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Dynamic Models of Auctions

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Notes on Donald et alias (2006)

Introduction and Motivation

• Auctions at which only one object is sold to potential buyers demanding but one unit have been studied extensively.

• In reality, however, many auctions involve the sale of multiple units of the same good to buyers who may demand several units, or many different objects.

• Hence, economists make the distinction between multi-unit and multi-object auctions.

• A multi-unit auction involves the sale of, say, several identical oranges; a multi-object auction involves the the sale of, say, an apple, an orange, and a pear.

• At multi-unit auctions it is typically reasonable to assume non-increasing marginal utility across units, which provides some extra structure that can be levered in the analysis.

• Understanding how such changes in the economic environment affect bidding behaviour as well as the process of price formation are open research questions.

Weber's Classification of Multiple-Object Auctions

• *Simultaneous-dependent* auctions at which bidders are required to take a single action that determines both the allocation of the objects and the payments to the seller.

• Weekly auctions of U.S. Treasury bills are an example of this type of multiple-object auction.

• A special case of the simultaneous-dependent auction, viewed as distinct by Weber, is the *simultaneous-independent* auction at which the sale of one object does not depend upon the outcome of other sales.

• Sales of mineral rights such as oil and gas leases are good examples of such auctions.

• The final category described by Weber is the *sequential* auction where one item (or lot containing several of the same item) is sold at a time.

• Art, stamps, and coins as well as cattle, fish, flowers, vegetables, and timber are often sold at sequential auctions.

Some Results in the Literature

• Within the independent private-values paradigm, when buyers are risk neutral and demand but one unit, Weber (1983) has shown that the winning price path should follow a martingale.

• Within the affiliated private-values paradigm and assuming risk aversion, Laffont, Loisel, and Robert (1994), who examined descending-price auctions, have derived inverted U-shaped price pathes for the expected winning price.

• Within the common-value paradigm, McAfee and Vincent (1992) have shown that the expected winning price path should decline across successive lots.

• McAfee and Vincent (1992) have also shown that the equilibrium strategy is a mixed one, which means that the auction may not yield efficient allocations.

What Do We Do?

• We investigate strategic behaviour of bidders with multi-unit demand at sequential, oral, ascending-price, open-exit auctions within the independent private-values paradigm.

• We confront this model with data from a specific sequential, oral, ascending-price, open-exit auction of timber export permits in Siberia, Russia.

What is Our Contribution?

• If potential buyers desire more than one unit, then our model predicts that the winning prices ought to increase, on average.

• This occurs even though the units are identical, the auction induces competition, and the agents are risk neutral.

• Reduced-form econometric evidence is consistent with this finding.

Sales of Timber Export Permits in Siberia: Demand

- In the Krasnoyarsk Region of Siberia, the Russian government regulates timber exports abroad by selling export permits at auction.
- The potential bidders in the market for timber export permits are the firms harvesting timber in the Krasnoyarsk Region.
- To export timber, firms require a foreign buyer and an export permit.
- But without a buyer, an export permit has no value since such permits are non-transferable; that is, no secondary market in timber export permits exists.
- Moreover, an export permit will expire if not used within a fixed period of time, typically less than one year.

Sales of Timber Export Permits in Siberia: Supply

- The supply of timber export permits is set by the Ministry of the Krasnoyarsk Region.
- By the Wednesday of the week prior to a particular auction (which is typically held on either a Tuesday or a Thursday), potential exporters of timber from the Krasnoyarsk Region are invited to submit requests for timber export permits to the Ministry of the Region.
- While each potential exporter knows the volume of his request, he is ignorant of the total number of requests as well as the total volume of requests.
- On the following Tuesday or Thursday, the Ministry of the Region allocates for sale at auction a volume of timber export permits in lots of between 180 and 5000 cubic metres of timber of a particular type.

Oral, Ascending-Price, Open-Exit Auctions

- Potential bidders are required to pay a fee of 5000R (about $3 US at the time the data we use below were collected) to attend the auction.
- In addition, for each lot a reserve price per cubic metre exported is announced.
- The sale of a particular lot begins with the auctioneer calling out the lot number, describing the export permit, and then asking interested bidders to hold up the white cards they were issued when they paid the entry fee.
- By holding up a white card, a participant signifies his willingness to pay the reserve price of the export permit.
- As the price rises, bidders signify that they are no longer in the sale by dropping their white cards.
- The winner is the bidder who holds up his white card the longest; he pays the price at which his last opponent dropped his white card.

A Model of Sequential, Oral, Ascending-Price,
Open-Exit Auction with Multi-Unit Demand

• Consider the case of a seller who wants to sell T lots of a homogeneous good through a sequence of T oral, ascending-price, open-exit sales.

• Assume the absence of a reserve price.

• Each buyer holds up a card to signal his willingness to buy at the current price.

• As the price rises, bidders lower their cards and forgo the possibility of buying the lot currently on sale.

• When all but one of the bidders have withdrawn, the price stops rising and the the lot is awarded to the last remaining bidders.

• We assume that there are $\mathcal{N}$ potential bidders who may bid for the lots for sale.

• Some of the $\mathcal{N}$ may not bid.

• In fact, the seller faces a random number of participants N, each having some private values of the lots on sale.

• Potential buyers are risk neutral.

• The valuations for some participant i takes the form of a vector

$$\mathbf{W}^i = \{W_1^i, W_2^i, \ldots, W_{m_i}^i, 0, 0, \ldots\}$$

where W_j^i represents participants i's valuation of his j^{th} unit of the good, and m_i denotes the number of positive valuations for participant i.

• We assume that orders arrive according to a Poisson process having intensity parameter λ.

• The number of positive valuations drawn by each potential bidder M is a random variable that is distributed Poisson with mean λ and probability mass function

$$\Pr[M = m] = \frac{\lambda^m \exp(-\lambda)}{m!} \quad m = 0, 1, \ldots.$$

• When a potential bidder receives no orders ($M = 0$), we assume that he does not attend the auction, and hence signals that he will not bid.

• The number of participants N is distributed binomially with parameters $\mathcal{N}$ and $(1 - \Pr[M = 0]) = [1 - \exp(-\lambda)]$. Thus,

$$\Pr[N = n] = \binom{\mathcal{N}}{n}[1 - \exp(-\lambda)]^n \exp(-\lambda)^{\mathcal{N}-n}.$$

• We also assume that each valuation W is an independently- and identically-distributed draw from the cumulative distribution function $F(w)$.

• Thus, for potential buyer i, each of the m_i draws ranked in descending order represents another business opportunity, the value of which is the profit associated with an extra unit purchased.

• The vector $\mathbf{W}^i$ and the number of positive valuation m_i are assumed to be the private information of participant i.

• We assume that the actual number of participants n, the cumulative distribution function $F(w)$, and the intensity parameter of the Poisson process λ are common knowledge.

• The state of nature or realization of types can be represented by a vector of descending valuations which ranks valuations of all n participants for all lots

$$\mathbf{W} = \{W_1, W_2, \ldots, W_t, \ldots, W_{\sum_{i=1}^{n} m_i}\}$$

where $W_1 \geq W_2 \geq \ldots \geq W_t \geq \ldots \geq W_{\sum_{i=1}^{n} m_i}$.

• Here, W_t denotes the t^{th} highest valuation among all n participants.

• The expression $W_t = W_2^i$ means that participant i's second highest valuation is the t^{th} highest valuation overall.

- Given the above assumptions, the following properties hold:

- **A:** For all $j < k$,

$$\Pr[W_k^i \leq y | W_j^i, W_{j-1}^i, \ldots, W_1^i] = \Pr[W_k^i \leq y | W_j^i].$$

- **B:** For all $j < k$, and i, $\ell \in \mathbf{N} = \{1, 2, \ldots, \mathcal{N}\}$

$$\Pr[W_k^i \leq y | W_j^i = x] = \Pr[W_k^\ell \leq y | W_j^\ell = x] \qquad \textbf{symmetry}.$$

- **C:** For all $i \in \mathbf{N}$ and for all $j < k$,

$$\Pr[W_k^i \leq y | W_j^i = x] = \Pr[W_{k-j+1}^i \leq y | W_1^i = x].$$

• Property $\mathbf{A}$ follows directly from the assumption of independent draws and the properties of order statistics.

• Property $\mathbf{B}$ follows directly from the assumption of symmetry.

• Property $\mathbf{C}$ is more involved: It states that the conditional distribution is invariant to a re-indexing of the order statistics.

- Given this notation, properties $\mathbf{B}$ and $\mathbf{C}$ imply

- $\mathbf{D}$**:** For all $j < k$, i, $\ell \in \mathbf{N}$, and after all histories $h(t)$,

$$\Pr[V_k^i(h(t)) \leq y | V_j^i(h(t)) = x] = \Pr[V_k^\ell(h(t)) \leq y | V_j^\ell(h(t)) = x]$$

robust symmetry.

- $\mathbf{E}$**:** For all $j < k$, $i \in \mathbf{N}$, the function

$$G(y|x) \equiv \Pr[W_k^i \leq y | W_j^i = x]$$

is strictly decreasing in x for all $y < x$ such that $F(y) > 0$.

Theorem 4.1:

For all lots t and all vectors $\mathbf{V}^i(h(t))$, the following characterizes a strategic equilibrium:

(i) Whenever $t \geq n$, each bidder i remains active until all other bidders stop, or the price is equal to or is above:

$$\mathbf{E}[V_t^{-i} | V_2^j = V_2^i \ \forall \ j \in \mathbf{R}, \ \hat{\mathbf{V}}_2^S, V_1^k \geq V_2^i \ \forall \ k \in \mathbf{N_w} \cap \mathbf{S}]$$

where V_2's correspond to the bidders' second highest valuation after re-indexing, and $\hat{V}_2^S$ corresponds to the vector of second valuations for all bidders in $\mathbf{S}$ is which consistent with the prices at which they have withdrawn.

(ii) Whenever $t < n$, then the participants first bid up to their first (highest) value, until $t = r$. Afterward, each remaining bidder i remains active until all other bidders stop, or the price is equal to or is above:

$$\mathbf{E}[V_t^{-i} | V_2^j = V_2^i \ \forall \ j \in \mathbf{R}, \ \hat{\mathbf{V}}_2^{S_2}, \ \hat{\mathbf{V}}_1^{S_1}, \ V_1^k \geq V_2^i \ \forall \ k \in \mathbf{N_w} \cap \mathbf{S_2}]$$

where $\mathbf{S_1}$ is the subset of $n - t$ bidders who first withdraw, and $\mathbf{S_2}$ the subset of bidders who withdraw later.

Theorem 4.2:

(i) If t exceeds the number of active participants r and at least two participants are still bidding, then their bids are strictly monotonic and symmetric functions of their second (re-indexed) valuations.

(ii) The allocation induced by the equilibrium is efficient; that is, the T lots are allocated to the buyers with the highest valuations.

(iii) If all other participants follow their equilibrium strategy, then in order to win sale t participant i must pay the price $\mathbf{E}[W_{T-\ell_i}^{-i} | \Omega_t]$ where ℓ_i is the number of lots already won by participant i and Ω_t is the information available to participants at the end of lot t. Moreover, participant i needs to pay exactly $W_{T-\ell_i}^{-i}$ in order to win when $n > t$ and the highest second valuation among all other participants is below their t^{th} highest first valuation.

Theorem 5.1:

(i) The winning prices form a super-martingale

$$\mathbf{E}[(P_{t-1} - P_t)|P_t] > 0.$$

(ii) The expected difference $\mathbf{E}[(P_{t-1} - P_t)|P_t]$ is greater when the winner of sale t wins more than two extra lots after sale t and even greater when he wins sale $(t-1)$.

Data

• For a sample of 37 auctions held between May 1993 and May 1994 and involving 308 lots, we have the winning prices for each lot, the identities of the winning bidders, and the number of participants present.

• Thus, for example, for the lots sold at an auction held on April 5, 1994, we have the following information:

Date	Lot No.	Product	Volume	Reserve Price	No. of Bidders	Winning Bid	Winner's Id. No.
940405	1	4403	1000	2500	7	2800	73
940405	2	4403	1000	2500	7	2700	73
940405	3	4403	500	2500	7	2700	73
940405	4	4403	250	2500	7	2800	45

Structural Econometric Framework

• Given a sample of J auctions, at which $\{T_j\}_{j=1}^J$ lots are sold to $\{n_j\}_{j=1}^J$ participants for the winning prices $\{\{p_t^j\}_{t=1}^{T_j}\}_{j=1}^J$, one strategy to estimate the structural parameters $\underline{\theta} = [\lambda, \underline{\alpha}^\top]^\top$ would be to choose values for $\underline{\theta}$ which minimize the sum of squared residuals

$$S(\underline{\theta}) = \sum_{j=1}^{J} \sum_{t=1}^{T_j} \left(p_t^j - \mathbf{E}[P_t^j | \Omega_t^j]\right)^2 + \sum_{j=1}^{J} \left(n_j - \mathbf{E}[N_j]\right)^2.$$

Key Insight in the Estimation

• Unfortunately, even though the regression function $\mathbf{E}[N_j]$ does have a known functional form, the regression function $\mathbf{E}[P_t^j|\Omega_t^j]$ does not.

• Even though solving for the equilibrium of this auction is complicated, we know the outcome is efficient.

• The outcome at a generalized Vickrey auction (GVA) is also efficient.

• Thus, regardless of the estimation strategy, at any candidate parameter vector $\underline{\theta}$, we simulate the average winning bid and the average number of participant, and then take a step.

• In other words, we estimate the regression function for P_t^j using simulation methods. Our strategy is described below

• First, we fix the unknown parameter vector $\underline{\theta}$ at $\underline{\theta}^0$. Using a modification of the binomial formula discussed earlier.

• In particular, the probability mass function for N_j is now

$$\Pr[N_j = n; \underline{\theta}] = \binom{\mathcal{N}}{n} \Big[1 - \exp\Big(-\lambda[1 - F(r_j; \underline{\alpha})]\Big)\Big]^n \exp\Big(-\lambda[1 - F(r_j; \underline{\alpha})]\Big)^{\mathcal{N}-n}.$$

which includes the fact that valuations are truncated due to the presence of reserve price r_j at auction j, we calculate $\mathbf{E}[N_j]$, so the expected number of participants for auction j,

$$\mathbf{E}[N_j; \underline{\theta}] = \mathcal{N}\Big[1 - \exp\Big(-\lambda[1 - F(r_j; \underline{\alpha})]\Big)\Big].$$

• Second, for each of S simulations samples, and for each of the J auctions at which $\{T_j\}_{j=1}^J$ lots have been sold to $\{n_j\}_{j=1}^J$ participants, we simulate the winning price for lot t at auction j using the result of Theorem 4.2(iii) and the corresponding dominant-strategy implementation of the equilibrium.

• For each sample, we generate a list of valuations for all n_j participants. Then we use these valuations to determine the list and order of the winners. Following Theorem 4.1, when $t \geq n_j$ the winner is the bidder with the highest re-indexed second value; when $t < n_j$ the winner is the bidder with the highest re-indexed second value among all those whose first value is among the t^{th} highest ones. Once the winner of auction t is determined, say bidder i, we let $p_t^{js} = W_{T-\ell_i}^{-i}$, where ℓ_i is the number of units assigned to i prior to auction t.

• For example, consider the case where $T = 4$, $n = 3$ and where $\mathbf{W}^1 = \{5, 4, 2\}$, $\mathbf{W}^2 = \{6, 3, 1\}$ and $\mathbf{W}^3 = \{4.5, 2.5\}$. The list and order of winners is given by $\{1, 2, 3, 1\}$. For instance, in the third auction, the re-indexed values are given by: $\mathbf{V}^1 = \{4, 2\}$, $\mathbf{V}^2 = \{3, 1\}$ and $\mathbf{W}^3 = \{4.5, 2.5\}$; players bid upto their first value until player 2 drops out at price $p = 3$; then 3 and 1 drops out since their second value is less than 3 which will be the winning price for the last two sales. The simulated winning prices will be given by: $\{W_4^{-1}, W_4^{-2}, W_4^{-3}, W_3^{-1}\} = \{2.5, 2.5, 3, 3\}$.

- Third, we estimate the regression function $\mathbf{E}[P_t^j|\Omega_t^j]$ by the average of the S simulated winning prices $\{p_t^{js}(\underline{\theta}^0; n_j, T_j)\}_{s=1}^S$

$$\bar{p}_t^j(\underline{\theta}^0; n_j, T_j) = \frac{\sum_{s=1}^S p_t^{js}(\underline{\theta}^0; n_j, T_j)}{S}.$$

Finally, we take a step $\mathbf{d}\underline{\theta}$ to improve

$$\hat{S}(\underline{\theta}) = \sum_{j=1}^J \sum_{t=1}^{T_j} \left(p_t^j - \bar{p}_t^j(\underline{\theta}; n_j, T_j)\right)^2 + \sum_{j=1}^J \left(n_j - \mathbf{E}[N_j; \underline{\theta}]\right)^2.$$