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Multidimensional endogenous gridpoint method: Solving triangular dynamic stochastic optimization problems without root-finding operations

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HIGHLIGHTS

- Certain dynamic stochastic models can be solved without root fining operations.
- The paper describes a class of such models using five conditions.
- The solution method is multidimensional endogenous gridpoint method (EGM).
- Typical member of this class is a model of multiple stock dynamics.

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ABSTRACT

This paper defines the class of triangular dynamic stochastic optimization problems with multiple continuous choices which can be solved by the multidimensional generalization of the method of endogenous gridpoints without costly root-finding operations. The typical member of this class is a model of multiple asset dynamics, with potential applications in wealth, health and human capital accumulation, portfolio problems, multisector growth models, etc.

1. Introduction

The endogenous gridpoint method (EGM) introduced by Carroll (2006) to solve stochastic dynamic models with one continuous state variable and a single continuous control, proved to be very fast and accurate[1]—mainly due to its ability to solve the first order conditions directly without root-finding operations. The method had been generalized to a number of settings,[2] where it proved to retain its numerical advantage over the traditional dynamic programming approach. Yet, the applicability of EGM method had been shown only for a collection of particular economic models, whereas the class of models which can be solved without root-finding operations has not been rigorously characterized.[3]

This paper proposes five conditions which are jointly sufficient for a given problem to belong to the class of *triangular* dynamic optimization problems that can be solved by the multidimensional version of EGM in such a way that all root-finding operations are

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1 As a result, the computation of the solution for a classic consumption–savings model that could "easily done on a 386-series PC, taking 5–20 min per calculation" by Deaton (1991, p. 1229), takes only a fraction of a second on a modern laptop (Jørgensen, 2013, p. 289), with the speedup of approximately two orders of magnitude from the use of EGM when compared to traditional value functions iterations. Barillas and Fernandez-Villaverde (2007) and Iskhakov et al. (2015) find similar magnitude of the speedup with no effect on the solution accuracy.

2 See Barillas and Fernandez-Villaverde (2007), Hintermaier and Koeniger (2010), Ludwig and Schön (2014), Fella (2014) and Iskhakov et al. (2015) for various applications of EGM framework.

3 White (2015) develops a general theory of EGM, which does not however aim at defining the class of problems solvable without root-finding operations.

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avoided. The key insight is to ensure that the system of Euler equations that characterizes the solution has a triangular structure, and thus can be solved directly using forward substitution under standard EGM assumptions of analytical invertibility of marginal utilities.

To simplify the exposition, in this paper I only consider unconstrained choice problems. The standard approach of treating corner solutions which relies on the monotonicity of the policy function[4] or Inada (1963) derivative conditions can be generalized with additional assumptions, but general treatment based on Kuhn–Tucker conditions used by Hintermaier and Koeniger (2010) is also available for particular applications. It should also be noted that the proposed multidimensional EGM does not break the curse of dimensionality—multidimensional integration and interpolation on irregular grids are inevitable in these models.[5]

2. Two-dimensional illustrative example

Consider a discrete time[6] dynamic optimization problem with two state variables X_t and Y_t, and two continuous decision variables x_t and y_t, which represent, for example, the choice of consumption and the exercise intensity in the model of health and wealth dynamics. Denoting the instantaneous utility function $u(x_t, y_t)$ and the discount factor β, the Bellman equation of the problem is

$$V_t(X_t, Y_t) = \max_{x_t, y_t} \{u(x_t, y_t) + \beta E\left[V_{t+1}(X_{t+1}, Y_{t+1}) | X_t, Y_t, x_t, y_t\right]\}, \quad t \in \{1, \dots, T-1\}. \tag{1}$$

Assume that the transition rules for the states X_t and Y_t can be written as

$$\begin{aligned} X_{t+1} &= \chi\left(f(X_t, x_t), g(Y_t, y_t), \xi_{t+1}\right), \\ Y_{t+1} &= \psi\left(f(X_t, x_t), g(Y_t, y_t), \xi_{t+1}\right), \end{aligned} \tag{2}$$

in other words *sufficient statistics* $f(t) = f(X_t, x_t)$ and $g(t) = g(Y_t, y_t)$ contain all the information about period t that determines the distribution of values of the states in period $t+1$. Following the tradition in operations research (Powell, 2007, pp. 129–144), I refer to $\big(f(t), g(t)\big)$ as *post-decision states*. For simplicity, assume that functions χ, ψ, f and g are time invariant.[7] The expectation in (1) is taken over the joint distribution of idiosyncratic shocks $\xi_{t+1} \in \mathbb{R}^K$, and because of (2) can be written conditional on $\big(f(t), g(t)\big)$.

The solution of the problem given by the family of policy functions $\delta_t : (X_t, Y_t) \to (x_t, y_t)$, $t \in \{1, \dots, T-1\}$ satisfies the system of first order conditions for (1)

$$\begin{cases} u'_x(x_t, y_t) + \beta f'_x(t) \\ \quad \times E\left[\dfrac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_f + \dfrac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_f \middle| f(t), g(t)\right] = 0, \\ u'_y(x_t, y_t) + \beta g'_y(t) \\ \quad \times E\left[\dfrac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_g + \dfrac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_g \middle| f(t), g(t)\right] = 0. \end{cases} \tag{3}$$

By envelope theorem we have

$$\begin{aligned} \frac{\partial V_t}{\partial X_t} &= \beta f'_X(t) E\left[\frac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_f + \frac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_f \middle| f(t), g(t)\right] \\ &= -\frac{f'_X(t)}{f'_x(t)} u'_x(x_t, y_t), \\ \frac{\partial V_t}{\partial Y_t} &= \beta g'_Y(t) E\left[\frac{\partial V_{t+1}}{\partial X_{t+1}} \chi'_g + \frac{\partial V_{t+1}}{\partial Y_{t+1}} \psi'_g \middle| f(t), g(t)\right] \\ &= -\frac{g'_Y(t)}{g'_y(t)} u'_y(x_t, y_t). \end{aligned} \tag{4}$$

Combining (4) with (3), we derive the system of Euler equations

$$\begin{cases} u'_x(x_t, y_t) = \beta f'_x(t) E\left[\chi'_f \dfrac{f'_X(t+1)}{f'_x(t+1)} u'_x(x_{t+1}, y_{t+1}) \right. \\ \qquad \left. + \psi'_f \dfrac{g'_Y(t+1)}{g'_y(t+1)} u'_y(x_{t+1}, y_{t+1}) \middle| f(t), g(t)\right], \\ u'_y(x_t, y_t) = \beta g'_y(t) E\left[\chi'_g \dfrac{f'_X(t+1)}{f'_x(t+1)} u'_x(x_{t+1}, y_{t+1}) \right. \\ \qquad \left. + \psi'_g \dfrac{g'_Y(t+1)}{g'_y(t+1)} u'_y(x_{t+1}, y_{t+1}) \middle| f(t), g(t)\right], \end{cases} \tag{5}$$

which constitutes the set of necessary conditions linking the optimal choices (x_t, y_t) in period t to the optimal choices (x_{t+1}, y_{t+1}) in $t+1$, given $\big(f(t), g(t)\big)$. Note that (x_t, y_t) does not appear in the right hand side (RHS) in (5).

Given an appropriate terminal condition, for example $\delta_T(X_T, Y_T) = (X_T, 0)$ in the hypothetical model of health and wealth dynamics, the EGM method is implemented by fixing the grid over $\big(f(t), g(t)\big)$; computing RHS in (5) for each point of this grid, using the next period policy function δ_{t+1} to find (x_{t+1}, y_{t+1}); backing out the optimal decision by solving (5) for (x_t, y_t); and finally recovering the endogenous state point (X_t, Y_t) from post-decision states and optimal decision. Repeating the latter steps for all points in the grid over $\big(f(t), g(t)\big)$ yields an approximation of the policy function δ_t, and the backward induction continues to the period $t-1$.

The numerical efficiency of EGM in two-dimensional problems satisfying (2) hinges on whether the system (5) can be solved for (x_t, y_t) by direct computation. The original one-dimensional EGM relies on the analytical invertibility of the marginal utility function, which is enough to yield the optimal period t choice by direct computation when (5) only has one equation. Clearly, the invertibility of all the partial derivatives is insufficient in the multidimensional case. Because (5) is a system of non-linear equations, even the existence of the solution is not guaranteed without additional conditions on the utility function.

3. *M*-dimensional triangular dynamic optimization problems

Consider now an optimization problem with M continuous choice variables $x_t = (x_t^1, \dots, x_t^M) \in \mathbb{R}^M$ which govern the transitions of M continuous state variables $X_t = (X_t^1, \dots, X_t^M) \in \mathbb{R}^M$. Let $s_t \in \mathbb{R}^N$ denote additional state variables which follow *uncontrolled* N-dimensional Markov processes, and therefore will be referred to as exogenous states.[8] The following assumptions are needed to derive the main result below.

4 Used by Carroll (2006), Fella (2014) and Iskhakov et al. (2015).

5 Ludwig and Schön (2014) show that already in two-dimensional case the computational cost of Delaunay triangulation may offset the speed advantage of the EGM approach. However, White (2015) proposes an interpolation method which takes into account the ordinal information about the endogenous gridpoints produced by EGM. Orthogonal polynomial approximation (Judd, 1998, pp. 202–223) may be suggested as another useful interpolation method.

6 Both infinite and finite horizon cases are included assuming that the infinite horizon model is solved by time iterations indexed with t, and $T = +\infty$.

7 It is straightforward to extend the analysis to time varying specifications.

8 To simplify the exposition, they will be treated as discrete. In applications these states are either discrete or discretized as in Tauchen (1986).

Assumption A1. Transition rules for the continuous state variables $X^1_t, \dots, X^M_t, j \in \{1, \dots, M\}$ can be expressed in the form

$$X^j_{t+1} = \chi^j\Big(f^1(X^1_t, x^1_t, s_t), f^2(X^1_t, x^1_t, X^2_t, x^2_t, s_t), \dots, f^M(X_t, x_t, s_t), s_t, s_{t+1}, \xi_{t+1}\Big), \tag{6}$$

such that the post-decision states $f(t) = \big(f^1(t), \dots, f^M(t)\big) \in \mathbb{R}^M$ admit the following structure:

$$\begin{aligned} &f^1(t) = f^1(X^1_t, x^1_t, s_t),\\ &f^2(t) = f^2(X^1_t, x^1_t, X^2_t, x^2_t, s_t),\\ &\dots\\ &f^M(t) = f^M(X^1_t, x^1_t, \dots, X^M_t, x^M_t, s_t) = f^M(X_t, x_t, s_t), \end{aligned} \tag{7}$$

where $\chi^j(\cdot)$ and $f^j(\cdot)$ are deterministic differentiable functions, and $\frac{\partial f^j}{\partial x^j} \neq 0$ and $\frac{\partial f^j}{\partial X^j} \neq 0$, $\xi_{t+1} \in \mathbb{R}^K$ are idiosyncratic shocks.

Assumption A2. Transition probabilities $P(s_{t+1}|X_t, s_t) = P(s_{t+1}|s_t)$ are independent of X_t.

Assumptions A1 and A2 ensure that the post-decision states $f(t)$ form a set of sufficient statistics for the states and decisions in period t.

Assumption A3. Equations $f^j(X^1_t, x^1_t, \dots, X^j_t, x^j_t, s_t) = \hat{f}$ can be analytically solved for X^j_t, while holding other arguments fixed, for all $j \in \{1, \dots, M\}$ and arbitrary constant $\hat{f}$.

Assumption A4. There exist permutations $k^1 = (k^1_{(1)}, \dots, k^1_{(M)})$ and $k^2 = (k^2_{(1)}, \dots, k^2_{(M)})$ of the natural numbers 1 to M such that if J_{k^1} and J_{k^2} are the corresponding permutation matrices[9] and H is the Hessian of the utility function $u(x_t, s_t)$ for a fixed s_t, it holds that the permuted Hessian $H^P = J_{k^1} H J_{k^2}$ is lower triangular, the diagonal elements of H^P are non-zero, i.e. letting $i = k^1_{(l)}$ and $j = k^2_{(l)}$, $l \in \{1, \dots, M\}$, $\frac{\partial^2 u(x_t, s_t)}{\partial x^i_t \partial x^j_t} \neq 0$, the partial derivatives $\frac{\partial u(x_t, s_t)}{\partial x^i_t}$ are strictly monotone in x^j_t, and their inverses in x^j_t have analytical forms.

Assumption A4 essentially describes the situation when the Hessian of the utility function can be converted to a lower triangular form by re-labeling the variables (according to permutation k^2) and permuting the rows of the matrix (according to k^1). For additively separable utility $u(x_t, s_t) = \sum_{j=1}^M u_j(x^j_t, s_t)$ with strictly monotone and analytically invertible partial derivatives, A4 is satisfied with any permutation $k^1 = k^2$. A non-trivial example of utility function satisfying A4 is $u(x^1_t, x^2_t) = (x^1_t)^{\alpha_1}(x^2_t)^{\alpha_2}$ when $\alpha_1 = 1$ and $\alpha_2 \neq 1$.

Assumption A5. The utility function $u(x_t, s_t)$ is concave in x_t, and standard regularity conditions (9.4–9.7) in Stokey et al. (1989, pp. 260–263) hold.

Proposition 1. *Under Assumptions A1–A5 the dynamic optimization problem given by the Bellman equation*

$$V_t(X_t, s_t) = \max_{x_t} \Big\{ u(x_t, s_t) + \beta E\Big[V_{t+1}(X_{t+1}, s_{t+1}) | X_t, s_t, x_t\Big]\Big\}, \tag{8}$$

where $X_t \in \mathbb{R}^M$, $x_t \in \mathbb{R}^M$ and $s_t \in \mathbb{R}^N$, $t \in \{1, \dots, T-1\}$, and the terminal policy function $\delta_T : (X_T, s_T) \to x_T$, admits a solution method which avoids all root-finding operations.

Proof. The proof is by construction. The system of Euler equations for the problem complying with Assumptions A1 and A2 is composed of M equations[10]

$$u'_{x^j}(x_t, s_t) = \beta \sum_{k=j}^M \frac{\partial f^k(t)}{\partial x^j_t} E\left[\sum_{i=1}^M \frac{\partial \chi^i}{\partial f^k} \mathcal{L}^i_{t+1} \middle| s_t, f(t)\right], \tag{9}$$

where $j \in \{1, \dots, M\}$ and $\mathcal{L}^i_{t+1}$ is a linear function of marginal utilities evaluated at the optimal decisions in period $t+1$,[11]

$$\mathcal{L}^i_{t+1} = \frac{\frac{\partial f^i(t+1)}{\partial X^i_{t+1}}}{\frac{\partial f^i(t+1)}{\partial x^i_{t+1}}} u'_{x^i}(x_{t+1}, s_{t+1}) + \sum_{k=i+1}^M \left(\frac{\frac{\partial f^k(t+1)}{\partial X^k_{t+1}}}{\frac{\partial f^k(t+1)}{\partial x^k_{t+1}}} - \frac{\frac{\partial f^{k-1}(t+1)}{\partial X^{k-1}_{t+1}}}{\frac{\partial f^{k-1}(t+1)}{\partial x^{k-1}_{t+1}}} \right) u'_{x^k}(x_{t+1}, s_{t+1}). \tag{10}$$

To recover the policy functions δ_t, $t \in 1, \dots, T-1$ the multi-dimensional EGM algorithm performs the following steps:

1. Initialize the backward induction with policy function $\delta_T : (X_T, s_T) \to x_T$. Proceed backwards through time periods $t \in T-1, \dots, 1$:
2. Loop over all points s_t, for each point s_t:
3. Proceed through the points of the fixed M-dimensional grid over post-decision states $f(t) = \big(f^1(t), \dots, f^M(t)\big)$:
4. Conditional on the particular values of post-decision states $\hat{f} = \big(\hat{f}^1, \dots, \hat{f}^M\big)$, compute the right hand sides $\big(\text{RHS}^1(\hat{f}), \dots, \text{RHS}^M(\hat{f})\big)$ of the Euler equations (9) using (6) to find the values of the next period continuous states X_{t+1} and calculating the expectations over the idiosyncratic shocks ξ_{t+1}[12] and the transition probabilities of exogenous states s_{t+1}. The value $\mathcal{L}^i_{t+1}$ in (9) can be both computed directly using the analytical formula (10) or interpolated over the endogenous grid in $t+1$, depending on whether the computational cost or the accuracy of the solution is prioritized. Assumption A2 enables integrating over the transition probabilities of the exogenous state process. The optimal values of decision variables x_{t+1} are found using M-dimensional interpolation of the period $t+1$ policy function δ_{t+1}.
5. Solve the Euler equations in the system (9) for x_t in the order of permutation k^1. Namely start by computing $x^j_t = \left(\frac{\partial u(x_t, s_t)}{\partial x^i_t}\right)^{-1} \big(\text{RHS}^i(\hat{f}), s_t\big)$, where $i = k^1_1$ and $j = k^2_1$, and then compute the rest of the elements of vector x_t in the order of permutation k^2, substituting previously computed solutions into every equation. The necessary analytical inverses of the partial derivatives exist due to Assumption A4, allowing for direct computations of the solutions of all Euler equations in the system. Due to strong monotonicity of partial derivatives of the utility function, each equation has exactly one solution, thus ensuring that the system as whole has exactly one solution.[13]

9 Permutation matrix J is obtained from the identity matrix by permuting its rows; the product JA is given by same permutation of rows, and product AJ by same permutation of columns of matrix A.

10 See Appendix for the derivation.

11 It follows that $\mathcal{L}^i_{t+1}$ is a negative of the partial derivative of the period $t+1$ value function with respect to X^i_{t+1}, see (18) in the Appendix.

12 Both quadrature, Monte Carlo or other methods can be used, see also Judd (1998, pp. 251–331).

13 I assume existence of *a solution* for every equation (9), otherwise the solution set of problem (8) is empty.

6. Compute the values of continuous states X_t^j, $j \in \{1,\ldots,M\}$, from Eqs. (7) by forward substitution. The required implicit functions are analytic under Assumption A3.
7. Continue through all the points of the grid on post-decision states in step 3, and by collecting the associated pairs (X_t, x_t) obtain (an approximation of) the current period policy function $\delta_t : (X_t, s_t) \to x_t$ for given s_t.
8. Continue with the rest of the discrete state points s_t in step 2, and the rest of time periods in step 1 to recover the policy functions $\delta_t : (X_t, s_t) \to x_t$ for all s_t and t. The algorithm concludes after period $t = 1$ policy function is found (in the finite horizon case) or when two successive approximations of the policy functions are sufficiently close (in the infinite horizon case).

None of the steps 1–8 require root-finding operations. Yet, the algorithm consequently solves the first order conditions of the maximization problem in (8), and because by Assumption A5 the problem is concave, finds the solution to the dynamic optimization problem. □

4. Discussion and conclusions

The model of *multiple stock dynamics* is a typical member of the class of triangular dynamic optimization problems developed in this paper. Total wealth, various disaggregated financial assets, investments in housing and durables, overall and specific health, human capital and job specific human capital—are examples of stocks which dynamics can be modeled in various combinations with a triangular dynamic optimization problem.

The model of human capital accumulation by Imai and Keane (2004) is an example of two-dimensional triangular problem where choices of consumption C_t and labor supply h_t govern the dynamics of wealth A_t and human capital K_t. The transition rules are given by

$$\begin{aligned} K_{t+1} &= \xi_{t+1}\Big(k_0 + \delta K_t + g_1(t)(B_1+K_t) \\ &\qquad \times \big[(h_t+d_1)^\alpha - B_2(h_t+d_1)\big]\Big), \\ A_{t+1} &= (1+r)A_t + RK_t h_t - C_t, \end{aligned} \tag{11}$$

where r is interest rate, R rental rate of a unit of human capital, $\xi_{t+1} \in \mathbb{R}^1$ is idiosyncratic wage shock, and $(k_0, \delta, g_1(t), d_1, \alpha, B_1, B_2)$ are parameters (Imai and Keane, 2004, pp. 605–608). With $X_t = (K_t, A_t)$, $\chi^1 = \xi_{t+1} f^1(K_t, h_t)$ and $\chi^2 = f^2(K_t, h_t, A_t, C_t)$ Assumptions A1–A3 are clearly satisfied. The utility function in the model is concave and additively separable, thus satisfying A4 and A5. With the proposed approach, the models like this can be further enriched by including additional dimensions, for example health process or job-specific human capital, at relatively low computational cost.

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Appendix. Derivation of the Euler equations

Consider the problem given by (6) and (8). Due to A1–A2 the expectations can be written as $E\big(\cdot|s_t, f(t)\big)$. The first order and envelope conditions in (8) are, $j \in \{1,\ldots,M\}$

$$\begin{aligned} & u'_{x^j}(x_t, s_t) + \beta \sum_{i=1}^M E\left[\frac{\partial V_{t+1}(X_{t+1}, s_{t+1})}{\partial X^i_{t+1}}\right. \\ &\qquad \times \left.\sum_{k=j}^M \left(\frac{\partial \chi^i}{\partial f^k}\frac{\partial f^k(t)}{\partial x^j_t}\right)\right| s_t, f(t)\bigg] = 0 \\ &\Longrightarrow u'_{x^j}(x_t, s_t) + \beta \sum_{k=j}^M \frac{\partial f^k(t)}{\partial x^j_t} \\ &\qquad \times E\left[\sum_{i=1}^M \frac{\partial V_{t+1}(X_{t+1}, s_{t+1})}{\partial X^i_{t+1}}\frac{\partial \chi^i}{\partial f^k}\right| s_t, f(t)\bigg] = 0. \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial V_t(X_t, s_t)}{\partial X^j_t} &= \beta \sum_{i=1}^M E\left[\frac{\partial V_{t+1}(X_{t+1}, s_{t+1})}{\partial X^i_{t+1}}\right. \\ &\qquad \times \left.\sum_{k=j}^M \left(\frac{\partial \chi^i}{\partial f^k}\frac{\partial f^k(t)}{\partial X^j_t}\right)\right| s_t, f(t)\bigg] \\ &= \beta \sum_{k=j}^M \frac{\partial f^k(t)}{\partial X^j_t} \\ &\qquad \times E\left[\sum_{i=1}^M \frac{\partial V_{t+1}(X_{t+1}, s_{t+1})}{\partial X^i_{t+1}}\frac{\partial \chi^i}{\partial f^k}\right| s_t, f(t)\bigg]. \end{aligned} \tag{13}$$

Combination of (12) and (13) for $j = M$ gives

$$\frac{\partial V_t(X_t, s_t)}{\partial X^M_t} = -\phi^M_t u'_{x^M}(x_t, s_t), \tag{14}$$

where $\phi^M_t = \frac{\partial f^M(t)}{\partial X^M_t} \Big/ \frac{\partial f^M(t)}{\partial x^M_t}$. Taking the differences in (12) and (13) between $j \in \{1,\ldots,M-1\}$ and $j+1$ gives

$$\begin{aligned} & u'_{x^j}(x_t, s_t) - u'_{x^{j+1}}(x_t, s_t) + \beta \frac{\partial f^j(t)}{\partial x^j_t} \\ &\qquad \times E\left[\sum_{i=1}^M \frac{\partial V_{t+1}(X_{t+1}, s_{t+1})}{\partial X^i_{t+1}}\frac{\partial \chi^i}{\partial f^j}\right| s_t, f(t)\bigg] = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} & \frac{\partial V_t(X_t, s_t)}{\partial X^j_t} - \frac{\partial V_t(X_t, s_t)}{\partial X^{j+1}_t} \\ &\quad = \beta \frac{\partial f^j(t)}{\partial X^j_t} E\left[\sum_{i=1}^M \frac{\partial V_{t+1}(X_{t+1}, s_{t+1})}{\partial X^i_{t+1}}\frac{\partial \chi^i}{\partial f^j}\right| s_t, f(t)\bigg], \end{aligned} \tag{16}$$

which leads to $j \in \{1,\ldots,M-1\}$

$$\frac{\partial V_t(X_t, s_t)}{\partial X^j_t} = \frac{\partial V_t(X_t, s_t)}{\partial X^{j+1}_t} + \phi^j_t\big[u'_{x^{j+1}}(x_t, s_t) - u'_{x^j}(x_t, s_t)\big]. \tag{17}$$

Linearizing the recursive expression (17) using (14) as the base, we have

$$\frac{\partial V_t(X_t, s_t)}{\partial X^j_t} = -\phi^j_t u'_{x^j}(x_t, s_t) - \sum_{i=j+1}^M (\phi^i_t - \phi^{i-1}_t) u'_{x^i}(x_t, s_t). \tag{18}$$

The system of Euler equations (9) follows from (12) and (18).

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