Disequilibrium Play in Tennis

Axel Anderson, Georgetown University
Jeremy Rosen, Topspin Shot Research
John Rust, Georgetown University
Kin-Ping Wong, Digonex

DSE 2022
Australian National University

December 18, 2022

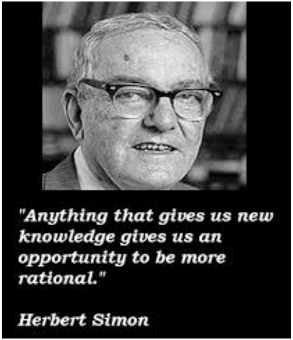
Bottom line of talk

- Are the serves of the world's best tennis pros consistent with the theoretical prediction of Nash equilibrium in mixed strategies? We analyze their serve direction choices (to the returner's left, right or body) with data from an online database called the Match Charting Project.
- Using a new methodology, we test and decisively reject a key implication of a mixed strategy Nash equilibrium, namely, that the probability of winning is the same for all serve directions.
- We use DP calculate optimal serve strategies for individual server-returner pairs, such as Novak Djokovic serving to Rafael Nadal.
- We show that the DP serve strategies significantly increase the probability of winning compared to the mixed strategies the pros actually use, which we estimate using flexible reduced-form logit models. Stochastic simulations verify that our results are robust to estimation error.

Three key axioms of economics

- Rationality
- Optimality
- Equilibrium
- These make eminent sense as key assumptions for theories of *perfectly rational behavior*
- But are they good assumptions for empirical models of how real people actually behave?

Friedman vs Simon



Satisficing vs Optimizing

- Simon's view that humans *satisfice* rather than *perfectly optimize* is a better approximation to reality.
- At the same time, he also contributed to the theory of fully rational decision making. He was also a founding father of *artificial intelligence.*
- In his 1978 Nobel prize speech, Simon noted that "by the middle 1950s, a theory of bounded rationality had been proposed as an alternative to classical omniscient rationality [and] a significant number of empirical studies had been carried out that showed actual business decision making to conform reasonably well with the assumptions of bounded rationality but not with the assumptions of perfect rationality."

Using rational tools to improve irrational behavior

- I started my own work in structural econometrics quite content to impose the big three axioms
- I am more cynical (maybe even misanthropic) as I get older since so much human behavior seems completely crazy. What's the purpose of using structural econometrics to "rationalize" patently irrational behavior?
- Like Simon I think the tools of rational decision making (e.g. econometrics, DP, game theory) are hugely valuable for *improving human decision making*

Background readings

- See my article: “Has Dynamic Programming Improved Decision Making?” *Annual Review of Economics* 2019
- Increasingly, there is fear that AI is starting to approximate rational decision making and could supplant humans in many contexts. Could it it even make us, structural econometricians, irrelevant?
- See my article with Fedor Iskhakov and Bertel Schjerning: “Structural Econometrics and Machine Learning: Contrasts and Synergies” *The Econometrics Journal* 2020.

Who is this guy?

Theory of Games and Economic Behavior

- Title of von Neumann and Morgenstern's (1944) seminal book. The forward to the 60th anniversary edition by Harold Kuhn states:
- "Although John von Neumann was no doubt the 'father of game theory' the birth took place after a number of miscarriages. From an isolated and amazing minimax solution of a zero-sum two-person game by Waldegrave in 1713 to sporadic considerations by Zermelo 1913, Borel 1924, and Steinhaus 1925, nothing matches the path-breaking paper by von Neumann 1928."
- "This paper, elegant though it is, might have remained a footnote to the history of mathematics were it not for the collaboration of von Neumann with Oskar Morgenstern in the early 40's."

The Minimax solution and Nash equilibrium

- The minimax solution to a two-person zero sum game is a special case of the more general concept of *Nash equilibrium* for N-person non-zero sum, non-cooperative games.
- There is no doubt that the concept of Nash equilibrium, along with Bellman's work on dynamic programmig, revolutionized economics.
- Yet, we can ask: does Nash equilibrium do a good job of describing human play in real games and behavior in other strategic situations?
- Colin Camerer's (2003) book *Behavioral Game Theory* summarizes a large literature on *human laboratory experiments* and concludes "In fact, the results are uniformly mixed, in a way that encourages the view that better theory is close at hand."

The standard critique of lab experiments

- A standard critique of violations of Nash equilibrium play in laboratory games is: 1) the experiments are often "artificial" and not games that people typically play in reality, 2) the subjects are inexperienced, and 3) the laboratory payoffs are too small and the duration of experiments too short to properly incentivize subjects and enable them to learn and converge to Nash equiibrium behavior.
- Nash equilibrium seems plausible intuitively, since if highly intelligent, highly motivated subjects were not playing Nash strategies, then by defintion, there are alternative strategies that one or more of the players could adopt in order to "beat" their opponents, i.e. to earn higher payoffs than they are receiving with their disequilibrium strategies.

Empirical analysis of “games in the wild”

- For these reasons, many economists are more convinced by tests of the Nash equilibrium hypothesis for games played by “professional” players in their natural setting, repeatedly, for very high stakes.
- A classic example of such a study was the 2001 study by Walker and Wooders *Minimax Play at Wimbeldon*
- They observed play of professional tennis servers and tested the hypothesis that their play should conform to von Neumann’s mixed strategy minimax solution, since tennis is a constant sum game.
- Via videos of serve directions, left of right, they tested a key implication of a mixed strategy Nash equilibrium: *the probability of winning should be the same for serving to the left or to the right.*

Implications of mixed strategy equilibrium

- Walker and Wooders concluded "The theory of mixed-strategy equilibrium has not been consistent with the empirical evidence gathered through more than 40 years of experiments involving human subjects. Conversely, the theory has performed far better in explaining the play of top professional tennis players in our data set."
- They also tested a related implication: *the directions of successive serves should be serially independent.* Though they did find mild serial dependence in serve directions, they concluded that "just as with our tests using players' win rates, the tests for randomness (and serial correlation in particular) reveal a striking difference between the theory's consistency with the data for top tennis players and its consistency with the data from experiments." (p. 1535)

What about other professional sports?

- Walker and Wooders' general conclusions have been replicated in other sports, e.g. Palacios-Huerta (2003) study of soccer penalty kicks, *Professionals Play Minimax.*
- So I would claim that "professionals play minimax" or more generally "professionals play Nash" has become the *conventional wisdom* in economics.
- I expect major pushback against our study that comes to the opposite conclusion after analyzing a much bigger data set on *entire tennis games* using a dynamic model of tennis (as opposed to a WW static analysis of individual tennis points, treated as a repeated two person two-action constant sum games) that considers three serve directions: 1) wide, 2) "down the T" and 3) to the body.
- We strongly reject the hypothesis of equal win probabilities and find evidence of serial correlation in successive serve directions.

Do humans play perfect chess?

- Before we dive into our analysis of tennis, however, I want to consider one other professional sport — chess. Do professional chess players (e.g. Magnus Carlson) play Nash equilibrium chess strategies?
- Answer: obviously not! Ever since *Deep Blue* beat Garry Kasparov in 1997, the world's best chess players have been computers!
- But do computers play "perfect" chess? **Answer:** not yet, but according to Jeremy Rutman's analysis we may get there by 2059.
- But first, step back and consider why chess is essentially an open and shut case that proves that humans do not play Nash equilibrium stategies in chess.

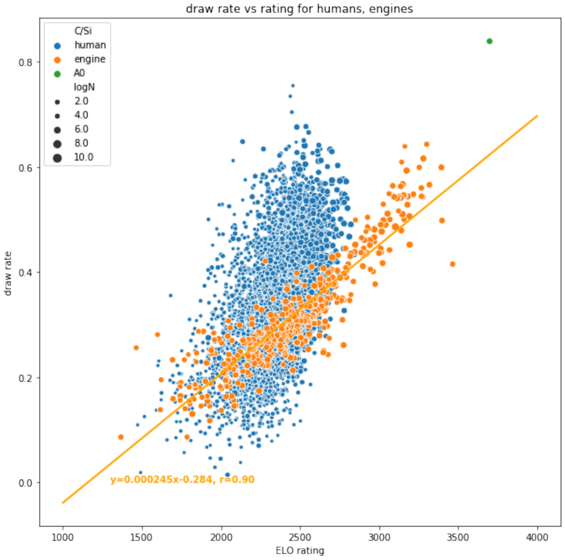
Zermelo's 1913 theorem

- **Zermelo's theorem** *In an alternating move, two player game of perfect information with no chance moves, if both players play "perfectly" (i.e. Nash equilibrium pure strategies) then there are only 3 possible outcomes:*
  - 1. the game always ends in a draw
  - 2. the first mover always wins
  - 3. the second mover always wins
- Example: tic tac toe played perfectly always ends in a draw
- Regardless of which outcome obtains, perfectly played chess is *boring.*
- But real chess is not boring, i.e. real chess players do not play perfectly, i.e. they do not play Nash equilbirium strategies.

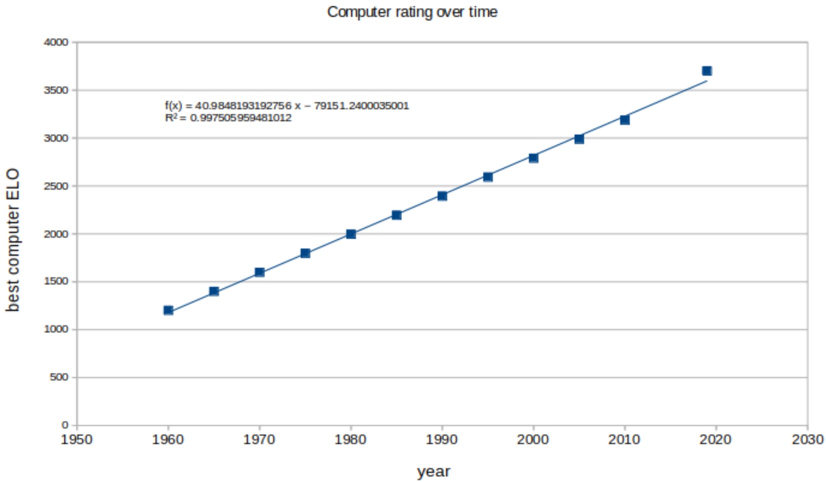
Which outcome holds for perfect chess?

- In 2007 Schaeffer *et. al. Science* solved checkers and showed *that it always ends in a draw.*
- The combinatorics of chess is still too large for us to “solve” it (though end game solutions with incresing numbers of remaining pieces have been solved).
- The empirical evidence suggests that perfectly played chess will also always end in a draw.
- Jeremy Rutman has compiled data on chess outcomes against the ELO score (a standard measure of ability in chess) and shown as the players’ ELO scores improve, the chance of a draw increased.
- Given the current technological rate of improvement in ELO score, he predicts that we will achieve “perfect chess” around 2059 with an ELO score of around 5200 (for comparison Magnus Carlson’s ELO score is 2840).

Draw rating vs ELO score in chess



Trend in ELO score over time



Chess versus tennis

- Both are alternating move, two player constant sum games.
- Does Zermelo's Theorem apply to tennis? No: there is a significant role of "chance" in tennis, both in terms of mixed serve strategies as well as in "imperfect muscle control" (e.g. faulting a first serve)
- Also there is no possibility of a tie in tennis: the rules of tennis lead to only 2 possible outcome of a game: 1) server wins, or 2) server loses.
- Brains vs brawn: unlike chess, which is purely a test of mental ability in reasoning and ability to "look ahead", physical ability plays a huge role in tennis.
- Can game theory acommodate differences in physical ability? Is it possible to quantify the relative roles of mental ability versus physical ability in success in tennis?

Tennis: a game of brawn over brain

- Even though strategies in chess are pure strategies whereas serve strategies in tennis are mixed, there is a clear sense in which we can say that tennis is "mentally simpler" than chess
- One measure is combinatoric: there are vastly more possible "board positions" (i.e. future states) to consider in chess compared to tennis
- A 2011 paper by Walker, Wooders and Amir showed tennis is a member of a class of *binary Markov games* which are alternating move dynamic games whose stage games (or "point games") have only two possible outcomes (win or loss) for the first mover. Under a plausible *Monotonicity condition* they proved a *decomposition theorem* that states that the equilibrium of the overall dynamic game (the "service game" in tennis) reduces to a simple repetition of myopic solutions to each point game.

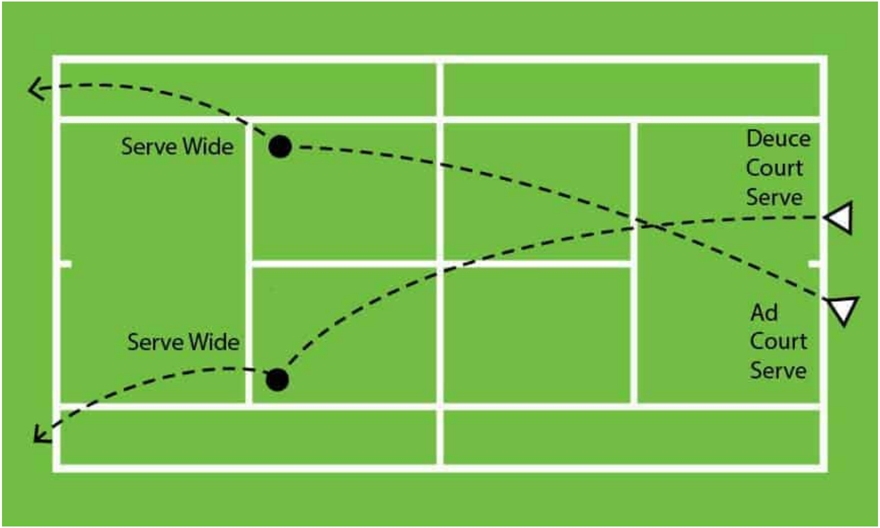
Myopic strategies can be optimal in tennis

- Players need only focus on finding strategies that maximize the probability winning each point of the service game, ignoring the future consequences of a winning or losing the current point.
- Each point requires only solving a two period backward induction: the server needs to consider the option value of the second serve when making the first serve
- Generally the first serve is as fast as possible, though with a higher probability of faulting. The second serve is slower, to lower the probability of faulting.
- When the Monotonicity Condition holds, players can ignore the effect of a win or loss on the current point on their probability of winning the overall service game. As long as the server maximizes the probability of winning each point, they also maximize the probability of winning the overall service game.

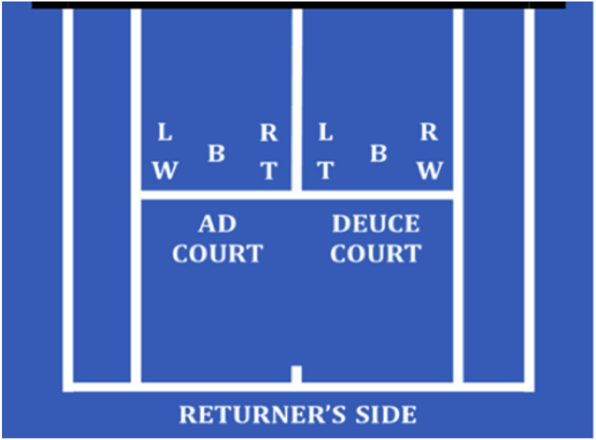
A review of the rules of tennis

- A *tennis match* consists of a series of *sets.* Typically the player who wins the majority of 3 or 5 sets wins the match.
- A *tennis set* consists of a sequence of *service games* and to win a set a player must win at least 6 games and be ahead by 2 games.
- Within each *service game* one of the players is the server. The server of the first game in a set is chosen by a coin flip, and alternates in successive games thereafter.
- A service game consists of a sequence of stage games called *points.* A point consists of a first serve plus the option for a second serve in the event of a faulted first serve.
- The service game ends when one of the players wins at least four points in total and at least 2 more than their opponent.

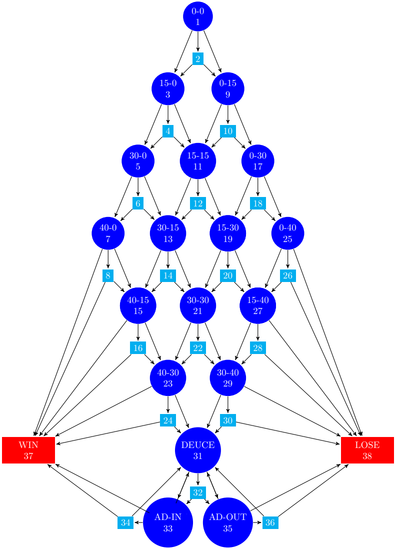
Serve directions: Wide, to the body, and down the T



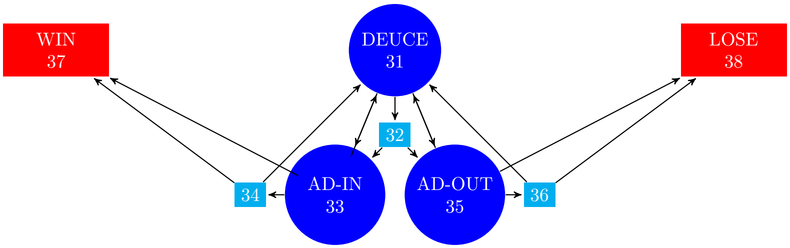
Serve directions from the returner's perspective



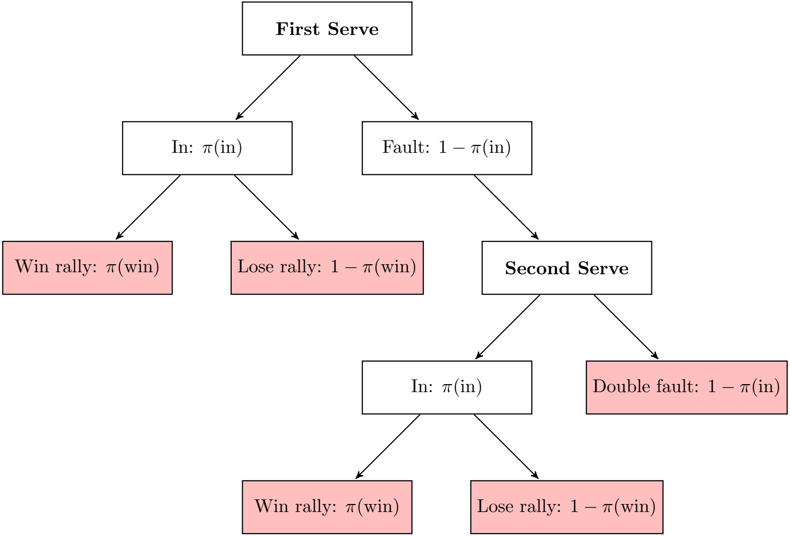
Graph of state transitions in the service game



Detail on the “deuce endgame”



Detail on the “point stage game”



Modeling tennis as a dynamic game

- Unlike Walker and Wooders, who focused on the point stage game as their unit of analysis, we focus on the entire service game.
- Let $x \in \{1, \ldots, 38\}$ denote the *score state.* Via our numbering of states in the previous graph, $x = 1$ is the initial node of the game, corresponding to a score of $0 - 0$.
- If the server wins the first serve, the state transits to $x = 3$ corresponding to a score of 15-0. If the server faults the first serve, the state transits to $x = 2$, corresponding to a second serve at score 0-0, etc.
- There are two *absorbing states:* $x = 37$ (server wins the game) and $x = 38$ (server loses the game).

Server and returner choices in tennis

- At each state x except the two absorbing states, the server chooses a *serve type* $t = (s, d)$ where $d \in \{l, r, b\}$ is the serve direction (left, right or to the body), and $s \in \mathcal{S} \subset R^2$ denotes *serve speed and spin.*
- The returner anticipates the direction choice of the server, and this includes observable choices (i.e. where to stand) and unobservable choices (i.e. where to focus his/her attention).
- We model anticipation with an attention vector $(a_l, a_r, a_b) \geq 0$ where a_d is the fraction of the returner's "attention budget" to anticipating a serve to direction d. We normalize the attention budget to 1, so $1 = a_l + a_r + a_b$.
- We assume that each serve the returner chooses a and where to stand, and the server moves next, choosing a serve type t.

Accounting for *muscle memory*

- Let $l(x, t)$ be the probability a serve lands in (i.e. is not a fault) given the score x and serve type t. Let $\omega(x, t, a)$ be the probability that the server wins the subsequent rally, given the serve is in when the score is x, the serve type is t and the returner's attention vector is a.
- We account for an additional dynamic in tennis via a *muscle memory state m*. "Muscle memory is a form of procedural memory that involves consolidating a specific motor task into memory through repetition, which has been used synonymously with motor learning." *Wikipedia*
- We account for a local of *short term version of muscle memory* where a tennis server can improve his/her chances of making a serve by repeating a recent serve of a similar type.

Operationalizing muscle memory in our model

- We operationalize muscle memory by setting $m = (d_{-2}, d_{-1})$, i.e. the vector of the previous two *first serve directions.* We have two lags because serves alternate between the deuce and ad courts and we hypothesize that the relevant muscle memory is for the previous serve to the same court.
- At the start of each service game we reset muscle memory to null, $m = (\emptyset, \emptyset)$. After each first serve to direction d muscle memory updates from $m = (d_{-2}, d_{-1})$ to $m' = (d_{-1}, d) = m'(m, d)$. The muscle memory for a second serve is the serve direction of the faulted first serve, but we do not update muscle memory after a second serve.
- Thus the key probabilities determining score state transitions are $l(x, m, d, s)$ and $\omega(x, m, s, a)$.

The stationarity assumption

Assumption

(Stationarity I) The conditional probabilities l and ω may vary across server-returner pairs, but do not vary over time (independent of the current state (x, m)) or across service games.

- Stationarity does not imply *equilibrium* — i.e. the server and return's strategies may not be mutual best replies.
- We assume stationarity in order to *pool observations on different service games played by specific server-returner pairs.*
- Stationarity rules out gradual learning or adaptation to the opponent's strategy, or slow evolution or aging in players' strategies over time.

Existence of equilibrium

- Though we do not assume play is in equilibrium, it is useful to know that an equilibrium to tennis exists. We assume each player's objective is to win the service game, resulting in a reward of 1 from winning and 0 from losing.
- Since tennis cannot end in a draw, it is a *recursive constant sum game* first analyzed by Everett (1957), who established the existence of minimax strategies.
- Let (σ_s, σ_r) denote the server and returner's (perhaps mixed and arbitrarily history-dependent) strategies in the service game. Let $W_s(x, m)$ be the set of probabilities that the server wins at state (x, m) induced by some pair of (not necessarily Markovian) subgame perfect equilibrium strategies.

Existence of equilibrium

Theorem

All sub-games of tennis have a unique value $W(x, m)$ and there exists a Markov Perfect Equilibrium (MPE) that result in this probability of winning in which the equilibrium strategies (σ_s, σ_r) depend on the current state (x, m) only.

- This Theorem can be regarded as a specialization of von Neumann's Minimax Theorem, but applied to tennis modeled as an extensive form game.

Point Outcome Probabilities (POPs)

- Our data, from the Match Charting Project, enables us to observe serve direction not serve speed or spin, s. Nor do we observe the location of the returner or the details of play during a rally, or the attention of the returner.
- Using Theorem 1, we can "project" the full strategies of the players onto the serve directions we can observe.
- Let $\rho(s|x,m)$ denote the probability distribution over speed and spin, and $\alpha(a|x,m)$ the probability disribution over returner attention in a mixed MPE of tennis.
- We define $\pi(in|x,m,d)$ and $\pi(win|x,m,d)$ denote the *point outcome probabilities* induced by the MPE strategies of the server and returner. That is, $\pi(in|,x,m,d)$ is the conditional probability a serve to direction d is in at state (x,m) and $\pi(win|x,m,d)$ is the probability the server wins the subsequent rally if the serve to direction d is in at state (x,m).

Point Outcome Probabilities (POPs)

Definition

If the stationarity assumption holds and (ρ, α) are implied by the MPE to the service game, then the *point outcome probabilities* (POPs) implied by these MPE strategies are given by

$$\begin{aligned} \pi(in|x, m, d) &\equiv \int I(x, m, d, s)\rho(ds|x, m) \\ \pi(win|x, m, d) &\equiv \int \int \omega(x, m, d, s, a)\rho(ds|x, m)\alpha(da|x, m) \end{aligned}$$

The induced dynamic program

- Now, assume that the play we observe in tennis is consistent with an MPE, i.e. we observe *equilibrium play.*
- We now show that the server's conditional win probability $W_s(x, m)$ is the solution to a dynamic program, which provides the optimal serve directions as a best response to the returner's strategy.
- If play is in equilibrium, we can treat the returner's mixed strategy as embodied, in reduced-form, via the POPs, and the server's strategy and win probability will be a "best reply" to these POPs.
- Bellman's equation for the server is

$$W_s(x, m) = \max_{d \in \{l,b,r\}} W_s(x, m, d)$$

where $W_s(x, m, d)$ is the server's conditional win probability from serving to direction d in state (x, m).

The server's Bellman equation

- Let $x^+(x)$ and $x^-(x)$ denote the successor point states conditional on the server winning and losing, respectively, the serve at point state x. Using these we can express $W_s(x, m, d)$ recursively as

$$\begin{aligned}
&W_s(x, m, d) = \\
&\pi(in|x, m, d)\pi(win|x, m, d)W_s(x^+(x), m'(d, m)) \\
&+\pi(in|x, m, d)[1 - \pi(win|x, m, d)W_s(x^-(x), m'(d, m)) \\
&+[1 - \pi(in|x, m, d)]W_s(x + 1, m'(d, m))
\end{aligned}$$

for a first serve state x, where $x + 1$ denotes the second serve state in the event of a faulted first serve and $m'(d, m)$ is the new muscle memory state after serving to direction m, $m'(d, m) = (d_{-1}, d)$ where $m = (d_{-2}, d_{-1})$.

The server's Bellman equation

- If x is a second serve state (i.e. x is an odd number in our score state numbering), then

$$
\begin{aligned}
&W_s(x, m, d) = \\
&\pi(in|x, m, d)\pi(win|x, m, d)W_s(x^+(x), m) \\
&+\pi(in|x, m, d)[1 - \pi(win|x, m, d)W_s(x^-(x), m) \\
&+[1 - \pi(in|x, m, d)]W_s(x^-(x), , m)
\end{aligned}
$$

  where we note that following a second serve, the muscle memory state is not updated and remains at m.

- The optimal serve strategy, $\sigma_s(x, m)$ is the set of maximizers of the Bellman equation

$$\sigma_s(x, m) = \underset{d \in \{l,b,r\}}{argmax}\, W_s(x, m, d).$$

Solving the server's DP problem

- We can write the Bellman equation for the server's conditional win probability more compactly as

$$W_s = \Gamma(W_s)$$

  where Γ is the *Bellman operator.* Even though there is no discounting in this DP problem, the POPs $(\pi(in|x, m, d), \pi(win|x, m, d))$ (which are generally strictly less than 1) play the equivalent role of a discount factor, ensuring the existence of a unique solution to the Bellman equation via the Contraction Mapping Theorem.
- In practice, we calculate the server's DP problem by *backward induction* but the induction is not *over time* but rather *over states* using an algorithm Iskhakov, Schjerning and Rust (IRS,2016) refer to as *state recursion.*

Directional Dynamic Games

- Tennis is a member of a class of dynamic games IRS called *directional dynamic games* which are dynamic Markovian games where there is a partial order over states that satisfies an intuitive notion of *directionality.*
- Tennis satifies this directionality condition for all states except the deuce endgame states since we have $x^{+}(x) > x$ and $x^{-}(x) > x$ for all non-endgame states x.
- However the endgame must be solved "simultaneously" since there are bi-directional transitions between the various endgame states (i.e. a player can go from deuce to advantage in or advantage out, and then back to deuce again).
- As a result, there is no *a priori* upper bound on the duration of a service game, though in practice most games end after fewer than 15 points (the longest game on record involved over 37 points).

Testable Implications of Nash Equilibrium

- Suppose we could calculate the server's conditional win probabilities $W_s(x, m, d)$. If the equilibrium is in mixed strategies, then a key implication of a mixed strategy Nash equilibrium is

Testable Implication of Equilibrium (1)

Equal win probabilities In all states (x, m) we have:

$$W_s(x, m, l) = W_s(x, m, b) = W_s(x, m, r)$$

Allowing for disequilibrium serve strategies

- Suppose σ_s is a mixed, stationary, Markovian strategy of the server that is also possibility a *disequilibrium strategy.*
- Let $P(s, d|x, m)$ denote the joint probability density that the server serves with speed/spin s and direction d in state (x, m), and let $P(d|x, m)$ be the marginal probability distribution over serve direction only.
- We make the following assumption about the actual (potentially disequilibrium) serve strategy and implied POPs

Assumption

(Stationarity II) The players use stationary, Markovian but potentially disequilibrium strategies, such that actual serve directions are given by a conditional probability $P(d|x, m)$ and the actual POPs are given by conditional probabilities $(\pi(in|x, m, d), \pi(win|x, m, d))$.

Valuing disequilibrium serve strategies

- Let $W_P(x,m)$ be the conditional win probability of a server whose potentially disequilrium Markovian serve direction strategy is $P(d|x,m)$. We have

$$W_P(x,m) = \sum_{d \in \{l,b,r\}} W_P(x,m,d)P(d|x,m)$$

  where $W_P(x,m,d)$ is the conditional probability of winning under the strategy P given the server serves to direction d in state (x,m).
- The values $W_P(x,m,d)$ are given by the same recursive equations as we derived for the optimal (equilibrium) serve strategy above in Bellman's equation.

Linear equation system for conditional win probabilities

- We can show that $W_P(x, m)$ is the solution to the following system of linear equations

$$W_P = w_P(P, \Pi) + M_P(P, \Pi) W_P \tag{1}$$

where $P = \{P(d|x, m)\}$, $\Pi = \{\pi(in|x, m, d), \pi(win|x, m, d)\}$ and $w_P(P, \Pi)$ is a 298×1 vector of probabilities of winning the service game directly (equal to zero for most states except the deuce endgame states and states where the server is within one shot of winning), and $M_P(P, \Pi)$ is a 298×298 Markov sub-transition probability matrix.

- Because $\|M_P(P, \Pi)\| < 1$, this system will have a unique solution W_P implied by any (P, Π).
- Note there are a total of $894 = 3 \times 298$ conditional win probabilities $W_P(x, m, d)$. Thus if we can estimate (P, Π) well using fewer parameters, we can derive more efficient estimates of $W_P(x, m, d)$.

Second Testable Implication of Nash Equilibrium

- Suppose we could estimate the server's conditional win probabilities $W_P(x, m, d)$ *without imposing optimality/equilibrium.* Then a key implication of a mixed strategy Nash equilibrium is

Testable Implication of Equilibrium (2)

One shot deviation principle In all states (x, m) and for all serve directions $d \in \{l, b, r\}$ we have:

$$W_P(x, m, d) = W_s(x, m, d)$$

- According to *Wikipedia* that One shot deviation principle "is the principle of optimality of dynamic programming applied to game theory. It says that a strategy profile of a finite extensive-form game is a subgame perfect equilibrium (SPE) if and only if there exist no profitable one-shot deviations for each subgame and every player."

Must serves be serially independent in equilibrium?

- NO! Especially in the presence of muscle memory, serve directions may be serially correlated in a MPE of tennis.
- Recall the Monotonicity Condition of Walker, Wooders and Amir. They ignored muscle memory so the only relevant state of tennis is the score state x.

Assumption

(Monotoncity Condition) Let $W_s(x)$ be the equilibrium conditional win probability for the server in a MPE without muscle memory. The equilibrium satisfies the Monotonicity Condition if we have for each $x\{1,\ldots,36\}$

$$W_s(x^+(x)) > W_s(x^-(x))$$

- That is, the probability of winning the service game is always higher if you win the current point game than if you lose it.

Walker, Wooders and Amir's Decomposition Theorem

Theorem

If there is no muscle memory and the Monotonicity Condition holds, then the MPE of the service game is a composition of the MPEs of each of the point subgames of tennis.

Corollary

If there is no muscle memory and the Monotonicity Condition holds, serve directions are serially independent. If the POPs are independent of of score state x, then serve directions are IID random variables.

Decomposition Theorem with muscle memory?

- Only if the following *Generalized Monotonity Condition* holds

Assumption

(Generalized Monotonicity Condition) Let $W_s(x, m)$ be the equilibrium conditional probability of winning for the serve in an MPE with muscle memory. The equilibrium satisfies the Generalized Monotonicity Condition if for each $x \in \{1, \dots, 36\}$ and all m we have

$$W_s(x^+(x), m'(m, d)) > W_s(x^-(x), m'(m, d')) \quad \forall d, d' \in \{l, b, r\}$$

- In words, it is always better for the server to win the current point regardless of how winning that point affects muscle memory. Intuitively, GMC will hold if muscle memory effects are sufficiently small.

Decomposition Theorem with muscle memory

Theorem

If there is muscle memory and the Generalized Monotonicity Condition holds, then the MPE of the service game is a composition of the MPEs of each of the point subgames of tennis.

Corollary

Serve directions will generally be serially correlated in the presence of muscle memory even if the Generalized Monotonicity Condition holds. As long as serve directions depend on m in equilibrium, then serve directions will exhibit serial dependence even in equilibrium play.

Summary of testable implications of equilibrium

- **Nash equilibrium** no other serve strategy strictly increases the server's probability of winning the service game relative to their current strategy P.
- **Mixed strategy equilibrium** the conditional win probability in any state (x, m) should be the same for all serve directions d played with positive probability.
- **Option value of the second serve** The serve strategy of the first and second serves should differ due to the option value of the second serve.
- **Optimality of myopic strategies** However if GMC holds, there is no extra advantage from considering what may happen in the service game beyond the current point.
- **Serial dependence** If GMC holds and there are no muscle memory effects, successive first serve directions will be serially independent. However if GMC fails or muscle memory effects are present, serve directions will generally be serially correlated.

Data for our study

- From the Match Charting Project, a crowd-sourced database that records play by play outcomes of thousands of tennis matches, including classifying each serve direction as 1) wide, 2) down the T or 3) to the body
- Even after restricting to matches played on hard courts, we end up with roughly ten times as many serves per server-returner pair than Walker and Wooders used in their analysis.
- We focus on elite professional tennis players who have all been ranked number one in the world and won multiple Grand Slams. These players are Roger Federer, Rafael Nadal, Novak Djokovic, Andy Murray, Pete Sampras, and Andre Agassi.
- Not only do we have the most observations on these players, but if we can show that they serve suboptimally, then even the best of the best are susceptible to strategic errors.

Mixed serve strategies of selected pros

Server → Returner	Games, serves Serves/game	1st serves 2nd serves	Serve directions L	B	R	Win prob (std) P-value: $P_1 = P_2$
Roger Federer →	523, 4732	3208	.4402	.1007	.4592	.7686 (.0184)
Rafael Nadal	8.36	1164	.2174	.2698	.5129	5.1×10^{-60}
Rafael Nadal →	519, 4081	3227	.6616	.2048	.1336	.8092 (.0172)
Roger Federer	7.86	854	.5937	.3208	.0855	6.3×10^{-12}
Roger Federer →	411, 3501	2524	.4521	.0939	.4540	.8200 (.0190)
Novak Djokovic	8.52	977	.4084	.3408	.2508	6.7×10^{-68}
Novak Djokovic →	407, 3653	2696	.4640	.1565	.3795	.8010 (.0198)
Roger Federer	8.98	957	.4389	.3365	.2247	1.0×10^{-33}
Rafael Nadal →	346, 2937	2230	.3964	.2825	.3211	.7197 (.0241)
Novak Djokovic	8.49	707	.4073	.5403	.0523	2.4×10^{-64}
Novak Djokovic →	356, 2877	2149	.4067	.1619	.4314	.7528 (.0222)
Rafael Nadal	8.08	728	.1484	.2940	.5577	1.2×10^{-40}
Novak Djokovic →	230, 1958	1447	.4651	.1244	.4105	.7696 (.0278)
Andy Murray	8.51	511	.2192	.4618	.3190	3.0×10^{-53}
Andy Murray →	230, 2141	1522	.3863	.0841	.5296	.7435 (.0288)
Novak Djokovic	9.31	619	.4233	.4782	.0985	5.8×10^{-122}
Pete Sampras →	140, 1275	884	.4434	.0724	.4842	.9000 (.0254)
Andre Agassi	9.11	391	.4680	.1765	.3555	5.3×10^{-8}
Andre Agassi →	135, 1125	825	.5127	.1115	.3758	.8666 (.0293)
Pete Sampras	8.33	300	.5766	.2700	.1533	7.2×10^{-16}

“Model-free” conclusions

- The evidence presented from a “model-free” analysis of the data seems broadly consistent with the predictions of game theory;
  - servers use mixed strategies,
  - these strategies differ across first and second serves (reflecting the effect of the option value of the 2nd serve discussed above),
  - servers appear to adjust their serve strategy to exploit the relative weaknesses of their opponents.
- The variation in win probabilities across server-returner pairs reflects differences in relative physical abilities of different players.

A flexible, agnostic reduced-form model of tennis

- In order to test the key necessary condition for a mixed strategy equilibrium — equality of conditional win probabilities for all serve directions — a deeper econometric analysis is required.
- If we had sufficient observations, we could non-parametrically estimate conditional win probabilities and test if they are equal.
- Walker and Wooders only considered 2 serve directions (excluding body serves), and they only analyzed 1st serves (not 2nd serves). We model the entire service game, but restrict our sample to hard courts.
- As we noted, once we account for muscle memory, there are a total of 894 conditional win probabilities $W_s(x, m, d)$. In order to estimate 894 probabilities with sufficient precision to have adequate power to test the hypothesis of equal win probabilities for all serve directions at all states (x, m), we would need roughly 10000 service games.

A flexible, agnostic reduced-form model of tennis

- Unfortunately in our data set we typically have only 100 to 200 service games per server-returner pair. However we can calculate *implied conditional win probabilities* as functions of flexibly parameterized reduced-form specifications of serve strategies $P(d|x, m)$ and the POPs $(\pi(in|x, m, d), \pi(win|x, m, d))$.
- Let $f(x, m, d)$ be a $1 \times K_P$ vector of indicators for various subsets of (x, m, d)-space, and θ_P a conformable parameter vector. Then we have

$$P(d|x, m, \theta_P) = \frac{\exp\{f(x, m, d)'\theta_P\}}{\sum_{\delta \in \{l,b,r\}} \exp\{f(x, m, \delta)'\theta_P\}}$$

- Similarly for $1 \times K_{in}$ and $1 \times K_{win}$ functions $g_{in}(x, m, d)$ and $g_{win}(x, m, d)$ and conformable parameter vectors θ_{in} and θ_{win} we can define the POPs

$$\begin{aligned} \pi(in|x, m, d, \theta_{in}) &= \frac{\exp\{g_{in}(x, m, d)'\theta_{in}\}}{1 + \exp\{g_{in}(x, m, d)'\theta_{in}\}} \\ \pi(win|x, m, d, \theta_{win}) &= \frac{\exp\{g_{win}(x, m, d)'\theta_{win}\}}{1 + \exp\{g_{win}(x, m, d)'\theta_{win}\}} \end{aligned}$$

Service outcomes

- There are 3 possible outcomes of a serve:
  - $o = 1$: serve is in and server wins the rally
  - $o = 2$: the serve is in and server loses the rally
  - $o = 3$: the serve is out, i.e. a fault
- Let $f(o|x, m, d, \theta_{in}, \theta_{win})$ be the conditional probability of these outcomes. We have

$$f(o|x, m, d, \theta_{in}, \theta_{win}) =$$
$$\begin{cases} \pi(in|x, m, d, \theta_{in})\pi(win|x, m, d, \theta_{win}) & \text{if } o = 1 \\ \pi(in|x, m, d, \theta_{in})[1 - \pi(win|x, m, d, \theta_{win})] & \text{if } o = 2 \\ 1 - \pi(in|x, m, d, \theta_{in}) & \text{if } o = 3 \end{cases}$$

Likeilhood function

- We estimate the parameter vector $\theta = (\theta_P, \theta_{in}, \theta_{win})$ by maximum likeihood using the log-likelihood function $L(\theta)$ given by

$$L(\theta) = \sum_{g=1}^{G} \sum_{s=1}^{S_g} \left[\log \left(P(d_{s,g} | x_{s,g}, m_{s,g}, \theta_P) \right) + \log \left(f(o_{s,g} | x_{s,g}, m_{s,g}, d_{s,g}, \theta_{in}, \theta_{win}) \right) \right],$$

  where G is the total number of service games observed for a particular server-returner pair, S_g is the number of serves in game g, and $(d_{s,g}, x_{s,g}, m_{s,g})$ is the observed direction of serve, game state and muscle memory state at serve s in game g.
- To avoid overfitting, we use AIC $= 2[K - L(\hat{\theta})]$ to select a preferred specification that balances the tradeoff between model flexibility (bias) and overfitting (variance).

Estimates of P, Djokovic and Federer

	Parameter Name	Djokovic $\to$ Federer Estimate	Djokovic $\to$ Federer Std Error	Federer $\to$ Djokovic Estimate	Federer $\to$ Djokovic Std Error
1	1st serve, deuce court, $d = l$	.836	(.102)	1.234	(.145)
2	1st serve, deuce court, $d = r$	.795	(.106)	1.402	(.143)
3	1st serve, deuce court, $d = d_{-2}$	.566	(.077)	.483	(.103)
4	2nd serve, deuce court, $d = l$	$-.208$	(.145)	$-.293$	(.132)
5	2nd serve, deuce court, $d = r$	$-.650$	(.151)	$-.531$	(.159)
6	2nd serve, deuce court, $d = d_{-1}$	$-.002$	(.168)	.137	(.144)
7	1st serve, ad court, $d = l$	.861	(.095)	1.367	(.133)
8	1st serve, ad court, $d = r$	.601	(.100)	1.319	(.132)
9	1st serve, ad court, $d = d_{-2}$	.294	(.079)	.396	(.086)
10	2nd serve, ad court, $d = l$	.462	(.146)	.551	(.152)
11	2nd serve, ad court, $d = r$	$-.386$	(.183)	$-.215$	(.171)
12	2nd serve, ad court, $d = d_{-1}$	$-.051$	(.136)	$-.143$	(.142)
	Observations, log-likelihood	2372, -2324.8		2333, -2265.06	
	AIC, BIC	4871.6, 4940.8		4554.1, 4623.2	

Estimates of $\pi(in|x, m, d)$, Djokovic and Federer

Parameter		Djokovic $\to$ Federer		Federer $\to$ Djokovic	
θ_{in}		Estimate	Std Error	Estimate	Std Error
1	1st serve, deuce court, $d = l$	.465	(.147)	.486	(.140)
2	1st serve, deuce court, $d = b$	.945	(.213)	.864	(.230)
3	1st serve, deuce court, $d = r$	.744	(.134)	.614	(.115)
4	1st serve, deuce court, $d = d_{-2}$	$-.093$	(.156)	$-.113$	(.145)
5	2nd serve, deuce court, $d = l$	3.468	(.572)	2.277	(.403)
6	2nd serve, deuce court, $d = b$	2.137	(.288)	3.249	(.459)
7	2nd serve, deuce court, $d = r$	2.150	(.445)	1.898	(.360)
8	2nd serve, deuce court, $d = d_{-2}$	.277	(.513)	.118	(.440)
9	1st serve, ad court, $d = l$	.604	(.159)	.122	(.148)
10	1st serve, ad court, $d = b$	.928	(.173)	.530	(.232)
11	1st serve, ad court, $d = r$	.813	(.142)	.422	(.128)
12	1st serve, ad court, $d = d_{-2}$	$-.138$	(.165)	.164	(.154)
13	2nd serve, ad court, $d = l$	2.214	(.406)	2.652	(.362)
14	2nd serve, ad court, $d = b$	1.917	(.352)	3.820	(.719)
15	2nd serve, ad court, $d = r$	1.430	(.374)	2.033	(.387)
16	2nd serve, ad court, $d = d_{-2}$	.294	(.452)	.329	(.499)
Observations, log-likelihood AIC, BIC		2333, -2403.9 4871.9, 5056.6		2372, -2324.8 4625.4, 4809.6	

Estimates of $\pi(win|x, m, d)$, Djokovic and Federer

	Parameter	Djokovic $\rightarrow$ Federer		Federer $\rightarrow$ Djokovic	
	θ_{win}	Estimate	Std Error	Estimate	Std Error
1	1st serve, deuce court, $d = l$	.641	(.182)	1.092	(.208)
2	1st serve, deuce court, $d = b$	.470	(.196)	.760	(.250)
3	1st serve, deuce court, $d = r$	.439	(.143)	.795	(.154)
4	1st serve, deuce court, $d = d_{-2}$	.456	(.188)	.314	(.210)
5	2nd serve, deuce court, $d = l$	.975	(.235)	.148	(.247)
6	2nd serve, deuce court, $d = b$	.593	(.195)	.073	(.182)
7	2nd serve, deuce court, $d = r$	.650	(.290)	.063	(.285)
8	2nd serve, deuce court, $d = d_{-2}$	$-.537$	(.273)	.038	(.270)
9	1st serve, ad court, $d = l$	.878	(.185)	1.397	(.232)
10	1st serve, ad court, $d = b$	.614	(.223)	1.055	(.305)
11	1st serve, ad court, $d = r$	.728	(.171)	.933	(.167)
12	1st serve, ad court, $d = d_{-2}$	$-.182$	(.194)	$-.392$	(.216)
13	2nd serve, ad court, $d = l$	$-.090$	(.221)	.292	(.221)
14	2nd serve, ad court, $d = b$	$-.282$	(.222)	$-.022$	(.222)
15	2nd serve, ad court, $d = r$	.530	(.295)	.142	(.263)
16	2nd serve, ad court, $d = d_{-2}$	.489	(.281)	.206	(.259)
	Observations, log-likelihood AIC, BIC	2333, -2403.9 4871.9, 5056.6		2372, -2324.8 4625.4, 4809.6	

Summary of reduced-form estimation results

- Our preferred specification (i.e. one with smallest AIC) has a total of 44 parameters 12 parameters θ_P and 16 parameters each for $(\theta_{in}, \theta_{win})$.
- Our preferred specification balances the tradeoff described above: it provides an accurate probability model of the entire service game for individual server-receiver pairs while avoiding the dangers of overfitting.
- Regarding muscle memory, we find significant positive serial correlation in first serve directions for both Federer and Djokovic, but not for 2nd serves.
- For the POPS, we do not find evidence of muscle memory in $\pi(in|x, m, d)$ but for $\pi(win|x, m, d)$ we find evidence that serving to the same direction increases the probability of winning given the serve is in.

Summary of reduced-form estimation results

- Conditional on a serve going in, Federer has a higher chance of winning the subsequent rally than does Djokovic when he serves to Federer.
- However we find that Djokovic has a lower probability of faulting relative to Federer. (An exception is for 2nd serves to the deuce court where Federer has a uniformly lower chance of faulting for all three serve directions.)
- Thus, our estimates reflect an intuitive trade-off: a faster serve has a higher chance of faulting, but conditional on it going in, the returner is at a disadvantage that reduces his chance of returning it successfully or winning the subsequent rally.

Testing the stationarity assumption

Server → Returner	No muscle memory			Muscle Memory		
	Restricted	Unrestricted	LR test	Restricted	Unrestricted	LR test
	LL, AIC	LL, AIC	df, P value	LL, AIC	LL, AIC	df, P value
Roger Federer →	-3132.5	-3110.4	32	-3104.7	-3077.6	44
Rafael Nadal	**6329.1**	6348.8	.074	**6297.5**	6331.2	.138
Rafael Nadal →	-3047.8	-3012.36	32	-3043.61	-2995.9	44
Roger Federer	6159.7	**6152.7**	8.9×10^{-5}	6175.2	**6167.8**	1.2×10^{-5}
Roger Federer →	-3352.99	-3320.0	32	-3325.7	-3285.7	44
Novak Djokovic	6770.0	**6768.1**	3.9×10^{-4}	**6739.4**	6747.3	7.2×10^{-4}
Novak Djokovic →	-3627.1	-3592.3	32	-3600.6	-3559.2	44
Roger Federer	7318.2	**7312.62**	1.3×10^{-4}	**7289.3**	7293.5	3.6×10^{-4}
Rafael Nadal →	-3577.0	-3569.0	32	-3566.4	-3554.8	44
Novak Djokovic	**7218.0**	7266.0	.992	**7220.8**	7285.7	.996
Novak Djokovic →	-3152.3	-3136.29	32	-3131.3	-3111.7	44
Rafael Nadal	**6368.6**	6400.6	.466	**6350.5**	6399.4	.683

Results of the stationarity tests

- We are not able to reject the stationarity assumption at the 5% critical level for Federer serving to Nadal, Nadal serving to Djokovic, or Djokovic serving to Nadal.
- But for the other three pairs the test does reject the restrictions implied by stationarity.
- In our preferred specification with muscle memory, imposing stationarity reduces AIC except for Nadal serving to Federer.
- We conclude that stationarity is a reasonable assumption, that enables us to pool sufficient numbers of service games to get the most reliable possible estimates of serve probabilities and the POPs.

Estimated serve and conditional win probabilities

Server → Returner	Est.	Win prob 1st serve	Conditional win probability, 1st serve			Spec test P-value
			L	**B**	**R**	
Roger Federer → Rafael Nadal	NP	.796 (.026)	.816 (.025)	.650 (.030)	.803 (.025)	.004
	RF	.829 (.023)	.828 (.024)	.819 (.027)	.833 (.022)	
Rafael Nadal → Roger Federer	NP	.786 (.026)	.748 (.028)	.896 (.020)	.762 (.028)	.107
	RF	.807 (.023)	.808 (.023)	.807 (.025)	.803 (.025)	
Roger Federer → Novak Djokovic	NP	.810 (.024)	.844 (.022)	.867 (.021)	.767 (.025)	.504
	RF	.818 (.020)	.826 (.020)	.812 (.023)	.813 (.021)	
Novak Djokovic → Roger Federer	NP	.782 (.025)	.769 (.026)	.710 (.028)	.815 (.024)	.910
	RF	.781 (.022)	.792 (.022)	.769 (.026)	.774 (.024)	
Rafael Nadal → Novak Djokovic	NP	.712 (.035)	.685 (.036)	.726 (.035)	.750 (.034)	.992
	RF	.712 (.034)	.712 (.034)	.701 (.035)	.718 (.034)	
Novak Djokovic → Rafael Nadal	NP	.829 (.029)	.868 (.026)	.735 (.034)	.833 (.029)	.278
	RF	.848 (.023)	.854 (.023)	.830 (.027)	.849 (.023)	
Novak Djokovic → Andy Murray	NP	.794 (.034)	.759 (.036)	.750 (.036)	.841 (.031)	.871
	RF	.791 (.029)	.796 (.031)	.758 (.034)	.799 (.029)	
Andy Murray → Novak Djokovic	NP	.721 (.038)	.816 (.033)	.500 (.042)	.701 (.038)	.675
	RF	.717 (.036)	.735 (.036)	.712 (.039)	.703 (.037)	
Pete Sampras → Andre Agassi	NP	.885 (.028)	.894 (.027)	1.00 (.000)	.859 (.030)	.150
	RF	.866 (.024)	.866 (.025)	.839 (.029)	.872 (.024)	
Andre Agassi → Pete Sampras	NP	.874 (.029)	.907 (.026)	.867 (.030)	.852 (.032)	.362
	RF	.859 (.024)	.861 (.026)	.853 (.026)	.859 (.024)	

Summary of specification tests of win probabilities

- The calculated win probabilities $W_P(1, 1)$ are close to the non-parametric estimates of these quantities, and almost always within a standard deviation of each other.
- The P-values of the Hausman specification tests in the final column of the table show that for all servers except Federer serving to Nadal we are unable to reject the reduced-form specification and its implied win probabilities.
- The middle columns compare the non-parametric estimates of the conditional win probabilities to the corresponding estimates implied by the reduced-form model, $W_P(1, 1, d)$ for $d \in \{l, b.r\}$. Again we see that the two estimates are generally close to each other.

Tests of equal conditional win probabilities

Server → Returner	Estimator	P-value, df Wald test equal win probs, 1st serves	P-value, df Omnibus test equal win probs, all serves
Roger Federer → Rafael Nadal	NP RF	.048, 2 .614, 2	0, 36
Rafael Nadal → Roger Federer	NP RF	.148, 2 .891, 2	0, 37
Roger Federer → Novak Djokovic	NP RF	.402, 2 .237, 2	0, 37
Novak Djokovic → Roger Federer	NP RF	.382, 2 .067, 2	0, 35
Rafael Nadal → Novak Djokovic	NP RF	.780, 2 .267, 2	0, 36
Novak Djokovic → Rafael Nadal	NP RF	.351, 2 .235, 2	0, 36
Novak Djokovic → Andy Murray	NP RF	.643, 2 .058, 2	0, 39
Andy Murray → Novak Djokovic	NP RF	.002, 2 .057, 2	0, 39
Pete Sampras → Andre Agassi	NP RF	.471, 2 .037, 2	0, 38
Andre Agassi → Pete Sampras	NP RF	.889, 2 .850, 2	0, 37

Summary of tests of equal win probabilities

- The omnibus tests of equal win probabilities have much greater power and we see that for all server pairs analyzed, the test decisively rejected this key implication of mixed strategy Nash equilibrium.
- The omnibus Wald tests 648 equality restrictions on the conditional win probabilities, which are implicit functions of (P, Π) and the latter are functions of the 44-dimensional parameter vector $\hat{\theta} = (\hat{\theta}_P, \hat{\theta}_{in}, \hat{\theta}_{win})$.
- For comparison, we also present Wald tests of equal win probabilities restricted to the first serve of each game, i.e. we test the two restrictions $W_P(1, 1, l) = W_P(1, 1, b)$ and $W_P(1, 1, b) = W_P(1, 1, r)$, so the relevant Wald statistic is asymptotically Chi-squared with two degrees of freedom under the null hypothesis.

Summary of tests of equal win probabilities

- We present the test statistics for both the non-parametric estimates of the conditional win probabilities (the rows labelled NP) as well as the conditional win probabilities implied by the reduced form estimates (the rows labelled RF).
- We see from the generally high P-values that these more limited Wald tests have much lower power in detecting deviations from the null hypothesis. Indeed, in only three cases we can reject the null hypothesis of equal win probabilities for all serve directions at the very first serve at the 5% level.
- Overall, we conclude that our approach to testing for equal win probabilities, combined with the much greater number of observations of service games compared to WW's original analysis explains why we are able to decisively reject the key implication of a mixed strategy Nash equilibrium.

Tests for presence of muscle memory

Server → Returner	Model	No muscle memory		Muscle memory		LR test
		AIC	**LL**	**AIC**	**LL**	P-value
Roger Federer → Rafael Nadal	Serves	3764.7	-1874.4	3713.5*	-1844.8	4.3×10^{-12}
	POPs	3928.3*	-1940.1	3932.6	-1934.3	.170
Rafael Nadal → Roger Federer	Serves	3397.8*	-1690.9	3400.3	-1688.2	.249
	POPs	3814.5*	-1883.3	3824.9	-1880.9	.779
Roger Federer → Novak Djokovic	Serves	4603.7	-2293.9	4554.1*	-2265.1	9.3×10^{-12}
	POPs	4617.4*	-2284.8	4625.4	-2280.7	.414
Novak Djokovic → Roger Federer	Serves	4925.5	-2454.8	4871.6*	-2423.8	1.1×10^{-12}
	POPS	4871.3*	-2411.7	4871.9	-2403.9	.048
Rafael Nadal → Novak Djokovic	Serves	2891.0	-1437.5	2889.3*	-1432.6	.044
	POPs	2879.6*	-1415.8	2892.4	-1414.2	.921
Novak Djokovic → Rafael Nadal	Serves	2744.8	-1364.2	2718.9*	-1347.4	9.0×10^{-7}
	POPs	2656.9*	-1304.5	2668.1	-1302.1	.779
Novak Djokovic → Andy Murray	Serves	2458.0	-1221.0	2427.1*	-1201.6	7.7×10^{-8}
	POPs	2425.5*	-1188.7	2430.3	-1183.2	.202
Andy Murray → Novak Djokovic	Serves	2525.9	-1254.9	2524.0*	-1250.0	.044
	POPs	2623.9	-1287.9	2524.1*	-1280.1	.049
Pete Sampras → Andre Agassi	Serves	2208.8	-1096.4	2194.9*	-1085.4	2×10^{-4}
	POPs	2296.2*	-1124.1	2299.7	-1117.9	.134
Andre Agassi → Pete Sampras	Serves	1906.5	-945.3	1887.6*	-931.8	2×10^{-5}
	POPs	2113.2*	-1032.6	2126.2	-1031.1	.934

Summary of tests for muscle memory

- Our LR tests of the hypothesis of "no muscle memory effects" in the last column of the table show that except for the case of Nadal serving to Federer, we can reject the hypothesis of no muscle memory in serves at the 5% significance level.
- There is far weaker evidence of serial correlation in the POPs. For most server-returner pairs we are unable to reject the hypothesis of no muscle memory effects.
- Why would that be the case? We think it may have to do with the returner's behavior. Specifically, if muscle memory effects are real, and the returner shifts his position accordingly, then the returner would effectively cancel out any effect that muscle memory would impart on the POPs.
- As a result, we would observe serial correlation in the server's directional choices but not in the POPs. This would be consistent with a Nash equilibrium, as we demonstrate in appendix B.

Dynamic structural model of tennis

- Our flexible agnostic model of tennis decisively rejects the key implication of a mixed strategy Nash equilibrium: namely that the probability of winning the game is the same regardless of serve direction.
- Now we try to get more insight into *why* this might be. We have shown that tennis is a “mentally simple” game and under a plausible Monotonicity Condition, it is sufficient to only solve a two period DP problem, i.e. to find an equilibrium in the “point game” consisting of a first serve and the option of a 2nd serve in the event of a fault.
- Thus we do not believe that tennis pros are too dumb to solve a 2 period DP problem. There must be some other explanation for the suboptimality of their serve strategies.

Modeling how far to look ahead

- To get deeper insight into the behavior of elite pro servers, we estimate three different structural models of serve behavior that impose the restriction that serve directions are chosen to maximize the server's probability of winning the game.
- The three structural models differ in how far a tennis pro "looks ahead":
  - 1) a *fully dynamic model* that assumes the server chooses a strategy that maximizes the probability of winning the entire *service game;*
  - 2) a *myopic model* that assumes the server chooses a strategy that maximizes the probability of winning each *point;*
  - 3) a *fully myopic model* that assumes the server chooses a strategy the maximizes the probability of winning each *serve.*

Recovering subjective beliefs of the POPs

- We estimate structural discrete choice models of the mixing probabilities over serve directions $P(d|x, m)$. A key feature of our structural approach is that we relax the assumption of *rational expectations* about the POPs, Π.
- Instead we are able to estimate a tennis server's *subjective beliefs* of the POPs. One explanation of the suboptimality of serve strategies is that tennis pros have subjective beliefs about their own strengths and weaknesses, and those of their opponent, that do not fully square with reality.

Incorporating "trembles" that affect serve directions

- We assume that at the moment each serve is made, the server's choice of direction reflects *trembles.*
- These are *IID* shocks that affect their perception of the probability of winning when serving to different directions d.
- We assume that these trembles or preference shocks are observed only by the server but not by the returner or the econometrician. For example, the server may feel more comfortable hitting to a certain direction at some point in the match due to a psychological factor.
- Let $\epsilon(d)$ be the tremble associated with serving to direction d. We assume the trembles are independently distributed across all three possible serve directions, $\{l, b, r\}$ and *IID* across successive serves, and have a Type 1 extreme value distribution with location parameter normalized so that $E\{\max_d \epsilon(d)\} = 0$ and scale parameter $\lambda \geq 0$.

Serve strategies with trembles

- Let $\sigma_{FD}(x, m, \epsilon)$ be the serve strategy under the fully dynamic structural model as function of the observed state (x, m) and the unobserved trembles $\epsilon = (\epsilon(l), \epsilon(b), \epsilon(r))$.
- The fully dynamic model presumes that for each (x, m, ϵ) the server chooses the serve direction that maximizes the probability of winning the game, given by

$$\sigma_{FD}(x, m, \epsilon) = \underset{d \in \{l,b,r\}}{argmax}[\lambda\epsilon(d) + V_\lambda(x, m, d)]$$

where the value function $V_\lambda(x, m, d)$ is the analog of the conditional win probability $W_S(x, m, d)$, given by

$$V_\lambda(x, m) = \lambda \log \left(\sum_{d \in \{l,b,r\}} \exp \left\{ V_\lambda(x, m, d)/\lambda \right\} \right).$$

Structural serve direction probabilities

- The serve direction probabilities implied by the fully dynamic model, denoted by $P_{FD}(d|x,m)$, is given by

$$\begin{aligned} P_{FD}(d|x,m) &= Pr\left\{d=\sigma_{FD}(x,m,\epsilon)|x,m\right\} \\ &= \frac{\exp\{V_\lambda(x,m,d)/\lambda\}}{\sum_{d'\in\{l,b,r\}}\exp\{V_\lambda(x,m,d')/\lambda\}}. \end{aligned}$$

- Though $\sigma_{FD}(x,m,\epsilon)$ is a pure strategy from the standpoint of the server, it appears to be a mixed strategy from the standpoint of someone who does not observe ϵ. This allows us to rationalize observed mixed serve strategies without imposing equal win probabilities, i.e. imposing equality of $V_\lambda(x,m,d)$ over serve directions d.

Convergence to mixed serve strategies

- Since the trembles are *IID* across serves, it would appear that this model should also imply conditional independence in serve directions across successive first and second serves. However, that will actually only be true if there is no muscle memory, i.e. the variable m does not enter $V_\lambda(x, m, d)$. When there is muscle memory serves can be serially correlated even though trembles are *IID.*
- The following limit holds uniformly over (x, m, d) as $\lambda \downarrow 0$:

$$W_S(x, m, d) = \lim_{\lambda \downarrow 0} V_\lambda(x, m, d)$$

so the only way for $P_{FD}(d|x, m)$ to converge to a mixed strategy as $\lambda \downarrow 0$ is when conditional win probabilities $W_s(x, m, d)$ obey the equal win probability constraints,

$$W_S(x, m, l) = W_S(x, m, b) = W_S(x, m, r) \quad \forall (x, m).$$

Myopic structural models of serve behavior

- The myopic and fully myopic models have the same general structure as the fully dynamic model, so the serve strategies, value functions, and choice (mixing) probabilities are given by the same equations as above, except for how the value function V_λ is defined.
- In the fully myopic model we have

$$V_\lambda(x, m, d) = \pi(\text{in}|d, x, m)\pi(\text{win}|d, x, m),$$

so $V_\lambda(x, m, d)$ is the probability of winning the serve. Thus, a fully myopic server chooses a serve strategy to maximize the probability of winning each serve, without any concern about the effect of winning or losing on the future state of the game.

Myopic vs fully myopic models of serve behavior

- The myopic server's objective is to win each *point,* but the server does recognize the option value provided by the second serve in the event of a faulted first serve.
- This requires a two period backward induction calculation. If x is a second serve state (i.e. x is an even number between 2 and 36 in our numbering of tennis states), then $V_\lambda(x,m,d)$ coincides with the fully myopic value function given above. However for any non-terminal first serve state (any odd value of x from 1 to 35), V_λ is given by

$$\begin{aligned} V_\lambda(x,m,d) &= \pi(\text{in}|d,x,m)\pi(\text{win}|d,x,m) + \\ & [1-\pi(\text{in}|d,x,m)]V_\lambda(x+1,m') \end{aligned}$$

  where $m' = (d^{-1}, d)$ and $V_\lambda(x,m)$ is the the expected maximum of the fully myopic values given above.
- As we noted above, the myopic serve strategy coincides with the fully dynamic serve strategy in the limit as $\lambda \downarrow 0$ when the GMC holds.

Comments on the structural models

- Note that all three structural models have mixed serve probabilities that are implicit functions of the POPs. Thus, the mixed serve directions for the three structural models are entirely determined by the POPs and the single parameter λ controlling the magnitude of the trembles.
- In comparison, the reduced-form model of serve directions is estimated separately from the POPs with flexible parameterization of serve directions.
- The structural models can be viewed as restricted special cases of the most flexible specification of the reduced form serve model. This enables us to conduct likelihood ratio specification tests for the three structural models relative to the unrestricted reduced form specification.

Structural estimation results

Player pair	Reduced form	Fully myopic	Myopic	Fully dynamic
Server $\rightarrow$ Returner	LL, N BIC	LL, $\hat{\lambda}$ AIC, LR P-value	LL, $\hat{\lambda}$ AIC, LR P-value	LL, $\hat{\lambda}$ AIC, LR P-value
Roger Federer $\rightarrow$ Rafael Nadal	-3779.1, 2011 7646.1	-3788.2, 7.9×10^{-4} 7642.7, .074	-3783.8, 5.8×10^{-3} **7633.7, .571**	-3817.3, 1.1×10^{-4} 7700.7, 6.9×10^{-12}
Rafael Nadal $\rightarrow$ Roger Federer	-3569.1, 1882 7226.2	-3571.3, 6.1×10^{-3} 7208.6, .957	-3570.6, 2.7×10^{-3} **7207.3, .990**	-3632.4, 2.2×10^{-4} 7330.7, 8.8×10^{-22}
Roger Federer $\rightarrow$ Novak Djokovic	-4545.8, 2333 9179.5	-4551.2, .010 **9168.4, .457**	-4552.2, 4.5×10^{-3} 9170.4, .300	-4576.0, 9.1×10^{-4} 9128, 7.5×10^{-9}
Novak Djokovic $\rightarrow$ Roger Federer	-4827.7, 2372 **9743.5**	-4840.0, .011 9746.0, .010	-4842.0, 1.9×10^{-3} 9750.0, 2.6×10^{-3}	-4844.8, 2.4×10^{-4} 9755.7, 3.3×10^{-4}
Rafael Nadal $\rightarrow$ Novak Djokovic	-2846.8, 1405 5781.7	-2853.8, 1.1×10^{-4} 5773.7, .232	-2853.2, 8.2×10^{-5} **5772.4, .310**	-2864.5, 1.2×10^{-5} 5795.0, 2.2×10^{-4}
Novak Djokovic $\rightarrow$ Rafael Nadal	-2649.5, 1344 5387	-2659.9, .070 5385.9, .035	-2656.1,.097 5378.2, .285	-2656.7, 1.7×10^{-5} **5375.3, .505**
Novak Djokovic $\rightarrow$ Andy Murray	-2384.7, 1201 4857.5	-2396.2, 9.9×10^{-3} **4858.3, .018**	-2396.9, .044 4859.8, .011	-2413.0, 5.5×10^{-4} 4892.0, 4.0×10^{-8}
Andy Murray $\rightarrow$ Novak Djokovic	-2649.5, 1328 5387.0	-2536.4, .014 **5138.9, .309**	-2539.8, 6.9×10^{-3} 5145.7, .051	-2556.3, $9.7 \times 1-^{-5}$ 5350.0, 2.2×10^{-7}
Pete Sampras $\rightarrow$ Andre Agassi	-2203.3, 1181 **4494.6**	-2219.6, .031 4505.3, 5.9×10^{-4}	-2217.7, .037 4501.4 2.5×10^{-3}	-2240.2, 1.9×10^{-6} 4546.4, 2.3×10^{-11}
Andre Agassi $\rightarrow$ Pete Sampras	-1962.9, 1050 4013.8	-1973.0, 1.8×10^{-3} 4011.9, .043	-1970.8, 1.9×10^{-5} **4007.6, .145**	-2004.7, 5.7×10^{-5} 4075.5, 2.8×10^{-13}

Summary of structural estimation results

- Note that the scale parameters $\hat{\lambda}$ for all specifications are uniformly small, which means that the data finds limited role for "trembles" to explain the observed mixed serve strategies of these players.
- The estimated $\hat{\lambda}$ is very close to zero for the fully dynamic specification, so as noted above the implied value functions satisfy the equal win probability constraint.
- Note that the λ estimates decline for the structural models that require increasingly "far sighted" calculations by the server: the λ values for the fully dynamic model are typically very close to zero; those for the myopic model are small but somewhat larger; and the λ estimates for the fully myopic serve model are typically the largest.

Summary of structural estimation results

- The maximum likelihood estimates of the POPs, $(\hat{\theta}_{in}, \hat{\theta}_{win})$ are distorted in a manner that results in conditional win probabilities much closer to equality than the ones implied by the reduced form estimates of the POPs.
- We see that the best model selected by AIC is generally also the model for which there is the least evidence for rejecting it in favor of the reduced form model via the likelihood ratio test.
- AIC selects the myopic two-period DP model as the best model for four of the servers, and it selects the fully myopic model for three of the servers. In two cases, Djokovic serving to Federer and Sampras serving to Agassi, the AIC criterion selects the reduced form model and the likelihood ratio test strongly rejects all three structural models.
- AIC selects the full DP model in only one case, Djokovic serving to Nadal.

Omnibus Wald tests of equal win probabilities

Player pair	**Reduced form**	**Fully myopic**	**Myopic**	**Fully dynamic**
Server $\rightarrow$ Returner	Wald stat, df P-value	Wald stat, df P-value	Wald stat, df P-value	Wald stat, df P-value
Roger Federer $\rightarrow$ Rafael Nadal	2.6×10^6, 36 0	6135, 28 0	12258, 28 0	0.8, 29 1.000
Rafael Nadal $\rightarrow$ Roger Federer	199, 37 4.9×10^{-24}	4717, 29 0	20, 27 .825	11, 29 .999
Roger Federer $\rightarrow$ Novak Djokovic	105946, 37 0	3412, 29 0	931, 28 6.0×10^{-178}	117, 29 1.9×10^{-12}
Novak Djokovic $\rightarrow$ Roger Federer	1.1×10^6, 35 0	37094, 29 0	12, 28 .995	37, 29 .142
Rafael Nadal $\rightarrow$ Novak Djokovic	11511, 36 0	68074, 29 0	.0001, 25 1.0000	1.0×10^{-5}, 25 1.000
Novak Djokovic $\rightarrow$ Rafael Nadal	146629, 36 0	219705, 29 0	247715, 29 0	.008, 27 1.000
Novak Djokovic $\rightarrow$ Andy Murray	4152, 39 0	7021, 29 0	109771, 29 0	8.5, 29 1.000
Andy Murray $\rightarrow$ Novak Djokovic	94519, 39 0	5784, 29 0	30, 29 .380	1.1, 28 1.000
Pete Sampras $\rightarrow$ Andre Agassi	2268, 38 0	853141, 29 0	138153, 29 0	33, 29 .265
Andre Agassi $\rightarrow$ Pete Sampras	44190, 37 0	177305, 29 0	1.9×10^{-5}, 24 1.000	5, 28 1.000

Summary of tests for equal win probabilities

- Due to the larger estimated values of λ for the fully myopic model, we strongly reject the hypothesis of equal win probabilities for this specification, just as for the reduced-form model.
- However for the fully dynamic specification, which has the smallest estimated values of λ, we are unable to reject the hypothesis of equal win probabilities for any of the 10 elite server-returner pairs except for Federer serving to Djokovic. (Note that the fully dynamic specification with the highest estimated $\hat{\lambda}$ is for Federer serving to Djokovic).

Win probabilities and Hausman specification tests

Player pair	Nonparametric win probability	Reduced form	Fully myopic	Myopic	Fully dynamic
Server $\rightarrow$ Returner	$\hat{W}(1,1)$	$W(1,1)$ P-value	$W(1,1)$ P-value	$W(1,1)$ P-value	$W(1,1)$ P-value
Roger Federer $\rightarrow$ Rafael Nadal	.796 (.026)	.829 (.023) .004	.825 (.021) .050	.830 (.021) .025	.749 (.021) 1.3×10^{-3}
Rafael Nadal $\rightarrow$ Roger Federer	.786 (.026)	.807 (.023) .107	.806 (.022) .147	.807 (.022) .132	.641 (.022) 1.0×10^{-22}
Roger Federer $\rightarrow$ Novak Djokovic	.810 (.024)	.818 (.020) .504	.818 (.020) .506	.818 (.020) .509	.759 (.020) 1.0×10^{-4}
Novak Djokovic $\rightarrow$ Roger Federer	.781 (.025)	.781 (.023) .910	.779 (.022) .838	.778 (.022) .756	.746 (.021) .011
Rafael Nadal $\rightarrow$ Novak Djokovic	.712 (.035)	.712 (.034) .992	.703 (.033) .485	.706 (.033) .675	.650 (.032) 4.1×10^{-5}
Novak Djokovic $\rightarrow$ Rafael Nadal	.829 (.029)	.848 (.023) .278	.846 (.023) .318	.850 (.023) .231	.797 (.024) .052
Novak Djokovic $\rightarrow$ Andy Murray	.794 (.034)	.792 (.029) .871	.791 (.030) .840	.791 (.030) .844	.750 (.029) .012
Andy Murray $\rightarrow$ Novak Djokovic	.721 (.038)	.717 (.036) .675	.717 (.034) .792	.718 (.035) .825	.584 (.032) 1.6×10^{-11}
Pete Sampras $\rightarrow$ Andre Agassi	.885 (.028)	.866 (.024) .150	.863 (.024) .130	.866 (.024) .187	.757 (.028) 0
Andre Agassi $\rightarrow$ Pete Sampras	.874 (.029)	.859 (.024) .362	.854 (.026) .183	.853 (.026) .150	.715 (.028) 1.4×10^{-58}

Summary of specification tests of win probabilities

- The Hausman tests strongly reject the fully dynamic specification, with the exception of Djokovic serving to Nadal (AIC selects the fully dynamic model as the preferred specification for Djokovic serving to Nadal).
- For the other servers, we note that the fully dynamic specification typically significantly underestimates the true win probability.
- This is caused by the need to distort the POPs to rationalize serve behavior as a best response to the estimated POPs in the fully dynamic specifications. The fully dynamic POPs generally imply higher probability of faults and a lower probability of winning the rally given a serve is in compared to the reduced form POPs.
- In contrast, the specification tests are generally unable to reject the fully myopic and myopic serve models.

Calculating best response serve strategies

- We conclude by providing a more powerful direct test of Nash equilibrium play in tennis: we construct alternative *deviation* serve strategies that significantly increase a server's chance of winning the game compared to the mixed strategy they are currently using.
- If the hypothesis of Nash equilibrium is correct, it should be impossible to construct any such deviation strategies. We construct deviation strategies using numerical DP, so they are pure strategies.
- The DP serve strategies exploit the unequal win probabilities captured by our reduced form estimates of the POPs. At each stage of the game, the DP strategy chooses the serve direction that has the maximum conditional win probability.

Calculating best response serve strategies

- We also calculate potentially suboptimal serve strategies based on the myopic and fully myopic specifications described at the start of this section. However, in all three cases, we do the calculations using the reduced form estimates of the POPs, not the structural estimates of the POPS which we showed in the previous section were distorted estimates of the true POPs.
- We also do the calculations with $\lambda = 0$, i.e. we do not allow for any "trembles" in our calculated serve strategies.
- Note that when GMC holds, the optimal myopic serve strategy (calculated via a 2 period DP within for each point separately, without considering the option value of the future state of the game that the full DP calculation accounts for) coincides with the full DP solution. Therefore, our calculations will be able to reveal when the GMC holds and does not hold.

Direct test of Nash equilibrium

- Our test of Nash equilibrium serve strategies appeals to the *one shot deviation principle* which states that there is *no deviation at any stage of a dynamic game that can increase the server's chance of winning, given the strategy of the returner.*
- We find that there are profitable one shot deviations at many stages of the game. While each such deviation yields a modest improvement in the win probability, the cumulative effect of all profitable deviations is often a large improvements in the overall game win probability.
- Of course, if a server were to switch to the optimal serve strategy we estimate, the returner may detect the change and adjust their own strategies, which would then result in changes in the POPs, likely mitigating the gain we estimate.

Shortcoming of direct tests of Nash equilibrium

- It is important to acknowledge one key shortcoming of our approach to testing the hypothesis of Nash equilibrium: we only have *estimates* of the POPs rather than the *true* POPs.
- Estimation error in the POPs could result in spurious, upward biased, estimates of the win probability when we use a noisy estimate of the POPs to calculate a best response strategy via DP instead of using the true POPs.
- Appealing to the usual common knowledge assumption underlying Nash equilibrium, the server and returner play with knowledge of the true POPs and true serve strategy. As econometricians we only have noisy estimates of the POPs and the server's mixed serve strategy.

A test of Nash equilibrium robust to estimation error

- To account for estimation error, we use the asymptotic distribution of our maximum likelihood parameter estimates to calculate an approximate probability distribution for the true POPs based on the data we observe.
- The true POPs are asymptotically distributed about the maximum likelihood estimate of the POPs according to the normal asymptotic distribution of the reduced form POP parameters $(\hat{\theta}_{in}, \hat{\theta}_{win})$.
- Via stochastic simulation, we can draw from the distribution of the true POPs by drawing values of $(\tilde{\theta}_{in}, \tilde{\theta}_{win})$ from an asymptotic normal distribution centered at the MLE $(\hat{\theta}_{in}, \hat{\theta}_{win})$ with a covariance matrix equal to the asymptotic covariance of the MLE.

A test of Nash equilibrium robust to estimation error

- This gives is a probablity distribution over the true POPs that the server might actually be facing (and knows, via the common knowledge assumption underlying Nash equilibrium), and we evaluate our calculated best response serve strategy using a robust control approach, where we calculate the win probability for our DP serve strategy for a random sample of POPs that the DP strategy was not "expecting".
- Recall that σ_S was used in Section 2 to denote the optimal serve strategy, which is an implicit function of the POPs, Π, which we now make explicit by writing $\sigma_S(\Pi)$.
- As econonetricians we do not know the true Π, though if the Nash equilibrium hypothesis is true, the server and returner both have common knowledge of it.

A test of Nash equilibrium robust to estimation error

- Assume a Nash equilibrium and let Π^* and P^* denote the true equilibrium POPs mixed serve strategy, respectively. By assumption, the players have common knowledge of these POPs.
- While we, as econometricians, do not directly observe Π^* and P^*, we can consistently estimate them with sufficient data. In particular, the hypothesis of Nash equilibrium implies that for any alternative serve strategy σ we have

$$W_S(P^*, \Pi^*) \geq W_S(\sigma, \Pi^*).$$

A test of Nash equilibrium robust to estimation error

- Let $\sigma_S(\Pi^*)$ be the optimal dynamic serve strategy (generally a pure strategy) calculated by dynamic programming for the true Nash equilibrium POPs, Π^*. Then by definition of optimality we have

$$W_S(\sigma_S(\Pi^*), \Pi^*) \geq W_S(P^*, \Pi^*) \geq W_S(\sigma, \Pi^*)$$

for all stationary Markovian serve strategies σ. The two inequalities above imply

$$W_S(P^*, \Pi^*) = W_S(\sigma_S(\Pi^*), \Pi^*),$$

that serves as the basis for our robust direct test of a mixed strategy Nash equilibrium in tennis: the optimal DP serve strategy should not result in a higher win probability compared to the mixed serve strategy P^* that the server actually uses.

A test of Nash equilibrium robust to estimation error

- To illustrate the problem with comparing the win rate given the optimal best response to the *estimated POPs* $\hat{\Pi}$ to the win rate given the estimated strategy $\hat{P}$ and the estimated POPs, note that by definition of optimality we have

$$W_S(\sigma_S(\hat{\Pi}), \hat{\Pi}) \geq W_S(\hat{P}, \hat{\Pi})$$

  That is, the optimal win probability using the estimated POPs will always be at least as high as the win probability implied by the estimated mixed serve strategy $\hat{P}$ and the estimated POPs, $\hat{\Pi}$.

- To develop a meaningful test of the key equality $W_S(P^*, \Pi^*) = W_S(\sigma_S(\Pi^*), \Pi^*)$ we rely on the Continuous Mapping Theorem and the fact that $W_S(\sigma, \Pi)$ is a continuous function of the serve strategy σ and POPs Π.

Asymptotic normality of estimated win probabilities

- Under the null hypothesis that P^* is a Nash equilibrium best response serve strategy we have

$$\sqrt{N}\left[W_S(\hat{P},\hat{\Pi}) - W_S(P^*,\Pi^*)\right] \Longrightarrow N(0,\Omega(P^*,\Pi^*)),$$

and

$$\sqrt{N}\left[W_S(\sigma_S(\hat{\Pi}),\hat{\Pi}) - W_S(\sigma_S(\Pi^*),\Pi^*)\right] \Longrightarrow N(0,\Omega(\sigma_S(\Pi^*),\Pi^*)),$$

where $\Omega(P^*,\Pi^*)$ and $\Omega(\sigma_S(\Pi^*),\Pi^*)$ are the asymptotic variances of the win probability, which can be calculated from the asymptotic variance covariance matrix of the reduced-form estimates of the serve parameters, $\hat{\theta}_P$ and POP parameters $(\hat{\theta}_{in},\hat{\theta}_{win})$ using the delta method.

A test of the Nash null based on randomization

- Let $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ be two *independent* random POPs that are drawn from $N(\hat{\Pi}, \Omega(\hat{\Pi}))$, the normal asymptotic distribution of the POPs, $\hat{\Pi}$.
- If H_o (Nash equilibrium) holds, then under H_o we have

$$\sqrt{N}\left[W_S(\hat{P}, \tilde{\Pi}_1) - W_S(P^*, \Pi^*)\right] \Longrightarrow N(0, \Omega(P^*, \Pi^*)),$$

$$\sqrt{N}\left[W_S(\sigma_S(\hat{\Pi}), \tilde{\Pi}_2) - W_S(\sigma_S(\Pi^*), \Pi^*)\right] \Longrightarrow N(0, \Omega(\sigma_S(\Pi^*), \Pi^*)),$$

The independence of the simulated POPs $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ implies

$$\sqrt{N}\frac{W_S(\hat{P}, \tilde{\Pi}_1) - W_S(\sigma_S(\hat{\Pi}), \tilde{\Pi}_2)}{\sqrt{\Omega(\hat{P}, \hat{\Pi}) + \Omega(\sigma_S(\hat{\Pi}), \hat{\Pi})}} \Longrightarrow N(0, 1),$$

A test of the Nash null based on randomization

- Using randomly drawn POPs circumvents the tautological aspect of the inequality $W_S(\sigma_S(\hat{\Pi}), \hat{\Pi}) \geq W_S(\hat{P}, \hat{\Pi})$ which indicates that the DP best-response must *always* outperform the estimated serve strategy $\hat{P}$.
- While $\sigma_S(\hat{\Pi})$ is a best response to the reduced form POPs $\hat{\Pi}$, it is not necessarily a best response to the *randomly drawn POPs* $\tilde{\Pi}$; and thus, it is possible that $W_s(\hat{P}, \tilde{\Pi}) > W_S(\sigma_S(\hat{\Pi}), \tilde{\Pi})$.
- That is, it is possible that the estimated reduced form serve strategy $\hat{P}$ will have a higher win probability than the DP serve strategy $\sigma_S(\hat{\Pi})$ for a randomly drawn POP $\tilde{\Pi}$.

A test of the Nash null based on randomization

- However if the Nash equilibrium hypothesis is true, and the two independently drawn POPs $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ are drawn from the normal asymptotic distribution for $\hat{\Pi}$, then both win probabilities will be close to each other and close to the common Nash equilibrium win probability.
- Thus, our robust randomized Nash test statistic will have an asymptotic $N(0,1)$ under the Nash null hypothesis.
- We do not need to rely on only a single pair of randomly drawn POPs, $(\tilde{\Pi}_1, \tilde{\Pi}_2)$ we can use T *IID* randomly drawn pairs of POPs $\{(\tilde{\Pi}_{1,t}, \tilde{\Pi}_{2,t})\}$ (each drawn from the asymptotic normal distribution of the reduced form estimate of the POPs, $\hat{\Pi}$) to obtain the following test statistic that has an asymptotic χ^2 distribution with T degrees of freedom

$$N \sum_{t=1}^{T} \frac{[W_S(\hat{P}, \tilde{\Pi}_{1,t}) - W_S(\sigma_S(\hat{\Pi}), \tilde{\Pi}_{2,t})]^2}{\Omega(\hat{P}, \hat{\Pi}) + \Omega(\sigma_S(\hat{\Pi}), \hat{\Pi})} \Longrightarrow \chi^2(T).$$

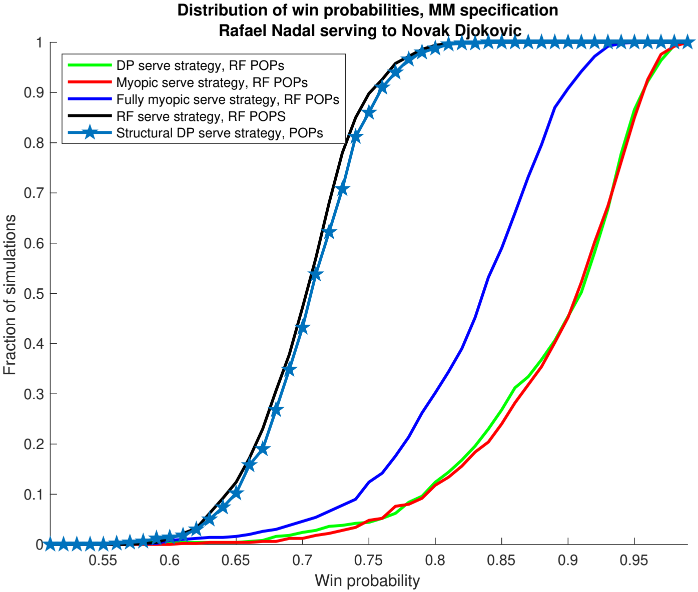
A test of the Nash null based on randomization

- The advantage of using many randomly drawn POPs $\{(\tilde{\Pi}_{1,t}, \tilde{\Pi}_{2,t})\}$ is that this provides a strong test of *robustness* of the estimated mixed serve strategy and the calculated DP serve strategy over a wide range of environments that these strategies were not "expecting."
- We can calculate the empirical CDF of win probabilities implied by $\hat{P}$ and $\sigma_s(\hat{\Pi})$ (i.e. the DP best response serve strategy) for different randomly drawn POPs $\tilde{\Pi}$ drawn from the asymptotic distribution of the MLE, $\hat{\Pi}$.
- We show an even stronger form of robustness: *the empirical CDF of win probabilities implied by the DP serve strategy $\sigma_s(\hat{P})$ stochastically dominates the empirical CDF of win probabilities implied by the estimated serve strategy $\hat{P}$.* Thus we conclude that our calculated DP best response serve strategy is robustly better than the serve strategies that the pros actually use.

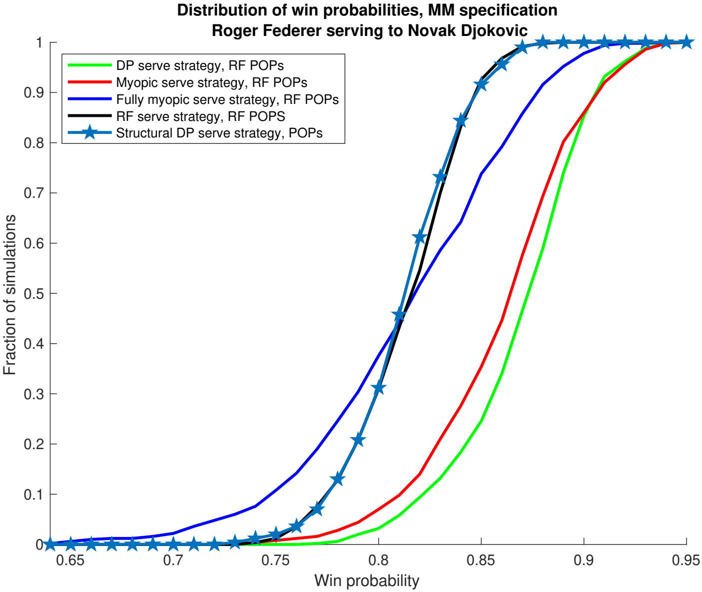
Improvements in simulated win probabilities

Player pair	Reduced form	Fully myopic	Myopic	Fully dynamic
Server→ Returner	$W(1,1)$	$W(1,1)$ P-value	$W(1,1)$ P-value	$W(1,1)$ P-value
Roger Federer → Rafael Nadal	.821 (.035)	.850 (.037) 0	.888 (.029) 0	.890 (.028) 0
Rafael Nadal → Roger Federer	.798 (.045)	.830 (.052) 0	.870 (.049) 0	.867 (.048) 0
Roger Federer → Novak Djokovic	.810 (.028)	.816 (.049) 0	.865 (.036) 0	.869 (.033) 0
Novak Djokovic → Roger Federer	.776 (.027)	.843 (.038) 0	.856 (.032) 0	.861 (.034) 0
Rafael Nadal → Novak Djokovic	.704 (.042)	.830 (.059) 0	.888 (.071) 0	.886 (.073) 0
Novak Djokovic → Rafael Nadal	.838 (.029)	.929 (.023) 0	.908 (.043) 0	.931 (.022) 0
Novak Djokovic → Andy Murray	.780 (.040)	.890 (.034) 0	.893 (.031) 0	.893 (.032) 0
Andy Murray → Novak Djokovic	.709 (.045)	.833 (.056) 0	.858 (.057) 0	.858 (.062) 0
Pete Sampras → Andre Agassi	.857 (.031)	.921 (.052) 0	.940 (.028) 0	.939 (.027) 0
Andre Agassi → Pete Sampras	.843 (.035)	.895 (.059) 0	.921 (.041) 0	.920 (.042) 0

Distributions of win probs: Nadal serving to Djokovic



Distributions of win probs: Federer serving to Djokovic



Interpreting the DP serve strategies

- In the case of Djokovic serving to Nadal the DP serve strategy generally entails serving to Nadal's right (i.e. backhand since Nadal is a lefty) on first serves, whereas on second serves, the optimal direction depends on the whether Djokovic is serving to the deuce or ad court.
- To the deuce court, Djokovic should serve to Nadal's backhand, whereas to the ad court, he should serve to Nadal's forehand. In other words, Djokovic should hit his second serve wide.
- Under the optimal strategy, the loss from one shot deviations from the recommended serve direction are typically small. For example, on first serves, the conditional probability of winning if Djokovic serves to Nadal's right (the optimal choice) is 0.937, but the worst choice, to serve to Nadal's body, still results in a conditional win probability of 0.921.

Small one shot gains cumulate to big overall gains

- The reason one shot deviations are not larger is that the full DP strategy entails an automatic recovery from “mistakes” at subsequent stages of the game and provided Djokovic follows the strategy most of the time, the expected losses from an occasional deviation from the optimal strategy are not too large.
- But when we consider the conditional win probabilities implied by the suboptimal mixed reduced form strategy that Djokovic uses, we see bigger gains at each stage.
- For example at the first serve of the service game, the direction with the highest win probability is to Nadal’s left, 0.854. The direction with the lowest win probability is to Nadal’s body, 0.831. Thus, there is a bigger penalty in terms of forgone win probability from serving to Nadal’s body, yet Djokovic serves to Nadal’s body with probability 0.155.

Small one shot gains cumulate to big overall gains

- Though the gain in win probability, *on the first serve* to serving to Nadal's left is only 0.023, this presumes that Djokovic reverts to his suboptimal mixed serve strategy for the remainder of the game.
- Dynamic programming exploits the "profitable deviations" in serve directions at every stage of the game and these gains cumulate to a much larger overall increase in win probability at the start of the game.
- As we showed above the total gain in win probability from switching from Djokovic's current mixed serve strategy to the optimal DP serve strategy is nearly 10 percentage points, from .831 to .938.

Conclusions

- Suppose you are convinced by our analysis that tennis pros don't play minimax. Why should you care? After all tennis is "just a game"
- One reason to care is that Nash equilibrium is a key part of the foundation not only of economic theory, but much of our empirical work as well, especially in industrial organization.
- To the extent that firms face vastly harder DP and equilibrium problems than tennis players face, one might worry that assuming rational Nash behavior when this assumption is not true could seriously distort our empirical work and policy conclusions.

Rediscovering Simon

- But finding ways to relax and test rationality assumptions is not a threat — instead it opens the door to a productive area for policy analysis: helping firms and individuals do better.
- But this can be regarded as a type of *empirical operations research* —- the sort of research Herbert Simon and others had already pioneered over 5 decades ago.
- Economists have been mislead by the Friedmanian “perfect rationality” mantra. While we are not suggesting that all economic agents are stupid, it is just as foolish to assume that all of them are unboundedly rational.
- The advent of AI and “big data” has spawned a rapidly growing new industry: *analytics.* Though like any other industry there are con artists and frauds, economists could have much to contribute to improving analytical models and advice, and in this way improve the economy.

Sports analytics

- One of the hot new areas is *sports analytics.* The (2003) book *Money Ball* by Lewis generated huge interest in this, as it suggested that sports analytics could improve the performance of entire baseball teams and generate millions of dollars in “deviation gains.”
- The 2006 JPE article by Romer on pro football strategy for 4th downs (vs field goals) also appears to have a measurable effect on how that game is played.
- Our findings suggest that even the elite pro tennis players may have inadequate statistical knowledge or an inadequate mental model of the POPs, the point outcome probabilities that implicitly embody their own strengths and weaknesses as tennis players as well as their opponents.
- Could our work be the foundation for “Money Ball for Tennis”?

The challenge: understanding how we learn and adapt

- We stress the "one shot deviation" nature of our results. If a server were to adopt our calculated best response serve strategy, since they are pure strategies it is likely that their opponents would quickly recognize and adapt to the change in serve strategy, partially or possibly entirely offsetting the deviation gains we have calculated.
- Thus we need to tackle the harder question of modeling how players learn and adapt. This is a fundamentally non-stationary analysis, so there are tremendous data challenges as well.
- What frictions prevent or impede players from learning, adapting, and responding optimally to their environment?
- Under which circumstance will there be a "steady state outcome" of a process of learning and experimentation that more closely approximates Nash equilibrium play?