasing in s_n , then observed bids can be rationalized by some IPV model. In either case, to identify the rationalizing valuations, we need simply apply the relevant inverse-bid mapping (4.7) or (4.8).

4.4.2 AV Model

If the hypothesis of affiliated private-values is relaxed to affiliated values, then the winner's curse can obtain, which obviously complicates the equilibrium mapping between primitives and bids. Nevertheless, as shown by Balat et al. [2016], the inverse-bid function of equation (4.8) has a clear interpretation. Specifically, applying that equation, one now recovers bidder n 's pivotal valuation $w_n(x_n)$ rather than bidder n 's pri-

vate value v_n :

$$w_n(x_n) = s_n + \frac{G_{(1:-n)}^0(s_n|s_n)}{g_{(1:-n)}^0(s_n|s_n)}, \quad n = 1, \dots, N. \quad (4.9)$$

Unfortunately, as at second-price auctions, knowledge of pivotal valuations is no longer sufficient to identify the model. In fact, given only knowledge of $w_n(x_n)$, one cannot even reject the hypothesis that $w_n(x_n) = x_n = v_n$, hence the conclusion of Laffont and Vuong [1996]: Without additional information, every AV model is observationally equivalent to some APV model.

As noted above, however, the crucial question is: What additional information might permit further empirical progress in settings with potential common values? In practice, answers to this question have been proposed along three primary lines: (1) assuming pure common values; (2) assuming a binding reserve price; and (3) assuming exogenous variation in competition. We next discuss each of these in turn.

Pure Common Values

When potential buyers have pure common values, $V_n = V_0$ for all bidders $n \in \mathcal{N}$, which is a plausible structure in auctions of mineral rights, where bidders have similar extraction costs but heterogeneous assessments regarding the amount of the resource present. Hendricks and Porter (and sometimes with coauthors) applied this pure CV model in the context of auctions of drilling rights on the Outer Continental Shelf of the United States.

Under the hypothesis of pure common values, the identification problem reduces to recovering the joint distribution $F_{\mathbf{X}V_0}^0$ of the signal-value vector (X_1, \dots, X_N, V_0) . Unfortunately, given only data concerning bids, identification remains impossible: Without additional information, one cannot hope to identify the joint distribution of $(N + 1)$ random variables from only N bids. Therefore, establishing identification requires either additional data concerning V_0 or additional structure on the relationship between V_0 and (X_1, \dots, X_N) . Here, we focus primarily on the former; see Li, Perrigne, and Vuong [2000] for an example of the latter.

In particular, suppose that in addition to the data assumed so far, the econometrician also observes an *ex post* measure of the unknown common value V_0 . Of course, whether this is reasonable will depend critically on the context as well as the data at hand, but in some applications it may be plausible. For instance, at auctions of drilling rights for oil on the Outer Continental Shelf, which were studied by Hendricks and Porter [1988] as well as Hendricks, Joris Pinkse, and Porter [2003], the government monitors the value of resources extracted from each tract. Assuming that such *ex post* measures are sufficiently detailed, it may be reasonable to model V_0 as observed.

Clearly, observing V_0 renders identification straightforward. For example, without loss of generality, suppose that the signals (X_1, \dots, X_N) are normalized to have uniform marginal distributions. Because bids are strictly increasing in signals, by definition $x_n = G_n^0(s_n)$ for all n ; that is, x_n must equal the quantile of s_n within the equilibrium distribution of S_n . One can then recover realizations of (X_1, \dots, X_N) up to an economically irrelevant normalization; combined with data concerning realizations of V_0 , identification of $F_{\mathbf{X}V_0}^0$ is established.

When only \tilde{V}_0 , an imperfect proxy of V_0 , is observed, establishing identification is unclear. At a minimum, one would require further structure on the relationship between \tilde{V}_0 and V_0 , but even then identification could remain elusive; see Athey and Haile [2002] for additional discussion.

Binding Reserve Price

Consider a symmetric AV model under pay-your-bid pricing with a binding reserve price $r > \underline{v}$. As shown by Milgrom and Weber [1982], participation is determined by a signal threshold x^* such that every potential buyer with $x_n > x^*$ tenders a bid, where the expected value of a participant having signal x^* just equals the reserve price:

$$x^* = \inf \left\{ x : \mathbb{E}(V_n | X_n = x, \max_{m \neq n} X_m \leq x) \geq r \right\}.$$

The equilibrium bid of a participant having signal x^* satisfies

$$\sigma_n(x^*) = w_n(x^*) = \mathbb{E}(V_n | X_n = x^*, \max_{m \neq n} X_m = x^*).$$

If the true model satisfies private values, then rival signals are irrelevant for V_n conditional on x_n . Hence, if r is binding, then

$$\begin{aligned}\sigma_n(x^*) &= \mathbb{E}(V_n | X_n = x^*, \max_{m \neq n} X_m = x^*) \\ &= \mathbb{E}(V_n | X_n = x^*, \max_{m \neq n} X_m \leq x^*) = r.\end{aligned}$$

In contrast, if the true model has a common-value element, then rival signals convey information about one's own value. In this case, assuming as above that bidder n 's full-information value function $\omega_n(x_n, \mathbf{x}_{-n}) = E[V_n | X_n = x_n, \mathbf{X}_{-n} = \mathbf{x}_{-n}]$ is strictly increasing in rival signal realizations \mathbf{x}_{-n} , we will have

$$\begin{aligned}\sigma_n(x^*) &= \mathbb{E}(V_n | X_n = x^*, \max_{m \neq n} X_m = x^*) \\ &> \mathbb{E}(V_n | X_n = x^*, \max_{m \neq n} X_m \leq x^*) = r.\end{aligned}$$

As pointed out by Hendricks et al., this fact can be used to test the hypothesis of private values. In particular, if the reserve price is binding and the data are generated by a private-values model, then the lowest bid should equal the reservation price. In contrast, in any setting involving common values, the lowest bid should strictly exceed the reservation price.

Hill and Shneyerov [2013] proposed a formal test based on this idea—finding significant evidence of common values in Canadian timber markets.

Exogenous Variation in Competition

The crucial difference between settings having common and private values is the possibility of the winner's curse: What does a bidder learn about his own valuation from the fact that he has won the auction? Furthermore, the strength of this potential winner's curse depends on the set of competitors faced. Learning that one has outbid many opponents conveys a relatively stronger negative signal than if one has outbid just a few opponents. This turns out to be a critical insight supporting identification and testing within the general AV model.

Balat et al. [2016] showed that variation in the number of potential buyers could provide a basis for a nonparametric test of the null of

private values. Specifically, consider the symmetric AV model, where N varies between $N = \underline{N}$ and $N = \overline{N}$, with $\overline{N} > \underline{N}$. Assume further that the variation in N is exogenous in the sense that the joint distribution of $(V_1, \dots, V_{\underline{N}}, X_1, \dots, X_{\underline{N}})$ equals the relevant marginal distribution of $(V_1, \dots, V_{\overline{N}}, X_1, \dots, X_{\overline{N}})$. As above, let $w_n(x_n; N)$ denote the pivotal valuation of bidder n with signal x_n , now conditional on the strength of potential competition N :

$$w_n(x_n; N) = \mathbb{E}(V_n | X_n = x_n, \max_{m \neq n} X_m = x_n; N).$$

By definition, under the null hypothesis of private values, $w_n(x_n; \underline{N}) = v_n = w_n(x_n; \overline{N})$. Meanwhile, in any AV model with nonprivate values, Haile et al. showed that

$$w_n(x_n; \underline{N}) > w_n(x_n; \overline{N}) \quad \text{for all } n. \quad (4.10)$$

It follows that $w_n(X_n; N)$ will be invariant to N in any private-valued environment, but stochastically decreasing in N in any common-valued environment. Combined with the insight that realizations of $w_n(X_n)$ are directly identified through the inverse-bid function defined in equation (4.9), this result in turn supports nonparametric testing of the private-values hypothesis.

Identification with Exogenous Variation in Competition

Somaini [2015] demonstrated that variation in competition may provide a basis not just for testing, but for the identification of the AV model as well. Specifically, consider a general asymmetric AV model, where in addition to the data assumed above, the econometrician also observes a vector of bidder-specific preference instruments $\mathbf{z} = (z_1, \dots, z_N)$ that are assumed common knowledge to all bidders. Assume further that bidders play only one equilibrium at each realization of \mathbf{z} ; this is necessary since with both affiliated-values and asymmetries, equilibria may not be unique. Let $\omega_n(x_n, \mathbf{x}_{-n}; \mathbf{z})$ denote bidder n 's unknown full-information valuation function at instrument realization \mathbf{z} :

$$\omega_n(x_n, \mathbf{x}_{-n}; \mathbf{z}) \equiv \mathbb{E}(V_n | X_n = x_n, \mathbf{X}_{-n} = \mathbf{x}_{-n}; \mathbf{z}).$$

Note that $[\omega_1(x_1, \mathbf{x}_{-1}; \mathbf{z}), \dots, \omega_N(x_N, \mathbf{x}_{-N}; \mathbf{z})]$ summarize all features of $F_{\mathbf{V}\mathbf{X}}$ relevant for payoffs and bidding. Therefore, identification of $\omega_n(\cdot, \cdot; \mathbf{z})$ for all n is effectively equivalent to identification of the model.

Assume the preference instruments \mathbf{z} satisfy the following two conditions: (1) preference instruments and signals are independent, so $F_{\mathbf{X}|\mathbf{Z}}^0 = F_{\mathbf{X}}^0$; (2) the full-information valuation of bidder n does not depend on rival preference instruments \mathbf{z}_{-n} :

$$\omega_n(x_n, \mathbf{x}_{-n}; \mathbf{z}) = \omega_n(x_n, \mathbf{x}_{-n}; z_n).$$

These conditions are analogous to the exogeneity conditions of Balat et al. [2016]. Given sufficient variation in rival preference shifters \mathbf{z}_{-n} , Somaini established that the unknown full-information valuation function for each bidder $n \in \mathcal{N}$ is nonparametrically identified.

To get some notion of how this works, suppose that only two bidders exist. Without loss of generality, normalize signals X_1 and X_2 to be marginal uniform distributions, and consider the identification of $\omega_1(x_1, x_2; z_1)$. Denote by $G_n^0(s_n|z_1, z_2)$ the true marginal cdf of S_n , conditional on public information $\mathbf{z} = (z_1, z_2)$. Because bids are strictly increasing in strategies and because the signals X_1 and X_2 are normalized to be marginally uniform, $x_n = G_n^0(s_n|z_1, z_2)$ for any (z_1, z_2) . Substituting this identity into the definition of the pivotal valuation $w_1(x_1; z_1)$,

$$\begin{aligned} w_1(x_1; z_1) &\equiv \mathbb{E}[V_1|X_1 = G_1^0(s_1|z_1, z_2), X_2 = G_2^0(s_1|z_1, z_2); z_1] \\ &\equiv \omega_1[G_1^0(s_1|z_1, z_2), G_2^0(s_1|z_1, z_2); z_1]. \end{aligned} \quad (4.11)$$

Recall that for any observed (z_1, z_2) bidder 1's pivotal valuation $w_1(x_1, x_1; z_1)$ is identified directly from the inverse-bid function of equation (4.9). Therefore, it follows that bidder 1's full-information valuation $\omega_1(x_1, x_2; z_1)$ is identified for any pair of signals (x_1, x_2) such that there exists a realization s_1 of S_1 and a preference shifter z_2 for bidder 2 satisfying the pair of equations

$$x_1 = G_1^0(s_1|z_1, z_2) \quad \text{and} \quad x_2 = G_2^0(s_1|z_1, z_2).$$

Put another way, $\omega_1(x_1, x_2; z_1)$ is identified at (x_1, x_2) if there exists a realization of the preference shifter z_2 such that bidder 2 with signal x_2

bids like bidder 1 with signal x_1 conditional on common information (z_1, z_2) . If, for any $(x_1, x_2) \in [0, 1] \times [0, 1]$, one can find a z_2 where this condition holds, then the model is identified. In this case, sufficient variation in z_2 exists to identify $\omega_1(x_1, x_2; z_1)$, and vice versa for identifying $\omega_2(x_2, x_1; z_2)$.

With three or more bidders, the argument is substantially more complicated as “pivotal” events for bidder 1 may now involve ties with any potential opponent. Somaini showed that, given sufficient variation in characteristics of all opponents, one can still recover complete information costs.

4.5 Dutch Auctions

Although Dutch auctions are pay-your-bid auctions, establishing identification is substantially more difficult because only the winning bid is ever observed. As in models of English auctions, identification in models of Dutch auctions hinges critically on independence of valuations.

Within the IPV model of a Dutch auction, identification obtains as follows: First, if the distribution of bids G_n^0 submitted by each bidder n is observed, then using the GPV inverse-bid function, values can be recovered as

$$v_n = s_n + \frac{\prod_{m \neq n} G_m^0(s_n)}{\prod_{m \neq n} g_m^0(s_n)}. \quad (4.12)$$

In practice, however, $G_{(1:N)}^0$, the distribution of the maximum bid, is what is typically observed. If bidders are symmetric and N is observed, then $G_m^0 \equiv F_S^0$ can be recovered from $G_{(1:N)}^0$ directly using the following identity:

$$G_{(1:N)}^0(s) = F_S^0(s)^N$$

or equivalently

$$F_S^0(s) = G_{(1:N)}^0(s)^{\frac{1}{N}}.$$

If potential buyers are asymmetric, then identifying (G_1^0, \dots, G_N^0) from $G_{(1:N)}^0$ is equivalent to identifying a statistical competing risks model with independent non-identically distributed risks. Simeon M. Berman [1963] demonstrated that this model is nonparametrically

identified using the following identity:

$$G_n^0(s) = \exp \left(\int_{-\infty}^s \left[\sum_{m=1}^N G_m^w(t) \right]^{-1} dG_n^w(t) \right),$$

where $G_n^w(s) \equiv \Pr(S_n \leq s, S_n \geq S_m \forall m)$ is the known true probability that bidder $n = 1, \dots, N$ wins with a bid below s . Mapping the individual bid distributions (G_1^0, \dots, G_N^0) , thus recovered through the inverse-bid function in equation (4.12), yields identification of (F_1^0, \dots, F_N^0) as above.

4.6 Complications

Although many complicating factors exist, we focus on just four: (1) a binding reserve price; (2) unknown numbers of potential buyers; (3) auction-specific unobserved heterogeneity; (4) risk aversion; and (5) endogenous participation.

4.6.1 Binding Reserve Price

Except for a brief discussion concerned with testing for common values, we have assumed that the seller sets a nonbinding reserve price $r \leq v$. What if the reserve price is binding, so $r \geq v$? Clearly, only partial identification can be established: Because no data concerning values below the reservation price are observed, no hope exists of identifying the distribution of values in this range. Nevertheless, substantial progress can be made.

To illustrate the principles involved, consider the simplest possible case: The symmetric IPV model in which the econometrician knows the number of potential buyers N , whether the auction resulted in a sale, and the transaction price in the event of sale. Inverting equilibrium strategies as above, it is easy to see that bidding data will reveal the distribution of values among bidders with $v \geq r$. Denote this truncated distribution by $F_V^0(v|r)$. Observe that $F_V^0(v|r)$ is related to the unknown primitive distribution F_V^0 by the identity:

$$F_V^0(v|r) = \frac{F_V^0(v) - F_V^0(r)}{[1 - F_V^0(r)]}.$$

Although recovering $F_V^0(v|r)$ is not the same as recovering F_V^0 , knowledge of $F_V^0(v|r)$ can be sufficient for some policy questions of interest. For instance, given $F_V^0(v|r)$, one can simulate counterfactual revenue corresponding to any reserve price $r' \geq r$.

Similar insights apply in settings with affiliated private-values and/or asymmetric bidders: Provided the set of potential buyers is observed, information on values above r in settings with $r > \underline{v}$ can typically be recovered.

The main exception to this rule—and the main practical complication induced by a binding reservation price—comes from incomplete data. In many cases, we only observe bidders who actually submit bids, not those who were present but drew valuations below the reservation price. This complication is sufficiently important to warrant a separate discussion, which we consider next.

4.6.2 Unobserved Numbers of Potential Buyers

In our discussion to this point, we have assumed that the econometrician observes the full set of participating bidders; in applications, this may not be the case. For example, when the seller sets a binding reservation price, we may observe the set of bids submitted, but not the number of potential buyers who drew valuations below the reserve price. Similarly, at English or Dutch auctions, we typically observe the transaction price, but not the full history of bids or the full set of bidders present. In both cases, the number of potential buyers N may be unobserved.

Fortunately, even when N is unobserved, we still have substantial opportunities for identification. For example, in sealed, pay-your-bid auctions, Yonghong An, Yinghao Hu, and Shum [2010] have shown that if bidders have independent private valuations, then the truncated distribution $F_V^0(v|r)$ is identified even when N is unknown to the econometrician. Similarly, under either the second-price rule or the clock model, Unjy Song [2015] demonstrated that if bidders have independent private valuations and the econometrician observes any two order statistics of valuations, then $F_V^0(v|r)$ is identified even when N

is unknown.² In more recent work, Joachim Freyberger and Bradley J. Larsen [2017] extended this result to settings with auction-specific unobserved heterogeneity, such as those we describe in detail below, provided the econometrician additionally observes a reserve price r which can be used as an additional proxy for the auction-specific unobserved characteristic U .

Note that, to compensate for lack of knowledge concerning N , we require greater structure on either valuations or bids. For instance, to apply Song’s results on identification in models of English auctions with unknown N , the econometrician must observe at least one bid below the transaction price, whereas with known N only the transaction price is required. Furthermore, bids must truthfully reveal valuations—thereby ruling out concerns about jump bidding, which could be more severe at bids below the transaction price. Similarly, to obtain identification in models of sealed-bid auctions with unknown N , following An et al. [2010], independence of valuations is pivotal—absent this assumption, very little regarding identification is known.

Mismeasurement of N could also arise from more subtle channels. For instance, at online, ascending-price auctions Hickman et al. [2017] pointed out that early bids by high-value bidders could lead to endogenous censoring of low-value bidders, even when full bid histories are observed. More precisely, bidders who arrive late in the auction may face a standing price greater than their valuation and, therefore, may not bid. Hence, the number of bidders may again provide only a lower bound on the set of actual participants, which in turn could confound identification based on the order statistic inversion (4.1) described above.

To address this problem, Hickman et al. [2017] proposed a filtering algorithm that corrected for such potential mismeasurement; we refer interested readers to that paper for details. For further discussion of the relationship between bid timing and model identification in cases where

²Song’s results also apply within the IPV model under the pay-your-bid pricing rule, where bidders do not observe N . In this case, the bids are independent as well as identically-distributed draws from a common unknown distribution, the key condition necessary to apply Song’s result. If bidders observe N , then pay-your-bid strategies will depend on N and Song’s methods do not apply.

N is unknown, see, for example, the work of Kyoo il Kim and Joonsuk Lee [2014], who used the timing of bids to test the symmetric IPV assumption, as well as Jose J. Canals-Cerda and Jason Pearcy [2013], who developed and estimated a parametric model of eBay bidding using rich data concerning the timing of bids. Although these researchers have made some headway, there remains much to do in this area of research; we encourage others to investigate this area.

4.6.3 Auction-Specific Unobserved Heterogeneity

A third significant challenge to identification of auction models is auction-specific unobserved heterogeneity. To fix ideas, suppose that, in addition to their private signals (X_1, \dots, X_N) , bidders observe an auction-specific random variable U , where the conditional distribution of $(V_1, \dots, V_N, X_1, \dots, X_N)$ depends on U . Realizations of U will then clearly affect equilibrium bidding behavior, both directly through the distribution of (X_1, \dots, X_N) and indirectly through equilibrium bidding strategies. If U were observed, this would be no problem: Researchers could simply apply the arguments above conditional on the observed realization of U . In practice, however, U may be unknown to the econometrician. In this case, naïvely applying the methods above would mis-specify the true equilibrium bidding relationship: Whereas bidders condition their strategies on specific realizations of U , the econometrician observes only the distribution of bids averaged across all realizations of U . If unaccounted for, such unobserved heterogeneity can substantially mislead the econometric analysis.

Broadly speaking, strategies for dealing with auction-specific unobserved heterogeneity fall into two main approaches. Under the first, applicable primarily under the pay-your-bid pricing rule, it is assumed that values (V_1, \dots, V_N) are conditionally independent given realizations of the auction-specific unobservable U . Under the second, labeled the control function approach, it is assumed that the unobserved U is deterministically related to a variable that is observed, such as the number of potential buyers N or the reserve price r . We discuss both of these approaches briefly in turn.

Conditional Independence

Consider a pay-your-bid pricing rule, where the (V_1, \dots, V_N) are conditionally independent given U , so

$$F_{\mathbf{V}}^0(v_1, \dots, v_N|u) = \prod_{n=1}^N F_n^0(v_n|u).$$

Since bidders observe realizations of U prior to bidding, bidder n 's valuation v_n and bid b_n will be related by the following conditional inverse-bid function:

$$v_n = s_n + \frac{G_{(1:-n)}^0(s_n|u)}{g_{(1:-n)}^0(s_n|u)}. \quad (4.13)$$

If the econometrician also observed U , then identification of $F_n^0(v_n|u)$ would follow from equation (4.13). But in practice, the econometrician observes only the joint distribution of bids S_1, \dots, S_N , that is,

$$g(s_1, \dots, s_N) = \int \prod_{n=1}^N g_n(s_n|u) f_u^0(u) du.$$

Fortunately, under the hypothesis of conditional independence, one can often identify both the marginal distribution of U and the conditional distributions of (S_1, \dots, S_N) given U from the unconditional joint distribution of bids (S_1, \dots, S_N) . Plugging these conditional distributions back into the inverse-bid function (4.13) yields identification of the unknown primitives of interest—namely, $[F_1^0(v_1|u), \dots, F_N^0(v_N|u)]$.

The key step in this argument is, of course, to establish identification of $[g_1(s_1|u), \dots, g_N(s_N|u)]$ and $f_u^0(u)$ from the unconditional joint distribution of bids (S_1, \dots, S_N) . Toward this end, two main approaches exist. The first, due to Elena Krasnokutskaya [2011], applies when valuations are either additively or multiplicatively separable in U , that is, if either $V_n = X_n + U$, or $V_n = UX_n$. Krasnokutskaya [2011] first noted that separability of valuations in U implies separability of equilibrium bids in U : If $V_n = X_n + U$, then $\sigma_n(X_n, U) = \sigma_n(X_n, 0) + U$, and if $V_n = UX_n$, then $\sigma_n(X_n, U) = U\sigma_n(X_n, 1)$. Levering a statistical result of Ignacy I. Kotlarski [1966] on identification with additively separable measurement error, she then established the desired conditional identification result.

More recently, Hu, David McAdams, and Shum [2013] considered identification under the pay-your-bid pricing rule, with conditionally independent private values, but nonseparable unobserved heterogeneity. Under an additional completeness condition, similar to those seen in work on nonparametric instrumental variables, Hu et al. established that $[g_1(s_1|u), \dots, g_N(s_N|u)]$ and $f_u^0(u)$ are nonparametrically identified from data on (S_1, \dots, S_N) .³ The key to this result is the observation that, given the hypothesis of conditionally independent private values, the bidding model is analogous to models with conditionally independent nonclassical measurement error, for which Hu [2008] as well as Hu and Schennach [2008] had previously studied identification.

The result of Hu et al. is a substantial generalization of identification under separability, but involves two additional practical costs: (1) the additional completeness condition required, which is challenging to verify unless U is either separable or discrete; (2) nonparametric estimation without separability is presently not well understood. Hence, applied researchers have tended to focus on separable structures following Krasnokutskaya [2011].

Note that, because order statistics are not independent, approaches based on conditional independence of valuations are typically valid only when the econometrician observes all bids. In practice, this means we can use such methods to account for unobserved auction-specific heterogeneity in Vickrey auctions and sealed, pay-your-bid auctions, but typically not in English or Dutch auctions. One exception to the latter rule is when the econometrician also observes a secret or nonbinding reserve price r , in which case one can apply an identification strategy paralleling Krasnokutskaya [2011] under the assumption that both r and valuations are additively or multiplicatively separable in U . See Freyberger and Larsen [2017] for an example of the latter approach in ascending auctions, which additionally allows for unknown numbers of bidders following Song [2015].

³See, for example, Whitney K. Newey and James L. Powell [2003], Hu [2008] as well as Hu and Susanne M. Schennach [2008] for further discussions of completeness.

Control Function Approaches

An alternative strategy for dealing with unobserved heterogeneity is to assume that the unobserved auction-specific characteristic U is deterministically related to some covariate that the econometrician observes. For example, Balat et al. [2016] assumed that the number of potential buyers N is strictly increasing in U .⁴ Similarly, Roberts [2013] considered identification assuming that the reserve price r is a strictly increasing function of U . As one way to motivate the latter hypothesis, Roberts [2013] demonstrated that within the optimal auction of Myerson [1981], the seller's revenue-maximizing reserve price is strictly monotone in U . Note, however, that seller optimality is not required—choosing r according to any strictly monotone function of U would also suffice.

In either case, the main idea is the same: Insofar as r (or N) has a one-to-one relationship with the unknown U , conditioning on r (or N) will be as good as conditioning on U . For instance, under the pay-your-bid pricing rule, with affiliated private-values, when r is strictly monotone in U , one can rewrite the latent conditional inverse-bid function as

$$v_n = s_n + \frac{G_{(1:-n)}^0(s_n|s_n; u)}{g_{(1:-n)}^0(s_n|s_n; u)} \equiv s_n + \frac{G_{(1:-n)}^0(s_n|s_n; r)}{g_{(1:-n)}^0(s_n|s_n; r)}.$$

Since all terms in the final expression are identified, the distribution of V_n is identified above r . Furthermore, importantly, this does not require the assumption of conditional independence; if monotonicity is correct, then one can proceed effectively “as if” U were known.

Of course, the key to the control function approach is the strict monotonicity hypothesis, which is likely to be most plausible when the control variable considered is determined endogenously by an agent (or agents) who also observe U . As an example, if the auctioneer chooses r in response to U , or N obtains from an entry game played by bidders

⁴Because N is a discrete variable, strict monotonicity of N in U (as originally proposed by Balat et al.) can only hold under fairly strong restrictions on U . In particular, U must be a discrete random variable whose consecutive values are all sufficiently distinct to induce different realizations of N .

observing U . Therefore, it is often helpful to specify an explicit economic model that gives rise to the monotonic relationship of interest. Although typically not the main focus of the analysis, such a model may also help to clarify the conditions needed to apply control function strategies within a specific institutional context considered. Given this monotonicity, however, control function approaches may be used to correct for unobserved heterogeneity in any auction format considered above.

4.6.4 Risk Aversion

A fourth potential challenge to identification is risk aversion among potential buyers. As is well known, risk aversion can have substantial effects on auction design and performance. For instance, within the IPV model, pay-your-bid auctions revenue-dominate second-price auctions when potential buyers are risk averse. Therefore, in applications, it may be of interest to allow preferences to exhibit risk aversion.

Note that empirical analyses of risk aversion essentially require data concerning sealed, pay-your-bid or Dutch auctions because at English and Vickrey auctions it is a weakly dominant strategy for players to bid their values regardless of risk preferences. Therefore, to fix ideas, consider the symmetric IPV model under the pay-your-bid pricing rule, where potential buyer n evaluates final wealth $y \equiv (v - s)$ according to a bounded, twice differentiable von Neumann-Morgenstern utility function $u_0(y)$. Without loss of generality, normalize $u_0(0)$ to be zero, and assume that potential buyers are weakly risk averse, so $u_0''(y) \leq 0$, with $u_0''(y) = 0$ corresponding to the special case of risk neutrality. The goal is to establish identification of both the latent distribution of values F_V^0 and the unknown utility function u_0 .

Toward this end, consider the problem of potential buyer n who has value v_n and bids against $(N - 1)$ symmetric opponents. Let $F_S^0(s|N)$ denote the distribution of bids submitted by each rival $m \neq n$ in equilibrium, and $f_S^0(s|N)$ denote the associated pdf. Bidder n 's bid s_n must then satisfy the following necessary first-order condition:

$$\frac{u_0(v_n - s_n)}{u_0'(v_n - s_n)} = \frac{F_S^0(s_n|N)}{(N - 1)f_S^0(s_n|N)}. \quad (4.14)$$

Now define a new function $\lambda_0(y) = u_0(y)/u'_0(y)$ and note that $\lambda_0(y)$ is invertible because $\lambda'_0(y) = 1 - u''_0(y)/u'_0(y)^2 \geq 1$. Plugging the definition $\lambda_0(y) = u_0(y)/u'_0(y)$ into the first-order condition given in equation (4.14); re-arranging the terms yields the following:

$$v_n = s_n + \lambda_0^{-1} \left[\frac{F_S^0(s_n|N)}{(N-1)f_S^0(s_n|N)} \right]. \quad (4.15)$$

Finally, denote the quantile functions of the distributions $F_V^0(v)$ and $F_S^0(s|N)$ by $v^0(a)$ and $s^0(a|N)$, respectively. Since bids are strictly increasing in values, if v_n is equal to the a^{th} quantile of $F_V^0(\cdot)$, then s_n must be equal to the a^{th} quantile of $F_S^0(\cdot|N)$. Therefore, one can rewrite equation (4.15) as

$$v^0(a) = s^0(a|N) + \lambda_0^{-1} [R^0(a|N)], \quad (4.16)$$

where the function $R^0(a|N)$ indexes the argument to λ_0^{-1} at competition N :

$$R^0(a|N) \equiv \frac{a}{(N-1)f_s^0(s^0(a|N)|N)}.$$

What does the final inverse-bid function given in equation (4.16) imply? Both $s^0(a|N)$ and $R^0(a|N)$ are functions of the bid distribution $F_S^0(\cdot|N)$, so those are clearly identified. Furthermore, as established by Guerre, Perrigne, and Vuong [2009], knowledge of λ_0^{-1} identifies u_0 up to location and scale. The question then is whether one can recover the unknown primitives $v^0(a)$ and λ_0^{-1} through equation (4.16), given knowledge of $s^0(a|N)$ and $R^0(a|N)$.

Unfortunately, without additional information, the answer to this question is no. As pointed out by Guerre et al., any candidate λ^{-1} such that $s^0(a|N) + \lambda^{-1}[R^0(a|N)]$ is strictly increasing in a is potentially consistent with the data. More problematically, if bidders are truly risk neutral, then one is unable to exclude any candidate λ^{-1} from the identified set.⁵ Therefore, practically speaking, empirical analysis of risk aversion requires additional identifying restrictions. From where might these come?

⁵The quantile function $s^0(a|N)$ is increasing by definition, so if $s^0(a|N) + \lambda^{-1}[R^0(a|N)]$ is decreasing it can be only through the second term $\lambda^{-1}[R^0(a|N)]$. But if the true data generating process satisfies risk neutrality, then one must have $v^0(a) = s^0(a|N) + R^0(a|N)$ and, hence, $s^0(a|N) + R^0(a|N)$ increasing by con-

Exogenous Variation in Auction Format

First, following Lu and Perrigne [2008], suppose that the econometrician observes bid data generated under both pay-your-bid and English auctions. Assume further that the variation in auction format is exogenous in the sense that bidders in both pay-your-bid and English auctions draw values from the same latent distribution F_V^0 . As Lu and Perrigne [2008] noted, bids observed under the English clock format will directly reveal values for any risk preferences, thus permitting recovery of the unknown latent distribution F_V^0 . In turn, given knowledge of F_V^0 (and hence v^0), one can solve for λ_0^{-1} directly through the pay-your-bid inverse-bid condition given in equation (4.16). Hence, exogenous variation in auction format is sufficient to establish nonparametric identification in settings when bidders are potentially risk averse.

Exogenous Variation in Competition

Alternatively, suppose (as in Guerre et al. [2009]) that the econometrician observes data under the pay-your-bid pricing rule, but that the number of potential buyers N varies exogenously in the sense that F_V^0 is invariant to N . Under this hypothesis, Guerre et al. demonstrated a very powerful result: Even in the presence of arbitrary nonparametric risk aversion, observing data at any two distinct competition levels, $N_1 \neq N_2$, is sufficient to establish identification. This argument turns on the following compatibility condition: When, in fact, F_V^0 is invariant to N , from equation (4.16), for any N_1 and N_2 ,

$$s^0(a|N_1) + \lambda_0^{-1} [R^0(a|N_1)] = s^0(a|N_2) + \lambda_0^{-1} [R^0(a|N_2)]. \quad (4.17)$$

Without loss of generality, suppose that $N_2 > N_1$. Since more competitive auctions lead to more aggressive bidding, we then have $s^0(a|N_1) < s^0(a|N_2)$ for all $a \in (0, 1]$, which (from (4.17)) implies $R^0(a|N_1) > R^0(a|N_2)$ for all $a \in (0, 1]$. Furthermore, since the bidder

struction. Since $\lambda^{-1'} \leq 1$ clearly implies $\lambda^{-1'}[R^0(a|N)] \times |R^{0'}(a|N)| \leq |R^{0'}(a|N)|$, it follows that there cannot exist a candidate λ^{-1} for λ_0^{-1} such that $s^0(a|N) + \lambda^{-1}[R^0(a|N)]$ is decreasing.

with the lowest valuation bids her value regardless of N , we must have both $s^0(0|N_1) = s^0(0|N_2)$ and $R^0(0|N_1) = R^0(0|N_2)$.

Now choose any point $r \in (0, \max_a R^0(a|N_1))$, and consider identification of $\lambda_0^{-1}(r)$ at r . Let $\alpha_0 \in [0, 1]$ denote any quantile such that $r = R^0(\alpha_0|N_1)$; since $r \in [0, \max_a R^0(a|N_1)]$ and both $R^0(a|N_1)$ and $R^0(a|N_2)$ are continuous, at least one such α_0 must exist. Rearranging the compatibility condition (4.17) at $a = \alpha_0$, we obtain

$$\lambda_0^{-1}(r) = s^0(\alpha_0|N_2) - s^0(\alpha_0|N_1) + \lambda_0^{-1}[R^0(\alpha_0|N_2)]. \quad (4.18)$$

The bid quantile functions $s^0(\alpha_0|N_1)$ and $s^0(\alpha_0|N_2)$ are both identified, so the only unknown on the right-hand side is $\lambda_0^{-1}[R^0(\alpha_0|N_2)]$. But observe that, since $R^0(\alpha_0|N_2) \in (0 \equiv R^0(0|N_1), R^0(\alpha_0|N_1))$ and $R^0(\cdot|N_1)$ are continuous, there must exist some quantile $\alpha_1 \in (0, \alpha_0)$ such that $R^0(\alpha_1|N_1) = R^0(\alpha_0|N_2)$. Choosing any such α_1 , we may substitute in (4.18) to obtain

$$\lambda_0^{-1}(r) = s^0(\alpha_0|N_2) - s^0(\alpha_0|N_1) + \lambda_0^{-1}[R^0(\alpha_1|N_1)]. \quad (4.19)$$

But again rearranging (4.17), this time at $a = \alpha_1$, we also must have

$$\lambda_0^{-1}[R^0(\alpha_1|N_1)] = s^0(\alpha_1|N_2) - s^0(\alpha_1|N_1) + \lambda_0^{-1}[R^0(\alpha_1|N_2)].$$

Substituting from this identity in (4.19) yields

$$\lambda_0^{-1}[R^0(\alpha_1|N_1)] = \sum_{i=0}^1 [s^0(\alpha_i|N_2) - s^0(\alpha_i|N_1)] + \lambda_0^{-1}[R^0(\alpha_1|N_2)].$$

Again, the only unknown on the right-hand side is $\lambda_0^{-1}[R^0(\alpha_1|N_2)]$. But by the same arguments as above, we can find some quantile $\alpha_2 < \alpha_1$ such that $\lambda_0^{-1}[R^0(\alpha_2|N_1)] = \lambda_0^{-1}[R^0(\alpha_1|N_2)]$. Proceeding recursively in this fashion, we ultimately obtain a sequence of quantiles $\{\alpha_i\}_{i=1}^{\infty}$ such that, for all i , $\alpha_{i+1} < \alpha_i$ and $\lambda_0^{-1}[R^0(\alpha_{i+1}|N_1)] = \lambda_0^{-1}[R^0(\alpha_i|N_2)]$. Recursively substituting into the compatibility condition (4.17) along this sequence as above, we ultimately conclude that

$$\lambda_0^{-1}(r) = \sum_{i=0}^1 [s^0(\alpha_i|N_2) - s^0(\alpha_i|N_1)] + \lim_{i \rightarrow \infty} \lambda_0^{-1}[R^0(\alpha_i|N_2)],$$

where, recalling that both $R^0(0|N_2) = 0$ and $\lambda_0^{-1}(0)$, the trailing term vanishes since $\lim_{i \rightarrow \infty} \alpha_i = 0$. This in turn implies identification of $\lambda_0^{-1}(r)$. But r was arbitrary, from which we conclude that $\lambda_0^{-1}(\cdot)$ is non-parametrically identified on its equilibrium domain $(0, \max_a R^0(a|N_1))$.

Identification of λ_0^{-1} on the equilibrium domain $(0, \max_a R^0(a|N_1))$ implies identification of v^0 through the inverse-bid function (4.16), with u_0 determined by λ_0^{-1} up to location and scale. Guerre et al. demonstrated further that these identification insights extend naturally to settings with affiliated private values, asymmetric bidders, and binding reserve prices. Exogenous variation in competition, therefore, provides a powerful basis for identification and testing under the pay-your-bid pricing rule when bidders are potentially risk averse.

4.6.5 Endogenous Participation

Finally, we consider identification in settings with endogenous participation. For the moment, we focus on *ex ante* symmetric bidders with independent, private values. Over this simple foundation, however, we layer the highly general simultaneous entry model introduced in section 2.4.4. In this model, N symmetric potential buyers first observe private pre-entry signals \mathbf{t} and make simultaneous entry decisions based on these. The M bidders choosing to enter then observe realizations of their private values \mathbf{v} and submit bids based on these. Since monotone signal transformations preserve information, without loss of generality, one can normalize T to a standard uniform distribution.

As in section 2.4.4, we allow both entry costs $c^0(t)$ and post-entry value distributions $F_{V|T}^0(\cdot|t)$ to depend on pre-entry signal realizations t , where higher t yields both lower entry costs $c^0(t)$ and stochastically higher valuations. We thereby nest a wide variety of structures considered in the endogenous participation literature, including in particular the mixed-strategy entry model of Levin and Smith [1994], the perfectly selective entry model of Samuelson [1985], the private-costs entry model of Lu [2008, 2010], Moreno and Wooders [2011], and Li and Zheng [2009] among others, and the arbitrarily selective or affiliated signal models of Ye [2007], Marmer et al. [2013], Roberts and Sweeting [2013], Gentry and Li [2014], and Gentry et al.

[2017].

Following Gentry and Li [2014], consider identification assuming that the econometrician has access to the number of potential buyers N as well as the number of entrants M , and all submitted bids. Assume further that the number of potential buyers N varies exogenously in the sense that the distribution of (V, T) is independent of N . The goal is to translate exogenous variation in N into identified bounds on latent primitives $F_{V|T}^0$ and $c^0(t)$.⁶

Extending the arguments in section 2.4.4, it is straightforward to show that equilibrium at potential competition N will involve a signal threshold t_N^* such that potential buyer n enters when $t_n \geq t_N^*$. This equilibrium threshold t_N^* will be uniquely determined by the break-even condition

$$\int_0^{\bar{v}} [1 - F_{V|T}^0(v|t_N^*)] \times [t_N^* + (1 - t_N^*)F^*(v|N)]^{N-1} dv = c(t_N^*), \quad (4.20)$$

where $F^*(v|N)$ denotes the equilibrium distribution of values among bidders choosing to enter at competition N :

$$F^*(v|N) = \frac{1}{1 - t_N^*} \int_{t_N^*}^1 F_{V|T}^0(v|t) dt. \quad (4.21)$$

For simplicity, suppose that the bidding mechanism is a second-price auction, so that bids directly reveal values; other formats will of course require a slightly more involved mapping from bids to values, but the argument is otherwise the same. The observed distribution of bids will then directly identify the post-entry value distribution $F^*(v|N)$. Furthermore, in view of the normalization, where T is distributed uniformly on $[0, 1]$, the observed probability of entry at N will identify the equilibrium entry threshold t_N^* . Therefore, the practical problem is

⁶Gentry and Li [2014] also considered identification based on an observed auction-specific covariate \mathbf{z} assumed to affect entry costs but not latent distributions of private information. Assuming that all bidders in a given auction have common entry costs—that is, $c^0(t; \mathbf{z}) = c(\mathbf{z})$ once cost shifter \mathbf{z} is admitted—they showed that continuous variation in \mathbf{z} yields point identification of model primitives. Of course, such additional variation would also be useful here in that (just as in Gentry and Li [2014]) it would point identify $F_{V|T}^0$. But when $c^0(t; \mathbf{z})$ depends on t , it would not establish point identification of $c^0(t; \mathbf{z})$.

to translate knowledge concerning $F^*(v|N)$ and t_N^* into bounds on the latent primitives $F_{V|T}^0$ and $c^0(t)$.

Toward this end, first consider bounds on $F_{V|T}^0$. Re-arranging from (4.21) and recalling that $t_{N+1}^* > t_N^*$, for any competition levels N , $N+1$:

$$(1 - t_N^*)F^*(v|N) - (1 - t_{N+1}^*)F^*(v|N+1) = \int_{t_N^*}^{t_{N+1}^*} F_{V|T}^0(v|t) dt.$$

This expression can be rearranged to obtain an identified lower bound on $F_{V|T}^0(\cdot|t)$ at $t = t_N^*$:

$$F_{V|T}^0(v|t_N^*) \geq \frac{(1 - t_N^*)F^*(v|N) - (1 - t_{N+1}^*)F^*(v|N+1)}{t_{N+1}^* - t_N^*} \equiv F_{V|T}^-(v|t_N^*).$$

Similarly, considering competition levels $(N-1)$ and N , an identified upper bound on $F_{V|T}^0(\cdot|t)$ at $t = t_N^*$ is

$$F_{V|T}^0(v|t_N^*) \leq \frac{(1 - t_{N-1}^*)F^*(v|N-1) - (1 - t_N^*)F^*(v|N)}{t_N^* - t_{N-1}^*} \equiv F_{V|T}^+(v|t_N^*).$$

Finally, consider bounds on the unknown entry cost $c^0(t)$. By definition, the equilibrium threshold t_N^* must satisfy the break-even condition (4.20). The only unknown on the left-hand side of (4.20) is $F_{V|T}^0(\cdot|t_N^*)$, and the left-hand integral is decreasing in $F_{V|T}^0(\cdot|t_N^*)$. Plugging in $F_{V|T}^+(\cdot|t_N^*)$ and $F_{V|T}^-(\cdot|t_N^*)$ for $F_{V|T}^0(\cdot|t_N^*)$ thus immediately yields identified bounds on $c^0(t)$ at $t = t_N^*$:⁷

$$c^+(t_N^*) = \int_0^{\bar{v}} [1 - F_{V|T}^-(v|t_N^*)] \times [t_N^* + (1 - t_N^*)F_V^*(v|N)]^{N-1} dv$$

and

$$c^-(t_N^*) = \int_0^{\bar{v}} [1 - F_{V|T}^+(v|t_N^*)] \times [t_N^* + (1 - t_N^*)F_V^*(v|N)]^{N-1} dv.$$

⁷Note that the lower and upper bounds $F_{V|T}^-(\cdot|t_N^*)$ and $F_{V|T}^+(\cdot|t_N^*)$ are, in fact, well-defined distributions over V . Plugging $F_{V|T}^-(\cdot|t_N^*)$ and $F_{V|T}^+(\cdot|t_N^*)$ back into (4.20) is equivalent to taking the expectation of post-entry profit $\pi(v|t_N^*, N)$ (an increasing function) with respect to candidate distributions over V , one of which stochastically dominates and the other of which is stochastically dominated by $F_{V|T}^0(\cdot|t_N^*)$.

Thus, under only minimal assumptions on the entry process, one can obtain robust nonparametric bounds on unknown model primitives. Of course, with additional structure, one can say much more—in particular, given either mixed-strategy entry as in Levin and Smith [1994] or perfectly selective entry as in Samuelson [1985], point identification follows even without variation in N . As shown by Gentry and Li [2014], however, the fully robust analysis above yields tight bounds on both model primitives and other key counterfactual objects such as expected seller revenue. Because the assumptions maintained here formally embed all four widely applied empirical models of entry, we view this as quite encouraging evidence regarding what can be learned—even absent strong restrictions on the entry model.

5

Two Approaches

The empirical literature concerned with auctions is by now relatively large, at least when compared to the late 1980s when the structural approach was in its infancy. That literature can be decomposed into two main approaches: the reduced-form and the structural.

Roughly speaking, under the reduced-form approach, researchers focus on measuring and testing whether some comparative static prediction of the theory is observed in field data—observational data. For example, does the RET hold? Is the linkage principle valid? Is the bid distribution of the informed participants to the right of that of the uninformed bidders? And so forth.

Under the structural approach, to the extent possible, the details of the auction institution are embedded in the decision problems of agents, which are then used to derive equilibrium bid functions; the subsequent characterization of the distribution of observed bids sometimes allows one to identify and then to estimate the primitives that are necessary to conduct counterfactual policy analysis—for instance, comparative institutional design.

Even though the structural approach is typically much more demanding to implement than the reduced-form approach, the reduced-

form approach cannot deliver the primitives necessary when conducting comparative institutional design, while the structural approach is relatively impervious to the Lucas critique. As economists, we are deeply interested in policy applications, such as comparative institutional design. For this reason, even though we agree that something can be learned from the reduced-form approach, in this review, we focus almost exclusively on the structural approach.

At this point, we would be remiss if we did not report that at some point or another each of us has heard the following criticism, often from economic theorists, but sometimes from theoretical econometricians as well: The structural approach takes economic theory too seriously by imposing incredible structure on the data, which in many cases cannot be tested. First of all, it is unclear to at least one of us how to take economic theory flippantly. Putting that aside, it appears that some folks misunderstand one important purpose of the structural approach, which is an identification strategy. In other words, absent the structure imposed, it is typically impossible to reverse-engineer unobserved types from observed actions; that is, a link between the observable actions of auction participants and the latent, unobserved signals/valuations only exists when economic theory is invoked. Obviously, one cannot test an identification strategy because, well, it is imposed to identify the unobservables.

Each of us probably agrees that the simple models that economic theorists have developed may in some cases do grievous harm to the data. That said, if the reduced-form approach could circumvent this harm, then each of us would certainly, readily adopt that approach. The problem is that the reduced-form approach often does not permit an interpretation of the data, while the structural approach, subject to caveats, does. In any case, under the structural approach, every one is clear about what the maintained hypotheses are, whereas under the reduced-form approach the maintained hypotheses are often not spelled out clearly, so considerable room exists for confusion as well as needless disagreements.

In their review, Hendricks and Paarsch [1995] provided a comparison of the two approaches, which we encourage the interested to read.

Here, we simply describe some examples of the two approaches.

5.1 Reduced-Form Approach

Under the reduced-form approach, researchers typically choose a particular property that theory has delivered, and then examine whether that property holds in the data. We illustrate the approach by example—using three from the literature.

5.1.1 Examining the RET

In auction theory, the most compelling theoretical property to examine is the RET. No great surprise then that this was the first property that empirical economists investigated. Although oral and sealed auctions are not always used simultaneously in the same industry, but presumably because timber is an important commodity sold at both oral and sealed auctions, Walter J. Mead [1967] chose to examine sales of standing timber by the United States Forest Service in the Pacific Northwest—specifically some timber sales held in Oregon and Washington. Unfortunately, in addition to different auction formats and pricing rules, standing timber possesses considerable heterogeneity—for example, the location of the timber as well as its quality and species—some observed, some unobserved. Moreover, because the demand for standing timber is a derived demand (logs bucked from the felled timber are used as an input when milling lumber), lumber prices are important, too. Rather than examining the entire implications of the theory, Mead just focused on the main reduced-form implication of the theory—namely, that having controlled for other factors (in Mead’s case, the volume of timber), the average winning bid at oral auctions should equal that at sealed tenders.

A natural way to implement such a notion involves the following empirical specification:

$$W_t = \mu_Y(\mathbf{x}_t) + \delta_{\text{Auction}} \text{Auction}_t + \varepsilon_t \quad t = 1, 2, \dots, T$$

where Auction_t denotes an indicator variable concerning the auction form—for example, the variable may equal one when the auction form

is sealed, pay-your-bid, and zero when that form is oral, second-price. Within this empirical specification, $\mu_Y(\mathbf{x}_t)$ is a short-hand notation that allows the mean to vary depending on \mathbf{x}_t a vector of observed covariates pertaining to auction t . A common way in which to model $\mu_Y(\mathbf{x}_t)$ is as a single-index $\mathbf{x}_t\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is a vector of unknown parameters that is conformable to \mathbf{x} . The unknown parameter δ_{Auction} measures the average difference that obtains as a result of the difference in auction format and pricing rule.

Using regression methods (specifically, the method of least squares), Mead estimated a positive δ_{Auction} . Within the symmetric IPV model, this is evidence consistent with risk aversion, which is unobserved. Alternatively, because collusion is easier to support in environments where more information is released, the result could be explained by less collusion at sealed-bid timber sales.

Johnson [1979] investigated whether bidder asymmetries could explain the differences between oral and sealed auctions, again using regression methods and similar United States Forest Service data, although this time from some sales held in Idaho and Montana, and including many more covariates. Johnson conjectured that small businesses, who are sometimes treated specially, are different from larger sawmills. In the presence of asymmetries, expected revenues can be higher at sealed, pay-your-bid auctions than at oral, second-price auctions; see, for example, René Kirkegaard [2012]. Despite the evident care that Johnson took when carrying out his empirical work, it remains unclear whether this explanation is supported by the data because the structure of economic theory is not imposed.

Robert G. Hansen [1985, 1986] then pointed out that a selection bias could exist—caused by the way the Forest Service chose which auction format and pricing rule to employ for each timber sale. Having corrected for this sample-selection bias, using the methods pioneered by Heckman [1978, 1979], Hansen reported empirical evidence consistent with the RET.

At the risk of belaboring the obvious, the presence of heterogeneity, either observed or unobserved, across auctions makes testing the RET difficult: if one rejects, then is this because the RET does not hold, or

because confounding factors have not been accounted for appropriately in the regression. If one does not reject, then how powerful is the test against interesting, alternative hypotheses? Under the reduced-form approach, alternative hypotheses are usually not spelled out, so it is difficult to say anything about power when the null hypothesis is not rejected.

By conducting laboratory and field experiments, in which some confounding factors can be controlled, researchers have sought to circumvent problems associated with heterogeneity. For instance, in laboratory experiments where independent, private values were generated, Vicki M. Coppinger, the Nobel laureate Vernon L. Smith, and John A. Titus [1980] as well as James C. Cox, Bruce Robertson, and Smith [1982, 1983] reported that the average winning bid at sealed, pay-your-bid auctions exceeded that at Dutch auctions. Using similar protocols, Kagel and Levin [1993] reported that the average winning bid at second-price auctions exceeded what theory predicted.

One criticism of laboratory experiments is a lack of realism: the stakes are often not high enough to induce the behavior predicted by economic theory. To circumvent such problems, David Lucking-Reiley [1999] employed field experiments to test the RET—using cards from the game *Magic: The Gathering*. He reported that, on average, the Dutch auction generated thirty percent higher revenues than the sealed, pay-your-bid auction, a violation of the RET and a reversal of previous laboratory results; English and Vickrey auctions generated roughly equivalent average revenues.

As one can see, when confronting theory with evidence, at least three sources of data exist: archival as well as laboratory and field experiments. We have chosen to focus on observational data (for the most part, those from archives) because such data are often the only data initially available. Sometimes, when observational data provide evidence worthy of further study, either laboratory or field experiments are then conducted—as for example in the research reported by Bajari and Hortaçsu [2005].

5.1.2 Examining the Linkage Principle

Sung-Jin Cho, Paarsch, and John Rust [2014] used observational data to examine the linkage principle—specifically, the prediction that within AV environments the expected revenues generated at open-outcry, ascending-bid auctions are higher than those under auction forms that reveal less information to participants. Their data were obtained from a large seller of used automobiles who experimented with different auction formats and pricing rules. Specifically, the firm conducted computerized pay-your-bid auctions over the Internet as well as oral, ascending-price auctions conducted by an auction house located centrally among potential buyers.

At the Internet auctions, potential buyers submitted bids electronically over the course of an auction that was two minutes in duration. Participants at the Internet auctions were strictly anonymous. A bidder could only see a single piece of information: whether his bid was the highest competing bid at the auction. Participants could not observe the bids of their opponents. In fact, an individual bidder did not even know what the highest bid was at any time during the auction, unless the bidder himself had the current highest bid.

The auction house employed an oral, ascending-bid auction that was virtually identical to an economist's notion of an English auction. In particular, unlike the firm's Internet auctions, a bidder at an English auction conducted by the auction house could see the other participating bidders as well as their bids at each stage in the auction, including the highest bid at any point in the auction.

Although individual-specific, private-value components are important in any automobile purchase, Cho et al. [2014, p. 351] argued that common-value elements must surely exist, too, writing

Specifically, a pre-owned vehicle's true quality is uncertain because the intensity with which it has been used and the care shown it by previous drivers are unknown. This unknown quality is basically the same to all potential buyers, but will remain undiscovered until the vehicle has exchanged hands and the new owner has experienced it on the road.

Using reduced-form, regression methods, Cho et al. reported that average revenues were significantly higher under an English auction than under the Internet, pay-your-bid that revealed less information to bidders, evidence consistent with the linkage principle—thus supporting the hypothesis of affiliation among signals/valuations.

Empirical tests of the linkage principle had been carried out using data from controlled laboratory experiments before Cho et al.; for instance, Kagel, Ronald M. Harstad, and Levin [1987] reported that the average winning bid at sealed, pay-your-bid auctions exceeded that at English auctions. In several important papers, John H. Kagel and Levin [1986] as well as Levin, Kagel, and Jean-François Richard [1996] analyzed the behavior of laboratory subjects at English and pay-your-bid auctions in situations where a common-valued component existed in their experimentally-generated values. The results of these experiments, which have been summarized by Kagel and Levin [2002], were mixed: For relatively inexperienced subjects, they found a pronounced winner's curse caused by overbidding at sealed auctions relative to English ones. On average, the overbidding caused the seller's revenues to be higher at sealed auctions than at English auctions, contrary to the prediction of the linkage principle. In experiments involving experienced bidders, however, the winner's curse was ameliorated and the English auctions generated higher expected revenues than the sealed ones, evidence consistent with the linkage principle.

5.1.3 Examining the Effects of Asymmetric Information

Dating back to Vickrey's [1961] paper, economists have been aware that within the IPV model asymmetries can result in inefficient outcomes at pay-your-bid auctions, either open or sealed. That is, weakness can breed aggression, which means that the highest-valuation bidder can lose the auction; the winning bid need not bear any relation to the opportunity cost, either in realization or in expectation.

What about the CV model? Suppose that two types of bidders exist—the informed and the uninformed, where the informed have more precise signals than the uninformed. In a Bayes-Nash equilibrium, the informed bidders are predicted to win more often than the uninformed.

More importantly, on average, the informed bidders are predicted to earn positive profits, while the uninformed are predicted to earn zero profits.

A naïve person might ask: Why would the uninformed ever bid? They know less. Such naïve beliefs would not constitute a Bayes-Nash equilibrium. In a Bayes-Nash equilibrium, the uninformed bid to discipline the informed, but the uninformed only bid to the point where zero expected profits obtain.

Using reduced-form methods, Hendricks and Porter [1988] examined empirically the implications of Bayes-Nash equilibrium within the CV model for informed and uninformed bidders who participated at sealed, pay-your-bid auctions conducted by United States Department of the Interior for the right to drill for oil and gas reserves on drainage leases on the Outer Continental Shelf. Drainage leases differ from wild-cat leases in that some of the firms have drilled wells on adjacent land (are neighbor firms in the vocabulary of Hendricks and Porter) and are, thus, better informed than those firms who have not (non-neighbor firms in their words). The main reduced-form implications of the theory outlined above are borne out in the data that Hendricks and Porter examined.

The paper by Hendricks and Porter is important because the empirical work examined an interesting economic phenomenon—the effects of informational asymmetries in games of incomplete information—which is neither obvious nor intuitive to most, and finds empirical evidence supporting that phenomenon.

5.1.4 Limitations

Perhaps the most damning critique of the reduced-form approach is that it cannot recover the primitives of the economic environment, which (in light of the Lucas critique) are central to implementing comparative institutional design. That is, one cannot conduct counterfactual policy analysis using empirical results derived from a reduced-form analysis of auction data. Of course, advocates of the reduced-form approach often counter that this is requiring too much of these empirical methods. For instance, if one only wants to investigate the broad predic-

tions of economic theory, then the reduced-form approach is adequate to the purpose.

The problem with this modest objective is that, because researchers of the reduced-form approach typically do not specify what the alternative hypothesis is, tests involving the reduced-form approach do not have much power. Moreover, it is unclear how one might include reasonable models of alternative behavior within the reduced-form approach. Some would counter this criticism by pointing out that much of empirical research in economics suffers from this criticism.

5.2 Structural Approach

In a nutshell, the structural approach builds on the simple notion that under the equilibrium hypothesis the observed actions are monotonic functions of the costs, signals, values—whatever constitutes the incomplete information that is summarized by random variables. Therefore, under suitable identification conditions, one can then reverse-engineer the distribution of unobserved, latent types from the distribution of observed actions. Having reverse-engineered the distribution of types, one can undertake comparative institutional design.

5.2.1 CV Model

Within the pure CV model, Albert K. Smiley was the first to note this fact in his doctoral dissertation submitted to the Department of Economics at Princeton University and published as Smiley [1979]. Smiley was concerned with estimating the value of oil tracts from bid data. As noted above, the tenders at sealed, pay-your-bid auctions are functions of the signals, but are not the true unknown values. In order to reverse-engineer estimates of those true values, Smiley wrote down explicit models of the signal distributions. In order to get tractable solutions for the equilibrium bid function, he then assumed the prior densities of the unknown values came from particular parametric families. Based on this, for an observed bid, Smiley could then invert the equilibrium bid function to recover the signal consistent with that bid, and then estimate the distribution of signals, which was centered about the true, but

unknown value. As noted above, for computational tractability, Smiley focused on proportional bid functions. Despite this limitation, his research is obviously a pioneering first step in the structural approach.

5.2.2 IPV Model

Paarsch [1997] was the first to apply the structural approach within the IPV model, and then to use his estimates to conduct comparative institutional design—specifically, estimating the optimal reserve price of timber sold by the Ministry of Forests in the province of British Columbia, Canada.

Riley and Samuelson [1981] showed that expected revenues as a function of the reserve price r can be written as

$$N \int_r^{\bar{v}} [uf_V(u) + F_V(u) - 1] F_V(u)^{N-1} du. \quad (5.1)$$

With a binding reserve price, some fraction of the time $F_V(r)^N$ the object goes unsold, in which case the seller just gets his valuation, which is assumed to be v_0 . Thus, the expected return to the seller of reserve price r is

$$v_0 F_V(r)^N + N \int_r^{\bar{v}} [uf_V(u) + F_V(u) - 1] F_V(u)^{N-1} du. \quad (5.2)$$

Differentiating the function given in equation (5.2) with respect to the reserve price r yields the following first-order condition:

$$\begin{aligned} 0 &= N v_0 F_V(r^*)^{N-1} f_V(r^*) - N [r^* f_V(r^*) + F_V(r^*) - 1] F_V(r^*)^{N-1} \\ 0 &= v_0 f_V(r^*) - [r^* f_V(r^*) + F_V(r^*) - 1] \\ r^* &= v_0 + \frac{[1 - F_V(r^*)]}{f_V(r^*)}. \end{aligned}$$

What is notable about the optimal reserve price r^* is that it does not depend on the number of potential buyers N . This is not to say that expected revenues do not depend on N : in Figure 5.1, using the running example from the Weibull family, where we have assumed $v_0 = 0$, we depict expected revenues as a function of the reserve price r —the function in equation (5.2)—when $N = 2$ and when $N = 5$.

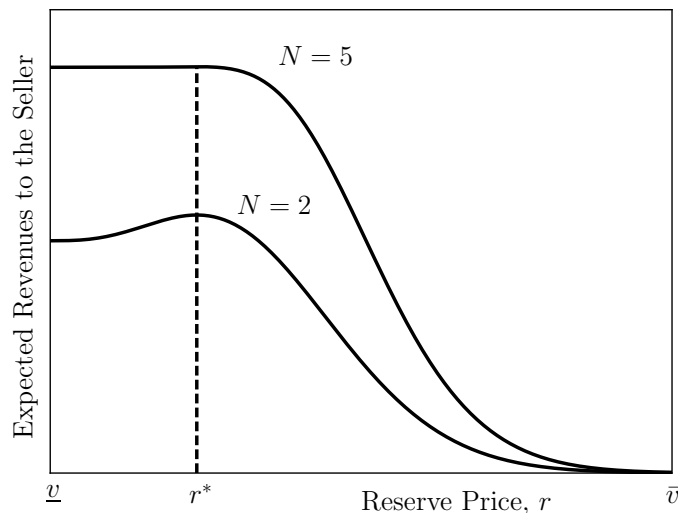


Figure 5.1: Expected Revenues as Function of the Reserve Price

In both the $N = 2$ and the $N = 5$ cases, the optimal reserve price is $1/\sqrt{2}$, but the functions have distinctly different shapes: When $N = 5$, the expected revenue function is basically flat to the left of r^* , whereas when $N = 2$, that function falls a bit. Suffice it to note that the expected revenue function is not a symmetric function about r^* , which is important when conducting inference concerning the optimal reserve price—as Kim [2013] has pointed out.

The reason why the shape of the expected revenue function is particularly relevant in the case studied by Paarsch is that over one quarter of the auctions in his data set had just one participant, and the average number of participants was around four. In short, too low a reserve price could be costly relative to the optimal reserve price.

For his empirical work, Paarsch did archival research and collected the entire set of bid sheets for a sample of 129 English auctions of standing timber. He then focused on the final bids tendered by each active participant, using that information to reverse-engineer the distribution

of valuations. Paarsch faced at least four problems: (1) the winning bid does not reveal complete information concerning the winner's true valuation; (2) in the presence of a reserve price, the empirical distribution of observed bids represents a truncated sample of data—only those potential buyers whose valuations exceed the reserve price will choose to bid; (3) as in all empirical studies of auctions, the joint distribution of bidding and nonparticipation depends on the number of potential buyers N , but finding a credible measure of potential competition is often difficult—when a measure can be found, that proxy is often inaccurate; (4) at many auctions the timber for sale is not homogeneous: covariate heterogeneity is important.

In addition to the the number of actual bidders and their final recorded bids as well as reserve prices, Paarsch also gathered information concering the volume of timber by species as well as the distance to the nearest timber processing facility. From other sources, he also derived measures for the price of logs—timber that has been felled, limbed, and bucked.

Employing the clock model, Paarsch used the result that it is a dominant strategy for nonwinners to bid up to the value of the timber. That strategy is unaffected by attitudes toward risk, which some think are important in the logging industry. By focusing on English auctions within the IPV model, Paarsch avoided the stronger common-knowledge-of- F_V assumption needed to solve for the Bayes-Nash equilibrium at sealed, pay-your-bid auctions.

Paarsch developed an empirical framework that allowed him to estimate the probability law of valuations using the empirical distribution of bids from a truncated and potentially incomplete sample of data in the absence of a measure of potential competition and with considerable observed covariate heterogeneity. Central to the empirical specification of Paarsch's study was the joint density of bidding and nonparticipation:

$$\begin{aligned} & \left\{ F_V(r)^N \right\}^{\mathbf{1}(M=0)} \left\{ N F_V(r)^{N-1} [1 - F_V(r)] \right\}^{\mathbf{1}(M=1)} \\ & \left\{ \binom{N}{m} F_V(r)^{(N-m)} m! \prod_{i=2}^m f_V[b_{(i:m)}] [1 - F_V(w)] \right\}^{\mathbf{1}(M \geq 2)}. \end{aligned} \quad (5.3)$$

The first part in braces, with the exponent $\mathbf{1}(M = 0)$, concerns timber that went unsold; the second part in braces, with the exponent $\mathbf{1}(M = 1)$, concerns auctions at which only one bidder participated; the third part in braces, with the exponent $\mathbf{1}(M \geq 2)$, concern timber sales with two or more bidders.

As mentioned, accurate and reliable measures of N are notoriously difficult to obtain. Even under the best of conditions, measurement error in N is still possible. This can result in mismeasuring the number of nonparticipants $(N - m)$ in the term $F_V(r)^{(N-m)}$. Thus, Paarsch chose to focus on the conditional (truncated) distribution of values because in such a specification $(N - m)$ would be absent. In short, conditioning on only those sales for which two or more potential buyers participated, which involved scaling by the pmf of participation from equation (2.14).

To estimate the optimal reserve price, one must recover information concerning the latent unobserved variable V . A natural way to proceed in recovering an estimate of V 's distribution $F_V(v)$ would be to examine the empirical distribution of bids and then to map back to the distribution of V . For example, as was shown earlier, in the absence of observed covariates and reserve prices and given a large enough sample, one could perform this exercise nonparametrically. The presence of a reserve price complicates matters because parts of the valuation distribution are unobserved. Because it is potentially necessary to have the entire valuation distribution in order to calculate the optimal reserve price, an alternative method to nonparametric estimation must be sought.

Paarsch assumed that $F_V(v)$ belonged to a parametric family of distributions, the Weibull. Implicit in any parametric assumption concerning $F_V(v)$ is an assumption concerning how a differential equation in V behaves locally. Paarsch assumed that this local behavior could be extended into regions he could not observe in order to get an estimate of $F_V(v)$ in that region.

Paarsch employed the method of maximum likelihood to estimate the unknown parameters of $F_V(\cdot)$, and then the optimal reserve prices for sales in his sample. While the average of historical reserve prices

in the sample was between \$2 and \$3 per cubic metre of timber, the average of estimated optimal reserve prices was between \$8 and \$12, depending on what he assumed for v_0 .

It is also interesting to note that in April 1994, after the release of Paarsch's working paper, which was published as Paarsch [1997], the Ministry of Forests more than doubled the stumpage rates it charged firms, suggesting that Paarsch's findings had real-world policy relevance.

5.3 Summary

From the few examples we have presented above, we hope to have provided the reader with some food for thought, at least concerning when to use the reduced-form approach and when to use the structural approach. Specifically, if the intent of the research is to provide a summary description of a market institution, showing that the broad predictions of economic theory are more or less consistent with the data (or not), then the reduced-form approach is adequate to the task. On the other hand, if the intent of the research is to provide relevant policy advice that is relatively robust to the Lucas critique, then the structural approach is the only way to go. In the remainder of this paper, we review parts of the literature concerned with the structural approach.

6

Estimation Strategies

The structural econometrics of auctions has made broad use of modern methods of data analysis. Each of the important estimation strategies (nonparametric, semi-nonparametric, maximum-likelihood, method-of-moments, quantile-regression, simulation—both frequentist and Bayesian) has been used in a variety of contexts. In this section, we review those methods that appear to have been most commonly used.

6.1 Nonparametric Methods

Nonparametric methods are statistical techniques that do not require the researcher to specify functional forms for objects being estimated—for instance, the cdf of bids, or valuations. Instead, one takes comfort from results in large-sample (asymptotic) theory—for instance the Glivenko-Cantelli lemma—that the empirical distribution function (edf) converges almost surely to the true population cdf in the sup norm. For reasons that will become clear in a moment, the most commonly-used method of estimating the pdf is referred to as kernel-

smoothing.¹ Although the method of kernel-smoothing is popular, it works best on large data sets, when not much observed heterogeneity in covariates exists—for example, fewer than, say, five covariates.

6.1.1 Vickrey Auctions

In the context of estimating $F_V^0(\cdot)$, the true population cdf of V , from a sample of T Vickrey auctions, the edf of bids is a natural candidate to use. Here, assume $\Pr(V_{nt} \leq v | N_t) = \Pr(V_{nt} \leq v) = F_V^0(v)$. Then, for the observed data $\{(\mathbf{b}_t, N_t)\}_{t=1}^T$, an independent and identically-distributed sample, by the Glivenko-Cantelli lemma, one knows that

$$\|\tilde{F}_B(v) - F_V^0(v)\|_\infty = \left\| \frac{1}{\sum_{t=1}^T N_t} \sum_{t=1}^T \sum_{n=1}^{N_t} \mathbf{1}(b_{nt} \leq v) - F_V^0(v) \right\|_\infty \xrightarrow{\text{as}} 0,$$

where $\xrightarrow{\text{as}}$ denotes almost-sure convergence, and $\|\cdot\|_\infty$ denotes the sup norm. Of course, a problem exists with $\tilde{F}_B(\cdot)$: because the edf is discontinuous, it is nondifferentiable. In short, $\tilde{F}_B(\cdot)$ is not a good candidate to use when estimating the optimal reserve price, for example.

An alternative estimator involves smoothing the indicator function $\mathbf{1}(\cdot)$. For instance, approximate $\mathbf{1}(\cdot)$ using a cdf like $\Phi(\cdot)$, the standard normal cdf, which delivers the following alternative estimator:

$$\hat{F}_B(v) = \frac{1}{\sum_{t=1}^T N_t} \sum_{t=1}^T \sum_{n=1}^{N_t} \Phi\left(\frac{v - b_{nt}}{h}\right),$$

where h is referred to as the smoothing (or bandwidth) parameter. When h is small, $\Phi(\cdot)$ changes very quickly from zero to one as v passes through b_{nt} , while when h is large, the change in $\Phi(\cdot)$ is more gradual. An estimate of $f_V^0(v)$, the derivative of $F_V^0(v)$, is then

$$\hat{f}_B(v) = \frac{1}{h \sum_{t=1}^T N_t} \sum_{t=1}^T \sum_{n=1}^{N_t} \phi\left(\frac{v - b_{nt}}{h}\right),$$

where $\phi(\cdot)$ is the standard normal pdf, which is often referred to as the Gaussian kernel function. Importantly, even though we have used

¹The primer written by Jeffrey S. Racine [2008] for the *Foundations and Trends*[®] of *Econometrics* series is an excellent place to learn about kernel-smoothing.

the standard normal pdf as the kernel, any non-negative function that integrates to h and is symmetric about zero could have been used instead.²

If the bandwidth parameter h shrinks as $\sum_{t=1}^T N_t$ gets large according to a particular rate, then

$$\hat{F}_B(v) \xrightarrow{\text{as}} F_V^0(v) \quad \text{and} \quad \hat{f}_B(v) \xrightarrow{\text{as}} f_V^0(v).$$

Evarist Giné and Richard Nickl [2008] demonstrated that, after a suitable normalization, $\hat{F}_B(v)$ is distributed asymptotically normal, in symbols

$$\sqrt{\sum_{t=1}^T N_t} [\hat{F}_B(v) - F_V^0(v)] \xrightarrow{d} \mathcal{N}(0, F_V^0(v)[1 - F_V^0(v)]).$$

Here \xrightarrow{d} denotes convergence in distribution, while \mathcal{N} denotes the Gaussian law, in this case having mean 0 and variance $F_V^0(v)[1 - F_V^0(v)]$. Thus, a consistent estimate of the optimal reserve price can be calculated by solving the following equation:

$$\hat{r}^* = v_0 + \frac{[1 - \hat{F}_B(\hat{r}^*)]}{\hat{f}_B(\hat{r}^*)}.$$

6.1.2 English Auctions

In the case of English auctions, under the clock model, when only the winning bid is observed, provided N does not vary across auctions, a smoothed estimate of $F_W^0(v)$ can be calculated using

$$\hat{F}_W(v) = \frac{1}{T} \sum_{t=1}^T \Phi\left(\frac{v - W_t}{h}\right).$$

Now,

$$F_W^0(w) = \frac{N!}{(N-2)!(2-1)!} \int_0^{F_V^0(w)} u^{N-2} (1-u)^{2-1} du,$$

so

$$\hat{F}_V(v) = Q_{(2:N)} [\hat{F}_W(v)].$$

²Alternative kernels have names like uniform (boxcar), triangular, Epanechnikov (parabolic), and triweight, among others. An advantage of the Gaussian kernel is that it has infinitely many derivatives. Several kernels have bounded support.

6.1.3 Sealed, Pay-Your-Bid Auctions

At sealed, pay-your-bid auctions, the link between the distribution of bids and the distribution of valuations is not as direct. In this case, Guerre et al. [2000] proposed a two-stage estimation strategy: in the first, estimate $F_S^0(s|N_t = m)$ and $f_S^0(s|N_t = m)$ using kernel-smoothing methods and all of the auctions for which $N_t = m$. In symbols,

$$\hat{F}_S(s|N_t = m) = \frac{1}{\sum_{t=1}^T \mathbf{1}(N_t = m)} \sum_{t=1}^T \mathbf{1}(N_t = m) \sum_{n=1}^{N_t} \Phi\left(\frac{s - s_{nt}}{h}\right).$$

Using $\hat{F}_S(s|N_t = m)$ and $\hat{f}_S(s|N_t = m)$, for every $N_t = m$, calculate the pseudo-values according to equation (4.7), namely,

$$\hat{v}_{nt} = s_{nt} + \frac{\hat{F}_S(s_{nt}|N_t = m)}{(m-1)\hat{f}_S(s_{nt}|N_t = m)}.$$

Subsequently, in a second step, using all of the estimated pseudo-values $\{\{\hat{v}_{nt}\}_{n=1}^{N_t}\}_{t=1}^T$, Guerre et al. then proposed estimating $f_V^0(v)$ using

$$\hat{f}_V(v) = \frac{1}{h \sum_{t=1}^T N_t} \sum_{t=1}^T \sum_{n=1}^{N_t} \phi\left(\frac{v - \hat{v}_{nt}}{h}\right).$$

An important contribution of Guerre et al. was to derive a bandwidth parameter h^* that achieves the optimal rate of convergence.

Boundary Correction

One problem faced by Guerre et al. is that in a neighborhood of the upper bound of support $\bar{s}(m)$, where the winning bid is more likely to obtain, kernel density estimators are known to be inconsistent and to exhibit bias near the extreme observations in finite samples. Kernel density estimators at a given point rely on observations within a bandwidth, but data that are within a bandwidth of the support endpoints pose a challenge since observations outside of a compact support are impossible. To deal with this, Guerre et al. proposed trimming the sample—discarding observations within a bandwidth of the sample extrema. Because the estimator is built in two stages of kernel-smoothing, boundary effects are compounded: The estimates are biased

downward in both stages. Moreover, the bandwidth selection rule becomes problematic—the econometrician must be concerned with both mean integrated squared error (the standard criteria for optimal bandwidth choice) and also the amount of data lost for the second stage because of trimming, the latter of which is neglected.

Hickman and Hubbard [2015] proposed dealing with these issues by replacing the standard kernel-smoothing and sample-trimming approach with a boundary-corrected, kernel-smoothed estimator developed by Shunpu Zhang, Rohana J. Karunamuni, and M. Chris Jones [1999] and improved on by Karunamuni and Zhang [2008]. The boundary-corrected, kernel-smoothed estimator is essentially a hybrid of two coping techniques for addressing the boundary effect—a reflection method which mirrors data near a boundary outside the support and a transformation method which maps the data onto an unbounded support. The modified estimator achieves uniform consistency on the entire support and does so at the same rate of convergence as standard kernel density estimators.

Hickman and Hubbard demonstrated that substantial improvements in finite-sample performance were to be had from using the boundary-corrected, kernel-smoothed estimator as well as the versatility of the approach—it is easily portable to any setting in which the multi-stage strategy of Guerre et al. has been extended. Hickman and Hubbard also revisited a setting first considered by Sandra Campo, Perrigne, and Vuong [2003] involving asymmetric bidders with affiliated private values; using the same data, the results are remarkably different, especially with respect to the level of informational rents and inefficiencies.

6.1.4 Dutch Auctions

At Dutch auctions, use the smoothed edf

$$\hat{F}_W(w|N_t = m) = \frac{1}{\sum_{t=1}^T \mathbf{1}(N_t = m)} \sum_{t=1}^T \mathbf{1}(N_t = m) \Phi\left(\frac{w - w_t}{h}\right)$$

for every $N_t = m$, to calculate

$$\hat{f}_W(w|N_t = m) = \frac{1}{h \sum_{t=1}^T \mathbf{1}(N_t = m)} \sum_{t=1}^T \mathbf{1}(N_t = m) \phi\left(\frac{w - w_t}{h}\right).$$

Whence calculate the pseudo-values according to

$$\hat{z}_t = w_t + \frac{m}{(m-1)} \frac{\hat{F}_W(w_t|N_t = m)}{\hat{f}_W(w_t|N_t = m)}.$$

Subsequently, in a second step, using these estimated pseudo-values, estimate $F_Z^0(z)$ using

$$\hat{F}_Z(z|m) = \frac{1}{\sum_{t=1}^T \mathbf{1}(N_t = m)} \sum_{t=1}^T \mathbf{1}(N_t = m) \Phi\left(\frac{z - z_t}{h}\right).$$

When the number of potential buyers is constant across T , these nonparametric methods are easy to implement, and reasonably accurate if T is, say, greater than one hundred, but with variability in N_t , nonparametric methods have large sampling variability.

6.1.5 Limitations

One of the drawbacks of the nonparametric approach, at least when applied to richer informational environments than the IPV model, for example the APV model, is that the restrictions required by affiliation are not respected by kernel-smoothing. Yet the first-order condition on which the approach is based is really only valid when, say, affiliation (or a like condition) holds.

6.2 Semi-Nonparametric Methods

A. Ronald Gallant and Douglas W. Nychka [1987] proposed an alternative to kernel-smoothing: the semi-nonparametric (SNP) approach.³ Under the SNP approach, the idea is to approximate flexibly an unknown pdf by orthogonal polynomials—such as Chebyshev, Laguerre,

³See, too, Brian J. Eastwood and Gallant [1991] as well as Victor M. Fenton and Gallant [1996].

or Hermite polynomials. We present a brief description of these polynomials in Appendix A.3; for a good discussion, see the textbook by Kenneth L. Judd [1998].

In their application, Bjarne O. Brendstrup and Paarsch [2006] chose the Laguerre polynomial because in their problem the domain was $[0, \infty)$. That is, Brendstrup and Paarsch assumed that the marginal utility of the good at auction was non-negative; when they admitted covariates, they used Hermite polynomials to approximate the pdf of the logarithm of the marginal utility, conditional on the covariates.

How does the SNP approach work? To begin, assume that the pdf f_V lives in the space \mathcal{F} which consists of densities having several properties. In describing these properties, introduce some additional notation: First, let d denote the number of derivatives for the unknown but true pdf f_V^0 on $[0, \infty)$. For some integer $d_0 (> \frac{1}{2})$, for some bound \mathcal{D}_0 , and for some $\varepsilon_0 (> 0)$ the space \mathcal{F} consists of the pdfs having the following form:

$$f_V(v) = [h_V(v)]^2 + \varepsilon \exp(-v)$$

with $\|h_V\|_{d+d_0,2,\mu}$ being less than \mathcal{D}_0 and ε being greater than ε_0 where μ equals $(1+v^2)^{\delta_0}$ and $\|h\|_{d+d_0,q,\mu}$ is the Sobolev norm; that is,

$$\|h\|_{d+d_0,q,\mu} = \left(\sum_{|\alpha| \leq d+d_0} |D^\alpha h(v)|^q \mu(v) dy \right)^{\frac{1}{q}} \quad q > 0,$$

where D^α is the differential operator. The bound \mathcal{D}_0 imposes a restriction on the densities in \mathcal{F} by constraining the tails of these densities from above. This restriction is needed to ensure that the space \mathcal{F} is compact.

The term $\varepsilon \exp(-v)$ is a lower bound on the pdf used to avoid $\log[f_V(v)]$ going to $-\infty$ and $\int \log[f_V(v)] f_{V'}(v) dv$ going to $-\infty$ for any two elements f_V and $f_{V'}$ in \mathcal{F} .

To make the discussion described above concrete, consider the following: Any pdf $f_V \in \mathcal{F}$ can be written in terms of an infinite-order polynomial of the form

$$f_V(v) = \left[\sum_{i=0}^{\infty} \theta_i \mathbb{L}_i(v) \right]^2 \exp(-v) + \varepsilon \exp(-v)$$

where $\mathbb{L}_i(v)$ is the Laguerre polynomial of order i . Approximate the infinite-order polynomial above by a finite-order polynomial of the form

$$f_V^{I_T}(v) = \left[\sum_{i=0}^{I_T} \theta_i \mathbb{L}_i(v) \right]^2 \exp(-v) + \varepsilon \exp(-v).$$

Of course, when truncating an infinite-order polynomial to obtain a finite-order one, an error is introduced. By letting the degree of the approximation get better as the sample size increases (that is, by letting I_T increase at a rate that is slower than the rate at which the sample size T increases), the approximation will converge to the truth. To be strictly nonparametric, the degree of the polynomial must tend to infinity as the sample size increases to infinity.

A natural way to implement this finite-order approximation is the method of quasi-maximum likelihood. For example, at an Vickrey auction, define the estimator \hat{f}_V by

$$\hat{f}_V = \operatorname{argmax}_{f_V \in \mathcal{F}_T} \frac{1}{\sum_{t=1}^T N_t} \sum_{t=1}^T \sum_{n=1}^{N_t} \log f_V(b_{nt} | \boldsymbol{\theta}).$$

6.2.1 Limitations

Like the nonparametric approach, a drawback of the SNP approach when applied to richer informational environments is that the restrictions required by, say, affiliation are not respected. In addition, the method is best suited to English and Vickrey auctions; it does not extend naturally to auctions held under the pay-your-bid rule.

6.2.2 Introducing Covariates

When introducing observed covariate heterogeneity (denoted below by the vector \boldsymbol{x}) into econometric models of auctions, only certain functional forms will lead to tractable empirical specifications. In particular, two different structures have typically been used to introduce observed covariates into the valuations (denoted by V s, below) of potential buyers. The first is an additive form, like

$$V_{nt} = \mu(\boldsymbol{x}_t) + \varepsilon_{nt}$$

for the n^{th} potential buyer at the t^{th} sale where $\mu(\cdot)$ is some (typically unknown) function, while the second is a multiplicative form, like

$$V_{nt} = \mu(\mathbf{x}_t)\varepsilon_{nt}$$

where $\mu(\cdot)$ is some (typically unknown) function. Here, ε_{nt} denotes the unobserved bidder-specific heterogeneity in valuations.

Under these functional-form assumptions, the Bayes-Nash equilibrium bid function is of the form

$$\sigma(V_{nt}) = \mu(\mathbf{x}_t) + \sigma(\varepsilon_{nt})$$

in the first case, and

$$\sigma(V_{nt}) = \mu(\mathbf{x}_t)\sigma(\varepsilon_{nt})$$

in the second. When it comes to implementing these specifications, researchers typically assume a single-index structure, like

$$\mu(\mathbf{x}) = \mathbf{x}\boldsymbol{\delta} \quad \text{or} \quad \log[\mu(\mathbf{x})] = \mathbf{x}\boldsymbol{\delta},$$

where $\boldsymbol{\delta}$ is a vector of unknown parameters conformable to \mathbf{x} .

To admit observed covariates, imagine that at the t^{th} auction the draw of bidder n is written as

$$\log V_{nt} = \mathbf{x}_t\boldsymbol{\delta} + U_{nt}$$

where $F_U(u)$ is the cdf of U_{nt} and $\mathbf{x}_t\boldsymbol{\delta}$ represents how the location is shifted as a result of the observed covariate vector \mathbf{x}_t at auction t and the conformable unknown vector $\boldsymbol{\delta}$. When U_{nt} s are independent of the \mathbf{x}_t , incorporating the covariate vector \mathbf{x} into this quasi-maximum likelihood framework simply involves optimizing with respect to the number of additional parameters in the vector $\boldsymbol{\delta}$. (By the way, the logarithmic transformation of V is used to guarantee that the marginal utility, the valuation, of the good is positive.) In this case, approximate $f_U(u)$ by an infinite-order polynomial of the form:

$$f_U(u) = \left[\sum_{i=0}^{\infty} \omega_i \mathbb{H}_i(u) \right]^2 \exp(-u^2/2) + \varepsilon \exp(-u^2/2)$$

because the support for the distribution of the U s is potentially the entire real line. Here, $\mathbb{H}_i(u)$ denotes an Hermite polynomial of order i .

6.2.3 eBay Auctions

As mentioned in section 2.4.5, on eBay as well as many other EA sites, the highest bidder is awarded the object, but the price the winner pays is determined by the smaller of two amounts: the price tendered by the highest bidder or the second-highest price submitted plus some fixed amount known as the bid increment. Bid increments may vary with the standing price or tendered amounts, but a schedule prescribing how the bid increment evolves is known by bidders before the auction begins. Because the final price is either the amount tendered by the winning bidder or an amount that is dictated by the highest-losing bid, these auctions are essentially a hybrid between pay-your-bid and second-price auctions. Hickman [2010] demonstrated that, in equilibrium, because there is a positive probability that the winner pays his own bid, participants will shave their bids in equilibrium. This challenged the common assumption in the empirical literature which maintained that these electronic auctions were second-price auctions at which each bidder would tender his valuation.

By not accounting for bid increments, but assuming a second-price rule instead, an econometrician mis-specifies the pricing rule, thus introducing a potential bias when estimating the model. Hickman, Hubbard, and Paarsch [2017] showed that even though bid increments are generally quite small, ranging from one to ten percent of the highest dollar amount relevant for a bid increment, this mis-specification has important statistical and economic consequences. The biased estimates of the latent valuation distribution compromises inference, which leads to biased estimates of the optimal reserve price and an underprediction of expected revenues under various counterfactual experiments—for example, the revenues that would be earned were another bidder to participate.

Beyond documenting the consequences of ignoring bid increments in the pricing rule, Hickman et al. proposed a way of accounting for bid increments as well as other real-world challenges when estimating structural models of EAs. One example of a real-world challenge is that neither the econometrician nor the bidders observes the total number of bidders, just activity from a subset of bidders who submit tenders—

a lower bound. The authors proposed a “filter process” representing intra-auction dynamics. Essentially, bidders receive valuations as well as index numbers indicating when their turn is to participate. When a bidder’s index is realized, the bidder is confronted with a standing price at the auction, and only participates if he is willing to tender an amount that exceeds this value. At the conclusion of the filter process, only a subset of the bidders will have elected to participate. This process can be simulated to compute the probability of observing a specific number of bidders, conditional on a specific total number of potential buyers.⁴ They demonstrated that this filter process admits nonparametric identification of the distribution of the number of potential buyers, provided the number of actual bidders is observed in the data.

Hickman et al. used data from eBay auctions of used laptop computers to estimate a structural model that accounted for bid increments and the filter process described to address selected participation. They employed a parametric assumption (generalized Poisson distribution) for the number of potential buyers and used B-splines to represent the latent valuation distribution.

B-splines have a number of attractive properties in any application: they are locally low dimensional and numerically stable; also, adjustments to improve the fit at one point have little or no effect elsewhere, and so forth; B-splines are particularly attractive in their application because they admit shape restrictions and can easily accommodate transition points at which the bid increment changes value.

Estimating both the valuation distribution and participation rates permitted model simulations. Hickman et al. used their model to understand why reserve prices are rarely used by sellers on eBay: revenues would increase by only a few cents in their application. In contrast, increasing the expected number of bidders who participate has a much larger effect on the sale price.

⁴Although Hickman et al. simulated these conditional probabilities, Brennan C. Platt [forthcoming] provided an analytic characterization of the expected number of bidders when the number of potential buyers is described by a Poisson distribution.

6.3 Parametric Methods

In higher dimensions (for example, say, five or more), nonparametric methods are infeasible: even in a world of Big Data there simply aren't enough observations. For although in the era of Big Data, the sample size T can be huge, so the sampling variability of nonparametric methods should be low, the presence of observed covariate heterogeneity (referred to as feature heterogeneity by others in data science), means that T must be very large, indeed. For example, consider the following: Assume two dozen different feature variables, which would be considered modest, at least in the Big Data era. Suppose each feature variable can take on just four discrete values. In this world, 4^{24} or 2^{48} different bins exist for all of the combinations of the covariates, so one needs about 2,814,749,767,106,560 observations, or more, just to populate an average bin with ten observations—that is, a sample size of 2.81×10^{15} . That's a lot of observations: a thousand is 10^3 ; a million is 10^6 ; a billion is 10^9 , a trillion is 10^{12} , so nearly three quadrillion observations.

When empirical work concerned with auctions was in its infancy, sample sizes were especially small, mostly because each datum was collected by hand. Thus, it was natural to impose some additional structure on the data generating process. Moreover, in some applications, the measurement equation is only identified under certain restrictions (for example, affiliation must hold), but standard applications of nonparametric methods (or SNP methods) do not impose those restrictions, and it is often difficult to impose the restrictions. Imposing such restrictions is often much easier within a parametric model, as we will see in section 8.

Thus, even though nonparametric methods are preferred, situations exist when parametric methods are the only real alternative. At least two other reasons also exist to pursue parametric methods: First, interpreting nonparametric estimates can be difficult to do, unlike in the case of parametric methods. Second, the sampling variability of nonparametric estimates is higher than parametric methods, sometimes much higher. At a very high level, in terms of the mean squared error of prediction, a trade off clearly exists between variance and bias. Ob-

viosuly, nonparametric methods have low bias, but they are remarkably variable as well. By placing additional structure on the data generating process, which obviously increases bias, one can reduce sampling variability. Whether the trade-off is worth it is an empirical matter, which depends on the loss function of the decision maker.

6.3.1 Maximum Likelihood Estimation

Initially, abstract from the presence of covariates and assume that $F_V^0(\cdot)$ belongs to a parametric family of distributions that does not depend on covariates, so it can be written as

$$F_V^0(v) = F_V(v|\boldsymbol{\theta}^0)$$

where $\boldsymbol{\theta}^0$ is a $(P \times 1)$ vector of unknown parameters.

Vickrey Auctions

Since the pdfs of B and V are one and the same, a parametric assumption concerning $F_V^0(\cdot)$ is an assumption concerning $F_B^0(\cdot)$, namely, $F_B^0(b) = F_V(b|\boldsymbol{\theta}^0)$. From a sample of T observations $\{\mathbf{b}_t\}_{t=1}^T$, one can construct the following likelihood function:

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_T) = \prod_{t=1}^T \prod_{n=1}^{N_t} f_V(\boldsymbol{\theta}|b_{nt}).$$

Estimation under the method of maximum likelihood (ML) proceeds in a straightforward manner; incorporating covariates (particularly as a single index) is equally straightforward. In the presence of a reserve price, either the truncated or the censored likelihood function must be considered, but economists are well acquainted with Tobit-type models.

English Auctions

In section 5, equation (5.3), we specified the joint density of bidding and nonparticipation at an English auction under the clock model when discussing the research of Paarsch [1997], who employed parametric methods in his investigation of timber sales in British Columbia, Canada, so we simply remark that estimation of this sort of model involves the straightforward application of numerical optimization.

Pay-Your-Bid Pricing Rule

As mentioned in section 3.1.2, the support of both the individual bid and the winning bid distributions depends on $F_V^0(\cdot)$ and N as well as \underline{v} and \bar{v} . Thus, when $F_V^0(v) = F_V(v|\boldsymbol{\theta}^0)$, because $F_V(\bar{v}|\boldsymbol{\theta}^0) = 1$,

$$\begin{aligned}\sigma(\bar{v}) &= \bar{v} - \frac{\int_{\underline{v}}^{\bar{v}} F_V(u|\boldsymbol{\theta}^0)^{N-1} du}{F_V(\bar{v}|\boldsymbol{\theta}^0)^{N-1}} \\ &= \bar{v} - \int_{\underline{v}}^{\bar{v}} F_V(u|\boldsymbol{\theta}^0)^{N-1} du \\ &= \bar{s}(\boldsymbol{\theta}^0, N).\end{aligned}$$

Thus, one of the regularity conditions typically assumed under the method of ML no longer holds. Moreover, the standard way to demonstrate parameter consistency in econometrics is unsatisfactory in this context.

Donald and Paarsch [1996] proposed an alternative strategy, focusing only on the winning bids, but the methods they proposed would apply were all bids used. A simple example will illustrate the nonstandard nature of the estimation problem encountered when attempting to estimate parametric structural models under the pay-your-bid rule within the IPV model.

Consider a random sample of size T for a random variable W that is distributed uniformly on the interval $[0, \theta^0]$, where θ^0 is an unknown parameter. The pdf of W is

$$f_W(w|\theta^0) = \frac{\mathbf{1}(0 \leq w \leq \theta^0)}{\theta^0} \quad \theta^0 > 0,$$

where $\mathbf{1}(\cdot)$ is the indicator function of the event argument. The standard approach to finding the ML estimator of θ^0 would involve maximizing the following logarithm of the likelihood function with respect to θ :

$$\frac{1}{T} \log[\mathcal{L}_T(\theta|W_1, W_2, \dots, W_T)] = -\log \theta + \frac{1}{T} \sum_{t=1}^T \log[\mathbf{1}(0 \leq W_t \leq \theta)].$$

Donald and Paarsch demonstrated parameter consistency by recasting the above optimization problem as the following constrained optimiza-

tion problem:

$$\max_{\theta} \frac{1}{\theta^T} \quad \text{subject to} \quad \begin{cases} w_1 \leq \theta \\ w_2 \leq \theta \\ \vdots \\ w_T \leq \theta, \end{cases}$$

which is more in line with the way the estimator is calculated in practice. The solution of the ML estimator in this example, which is depicted in Figure 6.1, involves one binding constraint—namely,

$$\hat{\theta} = \max(W_1, W_2, \dots, W_T) = W_{(1:T)}.$$

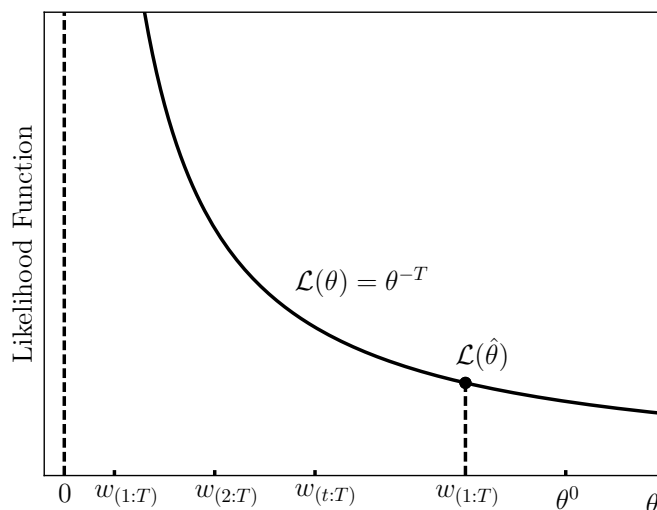


Figure 6.1: Likelihood Function in Uniform Example

Note, too, that the conventional methods used to determine the asymptotic distribution of $\hat{\theta}$ do not apply. Specifically, only the largest W_t is important in determining the distribution of $\hat{\theta}$. In short, an appropriately transformed $\hat{\theta}$ will follow the exponential distribution—one

of three different extreme-value distributions; see Appendix A.2 for additional discussion.

Having illustrated the main technical problem in the application of ML estimation using an example, now admit covariates—the principal motivation for going to a parametric approach in the first place. Assume that $F_V(v|\boldsymbol{\theta}^0)$ depends on a vector of covariates \mathbf{x} , denoted $F_V(v; \boldsymbol{\theta}^0, \mathbf{x})$, which then implies that the observed bids must lie in the interval $[\underline{v}, \bar{s}(\boldsymbol{\theta}^0, N, \mathbf{x})]$.

Let \mathbf{z}_t denote (N_t, \mathbf{x}_t) , the vector of all covariates for observation t , and let \mathbf{z}_t denote a particular realization of \mathbf{Z}_t . Donald and Paarsch assumed that the (W_t, \mathbf{Z}_t) s are independent as well as identically distributed, and focused on the case where the covariates are discrete.

The ML estimator $\hat{\boldsymbol{\theta}}$ is computed by solving the following constrained optimization problem:

$$\max_{\boldsymbol{\theta}} \sum_{t=1}^T \log[f_W(\boldsymbol{\theta}|w_t, \mathbf{z}_t)] \quad \text{subject to} \quad \begin{cases} w_1 \leq \bar{s}(\boldsymbol{\theta}, \mathbf{z}_1) \\ w_2 \leq \bar{s}(\boldsymbol{\theta}, \mathbf{z}_2) \\ \vdots \\ w_T \leq \bar{s}(\boldsymbol{\theta}, \mathbf{z}_T). \end{cases}$$

In practice, solving for the ML estimator will involve maximizing the following Lagrangean:

$$\Lambda_T(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \sum_{t=1}^T (\log[f_W(\boldsymbol{\theta}|w_t, \mathbf{z}_t)] + \lambda_t [\bar{s}(\boldsymbol{\theta}, \mathbf{z}_t) - W_t])$$

with respect to the vector $\boldsymbol{\theta}$, where $\boldsymbol{\lambda}$ collects the Lagrange multipliers in a vector $(\lambda_1, \dots, \lambda_T)^\top$. Under the assumptions stated by Donald and Paarsch, the Kuhn-Tucker conditions are both necessary and sufficient for an optimum. At most, P of the T constraints will ever bind at one time; that is, $(T - P)$ of the Lagrange multipliers will be zero at the optimum.

Because the solution to the nonlinear program typically obtains at the intersection of P constraints, the properties of the perturbed optimum are determined solely by the constraints. Under the assumptions made by Donald and Paarsch, a finite number of constraints exist asymptotically, which allowed them to demonstrate that the limiting

distribution of the largest winning bid for each possible covariate value will be exponential with intensity parameter equal to one. By an application of the delta method, so too is the distribution of the ML estimator; that is, the limiting distribution of the ML estimator, premultiplied by the inverse of a Jacobian matrix, is exponential

$$T\mathbf{A}_T^{-1}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0) \xrightarrow{d} \text{Exponential}(\boldsymbol{\iota}).$$

Two key points should be noted: (1) the standardization is proportional to T , unlike the usual rate \sqrt{T} encountered in most of econometrics, so the estimator converge at the rate T , which is sometimes referred to as superconsistency; (2) the limiting distribution is that of a vector of independent exponential random variables. Note that the result does not imply that the estimators themselves have one-sided distributions, just that there is a linear transformation of the estimators that has a one-sided distribution. The superconsistency of the ML estimator is a real windfall when samples sizes are small. Unfortunately, the asymptotic distribution characterized by Donald and Paarsch is difficult to use in practice, which is why they pursued other alternatives; see Donald and Paarsch [2002].

6.3.2 Generalized Method-of-Moments

Because the generalized method-of-moments (GMM) estimation strategy is very well known and understood by economists, this section is deliberately brief. In short, we simply outline two examples (one simple, one elaborate) of how the method has been applied.

The first application of GMM to a structural econometric model of an auction was by Paarsch [1992], who referred to it as nonlinear least squares (NLS). In his application, Paarsch examined procurement auctions of tree-planting services by the Ministry of Forests in the province of British Columbia, Canada—within the common-cost model, where the expected winning bid at auction t was a function of the number of potential suppliers at that auction N_t ,

$$W_t = \mu_1(N_t) + U_{1t}.$$

In order to reduce sampling variability, Paarsch also considered the

second raw moment of the winning bid,

$$W_t^2 = \mu_2(N_t) + U_{2t}.$$

He then stacked the moments, and solved for the NLS estimator. Given the objective function Paarsch chose, others have noted that the first-order condition that defined the estimator is essentially the GMM one, albeit without the optimal weighting matrix.

A more elaborate application of GMM was undertaken by Donald, Paarsch, and Robert [2006], who constructed a simple theoretical model of the multi-unit, sequential, Milgrom-Weber clock auction in which potential buyers could have multi-unit demand. Their model was developed to interpret data concerning the outcomes at 37 multi-lot, sequential, English auctions of timber export permits held between May 1993 and May 1994 in the Krasnoyarsk Region of Russia.

These auctions had two important empirical regularities: (1) bidders often won more than one lot; (2) different numbers of bidders attended different auctions. In addition, only a small amount of information was recorded—for example, the reserve price, the number of lots on sale, the number of participants at the auction, the winning bid for each lot sold, and the number of lots won by each participant at the auction.

Donald et al. assumed that the reserve price was known to everyone prior to auction. Although the participants at the auction observed the prices at which their opponents dropped out, as researchers, they assumed they did not. Thus, an asymmetry of information exists here between the decision makers (the participants at the auction) and the researchers, too. They incorporated this reality into their model by ruling out the econometrician's ability to observe dropout prices.

Donald et al. considered a seller seeking to dispose of K units of a homogeneous good through a sequence of K Milgrom-Weber clock auctions. They assumed that the seller is committed to selling all K units and that the auction process is repeated until all K units are sold. In order to avoid carrying around an unobserved state variable, unfulfilled demand, Donald et al. also assumed that the unfulfilled demands of participants at any auction disappeared (evaporated) completely after the K units were sold.

Milgrom and Weber analyzed the case where only one good is for

sale. In this case, at every price, a bidder must ask himself whether he should drop out of or remain in the auction. Because a bidder can always drop out at the next (infinitesimal) higher price, his calculation is myopic. Nevertheless, he must answer the following simple question: If all other bidders were to drop out simultaneously at the current price, would I prefer to win the auction or would I prefer to drop out now?

Within the IPV model, a bidder's best strategy is to drop out at his valuation. When two or more identical units of a good are sold sequentially using this auction format, where each participant may demand more than one unit, the strategic analysis becomes complex. Donald et al. computed a symmetric, perfect-Bayesian equilibrium of such a game. At any auction, they assumed a known number of risk-neutral potential buyers N who may bid for the K units for sale. They denoted by \mathbf{V}^n the vector of ordered valuations $(V_1^n, V_2^n, \dots, V_K^n)$ for potential buyer n where V_k^n represents bidder n 's valuation for the k^{th} unit of the good, so for all n , decreasing marginal utility obtains. Solving the sequential game under these assumptions is difficult. Most important, many equilibria can exist to this two-stage game, which is the kiss of death in the structural econometrics of auctions. Under the assumption of symmetry, the calculation is simplified: bidder n should calculate his dropout price p^n assuming active bidder m has the same one; that is, assuming V_1^m equals V_1^n .

The problem is maintaining symmetry in the auction as winners drop out. In order to solve the auction game, Donald et al. needed to specify the process generating the random vector of decreasing valuations for the n^{th} potential buyer. They assumed that the number of valuations drawn by each potential buyer is a Poisson random variable. When that random variable is positive, each of the positive valuations is drawn independently and identically from some twice-differentiable cumulative distribution F_V . The above stochastic process provided Donald et al. with a useful device to generate aggregate demand because it guaranteed certain properties sufficient to solve the auction game.

A bidding strategy in this model specifies a stopping rule that indicates at what price a participant should withdraw from the current sale, but that policy function depends on the number of units won by

others and left to be sold; information obtained from previous sales is superfluous because of the Poisson assumption.

Because the equilibrium of their model is difficult—some might say impossible—to calculate, Donald et al. proposed to estimate its average behavior by simulating another auction format, the generalized Vickrey auction, which is also efficient, so the expected price will be the same, but whose equilibrium one can easily calculate. In order to simulate on a computer, pseudo-random numbers from the distribution F_V are required. Without an explicit assumption concerning F_V , generating pseudo-random numbers on a computer is impossible. Thus, in addition to assuming that demand follows the Poisson law Donald et al. also assumed that F_V followed the Weibull law.

The theory of Donald et al. delivers the first moment of both the participation equation for the auction (the mean of a binomial random variable) and the winning price for the sale of unit k at the auction (the average winning price for unit k at a generalized Vickrey auction). Thus, a natural strategy for estimating the unknown parameters would be to choose some estimate that minimizes the distance between the observed data and the mean of the theoretical processes evaluated at a candidate estimate. But how should the distance be chosen? Donald et al. chose GMM.

Specifically, the mean of participation equation is

$$\mathbb{E}(M|r, N, \lambda, \alpha_1, \alpha_2) = \mu_0(r, N, \lambda) = N(1 - \exp[-\lambda \exp(-\alpha_1 r^{\alpha_2})]),$$

where λ denotes the Poisson mean and (α_1, α_2) denote the parameters of the Weibull. For a sample of T observations, in which there are up to $\hat{K} = \max(K_1, K_2, \dots, K_T)$ units, the population moment conditions are as follows:

$$\begin{aligned} \mathbb{E}(U_{0t}|r_t) &= \mathbb{E}[M_t - \mu_0(r_t, \theta^0)|r_t] = 0 \\ \mathbb{E}(U_{kt}|\Omega_{kt}) &= \mathbb{E}[P_k^t - \mu_k(z_t, p_{k-1}^t, p_{k-2}^t, \dots, \theta^0)|\Omega_{kt}] = 0 \\ &\text{for } k = 1, \dots, K_t \end{aligned}$$

where Ω_{kt} is the information set at sale k in the auction and includes z_t , which equals $(r_t, K_t, m_t)^\top$ plus, at the very least, all winning prices for prior lots. The U_{kt} s have the obvious interpretation of residuals.

Here, $\boldsymbol{\theta}^0$ denotes the true, but unknown, value of the parameter vector $(\lambda, \alpha_1, \alpha_2)^\top$ that Donald et al. sought to estimate. Donald et al. constructed the following instrument sets for each residual:

$$\begin{aligned} \mathbf{x}_{0t} &= r_t \\ \mathbf{x}_{1t} &= (r_t, K_t, m_t)^\top = \mathbf{z}_t \\ \mathbf{x}_{2t} &= (\mathbf{z}_t, p_1^t)^\top \\ &\vdots \\ \mathbf{x}_{K_t t} &= (\mathbf{z}_t, p_1^t, \dots, p_{K_t-1}^t)^\top, \end{aligned}$$

so that

$$\mathbb{E}(U_{kt} | \mathbf{x}_{kt}) = 0$$

because $\mathbf{x}_{kt} \subset \Omega_{kt}$. (The number of potential buyers N is not included as it did not vary in the data that Donald et al. collected.) Residuals can be constructed as a function of data and parameters as:

$$\begin{aligned} U_{0t}(\boldsymbol{\theta}) &= M_t - \mu_0(r_t, \boldsymbol{\theta}) \\ U_{kt}(\boldsymbol{\theta}) &= P_k^t - \mu_k(\mathbf{z}_t, p_{k-1}^t, p_{k-2}^t, \dots, \boldsymbol{\theta}). \end{aligned}$$

From these, Donald et al. then formed the following moment functions:

$$\begin{aligned} \mathbf{g}_{0t}(\boldsymbol{\theta}) &= \mathbf{x}_{0t} U_{0t}(\boldsymbol{\theta}) \\ \mathbf{g}_{kt}(\boldsymbol{\theta}) &= \mathbf{x}_{kt} U_{kt}(\boldsymbol{\theta}) \text{ if } k \leq \hat{K}, \text{ otherwise } 0; \end{aligned}$$

whence they defined the average sample moment functions as

$$\begin{aligned} \bar{\mathbf{g}}_0(\boldsymbol{\theta}) &= \frac{1}{T} \sum_{t=1}^J \mathbf{g}_{0t}(\boldsymbol{\theta}) \\ \bar{\mathbf{g}}_k(\boldsymbol{\theta}) &= \frac{1}{T} \sum_{t=1}^J \mathbf{g}_{kt}(\boldsymbol{\theta}) \quad \text{for } k = 1, \dots, \hat{K}. \end{aligned}$$

(Note that some of the terms in the latter are zero if, for any auction t , there are no units for sale.)

Given this structure, the standard way to define $\hat{\boldsymbol{\theta}}$, the GMM estimator of $\boldsymbol{\theta}^0$, is the following:

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \bar{\mathbf{g}}(\boldsymbol{\theta})^\top \mathbf{W} \bar{\mathbf{g}}(\boldsymbol{\theta}),$$

where \mathbf{W} is the appropriately-chosen weighting matrix.

7

Asymmetries

Initially, when the structural econometrics of auctions began, the data sets available were quite small—especially relative to those data sets used in such fields as finance or labor, where thousands of observations are often available. For instance, Jean-Jacques Laffont, Hervé Ossard and Vuong [1995] collected by hand 81 observations concerning winning bids at Dutch auctions of eggplant in Marmande, France. In cases like this, a structural model is extremely helpful in both interpreting the data and reducing sampling variance, especially in the presence of feature variables (covariates).

The twenty-first century ushered in new ways of measuring and capturing what were in the past invisible aspects of people’s lives. Cheap disk storage paved the way for Big Data. Auction data improved as well, especially because some auctions became electronic, reducing the cost of gathering information concerning each sale and every bid. In short, sample sizes have increased, and data concerning specific bidders can now be tracked cheaply—across auctions and over time.

The models used to interpret such data have been enriched, too. Admitting bidders who are asymmetric—drawing signals or valuations from different urns—is a natural way to extend the baseline models

described above. If bidders can be classified according to some observed characteristic, then added flexibility can be introduced into econometric models.

Researchers have introduced asymmetries in all sorts of applications: When allocating the rights to pursue oil and gas reserves on the Outer Continental Shelf, the United States Department of the Interior permitted firms to partner with each other when bidding at wildcat sales. Thus, Campo et al. [2003] distinguished between bids tendered by an individual firm versus joint bids submitted by a collection of firms.

In other settings, experience can be an important factor: At procurement auctions conducted by the Oklahoma Department of Transportation, Dakshina G. De Silva, Timothy Dunne, and Georgia Kosmopoulou [2003] noted important differences in the behavior of new entrant firms relative to that of established incumbent firms. They found the observed differences in behavior were consistent with a model in which the realized costs of entrants and incumbents were drawn from different distributions.

When investigating snow removal contracts in Montréal, Québec, Canada, Véronique Flambar and Perrigne [2006] noted that some firms had locational advantages over others. As in the doctoral dissertation of Bajari [1997], which was submitted to the Department of Economics at the University of Minnesota, a firm's distance from a given job is an important source of asymmetry.

At timber auctions conducted by the United States Forest Service Athey, Jonathan Levin, and Enrique Seira [2011] modeled loggers and sawmills as having latent valuations that followed different distributions.

In Texas, the Department of Transportation offers disadvantaged businesses the opportunity to complete a bidder training program: De Silva, Hubbard, and Kosmopoulou [2017] found that such differences generated differences in bidding behavior on future contracts.

In some instances, even if bidders can be thought of as *a priori* symmetric, either the rules that determine the winner or the behavior of the bidders requires an asymmetric auction model. As an example

of the former, consider a bidder preference program, which is one way an auctioneer might use a scoring rule to determine the winner at an auction. In general, bid preference policies inflate (discount in the procurement case) the bids from certain groups for the purposes of evaluation only. For example, if two bids $s_1 < s_2$ are tendered, then, at a standard pay-your-bid auction, bidder 2 would win the object at price s_2 . If bidder 1 is treated special under a preference policy, then s_1 is evaluated as $(1 + \rho)s_1$, where $\rho > 0$ is the preference rate. Such policies induce asymmetries: Bidders will respond in different ways to $\rho > 0$, depending on whether they are eligible for the preference.

In the United States, bid preference policies have been employed at all levels of government in procurement auctions—by municipalities, by states, and even at the federal level under the Buy American Act; see, for example, Justin Marion [2007], Hubbard and Paarsch [2009], as well as Krasnokutskaya and Katja Seim [2011] among others.

In other settings, even when the assumption of symmetric bidders appears reasonable, collusion among a subset of bidders can again imply that an asymmetric model is appropriate. In some cases, colluding bidders often meet before an auction to determine which member will win the object for sale, and how all colluding bidders should behave. If the cartel is efficient, then the bidder with the highest valuation (lowest cost in a procurement setting) will be selected and will compensate conspirators for not competing aggressively. In short, efficient cartels will select an extreme order statistic from the realized valuations (costs) of the colluding members, meaning the cartel's latent type comes from a distribution different from that of noncartel members.

In a similar vein, at multi-unit auctions, if bidders are *ex ante* symmetric, then as the auction progresses an asymmetry can arise endogenously in later rounds when a participant demands more than one unit of the good for sale: Those who win early, fulfill their high valuations, so the distributions of remaining valuations are different from those who have won nothing.

In general, adopting an asymmetric model makes sense when the data permit such distinctions. As the field has evolved from one in which nonparametric approaches have replaced parametric assump-

tions, whenever possible, admitting asymmetries reduces the number of assumptions an econometrician needs to make. Bjarne O. Brendstrup and Paarsch [2006] suggested evaluating the frequency of wins over time as a simple way to determine whether the symmetric IPV assumption made sense: Within a symmetric IPV model, each bidder should win with the same frequency, allowing for a testable null hypothesis.

Researchers have long understood that asymmetries are an empirical reality, but that they often complicate bidder behavior and can make calculating equilibria extremely arduous computationally. Laffont et al. [1995] noted that their symmetric model did a poor job of explaining behavior of a large eggplant trader in the market. Indeed, they also recognized that admitting asymmetries at these Dutch auctions is “not trivial” because it precludes analytic solution of the equilibrium bid functions.

We would be remiss were we not to recognize that as data sets continue to become increasingly detailed, in parallel, auction theorists have proven the existence and uniqueness of equilibrium as well as other important properties of asymmetric auction models; see, for example, Bernard Lebrun [1996, 1999, 2006], Maskin and Riley [2000a,b, 2003] as well as Athey [2001].

In what follows, we discuss how the estimation strategies described in section 6 can be employed in models of asymmetric pay-your-bid and second-price auctions.

7.1 Asymmetric IPV Model

Critical to considering asymmetric auction models are data. More data are needed, especially when considering nonparametric approaches, not just because of the slow rates of convergence of those methods, but also because admitting heterogeneity across bidders often requires information concerning bidders’ identities. We consider the effects of introducing asymmetries under pay-your-bid and second-price rules in the next two sections, respectively. Even though observable differences may exist that permit empirical researchers to partition the data along sensible lines, in the absence of such observables an asymmetric model could

be justified if it is reasonable that bidders are inherently different—for example, their budget constraints, attitudes towards risk, or other fundamental features of their preferences could be different. Therefore, although we begin by assuming bidder identities are observed, we then proceed to consider situations in which the econometrician believes asymmetries exist, but does not observe bidder identities. In our characterization, we assume there exists two groups, or classes, of bidders: group 1 and group 2, though what we present extends naturally to environments in which the number of groups exceeds two.

7.1.1 Pay-Your-Bid Rule

Fortunately, the standard identification and estimation results of Guerre et al. [2000] extend naturally to asymmetric environments, provided bidders can be classified into different groups according to the source of the asymmetry. Ignoring issues of endogenous entry, a common practice when estimating a symmetric IPV model is to assume the number of potential buyers N equals the number of actual bidders M and then to restrict attention to all m -bidder auctions. Note that this assumption is not even valid for a fixed N in an asymmetric model as the composition of m -bidder auctions is likely changing based on how many bidders are from each class.

Extending the two-bidder case presented in section 2.4.6 to an environment in which there exist m_1 and m_2 bidders from each group, yields two differential equations of the following form:

$$[\varphi_1(s_1) - s_1] \left[\frac{(m_1 - 1)f_1[\varphi_1(s_1)]\varphi_1'(s_1)}{F_1[\varphi_1(s_1)]} + \frac{m_2 f_2[\varphi_2(s_1)]\varphi_2'(s_1)}{F_2[\varphi_2(s_1)]} \right] = 1$$

characterizing behavior for bidders of group 1 and

$$[\varphi_2(s_2) - s_2] \left[\frac{m_1 f_1[\varphi_1(s_2)]\varphi_1'(s_2)}{F_1[\varphi_1(s_2)]} + \frac{(m_2 - 1)f_2[\varphi_2(s_2)]\varphi_2'(s_2)}{F_2[\varphi_2(s_2)]} \right] = 1$$

which characterizes the behavior of group 2 bidders. Importantly, note that the strategy of a bidder belonging to a particular group depends not just on the total number of bidders $m_1 + m_2 = m$ at auction, but also on the composition of the m bidders at auction. A group 1 bidder at an auction with two other group 1 bidders and two group 2 bidders

will behave differently from a bidder of group 1 at an auction with four group 2 bidders, although the total number of bidders is the same at these auctions ($m = 5$). Thus, a “binning” approach, as noted by Athey and Haile [2007] requires estimation be done separately for each set $\mathcal{A} = \{(m_1, m_2) | m_1 \in \mathcal{N}, m_2 \in \mathcal{N}\}$; that is, \mathcal{A} is the set of tuples where each element of the tuple belongs to \mathcal{N} . This is the downside of allowing flexibility and the cause for the complexity issues discussed earlier.

Technically, estimation is no more difficult than under the pay-your-bid rule within the symmetric IPV model. Following Flambard and Perrigne [2006], the two-stage nonparametric estimation procedure suggested by GPV can be extended to an asymmetric setting. In particular, in the first stage, estimation of the bid cdfs and pdfs allows the econometrician to recover pseudo types corresponding to each observed bid; in the second stage the latent type distributions and densities can then be estimated. Specifically, first-stage estimates $\{\hat{G}_1(s|\mathcal{A}), \hat{g}_1(s|\mathcal{A}), \hat{G}_2(s|\mathcal{A}), \hat{g}_2(s|\mathcal{A})\}$ must be constructed independently using only the bids from a given class over the set of \mathcal{A} auctions. Thus, the bid distributions can be estimated using the smoothed edfs

$$\hat{G}_i(s|\mathcal{A}) = \frac{1}{T_{\mathcal{A}}m_i} \sum_{t=1}^{T_{\mathcal{A}}} \sum_{\ell=1}^{m_i} \Phi\left(\frac{s - s_{i\ell t}}{h_i^{\mathcal{A}}}\right).$$

The number of auctions $T_{\mathcal{A}}$ represents the number of auctions in the sample for which N is fixed and involves m_i bids from group i . Thus, the observed bid $s_{i\ell t}$ represents the ℓ^{th} bid from a class i player at auction t which comes from a sample of $T_{\mathcal{A}}$ auctions in which (m, m_1, m_2) is the same for all auctions in this subsample. Our notation nests the symmetric model for a given m in which case $\mathcal{A} = \{(m, 0), (0, m)\}$. This is of practical importance as often symmetric auctions involving only bidders from a certain group are observed in real-world data. Likewise, the pdfs of bids can be estimated via

$$\hat{g}_i(s|\mathcal{A}) = \frac{1}{h_i^{\mathcal{A}}T_{\mathcal{A}}m_i} \sum_{t=1}^{T_{\mathcal{A}}} \sum_{\ell=1}^{m_i} \phi\left(\frac{s - s_{i\ell t}}{h_i^{\mathcal{A}}}\right)$$

where, again, ϕ is the Gaussian kernel function and $h_i^{\mathcal{A}}$ is a group-

specific bandwidth that will be different for different members of the set \mathcal{A} .

The estimates $\{\hat{G}_1(s|\mathcal{A}), \hat{G}_2(s|\mathcal{A}), \hat{g}_1(s|\mathcal{A}), \hat{g}_2(s|\mathcal{A})\}$ can be used to recover the pseudo-values using

$$\hat{v}_{ilt} = s_{ilt} + \left[\frac{1}{\frac{(m_{it}-1)\hat{g}_i(s_{ilt}|\mathcal{A})}{\hat{G}_i(s_{ilt}|\mathcal{A})} + \frac{m_{jt}\hat{g}_j(s_{ilt}|\mathcal{A})}{\hat{G}_j(s_{ilt}|\mathcal{A})}} \right]$$

for group i where m_{jt} denotes the number of group j bidders at auction t such that $(m_{1t}, m_{2t}) \in \mathcal{A}$. The term in brackets corresponds to the amount bidders of group i shave (markdown) their valuations.

The second stage is more direct: Recovery of pseudo-values strips the composition effects from the model, providing independent samples of valuations each from a respective pdf. As such, second-stage estimation of the latent distributions is much easier because, for a given group, data can be pooled from auctions not only across \mathcal{A} for a given number of bidders m , but also across all the realized values of m in the sample. Specifically, take the sample of group- i pseudo-values $\{\hat{v}_i\}$ and estimate the pdf of values using

$$\hat{f}_i(v) = \frac{1}{T_i} \sum_{t=1}^{T_i} \frac{1}{m_{it}} \sum_{\ell=1}^{m_{it}} \frac{1}{h_i^v} \phi\left(\frac{v - v_{ilt}}{h_i^v}\right)$$

where h_i^v is a second-stage group- i bandwidth, $\phi(\cdot)$ is, again, a kernel function, but now T_i represents the total number of auctions observed in the sample (across all \mathcal{A}) involving at least one group i bidder and m_{it} represents the number of group i bidders at auction $t \in \{1, 2, \dots, T_i\}$.

Because the econometric techniques developed in the symmetric setting extend naturally to an asymmetric one, we use this setting as an opportunity to discuss other challenges that may arise in asymmetric settings. In particular, we provide three examples: (1) endogenous entry; (2) differences in risk attitudes; (3) anonymity of the observed bids.

Among other things, when comparing sealed, pay-your-bid with English timber auctions, Athey et al. [2011] considered an asymmetric model. They modeled entry by assuming that bidders only know the distribution of valuations once they have decided to enter. They noted

that English auctions discourage participation by weak firms because the auctions are efficient in awarding the contract to the bidder with the highest value, but pay-your-bid auctions involve bid shaving, which gives weaker bidders an improved prospect of winning the auction. In estimation, the authors abandoned the nonparametric approach of the first stage in favor of a Gamma-Weibull specification of bids and unobserved heterogeneity. In the second stage, the distributions of valuations depend on unobserved heterogeneity so the authors average over realizations of unobserved characteristics. Athey et al. found that even though revenue equivalence breaks down in the presence of bidder asymmetries, the asymmetries alone led to only a minor advantage in expected revenues favoring pay-your-bid auctions. That said, admitting endogenous entry increased expected revenues relative to the pay-your-bid rule by nearly ten times when compared to the case in which bidder participation did not vary systematically with auction format, suggesting that entry is an important element in modeling auctions.

Some may find this result unsurprising: Bulow and Klemperer [1996] demonstrated in a symmetric IPV model that attracting another bidder to the auction does more for revenues than an optimally-set reserve price with one fewer bidder. Roberts and Sweeting [2016] discovered this result in a more general model that admits both bidder asymmetries and selective entry—a model in which bidders with higher values are more likely to participate.

Like Flambard and Perrigne [2006], Athey et al. [2011] assumed risk-neutral potential buyers. In contrast, Campo [2012] ignored endogenous entry and pursued instead a model with asymmetric bidders who might also differ in their attitudes towards risk. For example, modifying our two-group set of first-order conditions to admit risk-averse potential buyers, yields

$$\frac{u_1 [\varphi_1(s_1) - s_1]}{u'_1 [\varphi_1(s_1) - s_1]} \left[\frac{(m_1 - 1)f_1 [\varphi_1(s_1)] \varphi'_1(s_1)}{F_1 [\varphi_1(s_1)]} + \frac{m_2 f_2 [\varphi_2(s_1)] \varphi'_2(s_1)}{F_2 [\varphi_2(s_1)]} \right] = 1$$

and

$$\frac{u_2 [\varphi_2(s_2) - s_2]}{u'_2 [\varphi_1(s_2) - s_2]} \left[\frac{m_1 f_1 [\varphi_1(s_2)] \varphi'_1(s_2)}{F_1 [\varphi_1(s_2)]} + \frac{(m_2 - 1)f_2 [\varphi_2(s_2)] \varphi'_2(s_2)}{F_2 [\varphi_2(s_2)]} \right] = 1.$$

In such a setting, the econometrician seeks to estimate the unobserved model primitives $\{F_1(v), u_1(\cdot), F_2(v), u_2(\cdot)\}$. Campo noted that if two bidders share the same valuation characteristics, but exhibit different attitudes towards risk, then they may draw the same type, but submit different bids. This generates a “quantile equality” that allows the econometrician to distinguish the asymmetric risk-averse model from competing models. Thus, identification of bidders’ preferences is possible when these preferences are constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA), provided bidders’ degrees of risk aversion depend on some element which does not affect their valuation distribution.

Estimation follows the multi-stage approach of GPV whereby bid distributions, conditional on bidder- and auction-specific characteristics, are estimated first. Bids which then lead to the same quantile of the bid distribution for a given auction characteristic are next recovered and these estimates used in the quantile equality condition which can be used to estimate risk parameters. Lastly, bidders’ pseudo values and type distributions can be estimated using standard techniques.

Although the extension of GPV to an asymmetric model is reasonably straightforward, the econometrician must be able to identify bidders in the data and to classify them into respective groups. Laurent Lamy [2012] considered a setting in which bidders’ identities are unobserved or are only partially observed, but the econometrician knows the set of participants (or it is fixed across auctions). Even when the data do not admit grouping bidders, a researcher may still feel that an asymmetric model is most appropriate if he knows that a subset of bidders behave differently than other bidders. Lamy demonstrated that an asymmetric, affiliated model is unidentified, but that even with anonymous data, the asymmetric IPV model is identified up to a permutation of identities.

Critical to Lamy’s approach is employing a mapping from the order statistics of bids to the bidders’ bid distributions. In an IPV model, this inverse problem corresponds to the roots of a polynomial of degree N , whose coefficients are a linear combination of these estimated dis-

tributions of order statistics. Essentially, an N -equation system must be solved for N unknowns. In estimation, the complete vector of bids are used along with any partial information about bidders' identities. Specifically, for each possible bidder identity, a pseudo value is estimated for each observed bid. Estimation of private values then requires these pseudo values be merged with an estimate of the probability that each bid belongs to a particular bidder. Thus, a weighted vector of the N pseudo values from an auction is used to estimate the density of bidders' private values.

7.1.2 Procurement Auctions with Bid Preferences

In applied work, the policy being modeled often guides the structure that is imposed on a bidding model. Bid preferences are a good example of a policy that induces changes in equilibrium behavior because the allocation rule considers factors other than the dollar amounts of bids tendered. When the awarding rules specify that a set of factors will be aggregated in an explicit way to determine a ranking, and the bidder with the highest ranking is awarded the contract, auction researchers refer to the auction as a scoring auction. For example, a transportation department agency in a state might be interested in not just the dollar amount required to complete some task, but other factors as well, such as how long it will take a firm to complete the project, the reputation of the bidding firm, the quality of the materials, and so forth.

When the procuring agency states explicitly how each element contributes to the rule used to determine the winner, then this is an example of a scoring auction. By contrast, if an individual collects offers to complete some task and has available other information (such as a reputation rating or feedback score, reviews from past clients, and so forth), but the individual does not state explicitly how the winner will be determined—just that these factors are important—then this is like a beauty contest.

Bid preferences are an example of a scoring rule in which different bidders may be eligible to have their bids increased (decreased in a procurement setting) for the purposes of evaluation relative to the bids of other firms that are not qualified for special treatment. For example,

in a procurement setting, the bids of small firms; firms owned by a minority, woman, veteran; or if the firm is located domestically, have in some settings allowed the eligible firms' bids to be discounted for comparison to the bids of other firms.

When bidders know that a subset of firms qualify for special treatment, it induces changes in their equilibrium behavior, which must be accounted for in the structural approach. Hubbard and Paarsch [2009] modeled this environment, allowing for the entry behavior to vary across the different groups of bidders to show that such policies lead to differences in the behavior of bidders based on whether they are eligible for the preference policy (in which case they behave less aggressively and are more likely to enter) or not (in which case they behave more aggressively and are less likely to enter).

Justin Marion [2007] considered a reduced-form as well as a structural approach in considering the California Department of Transportation's (Caltrans) policy that awarded small firms a five percent bid preference on contracts using only state funds, but no preference on projects receiving federal aid. In his structural approach, Marion adopted the two-stage nonparametric approach of GPV, which was extended to an asymmetric environment by Flambard and Perrigne [2006]. Even if bidders are *ex ante* symmetric, the asymmetric model is essential because, by favoring a subset of firms, the policy leads to a divergence in equilibrium behavior among all of the firms. Marion found that the policy reduces firm profits, on average. Perhaps not surprisingly, Marion also found that the policy increases the number of inefficient allocations, which leads to an overall increase in the procurement costs of Caltrans.

After recovering estimates of the latent cost distributions, Marion also solved numerically for the equilibrium bid functions under alternative, hypothetical situations by representing the (inverse-) bid functions as polynomials—the coefficients of which are chosen to make the set of differential equations hold approximately on a grid of points. His counterfactual experiments investigated the effects of entry and alternative bid preference rates, in which he found that if participation were unaffected by bid preferences, a five percent rate would be close to optimal.

Because entry is such an important factor in determining revenues

at auction (costs in procurement settings), Krasnokutskaya and Seim [2011] revisited the application considered by Marion, taking his model to a different time period, but still using Caltrans data, and enhancing the model to include endogenous entry as well as accounting for unobserved project heterogeneity following Krasnokutskaya [2011]. They showed that accounting for participation decisions is crucial because these responses alter the assessment of bid-preference policies. Specifically, after recovering firms' project and entry costs, the estimates were used to consider the effect of the program on participation, cost to the government, and probability of the favored small firms winning a contract. Krasnokutskaya and Seim found that bidding responses across preference levels pale in comparison to the changes resulting from participation responses. Accounting for how firms will adjust their participation decisions is a crucial factor to account for in designing such programs. Moreover, they found that, if the goal of the procuring agency is to reduce its costs, then a lump-sum entry fee is more effective than a bid-preference policy.

7.1.3 Second-Price Rule

Under the second-price rule, when asymmetries exist, bidders have a weakly dominant strategy to bid their valuations. If the number of potential buyers is observed and bidders can be identified to belong to certain groups, then the econometrician can observe private values directly. To be clear, the bid data may include simply the amount paid by the winning bidder—the final price, but rarely contains information about the highest bid. Still, structural work just needs to reconcile the mapping of order statistics to their parent distribution. The asymmetric IPV model is identified and can be estimated using data from ascending auctions; see Theorem 2 of Athey and Haile [2002].

Tatiana Komarova [2013b] built on this work—establishing identification of the marginal distributions of bidders' valuations based on the existence and uniqueness of a system of nonlinear differential equations relating latent valuation distributions to observable data. She carefully characterized minimal requirements on observables that ensure point identification of the distribution functions. Importantly, independence

of bidder valuations is a sufficient condition in the asymmetric IPV environment, provided the winner's identity and the final price are observed auction outcomes.

To understand how equilibrium behavior admits identification and estimation in the asymmetric IPV model, consider an environment in which the econometrician observes the winning bid, the identity of the winner, and the number of potential buyers. Brendstrup and Paarsch [2006] demonstrated clearly how the pdf of the winning bid can be expressed as a mixture of the individual group's pdfs describing valuations. For example, denoting the permanent operator by Perm (see Appendix A.1 for a discussion of the permanent and its use finding the distributions of order statistics), in the two-group setting maintained in the previous section, with $n_1 = 2$ and $n_2 = 1$,

$$\begin{aligned} f_Y(y|F_1, F_2) &= \text{Perm} \begin{bmatrix} F_1(y) & F_1(y) & F_2(y) \\ f_1(y) & f_1(y) & f_2(y) \\ [1 - F_1(y)] & [1 - F_1(y)] & [1 - F_2(y)] \end{bmatrix} \\ &= 2 \left(F_1(y)[1 - F_1(y)]f_2(y) + F_1(y)[1 - F_2(y)]f_1(y) + \right. \\ &\quad \left. F_2(y)[1 - F_1(y)]f_1(y) \right). \end{aligned}$$

Nonparametric identification of this model can be established using an application of results from Meilijson [1981].

In the spirit of Gallant and Nychka [1987], Brendstrup and Paarsch [2006] proposed a SNP estimation strategy that involved approximating the unknown pdfs by unknown polynomials, the coefficients of which can be estimated using the method of quasi-ML. Representing the pdf as an infinite-order polynomial and following Gallant and Nychka, Brendstrup and Paarsch approximated the infinite-order polynomial by a finite-order polynomial, the degree of which increases at a rate that is slower than the rate at which the sample size increases.

Brendstrup and Paarsch proposed using Laguerre or Hermite (when admitting covariates) polynomials, which can be constrained to be non-negative over the support—in fact, the researchers employed other restrictions that ensured the pdfs live in a compact space. They applied their estimator to auctions of fish (plaice) held in north Jutland, Denmark, for which they distinguished between major and minor bidders

to compare revenues from the English auctions used each day at the *Grenå Fiskeauktion* to what might have been earned were the auctions conducted under a Dutch auction, which is the typical format employed at fish auctions in Denmark. What aided Brendstrup and Paarsch is that all seven (two major and five minor) potential buyers participated at virtually every auction, so N was known with relative certainty.

Rather than employ structural models to estimate the underlying distributions of valuations from bid data, Dominic Coey, Larsen, and Kane Sweeney [2014] demonstrated that a number of empirical questions can be evaluated just by considering what they called the “bidder exclusion effect.” The bidder exclusion effect corresponds to the expected change in auction revenue when a random bidder is excluded from the auction. With $m > 2$ bidders at an ascending-price auction, with probability $\frac{m-2}{m}$ the excluded bidder is one of the $(m-2)$ lowest valuation bidders, meaning revenue is unaffected. With probability $(2/m)$ one of the two highest bidders is excluded. At an ascending-price auction, the bidder exclusion effect is

$$\delta(m) \equiv \mathbb{E}[B_{(2:m)}] - \mathbb{E}[B_{(3:m-1|m)}].$$

Thus, the expected fall in revenue, assuming all bids remain unchanged is

$$\gamma(m) \equiv \frac{2}{m} \mathbb{E}[B_{(2:m)} - B_{(3:m)}].$$

Given private values, even with asymmetric bidders, $\delta(m) = \gamma(m)$, and this effect can be estimated using conditional means, where a vector of auction-level covariates \mathbf{x} is controlled for, as

$$\gamma(m|\mathbf{x}) \equiv \frac{2}{m} \mathbb{E}[B_{(2:m)} - B_{(3:m)}|\mathbf{x}].$$

The insight of Coey et al. is powerful in that it helps quantify in an empirical setting the theoretical results of Bulow and Klemperer. Moreover, a researcher can invert the distribution of the second-order statistic to obtain estimates of buyer valuations. The valuation distributions can then be used to simulate revenue that can be compared to the estimated bidder exclusion effect to test whether bidders have independent private values.

As in models of pay-your-bid auctions, when bidder identities are unobserved, the approach suggested by Lamy [2012] can again be used under the second-price rule. A challenge, however, to adopting Lamy’s approach in second-price environments is that the data often come from English auctions, for which only winning bids are observed: his anonymous bidding approach requires a complete set of bids.

Instead, Coey, Larsen, Sweeney, and Caio Waisman [2017] required only the transaction price and the number of potential buyers to be observed. Like Lamy, their model allows for bidder asymmetries and unobserved bidder identities, but they demonstrated that admitting asymmetries can actually help with identification and estimation. If bidder identities can be observed in the data, then asymmetries can provide valuable information that improves estimated bounds on valuation distributions. Order statistics are preserved under random permutations of bidders’ values which provides an exchangeable random vector.

Coey et al. derived a novel exogeneity condition: The composition of bidder types participating in the auction does not change as the number of bidders changes. An implication of this assumption is that the expected share of bidders who belong to a certain group should be constant across auctions with varying numbers of potential buyers. If it is impossible to assign bidders to groups from the data, but there is still reason to believe that asymmetries exist, then the authors provided sufficient conditions for which estimated bounds are valid. The bounds tighten if asymmetries are accounted for by averaging over different sets of participating bidders. They found that ignoring bidder asymmetries led to inaccurate conclusions.

7.1.4 Computational Methods

The nonparametric approach of Guerre et al. as well as Flambard and Perrigne is particularly useful in the presence of asymmetries because equilibrium behavior is characterized by a system of differential equations that does not satisfy a Lipschitz condition needed for integration. The GPV transformation avoids the need to solve for private values in terms of bids and the bid distributions. In counterfactual experiments,

however, researchers often want to consider alternative environments, which require the solution to the system of differential equations. Solving this system is required no matter how the latent distributions of valuations or preference structures were estimated. The general system not only has no closed-form solution, but the Lipschitz condition fails in the neighborhood of \underline{v} , so numerical methods are required.

Hubbard and Paarsch [2014] described three common approaches to solving this system of differential equations, which can be characterized as a boundary-value problem: In a variety of settings, economic theory prescribes conditions that must be satisfied at the bounds of support of the type distributions. Initially, researchers considered shooting algorithms; that is, take a guess for the unknown highest bid ($\sigma_i(\bar{v}) = \bar{s}$), then integrate the system backwards to determine whether the guess should be increased or decreased, depending on how close the other boundary condition ($\sigma_i(\underline{v}) = \underline{v}$) is satisfied, or whether a bifurcation occurs.

Gadi Fibich and Nir Gavish [2011], however, proved that this approach is unstable—proposing an alternative that involved fixed-point iteration. The method of Fibich and Gavish requires transforming the system of equations and relevant boundary conditions, essentially using a tying function that relates players' valuations to each other. This avoids the unknown and endogenous free boundary value \bar{s} , but requires that one be able to transform the system—something that is possible in standard asymmetric IPV models. In richer settings, which might involve endogenous entry, it is unclear whether such a transformation exists.

Instead, the bid functions can be modeled as a finite combination of approximating functions, the coefficients of which are chosen to minimize some objective function, subject to constraints imposed by auction theory. For example, one could evaluate the first-order conditions comprising the system of differential equations on a grid of points and then seek to make these as close to satisfied as possible, subject to the boundary conditions holding. Squaring the residuals defined by the first-order conditions would then convert solving a system of differential equations for the (inverse-) bid functions to a constrained, nonlinear

least-squares problem in which the coefficients on the approximating functions are chosen to minimize the sum of squared residuals.

This method is consistent with the Mathematical Programming with Equilibrium Constraints (MPEC) approach advocated by the late Che-Lin Su and Judd [2012], and is extremely flexible in that it can be applied in a variety of settings beyond the standard asymmetric IPV model. For example, Hubbard and Paarsch [2009] modeled the bid functions using Chebyshev polynomials when investigating a problem that had both bid preferences and endogenous entry. Their model was extended to consider selective entry models by Sweeting and Bhattacharya [2015] and to test for collusion at asymmetric, pay-your-bid auctions by Aryal and Maria F. Gabrielli [2013]. Regardless of the numerical strategy that a researcher adopts, Hubbard, Kirkegaard, and Paarsch [2013] encouraged using economic theory to investigate and to verify the validity of the approximations.

8

Dependence

Consider a painting that is being sold at auction. Several factors are surely important in determining the painting's market value—its condition, how rare it is, the historical importance of the artwork, the medium used, the dimensions of the work, almost certainly its provenance. Different potential buyers surely have different valuations for the piece—depending on the style, the artist, the subject matter, and so forth. Should art be modeled as private-values?

Contrast these personal tastes for the painting with the notion that many collectors view paintings as assets—investment opportunities that have value in the resale market. If potential buyers expect the value of the artwork to appreciate, then their motives for bidding can be driven by beliefs that they will be able to resell the painting at a later date and then generate profits. When the identity of the winner does not affect the resale value, should not art then be modeled within a common-valued environment?

As another example, consider a government that seeks to hire a firm to complete a task—such as paving a road. In a standard procurement auction, bids are solicited and the opportunity to complete the work is awarded to the firm tendering to the lowest bid price. From the theory

of industrial organization, one knows that any pricing model will involve rules based on the cost structure of firms. As such, the production technology of the firm, costs of inputs, managerial ability of the firms' leadership, and the amount of work the firm has already committed to will be important determinants of the firms' costs. For instance, if the task is to be completed at a location that is near another job site for a firm, then this suggests a firm might anticipate lower costs relative to other rivals who might not have established work sites near that location. In contrast, if a firm has already agreed to complete a certain number of projects, then their best resources might already be employed, which means that the firm would realize higher costs for an additional project. This example makes clear that firm-specific obligations and resources are important considerations—perhaps suggesting a private-costs model.

That said, if the project involves paving a new road, then the firm performing the work will need asphalt, labor, and equipment. If the employees at various firms are drawn from the same labor market, then that suggests wages should be pretty comparable across firms. If the production technology is fairly standardized, then all firms are going to require similar equipment. Asphalt is petroleum based, meaning the price of this critical input is going to fluctuate with the world price of oil—something that certainly all firms will take as given. Such similarities in cost structures across firms suggest that a common-cost model is appropriate.

Both sets of examples are intended to demonstrate that the choice of model by the econometrician is often not an easy one. Fortunately, Milgrom and Weber [1982] put forth a general theory of auctions that nests the IPV and CV models as special cases, but includes a much richer set of intermediate models. This is important to consider as a host of real-world settings and items at auction often do not fit squarely into the IPV or CV models. Moreover, with dependence amongst bidders' valuations, many predictions of basic auction theory break down. For example, the strategic equivalence across the various second-price auction formats no longer holds as an oral (English) format often permits bidders' to observe the behavior of rivals, which can be important

when common-valued elements exist in their valuation structure; this is impossible under the sealed formats. From this, the standard auction formats can actually be ranked in terms of the expected revenues they generate, something that has been investigated empirically by Cho et al. [2014]. Most important to the work of Milgrom and Weber [1982] is the assumption that they maintain about the type of dependence characterizing bidders' valuations. Milgrom and Weber assumed that bidders' valuations are affiliated, meaning that if one bidder has a high valuation, it is likely that rivals' valuations are also high, too. We consider affiliation a starting point where structural econometricians have made substantial progress identifying and estimating models under such an assumption, but we then consider other forms of dependence.

Several researchers have also developed tests for affiliation—de Castro and Paarsch [2010] using discrete data; Li and Zhang [2010] using entry behavior; as well as Sung Jae Jun, Pinkse, and Yuanyuan Wan [2010] nonparametrically.

Given the convenience and flexibility of GPV's nonparametric identification and estimation strategy, researchers have sought to extend their approach to the APV model. For example, Li et al. [2000] applied the approach to a conditional IPV model, which is a special case of the APV model. In their model, bidders' private information can be decomposed into two independent components: one, an auction-specific element, common to all bidders; the second, bidder-specific. The model is referred to as conditionally independent since, given the common component, each bidder's private information is independent of the others. The unknown common element, however, introduces dependence in bidders' private information. Li et al. used ideas from the statistics literature concerned with measurement error (specifically deconvolution methods) to establish that their conditional IPV model is nonparametrically identified, employing empirical characteristic functions in the second-step of estimation.

Because of the dependence in bidders' values, even after conditioning on the information set, the general APV model is challenging to implement. Consider the following first-order condition of a represen-

tative bidder:

$$\sigma'(v) = [v - \sigma(v)] \frac{f_{(1:-n)|V}(v|v)}{F_{(1:-n)|V}(v|v)}$$

where $V_{(1:-n)}$ is the maximum rival valuation. When transforming this system into one involving observables, conditional bid cdfs and pdfs appear in the first-order conditions

$$v_n = s_n + \frac{G_{(1:-n)}(s_n|s_n)}{g_{(1:-n)}(s_n|s_n)},$$

which complicates matters, but still means that each private value can be expressed as a function of observables. Using this result, Li, Perrigne, and Vuong [2002] established nonparametric identification; estimation is still possible as the ratio of the conditional bid distribution to the conditional pdf of the bids can be replaced by the ratio of the joint bid distribution to its joint density. The joint density can be estimated nonparametrically by computing a product of univariate kernels of the bid of interest and the maximum of that bidder's rivals' bids.

Campo et al. [2003] continued this nonparametric approach by extending the symmetric APV model to one that admits asymmetries across bidders. In this model, the number of different groups of bidders determines the dimensionality of the joint bid density, which highlights a challenge in working in this environment: a curse of dimensionality arises as the numbers of groups gets large, meaning convergence can be slow. Moreover, affiliation is not imposed in nonparametric estimation strategies, which means that the first-order condition used in estimation need not constitute an equilibrium. One solution to this problem is to impose additional structure. In fact, Li et al. [2002] demonstrated that, in order for a sample of bids to be rationalized by an APV model, the joint distribution of bids must not only be affiliated, but the inverse-bid function must be strictly increasing.

Statisticians have developed ways of characterizing joint distribution functions that describe explicitly the dependence across random variables.

8.1 Copulas

A general way in which to characterize dependence among random variables involves using what is called a copula. Roger B. Nelsen [1999] provided a detailed introduction to the theory of copulas. In what follows, we repeat some basic facts that are relevant to our discussion.

For two variables, U_1 and U_2 both contained on the interval $[0, 1]$, a bivariate copula $\mathcal{C}(u_1, u_2)$ is a continuous function having the following properties:

1. $\text{Dom}(\mathcal{C}) = [0, 1]^2$;
2. $\mathcal{C}(u_1, 0) = 0 = \mathcal{C}(0, u_2)$;
3. $\mathcal{C}(u_1, 1) = u_1$ and $\mathcal{C}(1, u_2) = u_2$;
4. \mathcal{C} is a twice-increasing function, so

$$\mathcal{C}(u_1^1, u_2^1) - \mathcal{C}(u_1^0, u_2^1) - \mathcal{C}(u_1^1, u_2^0) + \mathcal{C}(u_1^0, u_2^0) \geq 0$$

for any $u_1^0, u_2^0, u_1^1, u_2^1 \in [0, 1]^2$, such that $u_1^0 \leq u_1^1$ and $u_2^0 \leq u_2^1$.

Because U_1 and U_2 are both defined on the unit interval, they can be viewed as uniform random variables with $\mathcal{C}(u_1, u_2)$ being their joint distribution function. Alternatively, U_1 and U_2 can be viewed as the cdfs of two random variables V_1 and V_2 . In this case, their marginal distribution functions $F_1(v_1)$ and $F_2(v_2)$ are linked to their joint distribution $F_{12}(v_1, v_2)$ by

$$F_{12}(v_1, v_2) = \mathcal{C}[F_1(v_1), F_2(v_2)].$$

One attractive feature of copulas is that the marginal cumulative distribution functions do not depend on the choice of the dependence function for the two random variables in question. When one is interested in the association between random variables, copulas are a useful device because the dependence structure is easily separated from the marginal cdfs.

From Sklar's theorem, one knows that a unique function \mathcal{C} , the copula, exists such that

$$F_{12}(v_1, v_2) = \mathcal{C}[F_1(v_1), F_2(v_2)].$$

Also, if one introduces the copulas

$$\mathcal{M}(u_1, u_2) = \min(u_1, u_2)$$

and

$$\mathcal{W}(u_1, u_2) = \max(0, u_1 + u_2 - 1),$$

then the following inequalities hold:

$$\mathcal{W}(u_1, u_2) = \max(0, u_1 + u_2 - 1) \leq \mathcal{C}(u_1, u_2) \leq \min(u_1, u_2) = \mathcal{M}(u_1, u_2),$$

also known as the Fréchet–Hoeffding bounds, in honor of the French mathematician Maurice R. Fréchet and the Finnish statistician Wassily Hoeffding. Note, however, that the Fréchet–Hoeffding bounds are quite wide, so do not restrict the shape of the copula very much.

Many different families of copulas exist, most of which are discussed by Nelsen. In empirical work, a frequently employed family of copulas is the Archimedean family, which is defined by

$$\mathcal{C}_\zeta(u_1, u_2) = \zeta^{-1}[\zeta(u_1) + \zeta(u_2)],$$

where the generating function, $\zeta(\cdot)$, is a convex, decreasing function. Note that $\zeta(1)$ must equal zero and $\zeta^{-1}(u)$ must be zero for any u exceeding $\zeta(0)$. These conditions are both necessary and sufficient for \mathcal{C}_ζ to be a distribution function. Copulas within the Archimedean family have joint density functions of the following form:

$$c_\zeta(u_1, u_2) = -\frac{\zeta''(F_{12})\zeta'(u_1)\zeta'(u_2)}{[\zeta'(F_{12})]^3}.$$

Members of the Archimedean family are both symmetric and associative, so

$$\mathcal{C}_\zeta(u_1, u_2) = \mathcal{C}_\zeta(u_2, u_1) \quad \forall \quad u_1, u_2 \in [0, 1],$$

and

$$\mathcal{C}_\zeta[\mathcal{C}_\zeta(u_1, u_2), u_3] = \mathcal{C}_\zeta[u_1, \mathcal{C}_\zeta(u_2, u_3)] \quad \forall \quad u_1, u_2, u_3 \in [0, 1].$$

Another attractive feature of the Archimedean family is that it is easy to identify whether it has points of singularity, mass points; see the paper by Christian Genest and Jock MacKay [1986].

Well known members of the Archimedean family, which depend on a single parameter θ , are presented in Table 8.1. Note that independence obtains for the Clayton and Gumbel members when θ is zero and, in the case of Frank copulas, when θ is one. For these members of the Archimedean family, the joint density functions are easy to compute. For example, in the case of Frank copulas,

$$c_\zeta(u_1, u_2) = \frac{(\theta - 1) \log(\theta) \theta^{u_1 + u_2}}{[(\theta - 1) + (\theta^{u_1} - 1)(\theta^{u_2} - 1)]^2}$$

where it is easy to show that this density converges to one as θ tends to one.

Member	$\mathcal{C}(u_1, u_2)$	$\zeta(u)$	Range of θ
Clayton	$(u_1^{-\theta} + u_2^{-\theta} - 1)^{\frac{1}{\theta}}$	$\frac{u^{-\theta} - 1}{\theta}$	$[0, \infty)$
Frank	$\log_\theta \left[1 + \frac{(\theta^{u_1} - 1)(\theta^{u_2} - 1)}{(\theta - 1)} \right]$	$\log_\theta \left(\frac{1 - \theta}{1 - \theta^u} \right)$	$[0, \infty)$
Gumbel	$\exp \left[\left([-\log(u_1)]^{\theta+1} + [-\log(u_2)]^{\theta+1} \right)^{\frac{1}{\theta+1}} \right]$	$[-\log(u)]^{\theta+1}$	$[0, \infty)$

Table 8.1: Members of Archimedean Family of Copulas

8.2 Using Copulas

By using a copula to characterize the joint distribution of valuations in a symmetric APV model, Hubbard, Li, and Paarsch [2012] did not impose affiliation, but allowed for it; they proved that the conditional distribution $F_{V_{-n}|V_n}(v_{-n}|v_n)$ can be computed by taking the partial derivative of the copula with respect to the n^{th} component, which meant that the first-order condition can be expressed as

$$\sigma'(v) = [v - \sigma(v)](N - 1)f_V(v) \frac{\mathcal{C}_{nm}[F_V(v), \dots, F_V(v)]}{\mathcal{C}_n[F_V(v), \dots, F_V(v)]}.$$

Under this specification, the GPV transformation of random variables yields

$$v = s + \frac{\mathcal{C}_n[F_S(s), \dots, F_S(s)]}{(N-1)f_S(s)\mathcal{C}_{nm}[F_S(s), \dots, F_0(s)]}.$$

Critical to this transformation is the fact that the copula function is invariant under strictly increasing transformations of its arguments. Hubbard et al. proposed a semiparametric, pseudo-ML estimator in which the marginal bid cdf and pdf is first estimated, then the copula parameters are estimated by maximizing the logarithm of the likelihood function. This allows pseudo values to be recovered using the equation above and the latent valuation distribution to be estimated using standard kernel-smoothing.

Li and Zhang [2015] extended the work of Hubbard et al. to admit two realistic aspects of auctions that we have highlighted earlier in this review: asymmetric bidders and bidder entry. Li and Zhang considered a model in which potential buyers initially only know their personal entry costs, but know the joint distribution of entry costs and the joint distribution of valuations. Li and Zhang adopted a copula-based approach, where the copula is used to model the joint distribution of entry costs and the joint distribution of private values, but entry costs and private valuations are independent of one another. They adopted a two-stage indirect inference method in estimation.

Li [2010] provided an overview of indirect inference estimation in structural econometric models. Because entry costs and valuations are independent and because Li and Zhang assumed that the bidders know the number of opponents when making bidding decisions, the distribution of private values can be estimated using only the observed bids. The indirect inference approach is then again applied in a second step to estimate entry costs given observed entry decisions. The auxiliary models are simple to compute and involve a linear regression of observed bids and a Poisson regression of the number of actual bidders. Moreover, simulation of the structural model is facilitated by the copula approach. (Li and Zhang considered a Clayton copula, which has a closed-form that permits simulation quite easily.)

Like Hubbard et al. [2012], Kim [2015] also noted that the empirical approach considered by a researcher may be incompatible with the

underlying paradigm. Affiliation can be violated if it is not formally imposed; any violation of theoretical shape restrictions would invalidate policy recommendations based on the underlying auction theory. Using B-splines, Kim specified the pdf of valuations to ensure that monotonicity of the bid function is satisfied. Unlike the multi-stage (indirect) approach of GPV, he used a series representation to specify the pdf of valuations, thus ensuring that he could impose shape restrictions. His direct approach means bidding monotonicity is automatically satisfied and he used properties of B-splines to impose affiliation on the joint density of valuations. Kim also introduced the Bayesian method with a simulated likelihood method into the auctions literature, providing a step-by-step guide for implementation. He specified the logarithm of the joint density in his two-bidder application as

$$\log[f_{\mathbf{V}}(v_1, v_2|\boldsymbol{\Theta})] = \sum_{i \in I} \sum_{j \in I} \theta_{ij} \varphi_i(v_1) \varphi_j(v_2) + c(\boldsymbol{\Theta})$$

where I is index set that determines the number of components (typically chosen by the Bayesian information criterion, Akaike information criterion, cross-validation, or using Bayesian model selection), $\boldsymbol{\Theta}$ the matrix of all coefficients, and $c(\boldsymbol{\Theta})$ a normalizing constant. The basis functions $\varphi_i(\cdot)$ are constructed using B-splines and he transformed the data using a monotone mapping $\tilde{F}(\cdot|\mu)$ onto $[0, 1]$ to spread out the probability mass more evenly (than the raw data) over the support. Under this transformation, Kim estimated

$$\begin{aligned} \log[f_{\mathbf{V}}(v_1, v_2|\boldsymbol{\Theta})] = & \log[\tilde{f}_{\mathbf{V}}(v_1|\mu)] + \\ & \log[\tilde{f}_{\mathbf{V}}(v_2|\mu)] \sum_{i \in I} \sum_{j \in I} \theta_{ij} \varphi_i(v_1) \varphi_j(v_2) + c(\boldsymbol{\Theta}), \end{aligned}$$

where this relationship is interpreted as first approximating the marginal pdf of valuations, with the additional terms serving to improve the approximation as well as to accommodate a flexible dependence structure. Finally, Kim introduced the Metropolis-Hastings algorithm into the structural econometrics of auctions literature.

9

Multi-Unit and Multi-Object Auctions

Without a doubt, the open research questions in the structural econometrics of auctions revolve around modeling equilibrium behavior when several objects are sold at auction. At the highest level, two cases exist: multi-unit and multi-object. At multi-unit auctions, it matters not which unit a bidder wins, but rather the aggregate number of units he wins, while at multi-object auctions it matters which specific object(s) a bidder wins. The next important feature is whether the units/objects are sold sequentially, or simultaneously. When several units of the same object are sold, the next important feature is whether potential buyers demand just one unit or several units. If several units of a good are simultaneously for sale, at least two important questions arise: Who will be the winning bidders? What price(s) will those winners pay? One pricing rule involves the winners paying the same price—specifically, the highest rejected bid; this is often referred to as the uniform-price rule. Another pricing rule involves allocating the available (say K) units to those bidders who tendered the K highest bids, with each winner then paying what he bid for the unit(s) he won—often referred as the discriminatory-price rule.

Even though outlining this taxonomy is helpful to understand what

research has already been completed, the issues associated with these models are complex—involving modeling and computational problems, typically both together.

9.1 Multi-Unit Auctions

At multi-unit auctions, when participants demand more than one unit, two cases exist: In the first, the value of successive units is either constant or declines, while in the second, the value of successive units increase. When the value of two units is greater than the value of the sum of the two units together, a supermodularity exists. In such cases, bundling units appropriately is important.

9.1.1 Testing for Best Responses

Deciding whether bidders at auctions are playing a best response is perhaps one of the most fundamental questions faced by empirical workers employing the structural econometric approach. Investigations of this question depend heavily on whether the true valuations of bidders are observed. For example, Ali Hortagsu and Steven L. Puller [2008] generalized the approach of Guerre et al. [2000] by exploiting detailed and accurate cost data to form an accurate estimate of values in their test of efficiency in electricity auctions in Texas.

When observational data are used and the true valuations are unobserved, as is more typically the case, testing for a best response may be impossible. For example, consider a single-object, sealed, pay-your-bid auction. When the monotone likelihood-ratio property (MLRP) holds for observed bids, there always exists a latent distribution of values that can rationalize actual bidding behavior. Thus, in this case, one cannot possibly reject the hypothesis that each bidder always plays a best response.

One might suspect the same to be true at multi-unit auctions (MUAs). Indeed, if the true values (or costs) are unobserved and a condition that generalizes the MLRP holds, then there always exists a latent distribution of values that can rationalize observed bidding behavior. When values are unobserved, it would seem that rejecting

the best-response hypothesis is equally hopeless in MUAs. Even when values are unobserved, however, a researcher may be willing to impose restrictions on the space of possible values. For instance, in some applications, it may be reasonable to assume that bidders have nonincreasing marginal values, that is, each bidder's value for a first unit is weakly greater than his value for a second unit, and so forth.

Building on the research of McAdams [2008], James T. E. Chapman, McAdams, and Paarsch [2005, 2006, 2007] developed an approach to decide whether observed bids are best-response violations when bidders' private values are unobserved in multi-unit discriminatory auctions under the assumption of nonincreasing marginal valuations. They also derived a bound on the expected profits lost when a best response is not adopted. Subsequently, they applied their framework to data from auctions of Receiver General term deposits conducted by the central bank in Canada, finding that best-response violations are relatively frequent. For most bidders, however, the median lower bound on the economic distance between the estimated best-response and the bid data is small—about the cost of a Venti® Latte.

9.1.2 Treasury Auctions

Since the late 1950s, when the late Nobel laureate Milton Friedman [1959] suggested that a uniform-price rule would dominate a discriminatory-price rule for treasury bill auctions in the United States, economists have been studying them. Friedman argued that under the discriminatory-price rule bidders would shave their bids, whereas under the uniform-price rule, because a bidder was less likely to influence the price at which he traded, he would bid closer to his valuation than under the pay-your-bid rule.

Because capital markets are important and because detailed data are readily available, researchers have investigated Friedman's claim (and many others) concerning treasury auctions in different countries—for example, Hortaçsu [2002] as well as Hortaçsu and McAdams [2010], who analyzed Turkish treasury auctions; Hortaçsu and Jakub Kastl [2012], who analyzed Canadian treasury auctions; Nuno Cassola, Hortaçsu, and Kastl [2013], who analyzed the effects of auctions on Euro-

pean Central Bank monetary policy; and Hortacısu, Kastl, and Allen Zhang [2016], who analyzed treasury auctions in the United States. Even though a good amount of work has been completed, much research remains to be done in this area.

9.1.3 Electricity Auctions

Another set of important markets is those for electricity. Following Hortacısu and Puller [2008], Hortacısu, Fernando Luco, Puller, and Dongni Zhu [2017] have investigated electricity markets in the United States, but others have studied markets in Australia as well as the United Kingdom. Once again, even though considerable work has been completed, much remains to be done.

9.1.4 Online Auctions of Advertising

The main way to monetize online content on the Internet is through advertising. The publisher of content (for example, online search results or real estate listings or restaurant reviews as well as the typical print content of newspapers and magazines) offers the content to users free, but by showing those users advertisements (ads), the published content is monetized through user interaction. The advertising platform is responsible for showing relevant user- and content-specific ads; the resulting advertising revenues are then shared between the advertising platform and the content publisher.

Historically, online ads were sold under negotiated contracts—very similar to those still used to sell print as well as radio and television advertising. Given the magnitude and scope of information available concerning online users of the Internet, however, ad personalization is possible. In short, it became clear quickly that negotiated contracts are too restrictive to optimize the effectiveness of advertising. In other words, negotiated contracts do not ensure high-quality matches between ads and users.

Online environments require that ads be allocated and priced dynamically on a user-by-user as well as time- and content-specific basis. Early experiments by Overture Services (that later helped power advertising platforms at Yahoo!, MSN and paved the path for monetization

at Google) demonstrated that online auctions with the pay-per-click rule are very successful at effective, fast, and scalable allocation of ads to users. Under these mechanisms, each time a user lands on a given page, the advertising platform determines (based on some measure of relevance) the set of ads that are potentially eligible to be shown to the user. For each ad, an advertiser specifies a bid—essentially the maximum amount the advertiser is willing to pay per user click. The advertising platform then takes all of the bids for all relevant ads and runs an auction whose outcome determines the allocations of ads to advertising slots as well as their prices.

In this section, we discuss the design of ad auctions as well as approaches to modeling and inference using data from those auctions. The specific focus is sponsored-search advertising—that is, the ads that appear alongside the search results in online search engines, such as Google, Bing, Baidu, and Yandex.

Sponsored-search auctions are multi-unit auctions at which the prices as well as the allocations of heterogeneous advertising slots are determined. Within this environment, the most commonly used multi-unit auction is the rank-based auction. Rank-based auctions are attractive because the mechanism requires only a single input from each participant—his bid. The prices as well as the allocation of the ads to slots are based on the relative ranks of the bids of all participants.

Perhaps the most commonly-used auction mechanism for sponsored search is the weighted generalized second-price (GSP) auction. Under this auction, each bidder n is assigned a score s_n , which typically corresponds to the probability that a consumer issuing the search query would click on the ad of bidder n were that ad placed in the first slot. The bidders are then allocated to slots according to the rank of their score-weighted bid. Specifically, if there are N bidders and J slots, with

$$s_1 b_1 \geq s_2 b_2 \geq \cdots \geq s_N b_N,$$

then the first J bidders in the ordering are assigned the J slots. In the presence of a reserve price r , J bidders are allocated to slots if $s_J b_J > r$; otherwise, only the bidders whose score-weighted bids exceed the reserve price are allocated.

As the name indicates, pricing under the GSP rule is based on a

generalization of the second-price auction: The price of bidder in slot j is determined by the bidder ranked $(j + 1)$, or the reserve price if the $(j + 1)^{\text{th}}$ score-weighted bid is below the reserve price. The price bidder n pays when $s_n b_n > s_{n+1} b_{n+1}$ is

$$p_n = \frac{s_{n+1} b_{n+1}}{s_n}.$$

In addition, a bidder only pays when a user clicks on his ad; if the ad of the bidder is allocated to a slot, but a user does not click on that slot, then the bidder pays nothing.

Under the weighted GSP rule, the actions available to bidders are simple: Each bidder submits a single bid b_n that is used both for allocation and pricing. The main difference between models of sponsored search auctions and the models of other auctions discussed earlier in this review is that the preferences of bidders are not drawn from a distribution. Instead, each bidder n is characterized by his valuation per click v_n which is assumed to be a fixed parameter of the bidder. In that case, if bidder n is placed in slot n , with price per click p_n , and the user clicks on n^{th} ad, then the bidder receives the payoff

$$\text{Payoff}(b_n, p_n | v_n) = v_n - p_n.$$

The rationale behind the fixed value per click assumption is that most ads are created for specific advertising campaigns, where the value of user engagement with the ad is a concrete number—for instance, the purchase of the advertised product, such as car insurance or a wireless plan.

In the literature concerning ad auctions, user searches have been typically modeled as an exogenous process, where each consumer's clicking behavior is random and fully characterized by the array $[c_{nj}]$ which is the probability that a consumer clicks on the ad of bidder n if it is allocated slot j . It is commonly assumed that $[c_{nj}]$ is a multiplicative function of a slot-specific click probability α_j and a bidder-specific click probability s_n . A typical normalization is that α_1 equals one—that is, the click probability in the first slot corresponds to the bidder-specific click probability.

No Uncertainty Model of Advertising Auctions

In the pioneering theoretical research concerned with advertising auctions, no uncertainty existed. The high-frequency nature of online environments makes them different from the sealed-bid structure of traditional auctions: With enough experimentation, advertisers can discover one another's bids. Moreover, as mentioned, the ads are typically created for a specific purpose, for example, within a particular advertising campaign. Consequently, it is difficult to justify private- or affiliated-value models, where the values of advertisers per click are drawn from a distribution. Therefore, in the no uncertainty settings, it is assumed that the bids are common knowledge and the values of bidders are fixed.

Consider a model of an ad auction where each of N advertisers simultaneously place per-click bids b_n on a single search phrase. The bids are then held fixed and applied to all of the users who enter that search phrase over a pre-specified time period—for example, a day or a week.

Because bidders have no uncertainty concerning one another's bids, the equilibrium concept is the classic Nash equilibrium. The structure of Nash equilibria in the score-weighted GSP mechanism has been considered by Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz [2007] as well as Hal R. Varian [2007].

To describe this, we can express the expected payoff of bidder n from occupying the slot j (where the expectation is taken over the uncertainty of user clicks) as

$$\bar{c}_{nj}(v_n - p_j) = \alpha_j s_n \left(v_n - \frac{s_k b_k}{s_n} \right), \text{ rank}(s_k b_k) = \text{rank}(s_n b_n) + 1.$$

In the no-uncertainty models of Edelman et al. as well as Varian, bidders are assumed to know the set of competitors as well as the score-weighted bids of the opponents, and they consider *ex post* equilibria, where each bidder's score-weighted bid must be a best response to the realizations of $\{s_j b_j, j \neq n, j = 1, \dots, N\}$, assuming that click probabilities \bar{c}_{nj} are common knowledge.

The set of bids constituting a full-information Nash equilibrium in the no uncertainty model, where each bidder finds it unprofitable to

deviate from his assigned slot, satisfy the following:

$$\begin{aligned}\alpha_n \left(v_n - \frac{s_k b_k}{s_n} \right) &\geq \alpha_\ell \left(v_n - \frac{s_m b_m}{s_n} \right), \ell = m - 1 \geq i = k - 1 \\ \alpha_n \left(v_n - \frac{s_k b_k}{s_n} \right) &\geq \alpha_\ell \left(v_n - \frac{s_m b_m}{s_n} \right), n + 1 = k \geq m = \ell + 1.\end{aligned}$$

It is more convenient to express these inequalities in terms of score-weighted values as follows:

$$\min_{\ell < n} \frac{s_\ell b_\ell \alpha_n - s_{n+1} b_{n+1} \alpha_\ell}{\alpha_\ell - \alpha_n} \geq s_n v_n \geq \max_{\ell > j} \frac{s_{n+1} b_{n+1} \alpha_j - s_{\ell+1} b_{\ell+1} \alpha_\ell}{\alpha_n - \alpha_\ell}.$$

These inequalities identify the range of values of bidder n that is rationalized by the observed bids in Nash equilibrium.

An equilibrium always exists, but it is typically not unique, and equilibria may not be monotone: bidders with higher score-weighted values may not be ranked higher.

Edelman et al. as well as Varian defined a refinement of the set of equilibria, which Edelman et al. refer to as envy-free: no bidder wants to exchange positions and bids with another bidder. The set of envy-free equilibria is characterized by a tighter set of inequalities:

$$s_n v_n \geq \frac{s_{n+1} b_{n+1} \alpha_n - s_{n+2} b_{n+2} \alpha_{n+1}}{\alpha_n - \alpha_{n+1}} \geq s_{n+1} v_{n+1}. \quad (9.1)$$

The term in between the two inequalities is interpreted as the incremental costs divided by the incremental clicks from changing position, or the “incremental cost per click” $\text{ICC}_{n,n+1}$:

$$\text{ICC}_{n,n+1} = \frac{s_{n+1} b_{n+1} \alpha_n - s_{n+2} b_{n+2} \alpha_{n+1}}{\alpha_n - \alpha_{n+1}}.$$

Envy-free equilibria are monotone, in that bidders are ranked by their score-weighted valuations, and have the property that local deviations are the most attractive; the equilibria can be characterized by incentive constraints that ensure that a bidder does not want to exchange positions and bids with either the next-highest or the next-lowest bidder.

Edelman et al. considered a narrower class of envy-free equilibria, the one with the lowest revenue for the auctioneer and the one that

coincides with Vickrey payoffs as well as the equilibrium of a related ascending-price auction game. They required the following:

$$s_n v_n \geq \text{ICC}_{n,n+1} = s_{n+1} v_{n+1}. \quad (9.2)$$

Edelman et al. showed that despite the fact that payoffs coincide with Vickrey payoffs, bidding strategies are not truthful: bidders shave their bids, trading off higher price per click from a better position against the incremental clicks they obtain from the higher position.

Model with Score and Entry Uncertainty

Although the no-uncertainty model is attractive because of its simplicity and equilibrium characterization, it has some drawbacks. Athey and Denis Nekipelov [2010] reported that the fixed-value assumption is rejected by the data for many advertisers considered over a period of time exceeding a day. In addition, Varian had noted previously that the empirical incremental cost-per-click curve is frequently nonmonotonic; he also suggested using adjustment factors that introduce small distortions to make the empirical incremental cost-per-click curve monotone.

A larger issue is that the auction design (in terms of the scores s_n , the reserve price, and the set of eligible bidders) changes user query by user query. Because auction parameters change at such a high frequency, they cannot be learned by the bidders over time and their variation over time can be treated as random by the advertisers. This observation is very important because in these settings bidders do not (and cannot) adjust their bids to bid in each individual auction. Instead, they consider an expectation over user queries, and optimize their bid to maximize their expected payoffs.

Athey and Nekipelov developed a model that has these features, assuming that a bidder does not observe the set of competitors in a given query and does not observe either his own or his competitors' score shocks. Instead, a bidder forms beliefs regarding the distribution of all the score shocks of all bidders who are eligible for a given query and beliefs regarding the distribution of realizations of the set of his competitors in a user query. They also assumed that a bidder can observe the actual bids of his competitors. Each bidder n maximizes the

expected payoff (with per click value) with the expectation taken with respect to the bidder's beliefs regarding the distribution of uncertainty of scores and sets of competitors.

Consider observing a large number of queries for a given set of potential bidders, and consider the question of whether the valuations of the bidders can be identified. The auction platform and the econometrician observe bids, the set of entrants, and the scores. In this case, one can define a function $Q_n(\cdot)$, equal to the probability of a click on the ad by bidder n in a search query, as a function of the profile of bids. One can also define the function $TE_n(\cdot)$, which equals the expected expenditure of bidder n in a search query as a function of the profile of bids. Then, one can define the expected payoff that bidder n with bid b_n receives in an auction by

$$v_n Q_n(b_n) - TE_n(b_n).$$

Assuming sufficient smoothness and support size of the distribution of scores, Athey and Nekipelov demonstrated that functions $Q_n(\cdot)$ and $TE_n(\cdot)$ are strictly increasing and differentiable in a player's own bid, so one can recover the valuation of each bid using the necessary condition for optimality of a bid

$$v_n = \frac{\partial TE_n(b_n, \mathbf{b}_{-n}) / \partial b_n}{\partial Q_n(b_n, \mathbf{b}_{-n}) / \partial b_n}, \quad (9.3)$$

given that all of the distributions required to evaluate functions $Q_n(\cdot)$ and $TE_n(\cdot)$ are assumed to be observable.

Equation (9.3) provides a simple, practical method for estimating values per click: for each bidder, one changes his bid by a small amount and then computes the change in the outcome of the auctions where the original bid of the bidder was applied. The evaluated change in the expenditure will serve as an estimator for the derivative of $TE_n(\cdot)$ and the evaluated change in the number of clicks will serve as an estimator for the derivative of $Q_n(\cdot)$.

Industrial organization provides a simple recipe for platform optimization subject to the inferred preferences of the bidders, *assuming that the platform is in equilibrium under the old and the new settings*; see Bajari et al. [2013] as well as Bajari, Hong, John Krainer, and

Nekipelov [2010]. First, using the data from the historical realizations of auctions, one can estimate the components of the first-order condition (9.3). Next, for each bidder, one can reverse-engineer the value of this bidder. Third, one can take the new proposed auction parameters, or consider a new auction format. For each set of bids from bidders, one can then simulate the auction outcomes under the new auction parameters. Those new auction outcomes will generate the counterfactual functions $\widetilde{T}E_n(b_n, \mathbf{b}_{-n})$ and $\widetilde{Q}_n(b_n, \mathbf{b}_{-n})$ for each bid profile of participating bidders. Finally, considering the Nash equilibrium under new auction parameters, one knows that the bidders set their bids to best respond to the bids of their opponents. That means that one can compute the equilibrium bid profile $\mathbf{b}^* = (b_n^*, \mathbf{b}_{-n}^*)$ under the new auction settings as a solution of the following system of nonlinear equations:

$$v_n = \frac{\partial \widetilde{T}E_n(b_n^*, \mathbf{b}_{-n}^*) / \partial b_n}{\partial \widetilde{Q}_n(b_n^*, \mathbf{b}_{-n}^*) / \partial b_n} \quad n = 1, \dots, N, \quad (9.4)$$

using values inferred from the data. Unlike equation (9.3), where one used actual bids to obtain the values, equation (9.4) takes the bids as inputs to solve for \mathbf{b}^* that make the equalities hold for each bidder.

Under general conditions, no guarantee exists that system (9.4) has the structure that makes simple search algorithms suitable for finding its solutions quickly. Under additional constraints, such as strict monotonicity of functions $\widetilde{T}E_n(\cdot, \cdot)$ and $\widetilde{Q}_n(\cdot, \cdot)$, one can apply numerical continuation methods for finding solutions. As suggested by Athey and Nekipelov, numerical continuation methods can make use of the homogeneity of functions $\widetilde{T}E_n(\cdot, \cdot)$ and $\widetilde{Q}_n(\cdot, \cdot)$ for rank-based auctions. This allows for the use of bids in settings where each bidder n is the only bidder who participates in the auction, providing starting values for the numerical continuation to help in computing the full solution \mathbf{b}^* .

Model with Algorithmic Learning

Real-world advertising platforms are complex systems: thousands of bidders compete at the same auction, with bidders changing their tenders dynamically, entering and exiting the auction. In this context, the

information requirements for the bidders to derive the Bayes-Nash equilibrium profiles are truly astronomical since they are required to form beliefs over the actions of all of their thousands of opponents as well as the dynamic adjustment of the auction parameters by the advertising platform.

In practice, most bidders on large advertising platforms use algorithmic tools that allow them to adjust bids automatically and dynamically for multiple ads and advertising campaigns. The algorithmic solutions that are implemented in these tools take the advertisers' goals (in terms of the yield of the auction outcomes) as inputs and adjust the bids given the dynamic feedback from the auction outcomes. Such implementations can be associated with algorithmic learning, where the bidding strategy is treated as the goal of an online statistical learning procedure.

Recently, Tim Roughgarden [2009]; Vasilis Syrgkanis and Éva Tardos [2013]; as well as Jason Hartline, Syrgkanis, and Tardos [2015] have shown that some of the worst-case outcome properties of full-information pure Nash equilibria extend to outcomes when all players use no-regret learning strategies, assuming the game itself is stable. The assumption that players use no-regret learning to adjust their strategies is attractive for a number of reasons: No-regret learning outcomes generalize the Bayes-Nash equilibrium assumption. Rather than assuming that at each time step the actions of the players form a Bayes-Nash equilibrium, an assumption is made only about the aggregate behavior—each player has no-regret for any fixed action over a longer time period. If the behavior of all agents is stable over the time period, then this is exactly the Nash equilibrium assumption—however, evolving play is admitted.

Second, there are many well known learning strategies that guarantee that the player achieves no-regret—including the weighted majority algorithm of Sanjeev Arora, Elad Hazan, and Satyen Kale [2012], which is also known as hedge by Nick Littlestone and Manfred K. Warmuth [1994] as well as Yoav Freund and Daniel Hsu [2008], and regret matching by Sergiu Hart and Andreu Mas-Colell [2000], to name just a few simple ones.

Clearly, in the context of bidders using algorithmic strategies, one cannot directly estimate their preferences from their observed actions since they only reflect their preferences through the algorithmic implementation. Nekipelov, Syrgkanis and Tardos [2015] used the concept of no-regret learning in auctions and introduced the notion of a rationalizable set. Rather than assuming that players have absolutely no regret, this set contains pairs of valuations of players and regret constants that are compatible with the observable data. The expected regret of a player reflects the properties of a learning algorithm used.

The sponsored search auction is now placed in a dynamic setting where there is a sequence of T auctions and each auction t yields the expected payoff $u_{nt}(b_{nt}|v_n)$ to bidder n . A sequence of play that we observe has ϵ_n -regret for advertiser n if

$$\frac{1}{T} \sum_{t=1}^T u_{nt}(b_{nt}; v_i) \geq \frac{1}{T} \sum_{t=1}^T u_{nt}(b'; v_n) - \epsilon_n \quad (9.5)$$

for all feasible b' . This leads to the following definition of a rationalizable set under no-regret learning. Specifically, small regret learning is defined as follows: A pair (ϵ_n, v_n) of a value v_n and error ϵ_n is a rationalizable pair for player n if it satisfies equation (9.5). The set of such pairs is referred to as the rationalizable set, and is denoted by \mathcal{NR} .

The rationalizable set \mathcal{NR} is the identified set of possible values and average regrets of the learning strategy of the bidder. The rationality assumption of the inequality (9.5) models players who may be learning from experience while participating in the game. The strategies b_{nt} and realizations of payoff functions in each round $u_{nt}(\cdot, \cdot)$ (corresponding to the time variation of functions $Q_n(\cdot)$ and $TE_n(\cdot)$ considered previously) are assumed to be realized simultaneously, so bidder n cannot pick his strategy dependent on the payoff realization. This makes the standard of a single best strategy b_n natural. Beyond this, no assumptions concerning information available to the agents are made, nor how they choose their strategies.

One can specialize the definition of the rationalizable set in equation

(9.5) to sponsored search auctions by introducing functions

$$\Delta P(b') = \frac{1}{T} \sum_{t=1}^T [TE_{nt}(b') - TE_{nt}(b_{nt})]$$

and

$$\Delta C(b') = \frac{1}{T} \sum_{t=1}^T [Q_{nt}(b') - Q_{nt}(b_{nt})],$$

corresponding to an aggregate outcome across T auctions from switching to a fixed alternate bid b' . The ϵ -regret condition reduces to the following:

$$v \times \Delta P(b') \leq \Delta C(b') + \epsilon \quad (9.6)$$

for all $b' \in \mathbb{R}_+$. Hence, the rationalizable set \mathcal{NR} is an envelope of the family of half planes obtained by varying $b \in \mathbb{R}_+$ in equation (9.6). Under suitable assumptions regarding the expected auction outcomes $TE_n(\cdot)$ and $Q_n(\cdot)$ (such as continuity and monotonicity), one can establish some basic geometric properties of the rationalizable set, such as its convexity and closedness.

Nekipelov et al. [2015] found an elegant geometric characterization of the \mathcal{NR} set that also implies an efficient algorithm for computing that set. Because bounded, closed, convex sets are fully characterized by their boundaries, one can use the notion of the support function to represent the boundary of the set \mathcal{NR} . The support function of a bounded, closed, convex set \mathcal{X} is $h(\mathcal{X}, u) = \sup_{x \in \mathcal{X}} \langle x, u \rangle$, where in this case $\mathcal{X} = \mathcal{NR}$ is a subset of \mathbb{R}^2 —that is, value and error pairs (v, ϵ) —and then u is also an element of \mathbb{R}^2 .

An important property of the support function is the way it characterizes closed convex bounded sets. Denote by $d_H(\mathcal{A}, \mathcal{B})$ the Hausdorff distance between convex compact sets \mathcal{A} and \mathcal{B} . Recall that the Hausdorff norm for subsets \mathcal{A} and \mathcal{B} of the metric space E with metric $\rho(\cdot, \cdot)$ is defined as

$$d_H(\mathcal{A}, \mathcal{B}) = \max\left\{\sup_{a \in \mathcal{A}} \inf_{b \in \mathcal{B}} \rho(a, b), \sup_{b \in \mathcal{B}} \inf_{a \in \mathcal{A}} \rho(a, b)\right\}.$$

It turns out that $d_H(\mathcal{A}, \mathcal{B}) = \sup_u |h(\mathcal{A}, u) - h(\mathcal{B}, u)|$. Therefore, if one can find $h(\mathcal{NR}, u)$, this is equivalent to characterizing \mathcal{NR} itself.

The following result fully characterizes the support function of the set \mathcal{NR} based on the aggregate auction outcomes $\Delta P(\cdot)$ and $\Delta C(\cdot)$: Under monotonicity and continuity of $\Delta P(\cdot)$ and $\Delta C(\cdot)$, the support function of \mathcal{NR} is the following:

$$h(\mathcal{NR}, u) = \begin{cases} |u_2| \Delta C \left[\Delta P^{-1} \left(\frac{u_1}{|u_2|} \right) \right] & \text{if } u_2 < 0; \\ \frac{u_1}{|u_2|} \in [\inf_b \Delta P(b), \sup_b \Delta P(b)] (= +\infty) & \text{otherwise.} \end{cases}$$

This is an identification result for valuations and algorithm parameters for ϵ -regret algorithms. Unlike the identification result in the equilibrium settings, one cannot pin-point the values of players. At the same time, the characterization of the set \mathcal{NR} reduces to evaluation of two one-dimensional functions. One can use efficient numerical approximations to carry out such an evaluation.

In general, the shape of the set \mathcal{NR} will depend on the parameters of the particular algorithm used for learning. Thus, the analysis of the geometry of \mathcal{NR} can help one not only to estimate valuations of players, but the learning algorithm as well.

Inference concerning the set \mathcal{NR} reduces to characterizing its support functions, which only requires evaluation of $\Delta C(\Delta P^{-1}(\cdot))$, a one-dimensional function that can be estimated from the data via direct simulation.

Because the object of interest is the set \mathcal{NR} , one needs to characterize the distance between the true set \mathcal{NR} and the set $\widehat{\mathcal{NR}}$ that is obtained from subsampling the data. Nekipelov et al. [2015] showed that the characterization of the properties of the estimated rationalizable set reduces to the description of the properties of a single-dimensional function $f(\cdot) = \Delta C(\Delta P^{-1}(\cdot))$. Let $\widehat{f}(\cdot)$ be its empirical analog recovered from the data. The set \mathcal{NR} is characterized by its support function $h(\mathcal{NR}, u)$. Using the relationship between the Hausdorff norm and the sup-norm of the support functions, one can write

$$d_H(\widehat{\mathcal{NR}}, \mathcal{NR}) = \sup_{\|u\|=1} |h(\widehat{\mathcal{NR}}, u) - h(\mathcal{NR}, u)| \leq \sup_z |\widehat{f}(z) - f(z)|.$$

The empirical analog of $f(\cdot)$ can be directly estimated from the data via subsampling of auctions. The properties of the estimated set $\widehat{\mathcal{NR}}$ are thus determined by the properties of $f(\cdot)$. In particular, if function

f has derivatives up to order $k \geq 0$ and for some $\lambda \geq 0$, $|f^{(k)}(z_1) - f^{(k)}(z_2)| \leq \lambda|z_1 - z_2|^\alpha$, then the Hausdorff distance between the true and estimated rationalizable set can be bounded as

$$d_H(\widehat{\mathcal{NR}}, \mathcal{NR}) \leq \mathcal{O}((L^{-1} \log L)^{\gamma/(2\gamma+1)}), \quad \gamma = k + \alpha,$$

with probability approaching one as $L \rightarrow \infty$, where $L = N \times T$ is the total number of samples available (with T auctions and N players in each). This result imposes very mild restrictions on $f(\cdot)$ to guarantee the proximity of the estimated and true rationalizable sets. For instance, if $f(\cdot)$ is Lipschitz, then $k = 0$ and $\alpha = 1$, and the result provides $\mathcal{O}((L^{-1} \log L)^{1/3})$ convergence rate for the estimated set \mathcal{NR} .

Athey and Nekipelov [2010] established that if the bids were in *ex post* Nash equilibrium, and if $f(\cdot)$ is Lipschitz, then the estimated values of bidders are $\mathcal{O}(L^{-1/3})$ apart from the true ones. Thus, the learning environment only slows down the empirical convergence by a logarithmic factor.

The subsampling algorithm adapts to the underlying smoothness of $f(\cdot)$ —which can be verified by examining the auction outcome functions $\Delta P(\cdot)$ and $\Delta C(\cdot)$ —requiring fewer samples when auction outcomes are smoother in bids.

9.2 Multi-Object Auctions

Within the IPV model, assuming the clock model of an English auction, Brendstrup and Paarsch [2007] investigated sequential sales at which two different objects were sold—when the valuations across objects for a particular bidder are potentially dependent. They demonstrated that, in general, the model is unidentified, but a model within the Archimedean family of copulas is identified. Under the copula assumption, using only observed winning bids, they proposed a semiparametric estimation strategy to recover the joint distribution of valuations and implemented their methods using data from fish auctions held in Denmark.

The policy question that interested Brendstrup and Paarsch was whether bundling is expected-revenue enhancing; they found that the condition for bundling derived by Indranil Chakraborty [1999] holds

when fewer than five potential buyers exists. At the *Grenå Fiskeauktion*, which Brendstrup and Paarsch investigated, seven bidders typically attended each auction, so they concluded that selling the objects individually is optimal.

The topic of multi-object auctions is so important that Hortaçsu and McAdams [2016] have devoted an entire survey to it. Rather than duplicate that material, we direct the interested reader to their paper.

9.3 Long-Lived, Dynamic Auctions

Most of the structural modeling of auctions has assumed that bidders compete in one-shot, static auctions. Mireia Jofre-Bonet and Pesendorfer [2003] proposed a strategy to estimate a long-lived, dynamic auction game in which winning in one period affects competitiveness in later periods, in particular if bidders have capacity constraints.

Jofre-Bonet and Pesendorfer considered a procurement setting and were still interested in recovering privately known costs based on observed bids and contract characteristics, but recognized that bidding on a given contract does not happen in a vacuum: winning the current contract ties up a share of the firm's resources for a period of time which will mean the winning firm is not going to be as competitive bidding on future projects that would also employ those workers, equipment, and materials. Still, there is potential for experience to translate into expertise meaning that winning contracts could generate lower future costs after project completion. In a repeated bidding game, capacity constraints provide a link across periods. The first-order condition involves the expected discounted sum of future profits but depends only on the distribution of firms' bids conditional on bidders' states, allowing costs to be inferred in the traditional two-stage approach so long as bidders' state variables can be inferred from the data.

Jofre-Bonet and Pesendorfer assumed that contract characteristics were drawn from a known distribution before costs were realized in a given period. A transition function dictated that progress on awarded contracts happens consistently, and evenly—linearly, over the stated

contract period. This allowed the authors to determine the current state of all bidders for any point in time given they observed who won the contract, and how long the contract was specified to take. This state specifically consists of bidder-specific variables such as the amount of work remaining on projects won, the firm's plant locations and number of plants in a given region—though really only the first of these is important temporally as the locations are fixed.

Jofre-Bonet and Pesendorfer focused on symmetric Markovian strategies in which the bidding function of a given firm depends on the firm's cost, the contract characteristics, and the states of all firms. The first-order condition for optimal bids is the following:

$$c = s - \frac{1}{\sum_{m \neq n} h(s|\mathbf{x}_0, \mathbf{x}_m, \mathbf{x}_{-m})} + \rho \sum_{m \neq n} \frac{h(s|\mathbf{x}_0, \mathbf{x}_m, \mathbf{x}_{-m})}{\sum_{\ell \neq n} h(s|\mathbf{x}_0, \mathbf{x}_\ell, \mathbf{x}_{-\ell})} (V_n[\omega(\mathbf{x}_0, \mathbf{x}, n)] - V_n[\omega(\mathbf{x}_0, \mathbf{x}, m)])$$

where ω is the deterministic transition function of the state variable \mathbf{x} , \mathbf{x}_0 is the contract characteristics, V_n is the value function evaluated before contract characteristics and n 's cost are known, h is the hazard function of bids submitted by n for a given realization of the state, and ρ is the discount factor (which the authors are not able to identify). Critical to their work is the fact that the distribution of equilibrium bids determines the discounted sum of expected future profits, meaning the value function can be represented in terms of the distribution and density of bids only.

In estimation, Jofre-Bonet and Pesendorfer specified a Weibull pdf of bids for regular bidders—the ten largest firms (in terms of dollar value of contracts won), and a Beta pdf of bids for all other bidders. Using ML estimation, they estimated the model—claiming that the regularity concerns of Donald and Paarsch [1993] are assuaged for certain values of the distributional parameters, in which case ML estimation is consistent and efficient. The parameters are assumed to be linear functions of the contract and bidders' characteristics—including their states as well as distance from the contract location and number of plants in the contract region. Importantly, Jofre-Bonet and Pesendorfer constructed a bidder-specific backlog variable representing the dollar

amount of work remaining on previously won projects, which they standardized for each bidder by subtracting the bidder-specific mean and normalizing the difference by the bidder-specific standard deviation.

Jofre-Bonet and Pesendorfer [2003] found that the likelihood of submitting a bid is decreasing in the backlog variable—consistent with the notion of capacity constraints. Likewise, increasing the number of rivals or distance to the contract location decreases the probability of tendering a bid, while the more plants a firm has in a contract region, the more likely the firm is to bid. Similarly, the effect of backlog on the value function is also negative, as expected when capacity constraints are binding. Not surprisingly, the cost distribution of a constrained bidder first-order stochastically dominates that of an unconstrained bidder, perhaps because high backlogs increase costs for bidders due to shortages of workers and equipment. They decomposed mark-ups into two parts: the first is standard and due to the competition faced on a contract, while the second, is an option value of winning today versus winning later as winning today implies constraints are realized in later periods. Because capacity constraints have real effects on bidders, these propagate through to generate higher costs for the procuring agency—in the data considered by Jofre-Bonet and Pesendorfer when the regular bidders are capacity constrained, the resulting price is about 18 percent higher than when these bidders are unconstrained (as would be assumed by static models). Jofre-Bonet and Pesendorfer concluded that the effect of capacity constraints is important and play a critical role in bidding at procurement auctions.

10

Summary, Conclusions, and Suggestions

In a review of this length, it is obviously difficult to convey the enthusiasm we have for the many facets of the structural econometrics of auctions. Had we pursued a rah-rah strategy, the reader would be simply exhausted by now. Nevertheless, we hope to have conveyed three main features that attracted us to this approach in the first place: (1) because auction prices invariably form the basis of legal contracts (either explicitly or implicitly), typical explanations of the variation in data (such a measurement error or productivity shocks) are unlikely sources of observed variation in bids, but latent, agent-specific heterogeneity across participants provides a plausible explanation of this variation, and the latent heterogeneity has a natural interpretation in the theory of mechanism design; (2) economic theory informs the specification of empirical models, which then provides the structure to interpret estimates; (3) because structural econometric models of auctions are grounded in economic theory, the Lucas critique is addressed (subject to the usual caveats), so counterfactual policy analysis through comparative institutional design can be pursued.

By and large, the structural econometrics of auctions is an identification strategy: many different distributions of latent heterogeneity

could have given rise to the observed distribution of actions, but the twin hypotheses of optimization and equilibrium often uniquely identify a distribution of heterogeneity that has an economic interpretation and can, therefore, be used to conduct counterfactual policy analysis through comparative institutional design.

Structural models rely on assumptions about bidder behavior. The biggest criticism one faces is the rationality assumption, which some of our colleagues take issue with as well. In short, rationality implies that bidders seek to maximize expected utility; understand the relationship between their bids and the likelihood of winning the auction; and can maximize their expected utility based on their beliefs. Testing whether bid functions are increasing corresponds to a test of a necessary condition for rationality. Testing sufficient conditions using observational data is, however, challenging because bidder valuations (or costs) are typically unobserved. This makes work like that of Bajari and Hortagsu [2005] especially important because the authors used data from laboratory experiments in which the distribution of valuations and each bidder's realized valuation were known. In such environments, structural econometric methods can be applied to the bidding data and treated as if the latent distribution and bidder-specific types were unknown. Bajari and Hortagsu estimated four structural models of pay-your-bid auctions; they found that a model with risk-averse bidders built on the Bayes-Nash equilibrium concept fit the data best—relative to a model with risk-neutral potential buyers, a model built on adaptive learning, and a quantal-response equilibrium model.

Kirill Chernomaz and Hisayuki Yoshimoto [2017] extended this approach to a setting with asymmetric bidders competing at pay-your-bid auctions. They found that estimated valuation distributions from a Bayes-Nash equilibrium could not be rejected as different from the true distributions, again supporting a structural approach.

In our opinions, these applications of structural models to experimental economics are invaluable—subject of course to the usual caveat that bidders in laboratory experiments are often less experienced and compete in games where the stakes are far lower than that of real-world bidders. Since structural research depends on the validity of the

underlying model, testing the theory is especially important.

In the same vein as Chapman et al., Hortag̃su and Puller [2008] employed accounting data from bidders at multi-unit auctions for electricity to compare bidder mark-ups inferred from structural econometric auction models to actual mark-ups—finding that larger firms performed close to the profit maximization benchmark.

In cases where several equilibria exist, some way to decide which equilibrium is being played, and whether that equilibrium is always played are clearly worthy, first-order important problems that must be solved before anything can be contemplated regarding the structural econometrics of auctions. In most models with endogenous entry, multiple equilibria can exist. Relaxing certain assumptions can also lead to multiple equilibria in standard auction settings; see, for example, Todd R. Kaplan and Shmuel Zamir [2015], who considered asymmetric pay-your-bid auctions. Often, multiple equilibria arise when considering asymmetric equilibria, whereas most of the structural econometric work we have discussed has focused on symmetric equilibria, where uniqueness is typical. Marmer et al. [2013] proposed a binomial test that can detect the presence of asymmetric equilibria in observational data. More research either incorporating equilibrium selection mechanisms into an econometric model, using identification strategies that allow for multiple equilibria, or employing bounds estimation strategies—as in the research of Federico Ciliberto and Tamer [2009]—appear to be promising areas for future research.

Perhaps the most fruitful area of future research in the structural econometrics of auctions involves multi-unit and multi-object auctions. Two daunting, outstanding problems exist: (1) characterizing a unique equilibrium; (2) calculating that equilibrium. Hortag̃su [2011] described some recent progress made from an empirical perspective in studying multi-unit auctions. In these environments, economic theorists have been unable to provide the same guidance as in models of single-unit auctions. Revenue rankings and allocative efficiency criteria are poorly understood in such settings, making empirical work all the more important. Some of the most economically important auctions to society involve multiple units or multiple objects—treasury securities, whole-

sale electricity, online advertising (sponsored search and display), and electromagnetic spectrum auctions to name just a few. In addition, many states and countries are contemplating selling tradable emissions permits at auction. Thus, many opportunities exist for applications, yet a real need exists for the development of both economic theory and econometric methods in such settings.

Our focus has been primarily on methodological research that has considered identification and estimation, yet we also see a real need for applications of the methods described as well. In some sense, structural research on empirical auctions has suffered from a curse: beautiful models have been constructed that capture the idiosyncrasies of a given setting so well that it is then difficult to take the model off the shelf and apply it to a new setting. Given general theoretical results are difficult to obtain, policy advice is typically case specific. Therefore, applied research, even without major methodological contributions, is an important area for future work.

One strategy for applied work is for researchers to combine multiple elements of the various topics we have discussed. For example, Balat [2017] investigated the effects of the American Recovery and Reinvestment Act on prices paid for highway construction projects in California. He developed a dynamic auction model that allows for unobserved heterogeneity at the contract level as well as endogenous participation, with an intertemporal link in firms' marginal costs. He showed that his model is identified nonparametrically and estimated that the stimulus package increased project costs for the government by over 6 percent.

As another example, Yunmi Kong [2017] used the structural approach to investigate sequential auctions of adjacent oil and gas leases in New Mexico. She levered the pairwise structure of these auctions, where one lease sold first under a pay-your-bid rule and then later in the day the adjacent (partner) lease was sold at an English auction. This structure allowed her to disentangle synergies (a bidder who won the first lease will have a higher value for the second) from affiliation (bidders have *a priori* similar values for the two leases). Kong was able to quantify the role these two concepts, which can lead to opposing policy recommendations, play—finding in counterfactual simulations

that bundling the pair of leases would increase auction revenues, but decrease allocative efficiency only slightly. Thus, combining elements of the various econometric techniques we have described is an extremely promising and valuable way to approach new applications.

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Appendices

In these appendices, we present material too cumbersome for inclusion in the text of the paper.

A.1 Permanent and Order Statistics with Different Urns

Deriving the distribution of order statistics from observations drawn from different distributions can be tedious. Thus, statisticians, borrowing from developments made by mathematicians researching group theory, have developed a way to do this that reduces errors—the permanent operator. In this appendix, we define the permanent operator, and illustrate how it can be used to find the distribution of order statistics.

To begin, consider a simple example of three independent draws, each from a different distribution $F_1(v_1)$, $F_2(v_2)$, and $F_3(v_3)$.

Initially, focus on the highest order statistic $Z = V_{(1:3)}$. In this case,

$$\begin{aligned} F_Z(z) &= \Pr[(V_1 \leq z) \cap (V_2 \leq z) \cap (V_3 \leq z)] \\ &= F_1(z)F_2(z)F_3(z), \end{aligned}$$

so

$$\begin{aligned} f_Z(z) &= \frac{dF_Z(z)}{dz} \\ &= f_1(z)F_2(z)F_3(z) + F_1(z)f_2(z)F_3(z) + F_1(z)F_2(z)f_3(z). \end{aligned}$$

Consider next the second-highest order statistic $Y = V_{(2:3)}$. For Y to be the second-highest order statistic, one of the following six, mutually exclusive events must obtain: The draw from the first distribution is highest *and* the draw from the second is in the middle, and the draw from the third is lowest *or* the draw from the second is lowest, and the draw from the third is in the middle; *or* the draw is from the second distribution is highest *and* the draw from the first is in the middle, and the draw from the third is lowest *or* the draw from the first is lowest, and the draw from the third is in the middle; *or* the draw from the third distribution is highest *and* the draw from the first is in the middle, and the draw from the second is lowest *or* the draw from the first is lowest, and the draw from the second is in the middle. Thus, for any interval $[y, y + \Delta y)$, one can write this as

$$\begin{aligned} F_Y(y + \Delta y) - F_Y(y) &= \Pr\{Y \in [y, y + \Delta y)\} = \\ &\Pr(V_1 \geq y + \Delta y) [\Pr(V_2 \leq y + \Delta y) - \Pr(V_2 \leq y)] \Pr(V_3 \leq y) + \\ &\Pr(V_1 \geq y + \Delta y) [\Pr(V_3 \leq y + \Delta y) - \Pr(V_3 \leq y)] \Pr(V_2 \leq y) + \\ &\Pr(V_2 \geq y + \Delta y) [\Pr(V_1 \leq y + \Delta y) - \Pr(V_1 \leq y)] \Pr(V_3 \leq y) + \\ &\Pr(V_2 \geq y + \Delta y) [\Pr(V_3 \leq y + \Delta y) - \Pr(V_3 \leq y)] \Pr(V_1 \leq y) + \\ &\Pr(V_3 \geq y + \Delta y) [\Pr(V_1 \leq y + \Delta y) - \Pr(V_1 \leq y)] \Pr(V_2 \leq y) + \\ &\Pr(V_3 \geq y + \Delta y) [\Pr(V_2 \leq y + \Delta y) - \Pr(V_2 \leq y)] \Pr(V_1 \leq y). \end{aligned}$$

Now,

$$f_Y(y) = \lim_{\Delta y \rightarrow 0} \frac{[F_Y(y + \Delta y) - F_Y(y)]}{\Delta y},$$

so

$$\begin{aligned} f_Y(y) &= S_1(y)f_2(y)F_3(y) + S_1(y)f_3(y)F_2(y) + \\ &\quad S_2(y)f_1(y)F_3(y) + S_2(y)f_3(y)F_1(y) + \\ &\quad S_3(y)f_1(y)F_2(y) + S_3(y)f_2(y)F_1(y), \end{aligned}$$

where $S_n(y) = [1 - F_n(y)]$.

As one can imagine, writing out all of these permutations when the number of distributions is large, for arbitrary order statistics, can be tedious, so statisticians use the permanent operator, denoted by the symbol Perm, to evaluate them.

The permanent is similar to the determinant, both of which are special cases of the immanant. Calculating the permanent requires the same calculations as the determinant, except that all the principal minors have a positive sign. An example for a (3×3) matrix is

$$\text{Perm} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei + fh) + b(di + fg) + c(dh + eg).$$

Like the determinant, which in the transformation of random variables ensures that a pdf integrates to one, the permanent is a counting device that ensures that the pdf integrates to one by keeping track of all the permutations of the cdfs and pdfs that make up the distribution of the order statistic. Hence, expanding along the third row of the matrix below,

$$\begin{aligned} f_Y(y|F_1, F_2, F_3) &= \text{Perm} \begin{bmatrix} F_1(y) & F_2(y) & F_3(y) \\ f_1(y) & f_2(y) & f_3(y) \\ S_1(y) & S_2(y) & S_3(y) \end{bmatrix} \\ &= S_1(y)f_2(y)F_3(y) + S_1(y)f_3(y)F_2(y) + \\ &\quad S_2(y)f_1(y)F_3(y) + S_2(y)f_3(y)F_1(y) + \\ &\quad S_3(y)f_1(y)F_2(y) + S_1(y)f_2(y)F_1(y), \end{aligned}$$

which is the same as what was derived above.

In the case involving the n^{th} order statistic from a sample of size N where $\{V_n\}_{n=1}^N$ of independently-distributed random variables having cdfs $\{F_n(v)\}_{n=1}^N$ and corresponding pdfs $\{f_n(v)\}_{n=1}^N$. Collect the F_n s in the vector \mathbf{F} . Again, order the sequence from smallest to largest. In general, the pdf of n^{th} highest order statistic from N independent draws $X = V_{(n:N)}$ is

$$f_X(x|\mathbf{F}) = \frac{1}{(N-n)!(n-1)!} \text{Perm} \begin{bmatrix} F_1(x) & \dots & F_N(y) \\ \vdots & \ddots & \vdots \\ F_1(x) & \dots & F_N(x) \\ f_1(x) & \dots & f_N(x) \\ [1 - F_1(x)] & \dots & [1 - F_N(x)] \\ \vdots & \ddots & \vdots \\ [1 - F_1(x)] & \dots & [1 - F_N(x)] \end{bmatrix},$$

where the matrix above, on the right-hand side of the equal sign, is $(N \times N)$, with each column corresponding to a particular type, the first $(N - n)$ rows being the cdfs of the N different types, while the last $(n - 1)$ rows being the survivor functions $[1 - F_n(y)]$ of the N different types, and the n^{th} row being the pdfs of the N types.

A.2 Extreme-Value Distribution

Consider a random variable W having pdf $f_W(w)$ and cdf $F_W(w)$ where the mean $\mathbb{E}(W)$ and variance $\mathbb{V}(W)$ both exist. In section 3.1.2, we showed that Z , the maximum of $\{W_1, W_2, \dots, W_T\}$ a sequence of T independent and identically-distributed values concerning W , has cdf

$$F_Z(z) = F_W(z)^T$$

and pdf

$$f_Z(z) = \frac{dF_Z(z)}{dz} = T F_W(z)^{T-1} f_W(z).$$

For small T , using the exact distribution can be useful, but when T is large, analysts often employ an asymptotic approximation, which is referred to as an extreme value distribution. For just as the distribution of the sum $\sum_{t=1}^T W_t$ has a limiting distribution that is independent of $F_W(w)$ as T gets large, so too does the maximum Z . For example, with the sum, one would introduce two sequences $\{a_T\}_{T=1}^\infty$ and $\{b_T\}_{T=1}^\infty$ such that

$$\frac{\sum_{t=1}^T W_t - b_T}{a_T} = \frac{\sum_{t=1}^T W_t - \sum_{t=1}^T \mathbb{E}(W_t)}{\sqrt{\sum_{t=1}^T \mathbb{V}(W_t)}} = \frac{\sum_{t=1}^T W_t - T\mathbb{E}(W)}{\sqrt{T\mathbb{V}(W)}}$$

and then demonstrate that the pdf of this expression has limit $\phi(\cdot)$, the pdf of a standard normal random variable—that is, a Gaussian random variable having mean zero and variance one. In other words, a central limit theorem.

In this appendix, we describe a similar theorem for the extrema of large samples. In the case of the maximum, one must first choose two different sequences $\{a_T\}_{T=1}^\infty$ and $\{b_T\}_{T=1}^\infty$, and then demonstrate that the statistic

$$E = \frac{Z - b_T}{a_T}$$

has a cdf that converges to

$$G(e|\mu, \tau, \xi) = \exp \left(- \left[1 + \xi \left(\frac{e - \mu}{\tau} \right) \right]^{-1/\xi} \right),$$

where μ and τ denote the location and scale of E . The above is referred to as the generalized extreme-value distribution. In other words, an extreme value theorem.

Depending on the value of ξ , $G(e|\mu, \tau, \xi)$ has three limiting distributions: the Weibull, $\xi < 0$; the Gumbel, $\xi = 0$; and the Fréchet, $\xi > 0$. Which of the three distributions obtains depends on the features of the underlying $F_W(w)$. If the pdf $f_W(w)$ is strictly positive at the upper bound of support (as in the case of the winning bid at a pay-your-bid auction, where the upper bound of support is \bar{s}), then the limiting distribution is exponential, a special case of the Weibull.

A.3 Approximation Methods

Sometimes, one is interested in approximating a function $f^0(x)$ using a simpler function $\hat{f}(x)$. In this appendix, we describe some methods used to approximate functions.

Under certain conditions, $f^0(x)$ can be represented exactly by a Taylor series expansion of the following form:

$$f^0(x) = f^0(x_0) + \sum_{j=1}^{\infty} \frac{d^{(j)} f^0(x_0)}{dx^j} \frac{(x - x_0)^j}{j!}.$$

This polynomial is troubling in practice: how do you represent ∞ on a digital computer? You can't, so truncation errors will occur. Also, in many instances, the derivatives of $f^0(x)$ are unknown, because $f^0(x)$ is unknown, and needs to be estimated. What to do?

Suppose one is willing to truncate $f^0(x)$ at some high order K , so

$$f^0(x) = f^0(x_0) + \sum_{k=1}^K \frac{d^{(k)} f^0(x_0)}{dx^k} \frac{(x - x_0)^k}{k!} + U(x_0).$$

Suppose the values of $f^0(x)$ are known at a finite set of $(K + 1)$ points (x_1, \dots, x_{K+1}) ; denote them $y_k = f^0(x_k)$. One can estimate these derivatives according to the following linear system:

$$\begin{aligned}
y_1 &= \alpha_0 + \alpha_1(x_1 - x_0) + \alpha_2(x_1 - x_0)^2 \dots + \alpha_K(x_1 - x_0)^K + U_1 \\
y_2 &= \alpha_0 + \alpha_1(x_2 - x_0) + \alpha_2(x_2 - x_0)^2 \dots + \alpha_K(x_2 - x_0)^K + U_2 \\
&\vdots = \vdots \\
y_{K+1} &= \alpha_0 + \alpha_1(x_{K+1} - x_0) + \alpha_2(x_{K+1} - x_0)^2 \dots + \alpha_K(x_{K+1} - x_0)^K + U_{K+1}
\end{aligned}$$

or, in matrix notation,

$$\mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{U}$$

so

$$\hat{\boldsymbol{\alpha}} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}.$$

Suppose x_0 is zero. A matrix of the following form:

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^K \\ 1 & x_2 & x_2^2 & \dots & x_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{K+1} & x_{K+1}^2 & \dots & x_{K+1}^K \end{bmatrix}$$

is referred to as a Vandermonde matrix. When K is quite large, solving for $\hat{\boldsymbol{\alpha}}$ is a delicate problem numerically because the matrix $(\mathbf{X}^\top \mathbf{X})$ will be ill-conditioned. This occurs because the column of x_k^j s and the column of x_k^{j+1} s are almost collinear—that is, nearly linear combinations of one another.

What numerical analysts suggest is that one use an alternative basis, and replace the x_k^j s by orthogonal polynomials. A variety of different orthogonal bases exist. For example, when x is contained on $[-1, 1]$ Chebyshev polynomials are proposed, whereas when x is contained on $[0, \infty)$ Laguerre polynomials are proposed, and when x is contained on $(-\infty, \infty)$ Hermite polynomials are proposed. In general, one approximates $f^0(x)$ by a polynomial

$$\hat{f}_K(x) = \sum_{k=0}^K \alpha_k \mathbb{P}_k(x).$$

Typically, the polynomial terms $\{\mathbb{P}_k(x)\}_{k=0}^\infty$ can be constructed recursively. For example, in the case of Chebyshev polynomials $\mathbb{T}_0(x)$ is one, whereas $\mathbb{T}_1(x)$ is x . Subsequently,

$$\mathbb{T}_{k+1}(x) = 2x\mathbb{T}_k(x) - \mathbb{T}_{k-1}(x) \quad k = 1, 2, \dots$$

In the case of Laguerre polynomials $\mathbb{L}_0(x)$ is one, whereas $\mathbb{L}_1(x)$ is $(1 - x)$, and

$$\mathbb{L}_{k+1}(x) = \frac{1}{k+1}(2k+1-x)\mathbb{L}_k(x) - \frac{k}{k+1}\mathbb{L}_{k-1}(x) \quad k = 1, 2, \dots$$

In the case of Hermite polynomials, $\mathbb{H}_0(x)$ is one, whereas $\mathbb{H}_1(x)$ is $2x$ and

$$\mathbb{H}_{k+1}(x) = 2x\mathbb{H}_k(x) - 2k\mathbb{H}_{k-1}(x) \quad k = 1, 2, \dots$$

A.4 B-Splines

In this appendix, we describe methods of fitting a continuous nonparametric function $f : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}$ with an O -order B-spline, $O \in \mathbb{N}$. B-splines can be computed using different basis functions; Hickman et al. [2017] used the Cox-de Boor recursion formula. This formula requires one to pre-specify a knot vector $\mathbf{k}_K = \{k_1 < k_2 < \dots < k_K < k_{K+1}\}$ that partitions the domain into K subintervals, with $k_1 = \underline{x}$ and $k_{K+1} = \bar{x}$. Given knot vector \mathbf{k}_K and order O , one seeks to compute a set of $(K + O - 1)$ basis functions of degree $D \equiv (O - 1)$, denoted $\mathbb{B}_{\mathbf{k}_K, D} = \{\mathcal{B}_{i,D} : [\underline{x}, \bar{x}] \rightarrow \mathbb{R}, i = 1, \dots, K + D\}$. One begins by computing an extended knot vector with D additional copies of the endpoints added in,

$$\bar{\mathbf{k}}_K = \{k_{1-D} = \dots = k_1 < k_2 < \dots < k_K < k_{K+1} = \dots = k_{K+1+D}\}.$$

In order to define basis $\mathbb{B}_{\mathbf{k}_K, D}$, the Cox-de Boor recursion formula requires that one must first compute all lower-order bases $\mathbb{B}_{\mathbf{k}_K, d}$ for each $d = 0, \dots, D$. The recursion begins by defining the zero-degree basis functions as simply piecewise constant—that is, for each $i = 1, \dots, K$

$$\mathcal{B}_{i,0}(x) = \begin{cases} 1 & \text{if } x \in [k_i, k_{i+1}) \cup \bar{x}, \\ 0 & \text{otherwise.} \end{cases}$$

From the above formula, note that exactly K exist. One can now define,

for each $d = 1, \dots, D$ and for each $i = 1, \dots, K + d$,

$$\mathcal{B}_{i,d+1}(x) = \begin{cases} \mathcal{B}_{i,d}(x) \left(\frac{x-k_{i-d}}{k_i-k_{i-d}} \right) + \mathcal{B}_{i+1,d}(x) \left(\frac{k_{i-d+1}-x}{k_{i+1}-k_{i-d+1}} \right) & \text{if } (k_i - k_{i-d})(k_{i+1} - k_{i-d+1}) \neq 0, \\ \mathcal{B}_{i,d}(x) \left(\frac{x-k_{i-d}}{k_i-k_{i-d}} \right) & \text{if } (k_{i+1} - k_{i-d+1}) = 0, \\ \mathcal{B}_{i+1,d}(x) \left(\frac{k_{i-d+1}-x}{k_{i+1}-k_{i-d+1}} \right) & \text{if } (k_i - k_{i-d}) = 0 \end{cases}$$

From this recurrence relation, one can see that the order 2 (degree 1) basis $\mathbb{B}_{\mathbf{k}_K,1}$ results in a set of $(K + 1)$ piecewise linear functions. Similarly, the order 3 (degree 2) basis $\mathbb{B}_{\mathbf{k}_K,2}$ results in a set of $(K + 2)$ piecewise quadratic functions, and the order 4 (degree 3) basis $\mathbb{B}_{\mathbf{k}_K,3}$ results in a set of $(K + 3)$ piecewise cubic functions.

Several other properties of the basis functions are also worth noting: first, each basis function is globally defined, but any given function in $\mathbb{B}_{\mathbf{k}_K,D}$ is nonzero on at most $O = (D + 1)$ subintervals defined by \mathbf{k}_K ; second, on each subinterval, exactly O of the $(K + D)$ basis functions are nonzero; and, third, since B-spline functions are linear combinations of B-spline bases, their derivatives are also linear combinations of B-spline basis functions.

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