JOURNAL OF APPLIED ECONOMETRICS
J. Appl. Econ. **21**: 1221–1247 (2006)
Published online in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/jae.854

AN EMPIRICAL MODEL OF THE MULTI-UNIT, SEQUENTIAL, CLOCK AUCTION

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SUMMARY

We construct a model of participation and bidding at multi-unit, sequential, clock auctions when bidders have multi-unit demand. We describe conditions sufficient to characterize a symmetric, perfect-Bayesian equilibrium and then demonstrate that this equilibrium induces an efficient allocation. We propose an algorithm, based on the generalized Vickrey auction, to calculate the expected winning bid for each unit sold. This algorithm allows us to construct a simulation-based estimator of the parameters for both the participation process and the distribution of latent valuations. We apply our method to data from 37 multi-lot, sequential, English auctions of export permits for timber held in Russia. Copyright © 2006 John Wiley & Sons, Ltd.

Received 10 February 2003; Revised 13 January 2005

1. INTRODUCTION

We construct a simple theoretical model of the multi-unit, sequential, Milgrom–Weber (1982) clock auction in which potential bidders may have multi-unit demand. This model was developed to interpret data concerning the outcomes at 37 multi-lot, sequential, English auctions of export permits for timber held between May 1993 and May 1994 in the Krasnoyarsk Region of Russia. These auctions had two important empirical regularities. First, bidders often won more than one lot. Second, different numbers of bidders attended different auctions. In addition, only a small amount of information was recorded; e.g., the reserve price, the number of lots on sale, the number of participants at the auction, the winning bid for each lot sold, and the number of lots won by each participant at the auction.

We ask the following question: Under what conditions can one learn about the distribution of latent valuations across potential bidders using these commonly available data? In answering this question, we describe one set of probabilistic conditions under which the twin hypotheses of optimization and equilibrium are sufficient to recover the distributions of bidder participation and latent valuations. We then propose a method to estimate the distributions of bidder participation and valuations and then to investigate the sampling variability of this method. Finally, we illustrate the feasibility of our approach by applying it to our data.

Our paper is in four more parts. In the next section, we describe briefly the institutional environment which generated the data we analyse later in the paper; viz., the sale of export permits for timber in Russia. We feel that such a description provides the reader with a lens through which

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to view our later work. In Section 2, we also present certain relevant and important descriptive statistics concerning the available data. We end Section 2 by explaining why existing theoretical models are ill-suited to structuring our data and then go on, in Section 3, to construct a theoretical model that can be used to interpret our data. In our model, the symmetric, perfect-Bayesian equilibrium of the auction game yields an efficient allocation; the equilibrium of a generalized Vickrey auction is also efficient. By a revenue-equivalence result, we argue that the mean of the winning bid for each unit sold at a generalized Vickrey auction is the same as that at our sequential, clock auction. In Section 4, we propose an estimation strategy based on the generalized Vickrey auction as well as simulated generalized method-of-moments which we implement using the data described in Section 2. In Section 5, we summarize our results and conclude the paper.

2. AUCTIONS OF EXPORT PERMITS FOR TIMBER IN SIBERIA

The geographic region of Siberia is one of the richest resource areas of Russia. While exploiting the vast wealth of minerals and furs is of considerable interest to the Russian government, exploiting the region's timber is no less important. The bulk of the timber from Siberia is marketed either within Russia or in the other republics of the former Soviet Union, but some is also exported beyond these borders. With the increasing importance of foreign hard currencies in the development of investment opportunities within Russia, timber exports have taken on a special significance. In the Krasnoyarsk Region, also known as *Krasnoyarsk Kray*, the government regulates timber exports by selling export permits at auction.[1] This auction market, which is overseen by the Ministry of the Krasnoyarsk Kray, is located in the city of Krasnoyarsk, which is located on the Yenesei (sometimes spelt Yenisey) River some 2100 miles, as the crow flies, east of Moscow.

The potential bidders in the market for export permits of timber are the firms harvesting timber in the Krasnoyarsk Kray. While the exact number of firms harvesting timber is reportedly unknown to the authorities of the Kray, the total number of firms exporting timber is greater than 20, but probably less than 40. Some of these firms are privately owned, but the bulk are state-owned enterprises. In either case, these firms have stands of timber, and the logs produced from harvesting this timber can potentially be exported. To export, however, these firms require a foreign buyer and an export permit. But without a buyer an export permit has no value since such permits are non-transferable; i.e., no secondary market in export permits exists. Moreover, an export permit will expire if not exercised within a fixed period of time, typically less than one year. Thus, the maximum value of an export permit to a potential bidder is the profit that firm could make by exporting timber from Siberia in the near future. To a first approximation, it would appear that such valuations are firm-specific; later, we shall also assume that these valuations are independent draws from a common distribution of valuations.

As alluded to above, the supply of export permits for timber is set by the Ministry of the Krasnoyarsk Kray. By the Wednesday of the week prior to a particular auction, which is typically held on either Tuesday or Thursday, potential exporters of timber from the Krasnoyarsk Kray are invited to submit requests for export permits to the ministry. While each potential exporter knows the volume of his request, he knows neither the total number of requests nor the total volume of requests. On the following Tuesday or Thursday, the ministry sells the export permits for timber (as well as other commodities) in lots at an auction house, the *Birch*. Below, we focus on the sale

[1] Krasnoyarsk Kray is to Siberia what Kansas is to the Great Plains or British Columbia is to the Pacific Northwest.

DOI: 10.1002/jae

Table I. Relative frequencies: volume per lot in cubic metres

<500	500	1000	1500	2000	3000	5000
0.0325	0.3021	0.5292	0.0032	0.0942	0.0194	0.0194

of export permits for a product known as 4403, saw logs. The most common lot size for saw logs was 1000 cubic metres. In fact, in our sample, over half of the lots involved 1000 cubic metres of logs. In Table I, we present the relative frequencies of volumes per lot for our entire sample. In Russia, one railway car holds about 100 cubic metres of logs.

To attend the auction at the Birch, potential bidders were required to pay a fee of 5000 Roubles (R), about \$3 US at the time the data were collected. In addition, for each lot, a minimum bid price, often referred to in the auction literature as the *reserve price*, was announced. Reserve prices were relatively high, either 1500R or 2500R per cubic metre in our sample, but these reserve prices were the same per cubic metre for all lots at a particular auction.

Several different ways exist to sell T lots to $\mathcal{N}$ potential bidders. For example, under one format, the auctioneer could require that each potential bidder simultaneously submits in secret a bid for each lot on sale. The auctioneer would then tabulate these bids, allocating a particular lot to the bidder who tendered the most for that lot, providing that the bid exceeded the reserve price. A number of different pricing rules might then be employed. For example, the highest bidder for a particular lot might then be required to pay his bid for that lot. This is often referred to as the *multi-unit, simultaneous, discriminatory-price, sealed-bid auction*. An alternative pricing rule might involve the highest bidder for a particular lot paying what his nearest opponent had bid for that lot. Yet a third rule might involve the highest bidders all paying the same price across lots, with that price being chosen to ensure that all T lots were sold. Of course, other auction formats and pricing rules also exist; the interested reader should consult Weber (1983) who has provided a detailed description of them.

One potential problem with the sealed-bid format described above, regardless of the pricing rule, is that a bidder might win too many or too few lots relative to his needs. Because no resale market existed for export permits of timber, bidder behaviour could be affected by a potential dearth or excess of lots won. For example, when faced with the possibility of winning too many lots, a potential bidder might be more cautious than in the case of resale, while the possibility of winning too few lots might induce a potential bidder to be more aggressive than in the case of resale. The net effect is difficult to predict because the benefits and costs of winning too many and too few lots are difficult to quantify, in general.

At the Birch in Krasnoyarsk, lots were sold sequentially, rather than simultaneously. The sale of a particular lot began with the auctioneer announcing the lot number, describing the export permit, and then asking interested bidders to hold up the white cards they were issued when they paid the entry fee. By holding up a white card, a participant signified his willingness to pay the reserve price of the export permit. The selling price was then increased verbally, more or less continuously, in small increments; 100R per cubic metre, about \$0.30 US, appears to have been the common increment. As the price rose, bidders chose to exit the sale by dropping their white cards. The winner was the bidder who held up his white card the longest; he paid the price at which his last opponent dropped his white card. Thus, the auction was a multi-lot, sequential, oral ascending-price (English) auction. Because many bidders won several lots, we believe multi-unit demand is important. We also believe that, within this environment, exit was observed by both the seller and the participants.

DOI: 10.1002/jae

Table II. Information for August 24, 1993 auction

Date	Lot no.	Volume	Reserve	No. bidders	Winning bid	Id. no.
93.08.24	1	1000	1 500 000	7	4 100 000	1
93.08.24	2	1000	1 500 000	7	4 400 000	2
93.08.24	3	1000	1 500 000	7	4 000 000	1
93.08.24	4	1000	1 500 000	7	4 400 000	2
93.08.24	5	1000	1 500 000	7	4 400 000	1
93.08.24	6	1000	1 500 000	7	4 300 000	3
93.08.24	7	500	750 000	7	2 050 000	4

For a sample of 37 auctions held between May 1993 and May 1994 that involved the sale of 308 lots, we have data concerning the number of lots for sale and their respective volumes, the reserve price of each lot, the winning bid for each lot, the number of participants present at each auction, and the number of lots won by each auction participant. Thus, for example, in Table II we have tabulated information concerning the lots sold at an auction held on August 24, 1993. At this auction, the reserve price for each lot was 1500R per cubic metre; seven lots were for sale, the first six involving 1000 cubic metres and the last 500. As it happened, seven bidders chose to attend the auction. A bidder, having identification number 1, won three of the lots; another bidder, who had identification number 2, won two; while two other bidders, one having identification number 3 and the other identification number 4, won one each. Three of the participants at the auction won no export permits. While the identification number of a winning bidder was unique to him at any auction, unfortunately, these identification numbers often appeared to change randomly across auctions. For example, identification number 1 at one auction could identify Firm A, which might then have had identification number 81 at another auction, but we just do not know. Apparently, tracing bidders across auctions proved impossible for our data-gatherers.

With regard to bidder participation, August 24 was unremarkable. On average, for the 37 auctions in our sample, only 6.56 bidders showed up. Since around 30 potential bidders existed at the time in the Krasnoyarsk Kray, one can see that a majority of potential bidders did not attend. The average number of lots on sale in our sample was 8.32, and about 47% of participants at any given auction won some lots. In Table III, we have tabulated the number of lots won by participants for each of the 37 auctions. Note that three of the auctions had but one lot for sale, while the other 34 had more than one, one as many as 32. At all but three of the multi-lot auctions, at least one of the auction participants won more than one lot. In some cases, one participant won all of the lots at auction. Below, we develop a theoretical framework that admits differential participation across auctions as well as some bidders winning several lots, while others win none, even though apparently enough lots were on sale so that each auction participant could have won at least one.

Heterogeneity in lot volumes existed at our auctions but, as demonstrated in Table I, a majority of sales involved export permits for 1000 cubic metres. Because a majority of lots at our auctions contained 1000 cubic metres, we think of each sale as one unit, but adjust winning bids to control for lot-volume heterogeneity by scaling winning bids by the volume for a particular lot.

In Figure 1, we present a scatterplot of the average winning bid per cubic metre versus the lot number. Thus, for example, a lot number of 2 would mean that this was the second sale of an auction. For our auctions, the total number of lots on sale ranged from 1 to 32. As the reader can see, considerable heterogeneity in average winning bids across lots existed, so it is difficult to decide what average price profiles across lots would look like. At least two possible

Table III. Number of lots won

Date	n	Frequency of lots won	w	T
93.05.06	15	1, 1, 1, 1, 1, 1, 2, 2, 2, 3, 4, 4	12	23
93.05.25	8	2, 2, 4, 6, 6	5	20
93.06.04	8	1, 1	2	2
93.07.14	14	1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4	13	32
93.07.28	12	1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 10	11	31
93.08.06	13	1, 2, 2, 4	4	9
93.08.12	14	1, 2, 3, 4, 4	5	14
93.08.17	11	1	1	1
93.08.24	11	1, 1, 2, 3	4	7
93.08.26	4	1, 1, 4	3	6
93.09.02	4	1, 2	2	3
93.09.16	4	1, 2, 2, 5	4	10
93.09.23	8	1, 3, 7	3	11
93.09.30	6	2, 2	2	4
93.10.07	8	1, 2	2	3
93.10.14	8	1, 1, 2, 3	4	7
93.10.21	6	1, 2, 2, 3	4	8
93.10.26	2	1, 1	2	2
93.10.28	4	5	1	5
93.11.02	5	2, 4, 4	3	10
93.11.09	4	1	1	1
93.11.16	4	7	1	7
93.11.23	5	1, 2	2	3
94.03.01	6	1, 1, 2, 2, 3	5	9
94.03.03	3	2	1	2
94.03.10	6	1	1	1
94.03.15	6	1, 1, 10	3	12
94.03.18	3	7	1	7
94.03.22	5	2, 2	2	4
94.03.24	5	4, 9, 10	3	23
94.03.31	5	2, 3	2	5
94.04.05	4	1, 1	2	2
94.04.07	4	7	1	7
94.04.14	3	3, 5	2	8
94.04.21	4	2	1	2
94.05.05	7	1, 3	2	4
94.05.12	4	3	1	3

n: number of participants; w: number of winners; T: number of lots.

observable explanations exist for this bid heterogeneity—differences in the number of lots for sale and differences in the number of participants. In addition, at least two unobserved explanations could also exist—differences in the number of lots demanded by auction participants as well as differences in their relative values.

In Figure 2, we present scatterplots of the average winning bids versus the lot number for two specific auctions, one held on September 16, 1993 and the other held on September 23, 1993. At the earlier auction, the profile appears to fall, on average, while at the other one, merely a week later, it appears to rise, on average.

During the past four decades, economists have systematically investigated simple models of behaviour at auctions where only one object is sold to buyers demanding at most one object each.[2]

[2] For an up-to-date summary of this theoretical research, see Krishna (2002).

DOI: 10.1002/jae

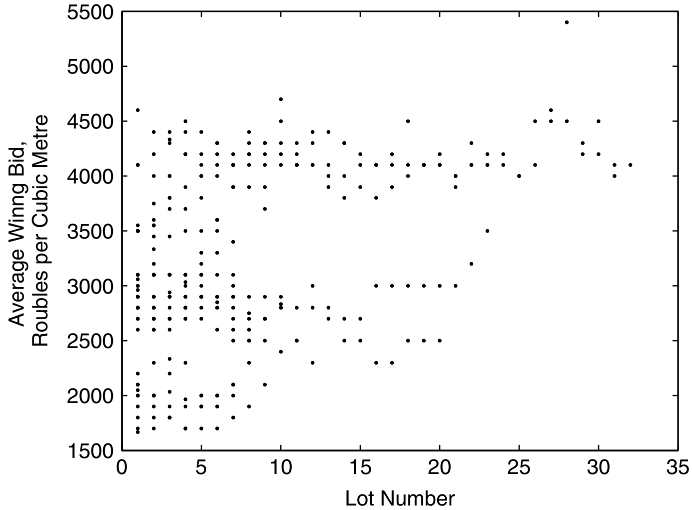

Figure 1. Average winning bid versus lot number

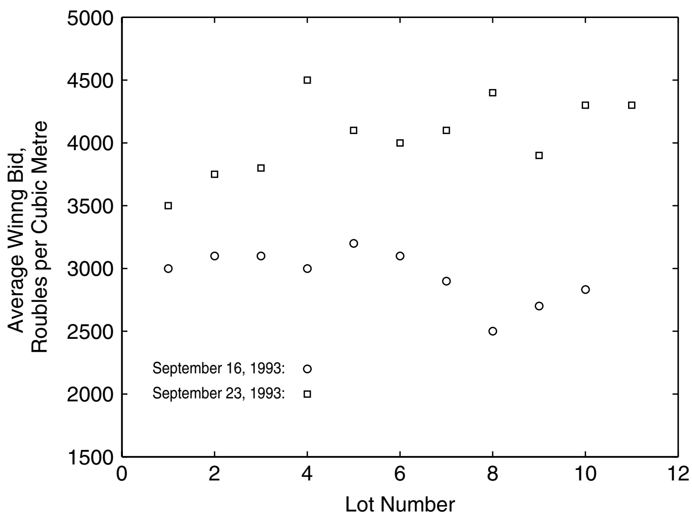

Figure 2. Average winning-bid profile: two auctions, one week apart

We argue here that the auctions of export permits for timber in the Krasnoyarsk Kray involved the sale of several units of the same good to buyers who often demanded several units each. As such, we make a distinction here between multi-object auctions and multi-unit auctions. An example of a multi-object auction would involve the sale of an apple, an orange and a pear; an example of a multi-unit auction would involve the sale of three identical apples. The key difference is that at a multi-unit auction it matters not which unit a bidder wins but rather the aggregate number of units he wins, while at a multi-object auction it matters which specific object(s) the bidder wins. From a modelling perspective, multi-unit auctions are much easier to analyse than multi-object auctions.

DOI: 10.1002/jae

In most recent research concerning multi-unit auction models, economists have assumed that each potential bidder demands at most one unit. For example, in the early 1980s, Weber (1983) as well as Milgrom and Weber (2000) initiated the investigation of multi-unit auctions when bidders have single-unit demand, referred to by Milgrom (2004) as *singleton-demand*. Within the independent private-values paradigm, where each potential bidder draws an independent valuation which is specific to him, Weber (1983) has shown that the winning bids at a sequence of English auctions should follow a martingale. To wit, the expected price of unit $(t+1)$, given that unit t sold for p follows:

$$\mathcal{E}(P_{t+1}|P_t = p) = p$$

Within the common-value paradigm, where the object's value is unknown before the auction but the same to all after it, McAfee and Vincent (1993) have provided an explanation of the declining-price anomaly. Assuming risk-averse bidders within the affiliated-values paradigm, where a bidder's valuation depends on the private information of others, Laffont and Robert (2000) have derived inverted U-shaped expected price paths. Because several bidders in our sample have won several lots, singleton-demand models do not appear applicable to us. Thus, later in Section 3, we construct a theoretical model that admits differential participation across auctions as well as some bidders winning several lots. One prediction of this model is that the expected price path should follow a submartingale: on average, bid prices should rise across successive lots sold at the auction.

To understand the basic intuition of why bid prices may increase, on average, at least within the independent private-values paradigm, consider first the model of Weber (1983) in which each buyer demands at most one unit and there are fewer units T for sale than there are participants n at the auction. Under these assumptions, the winning bid will be constant: each participant will bid up to his first valuation until the $(n-T)$th participant drops out. Then the T remaining participants will withdraw simultaneously. The winning bid will correspond to the $(T+1)$th-highest valuation, which we denote V_{T+1}. In the next sale, the winner, having satisfied his unit demand, withdraws from the remainder of the auction, so the bidding will stop as the price reaches the Tth next-highest valuation, which is again V_{T+1}. Here, there is one fewer participant and one fewer unit for sale, so these two factors exactly offset one another.

When participants potentially demand more than one unit, there is not necessarily one fewer participant each time a unit is sold because the most recent winner may still demand another unit. Thus, in contrast to the case where participants have singleton-demand, one less unit for sale is not offset by one less buyer. The reader can object and claim that bidders may be more aggressive in the earlier sales in order to keep the option of buying all subsequent units on sale. This does not turn out to be an issue. If valuations are such that one buyer ought to win all units, then he will not need to bid aggressively at the first sale in order to win.

Before proceeding to the theoretical and econometric analyses of the paper, we present here some reduced-form evidence concerning the martingale hypothesis of Weber (1983) as well as our submartingale hypothesis. Using the price data from $j = 1, \ldots, 34$ auctions of 305 lots, we estimated the average percentage change in price from lot t to $(t+1)$.[3] The average of $\log(P^j_{t+1}/P^j_t)$ over 271 observations was 0.0055, an increase of 0.55% per lot, but the standard error was 0.0046, yielding a p-value of 0.23.[4]

[3] Three of the 37 auctions had but one lot for sale, so data from these auctions could not be used.
[4] To complete the differencing, another 34 observations were lost.

DOI: 10.1002/jae

Table IV. Lot-specific estimates of percentage changes

	Lot					
	1	2	3	4	5	6
Estimate	0.0317*	0.0066	−0.0363*	0.0124	−0.0180	0.0115
Std error	0.0127	0.0137	0.0148	0.0158	0.0166	0.0170
	7	8	9	10	11	12+
Estimate	0.0128	0.0206	0.0064	−0.0206	−0.0288	0.0122
Std error	0.0198	0.0214	0.0235	0.0263	0.0281	0.0088

Of course, the price changes could vary with lot order. Estimates from a regression of $\log(P^j_{t+1}/P^j_t)$ onto lot-specific dummy variables for the first 11 lots, with lots of 12 and greater being a separate category, are presented in Table IV.[5] For only two of these estimates, denoted by an asterisk, does the p-value of the test fall below 0.05; one is positive, 3.2%, while the other is negative, −3.6%. The cumulative sum of these 12 estimates is 0.0105, 1.05% over 12 lots, but its standard error is also large. These results provide only weak evidence against the martingale hypothesis and in favour of the submartingale hypothesis.

Another way to investigate the martingale hypothesis is to consider the following parametric implementation:

$$\mathcal{E}(P^j_{t-1}|P^j_t) = (1+\phi)P^j_t$$

where under the martingale hypothesis ϕ is zero, while under the submartingale hypothesis ϕ should be positive. Thus, an empirical specification is

$$(P^j_{t+1} - P^j_t) = \phi P^j_t + \varepsilon^j_t \tag{1}$$

where ε^j_t is a mean zero, but potentially heteroskedastic error term. The reader may be tempted to think of t as a time subscript and equation (1) as a non-stationary time-series process, but remember that t here denotes the number of lots, including the current one, already sold. The time interval between sales can and did vary widely in our sample. Also, the number of lots T_j never exceeded 32. Our estimate of ϕ in (1) was 0.00212 with a corresponding robust standard error of 0.00536. This suggests that ϕ is not significantly different from zero, a fact consistent with Weber's result. However, we know from Table III that many participants won more than one lot at a given auction. Thus, multi-unit demand appears relevant.

Note that, because the standard error of the estimate of ϕ is relatively large, the power of the test against interesting economic alternatives is relatively small. For example, if ϕ actually equalled 0.010, which implies a 10.5% increase in the average winning price over 10 lots sold at auction, the power of this test would be only about 50%. In fact, the region of low power, as defined by Andrews (1989), is quite wide—between 0.00212 and 0.0195. This latter value for ϕ would imply a price increase over 10 lots sold of about 21.3%, an economically relevant amount.

3. THEORETICAL MODEL

As in most economic-theoretic modelling exercises, where the main goal is to provide a structural-econometric framework within which to interpret data, modelling assumptions are often dictated

[5] We chose 12 because only six of the auctions had more than 12 lots for sale.

DOI: 10.1002/jae

by data availability. Ours is no different. For a sample of J equal 37 independent auctions indexed by $j = 1, \ldots, J$, where at the jth auction T_j units of a homogeneous good were sold sequentially to n_j participants, we observed the reserve price r_j and the winning bids $\{p_t^j\}_{t=1}^{T_j}$.

We assume that the reserve price is known to everyone prior to the auction. Although the participants at the auction observed the prices at which their opponents dropped out, as researchers, we did not. Thus, an asymmetry of information between the decision-makers, the participants at the auction, and we, the researchers, exists. We incorporate this reality into our model by ruling out the econometrician's ability to observe drop-out prices. We have an estimate of the number of potential bidders $\mathcal{N}$, 30 in our case.

3.1. Overview of the Model

Consider a seller who seeks to dispose of T units of an homogeneous good through a sequence of T oral, ascending-price sales. We assume that, for each unit, the selling procedure corresponds to the Milgrom–Weber (1982) clock model of an English auction. Specifically, the price of the unit for sale is raised continuously and bidders continuously affirm their participation in the auction by holding down a button or holding up a card. The price rises progressively and, as it does, bidders signal that they are dropping out of the auction by releasing the button or dropping the card until only one bidder remains; the last active bidder wins the unit for sale and pays the price at which his last opponent dropped out. While Milgrom and Weber described their clock model in terms of a mechanical device, at the Birch in Krasnoyarsk the prices were simply announced loudly. Nevertheless, each participant knew which of his opponents remained in the current sale because they were the ones holding up white cards. This feature of the Birch auction format is special; typically, it does not exist at English auctions. Because it existed at the Birch, we feel comfortable in assuming that bidders could observe the prices at which others dropped out and, thus, could infer information from the drop-out prices.

We assume that the above process is repeated until all units T are sold and that the seller is committed to selling all T units. In order to avoid carrying around an unobserved state variable—unfulfilled demand—we also assume that the unfulfilled demands of participants at any auction disappear (evaporate) completely after the T units have been sold.

Milgrom and Weber (1982) analysed the case where only one good is for sale. In this case, at every price, a bidder must ask himself whether he should *drop out of* or *remain in* the auction. Because a bidder can always drop out at the next (infinitesimal) higher price, his calculation is myopic. Nevertheless, he must answer the following simple question:

> If all other bidders were to drop out simultaneously at the current price, would I prefer to win the auction or would I prefer to drop out now?

Within the independent, private-values paradigm, a bidder's best strategy is to drop out at his valuation. On the other hand, within the affiliated-values paradigm, a bidder must calculate, at any given price, the expected value of the good on sale—given his own private information, the drop-out prices of those who have already withdrawn, and assuming that all remaining bidders were to drop out simultaneously at the current price. In this case, a bidder should drop out when the current price exceeds this expected value.

When two or more identical units of a good are sold sequentially using the above auction format, where each participant may demand more than one unit, the strategic analysis becomes

DOI: 10.1002/jae

complex. In this paper, we compute a symmetric, perfect-Bayesian equilibrium of such a game. At any auction, we assume a known number of risk-neutral potential bidders $\mathcal{N}$ who may bid for the T units for sale. We denote by $\boldsymbol{V}^i$ the vector of ordered valuations $(V_1^i, V_2^i, \ldots, V_T^i)$ for potential bidder i where V_t^i represents bidder i's valuation for the tth unit of the good. Hence, for all i, decreasing marginal utility obtains

$$V_1^i \geq V_2^i \geq \ldots \geq V_T^i$$

Two-Bidder, Two-Unit Example

To begin, we illustrate the intuition behind our analysis by considering the case of just two bidders, i and j, who are competing for two identical units that are sold sequentially using a Milgrom–Weber clock auction. Each bidder has private information. For example, for bidder i, it takes the form of the valuation pair (V_1^i, V_2^i). We use backward induction to help us construct our argument. Consider first behaviour involving the last unit sold: each bidder will drop out only when the price reaches his valuation for the unit on sale. If i has won the first unit, then he will bid up to V_2^i, his value of having a second unit, and pay V_1^j if he wins. Otherwise, when j wins the first sale, i will bid up to V_1^i and pay V_2^j in order to win. Bidder j will win both units when V_2^j is greater than V_1^i, while bidder i will win both units when V_1^j is less than V_2^i.

Now, at a two-unit auction, when the first unit is for sale, a bidder can wait until the second sale to purchase a unit, so as the price rises that bidder must ask himself the following question:

> If all other active bidders were to drop out simultaneously at the current price, would I prefer to win the current sale at the current price and then try to win a second unit in the next sale, or would I prefer to forgo the opportunity to buy two units and wait for the next sale in order to try to win my first unit?

A bidder must *compare* the value of winning the first unit at price p plus the expected gain from participating in the second sale having won the first *with* the expected gain from participating in the second sale having lost the first. Let $b_j(V_1^j, V_2^j)$ denote j's bidding function in the first sale. When the current price is p, bidder i should stay in the auction if

$$(V_1^i - p) + \mathcal{E}[\max(0, V_2^i - V_1^j)|p = b_j(V_1^j, V_2^j)] \geq$$
$$\mathcal{E}[\max(0, V_1^i - V_2^j)|p = b_j(V_1^j, V_2^j)]$$

Calculating the expected values in the expression above is the difficult part of solving this decision problem because the information contained in the drop-out prices depends on the bidding strategies of others and a bidder's strategy depends on the information contained in the drop-out prices. Not surprisingly, many equilibria can exist in this two-stage game.[6] However, under the assumption of symmetry, the calculation is simplified. Bidder i should calculate his drop-out price p^i assuming active bidder j has the same one; i.e., assuming V_1^j equals V_1^i. Under this assumption, because V_1^j equals V_1^i, which is greater than or equal to V_2^i, bidder i cannot expect to win profitably a second

[6] In fact, in his paper on two-stage sequential auctions with multi-unit demand, which builds on the work of Black and de Meza (1992), Katzman (1999) has presented a list of these possible equilibria.

DOI: 10.1002/jae

unit if he wins the first sale, while, because V_1^i equals V_1^j which is weakly more than V_2^j, he could expect to win the next sale if some other active bidder j wins the first sale. Hence, we have

$$V_1^i - p^i = V_1^i - \mathcal{E}(V_2^j|V_1^j = V_1^i)$$
$$p^i = \mathcal{E}(V_2^j|V_1^j = V_1^i) \leq V_1^i$$

Bidder i's drop-out price is the expected price he will need to pay in order to win the next sale, were he to lose the current sale, assuming V_1^j equals V_1^i. Note that the actual bidding function will depend on the distribution of V_2^j, winner j's second highest valuation, given V_1^j equals V_1^i.[7] But as long as the assumed distributions are symmetric across bidders, the bidding function will be symmetric (as assumed) and the above will constitute a symmetric, perfect-Bayesian equilibrium.

Two-Bidder, Three-Unit Example

The above logic can be extended to a sequence of T sales *only* when the distributions of valuations in every sale remain symmetric. In order to highlight this symmetry condition, consider a sequential, three-unit clock auction with two bidders.

Suppose that j has won the first sale, so he enters the game involving the last two sales having valuations (V_2^j, V_3^j), while i enters having valuations (V_1^i, V_2^i). Following the reasoning presented above, the bidding strategies in the second sale would be as follows. Bidder i will bid up to $\mathcal{E}(V_3^j|V_2^j = V_1^i)$, while j will drop out at price $\mathcal{E}(V_2^i|V_1^i = V_2^j)$. This would constitute a symmetric equilibrium only when the distribution of V_3^j, given V_2^j, is the same as the distribution of V_2^i, given V_1^i. Hence, a necessary condition is that the distribution of valuations must remain identical across players, regardless of the number of units they have purchased in previous sales; i.e., the distribution of V_{k+1}^i, given V_k^i, must be independent of the indices i and k.

When this symmetry condition holds, the equilibrium of the three-unit clock auction has a nice recursive structure. In this paper, we seek to exploit this recursive structure. Recall that j will drop out of the second sale, when he has won the first one, at price $\mathcal{E}(V_2^i|V_1^i = V_2^j)$. By the symmetry condition assumed above, this equals $\mathcal{E}(V_3^j|V_2^j)$. Hence, conditional on V_1^j equalling V_1^i, the price i will need to pay in order to win the second sale when j has won the first one, is $\mathcal{E}[\mathcal{E}(V_3^j|V_2^j)|V_1^j = V_1^i]$, which equals $\mathcal{E}(V_3^j|V_1^j = V_1^i)$. So in the first sale, when the price reaches p or $\mathcal{E}(V_3^j|V_1^j = V_1^i)$, conditional on j's having dropped out at price p, i is indifferent between winning at price p and losing to j, but winning the next sale. For lower prices, i will strictly prefer the former and, for higher prices, he will strictly prefer the latter. Dropping out at price p equal to $\mathcal{E}(V_3^j|V_1^j = V_1^i)$ forms i's best response when j follows the symmetric strategy. Hence, the equilibrium bidding strategy can be written as an expected value which turns out be the expected price he would pay were he to wait until the very last round in order to purchase one extra unit.

This result generalizes to the case of $\mathcal{N}$ potential bidders and T units for sale. In the remainder of this section, we first introduce a process to generate valuations that guarantees the critical symmetry

[7] Note that, as in the affiliated private-values paradigm, bids depend on information revealed by the bids of others within the current round of bidding. However, the reason for this dependence is different from that typically encountered with affiliated values. With affiliated values, the gains from winning depend on the private information of others. Here, the gains from winning depend on the willingness of others to buy. In all but the last sale, losing bidders can always win a later sale; the expected price they will need to pay in order to win this later sale will depend on the willingness of others to pay.

condition described above and then state a theorem for the case with $\mathcal{N}$ potential bidders and T units at auction. Finally, we describe some properties of the equilibrium.

3.2. Demand-Generation Scheme

In order to solve the auction game, we need to specify the process generating the random vector of decreasing valuations for the ith potential bidder; i.e., the demand curve $\boldsymbol{V}^i$ which for potential bidder i equals $(V_1^i, V_2^i, \ldots, V_T^i)$.

We assume that the number of valuations for participant i, denoted M_i, is random having support in $\{0, 1, 2, \ldots\}$; M_i may be higher or lower than T. In the former case, only the highest T values will matter; in the latter, some values of $\boldsymbol{V}^i$ will be zero. At any auction, we assume that the number of valuations drawn by each potential bidder is a Poisson random variable having mean λ and probability mass function

$$\Pr(M_i = m) = \frac{\lambda^m \exp(-\lambda)}{m!} \quad m = 0, 1, 2, \ldots$$

When M_i is positive, each of the positive valuations is drawn independently and identically from some twice-differentiable cumulative distribution $F(\cdot)$ having support on the interval $[0, \infty)$; these valuations are then ranked in descending order. We also assume these draws are independent across auctions. Thus, potential bidders are *ex ante* symmetric at a given auction and across auctions through time.

The above stochastic process provides us with a useful device to generate aggregate demand, for it guarantees certain properties sufficient to solve the auction game. Note that, by ranking the valuations in descending order, we bring attention to the fact that participants will exploit their most lucrative opportunities first. Under these assumptions, when M_i is less than T, the valuation vector of the ith participant takes the form of a T-dimensional vector with $(T - M_i)$ zeros of padding, so

$$\boldsymbol{V}^i = (V_1^i, V_2^i, \ldots, V_{M_i}^i, 0, \ldots, 0)$$

where $V_1^i \geq V_2^i \geq \ldots \geq V_{M_i}^i \geq V_{M_i+1}^i = 0$. Note too that the aggregate and individual demands are independent of T, the quantity supplied.

For all j less than k and $i \in \mathbf{N}$, where $\mathbf{N}$ denotes the set of participants, the function $G(y|x)$, which denotes $\Pr(V_k^i \leq y | V_j^i = x)$, is strictly decreasing in x for all y less than x when $F(y)$ is positive. This reflects the fact that order statistics are correlated: if the jth highest valuation of a bidder is high, then his kth highest valuation, where k exceeds j, is more likely to be high than low.

One of the realities when bringing theoretical models of bidding to data is that some of the $\mathcal{N}$ potential bidders do not participate at any given auction. At first-price, sealed-bid auctions non-participation does not typically introduce a problem in deriving the data-generating process of either the observed bids or the winning bid because the decision rule depends on the number of potential bidders $\mathcal{N}$ not the number of participants n. But at English auctions the number of participants n is critical to the price determination process. Having a random number of participants is one way to admit differential participation across auctions.

In our model, the number of participants at each auction is random. Specifically, when a potential bidder demands nothing, so M_i is zero, we assume that he does not attend the auction, hence signalling he will not bid. Conversely, we assume that all participants present at the auction have

DOI: 10.1002/jae

at least one positive valuation. Because, at any auction, the number of positive valuations drawn by each potential bidder M_i follows a Poisson process, the number of participants N (those potential bidders having M_i greater than zero) will be random and is distributed binomially with parameters $\mathcal{N}$ and $[1 - \Pr(M = 0)]$ or $[1 - \exp(-\lambda)]$, so its probability mass function is

$$\Pr(N = n) = \binom{\mathcal{N}}{n} [1 - \exp(-\lambda)]^n \exp(-\lambda)^{\mathcal{N}-n}$$

Another way to admit differential participation across auctions would be to introduce a reserve price. However, in the auction game we study, the presence of a reserve price, while sufficient to guarantee differential participation, does not provide enough probabilistic structure to solve the game. Thus, we initially assume no reserve price, so the reserve price is not the only reason for differential participation. We relax this assumption later when we develop the econometric estimation strategy.

Finally, note that the distribution of types has a structure that guarantees symmetry across the game. This condition follows from the Poisson assumption concerning M_i. Indeed, we have the following properties:

(A) For all $j < k$,

$$\Pr(V_k^i \le y | V_j^i, V_{j-1}^i, \ldots, V_1^i) = \Pr(V_k^i \le y | V_j^i)$$

(B) For all $j < k$ and $i, \ell \in \mathbf{N} = \{1, 2, \ldots, \mathcal{N}\}$,

$$\Pr(V_k^i \le y | V_j^i = x) = \Pr(V_k^\ell \le y | V_j^\ell = x)$$

(C) For all $i \in \mathbf{N}$ and for all $j < k$,

$$\Pr(V_k^i \le y | V_{j+1}^i = x) = \Pr(V_{k-j}^i \le y | V_1^i = x)$$

Property A follows directly from the assumption of independent draws and the properties of order statistics. Property B follows directly from the assumption of symmetry. Property C is more involved: it states that the conditional distribution is invariant to a re-indexing of the order statistics. This is not a general property of order statistics, but it follows here from the assumption that the number of valuation draws follows a Poisson process. A formal proof of this result, like the proofs of all our results, is contained in the Appendix.

As stated before, property C proves to be essential. Suppose that after a number of sales, bidder i has already won ℓ_i units, his valuation for the next unit on sale becomes $V_{\ell_i+1}^i$ rather than V_1^i as it was initially. Hence, his bidding strategy should depend on $V_{\ell_i+1}^i$. A similar argument applies for all bidders. Symmetry of the bidding function, regardless of who has won what in the previous sales, requires the following property:

(D) For all j and $i \in \mathbf{N}$ and $k > 1$, ℓ_i and ℓ_j,

$$\Pr(V_{k+\ell_i}^i \le y | V_{1+\ell_i}^i = x) = \Pr(V_{k+\ell_j}^j \le y | V_{1+\ell_j}^j = x)$$

Property D follows from properties B and C.

DOI: 10.1002/jae

3.3. Solving the Auction Game

We now construct an equilibrium of the auction game induced by the sequence of T sales. We assume that the vector $\boldsymbol{V}^i$ and the number of positive valuations M_i are the private information of participant i. The cumulative distribution function $F(\cdot)$, the intensity parameter of the Poisson process λ, and the number of bidders n present are assumed common knowledge. As mentioned above, we assume that valuation draws are independent across auctions and that all unfulfilled demands at a given auction evaporate at the end of the game, so are not carried away to the next game at a later date.

A bidding strategy at these sales specifies a stopping rule that indicates at what price a participant should withdraw from the current sale. Such a stopping rule can be contingent on the entire history of previous sales as well as the current one; i.e., contingent on the prices at which other participants have withdrawn, previous winning bids, private information concerning the valuation attached to each unit purchased, the information shared by all participants, and so on. Hence, the strategies may be quite complex not just because they will be history-dependent but because each participant may want to manipulate his actions to influence the future bidding behaviour of others. In this paper, we focus on an equilibrium with a relatively simple structure. This equilibrium, which is characterized in the following theorem, is a direct generalization of the two-bidder, three-unit example presented above.

Before stating our main theorem, we introduce some notation. Let ℓ_i denote the number of units already purchased by bidder i and collect these for all n bidders in $\mathbf{L}$, $\{\ell_1, \ell_2, \ldots, \ell_n\}$. Thus, $\mathbf{L}$ summarizes the past history of winners at the auction. Whenever i has purchased ℓ_i units, his valuation of the next unit is $V^i_{1+\ell_i}$, which we shall refer to as i's next-highest valuation. The next-highest valuation overall is given by $\max_j\{V^j_{1+\ell_j}\}$. A sale will be deemed efficient if its winner is the bidder with the highest next valuation overall. When all previous rounds were efficient and some bidder h has previously won a unit, then $V^h_{\ell_h} \geq \max_j\{V^j_{1+\ell_j}\}$. We denote by V^{-i}_k the kth highest valuation among all valuations of i's opponents. In particular, when q_i denotes $\sum_{j\neq i}\ell_j$, the number of units already purchased by i's opponents, we have $V^{-i}_{q_i} \geq V^{-i}_{1+q_i}$ which equals $\max_{j\neq i}\{V^j_{1+\ell_j}\}$.

We consider a bidding strategy for each bidder i that depends on $\mathbf{L}$ and $V^i_{1+\ell_i}$ as well as the observed bidding behaviour of others in the current sale.

Theorem 1 For every sale of a T-unit auction, bidder i remains active until all other bidders stop, or until the price equals or exceeds

$$\begin{aligned}
&\mathcal{E}(V^{-i}_{T-\ell_i} | V^{-i}_{1+q_i} = V^j_{1+\ell_j} = V^i_{1+\ell_i}, \forall j \in \mathbf{R}; V^h_{1+\ell_k} = v^h, \forall h \in \mathbf{S}) = \\
&\quad \mathcal{E}(V^{-i}_{T-q_i-\ell_i} | V^{-i}_1 = V^j_1 = V^i_{1+\ell_i}, \forall j \in \mathbf{R}; V^h_1 = v^h, \forall h \in \mathbf{S}) \qquad (2)
\end{aligned}$$

where $\mathbf{R}$ denotes the subset of bidders still active, $\mathbf{S}$ denotes the subset of bidders that have withdrawn for the current sale, and v^h denotes the valuations for all bidders in $h \in \mathbf{S}$ that is consistent with the price at which h has withdrawn in the current sale and the equilibrium strategy.

Recall that a bidder's decision to drop out is always based on the conjecture that all currently-active bidders will simultaneously drop out at the current price. As the price converges to i's

DOI: 10.1002/jae

equilibrium drop-out price, i's conjecture will converge, under the assumption of symmetry, to the conjecture that all remaining bidders have the same next-highest valuation as his; i.e.,

$$V^{j}_{1+\ell_j} = V^{i}_{1+\ell_i}, \forall j \in \mathbf{R}$$

Hence, i's conjecture will be that

$$V^{-i}_{1+q_i} = \max_{j \neq i}\{V^{j}_{1+\ell_j}\} = V^{i}_{1+\ell_i}$$

In the last round, when only one unit remains to be sold, so $(T - \ell_i)$ equals $\left(1 + \sum_{j \neq i} \ell_j\right)$ or $(1+q)$, Theorem 1 prescribes that bidder i drops out at $\mathcal{E}(V^{-i}_{T-\ell_i} | V^{-i}_{1+q} = V^{i}_{1+\ell_i})$ which equals $V^{i}_{1+\ell_i}$, his next-highest valuation. Thus, the prescribed strategy indeed forms an equilibrium in the last round.[8] Note that no information from past auctions, other than the number of units won by others and left to be sold, enters the current bidding function. The basic idea is that all information obtained from previous sales is superfluous. Along the equilibrium path, the price at which inactive bidders have dropped out must be consistent with the previous play and the equilibrium. Hence, information from previous rounds is redundant. For active players, regardless of what occurred in the past, a bidder should always base his drop-out decision on the conjecture that all of his opponents will drop out simultaneously at the current price. From a strategic point of view, the fact that no information from past sales enters the current bidding function is important. No one can influence future bidding by deviating—dropping out earlier or staying longer than the equilibrium. Implicit in this solution is the property that bid manipulation designed only to signal is ignored. Along the equilibrium path, it turns out to be just the same as if he were to use all the previously available information.

Note that the value $V^{-i}_{T-\ell_i}$ corresponds to the price i would need to pay in order to get an extra unit were he to wait until the very last sale to purchase it. Theorem 1 states that bidder i should drop out at a price which equals the expected value of $V^{-i}_{T-\ell_i}$, conditional on all active bidders dropping out at the same price and on the drop-out prices of inactive bidders in the current sale.

3.4. Properties of the Equilibrium

The symmetric, perfect-Bayesian equilibrium has several important properties listed in the following theorem.

Theorem 2

(i) Bids are strictly monotonic and symmetric functions of next-highest valuations, $V^{i}_{1+\ell_i}$.

[8] When the number of bidders still active in the current sale exceeds the number of units left to be sold t, i conjectures that

$$V^{-i}_{T-\ell_i} = V^{-i}_{t+q} = V^{-i}_{r+q} = \max_{j \neq i}\{V^{j}_{1+\ell_j}\} = V^{i}_{1+\ell_i}$$

and

$$\mathcal{E}(V^{-i}_{T-\ell_i} | V^{-i}_{1+q} = V^{i}_{1+\ell_i}) = V^{i}_{1+\ell_i}$$

Bidder i will bid up to his next-highest valuation.

DOI: 10.1002/jae

(ii) The allocation induced by the equilibrium is efficient; i.e., the T units are allocated to the buyers with the highest valuations.

(iii) If all other participants follow their equilibrium strategies, then in order to win a sale, participant i must pay the price

$$\mathcal{E}(V_{T-\ell_i}^{-i}|\Omega_t)$$

where Ω_t denotes the information available to participants at the end of sale t.

After some history summarized by $\mathbf{L}$, each participant i bids a function of his next-highest valuation $V_{1+\ell_i}^{i}$. As shown in the formal proof in the Appendix, the bidding function in (2) is monotonic and symmetric, so the winners in each round will be those with the next-highest valuations overall. It follows immediately that the allocation induced by the equilibrium behaviour is efficient. Note that efficiency is not an intrinsic property of the sequential auction game. The efficiency result depends on crucial assumptions we have made. First, we have assumed that all bidders are *ex ante* symmetric; i.e., their valuations are independent draws from the same law. Second, we have made assumptions leading to property D—robust symmetry. These insure that those who have won previous sales behave in the remaining sales like the others. Without the above assumptions, computing the equilibrium would have been extremely difficult, perhaps impossible. This is why, in most game-theoretic papers concerning sequential auctions, researchers have limited their attention to simple cases—such as two-unit, two-player examples—or to cases where bidders have singleton demand.

Finally, the expected price paid by buyer i for his $(\ell_i + 1)$th unit equals the expected value of $V_{T-\ell_i}^{-i}$, the $(T - \ell_i)$th highest valuations among all valuations of the other participants. The result in Theorem 2(iii) is reminiscent of the dominant-strategy implementation of the efficient allocation, the generalized Vickrey auction for multi-unit demand. Consider the following mechanism: first, each participant i is asked to reveal his willingness to pay $\boldsymbol{V}^i$ equal to $\{V_1^i, V_2^i, \ldots, V_{M_i}^i, 0, \ldots, 0\}$; second, the T units are allocated to the participants with the T highest (revealed) values; and third, each winner i pays V_T^{-i} for his first unit, V_{T-1}^{-i} for his second unit, $V_{T-\ell}^{-i}$ for his $(\ell + 1)$th unit, and so forth. One can verify that it is a dominant strategy for each participant to reveal his actual valuations and that it implements the efficient allocation.[9] Theorem 2(iii) implies that the expected price paid by each winner in the sequential, clock auction equals the expected price he would pay in the above mechanism. Hence, we can use the dominant-strategy implementation of the efficient allocation in conjunction with simulation methods to calculate the expected winning bid for the sale of the tth unit. Note that the vector of winning bids at the sequential, clock auction need not correspond to the vector of generalized-Vickrey prices; however, on average, they are equal.

[9] Let V_ℓ denote the ℓth highest value announced by some participant i. If $V_{\ell-1}$ is less than $V_{T-\ell+2}^{-i}$, then for all announced V_ℓ less than $V_{\ell-1}$, i will not receive the ℓth unit, so i cannot gain by misreporting V_ℓ^i. Now, suppose $V_{\ell-1}$ weakly exceeds $V_{T-\ell+2}^{-i}$, so that i receives at least $(\ell - 1)$ units. Bidder i cannot gain by overstating his ℓth value because this will not affect the prices he will pay for any of his units, and it could only make a difference if V_ℓ weakly exceeds $V_{T-\ell+1}^{-i}$, which is greater than V_ℓ^i since he will be awarded the ℓth unit and he will pay a price above his valuation for this unit. Similarly, i cannot gain by understating his ℓth value since i may only lose the chance of purchasing the ℓth unit at a price below his valuation. This argument applies for all ℓ, and all vectors $\boldsymbol{V}^{-i}$, so i can never gain by misreporting his private information.

3.5. Pattern of Winning Bids

Using the equilibrium constructed in the previous section, we can make a prediction concerning the pattern of winning bids. In expected terms, the winning bids increase. The likelihood and the magnitude of these expected increases depend on the identity of the winners, specifically on the presence of repeated purchases by individual participants.

The winning bid at the first sale P_1, conditional on the information set Ω_1, is given by $\mathcal{E}(V_T^{-i}|\Omega_1)$ whenever participant i is the winner of the first sale. The winning bid is participant i's expected value of the Tth highest value of the valuations for the other participants, conditional on his current information. The winning bid at the last sale is given by V_{T+1}, the highest extra-marginal valuation. If participant i has ℓ_i values above V_{T+1}, then V_T^{-i} is weakly less than $V_{T+\ell_i}$. Conditional on i winning the first sale, ℓ_i weakly exceeds one. It follows that

$$P_1 = \mathcal{E}(V_T^{-i}|\Omega_1) \leq \mathcal{E}(V_{T+\ell_i}|\Omega_1) \leq \mathcal{E}(V_{T+1}|\Omega_1) = \mathcal{E}(P_T|\Omega_1)$$

where the last winning bid P_T is greater than the price of the first sale P_1 in conditional expectation. The upward drift in bids approximately equals the expected difference between V_{T+1} and $V_{T+\ell_i}$, where ℓ_i is the expected number of units won by i, the winner of the first sale. Using the notation established above, where P_t and P_{t+1} denote the winning bids for sales t and $(t+1)$, we state the following theorem:

Theorem 3 Under the conditions assumed above, the winning bids form a submartingale, so

$$\mathcal{E}[(P_{t+1} - P_t)|P_t] > 0$$

3.6. Binding Reserve Price

The theoretical results presented above can easily be extended to admit a binding reserve price r which is the same for each lot on sale at a given auction. A potential bidder will participate in a sale if and only if his highest next valuation is higher than the reserve price; i.e., when $V_{1+\ell_i}^i$ weakly exceeds r. If $V_{1+\ell_i}^i$ does exceed r, then a participant should withdraw from the auction when the current price reaches

$$\mathcal{E}[\max(r, V_{T-\ell_i}^{-i})|V_{1+q_i}^{-i} = V_{1+\ell_j}^j = V_{1+\ell_i}^i, \forall j \in \mathbf{R}; V_{1+\ell_k}^h = v^h, \forall h \in \mathbf{S}]$$

Note that $\max(r, V_{T-\ell_i}^{-i})$ corresponds to the price bidder i would need to pay were he to wait until the very last period to win an extra unit. As before, the optimal strategy is for a participant to bid up to the expected value of that price given the dropping prices of inactive bidders and under the conjecture that active bidders simultaneously drop out.

Note too that a potential bidder will not participate at the auction when none of his valuations exceeds the reserve price r; the probability of this event is then $(1 - \exp\{-\lambda[1 - F(r)]\})$. The number of valuations above r now follows a Poisson process with mean $\lambda[1 - F(r)]$. The probability mass function of N is then given by

$$\Pr(N = n) = \binom{\mathcal{N}}{n} (1 - \exp\{-\lambda[1 - F(r)]\})^n \exp\{-\lambda[1 - F(r)]\})^{\mathcal{N}-n}$$

4. ECONOMETRIC MODEL AND ESTIMATION STRATEGY

Because the equilibrium of our model is difficult, some might say impossible, to calculate, we propose to estimate its average behaviour by simulating another auction format, the generalized Vickrey auction in our case, which is also efficient, so the expected price will be the same, but whose equilibrium we can easily calculate. In single-object auctions, such a strategy was first proposed by Laffont *et al*. (1995), who used the revenue equivalence of Dutch and Vickrey auctions within the IPVP when potential bidders are risk-neutral. In order to simulate on a computer, pseudo-random numbers from the distribution $F(\cdot)$ are required. Without an explicit assumption concerning $F(\cdot)$, generating pseudo-random numbers is impossible. Thus, in addition to assuming that M_i follows the Poisson law with mean λ, we also assume that $F(v)$ follows the Weibull law having parameters α_1 and α_2, so

$$F(v; \alpha_1, \alpha_2) = [1 - \exp(-\alpha_1 v^{\alpha_2})] \quad \alpha_1 > 0, \alpha_2 > 0$$

The theory presented above delivers the first moment of both the participation equation for the auction (the mean of a binomial random variable) and the winning price for the sale of unit t at the auction (the average winning price for unit t at a generalized Vickrey auction). Thus, a natural strategy for estimating the unknown parameters λ as well as α_1 and α_2, which we collect in $(\lambda, \alpha_1, \alpha_2)$ and denote by the unknown vector $\boldsymbol{\theta}$, would be to choose some estimate $\tilde{\boldsymbol{\theta}}$ which minimizes the distance between the observed data and the mean of the processes evaluated at $\tilde{\boldsymbol{\theta}}$. But how should the distance be chosen? Laffont *et al*. chose the sum of squared residuals, adjusted appropriately for pre-estimation error when the mean function of the winning bid is unknown and must be estimated using simulation methods. We propose a strategy based on the generalized method-of-moments (GMM).

We have a participation equation, the mean of which, in closed-form, is

$$\mathcal{E}(N; r, \mathcal{N}, \lambda) = \mu_0(r, \mathcal{N}, \lambda) = \mathcal{N}\{1 - \exp[-\lambda(1 - F(r)]\}$$

for some reserve price r. Under the Weibull assumption,

$$\mu_0(r, \mathcal{N}, \boldsymbol{\theta}) = \mathcal{N}\{1 - \exp[-\lambda \exp(-\alpha_1 r^{\alpha_2})]\}$$

We have data concerning J equal 37 auctions, at which there are up to 32 lots. The population moment conditions are as follows:

$$\begin{aligned}
&\mathcal{E}(U_{0j}|r_j) = \mathcal{E}[N_j - \mu_0(r_j, \boldsymbol{\theta}^0)|r_j] = 0\\
&\mathcal{E}(U_{tj}|\Omega_{tj}) = \mathcal{E}[P_t^j - \mu_t(z_j, P_{t-1}^j, P_{t-2}^j, \ldots, \boldsymbol{\theta}^0)|\Omega_{tj}] = 0 \quad \text{for } t = 1, \ldots, T_j
\end{aligned}$$

where Ω_{tj} is the information set at sale t in the auction and includes z_j which equals $(r_j, T_j, n_j)^\top$ plus, at the very least, all winning prices for prior lots. Here, $\boldsymbol{\theta}^0$ denotes the true, but unknown, value of the parameter vector $\boldsymbol{\theta}$ that we seek to estimate. We shall construct instrument sets for each residual

$$\begin{aligned}
\boldsymbol{X}_{0j} &= r_j\\
\boldsymbol{X}_{1j} &= (r_j, T_j, n_j)^\top = z_j\\
\boldsymbol{X}_{2j} &= (z_j, P_1^j)^\top
\end{aligned}$$

DOI: 10.1002/jae

$$\vdots$$

$$\boldsymbol{X}_{T_j j} = (z_j, P_1^j, \ldots, P_{T_j-1}^j)^\top$$

so that

$$\mathcal{E}(U_{tj}|\boldsymbol{X}_{tj}) = 0$$

because $\boldsymbol{X}_{tj} \subset \Omega_{tj}$. Initially, we shall assume that we can estimate μ_t arbitrarily well by driving up the number of simulations, so that we can get the residuals as a function of data and parameters. Hence,

$$U_{0j}(\boldsymbol{\theta}) = N_j - \mu_0(r_j, \boldsymbol{\theta})$$

$$U_{tj}(\boldsymbol{\theta}) = P_t^j - \mu_t(z_j, P_{t-1}^j, P_{t-2}^j, \ldots, \boldsymbol{\theta})$$

We can now form the moment functions

$$\boldsymbol{g}_{0j}(\boldsymbol{\theta}) = \boldsymbol{X}_{0j} U_{0j}(\boldsymbol{\theta})$$

$$\boldsymbol{g}_{tj}(\boldsymbol{\theta}) = \begin{cases} \boldsymbol{X}_{tj} U_{tj}(\boldsymbol{\theta}) & \text{if } t \leq T_j \\ 0 & \text{otherwise} \end{cases}$$

We introduce $\hat{T}$ to be $\max(T_1, \ldots, T_J)$ and then define the average moment functions as

$$\overline{\boldsymbol{g}}_0(\boldsymbol{\theta}) = \frac{1}{J} \sum_{j=1}^{J} \boldsymbol{g}_{0j}(\boldsymbol{\theta})$$

$$\overline{\boldsymbol{g}}_t(\boldsymbol{\theta}) = \frac{1}{J} \sum_{j=1}^{J} \boldsymbol{g}_{tj}(\boldsymbol{\theta}) \quad \text{for } t = 1, \ldots, \hat{T}$$

where we note that some of the terms in the latter will be zero if, for any auction j, there are not $\hat{T}$ lots. The standard way to define $\tilde{\boldsymbol{\theta}}$, the GMM estimator of $\boldsymbol{\theta}^0$, is as

$$\tilde{\boldsymbol{\theta}} = \underset{\langle \boldsymbol{\theta} \rangle}{\operatorname{argmin}} \; \overline{\boldsymbol{g}}(\boldsymbol{\theta})^\top \mathbf{W} \overline{\boldsymbol{g}}(\boldsymbol{\theta})$$

where

$$\overline{\boldsymbol{g}}(\boldsymbol{\theta}) = [\overline{\boldsymbol{g}}_0(\boldsymbol{\theta})^\top, \overline{\boldsymbol{g}}_1(\boldsymbol{\theta})^\top, \ldots, \overline{\boldsymbol{g}}_{\hat{T}}(\boldsymbol{\theta})^\top]^\top$$

where $\mathbf{W}$ is the weighting matrix. In its most general form, many moment conditions will exist, so $\mathbf{W}$ will be large. However, the submartingale structure of our equilibrium and the assumed independence across auctions simplifies matters considerably. In particular, the optimal weighting matrix will be the inverse of the variance–covariance matrix of $\sqrt{J}\overline{\boldsymbol{g}}(\boldsymbol{\theta})$. In our case, independence across auctions implies that

$$\mathcal{V}[\sqrt{J}\overline{\boldsymbol{g}}(\boldsymbol{\theta})] = \frac{1}{J} \sum_{j=1}^{J} \mathcal{E}[\boldsymbol{g}_j(\boldsymbol{\theta}) \boldsymbol{g}_j(\boldsymbol{\theta})^\top]$$

$$= \frac{1}{J} \sum_{j=1}^{J} \mathbf{C}_j = \overline{\mathbf{C}}$$

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To compute the variances and covariances, we do the following. First, we consider the covariances. Note that the structure above implies that

$$\boldsymbol{X}_{0j} \subset \boldsymbol{X}_{tj} \quad \text{for } t = 1, \ldots, T_j$$

and U_{0j} is known, given $\boldsymbol{X}_{tj}$ for $t = 1, \ldots, T_j$. Thus,

$$\mathcal{E}[\overline{\boldsymbol{g}}_{0j}(\boldsymbol{\theta}^0)\overline{g}_{tj\ell}(\boldsymbol{\theta}^0)] = 0$$

For the other covariances, we know that when t_1 is less than t_2,

$$\boldsymbol{X}_{t_1 j} \subset \boldsymbol{X}_{t_2 j} \quad \text{for } t = 1, \ldots, T_j$$

and $U_{t_1 j}$ is known, given $\boldsymbol{X}_{t_2 j}$ for $t = 1, \ldots, T_j$. Thus,

$$\mathcal{E}[g_{t_1 jk}(\boldsymbol{\theta}^0) g_{t_2 j\ell}(\boldsymbol{\theta}^0)] = 0$$

so $\mathbf{W}$ is block diagonal with blocks given by

$$\mathbf{W}_0 = \mathcal{E}[\boldsymbol{g}_{0j}(\boldsymbol{\theta})\boldsymbol{g}_{0j}(\boldsymbol{\theta})^\top]^{-1}$$
$$\mathbf{W}_t = \mathcal{E}[\boldsymbol{g}_{tj}(\boldsymbol{\theta})\boldsymbol{g}_{tj}(\boldsymbol{\theta})^\top]^{-1}$$

We can estimate these in the usual way by taking sample averages of the moment functions estimated with an initial consistent estimate of $\boldsymbol{\theta}^0$. Denote these by $\hat{\mathbf{W}}_t$, so

$$\tilde{\boldsymbol{\theta}} = \underset{\langle\boldsymbol{\theta}\rangle}{\operatorname{argmin}} \sum_{t=0}^{\hat{T}} \overline{\boldsymbol{g}}_t(\boldsymbol{\theta})^\top \hat{\mathbf{W}}_t \overline{\boldsymbol{g}}_t(\boldsymbol{\theta})$$

One practical problem with using averages is that, for some auctions, only one observation exists; e.g., when T_j is 32 it is impossible to create a full-rank weight matrix. Also, in several cases, the optimal weighting matrix was ill-conditioned. In such cases, to avoid numerical problems, we substituted the identity matrix of the appropriate dimension.

Of course, we do not know $\mu_t(\cdot)$ in closed-form, but we can form an estimate of it by simulating the average winning bids at a generalized Vickrey auction to get $\hat{\mu}_t(\cdot)$.[10] Note, however, that the simulation error is orthogonal to the instruments, so unlike in Laffont *et al.*, we do not need to adjust the objective function.

To undertake inference, two options are available—first-order asymptotic methods and bootstrap resampling methods. Because we only have 37 auctions, we chose to use bootstrap methods to calculate the standard errors of our parameter estimates. Under the conditions assumed

[10] For example, consider the case where T is four and n is three with $\boldsymbol{V}^1$ being $\{5, 4, 2, 0\}$, $\boldsymbol{V}^2$ being $\{6, 3, 1, 0\}$ and $\boldsymbol{V}^3$ being $\{4.5, 2.5, 0, 0\}$. The aggregate-demand vector in this example is then $\{6, 5, 4.5, 4, 3, 2.5, 2, 1\}$. Thus, the four units will be allocated to bidders having valuations $\{6, 5, 4.5, 4\}$. The list and order of winners is $\{2, 1, 3, 1\}$. The price at the first sale will be V_4^{-2} or 2.5, while the price at the second sale will be V_4^{-1} or 2.5, and the price at the third sale will be V_4^{-3} or 3, with the price in the last sale being V_3^{-1} or 3. Note that this price profile does not correspond exactly to the prices players would pay according to the equilibrium but, according to Theorem 2(iii), on average, this price will equal the average equilibrium winning price.

DOI: 10.1002/jae

Table V. SGMM parameter estimates, $\mathcal{N} = 30$

Parameter	Estimate
λ	1.336
α_1	0.147
α_2	1.711

Table VI. SGMM parameter estimates and standard errors

Parameter	Estimate	Std error
λ	1.615	0.527
α_1	0.119	0.029
α_2	1.842	0.452
$\mathcal{N}$	24	—

above, the regularity conditions necessary to apply bootstrap resampling methods are met; see Horowitz (2001).

To implement the simulation estimator, we chose the programming environment MATLAB. We chose to use 50 simulations. In Table V, assuming $\mathcal{N}$ is 30, we present estimates of the Poisson mean λ as well as the parameters of the Weibull distribution α_1 and α_2. Within this framework, however, we can identify and estimate $\mathcal{N}$ from the available data. Thus, we searched over $\mathcal{N}$, choosing parameter estimates for which the objective function was smallest. This strategy suggested that the best estimate of $\mathcal{N}$ was 24, rather than 30. In Table VI, we present estimates of the Poisson mean λ and the parameters of the Weibull distribution α_1 and α_2 as well as their bootstrap standard errors. Our bootstrap standard errors are based on 100 resamplings with replacement from the 37 auctions in the data set.

The empirical results in Table VI suggest that, on average, a bidder gets 1.615 business opportunities per week. The average value, in terms of 1000R per cubic metre, of an opportunity is estimated to be 2.8214, while in our sample of data the average winning-bid price per cubic metre, in 1000R, was 3.2492.

One question a reader might ask is: Can these structural-econometric estimates be consistent with the reduced-form ones presented in Section 2? In particular, the reduced-form results presented previously suggest little evidence of a submartingale in average winning-bid prices. To investigate this issue, we undertook two exercises. First, using our estimated parameters and $\mathcal{N}$ of 24, we simulated the winning prices for 1000 generalized Vickrey auctions when T was 4, 8 and 12 units. The averages of these winning prices for each unit—conditional on 4, 8 and 12 units for sale—are depicted in Figure 3. As one can see, the average winning prices are higher when 4 units are for sale than when 8 are, and both are greater than when 12 units are for sale. Also, the average winning-price profile in each case is quite flat. For example, in the case of T equal 8 units, the approximate average number of lots for sale in our data set, the difference from the first unit to the last is 1.26%, or an average of 16 basis points per unit. Perhaps, in a sample of 37 auctions, sampling variation masks the submartingale property and prevents us from estimating it with any precision.

DOI: 10.1002/jae

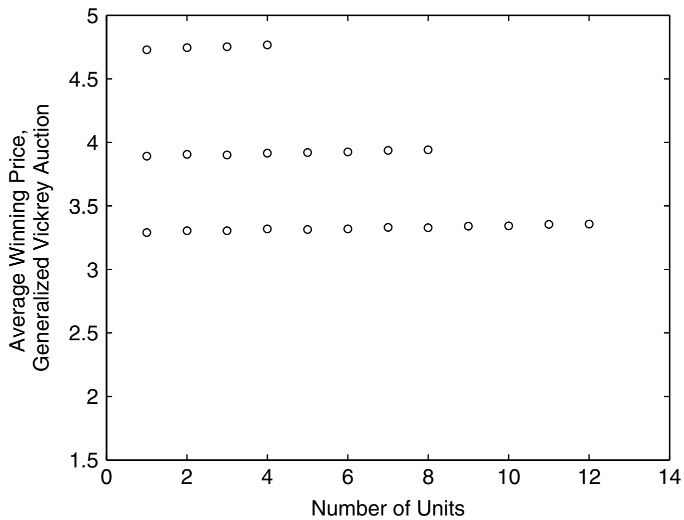

Figure 3. Simulated average winning-bid profile: generalized Vickrey auction

Second, using our 37 sample covariate pairs (r_j, T_j) as well as the estimated parameters and $\mathcal{N}$ of 24, we generated 100 samples of winning bids at generalized Vickrey auctions. We then estimated the reduced-form specifications considered in Section 2. As in Section 2, we found limited evidence of a submartingale; the empirical evidence was consistent with a martingale, even though many of our simulated bidders won more than one unit.

One of the empirical features of the data that the estimated parameters cannot mimic is the number of times that a single bidder won 5, 6 or 7 as well as 9 or 10 lots at an auction. The incidence of these events is unlikely given a Poisson parameter estimate of 1.615.

The empirical analysis presented here has at least two other limitations. First, the prices used in the empirical work are per-unit ones, not those for the entire lot, and the lots are of different sizes. We have adopted this approach because we have found it impossible to characterize optimal bidding behaviour when lots are of different sizes. Second, since the auctions were held almost weekly, orders could have been durable and thus would not evaporate, becoming part of the aggregate demand for the next period.

5. CONCLUSIONS

In this paper, we have developed a simple model of behaviour at a multi-unit, sequential, clock auction with multi-unit demand. Subsequently, we also described conditions sufficient to characterize a strategic equilibrium of the auction game and then demonstrated that this equilibrium induces an efficient allocation. From a theoretical perspective, this paper differs from other studies on sequential auctions in that we allow participants to demand more than one unit of the good for sale. Our main theoretical finding concerns the pattern of winning prices. When potential buyers demand at most one unit, the winning prices in a sequence of auctions are the same, equalling the highest extra-marginal valuation. On the other hand, when buyers demand more than one unit, our model predicts that winning prices follow a submartingale process; the average winning price will

DOI: 10.1002/jae

increase over the auction. This prediction illustrates how incomplete our understanding of price determination in markets is, for the units at auction in our model correspond, in all respects, to a homogeneous good in the Arrow–Debreu sense. The auction's design ensures competition among participants and generates considerable public information. We assume risk-neutrality, so risk-aversion does not limit attempts to exploit arbitrage possibilities. Yet, contrary to the conventional Walrasian wisdom, the price of each lot is not constant, either in actual terms or on average. One of the main modelling innovations, which was instrumental in delivering these conclusions, is the Poisson arrival process for demand by potential bidders.

The structure of the theoretical model has a number of attractive features from an empirical perspective. First, from the theoretical structure, we have proposed an algorithm that allows us to calculate the expected winning price for each unit sold. This algorithm forms the cornerstone of a simulation-based, structural-econometric strategy to estimate the parameters of both the participation process and the distribution of latent valuations using commonly available data. We have demonstrated that our approach is feasible by applying it to data from a small sample of multi-lot, sequential, English auctions of export permits for timber held in the Krasnoyarsk Kray of Russia, where we found that the main reduced-form prediction of the theoretical model is not rejected by regression estimates. We have also calculated structural parameter estimates of our model.

APPENDIX

In this Appendix, we collect the proofs of properties and theorems presented in the text of the paper.

Proof of Property C

We need to show that $\Pr(V_k^i \leq y | V_j^i = x, M_i \geq j)$ depends only on $\Delta \equiv (k-j)$, y and x and not on j and k specifically. We have

$$\begin{aligned}\Pr(V_j^i = x, M_i \geq j) &= \sum_{m=j}^{\infty} \frac{\lambda^m \exp(-\lambda)}{m!} \frac{m!}{(m-j)!(j-1)!} F(x)^{m-j}[1-F(x)]^{j-1} f(x) \\ &= \frac{\lambda^j [1-F(x)]^{j-1} f(x)}{(j-1)!} \sum_{m=j}^{\infty} \frac{\lambda^{m-j} F(x)^{m-j} \exp(-\lambda)}{(m-j)!} \\ &= \frac{\lambda^j [1-F(x)]^{j-1} f(x)}{(j-1)!} \exp(-\lambda)[1-F(x)] \qquad \text{(A.1)}\end{aligned}$$

and

$$\begin{aligned}&\Pr(V_j^i = x, M_i \geq j, V_k^i \leq y) \\ &= \sum_{m=j}^{k-1} \frac{\lambda^m \exp(-\lambda)}{m!} \frac{m!}{(m-j)!(j-1)!} F(x)^{m-j}[1-F(x)]^{j-1} f(x) \\ &+ \sum_{m=k}^{\infty} \frac{\lambda^m \exp(-\lambda)}{m!} \frac{m!}{(m-j)!(j-1)!} F(x)^{m-j}[1-F(x)]^{j-1} f(x)\end{aligned}$$

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$$\times \frac{(m-j)!}{(k-j-1)!(m-k)!}\int^{\frac{F(y)}{F(x)}} t^{m-k}(1-t)^{k-j-1}dt$$

$$= \frac{\lambda^j[1-F(x)]^{j-1}f(x)}{(j-1)!}\left[\sum_{m=j}^{k-1}\frac{\lambda^{m-j}\exp(-\lambda)}{(m-j)!}F(x)^{m-j}\right.$$

$$\left.+\sum_{m=k}^{\infty}\frac{\lambda^{m-j}\exp(-\lambda)}{(m-j)!}F(x)^{m-j}\frac{(m-j)!}{(k-j-1)!(m-k)!}\int^{\frac{F(y)}{F(x)}} t^{m-k}(1-t)^{k-j-1}dt\right] \quad \text{(A.2)}$$

Dividing (A.2) by (A.1), we obtain after simplification and using $\ell = (m-j)$, as stated above, an expression independent of j:

$$\Pr(V_k^i \le y | V_j^i = x, M_i \ge j) = \sum_{\ell=0}^{\Delta-1}\frac{[\lambda F(x)]^{\ell}\exp[-\lambda F(x)]}{\ell!}$$

$$+\sum_{\ell=\Delta}^{\infty}\frac{[\lambda F(x)]^{\ell}\exp[-\lambda F(x)]}{\ell!}\frac{\ell!}{(\Delta-1)!(\ell-\Delta)!}\int^{\frac{F(y)}{F(x)}} t^{\ell-\Delta}(1-t)^{\Delta-1}dt$$

Proof of Theorem 1

The proof proceeds by backward induction. At the last sale, the game corresponds to the usual single-unit, clock auction where bidder i's private value for the next unit is given by $V^i_{\ell_i+1}$. The equilibrium strategy for each participant i is to bid until the price reaches $V^i_{\ell_i+1}$. We have already argued in the text that this is the strategy prescribed by Theorem 1.

Introduce the history $\mathbf{L}$, which denotes $\{\ell_1, \ell_2, \ldots, \ell_n\}$, the vector of the number of units already purchased by each bidder. We now show that, if it is a best response for each player to follow the proposed equilibrium strategy at round $(t+1)$ and later, then it is his best response to follow at round t. Consider the case of bidder i who has already won ℓ_i units and let j be the bidder with the highest valuation among all bidders but i after some history. According to the equilibrium, no one but i would wish to out-bid j. We now show that i must be indifferent between the two following alternatives: first, participate until he wins sale t and withdraw from sale $(t+1)$ and let j win or, second, let j win sale t and win sale $(t+1)$ instead. Because bidding strategies depend only on how many units each bidder has won previously, the choice between these two alternatives does not affect the outcome of sale $(t+2)$ to T. The only thing that matters is the expected price that i will need to pay in order to win in sale t or $(t+1)$. Following Theorem 2(iii), in order to win sale t and out-bid j, i will need to pay $\mathcal{E}(V^{-i}_{T-\ell_i}|\Omega_t)$, where Ω_t is the information available at the end of sale t. In order to win sale $(t+1)$, when j has won sale t, i will need to pay $\mathcal{E}(V^{-i}_{T-\ell_i}|\Omega_{t+1})$. Conditional on Ω_t, the expected value of this latter price is given by $\mathcal{E}[\mathcal{E}(V^{-i}_{T-\ell_i}|\Omega_{t+1})|\Omega_t]$, which equals $\mathcal{E}(V^{-i}_{T-\ell_i}|\Omega_t)$. It follows that if both expected prices are the same, the first and second alternatives yield the same expected payoffs to bidder i.

Now, suppose i should win sale t, but he deviates and lets j win instead. Bidder i would then win the next sale should he use his equilibrium strategy. But we know from above that deviating in sale t and then following the equilibrium strategy in sale $(t+1)$ is not better than following the equilibrium strategy, winning in sale t, and withdrawing from sale $(t+1)$ to let j

DOI: 10.1002/jae

win. This strategy, in turn, under the assumption of equilibrium, cannot be better than following the equilibrium strategy in both sales. Hence, i has no incentive to bid less than prescribed by Theorem 1. Similarly, suppose that j were supposed to win sale t, but that i deviates and out-bids j in sale t. j would win the next sale if i follows in sale $(t+1)$ the equilibrium. Here again, we know that to deviate in sale t and then to follow the equilibrium strategy in sale $(t+1)$ is not better than to follow the equilibrium strategy in sale t and to deviate in order to win in sale $(t+1)$. Under the assumption, it is a best response for each player to follow the proposed equilibrium strategy at round $(t+1)$ and later; the latter alternative is no better than following the proposed strategy in both rounds. Hence, i has no incentive to bid more than prescribed by Theorem 1.

Proof of Theorem 2

Part (i)

(Symmetry) Consider the bid functions of bidders i and j, when i has won ℓ_i units and j has won ℓ_j units. Bidder i's drop-out price b^i will be a function of $V^i_{1+\ell_i}$, and bidder j's drop-out price b^j will be a function of $V^j_{1+\ell_j}$. Let $b^i(x)$ denote i's valuation when $V^i_{1+\ell_i}$ equals x. Suppose that both these valuations equal x. We then have

$$\begin{aligned} b^i(x) &= \mathcal{E}(V^{-i}_{T-\ell_i} | V^{-i}_{1+q_i} = V^j_{1+\ell_j} = x, \forall j \in \mathbf{R}; V^h_{1+\ell_k} = v^h, \forall h \in \mathbf{S}) \\ &= \mathcal{E}(V^{-i}_{T-q_i-\ell_i} | V^{-i}_1 = V^j_1 = x, \forall j \in \mathbf{R}; V^h_1 = v^h, \forall h \in \mathbf{S}) \\ &= \mathcal{E}(V^{-j}_{T-q_j-\ell_j} | V^{-j}_1 = V^i_1 = x, \forall i \in \mathbf{R}; V^h_1 = v^h, \forall h \in \mathbf{S}) = b^j(x) \end{aligned}$$

From property D, the conditional distribution of lower-order statistics given higher-order statistics is the same across bidders and both expressions have expected value based on the same conditional information. This implies symmetry.

(Strict monotonicity) We wish to show that bidder i's drop-out price b^i will be a strictly monotone function of $V^i_{1+\ell_i}$. The function $b^i(x)$ is an operator that evaluates $V^{-i}_{T-\ell_i}$, the $(T-\ell_i)$th highest valuation among those all bidders but i, given some information. Let $H_k(\boldsymbol{V}^{-i})$ denote the kth highest valuation among all valuations in $\boldsymbol{V}^{-i}$. H_k is continuous and non-decreasing in all its arguments. Now, some of the values in $\boldsymbol{V}^{-i}$ are known. In particular, for all active bidders $h \in \mathbf{R}$, we assume that $V^h_{1+\ell_h}$ equals $V^i_{1+\ell_i}$. Furthermore, the unknown valuations $\{V^h_{2+\ell_h}, V^h_{3+\ell_h}, \ldots, V^h_{T-\ell_i}\}$ for all $h \in \mathbf{R}$ are positively correlated with $V^h_{1+\ell_h}$ equal $V^i_{1+\ell_i}$ with support in $[0, V^i_{1+\ell_i}]$, so the function b^i is indeed strictly increasing in $V^i_{1+\ell_i}$.

Part (ii)

Since bids are monotonic and symmetric functions of each bidder's next-highest valuations, the $V^i_{\mathbf{1}+\ell_i}$'s, the winner after every history will then be the one with the next-highest individual value, so in the tth sale, the winner will be the bidder with the tth highest valuation over all.

Part (iii)

We prove (iii) given efficiency and part (i). Suppose that, during a sale, bidder i stays in until he wins the unit and that j is the last competitor to drop out. According to the equilibrium, j will drop out at the following price:

$$b^j = \mathcal{E}(V^{-j}_{T-\ell_i} | V^i_{1+\ell_i} = V^j_{1+\ell_j}, V^h_{1+\ell_h} = v_h, \forall h \in \mathbf{N}/\{i, j\}, V^h_k \geq V^j_{1+\ell_j}, \forall k < 1 + \ell_h)$$

$$= \mathcal{E}(V^{-i}_{T-\ell_i} | V^j_{1+\ell_j}, V^h_{1+\ell_h} = v_h, \forall h \in \mathbf{N}/\{i, j\}, V^h_k \geq V^j_{1+\ell_j}, \forall k < 1 + \ell_h)$$
$$= \mathcal{E}(V^{-i}_{T-\ell_i} | \{(V^h_{\mathbf{1}}, \ldots, V^h_{1+\ell_k})\}, \forall k \neq i) = \mathcal{E}(V^{-i}_{T-\ell_i} | \Omega_t)$$

The first expression represents the price at which bidder j will drop out of the auction when, in the end, only i and j remain. The first equality follows from symmetry (property D). The third equality is more involved. Following property A, which comes from the assumption of independent draws, all V^h_k for all h and k less than $(1 + \ell_h)$ are not useful to forecast $V^{-i}_{T-\ell_i}$. The last equality follows from the fact that Ω_t, all information available after sale t, is included in $[(V^k_1, \ldots, V^k_{1+\ell_k}), \forall k \neq i]$, but includes $[V^j_{1+\ell_j}, \{V^h_{1+\ell_h}\}h \in \mathbf{N}/\{i, j\}$ and $V^h_k \geq V^j_{1+\ell_j}, \forall k < 1 + \ell_h]$.

Proof of Theorem 3

Suppose that bidder i wins sale t and j is the last competitor to drop out of this sale. The winning price at sale t, P_t, is $\mathcal{E}(V^{-j}_{T-\ell_j} | V^i_{1+\ell_i} = V^j_{1+\ell_j}, \Omega_t)$. This is the price at which j will drop out. According to the equilibrium, in the next sale $(t + 1)$, either bidder j (if $V^i_{2+\ell_i}$ weakly exceeds $V^j_{1+\ell_j}$) or i wins (if $V^i_{2+\ell_i}$ exceeds $V^j_{1+\ell_j}$). Following Theorem 2(iii), if j wins, then he will need to pay $\mathcal{E}(V^{-j}_{T-\ell_j} | \Omega_{t+1})$ which equals $\mathcal{E}(V^{-j}_{T-\ell_j} | V^i_{1+\ell_i} \geq V^j_{1+\ell_j}, V^i_{2+\ell_i}, \Omega_t)$. If i wins, then he will need to pay $\mathcal{E}(V^{-j}_{T-\ell_j} | V^i_{1+\ell_i} \geq V^i_{2+\ell_i} = V^j_{1+\ell_j}, \Omega_t)$. Conditional on Ω_t, and the fact that i won sale t, the expected value of P_{t+1} is given by

$$\begin{aligned}
\mathcal{E}(P_{t+1} | V^i_{1+\ell_i} \geq V^j_{1+\ell_j}, \Omega_t) &= \mathcal{E}[\mathcal{E}(V^{-j}_{T-\ell_j} | \Omega_{t+1}) | V^i_{1+\ell_i} \geq V^j_{1+\ell_j}, \Omega_t] \\
&= \mathcal{E}(V^{-j}_{T-\ell_j} | V^i_{1+\ell_i} \geq V^j_{1+\ell_j}, \Omega_t) \\
&\geq \mathcal{E}(V^{-j}_{T-\ell_j} | V^i_{1+\ell_i} = V^j_{1+\ell_j}, \Omega_t) \\
&= P_t
\end{aligned}$$

P_t is the expected value of $V^{-j}_{T-\ell_j}$ given that $V^i_{1+\ell_i}$ equals $V^j_{1+\ell_j}$, while the expected value of P_{t+1} is the expected value of $V^{-j}_{T-\ell_j}$ given that $V^i_{1+\ell_i}$ weakly exceeds $V^j_{1+\ell_j}$. The inequality follows from the fact that the expected value of $V^{-j}_{T-\ell_j}$ is increasing in $V^i_{1+\ell_i}$.

ACKNOWLEDGEMENTS

Donald acknowledges support from the Alfred P. Sloan Foundation, while Paarsch acknowledges support from both the NSERC and the SSHRC of Canada, and Robert acknowledges support from the SSHRC, FCAR and CIRANO. The authors would like to acknowledge the contribution of Mikhail I. Golovanov who directed the collection, translation and tabulation of the information from which the data set was constructed. The authors also thank Han Hong and Sergei Izmalkov as well as two anonymous referees for helpful comments and useful suggestions on the penultimate version of this paper. However, they are most grateful to John Rust for his specific and detailed comments and suggestions which resulted in this final version of the paper. The authors are also grateful to M. Ryan Haley who allowed them to use some software he developed in the estimation. Earlier versions of this paper circulated as Paarsch and Robert (1995) as well as Donald *et al.* (1996).

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DOI: 10.1002/jae