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The (Neural) Dynamics of Stochastic Choice

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Abstract. The standard framework for modeling stochastic choice, the random utility model, is agnostic about the temporal dynamics of the decision process. In contrast, a general class of *bounded accumulation models* from psychology and neuroscience explicitly relate decision times to stochastic choice behavior. This article demonstrates that a random utility model can be derived from the general class of bounded accumulation models, and characterizes how the resulting distribution of random utility depends on response time. This relationship can bias the estimation of structural preference parameters. The bias can be alleviated via the inclusion of standard observables directly in the econometric specification, or through incorporating novel observables such as response time or neurobiological data. Examples of estimating risk and brand preferences are pursued.

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Keywords: neuroeconomics • stochastic choice • random utility • bounded accumulation • drift diffusion • scale heterogeneity • generalized multinomial logit

1. Introduction

The random utility model (RUM; Block and Marschak 1959, Becker et al. 1963) has become the standard framework for modeling discrete choice, providing a highly flexible empirical framework for relating observables to stochastic choice behavior (McFadden 2001). In a RUM, the probability, P_i, that an individual chooses an alternative i from a discrete choice set is the primary object of inquiry. Provided rather weak sufficient conditions, these choice probabilities can be represented by a maximization over a set of random utilities, with different choice probabilities implied by different distributional assumptions over this set (McFadden and Train 2000).

It has been well documented that these distributional assumptions have important, if not critical, implications for interpreting tests of behavioral theory (Loomes and Sugden 1995, Haile et al. 2008). For example, violations of the independence axiom found in lottery choice experiments can be reconciled by appending an additive error to the expected utility (EU) model (Wilcox 2008). Relating the variance of this additive error to observables (including response time) yields improved prediction (Hey 1995) and cautions against rejecting EU in favor of more general models (Buschena and Zilberman 2000). More broadly, a form of "scale heterogeneity" in the variance has been observed in a wide variety of empirical settings (Louviere et al. 2002, Fiebig et al. 2010). However, a theory for *how* the random utility distribution (particularly its variance and skew) relates to observables remains absent (Loomes 2005). The crucial role played by the error specification has led to calls for an empirical foundation for stochastic choice models (Louviere et al. 2002, Loomes 2005, Hey 2005), and to speculation that neuroeconomics can offer a richer, data-driven approach (Harrison 2008).

This article proposes a neurobiological foundation to a random utility model, grounded explicitly in the prevailing framework for modeling the dynamics of decision making found in the psychology and neuroscience literature. Over a half century of psychometric research has established empirically the joint distribution between perceptual judgements and response time in a wide variety of sensory domains.[1] If we denote response time as a continuous random variable $t^* \in \mathbb{R}^+$, and denote the revealed choice i^*, the focus of inquiry then shifts to the joint distribution $\Pr[i^* = i, t^* \leq t]$. To formally capture this joint distribution of time and choice, the dynamics of a decision are modeled directly. A general class of bounded accumulation models (BAMs) posits a decision variable for each choice alternative that accumulates stochastically toward a decision threshold, with the rate of accumulation determined by the strength of the stimulus (see Ratcliff and Smith 2004, Gold and Shadlen 2007, for reviews). Therefore, a BAM predicts which decision variable crosses a threshold first (the choice) and when it crosses (the response time), providing a richer data set that can be leveraged to improve choice prediction results.

In the domain of economic decision making, a well-known special case of this large class of models—the *drift diffusion model* (DDM) (Ratcliff 1978)—has recently been used to improve binary choice prediction (Milosavljevic et al. 2010; Krajbich et al. 2010, 2014;

Frydman and Nave 2017; Clithero 2018). From a normative perspective, this framework has also been extended to study the optimal learning of costly information when the utility (or quality) of alternatives is not known a priori (Woodford 2014, 2016; Natenzon 2018; Fudenberg et al. 2017).[2] While it is well known that a simple DDM and a special case of RUM both yield logit choice probabilities (Cox and Miller 1965, McFadden 1973), the formal relationship between the aforementioned extensions and the form (or even existence) of a RUM distribution remains unclear (e.g., Caplin and Dean 2015).

The BAM framework has also received much attention because it conforms with substantial neurobiological evidence. Electrophysiological methods (which record the activity of a neuron via a cellular electrode) routinely demonstrate an accumulation of neural activity that mirrors the accumulation of evidence in a decision task (e.g., Roitman and Shadlen 2002, Wang 2002, Mazurek et al. 2003, Gold and Shadlen 2007). Similar results have now also been established in human decision-making tasks via functional magnetic resonance imaging (fMRI) (Basten et al. 2010, Hare et al. 2011, Domenech et al. 2017). However, the combination of neural, behavioral, and response-time data suggests that the simple DDM framework may be too restrictive. As a result, recent attention has turned to accumulation models with competing decision variables that race to a fixed threshold, broadly referred to as *race* models (Usher and McClelland 2001; Kiani et al. 2008, 2014). In general, the class of BAMs currently pursued by the psychology and neuroscience literature is varied and actively debated (e.g., Tsetsos et al. 2012, Teodorescu and Usher 2013, Mullett and Stewart 2016). Again, it is unclear whether these various models yield a RUM representation, and if so, what form the random utility distribution might take.

The contributions provided by this study fall along two dimensions. First, Section 3 demonstrates that a general class of BAM implies a form of RUM in which the stochastic component is additively separable. The intuition is straightforward: the accumulation to threshold in a BAM implements the maximization required by a RUM, but with a stochastic element. This clarifies the formal relationship between these two classes of models. The various BAMs currently pursued in the neuroscience and economics literature each have a direct analogue in standard economic theory. The accumulation models studied by Woodford (2016), Fudenberg et al. (2017), and Usher and McClelland (2001), for example, all fit under the definition of a BAM; therefore, they satisfy the necessary conditions of a RUM representation.[3]

Second, this article characterizes how the distribution of random utility depends on the distribution of response time and other observables, a subject on which the discrete choice literature has thus far remained agnostic.[4] In particular, Section 4 specifies how restrictions placed on a BAM—whether from additional data on response time, neurobiological evidence, or normative reasoning—impact the distribution of stochastic choice through the distribution of response times. The random utility distribution implied by the DDM is derived and (perhaps surprisingly) it differs markedly from the independent Gumbel distribution. Beyond the DDM, the random utility distribution implied by a race model is derived and its variance is characterized as a function of response time. In contrast to the simple logit, this implies a form of scale heterogeneity in the variance whereby some choices are more stochastic than others. This result not only provides a theoretical justification for why such scale heterogeneity is observed (e.g., the "generalized multinomial logit"; Fiebig et al. 2010), but also suggests how this heterogeneity depends on observables such as response time.

Taken together, these results serve to specify a subset of random utility models that fit both the behavioral and neural evidence. From the perspective of a choice modeler, these results provide guidance on how an endogenous response-time variable should be incorporated into the specification of a random utility model. Moreover, the results yield falsifiable statements about how choice probabilities depend on observables *even if data on response time or neural activity is not observed*. The following example provides a simple demonstration.

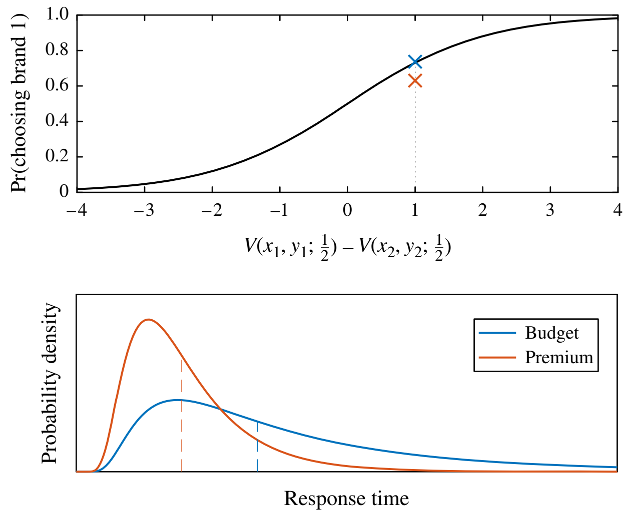
Consider consumers with valuations over two brands that differ along two attributes x and y, given by a simple linear model $V(x, y; \alpha) = \alpha x + (1-\alpha)y$. The parameter α—which reflects the "share" of value that a consumer assigns to the first attribute—is the subject of inference from a discrete choice data set. For simplicity, suppose consumers weight each attribute equally ($\alpha = 0.5$) and a data set is comprised of a large number of choices from vertically differentiated choice sets (Table 1).

For the comparison of $V(x_1, y_1; \alpha)$ and $V(x_2, y_2; \alpha)$ on a choice trial, a standard application of a RUM would specify some probability distribution over the differences in valuations. For example, the logistic distribution—with a given variance—would yield the probability of choosing brand 1 depicted in Figure 1. Inference about the preference parameter α could then be made via a logit model.

Table 1. Vertically Differentiated Choice Sets

	Brand 1		Brand 2		
	x_1	y_1	x_2	y_2	$V(x_1, y_1; \frac{1}{2}) - V(x_2, y_2; \frac{1}{2})$
Budget	2	1	0	1	1
Premium	7	4	4	5	1

Figure 1. Choice Probabilities and Response-Time Densities for Example Choice Sets in Table 1



Notes. (Top) The choice probabilities from a logistic (black line) and a race model (crosses). (Bottom) Response-time densities implied by the race model, with their means (dashed lines).

What if the underlying decision process of these consumers is given by an accumulation model? More precisely, suppose that on each choice trial, a consumer's decision process is given by an accumulation of the two valuations, over time, with the first to cross some threshold being chosen. If this accumulation is a DDM, the choice probabilities will be given by the logistic function with a constant variance; therefore, the logit inference problem is well specified.[5] However, Figure 1 also depicts the choice probabilities from a simple race model. The probability of choosing brand 1 is now lower in the premium choice set than in the budget choice set, even though the differences in valuations are the same, because the scale of the error variance differs over choice sets. This implies that the choice probabilities cannot be represented solely by a single distribution over value differences.[6] As a result, an application of the logit model to this data set is misspecified and returns a biased estimate for α of 0.33.

How are the results in this article useful? First of all, the accumulation model used to generate the choices implies different distributions of response time across the choice sets (Figure 1).[7] This additional information can be incorporated directly into the specification of the RUM to correct bias due to misspecification. However, some care must be taken. In an accumulation model, response time and choice are determined jointly from an underlying stochastic process; therefore, response time is an endogenous variable. To avoid bias induced by this endogeneity problem, a measure of the response-time distribution that is uncorrelated with the choice is required. One candidate, the mean response time, differs over the premium and budget sets and is uncorrelated with choice on any given trial. Therefore, it can be used to correct the variance specification of the RUM.

Second, the distribution of response times is determined by the underlying valuations. This means that a control function approach is feasible. Standard observables, such as the levels of the brand attributes, can be used to specify the variance *in place of response times*. Not only can these variables be used to correct the specification for parameter estimation, but whether they enter significantly (or not) can be used as a direct test of the underlying accumulation model. For instance, if the variance of the RUM was found not to depend on attribute levels, the data would be inconsistent with a race model. Conversely, if it did, this would be inconsistent with a simple DDM.

Since different BAMs have different implications for the distribution of random utility, the properties of structural parameter estimates depend crucially on the specification of the accumulation model. Section 5 provides another example within a well-known experimental paradigm for choice under uncertainty ("multiple price lists"; Holt and Laury 2002). If choices in the MPL are generated from a range of accumulation models currently studied in the literature, the distribution of random utility differs substantially from the logit. In particular, the variance decreases in response time and increases in the components of the lottery that enter the utility specification. These results imply that choice probabilities depend on both the relative difference *and* the magnitude of observables—in contrast to a model with constant variance—and conform with the existing empirical literature (Hey 1995, Buschena and Zilberman 2000). Moreover, standard methods for estimating the coefficient of relative risk aversion will again be biased, and this bias can be corrected using results established in Section 4. These results suggest that using the RUM framework to estimate risk preferences requires careful model specification, in addition to those already documented in recent literature (Wilcox 2008, Apesteguia and Ballester 2018).

Incorporating bounded accumulation models into the random utility framework therefore has important implications for discrete choice modeling. For the choice modeler, it maps out the equivalence between different modeling approaches pursued in neuroscience, psychology, and economic theory. This provides a means of constraining a stochastic choice model to the empirical features of the choice process—either through data on response time or directly from insight on the underlying choice process—thereby guiding applied research. In the other direction, these results clarify that it is possible to distinguish between the models in the large class of BAM using choice data alone. This interaction emphasizes the gains from modeling choice behavior at different levels of abstraction.

2. Models of Stochastic Choice

2.1. The Random Utility Model

A RUM is an implementation of stochastic revealed preference that seeks to identify whether distributions of observed choices are consistent with the principle of utility maximization (McFadden 2005). Consider a choice set comprised of n alternatives (indexed $i = 1 \ldots n$) and a vector of measured observables $\mathbf{x}_i$ for each alternative. The RUM posits a vector of random variables $\mathbf{u}$, with element u_i, such that $\Pr[u_i > u_j, \forall j \neq i] = P_i$. Conditions placed on P_i determine whether observed choices in a population can be represented by $\mathbf{u}$ (Block and Marschak 1959, McFadden 1973, Falmagne 1978).

For exposition, I return to the original interpretation of a "population" as consisting of multiple choices by the same individual in the same choice situation (Becker et al. 1963, McFadden 1981), as is common in the experimental literature on estimating structural preference parameters. In principle, the results in this article also apply to an environment in which choice stochasticity arises in a population of individuals (as in the example in the introduction). The issue of heterogeneity in preferences across individuals is addressed further in the conclusion.

The formulation of a RUM as an empirical tool arises from specifying a relation between the random utilities and observables. Let the function $V(\mathbf{x}_i, \alpha)$ denote a behavioral theory mapping observables $\mathbf{x}_i$ to unobservable valuations $\mathbf{v} = [v_1, \ldots, v_n]$. For example, $\mathbf{x}_i$ could be the attributes of a brand or a lottery over some reward space, and $v_i = V(\mathbf{x}_i)$ could be the (expected) utility of this brand or lottery. The analyst is interested in testing the model $V(\mathbf{x}_i, \alpha)$, or additionally, estimating some unobserved behavioral parameter(s), α.[8]

To bring the model to data, a specification for how the behavioral theory relates to choice probabilities is needed. One well-studied case, originally borrowed from psychophysics, appends an additively separable stochastic component in the specification of utility on a given trial

$$\mathbf{u} = V(\mathbf{x}_i, \alpha) + \boldsymbol{\eta}, \tag{1}$$

where the random vector $\boldsymbol{\eta} \equiv [\eta_1, \ldots, \eta_n]$ has some joint density $f(\boldsymbol{\eta})$ that may also have parameters of interest. The choice criterion is thus given by

$$i^* = \arg\max_i \{v_i + \eta_i\}. \tag{2}$$

This model has been widely applied throughout the discrete choice literature (McFadden 2001).[9] Its usefulness arises from the convenient fact that the probability of choosing an alternative i depends on the comparison between the magnitude of the difference $v_i - v_j$ and the (stochastic) difference $\eta_j - \eta_i$,

$$P_i(\mathbf{v}) = \Pr[v_i - v_j > \eta_j - \eta_i, \forall j \neq i]. \tag{3}$$

To calculate this probability, a specification for the joint density $f(\boldsymbol{\eta})$ is required.[10] In many applications only the choice probability function itself is of interest (e.g., forecasting policy counterfactuals; Berry et al. 1995, Berry and Haile 2014), so the assumptions on the form of $f(\boldsymbol{\eta})$ (indeed, even additive separability) are typically made for technical convenience. However, for the purposes of testing and estimating the primitives of a behavioral theory, the specification of $f(\boldsymbol{\eta})$ is crucial since there may not be a unique combination of $V()$ and $f(\boldsymbol{\eta})$ that yields any given choice probability function (Loomes and Sugden 1995, Haile et al. 2008). An example that nicely demonstrates the importance of correctly specifying this distribution can be found in the experimental literature on choice under uncertainty.

Suppose v_i is given by the expected utility of a lottery i over some reward space, and choices over sets of lotteries are observed. Under an additive RUM (3), choice behavior that deviates from EU—such as a violation of first-order stochastic dominance—is captured by the covariance structure placed on $\boldsymbol{\eta}$. Under this interpretation, the additive RUM is a model of choice errors. However, the pattern of EU violations typically observed in experiments cannot be reconciled by a $f(\boldsymbol{\eta})$ that is assumed to be independent over lotteries, have of constant variance (or scale, interchangeably), and symmetric about zero (Loomes 2005). Indeed, it has been demonstrated that the variance of the stochastic component depends on observable features of the set of lotteries (Buschena and Zilberman 2000)—and of particular relevance to this study—decreases with the time taken to make a decision (Hey 1995). In a thorough analysis of the literature, Wilcox (2008) concludes that the stochastic component of choice is "at least as important in determining sample properties as [model] structures are" (p. 275). The issues typified by this one example apply to the discrete choice literature more generally; namely, that the specification of the stochastic component is crucially important for testing behavioral theory.[11]

A first step in addressing the specification of the stochastic element is to allow the random vector $\boldsymbol{\eta}$ to depend on both valuations and time. Formally, let $\eta(\mathbf{v}, t)\colon \mathbb{R}_+^n \times \mathbb{R}_+ \to \mathbb{R}^n$ denote this stochastic vector mapping, with $\eta_i(\mathbf{v}, t)$ denoting the ith element. The following two conditions provide some natural conditions on how the random variable η_i should depend on $\mathbf{v}$ and t.

Condition 1 (Exchangeability).

$$\eta_i(\mathbf{v}, t) = \eta_i(\mathbf{v}_{-i}, t),$$

where $\mathbf{v}_{-i}$ is a permutation of all elements of $\mathbf{v}$ except i (i.e., the ith element of $\mathbf{v}$ and $\mathbf{v}_{-i}$ is v_i).

The exchangeability condition states that the relation between η_i and v_j, $\forall j \neq i$ depends on the magnitudes of those valuations, but not on their indexing over alternatives. This eliminates biases toward a choice alternative inherent to the stochastic structure. For instance, the probability of choosing an alternative when all valuations are equal should be $1/n$. Note, however, that η_i and η_j can still be correlated because of their relation to response time.

Condition 2 (Separability).

$$\eta_i(\mathbf{v}, t) = \eta_i(\mathbf{v}_{\alpha i}, t),$$

where $\mathbf{v}_{\alpha i}$ *replaces the ith element of* $\mathbf{v}$ *with* $\alpha \geq 0$.

The separability condition states that η_i does not depend directly on v_i, a natural extension of the original separability assumption (McFadden 1973, 1981) to allow for the influence of v_j on η_i, $\forall\, j \neq i$. However, η_i does still depend on response time; therefore, v_i can still influence the distribution of η_i through this (observable) channel.

While Conditions 1 and 2 place some structure on $\boldsymbol{\eta}$, the class of distributions that satisfy them is no doubt large. This presents an opportunity for insight from neuroeconomics, with specifications for $\boldsymbol{\eta}$ that are driven by both neural and behavioral data. The goal of this article is to do just this: to provide a foundation for the distribution of an additive RUM derived from the models of dynamic decision making currently at the fore of the neuroscience and cognitive psychology literature.

2.2. Bounded Accumulation Models

The general formulation for a BAM is composed of three elements: a stochastic accumulator, a stopping rule, and a choice criterion. Together, these components serve to compare valuations over time for the purpose of choice. The class of models that make up BAM is large, and our formal statement is designed to capture this class in a manner that is tractable to relate to a RUM. Special cases of the general formulation that correspond to familiar models in the neuroscience literature will be noted. A review of the neuroscience justifying this formulation, and the state of the empirical literature, can be found in Appendices A and B.

2.2.1. The Accumulation Process. A neuron outputs discrete, electrical impulses, and this activity of individual neurons in cortex is highly irregular. This stochasticity arises, at least in part, from the small-scale thermodynamic processes involved in the flow of ions between neurons (Mainen and Sejnowski 1995, Stevens 2003, Glimcher 2005). For this reason, it is widely accepted that information is encoded in the *rate* of stochastic neural activity over time (Shadlen and Newsome 1994, Rieke et al. 1997). A computation that calculates (or in the language of neuroscience "decodes") this rate information is an accumulation.

In an accumulation, each alternative i in the choice set is associated with a decision variable. In empirical practice, the decision variable is widely taken to be the activity level of a population of neurons associated with a given alternative (see Appendix B). In formal modeling, it is a vector-valued Markov process $\mathbf{Z}(t) \in \mathbb{R}^n$ with a continuous time index $t \in \mathbb{R}_+$.[12] This process accumulates the rate information, here assumed to be given by the subjective valuations $v_i > 0$, over time.[13] The general accumulation process is given by the following set of stochastic differential equations where the parameterization by $\mathbf{v}$ is made explicit:

$$d\mathbf{Z}(t;\mathbf{v}) = [\mathbf{v} + c(t)\mathbf{1} + \boldsymbol{\Gamma}(t)\mathbf{Z}(t;\mathbf{v})]\,dt + \boldsymbol{\sigma}(t)\,d\mathbf{B}(t), \qquad \mathbf{Z}(0) = \mathbf{0}. \tag{4}$$

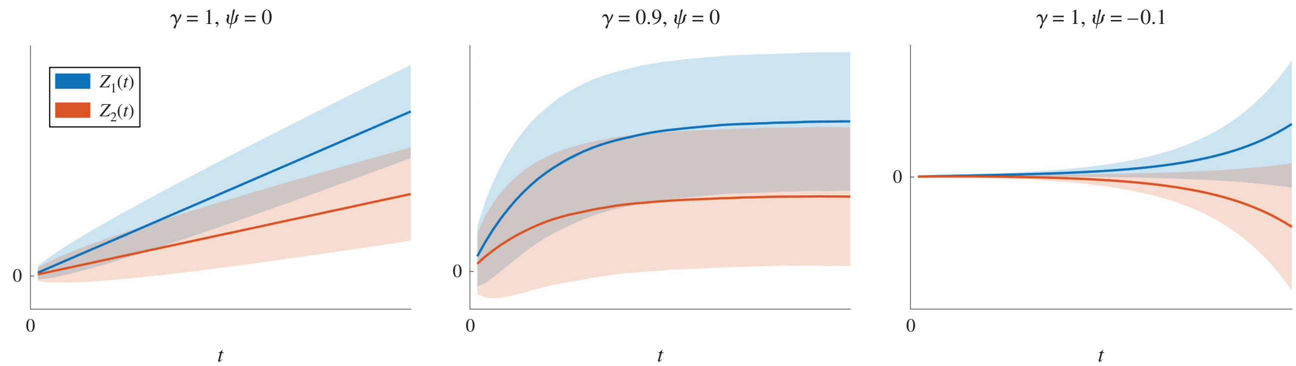
The stochastic element of this process, $\mathbf{B}(t)$, is a vector-valued Brownian motion.[14] This stochasticity is generally held to arise from two sources that are not separately identifiable by choice data: stochasticity within the accumulation computation itself (Shadlen and Newsome 1998), and stochasticity in the encoding (or sampling) of each v_i from upstream brain areas (e.g., Shadlen and Shohamy 2016). See Webb et al. (2016b) for more detail. To be as general as possible, the diffusion of $\mathbf{B}(t)$ is governed by the ($n \times n$ matrix-valued) function $\boldsymbol{\sigma}(t)$, allowing for time-varying correlation between the stochastic process for each alternative.

The ($n \times n$ matrix-valued) function $\boldsymbol{\Gamma}(t)$ allows for both a time- and state-varying relationship in the accumulators. The form of these relationships is proposed to arise in the organization of the cortical circuits that perform the accumulation. A leading example proposes a slow recurrent excitation of neurons for each alternative, balanced by a fast feedback inhibition from the neurons of other alternatives (Wang 2002, 2008). Of special interest is the case of binary choice with two accumulators governed by $\boldsymbol{\Gamma} = \left(\begin{smallmatrix}\gamma & \psi \\ \psi & \gamma\end{smallmatrix}\right)$, $\psi < 0$. Under this process, each v_i influences the rate of accumulation of $Z_j(t;\mathbf{v})$, $\forall\, j \neq i$ and are said to *mutually inhibit* each other (e.g., Usher and McClelland 2001). Throughout, we maintain the assumption that the parameters that govern the accumulation $\boldsymbol{\Gamma}(t)$ and $\boldsymbol{\sigma}(t)$ are symmetric over alternatives.

Finally, this general formulation also includes a time-dependent drift term $c(t) > 0$ that does not depend on $\mathbf{v}$ or i. This parameter is included to capture a universal ramping of neural activity over the course of a decision, particularly observed in cases of time pressure, and is often termed an "urgency" signal (Cisek et al. 2009).[15]

To clarify the general parametrization given in (4), it is instructive to consider special cases defined in terms of restrictions of the stochastic differential equations (see Figure 2). Unlike (4), the solution to each of these cases yields a Gaussian process; therefore, the posterior density of the decision signal can be calculated

Figure 2. Mean and Standard Deviation of Example Accumulation Processes with $v_1 > v_2$



Note. (Left) Brownian motion; (center) leaky accumulators; (right) mutually inhibited accumulators.

analytically (see Drugowitsch et al. 2012, Fudenberg et al. 2017, Woodford 2016, for examples that use these posterior densities to model costly information acquisition).

Definition 1. An accumulation process $Z_i(t;\mathbf{v})$ is uncoupled if $\mathbf{\Gamma}(t)$ is a diagonal matrix with element $\gamma(t)$.

This restriction implies that each stochastic process no longer depends on the processes governing the other alternatives (i.e., $\psi = 0$),

$$dZ_i(t;v_i) = [v_i + c(t) + \gamma(t)Z_i(t;v_i)]dt + \sigma(t)\,dB_i(t), \quad \forall\, i. \tag{5}$$

However, the statistics of the accumulation are still allowed to depend on both time and state. This is an important requirement. Woodford (2014) demonstrates that optimal accumulation given a capacity constraint yields an uncoupled process in which the optimal drift depends on the state of the accumulation.

Additionally, much empirical focus has been placed on whether the accumulation perfectly integrates v_i (Usher and McClelland 2001, Kiani et al. 2008). For instance, when $\gamma(t) < 0$, the accumulation is "leaky" in the sense that early realizations of the accumulation are discounted. Eliminating these possibilities brings us to a familiar process that is both time and state homogenous.

Definition 2. An accumulation process $Z_i(t;v_i)$ is a Brownian motion (with drift) if it is uncoupled, time homogeneous, and $\gamma = 0$:

$$dZ_i(t;v_i) = v_i\,dt + \sigma\,dB_i(t), \quad \forall\, i. \tag{6}$$

Brownian motion describes the continuous time properties of a discrete random walk. It is a Gaussian process with a closed-form solution for $Z_i(t;\mathbf{v})$, and its mathematical tractability greatly simplifies the derivation of choice probabilities and other features of the accumulation process. For this reason, it is often applied in practice.

2.2.2. Stopping Rules. The second element of a BAM is a stopping rule that determines when/whether each accumulator (or a function of each accumulator) has reached a decision threshold. In practice, the stopping rule provides a functional equation for the response time t^*. As with the accumulation process, many forms of stopping rules have been proposed. Though the main result of this article is valid for a general stopping rule, it will be useful to examine three rules that have received particular attention in the literature.

In the simplest form of a stopping rule, a fixed response time $\bar{t}$ is imposed. Experimental conditions that implement this stopping rule are called *interrogation protocols* since the subject is not free to determine the response time.

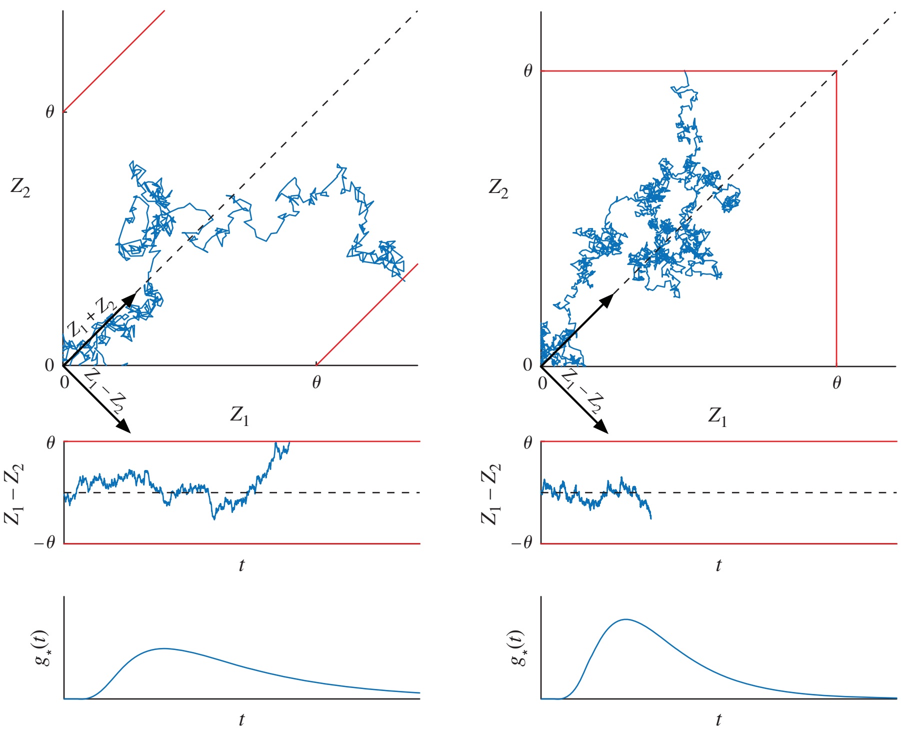
Definition 3. A *fixed* stopping rule is given by $t^* = \bar{t}$ for constant $\bar{t}$.

While a fixed stopping rule has the feature of accumulating each v_i over time via a stochastic process, thus introducing stochasticity in the choice, it does not yield the skewed distribution of response times that is typically observed when subjects are able to respond freely (Luce 1986). To address this, accumulation models implement a rule that terminates the decision once the accumulator reaches some threshold, or more generally, exits some region $\mathcal{R}(t) \subset \mathbb{R}^n$ (Smith 2000). Formally, the response time is a random variable given by

$$t^* = \inf\{t\colon \mathbf{Z}(t;\mathbf{v}) \notin \mathcal{R}(t)\}. \tag{7}$$

The various accumulation models proposed in the literature differ in how the region $\mathcal{R}(t)$ is specified and yield different properties for the distribution of response time, primarily how it depends on $\mathbf{v}$. I restrict the analysis to the following two stopping rules found in the literature. They are both "symmetric" regions in the sense that each accumulator has an identical threshold, though the result is conjectured to hold more generally.

Figure 3. (Color online) Sample Paths of a Brownian Motion Accumulator for the Differenced (Left) and Race (Right) Stopping Rules



Note. Paths of $Z_1(t;v_1) - Z_2(t;v_2)$ are also depicted as a function of time, with the associated density of response times $g_*(t)$.

Definition 4. A *differenced* stopping rule is given by Equation (7) and a region

$$\mathcal{R}(t) = \{r: |r_i - r_j| < \theta(t),\ \forall\, i \neq j\}. \tag{8}$$

The differenced stopping rule terminates the decision when one accumulator is larger than all others by $\theta(t)$ (Figure 3). Drugowitsch et al. (2012) and Fudenberg et al. (2017) show that this rule provides a normative solution for the special case of binary choice when information acquisition costs are linear.

Remark 1. For $n = 2$, if $Z_i(t;v_i)$ is a Brownian motion and the stopping rule is differenced, then $Z_1(t;v_1) - Z_2(t;v_2)$ is a one-dimensional Brownian motion to dual thresholds given by $\theta(t)$ and $-\theta(t)$,

$$d(Z_1(t;v_1) - Z_2(t;v_2)) = (v_1 - v_2)dt + \sigma d(B_1(t) - B_2(t)). \tag{9}$$

A formal statement can be found in Bogacz et al. (2006) with some intution in Figure 3; the differenced stopping rule projects $\mathbf{Z}(t;\mathbf{v})$ onto the one-dimensional coordinate axis, $Z_1(t;v_1) - Z_2(t;v_2)$. If we further restrict $\theta(t) = \theta$ to be constant in time, this model is commonly referred to as the *drift diffusion model* and has been largely successful in matching the properties of reaction time data in binary choice (Ratcliff 1978, Ratcliff and Smith 2004). In particular, as the difference in valuations $v_1 - v_2$ grows larger, the expected response time shrinks.

By definition, the DDM requires that the difference between accumulators is constant at t^* but puts no restriction on the magnitude of each accumulator. This feature is at odds with much of the neurobiological data (see Appendix A). The following stopping rule has the opposite feature: the threshold is fixed, but there is no constraint on the relative magnitudes.

Definition 5. A *race* stopping rule is given by Equation (7) and a region

$$\mathcal{R} = \{r_i: -\infty < r_i < \theta,\ \forall\, i\}, \quad \text{where } 0 < \theta < \infty. \tag{10}$$

The race stopping rule results in a decision when an accumulator surpasses a fixed threshold θ (Figure 3). Recent models that pair the race rule with various forms of the accumulator to account for both neural and behavioral data are detailed in Appendix A.

2.2.3. Choice Criterion. The final component of an accumulation model is the choice criterion. Given the response time t^*, the choice is determined by the largest accumulator at t^*, with

$$i^* = \arg\max_i \{Z_i(t^*;\mathbf{v})\}. \tag{11}$$

This criterion provides our first hint that the distribution of random utility will depend on the distribution of stopping time, since the stochastic choice is determined, in part, by the random variable t^*. This point is clarified in the following section.

3. Derivation of Additive Random Utility Model

In an accumulation model, the probability that an accumulator reaches a particular threshold (thereby implements a choice) depends on the vector $\mathbf{v}$. The open question is whether the choice probability function can be represented by an additive form of the RUM, and how this form relates to observables.

For the special case of the DDM, the answer is straightforward because the choice probabilities can be derived in closed form. It is well known that the DDM implies logistic choice probabilities (Cox and Miller 1965), and these choice probabilities are also implied by a RUM of the logit form (Luce and Suppes 1965, McFadden 1973).

Example 1. For $n=2$, if $Z_i(t)$ is a Brownian motion (6) and the stopping rule is differenced (8) with constant threshold $\theta(t)=\theta$, then the resulting choice probabilities are logistic

$$P_i(\mathbf{v}) = \frac{1}{1+e^{-2(v_i-v_j)\theta/\sigma^2}}. \tag{12}$$

Furthermore, they can be represented by an additive RUM in which the η_i follow the independent Gumbel distribution. □

Thus, the choice probabilities implied by a DDM are represented by an additive RUM that depends only on the difference in valuations $v_i - v_j$.[16] However, the Gumbel distribution is not unique in yielding the logit for binary choice. The actual distribution implied by the DDM will be taken up in Section 4.1.

For the general class of accumulation models and stopping rules given by Equations (4) and (7), closed-form expressions for the choice probabilities are not known. However, we can still demonstrate that a BAM yields an additive RUM. To provide some intuition, consider the simple case of a Brownian motion with drift.

Example 2. For $Z_i(t)$ given by a Brownian motion with drift (6), a response time t^*, and choice criterion (11), the resulting choice probabilities can be represented by an additive RUM in which $\eta_i(\mathbf{v},t^*) = \sigma B_i(t^*)/t^*$.

Proof. The goal is to isolate the deterministic quantity v_i in the choice criterion, and in doing so, characterize the distribution of the stochastic term to which v_i is compared. Begin by noting that the choice criterion (11) is preserved under a linear scaling, $\lambda > 0$, at time $t^* > 0$.[17]

$$i^* = \arg\max_i \{Z_i(t^*; v_i)\} = \arg\max_i \{\lambda Z_i(t^*; v_i)\}. \tag{13}$$

The solution to the differential equation (6) at time t^* is

$$Z_i(t^*; v_i) = v_i t^* + \sigma B_i(t^*). \tag{14}$$

This expression for $Z_i(t^*; \mathbf{v})$ is fully characterized by separable terms consisting of the exogenous value v_i and the realizations of the stochastic process $B_i(t)|_{t=0,\ldots,t^*}$. Substituting (14) into (13) yields the choice criterion

$$i^* = \arg\max_i \{\lambda v_i t^* + \lambda\sigma B_i(t^*)\}.$$

Now all that remains is to choose a suitable value for λ. Define $\lambda \equiv 1/t^* > 0$; therefore, the choice criterion becomes

$$i^* = \arg\max_i \left\{ v_i + \frac{\sigma B_i(t^*)}{t^*} \right\},$$

with choice probabilities given by

$$\Pr\left[v_i - v_j > \frac{\sigma(B_j(t^*) - B_i(t^*))}{t^*}, \forall\, j \neq i \right].$$

This expression has the form of an additive RUM (2) in which η_i is comprised of the location of the Brownian motion $B_i(t^*)$, scaled by the stopping time t^*,

$$\eta_i(\mathbf{v}, t^*) = \frac{\sigma B_i(t^*)}{t^*}. \tag{15}$$

Note that η_i does not depend directly on v_i (nor, in this special case, $v_j, \forall\, j \neq i$); therefore, η_i satisfies Condition 1. Condition 2 follows from the symmetry of the stopping rules. □

This derivation of $\eta_i(\mathbf{v}, t^*)$ for the special case of Brownian motion clarifies the link between the stochastic process governing accumulation and the choice probabilities from the implied RUM. More broadly, it can be stated for the general form of bounded accumulation models as a sufficient condition for an additive RUM.[18]

Proposition 1. *For the general accumulator given by (4), a response time t^*, and choice criterion (11), the resulting choice probabilities can be represented by an additive RUM that satisfies Conditions 1 and 2.*

Proof. The proof is provided in Appendix D. □

Moreover, the functional form for $\eta(\mathbf{v}, t^*)$ follows directly from assumptions on the stochastic process. The following remark notes some commonly studied cases.

Remark 2. If $\boldsymbol{\Gamma}(t) = \boldsymbol{\Gamma}$, $c(t) = 0$, $\boldsymbol{\sigma}(t) = \sigma\mathbf{I}$ and $n = 2$,

$$\eta(\mathbf{v}, t^*) = \frac{\int_0^{t^*} e^{-\gamma t} \sinh(\psi(t^*-t))\, dt}{\int_0^{t^*} e^{-\gamma t} \cosh(\psi(t^*-t))\, dt} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} + \frac{\int_0^{t^*} e^{-\gamma t} \begin{pmatrix} \cosh(\psi(t^*-t)) & \sinh(\psi(t^*-t)) \\ \sinh(\psi(t^*-t)) & \cosh(\psi(t^*-t)) \end{pmatrix} \sigma\, d\mathbf{B}(t)}{\int_0^{t^*} e^{-\gamma t} \cosh(\psi(t^*-t))\, dt},$$

if $\psi = 0$,

$$\eta_i(\mathbf{v}, t^*) = \gamma\sigma \frac{\int_0^{t^*} e^{-\gamma t}\, dB_i(t)}{1 - e^{-\gamma t^*}}, \quad \forall\, i,$$

and if $\gamma = 0$,

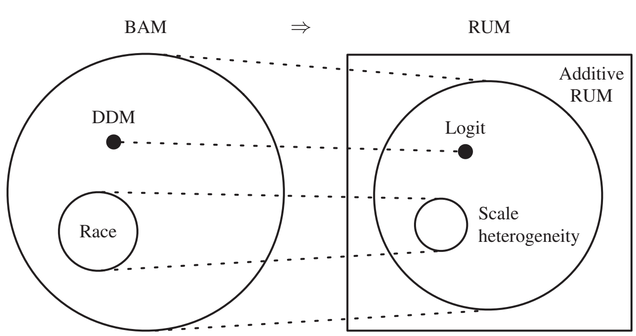
$$\eta_i(\mathbf{v}, t^*) = \frac{\sigma\, B_i(t^*)}{t^*}, \quad \forall\, i. \quad \square$$

The derivation of an additive RUM from a BAM has three important implications for discrete choice modeling, represented schematically in Figure 4. First, it is possible to assess whether existing results for models of optimal sequential sampling (Woodford 2016, Fudenberg et al. 2017) fall in the class of RUM. The answer is yes, because these models satisfy the condition of a BAM as defined in Section 2.2. This allows known results for RUM to be applied to the general class of BAM. For instance, it is not immediately apparent whether all models in the BAM class satisfy a *Regularity* condition (Block and Marschak 1959), because closed-form choice probabilities are not available for the majority of these models. However, Proposition 1 provides the answer. Since regularity is a necessary condition for a RUM, and a BAM is a RUM, the following corollary holds.

Corollary 1. *The choice probabilities implied by a BAM with general accumulator* (4), *a response time* t^*, *and choice criterion given by* (11) *satisfy regularity.*

Second, the choice probabilities implied by a BAM can be captured within the empirical RUM framework currently used in econometric application. Since the entire class of RUMs can be approximated by a Generalized Extreme Value (GEV) model (Dagsvik 1995, McFadden and Train 2000), this means that the choice probabilities resulting from a BAM can also be approximated by GEV with an appropriate mixing distribution, thus fit to choice data alone. Example 1 is an (exact) special case. However, the sufficiency argument of Proposition 1 places relatively weak structure on $\boldsymbol{\eta}$ for the general case.[19] Therefore, it still remains for the choice modeler to specify a suitable mixing distribution to approximate $\boldsymbol{\eta}$. As previously established in the literature, this is a daunting state of affairs with many degrees of freedom available to the researcher, and common specifications (such as a normal mixing distribution in a "mixed-logit") might be seriously misspecified (Louviere et al. 2002, Fiebig et al. 2010). This is where the third implication provides guidance.

Figure 4. Relation Between BAM and RUM



The dynamics of the accumulation process imply a distribution of $\boldsymbol{\eta}$ through the mapping $\eta(\mathbf{v}, t^*)$. For example, Proposition 1 demonstrates how the error term $\boldsymbol{\eta}$ is scaled by the response time t^*, while Example 1 details one special case in which it is not. This implies that data sets that collect response time can yield a more accurate characterization of the distribution of $\boldsymbol{\eta}$ and how it might be scaled as a function of observables. As the general class of BAMs is constrained (whether via direct observation of t^*, or via neural measurements of the bounded accumulation process), this guides the choice of structure to impose on the behavioral model. Moreover, in some cases the relationship between t^* and $\mathbf{v}$ arising from a particular stopping rule will allow the moments of $\tilde{\eta}_{ji}$ to be characterized in terms of the observables that determine $\mathbf{v}$; therefore, response time need not even be observed. These relationships have implications for econometric analysis that we explore in greater detail in the following sections.

4. The Distribution of Stochastic Choice

The distribution of stochastic choice that arises from a BAM depends on the form of accumulation (e.g., γ, ψ), the statistics of the stochastic process (e.g., σ), and the distribution of response time t^* that results from the stopping rule. This latter dependence on the random variable t^* requires particular consideration.

Some intuition can be gleaned from the case of a Brownian motion derived in (15) and a fixed stopping time. Let $\tilde{\eta}_{ji}$ denote the differenced error term. Therefore, $\tilde{\eta}_{ji}$ can be expressed in terms of the stochastic processes $B(t^*)$, scaled by the stopping time t^*,

$$\tilde{\eta}_{ji} \equiv \eta_j(\mathbf{v}, t^*) - \eta_i(\mathbf{v}, t^*) = \frac{\sigma\big(B_j(t^*) - B_i(t^*)\big)}{t^*}. \tag{16}$$

Under a fixed stopping rule (i.e., $t^* = \bar{t}$), we therefore have the following result for the distribution of $\tilde{\eta}_{ji}$.

Example 3. For a Brownian motion accumulator (6) and a fixed stopping time $\bar{t}$, $\tilde{\eta}_{ji}$ is distributed $\mathcal{N}(0, 2\sigma^2/\bar{t})$.

Proof. For some constant $\bar{t}$, $B_i(\bar{t}) \sim \mathcal{N}(0, \bar{t})$, $\forall\, i$, and $\tilde{\eta}_{ji} = \sigma(B_j(\bar{t}) - B_i(\bar{t}))/\bar{t}$. Therefore, $\tilde{\eta}_{ji} \sim \mathcal{N}(0, 2\sigma^2/\bar{t})$ with variance decreasing in $\bar{t}$. $\square$

This simple example demonstrates the role that time plays in determining the distribution of stochastic choice. A BAM in which the time allotted to a decision is exogenously varied yields a probit model with heteroskedasticity of known form: the variance is scaled down by time.[20]

In fact, the re-expression of $\tilde{\eta}_{ji}$ as a ratio in which t^* appears in the denominator also holds for more general processes and stopping rules (see the proof of Proposition 1), with the caveat that t^* is a random variable. Though the density of a ratio of (correlated) random variables is not easily characterized, the fact that the denominator increases in t^* will allow us to make statements about the distribution and/or moments of $\tilde{\eta}_{ji}$ for various cases of the general formulation. In particular, we can investigate how the distribution of t^* depends on $\mathbf{v}$ and how it impacts the choice probabilities for various formulations of BAM.

In doing so, it is convenient to introduce the concept of a *first passage time*. For each accumulator $Z_i(t)$, the random variable t_i is defined as the first time $Z_i(t)$ exits the region defined by the stopping rule. For example, under a race stopping rule, $t_i = \inf\{t\colon Z_i(t) \geq \theta_i\}$. The cumulative distribution function (CDF) of the first passage time distribution is denoted by $G(t; v_i)$, where the dependence on the valuation v_i is made explicit and depends on the choice of stopping rule. First passage times will prove useful in demonstrating the following results.

1. Example 1 demonstrated that a Brownian motion accumulator combined with the differenced stopping rule (i.e., the DDM) yields logistic choice probabilities. These closed-form choice probabilities are derived, in part, from the first passage time distributions (Cox and Miller 1965). Section 4.1 verifies that the formulation of $\eta(\mathbf{v}, t^*)$ resulting from the DDM recovers the logit model from a different class of distributions for $\boldsymbol{\eta}$ than the independent Gumbel distribution.

2. The first passage times provide a means to characterize t^* via the stopping rule. For instance, under the race stopping rule the density of t^* is determined by the first-order statistic $t^* \equiv \min\{t_1, \ldots, t_n\}$. Section 4.2 demonstrates how $\eta(\mathbf{v}, t^*)$ depends on $\mathbf{v}$ and t^* under this rule, and how this implies a form of scale heterogeneity in which the variance of the error $\eta(\mathbf{v}, t^*)$ depends on observables. This provides a theoretical justification for the observation of scale heterogeneity typically observed in the literature (Fiebig et al. 2010), and suggests a means to characterize its distribution.

3. For the general class of accumulators in (4), where no closed-form expressions for the distributions of $Z(t)$, t^*, or even t_i exist, numerical approximations of the first passage time distributions are still possible. Through sampling from these distributions, stopping rules given by Equation (7) can be implemented via simulation, and numerical methods can be used to approximate choice probabilities for estimation. How well this approximation captures the actual choice process—and choice data in particular—then becomes an empirical question. These methods are described in Section 4.3 before moving to an example application in Section 5.

4.1. Differenced Stopping Rule

It is instructive to begin with some clarification on how the density derived via $\eta(\mathbf{v}, t^*)$ (Proposition 1) relates to known results for the logit and the DDM (Example 1). Under a differenced stopping rule (8), the magnitude of the difference between the accumulators at t^* must be fixed at θ. If we let R denote a random variable that takes on θ if the upper threshold is crossed (alternative i is chosen), and $-\theta$ otherwise, then $Z_i(t^*) - Z_j(t^*) = R$, and

$$B_i(t^*) - B_j(t^*) = \frac{R - t^*(v_i - v_j)}{\sigma}. \tag{17}$$

For intuition, consider a diffusion with no drift ($v_i - v_j = 0$ and $\sigma = 1$). Then $B_j(t^*) - B_i(t^*)$ must either be θ or $-\theta$ for the accumulation to terminate (Figure B.1).

More generally, substituting (17) into (16) yields

$$\tilde{\eta}_{ji} = \eta_j(\mathbf{v}, t^*) - \eta_i(\mathbf{v}, t^*) = \frac{-R}{t^*} + (v_i - v_j),$$

with choice probabilities given by

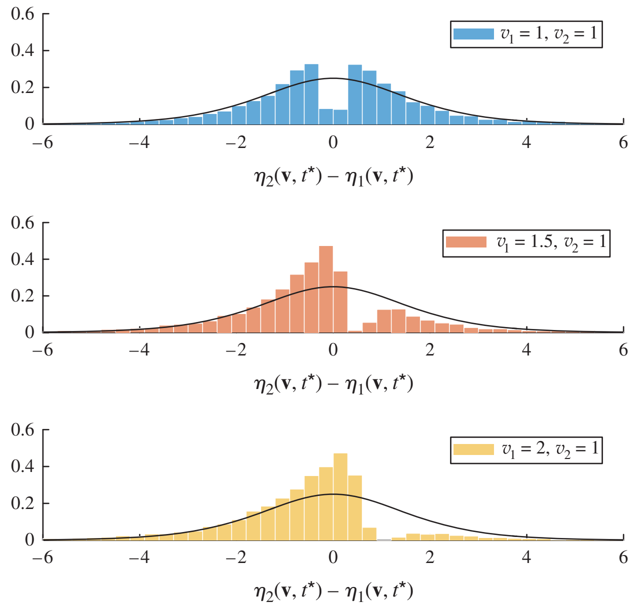
$$\begin{aligned} P_i(\mathbf{v}) &= \Pr\left[v_i - v_j > -R/t^* + (v_i - v_j),\ \forall\, j \neq i\right] \\ &= \Pr\left[R/t^* > 0,\ \forall\, j \neq i\right]. \end{aligned} \tag{18}$$

Since $t^* > 0$ and the probability that $R > 0$ is given by the logistic function in Equation (12), the logit model has been recovered, but from a different class of distributions for $\boldsymbol{\eta}$ than previously noted in the literature (i.e., the independent Gumbel; see Figure 5).[21]

How can two starkly different classes of distributions for η_i yield the same choice probabilities? The necessary and sufficient relationship between the Gumbel random variables η_i and the logit (McFadden 1973) applies only when η_i are independent, and only holds for cases in which there are more than two choice alternatives (Yellott 1977). Therefore, the necessary condition does not hold under the derivation from a DDM since $\eta_i(\mathbf{v}, t^*)$ and $\eta_j(\mathbf{v}, t^*)$ are not independent through their relation to t^*. Moreover, the existing closed-form results for the DDM choice probabilities are only relevant for the binary case, by definition. Therefore, multiple specifications of η_i can yield the logit choice probabilities.

With that said, the form for $\eta(\mathbf{v}, t^*)$ derived from the DDM seems particularly unusual compared to the general class of BAM. While $t^* > 0$ scales the random variable R in (18), it has no bearing on whether the ratio

Figure 5. (Color online) Density of $\eta_2 - \eta_1$ Simulated from the DDM for Different $\mathbf{v}$



Notes. For each simulation, $\sigma = 1$ and $\theta = 20$. The crossed threshold was divided by the realization t^* according to the calculation in Proposition 1. The corresponding distribution for t^* can be found in Figure 3. For comparison, the logistic density, derived from the difference of two Gumbel random variables (12), is depicted in black.

R/t^* is greater or less than 0. Therefore, under the differenced stopping rule and a Brownian motion, the distribution of t^* does not alter the choice probabilities from the logistic form. This property is not shared by other stopping rules, as demonstrated in the following section.

Beyond the simple DDM, closed forms for the choice probabilities under the differenced stopping rule are not available. Busemeyer and Townsend (1992) give an expression under a restricted version of (5) (where $\nu()$, $\gamma()$, and $\sigma()$ are constant) in terms of a ratio of definite integrals, and numerical simulations suggest these probabilities indeed are of the logistic form. Also see the special case studied by Woodford (2016). However, when the restrictions on these functions are relaxed, for instance, a drift rate or boundary $\theta(t)$ that varies in time, choice probabilities can deviate from the logit (Srivastava et al. 2017), if only slightly (Fudenberg et al. 2017).

Though no analytic statements for the choice probabilities are available for the differenced stopping rule and the general accumulation (4), approximations of the choice probabilities are available for the case of binary choice. Define $\hat{t}$ as the first stopping time at the upper boundary for the one-dimensional process $Z_1(t) - Z_2(t)$, conditional on not having crossed the lower. The first passage time density $g_{\hat{t}}(t)$ can then be approximated by numerical methods given in Smith (2000).[22] Since the process terminating at one boundary (versus the other) is mutually exclusive, the choice probability for alternative 1 is given by integrating $g_{\hat{t}}(t)$,

$$P_1(\mathbf{v}) = \int_0^{\infty} g_{\hat{t}}(t)\,dt.$$

If a data set also contains information on response times, likelihood methods can be used to incorporate this added information to yield a more efficient estimate via

$$P_1(\mathbf{v}, E[t^*]) = \int_0^{E[t^*]} g_{\hat{t}}(t)\,dt.$$

We will see an example of this in Section 5.

4.2. Race Stopping Rule

A race stopping rule is an example of a stopping rule in which the levels, not just the differences, of each accumulator are relevant to the decision. Importantly, it yields a straight-forward characterization of the stopping time as an order statistic of the first passage times for each accumulator, $t^* = \min[t_1, \ldots, t_n]$, with associated density $g_*(t; \mathbf{v})$. Given the expression for $\tilde{\eta}_{ji} = \eta_j(\mathbf{v}, t^*) - \eta_i(\mathbf{v}, t^*)$ derived via Proposition 1, it is therefore possible to derive the joint density of $\tilde{B}(t^*)$ and t^* (Appendix C).[23] For special cases of the accumulation process, this density yields insight into how the moments of $\tilde{\eta}_{ji}$ depend on $\mathbf{v}$. Importantly, this extends to more general cases.

As an example, consider two uncoupled accumulators governed by a simple Brownian motion with drift, and the race stopping rule. Figure 6 depicts the variance of $\tilde{\eta}_{21}$ resulting from such a model, calculated via (16) (Appendix C). In particular, note that $\mathrm{Var}(\tilde{\eta}_{21})$ scales with the magnitudes v_1, v_2, and $v_1 - v_2$. A model with constant variance (such as the simple DDM/logit) would exhibit none of these relationships.

This relationship between $\tilde{\boldsymbol{\eta}}$ and $\mathbf{v}$ has an important implication for choice sets in which the valuations are increased, yet the relative differences are constant. Consider two choice sets, both of size n, one with valuations $\mathbf{v}$ and the other increased uniformly to $\mathbf{v} + \alpha\mathbf{1}$ where $\alpha > 0$. Denote $P_{\bar{i}}$ as the probability of choosing the highest-valued alternative (i.e., $v_{\bar{i}} - v_j > 0$, $j \neq \bar{i}$), and define the set $D_{\mathbf{v}}$ as the realizations of the differenced error vector $\tilde{\boldsymbol{\eta}} \equiv [\tilde{\eta}_{1\bar{i}}, \ldots, \tilde{\eta}_{n\bar{i}}]$ for which $\bar{i}$ is chosen:

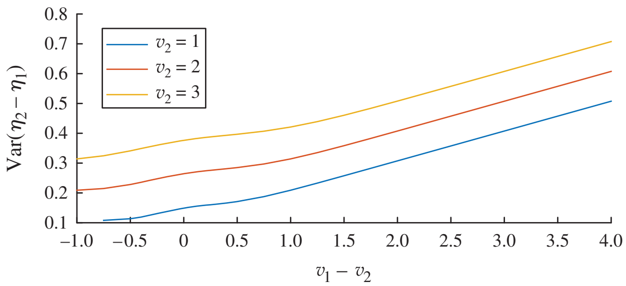
$$D_{\mathbf{v}} \equiv \{\tilde{\boldsymbol{\eta}}\colon v_{\bar{i}} - v_j > \tilde{\eta}_{j\bar{i}},\ \forall\, j \neq \bar{i}\}.$$

Proposition 2. *Any uniform increase of* $\mathbf{v}$ (*i.e.*, $\alpha > 0$) *that transfers probability density toward realizations of* $\tilde{\boldsymbol{\eta}} \notin D_{\mathbf{v}}$ *implies that* $P_{\bar{i}}(\mathbf{v}) > P_{\bar{i}}(\mathbf{v} + \alpha\mathbf{1})$.

Proof. The proof is provided in Appendix D. □

Figure 7 illustrates two such examples. The first is the Brownian motion process just considered. The second is a coupled accumulator ($\psi < 0$) that is at the forefront

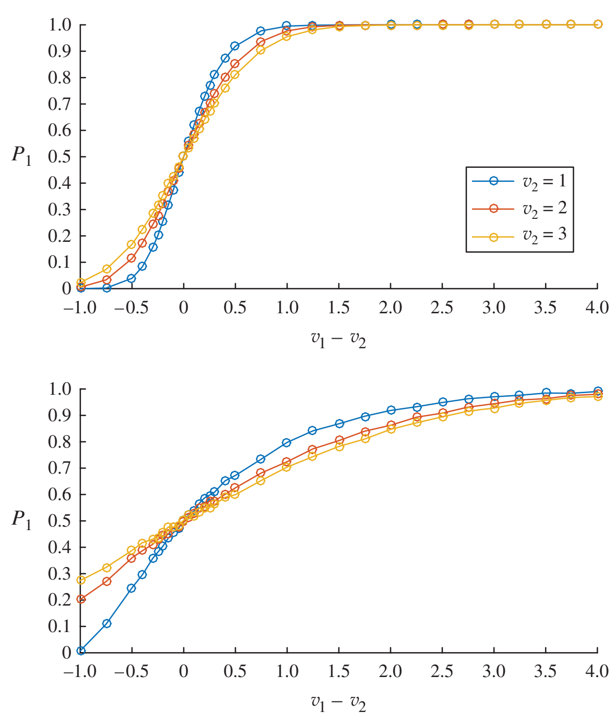
Figure 6. Variance of $\tilde{\eta}$ Resulting from Two Brownian Motion Processes and a Race Rule, Depicted as a Function of $\mathbf{v} = [v_1, v_2]$



of current practice in the neuroscience and psychology literature (the Leaky Competing Accumulator model; Usher and McClelland 2001).[24] In both cases, the choice probability of alternative 1 depends on both the differences in valuations *and* the magnitudes. An additive RUM with constant variance, such as the logit, does not share this property. It would predict choice probabilities that lie on a single curve for all v_2. The econometric implications of this result will be explored in Section 5

Finally, note that the scaling of the variance with $\mathbf{v}$ does not imply that preferences (in a stochastic sense) are reordered. Any increase in the value of an alternative will increase the probability that its accumulator will hit the threshold first, regardless of the stochasticity added by the decrease in the expected response time (Appendix C).

Figure 7. Choice Probabilities for the Race Stopping Rule Paired with a Brownian Motion Accumulator ($\psi = 0$, Top) or a Coupled Accumulator ($\psi < 0$, Bottom)



4.3. Numerical Methods for Approximating Choice Probabilities

Where closed-form expressions for the distributions of $Z(t)$, t^*, or t_i do not exist, as in the general accumulation (4), a numerical method for approximating the first passage time density is still possible. Since the race stopping rule allows easy characterization of the stopping time t^*, it provides a useful example relevant for the empirical demonstration in Section 5. Note that this method generalizes to other stopping rules, including the differenced rule as well as rules not pursued in this article. All that is required is a functional relation deriving the choice i^* from the first passage times $\{t_1, \ldots, t_n\}$.

In the case of the race stopping rule, this function is given by $i^* = \arg\min\{t_1, \ldots, t_n\}$. One immediate implication is that the random variables t_i can be directly interpreted as a random utility,

$$i^* = \arg\max_i\{u_i\} = \arg\max_i\{Z_i(t^*)\} = \arg\min_i\{t_i\},$$

since the random vector $[t_1, \ldots, t_n]$ represents the choice probabilities (Marley and Colonius 1992). Therefore, the result for the binary choice probability noted earlier extends to n alternatives,

$$P_i = \Pr[t_i < t_j, \forall i \neq j]. \tag{19}$$

Since the density $g_i(t; \mathbf{v})$ can be approximated numerically for a wide range of stochastic processes (Smith 2000), repeated sampling from this density yields a simulated approximation of P_i via (19), and the methods of maximum simulated likelihood can be applied (Train 2009).[25] An example of this method, and the results from the earlier sections, is now pursued in the context of choice under uncertainty.

5. Econometric Implications

As noted in Section 4, the formal mapping between the class of BAM and an additive RUM determines how the distribution of stochastic choice depends on $\mathbf{v}$ and response time. This has two implications for the applied researcher:

1. The BAM provides guidance on how to correctly specify an econometric analysis, including how to incorporate data on response time.
2. These insights can be applied to data sets for which response time is not observed. The relationship between t and $\mathbf{v}$ means that standard observables can be used in place of time in the empirical specification. This not only improves estimation, but allows the researcher to reject classes of BAMs even if data on response time is not observed.

These implications apply quite broadly to the applied literature on discrete choice. In contrast with linear models, it is well known that misspecification

Table 2. Lottery Choices in the Holt–Laury Risk Aversion Experiment

Lottery 1 (safe)				Lottery 2 (risky)						
$p_{\$2}$	(\$)	$p_{\$1.60}$	(\$)	$p_{\$3.85}$	(\$)	$p_{\$0.10}$	(\$)	EV_1 (\$)	EV_2 (\$)	$EV_1 - EV_2$ (\$)
0.1	2	0.9	1.60	0.1	3.85	0.9	0.10	1.64	0.48	1.17
0.2	2	0.8	1.60	0.2	3.85	0.8	0.10	1.68	0.85	0.83
0.3	2	0.7	1.60	0.3	3.85	0.7	0.10	1.72	1.23	0.49
0.4	2	0.6	1.60	0.4	3.85	0.6	0.10	1.76	1.60	0.16
0.5	2	0.5	1.60	0.5	3.85	0.5	0.10	1.80	1.98	−0.17
0.6	2	0.4	1.60	0.6	3.85	0.4	0.10	1.84	2.35	−0.51
0.7	2	0.3	1.60	0.7	3.85	0.3	0.10	1.88	2.73	−0.84
0.8	2	0.2	1.60	0.8	3.85	0.2	0.10	1.92	3.10	−1.18
0.9	2	0.1	1.60	0.9	3.85	0.1	0.10	1.96	3.48	−1.52
1	2	0	1.60	1	3.85	0	0.10	2.00	3.85	−1.85

of discrete choice models leads to inconsistent estimates (Greene 2003). For the range of BAMs studied here, the implications of misspecification apply generally to preference functions both linear and nonlinear in parameters, and across different choice domains. To demonstrate this point, let us consider an example from the domain of choice under uncertainty. The goal of this exercise is to provide a simple example in which the estimation of a structural preference parameter hinges crucially on model specification, and to highlight how insight into the underlying accumulation model can improve estimation.

5.1. Example: Measuring Risk Aversion

Issues with estimating risk preferences using the RUM framework have been previously documented (Apesteguia and Ballester 2018), and these include the observation that the misspecification of the stochastic structure can lead to incorrect model inference (Wilcox 2008, 2011). To illustrate the additional issues highlighted in Section 4, I now consider a simple example involving the estimation of a structural parameter for risk aversion from a well-known experiment on choice over uncertainty (Holt and Laury 2002).[26]

In Holt and Laury (2002), each subject was presented with a list of lottery pairs ordered by the difference in their expected value, commonly referred to as a "multiple price list" paradigm (Table 2). As each subject picks a lottery from each of the pairs, their willingness to choose a lottery with negative expected value is taken to reflect the degree of risk aversion in their preferences.

From choice data of this type, it is common to estimate a structural model of risk preferences using standard likelihood methods, provided an assumption on the distribution of stochastic choice (Camerer and Ho 1994, Hey and Orme 1994, Holt and Laury 2002, Harrison and Rutstrom 2008). Typically, the logistic function is assumed. However, the MPL contains a property that makes this assumption critical. Note that while the differences in the expected value of the lotteries are symmetric (around zero) in Table 2, the magnitudes of the expected values increase further down the list. If there is a relationship between the magnitude of the expected utilities and the degree of stochasticity in choice, this relationship should be accounted for in the specification of the model.

For instance, Section 4.2 describes a relationship between the magnitude of v_i and the variance of $\tilde{\eta}_{ji}$ under a race stopping rule. This would imply that the variance of the error term scales with the expected utility of the safe and risky lottery (EU_1 and EU_2, respectively) and therefore will *depend on the lottery pair* in the multiple price list experiment. This amounts to an additive RUM with heteroskedasticity of a form that, if ignored in estimation, will lead to misspecification of the choice probabilities and biased estimates of model parameters.

A Monte Carlo analysis demonstrates this result for two versions of the accumulator. Lottery choices were simulated for 20 risk-neutral subjects, with choice stochasticity introduced using a BAM with a race stopping rule. The accumulator was parameterized as both an independent accumulation ($\psi = 0$) and a coupled accumulation ($\psi = -0.1$), with v_i given by the expected value of the lottery, and the decision terminated via the race stopping rule with $\theta = 10$. The variance of the accumulation, σ, was normalized to 1. Risk preferences were then estimated via a utility curvature parameter α from a constant relative risk aversion (CRRA) utility function $u(x) = x^{\alpha}$. An unbiased estimator would thus yield an estimate $\hat{\alpha} = 1$.

Table 3 reports the mean of estimates from 1,000 simulated data sets, estimated under the standard assumption of logistic choice probabilities. For estimation purposes, we specify the choice probabilities $P_i = (1 + e^{(EU_1 - EU_2)/\sqrt{\mathrm{Var}(\tilde{\eta}_{21})}})^{-1}$ with a constant variance given by the exponential function,

$$H_0: \quad \mathrm{Var}(\tilde{\eta}_{21}) = e^{s} > 0.$$

Under this specification of constant variance, the average estimate $\hat{\alpha}$ for the independent accumulator is

Table 3. Average Estimates of the CRRA Coefficient $\hat{\alpha}$ for Multiple Specifications of the Stochastic Choice Distribution

	Race with independent accumulator ($\psi = 0$)						
	H_0: Logit	H_{1a}: Time	H_{1b}: Time	H_2: Diff	H_3: Magn	H_4: Both	H_5: Race
$\hat{\alpha}$	0.954	0.943	0.993	0.951	0.986	1.021	0.997
$\hat{s}$	−2.245	−2.62	−1.30	−1.78	0.181	−0.951	−3.185
$\hat{k}$				−0.723		−3.388	
$\hat{h}_1$					−1.793	−2.295	
$\hat{h}_2$					0.392	1.952	
$\hat{g}$		0.0685	−0.208				
	Race with coupled accumulator ($\psi = -0.1$)						
	H_0: Logit	H_{1a}: Time	H_{1b}: Time	H_2: Diff	H_3: Magn	H_4: Both	
$\hat{\alpha}$	0.849	1.517	1.013	0.864	0.999	1.022	
$\hat{s}$	−0.483	−61.017	1.642	−0.040	0.408	0.699	
$\hat{k}$				−0.795		−1.587	
$\hat{h}_1$					−0.979	−0.413	
$\hat{h}_2$					0.509	0.063	
$\hat{g}$		15.388	−0.329				

Notes. The estimates are calculated for 1,000 simulated Holt–Laury data sets of a risk-neutral chooser ($\alpha = 1$). The true error distribution is given by a race model ($\phi = 1$, $\theta = 10$, $\sigma = 1$), with either independent accumulators ($\psi = 0$) or coupled accumulators ($\psi = -0.1$).

biased to 0.954, with a type I error rate of 12.1% under a 5% level test. Since α enters the utility function nonlinearly, this degree of bias varies depending on the lottery, but it is consequential. For the 50/50 version of lottery 2, the risk premium amounts to ~ 2.5% of the expected value (Figure 8). The bias in the estimate of risk aversion is even larger if the stochastic process generating the choices results from a coupled accumulator ($\psi < 0$). The average estimate $\hat{\alpha}$ is 0.849, yielding a risk premium near 10% of the expected value of lottery 2.

The bias in the estimate of $\hat{\alpha}$ arises because the variance of $\tilde{\eta}_{ji}$ depends, in part, on the magnitude of the expected utilities of each of the lotteries. In the Holt and Laury experiment, the lotteries with the largest expected utilities are found in pairs where $EU_1 - EU_2 < 0$, increasing the choice stochasticity for these lotteries. As a result, the choice probabilities, as a function of $EU_1 - EU_2$, are not symmetric around zero (as implied by the logistic function), and the estimator attempts to compensate by biasing the estimate of α (Figure 9).

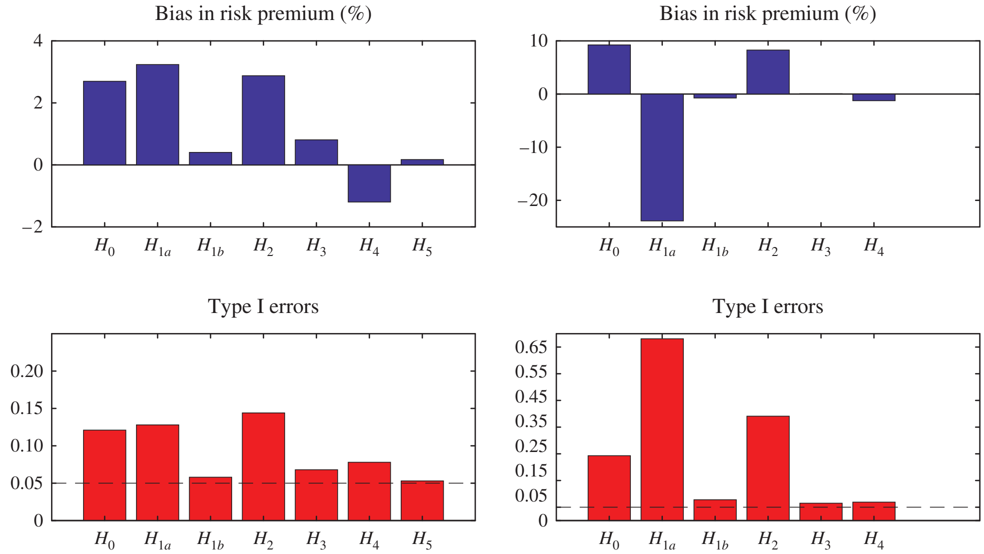
Fortunately, a range of methods are available to achieve less biased (or even unbiased) results, depending on the specification of choice stochasticity and the availability of response-time data. One possibility is to use data on response time directly. As noted in Section 4.2, the variance of the random utility model can be equivalently stated in terms of time: the variance of $\tilde{\eta}_{ji}$ increases as expected response times shorten. The following specification captures this relationship:

$$H_1: \quad \mathrm{Var}(\tilde{\eta}_{21}) = e^{s+gt}.$$

Two variations of this specification are considered. The first (H_{1a}) uses the response time on a given trial as proposed by Hey (1995). However, in an accumulation model, response time and choice are determined jointly from an underlying stochastic process. Therefore, response time is an endogenous variable that is correlated with the choice. Not surprisingly, directly including response time on a trial worsens the bias of the estimate $\hat{\alpha}$. To address this, the second specification (H_{1b}) instead uses the mean response time over repeated trials (i.e., $E[t^*]$) to specify the variance. Provided a large enough number of trials, this statistic will be uncorrelated with the choice on any given trial.[27] As expected, including exogenous information about the distribution of response times eliminates the bias in $\hat{\alpha}$ for both the independent and the coupled accumulator. Type I error rates are 5.8% and 7.8%, respectively, yielding substantial improvement in inference compared to the logit specification.

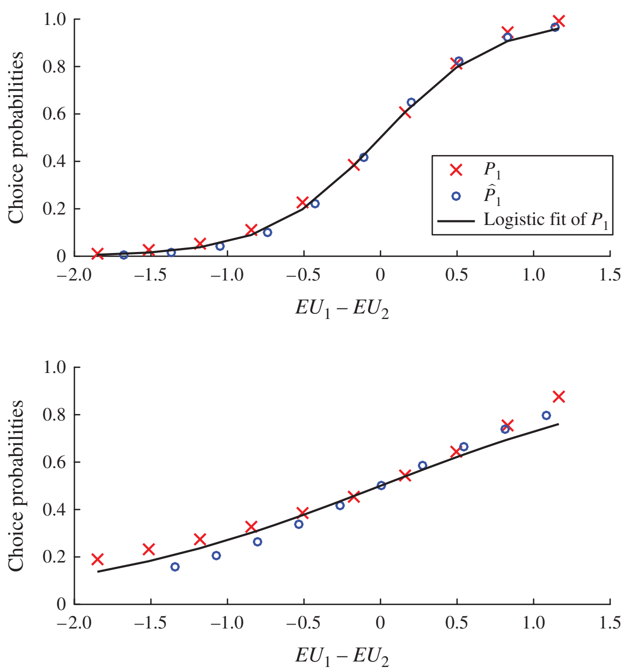
A specification that includes response time can also be used to distinguish between the accumulation models that might have generated the data. In particular, the estimated coefficient on response time ($\hat{g}$) can be used to test for the absence of heteroskedasticity predicted by the race model via a standard likelihood ratio test. For the Holt–Laury experimental design, the estimates from simulations are significantly negative (pooled $p = 0.00$); thus, time decreases the variance of the error term as predicted. This is consistent with the predictions of the race model, in contrast to the logit and simple DDM; thus, the DDM can be rejected as having generated the data. The available empirical

Figure 8. (Color online) Risk Premium and Type I Errors Implied by the Biased Estimate $\hat{\alpha}$, for Independent Accumulators ($\psi = 0$, Left) and Coupled Accumulators ($\psi = -0.1$, Right)



evidence suggests this is also the case in practice. For instance, Hey (1995) observed that 68 of 80 subjects exhibited a decreasing relationship between the time a decision is made and the variance of the error term; again, inconsistent with the DDM.

Figure 9. (Color online) Choice Probabilities from a BAM with Race Stopping Rule ($\psi = 0$, Top) and ($\psi = -0.1$, Bottom)



Notes. For comparison, the true choice probabilities (×) all lie above a logistic fit (black). An estimation that uses logit choice probabilities with constant variance (∘) yields a biased estimate of $\hat{\alpha}$ by shifting these points horizontally.

What if (repeated) data on response times is not available? One possibility, originally suggested by Buschena and Zilberman (2000), is to include the absolute value of the difference in expected utility in the specification.

$$H_2\colon\quad \mathrm{Var}(\tilde{\eta}_{21}) = e^{s+k|EU_1-EU_2|}$$

Another is to allow the variance to depend on the magnitudes, as suggested by the results in Section 4.2.

$$H_3\colon\quad \mathrm{Var}(\tilde{\eta}_{21}) = e^{s+h_1EU_1+h_2EU_2}$$

Results from these two specifications are reported in Table 3. While including the difference in expected utility yields little improvement, including the magnitudes reduces the bias of $\hat{\alpha}$ considerably for the independent accumulator ($\psi = 0$), and nearly eliminates the bias for the coupled accumulator ($\psi = -0.1$). Type I errors are reduced to 6.8% and 6.5%, respectively (Figure 8). Moreover, the estimated coefficients on the magnitude ($\hat{h}_1$ and $\hat{h}_2$) can be used to reject the logit model via a likelihood ratio test.

Again, these specifications can also be used to test the form of the underlying accumulation model. Significant parameter estimates for $\hat{k}$ and $\hat{h}$ will reject the hypothesis that the DDM generated the data, even if time is not directly observed in the data set. The full specification yields a similar pattern of results.

$$H_4\colon\quad \mathrm{Var}(\tilde{\eta}_{21}) = e^{s+k|EU_1-EU_2|+h_1EU_1+h_2EU_2}$$

If the underlying accumulation model is correctly specified, the methods in Section 4.3 can be used to arrive at estimates for α and θ, even if data on response times are not available. For example, Section 4.2 describes how the race stopping rule with independent accumulators yields stopping times that are distributed Inverse Gaussian with mean θ/EU_i and variance θ^2. Drawing a large number of samples from these distributions (for each lottery) yields an approximation of P_i via Equation (19) for a given α and θ.[28] As expected, a maximum simulated likelihood estimation of α is unbiased (H_5), with a type I error rate of 5.3%.

6. Conclusion

Significant progress has been made in understanding the dynamic neural processes underlying choice behavior. From this line of theoretical and empirical research, a class of bounded accumulation models have been developed to link both the behavioral and neural data. This article demonstrates that each BAM pursued in the economics and neuroscience literature has an analogue additive random utility model—the benchmark framework for discrete choice in econometric applications. This provides a neurobiological foundation for random utility maximization, and establishes that the large class of BAM models spanning economics, neuroscience, and psychology satisfy the necessary condition of this representation.

The relationship between these two classes of models also has important consequences for testing economic theory and predicting choice behavior. The specific parameterization of a BAM—particularly its dynamics, stopping rule, and implied distribution of response time—influences the resulting distribution of stochastic choice. This implies that both behavioral and neural data can be used to restrict the specification of additive RUMs, yielding improvements in choice prediction, and that choice data alone can be used to restrict the set of BAMs under consideration.

In the special case of a drift diffusion model for binary choice, the resulting choice probabilities are logistic but arise from a different class of error distributions than traditionally reported in the literature. At first glance, this is a promising result. It implies that a logit model can be applied to data generated by a DDM process without concern for bias due to misspecification, and that the joint distribution of choice and response times can improve the efficiency of model estimates. However, this result also suggests how restrictive a "necessary" version of Proposition 1 must be. Even in the simple case of the logit choice probabilities, there are two specifications of η_i that yield the same choice probabilities, only one of which is known to arise from an accumulation model. Therefore, while Proposition 1 clarifies which versions of BAM map to which specifications of an additive RUM—providing fruitful ground for empirical claims for both neural and behavioral data—the existence of a RUM representation (or, say, a particular form of choice probabilities like the logit) cannot be used to establish that a BAM must have generated the stochastic structure that we observe in choice data.

Beyond the DDM, more neurobiologically plausible models imply that there is scale heterogeneity in the variance of the error term, and this scaling will depend on observables. This provides a theoretical justification for the scale heterogeneity typically observed in a wide variety of empirical applications (Louviere et al. 2002, Fiebig et al. 2010). Webb et al. (2016a) describe a similar observation in the domain of IIA violations.[29] This result lies in contrast with the simple logit and has important implications for econometric applications. For instance, estimates of structural choice parameters will be biased if the logit or other common (mis)specifications are employed. Directly incorporating endogenous response-time data in the specification worsens the bias. Instead, multiple methods for correction are proposed. If response times are observed, a specification that uses average response times over repeated trials reduces the endogeneity problem. If response times are not observed, a specification that uses the levels of observables offers a correction via a control function approach. Both of these specifications can be used to test hypotheses on the underlying accumulation model using only behavioral data. Finally, a full correction is possible since the true choice probabilities can be approximated by simulating the stopping time distributions directly. Conveniently, these methods apply for the general formulation of a BAM.

The results in this article apply quite broadly. Misspecification of the choice probabilities can apply to any model in which the magnitude of utility might differ over choice sets, not just the domains of consumer choice or lotteries (Fiebig et al. 2010). Moreover, it is present for preference functions that are both linear and nonlinear in parameters and thus will apply to other contexts such as the estimation of time preference (as in Apesteguia and Ballester 2018). This suggests that, at a minimum, empirical researchers should incorporate average response times and/or valuations into the specification of variance in any analysis of structural preference parameters. In the domain of risk preferences, the small set of studies that do so find evidence for the form of scale heterogeneity implied by the race rule, in contrast to the logit and simple DDM (Buschena and Zilberman 2000, Hey 1995). This preliminary evidence suggests that collecting data on response times is particularly of value in future empirical applications.

The results are also applicable in domains beyond the estimation of individual-level preference parameters. Typically in economics and marketing, the stochastic component of a RUM, η_i, is interpreted as the (horizontal) preference for unobserved product attributes over a sample of individuals. This individual-level heterogeneity can be captured by the accumulation process (4). If the valuation for an individual v_i contains an additional (independent) stochastic element, then for each observation (now individuals, not trials), it will be accumulated in the stochastic process $\mathbf{B}(t)$ for that observation. One immediate implication is that the results presented carry through for the population if one assumes that the form and parameterization of the accumulation model is the same over individuals. For instance, the distribution of response times in the population sample could be used to correct the bias in (aggregate) preference parameters arising in estimation. More generally, this also raises an interesting application for modeling heterogeneity via Bayesian modeling techniques (Rossi et al. 2012). If the parameters of the accumulation were specified to vary over individuals according to some distribution, then, in principle, both the distribution of heterogeneity in v_i and in the accumulation parameters themselves can be estimated. Wiecki et al. (2013) provide an initial example of this method for the simple DDM. The further development and application of these tools to individual-level data would be of interest not only in economics and marketing, but also in more traditional applications of response-time data found in psychology and neuroscience.

The role of a stochastic accumulation process in generating stochastic choice behavior also bears further examination. While it is commonly observed that experimental subjects will switch their choices on repeated presentation of a choice set, recent evidence documents that this occurs even for repetitions of choice sets presented immediately one after the other (Agranov and Ortoleva 2017). This suggests that a hedging motive may be a predominant determinant of stochastic choice, with subjects preferring a mixture over choice alternatives. Of course, any mixture still requires a unique outcome realized on any particular trial. The role that stochasticity within an accumulation computation plays in implementing this mixture is initially considered by Webb and Dorris (2013).

Finally, this article also points to a methodological goal. Many authors have conjectured that knowledge of the choice process in the brain could help constrain the space of behavioral models one must consider (Spiegler 2008, Bernheim 2009, Sobel 2009, Dean 2013). There is no doubt that more research is required to provide a sharper specification of the stochastic choice distribution that results from the dynamic processes underlying decision. This would include further empirical study at both the neural and behavioral level. However, the formal relationship between these two levels of analysis implies that advances in neuroscientific modeling can reduce the degrees of freedom that behavioral researchers must consider, and vice versa. In essence, if one takes seriously the notion that behavior results from the brain, then modeling at the level of behavior and at the level of dynamic neural processes is a symbiotic exercise.

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Appendix A. State of BAM Literature
The pairing of an accumulator with a stopping rule defines a particular BAM. The definition of a general accumulation process (4) and a general stopping rule (7) ensures that our results will apply as widely as possible to the models explored in the neuroscience and psychology literature. However, it will be useful to describe the state of this literature in more detail.

An important feature of all bounded accumulation models is the prediction of a distribution for response times that matches empirical observations (Ratcliff 1978, Luce 1986, Roitman and Shadlen 2002). Typically, these distributions are positively skewed (e.g., Figure B.1). In the case of binary choice, the DDM (with the added feature of a stochastic initial condition) is considered the benchmark model (for a review, see Smith and Ratcliff 2004) and is particularly appealing if one assumes no opportunity cost of time. Under this assumption, the DDM implements a decision in the least amount of time for a given error rate, achieving a normative solution (Gold and Shadlen 2002).[30] However, when a cost of time is introduced—such that the decision maker must trade off between improving the accuracy of the current decision (by accumulating more evidence) versus beginning a new decision—the optimal stopping rule collapses the boundaries so that $\theta(t)$ is no longer constant in time (Drugowitsch et al. 2012, Fudenberg et al. 2017). Intuitively, the decision maker is willing to decrease accuracy in the current decision to arrive at a new decision problem earlier.

For the case of three (or more) alternatives, the optimal stopping rule(s) are not known. Various extensions of the differenced stopping rule have been proposed in the literature (McMillen and Holmes 2006, Krajbich and Rangel 2011, Niwa and Ditterich 2008, Ditterich and Churchland 2012); however, these approximate an optimal solution only asymptotically (as the degree of choice error goes to zero).[31]

This and other issues have led researchers to explore alternative accumulation processes and stopping rules. In particular, there is recognition that modeling binary choice as resulting from a single accumulator to dual threshold arises from mathematical convenience, not from neurobiological plausibility (Kiani et al. 2014). The neurobiological evidence suggests that multiple, competing accumulators $\mathbf{Z}(t)$ are the primitive objects underlying the neural processes of decision (Mazurek et al. 2003, Bogacz et al. 2007, Beck et al. 2008, Churchland et al. 2008, Furman and Wang 2008, Thevarajah et al. 2010). In particular, the evidence is incompatible with a stopping rule that depends on the difference between accumulators, and instead points to a fixed threshold that is constant over trials (Roitman and Shadlen 2002, Churchland et al. 2008, Niwa and Ditterich 2008; though see Heitz and Schall 2012). The race rule has this feature and, moreover, is easily generalizable to n choice alternatives.

A race stopping rule also has the feature that the "confidence" of a decision (the posterior belief that $v_1 > v_2$) can be inferred from the magnitude of $Z_1(t^*) - Z_2(t^*)$ at the time of decision (Drugowitsch and Pouget 2012). Intriguingly, neural measurements of this quantity correlate with the decision to opt out of an uncertain choice decision in favor of a certain option (Kiani and Shadlen 2009, Kiani et al. 2014). This observation is not compatible with a simple differenced stopping rule since $Z_1(t^*) - Z_2(t^*) = \theta$ is constant by definition.

However, for a simple Brownian motion accumulator, the race rule carries with it the odd prediction that response times do not depend on the magnitude difference between choice alternatives, an empirical regularity that has been observed in a wide range of experimental data sets (Luce 1986). For this reason, models that aim for both neurobiological plausibility and realistic characterization of response times use the more general stochastic process given in (4) in tandem with the race rule (Usher and McClelland 2001, Roxin and Ledberg 2008, Tsetsos et al. 2012). The coupling of the accumulators is grounded in principles of neural computation (i.e., mutual inhibition; Wang 2002, Wong and Wang 2006) and ensures that the accumulation is influenced by all of the choice alternatives, thus matching the same features of the response-time distribution as the DDM (Ratcliff and Smith 2004).

There is still much ongoing debate over the particulars of the dynamic process, the form of the boundary, and how a dynamic decision process can be implemented in neural architecture (e.g., Cisek 2006, Kiani et al. 2008, Tsetsos et al. 2010, Ditterich 2010, Hunt et al. 2012, Tsetsos et al. 2012, Ditterich and Churchland 2012, Heitz and Schall 2012, Liston and Stone 2013, Standage et al. 2013, Kiani et al. 2014). It is important to emphasize that in all of these implementations, the equation determining the choice is given by (11), and the difference between BAMs lies in the distribution of stopping times implied by the choice of accumulator and stopping rule. The fact that all BAMs can be written with this choice criterion and an appropriate distribution of stopping times allows a derivation of equivalence between bounded accumulation models and an additive RUM.

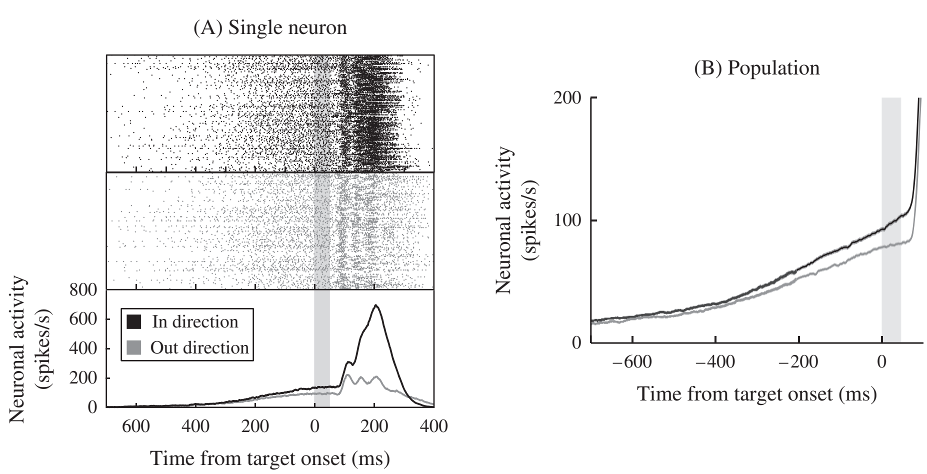
Appendix B. Neurophysiological Evidence for Accumulation

The dynamic process by which the brain reaches a decision is a topic of intense research interest in psychology and neuroscience. Originally, this work focused on the dynamics of *perceptual* decision making in which subjects are required to determine the state of an objectively known stimulus. For example, in the "random dot motion task" (Britten et al. 1992), a subject is tasked with determining the net direction of motion (left or right) of a number of erratically moving dots. The experimenter varies the number of dots moving in a particular direction as a means of varying the strength (cohesion) of the stimulus.

Electrophysiological recordings from these experiments revealed that neurons in the parietal cortex have the properties of an accumulation process (Roitman and Shadlen 2002, Mazurek et al. 2003). At the level of a single neuron, the rate of activity was larger for stimuli with higher coherence. At the population level, activity increased throughout the trial, with the rate of this increase determined by the strength of the stimulus. Moreover, the response of the animal conformed with the presence of a decision threshold: regardless of the motion strength, the response was executed when the neural activity reached a consistent level. These results served as initial neurophysiological evidence for an accumulation process and inspired a large number of studies focused on this experiment (Gold and Shadlen 2000, Kiani et al. 2008, Kiani and Shadlen 2009, Hanks et al. 2011, Shadlen and Kiani 2013, Kiani et al. 2014, Gold and Shadlen 2007, for an early review, see).

More recently, attention has turned to economic choice and whether the same accumulation mechanism can be observed

Figure B.1. Activity in Superior Colliclus Recorded During a Matching Pennies Game



Source. Reproduced from Thevarajah et al. (2010).
Notes. The neuron is associated with the In response. At time 0, the subject was free to choose either response. (A) A raster plot of a single neuron's spike activity, segregated by the choice of the subject (top and middle panes), and a smoothed average over trials (bottom pane). (B) The mean activity of all neurons in the sample, again segregated by the choice.

in this domain. One example is from a strategic decision-making task reported in Thevarajah et al. (2010). In this experiment, a monkey subject played the matching pennies game against an opponent. There were two actions in the game, *In* or *Out*, and the monkey received a reward if it correctly matched the opponent's action. In this game, the Nash equilibrium strategy for the monkey is to mix equally over the two actions on each trial. Observed behavior matched the predicted proportion, though with some correlation over trials.

During the trial, the monkey was required to fixate on a central location, and after a length of time, the subject was free to choose their response. The requirement for fixation, in which there were no other stimuli visible, allowed a clear measurement of valuation activity prior to the response without any interference from sensory stimuli.

Similar to the sensory domain, the neural activity observed in this experiment showed evidence for an accumulation. On trials in which the subject chose *In*, the activity of the neuron associated with the *In* response increased at a faster rate. Moreover, the rate of this activity varied with the valuation estimated using a standard learning model applied to strategic games. This learning model takes as arguments both the choice and outcome of previous trials, suggesting some dependence between choice trials. Further evidence for an accumulation model applied to this data set, and the relation between best response dynamics and stochastic choice, is discussed in Webb and Dorris (2013).

Appendix C. Relationship Between $\mathrm{Var}(\tilde{\eta})$ and $\mathbf{v}$ in a Binary Choice Race Model

To understand why the race stopping rule yields a relationship between $\tilde{\boldsymbol{\eta}}$ and $\mathbf{v}$, we consider the binary choice model (and thus drop the subscript ji on the scalar $\tilde{\eta}$). Given the expression for $\tilde{\eta} = \eta_2(\mathbf{v}, t^*) - \eta_1(\mathbf{v}, t^*)$ derived via Proposition 1, it is therefore possible to derive the joint density of $\tilde{B}(t^*)$ and t^*. Let $h_{\tilde{B}(t^*),t^*}(b,t)$ denote this density on the support $(b,t) \in \mathbb{R} \times \mathbb{R}_+$. Given the linear form of the accumulation, it can be expressed as

$$h_{\tilde{B}(t^*),t^*}(b,t) = h_{\tilde{Z}(t^*),t^*}(z,t), \quad \text{where } z = (v_2 - v_1)t + b.$$

Since the random variable $Z_2(t^*) - Z_1(t^*)$ takes on either a value $Z_2(t^*) - \theta$ or $\theta - Z_1(t^*)$, depending on which accumulator has won, $h(z,t)$ can be derived from the joint density that a Brownian motion accumulator i has reached the bound at time t and the losing accumulator is at location $z < \theta$. Moreno-Bote (2010) gives this density as the product of two "primitive" densities:

1. the density of the stopping time for the winning accumulator (i.e., the inverse Gaussian density with mean θ/v and variance θ^2/σ^2; Cox and Miller 1965, p. 221),

$$g(t;v) = \frac{\theta}{\sqrt{2\pi\sigma^2 t^3}} e^{\frac{-(\theta - vt)^2}{2\sigma^2 t}},$$

2. the location of the accumulator that did not win at time t,

$$f(z,t;v) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma^2 t}} \left[e^{\frac{-(z-vt)^2}{2\sigma^2 t}} - e^{\frac{2v\theta}{\sigma^2} - \frac{(z-2\theta-vt)^2}{2\sigma^2 t}} \right], & \text{if } z < \theta; \\ 0, & \text{otherwise.} \end{cases} \tag{C.1}$$

Therefore, $Z_2(t^*) - Z_1(t^*)$ takes on either a value $Z_2(t^*) - \theta$ or $\theta - Z_1(t^*)$ with corresponding density proportional to $f(z+\theta, t; v_2)$ or $f(\theta - z, t; v_1)$. This yields the desired joint density

$$h(z,t;[v_1,v_2]) = g(t;v_1) f(z+\theta, t; v_2) + g(t;v_2) f(\theta - z, t; v_1)$$

and the choice probability

$$P_1(v_1,v_2) = \Pr[t_1 < t_2] = \int_0^\infty g(\tau;v_1)(1 - G(\tau;v_2))\,d\tau, \tag{C.2}$$

where $G(t;v)$ is the CDF of the inverse Gaussian distribution.

While closed-form expressions for the integral P_1 are not available (Section 4.3 details how it can be approximated numerically), Figure C.1 depicts $h(z,t)$ for two exemplary cases of $\mathbf{v}$: $v_1 = v_2$ and $v_1 > v_2$.

In the case of $v_1 = v_2$, the joint density is given by

$$h(z,t;[v,v]) = g(t;v)(f(z+\theta,t;v) + f(\theta - z, t; v)),$$

where v is used to denote the value of v_1 and v_2. As v increases, more density is placed on smaller t (Figure C.1, top and middle panels). The fact that shorter stopping times enter the denominator of $\tilde{\eta}$ (via Proposition 1) is of primary relevance for the increasing relation between $\mathrm{Var}(\tilde{\eta})$ and v, stated formally in the following proposition.

Proposition 3. *For two Brownian motion accumulators, $Z_1(t)$ and $Z_2(t)$, with $v_1 = v_2 = v$, and a race stopping rule:*
a. $G_*(t;v)$ *is increasing in* v
b. $E[\tilde{\eta}] = 0$,
c. $\mathrm{Var}(\tilde{\eta}) = \int_0^\infty g_*(t;v)(1/t^2)\mathrm{Var}(\tilde{Z}_{ji}(t) \mid t^* = t)\,dt$,
d. $\mathrm{Var}(\tilde{Z}(t) \mid t^* = t)$ *is decreasing in* v.

Proof. The proof is provided in Appendix D. □

In particular, Proposition 3.c states the variance of $\tilde{\eta}$ in terms of the stopping time density $g_*(t;v)$ and the conditional variance of $\tilde{Z}(t)$. As v increases, the stopping time density $g_*(t;v)$ places greater weight on smaller values of t and increases $\mathrm{Var}(\tilde{\eta})$ via the factor $1/t^2$. This shrinking denominator outweighs the (small) decrease in $\mathrm{Var}(\tilde{Z}(t) \mid t^* = t)$ that arises from Proposition 3.d (Figure C.1, top panels). Note that $\mathrm{Var}(\tilde{Z}(t) \mid t^* = t)$ is decreasing in t, except for small values of t. This is demonstrated in Lemma 2 in Appendix D.

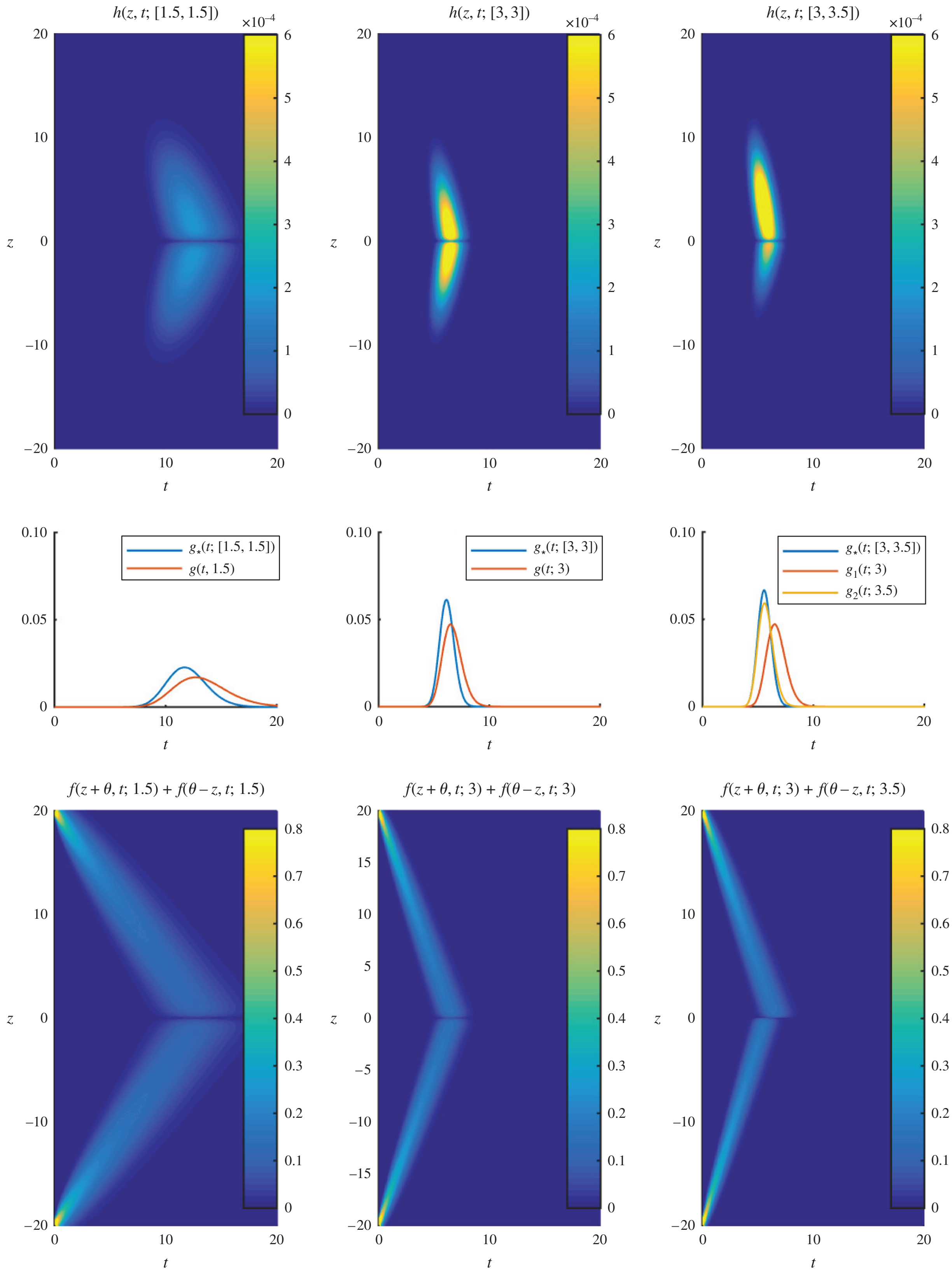
When $v_1 > v_2$, a formal statement is complicated by the fact that $E[\tilde{\eta}] \neq 0$; however, the intuition underlying the effect of increasing v_1 is similar: as more density is placed on shorter stopping times (see Lemma 1 in Appendix D), the variance of $\tilde{\eta}$ increases. Formally,

$$\begin{aligned} \mathrm{Var}(\tilde{\eta}) &= \int_0^\infty g_*(t;\mathbf{v}) \frac{1}{t^2} \mathrm{Var}(\tilde{Z}(t) \mid t^* = t)\,dt \\ &= \int_0^\infty g_*(t;\mathbf{v}) \frac{1}{t^2} E^2[\tilde{Z}(t) \mid t^* = t]\,dt \\ &\quad - \left[\int_0^\infty g_*(t;\mathbf{v}) \frac{1}{t} E[\tilde{Z}(t) \mid t^* = t]\,dt \right]^2. \end{aligned}$$

Note each term has a factor of t in the denominator of the integrand.

Finally, choice probabilities are still monotonic in valuations. If $P_i(\mathbf{v}) > P_j(\mathbf{v})$, then $P_i(\mathbf{v} + \alpha\mathbf{1}) > P_j(\mathbf{v} + \alpha\mathbf{1})$. Moreover, the increase in variance of $\tilde{\eta}_{ji}$ from increasing any v_i does *not* reduce the probability that i is chosen. This result can be stated explicitly for the case of a Brownian motion accumulator.

Figure C.1. Construction of the Density $h(z,t;\mathbf{v})$ from the "Primitive" Densities $g(t;v)$ and $f(\theta - z, t; v)$



Proposition 4. *For a Brownian motion accumulator (6) and a race stopping rule (10), then $P_i(\mathbf{v}\boldsymbol{\alpha}^\top) > P_i(\mathbf{v})$ for some $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n]$ where $\alpha_i > 1$ and $\alpha_j = 1, \forall\, j \neq i$.*

Proof. The proof is provded in Appendix D. □

Appendix D. Proofs

D.1. Proof of Proposition 1

Let the $n \times n$ matrix function $\boldsymbol{\Delta}(t)$ be the fundamental solution to the first-order linear differential system

$$\boldsymbol{\Delta}'(t) = \boldsymbol{\Gamma}(t)\boldsymbol{\Delta}(t), \quad \boldsymbol{\Delta}(0) = \mathbf{I}.$$

Then the solution to Equation (4) at t^* is

$$\begin{aligned}\mathbf{Z}(t^*) &= \boldsymbol{\Delta}(t^*)\int_0^{t^*} \boldsymbol{\Delta}^{-1}(t)[\mathbf{v} + c(t)]\,dt + \boldsymbol{\Delta}(t^*)\int_0^{t^*} \boldsymbol{\Delta}^{-1}(t)\boldsymbol{\sigma}(t)\,d\mathbf{B}(t)\\ &= \boldsymbol{\Delta}(t^*)\int_0^{t^*} \boldsymbol{\Delta}^{-1}(t)\,dt\,\mathbf{v}\\ &\quad + \boldsymbol{\Delta}(t^*)\int_0^{t^*} \boldsymbol{\Delta}^{-1}(t)[c(t)\,dt + \boldsymbol{\sigma}(t)\,d\mathbf{B}(t)]\\ &= \mathbf{D}(t^*)\mathbf{v} + \mathbf{C}(t^*),\end{aligned}$$

where for exposition we define $\mathbf{D}(t^*) \equiv \boldsymbol{\Delta}(t^*)\int_0^{t^*} \boldsymbol{\Delta}^{-1}(t)\,dt$ and the stochastic portion of the integral as $\mathbf{C}(t^*)$.

To proceed, observe that the solution $\boldsymbol{\Delta}(t)$ takes the form a matrix exponential

$$\boldsymbol{\Delta}(t) = e^{\int_0^{t^*} \boldsymbol{\Gamma}(t)\,dt} = \frac{1}{n} e^{\int_0^{t^*} \gamma(t) - \psi(t)\,dt}\mathbf{R}(t^*),$$

where the symmetric matrix $\mathbf{R}(t^*)$ has positive main diagonal elements $(n-1) + e^{n\int_0^{t^*} \psi(t)\,dt}$ and off-diagonal elements $e^{n\int_0^{t^*} \psi(t)\,dt} - 1$.

Note also that $\boldsymbol{\Delta}(t)^{-1} = \boldsymbol{\Delta}(-t)$ by property of the matrix exponential, with a positive main diagonal. Therefore, the main diagonal elements of $\mathbf{D}(t^*)$ are given by the scalar

$$\lambda(t^*) \equiv \operatorname{diag}(\boldsymbol{\Delta}(t^*))\operatorname{diag}\left(\int_0^{t^*} \boldsymbol{\Delta}(-t)\,dt\right) > 0.$$

Define $\mathbf{S}(t^*)$ as the matrix $\boldsymbol{\Delta}(t^*)\int_0^{t^*} \boldsymbol{\Delta}^{-1}(t)\,dt$ with its main diagonal replaced with zeros. Substituting these terms in to the solution to the differential equation yields

$$\mathbf{Z}(t^*) = \lambda(t^*)\mathbf{v} + \mathbf{S}(t^*)\mathbf{v} + \mathbf{C}(t^*),$$

where we now explicitly separate the portion of the accumulation of Z_i that depends on v_i from the portion that depends on the other alternative. Multiply through by $\lambda(t^*)^{-1}$, yielding

$$\lambda^{-1}(t^*)\mathbf{Z}(t^*) = \mathbf{v} + \lambda^{-1}(t^*)\mathbf{S}(t^*)\mathbf{v} + \lambda^{-1}(t^*)\mathbf{C}(t^*).$$

Finally, substituting into the choice criterion yields

$$\begin{aligned} i^* &= \arg\max\{\lambda^{-1}(t^*)\mathbf{Z}(t^*)\}\\ &= \arg\max\{\mathbf{v} + \boldsymbol{\eta}\},\end{aligned}$$

where

$$\boldsymbol{\eta} = \eta(\mathbf{v}, t^*) \equiv \lambda^{-1}(t^*)\mathbf{S}(t^*)\mathbf{v} + \lambda^{-1}(t^*)\mathbf{C}(t^*),$$

and η_i does not depend directly on v_i. The functional form for $\eta(\mathbf{v}, t^*)$ follows directly from the relevant assumptions on the stochastic process. □

D.2. Proof of Proposition 2

The choice probabilities are given by Equation (3), restated here as an integral over the set of realizations $\tilde{\boldsymbol{\eta}}$ for which $\bar{i}$ is chosen (Train 2009):

$$P_{\bar{i}}(\mathbf{v}) = \int_{\tilde{\boldsymbol{\eta}} \in D_{\mathbf{v}}} p_{\mathbf{v}}(\tilde{\boldsymbol{\eta}})\,d\tilde{\boldsymbol{\eta}},$$

where $p_{\mathbf{v}}(\tilde{\boldsymbol{\eta}})$ is the density of $\tilde{\boldsymbol{\eta}}$ implied by the stochastic structure placed on $\boldsymbol{\eta}$.

Since $\mathbf{v} + \alpha\mathbf{1}$ preserves the relative differences between alternatives,

$$D_{\mathbf{v}+\alpha\mathbf{1}} = \{\tilde{\boldsymbol{\eta}}\colon v_i + \alpha - v_j - \alpha > \tilde{\eta}_{ji}, \forall\, j \neq i\} = D_{\mathbf{v}},$$

the sets of realizations under a linear scaling are identical; therefore,

$$P_{\bar{i}}(\mathbf{v} + \alpha\mathbf{1}) = \int_{\tilde{\boldsymbol{\eta}} \in D_{\mathbf{v}}} p_{\mathbf{v}+\alpha\mathbf{1}}(\tilde{\boldsymbol{\eta}})\,d\tilde{\boldsymbol{\eta}}.$$

Moreover, $D_{\mathbf{v}}$ is a convex set. Therefore, $P_{\bar{i}}(\mathbf{v}) > P_{\bar{i}}(\mathbf{v} + \alpha\mathbf{1})$ if and only if the linear scaling $\mathbf{v} + \alpha\mathbf{1}$ (i.e., $\alpha > 0$) transfers probability density toward realizations of $\tilde{\boldsymbol{\eta}} \notin D_{\mathbf{v}}$. If the stochastic structure placed on $\boldsymbol{\eta}$ via $\eta(\mathbf{v}, t^*)$ increases the variance of η_i, $\forall\, i$, this latter condition holds. □

D.3. Proofs from Appendix C

We begin by establishing that an increase in the valuation of any alternative will result in a distribution of stopping times that is first-order stochastically dominated.

Lemma 1. *For a Brownian motion accumulator (6) and a race stopping rule (10), for any $\mathbf{v}'$ and $\mathbf{v}$ where $v'_j \geq v_j$, $\forall\, j$, and some i where $v'_i > v_i$, then $G_*(t; \mathbf{v}') > G_*(t; \mathbf{v})$.*

Proof of Lemma 1. The stopping time $t^*(\mathbf{v}) = \min\{t_1, \ldots, t_n\}$ is an order statistic with the following cumulative distribution function,

$$G_*(t; \mathbf{v}) = 1 - \prod_{i=1}^{n}(1 - G(t; v_i)), \tag{D.1}$$

where $G(t; v_i)$ is the CDF of the Inverse Gaussian distribution, with density function $g(t; v_i)$. For $v'_i > v_i$,

$$\begin{aligned}\frac{\delta[g(t; v_i)/(g(t; v'_i))]}{\delta t} &= \frac{\delta[e^{-(\theta(v-v') + (v'^2 - v^2)t)/(2\sigma^2)}]}{\delta t}\\ &= e^{-(\theta(v-v') + (v'^2 - v^2)t)/(2\sigma^2)}\left[\frac{(v'^2 - v^2)}{2\sigma^2}\right] > 0;\end{aligned}$$

therefore, the first passage time density $g(t; v)$ satisfies the monotone likelihood ratio property for increasing v. Therefore, $G(t; v'_i) > G(t; v_i)$. By means of (D.1), this implies $G_*(t; \mathbf{v}') > G_*(t; \mathbf{v})$. □

Before we proceed with a proof of Proposition 3, we must characterize how the density of the losing accumulator depends on v and t. Let $F(x; t, v)$ denote the conditional distribution at $x > 0$ for a given t and v, derived from integrating (C.1),

$$F(z; t, v) \equiv \frac{1}{\bar{F}}\int_0^z f(\theta - x, t; v)\,dx$$

where the normalization constant $\bar{F}$ is given by

$$\begin{aligned}\bar{F} &\equiv \int_0^\infty f(\theta - z, t; v)\, dz\\ &= \frac{1}{2}\left[\mathrm{Erfc}\left(\frac{tv-\theta}{\sqrt{2\sigma^2 t}}\right) - e^{2v\theta/\sigma^2}\mathrm{Erfc}\left(\frac{tv+\theta}{\sqrt{2\sigma^2 t}}\right)\right]\\ &= 1 - G(t; v).\end{aligned}$$

We now establish the following two Lemmas regarding the integral of the conditional density $f(\theta - z, t; v)/(1 - G(t; v))$.

Lemma 2. *$F(z; t', v) < F(z; t, v)$ for some $t' > t$.*

Proof of Lemma 2. As $t \to 0$,

$$\frac{f(\theta - z, t; v)}{1 - G(t; v)} \to \delta(z), \quad \text{where } \delta(z) = \begin{cases}\infty, & z = \theta;\\ 0, & z \neq \theta\end{cases}$$

is the Dirac delta function centered on θ. Therefore,

$$\lim_{t\to 0} F(z; t, v) = 1, \quad \forall z > \theta.$$

However, for some small $t' > 0$ and all $z > \theta$, it can be shown that $f(\theta - z, t'; v)/(1 - G(t'; v)) > 0$. Therefore, $\exists t' > t$ for which $F(z; t', v) < F(z; t, v)$. □

Lemma 3. *$F(z; t, v') > F(z; t, v)$ for all $v' > v$.*

Proof of Lemma 3. The ratio of the conditional densities is

$$\begin{aligned}&\frac{f(\theta - z, t; v)/(1 - G(t; v))}{f(\theta - z, t; v')/(1 - G(t; v'))}\\ &= \left[\frac{e^{-(z-vt)^2/(2\sigma^2 t)} - e^{2v\theta/\sigma^2 - (z-2\theta-vt)^2/(2\sigma^2 t)}}{e^{-(z-v't)^2/(2\sigma)^2 t} - e^{2v'\theta/\sigma^2 - (z-2\theta-v't)^2/(2\sigma^2 t)}}\right]\frac{1 - G(t; v')}{1 - G(t; v)}.\end{aligned}$$

Since this ratio is increasing in z,

$$\frac{\delta\left[\frac{f(\theta-z,t;v)/(1-G(t;v))}{f(\theta-z,t;v')/(1-G(t;v'))}\right]}{\delta z} = \left[\frac{(v'-v)}{\sigma^2} e^{\frac{(v'-v)(t(v'+v)+2(z-\theta))}{2\sigma^2}}\right]\frac{1-G(t;v')}{1-G(t;v)} > 0,$$

the densities satisfy the monotone likelihood ratio property. This implies $F(z; t, v') > F(z; t, v)$. □

Proof of Proposition 3. Result a. is a special case of Lemma 1 where $n = 2$.

From the derivation of $\tilde{\eta}_{ji}$ (via Proposition 1) and definition of $Z_i(t)$,

$$\begin{aligned}\mathrm{E}[\tilde{\eta}_{ji}] = \mathrm{E}\left[\frac{\tilde{B}_{ji}(t^*)}{t^*}\right] &= \mathrm{E}\left[\frac{\tilde{Z}_{ji}(t^*)}{t^*}\right] - (v_i - v_j)\\ &= \mathrm{E}_{t^*}\left[\frac{1}{t}E[\tilde{Z}_{ji}(t^*) \mid t^* = t]\right],\end{aligned}$$

where the last equality follows from the Law of iterated expectations. Moreover,

$$\begin{aligned}&E[\tilde{Z}_{ji}(t^*) \mid t^* = t]\\ &= \frac{1}{4(1 - G(t; v))}\int_{-\infty}^{\infty} z(f(z + \theta, t) + f(\theta - z, t))\, dz\\ &= 0,\end{aligned}$$

since the function given by $f(z + \theta, t) + f(\theta - z, t)$ is even $\forall t$. This yields result b.

Similarly, from Proposition 1 and the definition of $Z_i(t)$,

$$\mathrm{Var}(\tilde{\eta}_{ji}) = \mathrm{Var}\left(\frac{\tilde{B}_{ji}(t^*)}{t^*}\right) = \mathrm{Var}\left(\frac{\tilde{Z}_{ji}(t^*)}{t^*}\right).$$

From the definition of conditional variance,

$$\begin{aligned}&\mathrm{Var}\left(\frac{\tilde{Z}_{ji}(t^*)}{t^*}\right)\\ &= E_{t^*}\left[\mathrm{Var}\left(\frac{\tilde{Z}_{ji}(t)}{t} \,\middle|\, t^* = t\right)\right] + \mathrm{Var}_{t^*}\left(E\left[\frac{\tilde{Z}_{ji}(t)}{t} \,\middle|\, t^* = t\right]\right)\\ &= E_{t^*}\left[\frac{1}{t^2}\mathrm{Var}\left(\tilde{Z}_{ji}(t) \mid t^* = t\right)\right] + 0\\ &= \int_0^\infty g_*(t)\frac{1}{t^2}\mathrm{Var}(\tilde{Z}_{ji}(t) \mid t^* = t)\, dt.\end{aligned}$$

This yields result c.

For result d, a similar argument to b holds:

$$\begin{aligned}&\mathrm{Var}(\tilde{Z}_{ji}(t^*) \mid t^* = t)\\ &= \int_{-\infty}^{\infty} z^2 h(z \mid t^* = t)\, dz\\ &= \frac{1}{2(1 - G(t; v))}\left[\frac{1}{2}\int_{-\infty}^{0} z^2 f(z + \theta, t)\, dz + \frac{1}{2}\int_0^\infty z^2 f(\theta - z, t)\, dz\right]\\ &= \frac{1}{2(1 - G(t; v))}\int_0^\infty z^2 f(\theta - z, t)\, dz,\end{aligned} \tag{D.2}$$

where the final equality arises since $f(z + \theta) = f(\theta - (-z))$, $\forall z$ (i.e., $h(z, t)$ is symmetric about 0).

Since the density $f(\theta - z, t)/(1 - G(t; v))$ places less weight in its right tail for larger v (Lemma 3), this yields result d via (D.2). □

Proof of Proposition 4. Recalling the binary choice probability given in (C.2), the probabilities for the general case of n accumulators are given by

$$P_i(\mathbf{v}) = \int_0^\infty g(t; v_i)\prod_{j\neq i}(1 - G(t; v_j))\, dt$$

and

$$P_i(\mathbf{v}\boldsymbol{\alpha}^\top) = \int_0^\infty g(t; \alpha v_i)\prod_{j\neq i}(1 - G(t; v_j))\, dt.$$

If $G(t; v_i)$ is increasing in v_i, $g(t; \alpha v_i)$ places more density on smaller values of t than does $g(t; v_i)$. Since $1 - G(t; v_j)$ is decreasing in t, $P_i(\mathbf{v}\boldsymbol{\alpha}^\top) > P_i(\mathbf{v})$. □

Endnotes

[1] In the psychometrics literature, response time is defined as the duration from the onset of a stimulus until a decision is made. For a substantial early review, see Luce (1986). The potential role of response time in shaping the scale of the distribution of random utilities was noted by Block and Marschak (1959) on their introduction of the random utility model to economics from psychology: “The shorter the delay the larger is the difference between certain values .... A very long delay reveals a state of (almost) indifference, the ‘conflict situation’: Hamlet took a very long time to decide whether to kill his uncle” (p. 1).

[2] In these models, evidence arrives via a stochastic sampling process and is accumulated over time. These papers expand the original normative statement, which allowed beliefs over a binary state of the world (i.e., whether a sensory stimulus is “left” or “right”; Wald and

Wolfowitz 1948, Drugowitsch et al. 2012), to include a belief over utilities with full support on the reals.

[3] This is not obvious given the lack of closed-form choice probabilities required for a direct test of the necessary conditions (Block and Marschak 1959, Falmagne 1985).

[4] More accurately, standard models make distributional assumptions that ignore the role of response time explored in this article. The growth of digital data collection methods that collect response-time information by default, in both field and experimental conditions (e.g., Reutskaja et al. 2011, Rubinstein 2013, Krajbich and Dean 2015), only strengthens the need for examining this relationship.

[5] The joint distribution on choice and response time from the DDM can still be used to improve the efficiency of the parameter estimate (Clithero 2018).

[6] Since the levels of the attributes for the premium set are larger, the accumulation terminates earlier with a lower probability that brand 1 is chosen. The observation that a given utility difference is harder to discriminate for larger magnitudes is a well-known application of the principle of *diminishing sensitivity* (Weber 1834, Bordalo et al. 2013).

[7] Note that the form of these distributions will depend on the parameterization of the particular accumulation model. What is crucial is that the distributions are *different* across sets.

[8] No restrictions are placed on the form of $V()$. To focus on the role of the stochastic specification, the bulk of this article will examine the implications of changes in observables $\mathbf{x}_i$ by working with the resulting valuations $\mathbf{v}$ directly, thus suppressing $\mathbf{x}_i$ in the notation. In practice, since only $\mathbf{x}_i$ is observable, Section 5 will return to another example of the econometric issues involved in estimating α when $V()$ is specified to be the expected utility model.

[9] Since the additive formulation of (2) implies that a $\mathbf{u}$ can be defined for any $\mathbf{v}+\boldsymbol{\eta}$, every additive model is a RUM (Becker et al. 1963, Batley 2008). However, the reverse implication does not hold; for examples, see the literature on random preferences (Loomes and Sugden 1995, Apesteguia and Ballester 2018).

[10] For instance, if η_i follows an independent Gumbel distribution, the choice probabilities are given by the multinomial logit model and obey the axiom of independence of irrelevant alternatives (IIA) (Luce and Suppes 1965, McFadden 1973). Departures from this assumption induce choice probabilities that violate IIA (Debreu 1960, Hausman and Wise 1978, Train 2009); however, the class of such RUMs is large and can entail starkly different forms for the choice probabilities (McFadden and Train 2000). IIA violations rooted in the principles of neurobiology are explored in Webb et al. (2016a) and Louie et al. (2015). Note that the BAM models explored here need *not* satisfy IIA.

[11] For example, it extends to the domain of strategic choice. An equilibrium theory with a stochastic best response alters the equilibrium prediction away from Nash equilibrium (McKelvey and Palfrey 1993). However, a unique equilibrium prediction depends crucially on a unique specification of the random utility distribution. Otherwise, *any* observed outcome can be an equilibrium (Haile et al. 2008).

[12] The continuous time process can be derived as the limit of a discrete time random walk and has the advantage that it includes the class of Gaussian processes for which solutions can be derived in closed form. Discrete time may, in fact, be the more appropriate approach to modeling neural activity (Shadlen et al. 2006). The main result in this paper holds regardless.

[13] An additional note on the observability of $\mathbf{v}$ in neural activity might be helpful. Neural activity in the medial prefrontal cortex both is correlated with behavioral measures of valuation (e.g., Levy and Glimcher 2011, Bartra et al. 2013, Clithero and Rangel 2014) and can predict choice behavior (Webb et al. 2016b, Smith et al. 2014). However, choice requires more than a representation; it also requires a comparison of each v_i to identify the maximal element. The neural evidence suggests that this comparison takes place in dorsomedial and parietal regions via a bounded accumulation process (Basten et al. 2010, Hare et al. 2011, Domenech et al. 2017). Since $\mathbf{v}$ is interpreted to be a neural quantity, it is natural to assume $v_i > 0$. A BAM concerned with neural implementation imposes the additional constraint that $Z_i(t) \geq 0$ (e.g., Usher and McClelland 2001).

[14] The technical details of constructing the differential equations, and their solutions, can be found in Cox and Miller (1965) and Smith (2000).

[15] Some recent literature has begun to relax the assumption that the valuation $\mathbf{v}$ is constant in time in an effort to capture the role of attention (e.g., Krajbich et al. 2010). Certainly for the general case, an equivalence to a RUM will not hold since some of these models predict violations of the regularity axiom, a necessary condition for the RUM (e.g., Tsetsos et al. 2016). Assessing the conditions on a time-dependent valuation that still yield a RUM is an open question.

[16] Example 1 also highlights an identification issue that will be familiar to practitioners. If $v_i - v_j$ is known, it is clear that only the term $\beta \equiv -2\theta/\sigma^2$ is identified by choice data alone. Therefore, the variance of the stochastic process is typically normalized before a logit model can be applied. However, there is still something to be gained from the accumulation model. Incorporating a second observable—the response time t^*—identifies an additional parameter of the DDM. This results in a more efficient estimate of the relationship between $v_i - v_j$ and the logistic choice probabilities (12) than can be achieved through choice data alone (Clithero 2018).

[17] In practice, this is equivalent to observing that the scale of the random utility model is arbitrary and thus requires normalization.

[18] A derivation for coupled accumulators in discrete time can be found in a working paper.

[19] We address counterexamples to a necessary condition in Sections 4.1 and 6.

[20] A forthcoming manuscript by Davis-Stober et al. (2017) uses this fixed stopping time result to approximate the accumulation model of Tsetsos et al. (2016).

[21] The gaps in the distribution of $\tilde{\eta}_{ji}$ occur because of the skewed distribution for t^* in the denominator. For example, when $v_i = v_j$, the density for large t^* approaches 0; thus, the density of R/t^* also approaches 0.

[22] Computational methods for implementing these approximations are available under a BSD license at https://github.com/jdrugo/dm.

[23] Note that the density of $B(t^*)$, conditional on one of the processes having stopped at $t^* = t$, is decidedly *not* Gaussian. This is due to the threshold imposed by the stopping rule. Intuitively, the threshold truncates sample paths that would have crossed the threshold and then returned below the threshold.

[24] For reasons of neurobiological plausibility the LCA model restricts $Z_i(t) > 0, \forall i$.

[25] Computational methods for implementing these approximations are available under a BSD license at https://github.com/jdrugo/dm.

[26] Note that the issue of misspecification will be equally present in models with a random preference parameter (instead of an additive error term); however, the form of this misspecification, and its possible relation to accumulation models and the neurobiology of risk preference, remains to be formally addressed.

[27] For data sets with a small sample of repeated trials in which the mean response time might still be highly correlated with an individual trial, the mean response time of all trials, less the current trial, could also be used.

[28] The estimate $\hat{\theta}$ is reported as $1/e^{\hat{s}}$ to conform with the other specifications.

[29] In the absence of some mechanism for rescaling valuations, the race stopping rule implies more stochasticity in choice behavior for more valuable choice sets than would be predicted by a model with

constant variance. From the perspective of optimal choice, this suggests that the brain might have some mechanism to rescale, or normalize, valuations relative to a given threshold, reducing the number of errors but leading to a particular pattern of IIA violations observed in choice data.

[30] Consider a decision maker who receives a temporal sequence of independent, stochastic signals about a binary state of the world and must select an appropriate policy. The decision maker must accumulate the imperfect state information until sufficient evidence about the current state yields an efficient decision. The sequential probability ratio test (Wald and Wolfowitz 1948), which achieves a specified error rate with the smallest number of samples on average, is the normative solution to this problem. The DDM implements this solution exactly.

[31] Moreover, each of them requires determining which accumulator is greatest at every t then comparing it to other alternatives. From a computational standpoint, if the role of a BAM is to implement a "max" operator on subjective value, it would be (infinitely) recursive to embed a second "max" operation inside of the stopping rule.

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