Identifying Present-Biased Discount Functions in Dynamic Discrete Choice Models*

Jaap H. Abbring[†] Øystein Daljord[‡] Fedor Iskhakov[§]

August 2, 2019

Abstract

We derive conditions for identification of sophisticated, quasi-hyperbolic time preferences in a finite horizon, dynamic discrete choice model under a set of economically motivated exclusion restrictions. Identification is reduced to characterizing of the zero set of two bivariate polynomial moment conditions. The number of discount function parameters in the identified set is bounded by known features of the data distribution. We show that, though the discount function parameters are formally identified, it is hard to precisely estimate these parameters separately.

*Jaap Abbring's research is financially supported by the Netherlands Organisation for Scientific Research (NWO) through Vici grant 453-11-002. Thanks to Hanming Fang for helpful discussions.

[†]CentER, Department of Econometrics & OR, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands. E-mail: jaap@abbring.org. Web: jaap.abbring.org

[‡]Booth School of Business, University of Chicago, 5807 South Woodlawn Avenue, Chicago, IL 60637, USA. E-mail: Oeystein.Daljord@chicagobooth.edu. Web: faculty.chicagobooth.edu/oystein.daljord

[§]1018 HW Arndt building The Australian National University, Canberra, ACT 0200, Australia E-mail: fedor.iskhakov@anu.edu.au. Web: http://fedor.iskh.me/

Keywords: discount factor, dynamic discrete choice, empirical content, hyperbolic discounting, identification, present bias.
JEL codes: C14, C25, C61.

1 Introduction

The empirical literature on present-biased time preferences is motivated by evidence of preference reversals (e.g. Frederick et al., 2002). In Thaler's (1981) classic example, subjects who prefer one apple today to two apples tomorrow tend to prefer two apples one year and one day from now to one apple one year from now. Such preference reversals are the defining feature of time-inconsistent preferences. Observed preference reversals are direct evidence of present-bias and would be at the core of an empirical identification strategy.

The literature distinguishes between *naive* and *sophisticated* present-bias (e.g. O'Donoghue and Rabin, 1999). Naive agents are not aware of their present-bias and consistently make present-biased decisions, while believing they will make time-consistent choices in the future. Sophisticated agents are fully aware of their present-bias and make savings choices strategically taking their future present-biased choices into account.[1]

Demonstrated demand for commitment devices has been taken as evidence of sophisticated present-bias, e.g. Malmendier and DellaVigna (2006). An agent who is at least partly aware of her own present-bias may look for commitment devices to achieve self control. For instance, a sophisticated agent may want to lock in her savings to avoid excessive spending by future present-biased selves, that is, to restrict her future choice sets without receiving a current period pay-off. Demand for commitment can be viewed as a response to anticipated preference reversals. We give conditions under which sophisticated present bias discount function parameters can be recovered from choice data.

There is a wealth of empirical lab studies of present biased discount functions in general and $\beta\delta$ discount functions in particular, see Urminsky and Zauberman (2015) for a survey. There are to our knowledge only three studies that estimate $\beta\delta$ discount functions from choice data. Mahajan and Tarozzi (2011) uses survey evidence on beliefs and preferences along with evidence of demand for commitment devices to identify partially naive discount functions in a non-stationary DDC model. The lack of evidence from choice data is partly because the discount factor is non-parametrically underidentified in the standard DDC model. Fang and Wang (2015) proposed a proof of identification for hyperbolic discount functions. Chan (2017) estimates a DDC model of welfare benefit choices with a $\beta\delta$ which uses the identification argument of Fang and Wang (2015). Abbring and Daljord (2019a) showed that its main identification claim is void— that is, has no implications for identification of the dynamic discrete choice model— and its main proof of identification incorrect and incomplete. Identification of sophisticated $\beta\delta$ preferences has therefore been an open question that we address in this paper.[2] We have not been able to establish identification for the partial naive case, which remains

[1] O'Donoghue and Rabin (1999) characterized *partial naivite*, an intermediate case where an agent may not be fully aware of her own present-bias.

[2] We emphasize that we do not believe that the incorrect results in Fang and Wang invalidates the results in Chan. On

an open question.

We consider the joint identification of a non-parametric utility function and a sophisticated $\beta\delta$ discount function using the behavioural model of Fang and Wang (2015).[3] Our identification argument relies on interpretable exclusion restrictions on primitive utilities, similar to the ones used by Chan, and follows the identification argument for time consistent preferences in Abbring and Daljord (2019b) closely. We show that the discount function parameters are formally identified as the zero set of a bivariate set of polynomial moment conditions. The number of discount function parameters in the identified set is finite and bounded above by known features of the data. The identification of the discount function parameters are shown to imply the identification of a non-parametric, normalized pay-off function, similar to the analogous results in Magnac and Thesmar (2002) for the discount factor.

After showing that the parameters are formally identified, we note that in practice, it is hard to separate the discount function parameters, i.e. in finite samples. Though the product $\beta\delta$ can be estimated to a high degree of precision, we show that due to how the parameters enter largely interchangeably in the identifying moment conditions, the parameters are likely to be estimated individually at low levels of precision. We find that the exclusion restriction approach that is effective in recovering a discount factor does not reflect an experimental design that is well suited to recover $\beta\delta$ preferences since it is not well-designed to elicit preference reversals. We show finally that a more modest goal of testing for the existence of present bias, i.e. for $\beta < 1$, is however feasible using exclusion restrictions. We illustrate these features theoretically and in a simulation.

2 Model

We study a finite horizon dynamic discrete choice model in which agents may suffer from sophisticated present-bias. Our model is similar to Fang and Wang's (2015), but is nonstationary, and assumes that agents are sophisticated. We discuss infinite horizon, stationary models and naïve agents in the appendices.

2.1 Primitives

Time is indexed by $t = 1, \ldots, T$; with $T < \infty$. In each period t, the agent chooses an action d_t from a finite set $\mathcal{D} = \{1, \ldots, K\}$. Prior to making this choice, the agent draws and observes vectors of state variables x_t and $\epsilon_t = \{\epsilon_{1,t}, \ldots, \epsilon_{K,t}\}$. The observable (to the econometrician) states x_t have finite support $\mathcal{X}$ and evolve as a controlled (by d_t) first order Markov process. For notational simplicity

the contrary, we think our results confirm that the model in Chan is formally identified.

[3]Though Abbring and Daljord (2019a) shows that Fang and Wang's identification results are incorrect, its empirical model of present biased preferences is very useful.

only, we take this process to be stationary, with Markov transition distribution Q_k if $k \in \mathcal{D}$ is chosen. The utility shocks $\epsilon_{k,t}$ are independent from x_t and prior states and choices, over time, and across choices, and have type 1 extreme value distributions.[4]

If, in period t and state x, the agent chooses k, she collects a flow of utility $u_{k,t}(x) + \epsilon_{k,t}$. We normalize $u_{K,t}(x) = 0$ for all $t \in 1, \dots, T$ and $x \in \mathcal{X}$. This normalization is substantive, but is standard in the literature and cannot be rejected by the type of observational data on choices and states that we will assume available in this paper.[5]

The agent's discount function has two parameters: a non-negative and finite *standard discount factor* δ and a *present-bias parameter* $\beta \in (0, 1]$. Since the horizon is finite, we do not require that the discount factor δ is smaller than one. If $\beta = 1$, the model reduces to one with standard geometric discounting. The present-bias parameter is bounded away from zero to distinguish present-bias from myopia.

2.2 Choices

Choices in dynamic discrete choice models are regulated by value functions. Since present-biased time preferences are time inconsistent, these value functions do not follow from a standard dynamic program. It is common to think about the values as summarizing the pay-offs to players in a Stackelberg-like game played between selves in different time periods (e.g. Elster, 1985).

Let $\tilde{\sigma}_t : \mathcal{X} \times \mathbb{R}^K \to \mathcal{D}$ be an arbitrary choice strategy and $\tilde{\boldsymbol{\sigma}}_t = \{\tilde{\sigma}_\tau\}_{\tau=t}^T$ an arbitrary strategy profile. The agent's *current choice specific value function*, which regulates the choices, is

$$w_{k,t}(x; \tilde{\boldsymbol{\sigma}}_{t+1}) = u_{k,t}(x) + \beta\delta \int v_{t+1}(x'; \tilde{\boldsymbol{\sigma}}_{t+1}) dQ_k(x'|x) \tag{1}$$

for $t < T$, with terminal value $w_{k,T}(x) = u_{k,T}(x)$. The agent trades off current utility versus future values by factor $\beta\delta$, but the stream of all future utilities are discounted geometrically by factor δ according to the *perceived long run value function*, which equals

$$\begin{aligned} v_{t+1}(x; \tilde{\boldsymbol{\sigma}}_{t+1}) = \mathbb{E}_{\epsilon_{t+1}} \Big[& u_{\tilde{\sigma}_{t+1}(x,\epsilon_{t+1}),t+1}(x) + \epsilon_{\tilde{\sigma}_{t+1}(x,\epsilon_{t+1}),t+1} \\ & + \delta \int v_{t+2}(x'; \tilde{\boldsymbol{\sigma}}_{t+2}) dQ_{\tilde{\sigma}_{t+1}(x,\epsilon_{t+1})}(x'|x) \Big] \end{aligned} \tag{2}$$

for $t + 1 < T$, with terminal value $v_T(x; \tilde{\boldsymbol{\sigma}}_T) = \mathbb{E}_{\epsilon_T} \left[u_{\tilde{\sigma}_T(x,\epsilon_T),T}(x) + \epsilon_{\tilde{\sigma}_T(x,\epsilon_T),T} \right]$.

[4] Our analysis straightforwardly extends to the case in which the vectors ϵ_t are independent over time, with *known* continuous distributions G_t on a common support $\mathbb{R}^K$. Note that the exact choice of G_t, for $t = 1, \dots, T$, does not impose testable restrictions on the type of data that we assume are available in this paper.

[5] The way $u_{K,t}$ is normalized affects the model's implied behavioural responses to many, but not all, counterfactual interventions (e.g. Norets and Tang, 2014; Aguirregabiria and Suzuki, 2014; Kalouptsidi et al., 2016).

The perceived long run value depends on the current self's perceptions of its future selves' strategies $\tilde{\sigma}_{t+1}$. At the time of decision, each of these future selves have present-biased preferences which are in conflict with the current self's time consistent long run time preferences.

Since the agent is sophisticated, her perceptions of her future strategies are correct in equilibrium. Thus, in a sophisticated intrapersonal equilibrium, her selves use a *perception perfect strategy* (O'Donoghue and Rabin, 1999), which is a strategy profile $\boldsymbol{\sigma}_1^*$ such that each σ_t^* is a best response to her perceived future strategy profile $\boldsymbol{\sigma}_{t+1}^*$:

$$\sigma_t^*(x, \epsilon_t) = \arg\max_{k \in \mathcal{D}} \{w_{k,t}(x; \boldsymbol{\sigma}_{t+1}^*) + \epsilon_{k,t}\}. \tag{3}$$

Here, $w_{k,T}(x; \boldsymbol{\sigma}_{T+1}^*)$ should be read as $w_{k,T}(x)$.

It is easy to show, by backward induction from time T, that a perception perfect strategy exists and is unique (up to the resolution of ties in the decision in (3)).

3 Identification

For given primitives $Q_1, \ldots, Q_K$; β; δ; and $u_{1,t}, \ldots, u_{K-1,t}$; $t = 1, \ldots, T$; the model implies unique conditional choice probabilities

$$p_{k,t}(x) = \Pr(d_t = k | x_t = x) = \mathbb{E}_{\epsilon_t}[\mathbb{1}\{\sigma_t^*(x, \epsilon_t) = k\}] \tag{4}$$

for all $k \in \mathcal{D}$; $t = 1, \ldots, T$; and $x \in \mathcal{X}$. Together with the state transition probabilities $Q_1, \ldots, Q_K$; these conditional choice probabilities fully determine the joint distribution of observed states and choices.

This paper studies the extent to which, conversely, the model primitives are uniquely determined— identified— from the state transition and choice probabilties. Observed state transitions directly identify Q_k and thus, because they were assumed rational, the agent's expectations. We therefore focus on the identification of the utility functions $u_{k,t}$ and the discount parameters β and δ from the conditional choice probabilities for given $Q_1, \ldots, Q_K$.

3.1 Basic Results

The choice probabilities only depend on the primitives through the value contrasts $w_{k,t}(x;\boldsymbol{\sigma}^*_{t+1}) - w_{K,t}(x;\boldsymbol{\sigma}^*_{t+1})$. In particular, (4) implies that[6]

$$\ln\left(\frac{p_{k,t}(x)}{p_{K,t}(x)}\right) = w_{k,t}(x;\boldsymbol{\sigma}^*_{t+1}) - w_{K,t}(x;\boldsymbol{\sigma}^*_{t+1}) \tag{5}$$

for all $k \in D/\{K\}$; $t = 1,\ldots,T$; and $x \in \mathcal{X}$. With the restriction that the choice probabilities add up to one over choices, (5) gives

$$-\ln(p_{K,t}(x)) = \ln\left(\sum_{k\in\mathcal{D}} \exp\left[w_{k,t}(x;\boldsymbol{\sigma}^*_{t+1}) - w_{K,t}(x;\boldsymbol{\sigma}^*_{t+1})\right]\right). \tag{6}$$

With $-\ln(p_{K,t}(x))$ in hand, (5) determines $p_{k,t}(x)$ from the value contrasts.

Conversely, as in the case without present-bias (Hotz and Miller, 1993), using (5), the current choice specific value contrasts can be uniquely recovered from the observed choice probabilities. Altogether, this implies that we can focus our identification analysis on the question to what extent the discount parameters and utilities are uniquely determined from the value contrasts $w_{k,t}(x;\boldsymbol{\sigma}^*_{t+1}) - w_{K,t}(x;\boldsymbol{\sigma}^*_{t+1})$, for given Q_k.

It is well known that the dynamic discrete choice model with geometric discounting ($\beta = 1$) is not identified (Rust, 1994, Lemma 3.3, and Magnac and Thesmar, 2002, Proposition 2). The underidentification carries over to its generalization with present-bias. Specifically, the following version of Magnac and Thesmar's (2002) Proposition 2 holds.

Theorem 1. For given $Q_1,\ldots,Q_K$; β; δ; and $p_{k,t}(x)$; $k \in \mathcal{D}$; $t = 1,\ldots,T$; and $x \in \mathcal{X}$; there exists unique utility functions $u_{1,t},\ldots,u_{K-1,t}$; $t = 1,\ldots,T$; such that (1), (2), (3), and (4) hold.

Proof. Using (5), $p_{k,T}$, $k \in \mathcal{D}$, gives the unique $w_{k,T} - w_{K,T}$, $k \in \mathcal{D}$, that are consistent with (4). Using the terminal condition of (1) and the normalization $u_{K,T} = 0$, this gives $w_{k,T} = u_{k,T}$, $k \in \mathcal{D}$. The strategy σ^*_T follows up to ϵ-almost sure equivalence from (3). Finally, v_T follows from (2).

Next, iterate the following argument for $t = T-1,\ldots,1$. Suppose that we have constructed unique $u_{k,t+1}$, $k \in \mathcal{D}$, unique v_{t+1}, and unique (up to ϵ-almost sure equivalence) $\boldsymbol{\sigma}^*_{t+1} = (\sigma^*_{t+1},\ldots,\sigma^*_T)$ consistent with (1), (2), (3), and (4) and the choice probabilities. For each $x \in \mathcal{X}$, using (5), $p_{k,t}(x)$,

[6] The functional form of the mapping between value contrasts and choice probabilities is specific to the assumption that the $\epsilon_{k,t}$ have independent type 1 extreme value distributions, but the results given here extend to general known G_t.

$k \in \mathcal{D}$, gives the unique $w_{k,t}(x;\boldsymbol{\sigma}^*_{t+1}) - w_{K,t}(x;\boldsymbol{\sigma}^*_{t+1})$, $k \in \mathcal{D}$, that are consistent with (4). Using (1),

$$\begin{aligned} & w_{k,t}(x;\boldsymbol{\sigma}^*_{t+1}) - w_{K,t}(x;\boldsymbol{\sigma}^*_{t+1}) \\ & = u_{k,t}(x) - u_{K,t}(x) + \beta\delta \int v_{t+1}(x';\boldsymbol{\sigma}^*_{t+1}) \left[dQ_k(x'|x) - dQ_K(x'|x)\right], \ k \in \mathcal{D}. \end{aligned} \tag{7}$$

Because the last term in the right hand side of (7) is known at this point and $u_{K,t}(x)$ is normalized to zero, this determines $u_{k,t}$, $k \in \mathcal{D}$. The strategy σ^*_t follows up to ϵ-almost sure equivalence from (3). Finally, v_t follows from (2). □

Theorem 1 implies that β and δ can only be identified if further data are available or additional assumptions are made. In this paper, we explore identification under exclusion restrictions on the utility functions. Our analysis focuses on the identification of β and δ. Theorem 1 shows that, once β and δ are identified, unique utility functions can be found that rationalize the choice data.

3.2 Concentrating identification on the discount factors

Because $\mathcal{X}$ is finite— say it has J elements— it is convenient to express expectations in matrix notation. To this end, let $\boldsymbol{v}_t(\boldsymbol{\sigma}^*_t)$ be a $J \times 1$ vector that stacks the values of $v_t(x;\boldsymbol{\sigma}^*_t)$, $x \in \mathcal{X}$, and $\boldsymbol{Q}_k(x)$ a $1 \times J$ vector that stacks the values of $Q_k(x'|x)$, $x' \in \mathcal{X}$, in corresponding order. Then, (5) and (7), with the normalization $u_{K,t}(x) = 0$, give

$$\ln\left(\frac{p_{k,t}(x)}{p_{K,t}(x)}\right) = u_{k,t}(x) + \beta\delta\left[\boldsymbol{Q}_k(x) - \boldsymbol{Q}_K(x)\right]\boldsymbol{v}_{t+1}(\boldsymbol{\sigma}^*_{t+1}). \tag{8}$$

Recall that, given the transition distributions Q_k, (8) contains all information in the choice probabilities about the model's primitives.

We will concentrate the identification analysis on the discount factors by controlling the current period utility $u_{k,t}(x)$ in the right hand side of (7) with exclusion restrictions and expressing the continuation value in terms of the discount factors and data only. As $\boldsymbol{Q}_k(x)$ and $\boldsymbol{Q}_K(x)$ are data, this only requires that we express the perceived long run values $\boldsymbol{v}_{t+1}(\boldsymbol{\sigma}^*_{t+1})$ in terms of the discount factors and data. To this end, first substitute (1) and (3) into (2) to get

$$\begin{aligned} & v_{t+1}(x;\boldsymbol{\sigma}^*_{t+1}) \\ & = \mathbb{E}_{\epsilon_{t+1}}\left[\max_{k\in\mathcal{D}}\left\{w_{k,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) + \epsilon_{j,t+1}\right\} + \delta(1-\beta)\boldsymbol{Q}_{\sigma^*_{t+1}(x,\epsilon_{t+1})}(x)\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2})\right]. \end{aligned} \tag{9}$$

Next, as we can express the value contrast $w_{k,t+1} - w_{K,t+1}$ in terms of data using (5), we substract $w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2})$ from the first term in the right hand side of (9) and add it to the second term, which gives

$$v_{t+1}(x;\boldsymbol{\sigma}^*_{t+1}) = m_{t+1}(x) + w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) + \delta(1-\beta)\mathbb{E}_{\epsilon_{t+1}}\left[\boldsymbol{Q}_{\sigma^*_{t+1}(x,\epsilon_{t+1})}(x)\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2})\right], \tag{10}$$

where

$$m_{t+1}(x) = \mathbb{E}_{\epsilon_{t+1}}\left[\max_{k\in\mathcal{D}}\left\{w_{k,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) - w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) + \epsilon_{k,t+1}\right\}\right] \tag{11}$$

is the McFadden surplus (before observing ϵ_{t+1}) for the choice among $k \in \mathcal{D}$ with utilities $w_{k,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) - w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) + \epsilon_{k,t+1}$. Under our assumption that ϵ_{t+1} is extreme value distributed, the right-hand side of (11) reduces to the right-hand side of (6), so that $m_{t+1}(x) = -\ln(p_{K,t+1}(x))$ is known from the choice data.[7] The term $w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2})$ can be expressed recursively as

$$w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) = \beta\delta\boldsymbol{Q}_K(x)\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2}). \tag{12}$$

Finally, as the expectation over ϵ_{t+1} in the right hand side of (10) is effectively an expectation over implied actions $\sigma^*_{t+1}(x,\epsilon_{t+1})$, it can be expressed in terms of the observed choice probabilities using (3):

$$\mathbb{E}_{\epsilon_{t+1}}\left[\boldsymbol{Q}_{\sigma^*_{t+1}(x,\epsilon_{t+1})}(x)\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2})\right] = \sum_{k\in\mathcal{D}} p_{k,t+1}(x)\boldsymbol{Q}_k(x)\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2}). \tag{13}$$

Substituting (12) and (13) into (10) gives

$$v_{t+1}(x;\boldsymbol{\sigma}^*_{t+1}) = m_{t+1}(x) + \delta\left[\beta\boldsymbol{Q}_K(x) + (1-\beta)\overline{\boldsymbol{Q}}_{t+1}(x)\right]\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2}) \tag{14}$$

where $\overline{\boldsymbol{Q}}_{t+1}(x) = \sum_{k\in\mathcal{D}} p_{k,t+1}(x)\boldsymbol{Q}_k(x)$ is the expected state transition probability distribution under strategy σ^*_{t+1} in state x. This mixture represents an expectation over how the choices of present-biased future selves control future state transitions, choices which are in conflict with the current self's long term preferences.

Define the $J \times J$ matrix of probability mixtures

$$\boldsymbol{Q}^{pb}_t(\beta) = \beta\boldsymbol{Q}_K + (1-\beta)\overline{\boldsymbol{Q}}_t, \tag{15}$$

where $\overline{\boldsymbol{Q}}_t$ stacks $\overline{\boldsymbol{Q}}_t(x)$ and $\boldsymbol{Q}_K$ stacks $\boldsymbol{Q}_K(x)$. Then, we can write (14) as a recursive expression for

[7] More generally, given G, $m_{t+1}(x)$ is a known function of $w_{k,t+1}(x;\boldsymbol{\sigma}^*_{t+2}) - w_{K,t+1}(x;\boldsymbol{\sigma}^*_{t+2})$, $k \in \mathcal{D}$, and thus, using (5), of the choice probabilities (Arcidiacono and Miller, 2011).

$\boldsymbol{v}_{t+1}(\boldsymbol{\sigma}^*_{t+1})$ in vector notation:

$$\boldsymbol{v}_{t+1}(\boldsymbol{\sigma}^*_{t+1}) = \boldsymbol{m}_{t+1} + \delta \boldsymbol{Q}^{pb}_{t+1}(\beta)\boldsymbol{v}_{t+2}(\boldsymbol{\sigma}^*_{t+2}).$$

Completing the recursion until the end of time T expresses

$$\boldsymbol{v}_{t+1}(\boldsymbol{\sigma}^*_{t+1}) = \boldsymbol{m}_{t+1} + \sum_{\tau=t+2}^{T} \delta^{\tau-t-1} \left(\prod_{r=t+1}^{\tau-1} \boldsymbol{Q}^{pb}_{r}(\beta) \right) \boldsymbol{m}_{\tau}, \tag{16}$$

in terms of the discount factors and data only. Substituting (16) into (8) gives

$$\begin{aligned} \ln\left(\frac{p_{k,t}(x)}{p_{K,t}(x)}\right) =& u_{k,t}(x) + \\ & \beta\delta \left[\boldsymbol{Q}_k(x) - \boldsymbol{Q}_K(x)\right] \left[\boldsymbol{m}_{t+1} + \sum_{\tau=t+2}^{T} \delta^{\tau-t-1} \left(\prod_{r=t+1}^{\tau-1} \boldsymbol{Q}^{pb}_{r}(\beta) \right) \boldsymbol{m}_{\tau} \right]. \end{aligned} \tag{17}$$

The log choice probability ratio in the left hand side of (17) measures the observed propensity to choose k over K in state x. The right hand side of (17) explains this observed propensity by the current period's utility difference $u_{k,t}(x) - u_{K,t}(x) = u_{k,t}(x)$ and a difference in continuation values, which is a polynomial in β and δ with coefficients that are fully determined by the choice and transition data. We study identification from variation in these continuation values, under exclusion restrictions on primitive utility that control the effects of variation in the current period's utility. This formalizes the common intuition that holding current period utilities constant, current choice responses to variation in future values are informative about time preferences.

3.3 Exclusion restrictions

Our identification argument holds for exclusion restrictions on utilities between pairs of time periods, choices, states, or any combinations of the three. To simplify the exposition, we however focus on exclusion restrictions on utilities between pairs of states. We primarily focus on the case in which we have two such exclusion restrictions, which is the minimum needed to identify the two unknown discount factors, β and δ. In applications, intuition for exclusion restrictions would typically deliver a *variable* that affects continuation values, but not the current period's utility. Such a excluded variable would typically imply more than two exclusion restrictions on states, which would further restrict the identified set of discount factors.

So, consider two exclusion restrictions, indexed by a and b. Let $x_{a,1}, x_{a,2} \in \mathcal{X}$ and $x_{b,1}, x_{b,2} \in \mathcal{X}$

be two pairs of states such that $x_{a,1} \neq x_{a,2}$ and $x_{b,1} \neq x_{b,2}$. The exclusion restrictions are

$$u_{k,t}(x_{a,1}) = u_{k,t}(x_{a,2}) \text{ and } u_{k,t}(x_{b,1}) = u_{k,t}(x_{b,2}) \text{ for some } k \in \mathcal{D}/\{K\} \text{ and some } t < T-1. \tag{18}$$

Difference (17) corresponding to the indices of the exclusion restrictions to get the following bivariate polynomial system in β and δ

$$\begin{aligned}
&\ln\left(\frac{p_{k,t}(x_{a,1})}{p_{K,t}(x_{a,1})}\right) - \ln\left(\frac{p_{k,t}(x_{a,2})}{p_{K,t}(x_{a,2})}\right) = \\
&\beta\delta\left[\boldsymbol{Q}_k(x_{a,1}) - \boldsymbol{Q}_K(x_{a,1}) - \boldsymbol{Q}_k(x_{a,2}) + \boldsymbol{Q}_K(x_{a,2})\right]\left[\boldsymbol{m}_{t+1} + \sum_{\tau=t+2}^{T} \delta^{\tau-t-1}\left(\Pi_{r=t+1}^{\tau-1}\boldsymbol{Q}_r^{pb}(\beta)\right)\boldsymbol{m}_\tau\right]
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\ln\left(\frac{p_{k,t}(x_{b,1})}{p_{K,t}(x_{b,1})}\right) - \ln\left(\frac{p_{k,t}(x_{b,2})}{p_{K,t}(x_{b,2})}\right) = \\
&\beta\delta\left[\boldsymbol{Q}_k(x_{b,1}) - \boldsymbol{Q}_K(x_{b,1}) - \boldsymbol{Q}_k(x_{b,2}) + \boldsymbol{Q}_K(x_{b,2})\right]\left[\boldsymbol{m}_{t+1} + \sum_{\tau=t+2}^{T} \delta^{\tau-t-1}\left(\Pi_{r=t+1}^{\tau-1}\boldsymbol{Q}_r^{pb}(\beta)\right)\boldsymbol{m}_\tau\right].
\end{aligned} \tag{20}$$

The moment conditions (19) and (20) are bivariate polynomials of order $T-t$ in β and δ, with coefficients that are determined by known functions of the data. The moment conditions are independent of $\boldsymbol{u}$ and must hold exactly in the population. The identified set is consequently reduced to characterizing the zero set of the two moment conditions, independently of the other moment conditions in (5).

Assumption 1. Either $\frac{p_{k,t}(x_{a,1})}{p_{K,t}(x_{a,1})} \neq \frac{p_{k,t}(x_{a,2})}{p_{K,t}(x_{a,2})}$, or the following rank condition holds

$$\left[\boldsymbol{Q}_k(x_{a,1}) - \boldsymbol{Q}_K(x_{a,1}) - \boldsymbol{Q}_k(x_{a,2}) + \boldsymbol{Q}_K(x_{a,2})\right]\boldsymbol{m}_{t+1} \neq 0,$$

and either $\frac{p_{k,t}(x_{b,1})}{p_{K,t}(x_{b,1})} \neq \frac{p_{k,t}(x_{b,2})}{p_{K,t}(x_{b,2})}$, or the following rank condition holds

$$\left[\boldsymbol{Q}_k(x_{b,1}) - \boldsymbol{Q}_K(x_{b,1}) - \boldsymbol{Q}_k(x_{b,2}) + \boldsymbol{Q}_K(x_{b,2})\right]\boldsymbol{m}_{t+1} \neq 0.$$

Finally, we need some terminal conditions.

Assumption 2. $w_{K,T}(x) = 0$ and $v_T(x) = m_T(x)$ for all $x \in \mathcal{X}$.

Write the moment conditions (19) and (20) as $f_a(\beta,\delta) = 0$ and $f_b(\beta,\delta) = 0$, respectively, with f_a and f_b $(T-t)$'th order polynomials. Then, we say that the polynomials f_a and f_b have a *common factor* h if $f_a(\beta,\delta) = h(\beta,\delta)g_a(\beta,\delta)$ and $f_a(\beta,\delta) = h(\beta,\delta)g_a(\beta,\delta)$, with h a polynomial of of order one or higher and g_a and g_b polynomials.

A simple example of a common factor of (19) and (20) in the case that their left hand sides are zero (no choice responses) is $h(\beta, \delta) = \delta$. It turns out that the common factors of (19) and (20) characterize the exceptions to a result that (β, δ) is identified from these moment conditions up to a finite set. We first present the formal identification result before we comment on the exceptions.

Theorem 2. Suppose that the exclusion restrictions in (18) hold, that Assumptions 1 and 2 hold, and that (19) and (20) have no common factors. Then the identified set B is discrete with no more than $(T - t)^2$ points.

Proof. We need to show that under the stated assumptions, (19) and (20) are a system of non-constant multivariate polynomials on the domain of β and δ. Then by Bezout's Theorem, the system has no more than $(T-t)^2$ zeros in the complex plane, which is also an upper bound on the number of zeros on the domain of β and δ.

Suppose that B has at least one zero. We consider three cases. Suppose first that the left hand sides of (19) and (20) are both non-zero. Since the right hand sides are both zero at $\delta = 0$ for any β on its domain, then if the system has at least one zero on its domain, then the right hand sides are non-constant.

Suppose next that one left-hand side is zero, but not both. Since $\beta \in (0, 1]$, and the right hand sides are zero in $\delta = 0$, then δ must be strictly positive, and the right hand sides are non-constant.

Suppose finally that both left hand sides are zero. Then $\beta\delta$ is a common factor with exactly one zero for $\delta = 0$ for any β on its domain. Second, when the rank conditions in Assumption 1 hold, then the first derivatives of each moment condition with respect to both δ and β at $\delta = 0$ and for any β on its domain, are both non-zero, and both moments are non-constant.

$\square$

Bezout's theorem generalizes the fundamental theorem of algebra to multivariate polynomials, see e.g. Cox et al. (2015). Note that Theorem 2 does not guarantee a solution. The zero set may be empty, in which case the model is rejected.[8] Except for certain special cases, such as when one moment condition is a multiple of the other, we have not found obvious economic interpretations of common factors. The existence of common factors can however easily be verified on a case-by-case basis by calculating the resultant, the determinant of the Sylvester matrix of the moment conditions. The resultant is in general a polynomial in either β or δ,

[8] See Abbring and Daljord (2019b) for a discussion of the empirical content of dynamic discrete choice models under exclusion restrictions.

where any one of the two parameters can arbitrarily be chosen as a base. The real roots to this univariate polynomial on the unit are also roots to the moment conditions. The resultant is everywhere zero if and only if the moment conditions have common factors.

4 Relation to preference reversals

We showed above that the exclusion restriction approach that has been proposed to identify geometric time preferences in dynamic discrete choice models formally extends to present-biased time preferences. It is however less obvious how it captures preference reversals, the main feature of present-biased time preferences. The most common lab approach to measure time preferences is however to use contrasts between observed choices from menus of Sooner-Smaller (SS) and larger-later (LL) rewards. One famous example is Thaler's apples: while most people may prefer an apple today to two apples tomorrow, the same people would presumably prefer two apples one year and one day from now to one apple one year from now. Such choice contrasts are direct and intuitive measures of preference reversals.

The exclusion restrictions above however force the current period pay-off invariant across either a pair of states or a pair of choices, which by construction precludes the common lab design. Rather than comparing choice contrasts between different current period payoffs, such as in Thaler's apples, the identification relies on the contrasts in choice responses to variation in how continuation values are distributed across these states or choices. We next show that the identification of present bias in this design relies on much subtler mechanisms that, though informative about time preferences, are furthermore likely to be hard to estimate to a reasonable level of precision.

The mechanism that distinguishes the present bias model from the geometric model is related to the perceived long term value function in (14), which we repeat here for convenience

$$\boldsymbol{v}_{t+1}(\boldsymbol{\sigma}^*_{t+1}) = \boldsymbol{m}_{t+1} + \sum_{\tau=t+2}^{T} \delta^{\tau-t-1} \left(\prod_{r=t+1}^{\tau-1} \boldsymbol{Q}_r^{pb}(\beta) \right) \boldsymbol{m}_\tau, \tag{21}$$

where from (15)

$$\boldsymbol{Q}_t^{pb}(\beta) = \beta \boldsymbol{Q}_K + (1-\beta)\overline{\boldsymbol{Q}}_t, \tag{22}$$

For time consistent preferences ($\beta = 1$), the future choice contrasts are controlled by $\boldsymbol{Q}_K$ along the time-consistent optimal policy. For time-inconsistent preferences, the agent adjusts the perceived long run value function by the correction term $(1-\beta)\overline{\boldsymbol{Q}}_t$, which represents the weighted deviation from the current selves optimal strategies by future, present biased decision makers. In other words, this term represents the expected preference reversals of future selves. The sophisticated current self anticipates these preference reversals and make current choices in part to minimize the incentives of future selves to deviate from her desired long run choice path by controlling the state evolution.

The identification of present bias can therefore be said to be identified by the weight the long run value function assigns to the expected preference reversals. This is however a subtle mechanism which is less intuitive and transparent than the typical sooner-smaller larger-later design where preference reversals can typically be seen directly from the choice probabilities, e.g. Thaler's apples. Moreover, we show in the next section that though β is formally identified separately from δ, it may be hard to separate these parameters in finite samples to a meaningful level of precision.

5 Identification and inference in a three period model

In this example, we assume binary choice. We set $t_{a,1} = t_{a,2} = t_a$ and $t_{b,1} = t_{b,2} = t_b$ and assume the exclusion restrictions

$$u_{1,t_a}(x_{a,1}) = u_{1,t_a}(x_{a,2}) \tag{23}$$

$$u_{1,t_b}(x_{b,1}) = u_{1,t_b}(x_{b,2}) \tag{24}$$

The two exclusion restrictions lead to the two moment conditions

$$\begin{aligned}
&\ln\left(\frac{p_{1,t_a}(x_{a,1})}{p_{2,t_a}(x_{a,1})}\right) - \ln\left(\frac{p_{1,t_a}(x_{a,2})}{p_{2,t_a}(x_{a,2})}\right) = \\
&\beta\delta\left[\boldsymbol{Q}_1(x_{a,1}) - \boldsymbol{Q}_K(x_{a,1}) - \boldsymbol{Q}_1(x_{a,2}) + \boldsymbol{Q}_K(x_{a,2})\right]\left[\boldsymbol{m}_{t_a+1} + \delta\boldsymbol{Q}^{pb}_{t+1}\boldsymbol{v}_{t_a+2}\right]
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\ln\left(\frac{p_{1,t_b}(x_{b,1})}{p_{2,t_b}(x_{b,1})}\right) - \ln\left(\frac{p_{1,t_b}(x_{b,2})}{p_{2,t_b}(x_{b,2})}\right) = \\
&\beta\delta\left[\boldsymbol{Q}_1(x_{b,1}) - \boldsymbol{Q}_K(x_{b,1}) - \boldsymbol{Q}_1(x_{b,2}) + \boldsymbol{Q}_k(x_{b,2})\right]\left[\boldsymbol{m}_{t_b+1} + \delta\boldsymbol{Q}^{pb}_{t_b+1}\boldsymbol{v}_{t_b+2}\right]
\end{aligned} \tag{26}$$

For $\beta = 1$, the discounted continuation value is reduced to $\delta \boldsymbol{Q}_{t+1}^{pb} \boldsymbol{v}_{t+2} = \delta \boldsymbol{Q}_2 \boldsymbol{v}_{t+2}$, which is now state invariant since $\boldsymbol{Q}_2$ resets x with probability 1, so
$[\boldsymbol{Q}_1(x_1) - \boldsymbol{Q}_1(x_2)]\, \delta \boldsymbol{Q}_2 \boldsymbol{v}_{t+2} = 0$ for any pair $x_1, x_2 \in \mathcal{X}^2$. The moment conditions are generally higher order polynomials in both β and δ, and point identification is therefore generally lost.

Period $T - 1$

We first show that present-biased discount functions can not be identified from only two periods of observed choices and states. Let $t_a = t_b = T - 1$, then define

$$\Delta \boldsymbol{Q}_1(x_a) = [\boldsymbol{Q}_1(x_{a,1}) - \boldsymbol{Q}_K(x_{a,1}) - \boldsymbol{Q}_1(x_{a,2}) + \boldsymbol{Q}_K(x_{a,2})]$$

and analogously for $\Delta \boldsymbol{Q}_1(x_b)$. Next, define

$$\Delta \ln(p_{1,T-1}(x_a)) = \ln\left(\frac{p_{1,T-1}(x_{a,1})}{p_{2,T-1}(x_{a,1})}\right) - \ln\left(\frac{p_{1,T-1}(x_{a,2})}{p_{2,T-1}(x_{a,2})}\right).$$

The moment conditions in (19) can now be written

$$\Delta \ln(p_{1,T-1}(x_a)) = \beta\delta\Delta \boldsymbol{Q}_1(x_a) \boldsymbol{m}_T \tag{27}$$
$$\Delta \ln(p_{1,T-1}(x_b)) = \beta\delta\Delta \boldsymbol{Q}_1(x_b) \boldsymbol{m}_T \tag{28}$$

The two polynomials are clearly linearly dependent. Since the parameters β and δ are interchangeable in both moment conditions, they can not be separately identified with only two periods of data. Their product is however point identified.

Period $T - 2$

With three periods of data, the discount function parameters are formally set identified. Let $t_a = t_b = T - 2$. The moment conditions are

$$\Delta \ln(p_{1,T-2}(x_a)) = \beta\delta\Delta \boldsymbol{Q}_1(x_a) \left[\boldsymbol{m}_{T-1} + \delta \boldsymbol{Q}_{T-1}^{pb} \boldsymbol{m}_T\right]$$
$$\Delta \ln(p_{1,T-2}(x_b)) = \beta\delta\Delta \boldsymbol{Q}_1(x_b) \left[\boldsymbol{m}_{T-1} + \delta \boldsymbol{Q}_{T-1}^{pb} \boldsymbol{m}_T\right]$$

Writing out the terms, we get

$$\begin{aligned}\Delta \ln(p_{1,T-2}(x_a)) =& \beta\delta\Delta\boldsymbol{Q}_1(x_a)\boldsymbol{m}_{T-1} + \beta\delta^2\Delta\boldsymbol{Q}_1(x_a)\overline{\boldsymbol{Q}}_{T-1}\boldsymbol{m}_T + \\ & \beta^2\delta^2\Delta\boldsymbol{Q}_1(x_a)\left[\overline{\boldsymbol{Q}}_{T-1} - \boldsymbol{Q}_2\right]\boldsymbol{m}_T \end{aligned} \tag{29}$$

$$\begin{aligned}\Delta \ln(p_{1,T-2}(x_b)) =& \beta\delta\Delta\boldsymbol{Q}_1(x_b)\boldsymbol{m}_{T-1} + \beta\delta^2\Delta\boldsymbol{Q}_1(x_b)\overline{\boldsymbol{Q}}_{T-1}\boldsymbol{m}_T + \\ & \beta^2\delta^2\Delta\boldsymbol{Q}_1(x_b)\left[\overline{\boldsymbol{Q}}_{T-1} - \boldsymbol{Q}_2\right]\boldsymbol{m}_T \end{aligned} \tag{30}$$

We first note that the only term for which β and δ are not interchangeable in period $T-2$ is $\beta\delta^2\Delta\boldsymbol{Q}_1(x_a)\overline{\boldsymbol{Q}}_{T-1}\boldsymbol{m}_T$. The set identification of β and δ therefore relies on a higher order interaction term. These terms are furthermore likely to be highly correlated in finite samples which suggests that precise estimation of the two parameters separately may be hard to achieve. We illustrate this point with a simulation below.

6 Estimation routine

We estimate β and δ from the sample counterparts to the moment conditions in (19) and (20) by minimum distance. Holding the choice fixed at some $k \in \mathcal{D}\ K$, a pair of periods t_1 and t' and a pair of states x_1 and x_2 give the exclusion restriction $u_t(x_1) = u_{t'}(x_2)$. The corresponding moment is

$$\begin{aligned}\boldsymbol{\psi}(\beta,\delta;x,t) =& \frac{p_{k,t}(x_1)}{p_{K,t}(x_1)} - \frac{p_{k,t'}(x_2)}{p_{K,t'}(x_2)} - \\ & \beta\delta\left([\boldsymbol{Q}_k(x_1) - \boldsymbol{Q}_K(x_2)]\boldsymbol{v}_{t+1} - [\boldsymbol{Q}_k(x_2) - \boldsymbol{Q}_K(x_2)]\boldsymbol{v}_{t'+1}\right)\end{aligned} \tag{31}$$

where $\boldsymbol{v}_t = \boldsymbol{m}_t + \delta\boldsymbol{Q}_t^{pb}\boldsymbol{v}_{t+1}$. We denote the vector of moments which has one element for each exclusion restriction $\boldsymbol{\psi}(\beta,\delta;.,.)$. The minimum distance criterion is

$$S(\beta,\delta) = \boldsymbol{\psi}\boldsymbol{W}\boldsymbol{\psi}'$$

for a weight matrix $\boldsymbol{W}$. The gradient and the Hessian of the criterion function are given in the appendix.

7 Simulation

We set the number of states $J = 6$, for $T = 3$, and draw data for $N = 1000000$ agents. The discount parameters are set $\beta = 0.80$ and $\delta = 0.50$. The exclusion restrictions $u_{1,1}(x_1) = u_{1,1}(x_2) = 1.00$ are imposed in estimation. The utilities are

$$\boldsymbol{u}_1 = \begin{bmatrix} 1.00 & -1.00 & 1.00 \\ 1.00 & 2.00 & 1.00 \\ 1.00 & 2.00 & 4.00 \\ 1.00 & -1.00 & 4.00 \\ 4.00 & 2.00 & 1.00 \\ 1.00 & 5.00 & 3.00 \end{bmatrix}$$

and the transitions are drawn randomly from the true transitions

$$\boldsymbol{Q}_1 = \begin{bmatrix} 0.19 & 0.22 & 0.06 & 0.28 & 0.06 & 0.19 \\ 0.11 & 0.32 & 0.07 & 0.11 & 0.14 & 0.25 \\ 0.28 & 0.11 & 0.17 & 0.28 & 0.06 & 0.11 \\ 0.21 & 0.14 & 0.24 & 0.24 & 0.07 & 0.10 \\ 0.03 & 0.24 & 0.24 & 0.24 & 0.22 & 0.03 \\ 0.10 & 0.14 & 0.10 & 0.19 & 0.05 & 0.43 \end{bmatrix}$$
$$\boldsymbol{Q}_2 = \begin{bmatrix} 0.25 & 0.19 & 0.12 & 0.12 & 0.12 & 0.19 \\ 0.08 & 0.08 & 0.31 & 0.15 & 0.23 & 0.15 \\ 0.27 & 0.07 & 0.27 & 0.07 & 0.20 & 0.13 \\ 0.23 & 0.23 & 0.31 & 0.08 & 0.08 & 0.08 \\ 0.19 & 0.25 & 0.12 & 0.06 & 0.25 & 0.12 \\ 0.19 & 0.12 & 0.19 & 0.19 & 0.25 & 0.06 \end{bmatrix}$$

We first confirm that β and δ are identified. We use the true choice probabilities and true transition distributions to recover β and δ up to numerical precision at $\hat{\beta} = 0.80$ and $\hat{\delta} = 0.50$.

Figure 1 plots the criterion for β and δ, holding δ and β, respectively, at their true values, using the true choice data. The plot shows no clear basin around the minimum, but instead a banana shaped trough. The trough points to issues of inference in finite samples. A similar observation was made in Laibson et al. (2007) for a lifecycle consumption model with $\beta\delta$ preferences and continuous choices, see its Figure 1.

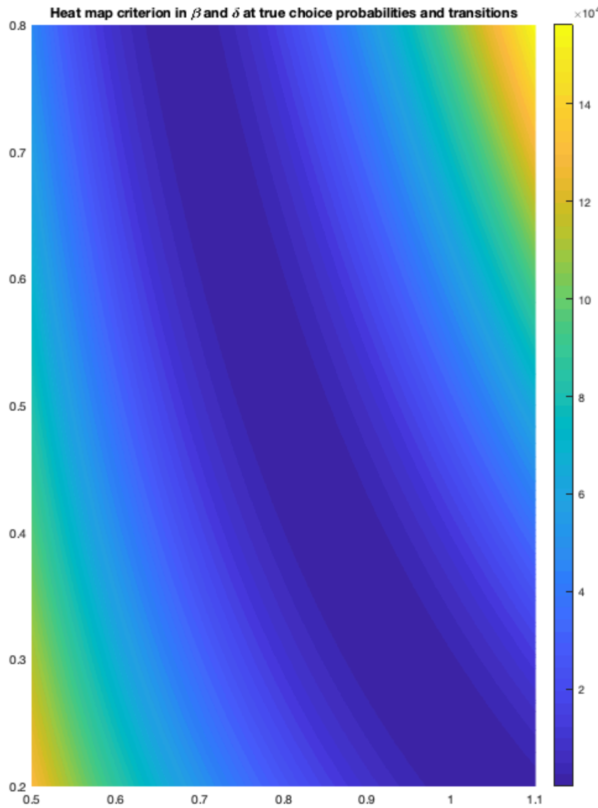

Figure 1: Heat map of the criterion function for the hyperbolic model using true choice data (no sampling variation).

We next use choice data with sampling variation. In Figure 2, we plot β and δ estimates from 100 data sets drawn from the same DGP. The estimates are seen to lie along a hyperbole that is implied by the product of their true values $\beta = \frac{0.80*0.50}{\delta}$, similar to the trough in the heat map in Figure 1. The scatterplot shows that though the parameters are imprecisely estimated separately (the swarm of points stretch along the hyperbole), the products of the parameters are relatively more precisely recovered (the variation around the hyperbole). This points to a practical difficulty in recovering hyperbolic discount function parameters precisely in observational data using our exclusion restrictions. Finally, we estimate an exponential discount function using data generated by a DGP with $\beta = 0.80$ and $\delta = 0.50$. We expect the estimate

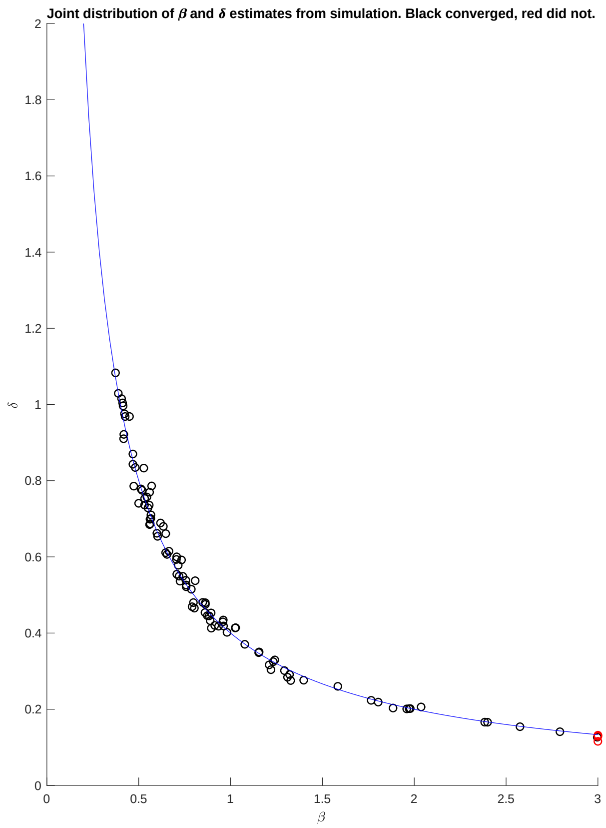

Figure 2: Estimates of β and δ from data with sampling variation.

of δ to be close to 0.80×0.50 and precisely estimated. The estimate is 0.40. The criterion is given in Figure 3.

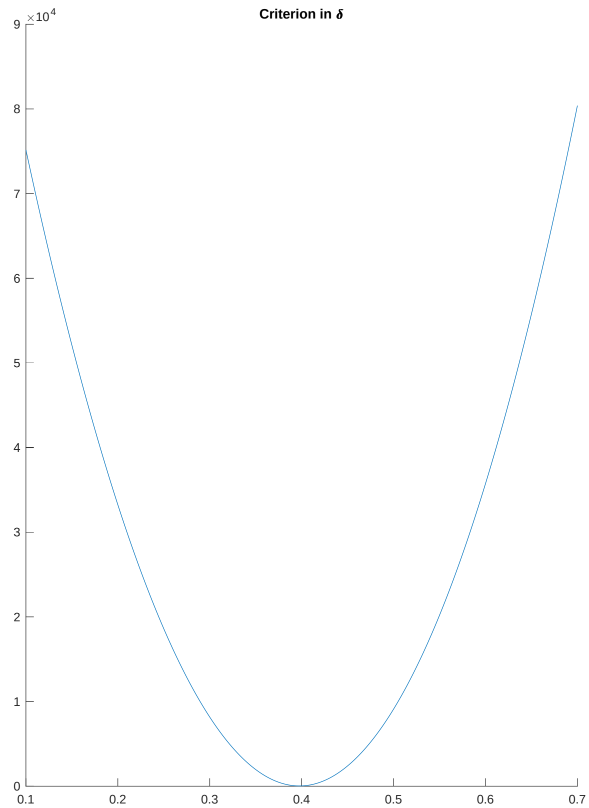

Figure 3: Plot of the criterion function for the geometric model using choice data with sampling variation.

References

Abbring, J. H. and Ø. Daljord (2019a). A comment on "Estimating dynamic discrete choice models with hyperbolic discounting" by Hanming Fang and Yang Wang. Working paper, Tilburg University and University of Chicago. arXiv:1905.07048 [econ.EM]. 2, 3

Abbring, J. H. and Ø. Daljord (2019b). Identifying the discount factor in dynamic discrete choice models. Working Paper 2018–55 (revised), Becker Friedman Institute, Chicago. arXiv:1808.10651 [econ.EM]. 3, 11

Aguirregabiria, V. and J. Suzuki (2014). Identification and counterfactuals in dynamic models of market entry and exit. *Quantitative Marketing and Economics 12*, 267–304. 4

Arcidiacono, P. and R. Miller (2011). CCP estimation of dynamic discrete choice models with unobserved heterogeneity. *Econometrica 79*, 1823–1867. 8

Chan, M. (2017). Welfare dependence and self-control: An empirical analysis. *Review of Economic Studies 84*(3), 98–128. 2, 3

Cox, D., J. Little, and D. O'Shea (2015). *Ideals, Varieties, and Algorithms*. Springer. 11

Elster, J. (1985). *The Multiple Self*. Cambridge University Press. 4

Fang, H. and Y. Wang (2015). Estimating dynamic discrete choice models with hyperbolic discounting, with an application to mammography decisions. *International Economic Review 56*(2), 565–596. 2, 3

Frederick, S., G. Loewenstein, and T. O'Donoghue (2002, June). Time discounting and time preference: A critical review. *Journal of Economic Literature 40*(2), 351–401. 2

Hotz, V. J. and R. A. Miller (1993). Conditional choice probabilities and the estimation of dynamic models. *Review of Economic Studies 60*(3), 497–529. 6

Kalouptsidi, M., R. Souza-Rodriguez, and P. Scott (2016, June). Identification of counterfactuals in dynamic discrete choice models. Working paper, NBER. 4

Laibson, D., A. Repetto, and J. Tobacman (2007, August). Estimating Discount Functions with Consumption Choices over the Lifecycle. NBER Working Papers 13314, National Bureau of Economic Research, Inc. 16

Magnac, T. and D. Thesmar (2002). Identifying dynamic discrete choice processes. *Econometrica 70*, 801–816. 3, 6

Mahajan, A. and A. Tarozzi (2011). Time inconsistency, expectations, and technology adoption: The case of insecticide treated nets. Mimeo, Duke University. 2

Malmendier, U. and S. DellaVigna (2006). Paying not go to the gym. *American Economic Review 96*(3), 674–719. 2

Norets, A. and X. Tang (2014). Semiparametric inference in dynamic binary choice models. *Review of Economic Studies 81*(3), 1229–1262. 4

O'Donoghue, T. and M. Rabin (1999, March). Doing it now or later. *American Economic Review 89*, 103–124. 2, 5

Rust, J. (1994). Structural estimation of Markov decision processes. In R. Engle and D. McFadden (Eds.), *Handbook of Econometrics*, Volume 4, pp. 3081–3143. Amsterdam: North-Holland. 6

Thaler, R. (1981). Some empirical evidence on dynamic inconsistency. *Economics Letters*, 201–207. 2

Urminsky, O. and G. Zauberman (2015). *The Psychology of Intertemporal Preferences*, pp. 141–181. John Wiley & Sons, Ltd. 2

A Gradient and Hessian of the criterion function

For each exclusion restriction in (31), the corresponding moment $\boldsymbol{\psi}$ has derivatives

$$\frac{\partial \boldsymbol{\psi}(\beta, \delta; x, t)}{\partial \beta} = -\delta \Bigg([\boldsymbol{Q}_k(x_1) - \boldsymbol{Q}_K(x_1)] \left[\boldsymbol{v}_{t+1}(\beta, \delta) + \beta \frac{\partial \boldsymbol{v}_{t+1}(\beta, \delta)}{\partial \beta} \right] - \\ [\boldsymbol{Q}_k(x_2) - \boldsymbol{Q}_K(x_2)] \left[\boldsymbol{v}_{t'+1}(\beta, \delta) + \beta \frac{\partial \boldsymbol{v}_{t'+1}(\beta, \delta)}{\partial \beta} \right] \Bigg) \tag{32}$$

$$\frac{\partial \boldsymbol{\psi}(\beta, \delta; x, t)}{\partial \delta} = -\beta \Bigg([\boldsymbol{Q}_k(x_1) - \boldsymbol{Q}_K(x_1)] \left[\boldsymbol{v}_{t+1}(\beta, \delta) + \delta \frac{\partial \boldsymbol{v}_{t+1}(\beta, \delta)}{\partial \delta} \right] - \\ [\boldsymbol{Q}_k(x_2) - \boldsymbol{Q}_K(x_2)] \left[\boldsymbol{v}_{t'+1}(\beta, \delta) + \delta \frac{\partial \boldsymbol{v}_{t'+1}(\beta, \delta)}{\partial \delta} \right] \Bigg) \tag{33}$$

The derivatives $\boldsymbol{v}_t(\beta, \delta) = \boldsymbol{m}_t + \delta \boldsymbol{Q}_t^{pb}(\beta) \boldsymbol{v}_{t+1}(\beta, \delta)$ are calculated recursively

$$\frac{\partial \boldsymbol{v}_t(\beta, \delta)}{\partial \beta} = \delta \left(\boldsymbol{Q}_t^{pb}(\beta) \frac{\partial \boldsymbol{v}_{t+1}(\beta, \delta)}{\partial \beta} + \delta [\boldsymbol{Q}_K - \overline{\boldsymbol{Q}}_t] \boldsymbol{v}_{t+1}(\beta, \delta) \right) \tag{34}$$

$$\frac{\partial \boldsymbol{v}_t(\beta, \delta)}{\partial \delta} = \boldsymbol{Q}_t^{pb}(\beta) \left[\boldsymbol{v}_{t+1}(\beta, \delta) + \delta \frac{\partial \boldsymbol{v}_{t+1}(\beta, \delta)}{\partial \delta} \right] \tag{35}$$

with terminal conditions $\frac{\partial \boldsymbol{v}_T}{\partial \beta} = \frac{\partial \boldsymbol{v}_T}{\partial \delta} = 0$. The gradient of the criterion

$$S = \boldsymbol{\psi} \boldsymbol{W} \boldsymbol{\psi}' \tag{36}$$

is then

$$g(\theta) = 2 \frac{\partial \boldsymbol{\psi}}{\partial \theta} \boldsymbol{W} \boldsymbol{\psi}', \tag{37}$$

where $\theta = [\beta, \delta]$.

The second derivatives of a given moment are

$$\frac{\partial^2 \boldsymbol{\psi}(\beta,\delta;x,t)}{\partial\beta\partial\delta} = -\frac{1}{\delta}\frac{\partial\boldsymbol{\psi}}{\partial\beta} - \delta\left[\boldsymbol{Q}_k(x_1) - \boldsymbol{Q}_K(x_1)\right]\left[\frac{\partial \boldsymbol{v}_{t+1}}{\partial\delta} + \beta\frac{\partial^2 \boldsymbol{v}_{t+1}}{\partial\delta\partial\beta}\right] - \left[\boldsymbol{Q}_k(x_2) - \boldsymbol{Q}_K(x_2)\right]\left[\frac{\partial \boldsymbol{v}_{t'+1}}{\partial\delta} + \beta\frac{\partial^2 \boldsymbol{v}_{t'+1}}{\partial\delta\partial\beta}\right] \tag{38}$$

$$\frac{\partial^2 \boldsymbol{\psi}(\beta,\delta;x,t)}{\partial\beta^2} = -\delta\Bigg(\left[\boldsymbol{Q}_k(x_1) - \boldsymbol{Q}_K(x_1)\right]\left[2\frac{\partial \boldsymbol{v}_{t+1}}{\partial\beta} + \beta\frac{\partial^2 \boldsymbol{v}_{t+1}}{\partial\beta^2}\right] - \left[\boldsymbol{Q}_k(x_2) - \boldsymbol{Q}_K(x_2)\right]\left[2\frac{\partial \boldsymbol{v}_{t'+1}}{\partial\beta} + \beta\frac{\partial^2 \boldsymbol{v}_{t'+1}}{\partial\beta^2}\right]\Bigg) \tag{39}$$

$$\frac{\partial^2 \boldsymbol{\psi}(\beta,\delta;x,t)}{\partial\delta^2} = -\beta\Bigg(\left[\boldsymbol{Q}_k(x_1) - \boldsymbol{Q}_K(x_1)\right]\left[2\frac{\partial \boldsymbol{v}_{t+1}}{\partial\delta} + \delta\frac{\partial^2 \boldsymbol{v}_{t+1}}{\partial\delta^2}\right] - \left[\boldsymbol{Q}_k(x_2) - \boldsymbol{Q}_K(x_2)\right]\left[2\frac{\partial \boldsymbol{v}_{t'+1}}{\partial\delta} + \delta\frac{\partial^2 \boldsymbol{v}_{t'+1}}{\partial\delta^2}\right]\Bigg) \tag{40}$$

The second derivatives of the value functions are calculated recursively

$$\frac{\partial \boldsymbol{v}_t}{\partial\delta\partial\beta} = \frac{\partial \boldsymbol{Q}_t^{pb}}{\partial\beta}\left[\boldsymbol{v}_{t+1} + \delta\frac{\partial \boldsymbol{v}_{t+1}}{\partial\beta}\right] + \boldsymbol{Q}_t^{pb}\left[\frac{\partial \boldsymbol{v}_{t+1}}{\partial\beta} + \delta\frac{\partial \boldsymbol{v}_{t+1}}{\partial\delta\partial\beta}\right] \tag{41}$$

$$\frac{\partial \boldsymbol{v}_t}{\partial\beta^2} = \delta\left[\frac{\partial \boldsymbol{Q}_t^{pb}}{\partial\beta}\left[\frac{\partial \boldsymbol{v}_{t+1}}{\partial\beta} + \delta\boldsymbol{v}_{t+1}\right] + \boldsymbol{Q}_t^{pb}\frac{\partial \boldsymbol{v}_{t+1}}{\partial\beta^2}\right] \tag{42}$$

$$\frac{\partial^2 \boldsymbol{v}_t}{\partial\delta^2} = \boldsymbol{Q}_t^{pb}\left[2\frac{\partial \boldsymbol{v}_{t+1}}{\partial\delta} + \delta\frac{\partial \boldsymbol{v}_{t+1}}{\partial\delta^2}\right] \tag{43}$$

where we note that $\frac{\partial \boldsymbol{Q}_t^{pb}}{\partial\beta} = \boldsymbol{Q}_K - \overline{\boldsymbol{Q}}_t$. The Hessian is

$$H(\theta) = 2\left(\frac{\partial^2\boldsymbol{\psi}}{\partial\theta^2}\boldsymbol{W}\left[\boldsymbol{I}_2 \otimes \boldsymbol{\psi}'\right] + \frac{\partial\boldsymbol{\psi}}{\partial\theta}\boldsymbol{W}\frac{\partial\boldsymbol{\psi}'}{\partial\theta}\right). \tag{44}$$