Hụi:
A Case Study of
A Sequential Double Auction of Capital

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Abstract

For many immigrants, raising capital through conventional financial institutions (such as banks) is difficult, even impossible. In such circumstances, alternative institutions are often employed to facilitate borrowing and lending within the immigrant community. Using the theory of non-cooperative games under incomplete information, we analyze one such institution—the hụi—which is essentially a sequential, double auction among the participants in a cooperative. Within the symmetric independent private-values paradigm, we construct the perfect Bayesian equilibrium of a sequential, first-price, sealed-bid auction game, and then use this structure to interpret field data gathered from a sample of hụi held in Melbourne, Australia during the early 2000s.

Keywords: Auctions; rotating savings and credit association; ROSCA; institutions in economic development.

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1 Introduction and Motivation

Among Vietnamese immigrants, in countries such as Australia and New Zealand, there exists a healthy distrust of formal institutions, including banks. Moreover, even when this distrust can be overcome, many of these immigrants simply do not have long enough credit histories to qualify for conventional small-business loans. Yet one of the principal ways in which immigrants accumulate capital is by starting and growing small businesses, such as laundries and restaurants as well as neighbourhood markets and repair shops. What to do? Using experience gathered by their ancestors over generations in their home countries, these immigrants often employ alternative institutions that allow them to borrow and to lend among themselves within their communities.

One such institution is the hụi which, as we shall argue later, is essentially a sequential double auction.[1] A hụi allows a group of immigrants to pool scarce financial resources, and then to allocate these resources among potentially lucrative investments. In a typical hụi, some N people form a cooperative; N can range from twenty to sixty. Each participant in the hụi must deposit a sum u with the *banker*, typically a trusted elder in the community. In many of the hụi for which we have data, u is between \$200 and \$500. On the final day that funds are collected, and in each month thereafter, until each participant has had his turn to win, a first-price, sealed-bid auction is held to determine the implicit interest rate paid; after the winner has been determined, only the winning bid is revealed. We refer to each auction in the hụi as a *round* of the hụi.

In each round, a participant must choose a bid variable (denoted below by s) which is the discount below the deposited amount u he would be willing to accept from each remaining participant in that round. The participant in round t who has submitted the highest bid w_t wins that round of the hụi, and is excluded from participating in all subsequent rounds. In exchange for relinquishing his right to participate in future rounds, the winner receives a sum that is the product of the number of participants in the round and the discounted sum outstanding, plus the deposit from each of the previous winners as well as his initial contribution to the hụi from the banker: to wit, in round t, a winner receives the capital $[t \times u + (N - t) \times (u - w_t)]$.

Sound confusing? Perhaps the following example can clear things up. For simplicity, suppose that N is four, while u is \$300, and that these four participants tender the following first-round bids: \$12, \$10, \$8 and \$6. In this event, the first bidder in the sequence (the one who bid \$12) wins this round and he receives \$1,134: \$288 from each of the other three participants in the round, plus \$300 from the banker as there are no previous winners in the first round. The winner can now use this capital to finance some business venture.

In the second round of the hụi, held a month later, the remaining three participants must decide what discount each would be willing to offer. For simplicity, suppose none of the bids has changed, so \$10, \$8, and \$6 remain the standing bids.[2] In this event, the winner receives \$1,180: each of his remaining two opponents pays him \$290, while the winner of the first round must pay

[1] We believe that the word hụi is pronounced like the *h* in <u>h</u>at along with the *uoy* in b<u>uoy</u>, but we have been informed by reliable native speakers of Vietnamese that this is a coarse approximation at best. In any case, we pronounce hụi as if it were the word *hoi* in English. The word hụi is probably derived from Chinese, where the *Guangyun* romanization of this particular form is *Piao-Hui*, bidding hui, to be distinguished from the *Lun-Hui*, rotating hui, and *Yao-Hui*, dice-shaking hui. These institutions are examples of Rotating Savings and Credit Associations and are related to credit cooperatives which evolved later in such countries as Germany during the nineteenth century. Robert T. Anderson [1966] has noted other English terms to describe this institution such as *contribution club*, *slate*, *mutual lending society*, *pooling club*, *thrift group*, and *friendly society*. We postpone our discussion of this until later in the paper.

[2] In the theory developed below in section **??**, we demonstrate that bids should, in fact, vary over rounds of the hụi, but we abstract from that here.

Table 1: Net Cash Flow

Bidder/Round	Banker	1	2	3	4	Final
Banker	$1,200	−$300	−$300	−$300	−$300	$0
1	−300	1,164	−300	−300	−300	−36
2	−300	−288	1,180	−300	−300	−8
3	−300	−288	−290	1,192	−300	14
4	−300	−288	−290	−292	1,200	30

him $300 and he, of course, gets $300 from the banker.

Consider now the third round of the hụi, held another month later with only two participants remaining. Again, suppose that the discounts are unchanged at $8 and $6. In this event, the winner is the first bidder who receives $292 from the other participant, plus $300 from each of the winners of the first and second rounds and, again, $300 from the banker—in short, a total of $1,192.

In the final round of the hụi, held another month later, the sole remaining participant gets $1,200: $300 from each of the previous three winners for a total of $900, plus $300 from the banker. The last remaining participant has no incentive to tender a positive bid. What would be the point? He faces no competitors; the reserve prices in hụi are zero.

In table 1, we present the payment streams for the banker as well as each of the four participants in the above example. As one can see from the net positions in the column headed by "Final" on the far right of table 1, some of the participants are net borrowers (for example, those with negative net cash positions), while other participants are net lenders (for example, those with positive net cash positions). It is in this sense that we argue that the hụi is effectively a double auction: as an economic institution, the hụi enables one side of the market to borrow from the other. Like many double auctions, the trades are executed sequentially over time. What is somewhat different in the hụi is that offers to lend are only implicit. Those participants with less attractive investment opportunities do not quote offers to lend, but simply bid less than those who have better investment opportunities. In short, those participants with higher-valued rates-of-return win the early rounds of the hụi, while those with lower-valued rates-of-return win later rounds. Under certain conditions, which we outline below, the hụi guarantees an efficient allocation of the scarce capital available to the cooperative.

The hụi obviously facilitates inter-temporal smoothing, and appears to be implementable under primitive market conditions, such as those present in developing countries. Presumably, the structure of the hụi accommodates an informational asymmetry that conventional banks cannot. Within immigrant communities, those of the same ethnic group typically have better information concerning what their fellow countrymen are doing than would the loan officer on Main Street. In addition, within these communities, the hụi is perhaps the only way in which some liquidity-constrained individuals are able to borrow small to medium amounts of capital. In our data from a suburb of Melbourne, Australia the average hụi has around forty members, each depositing as much as $500, so loans are on the order of $20,000 for three to four years. During our field work, we learned that those who default in a hụi are castigated within the community—cut-off from borrowing in the future. In addition, were a defaulter to leave the community, close (and even somewhat distant) relatives, who remain in the community, typically feel the need to defend clan honour by paying off a defaulter's debt. Thus, it is highly uncommon for participants in a hụi to default.[3]

The hụi that we study is a special case of a class of institutions referred to in the literature as *Rotating Savings and Credit Associations* (ROSCAs); these institutions have been studied extensively, first by anthropologists and sociologists, and then by economists. Most prior analyses have focused on a variety we shall refer to as the *household* ROSCA; it has been argued that one reason this mechanism exists is to help people save for important, one-time, indivisible purchases, such as consumer durables. Another variety of ROSCA is one we refer to as the *business* ROSCA; we believe that this mechanism exists to help small-business owners obtain capital for investments, often when it is costly or impossible to do so through other means.

Perhaps the first in-depth study of a ROSCA was completed by Sidney D. Gamble [1944], who investigated what he referred to as a *Chinese mutual savings society*, one example of which is exactly like the hụi we have described above. In *Guangyun* Chinese, the hụi we described above is referred to as *Piao-Hui*—bidding hui. Gamble, in fact, developed his anecdote involving Mr. Chang who lived in Hopei Province in China using the business bidding ROSCA (*Piao-Hui*) as a motivation. Later, William R. Bascom [1952] described another type of ROSCA where no bidding occurs, which is referred to as the *esusu* in the Yoruba of Nigeria, Africa. Under this institution, the winner is determined by the president of the *esusu*, who selects the order of rotation. Thus, Bascom focused his attention on the household, pre-determined rotation-order ROSCA in Africa. Clifford Geertz [1962] studied ROSCAs, conducted in eastern Java, that follow a pre-determined rotation order; he reported that the institution is referred to there as *arisan*—literally "cooperative endeavor" or "mutual help." In *Guangyun* Chinese, the *esusu* and the *arisan* would be referred to as *Lun-Hui*—rotating hui.

Shirley G. Ardener [1964] conducted an extensive study of ROSCAs in different regions of Africa, comparing and contrasting the different forms. The major remaining alternative way to determine the winner of any round is by lot drawn at random from the remaining participants. In *Guangyun* Chinese, this would be referred to as *Yao-Hui*—dice-shaking hui. In his study of Mexican-American immigrants in California, Donald V. Kurtz [1973] has reported that this institution is referred to as the *cundina*, while Kurtz and Margaret Showman [1978] have reported that it is referred to as the *tanda* in Mexico, where the word means "alternative order." Frits J. A. Bouman [1995] has provided a glossary of other names used in various countries throughout the world.

Several researchers (including Simon Ottenberg [1968], David H. Penny [1968], David Y. H. Wu [1974], and Girma Begashaw [1978]) have documented the importance of ROSCAs in societies with non-existent or limited formal financial institutions. In fact, Wu [1974] has attributed the financial success of the overseas Chinese in Papua New Guinea (prior to self-governance in 1973), in part, to the business bidding ROSCA (*Piao-Hui*) because it allowed these immigrants to circumvent the discriminatory lending practices of Europeans at the time. For these reasons, and others, economists have also been interested in ROSCAs.

One of the first researchers to focus on the economic importance of ROSCAs was Phillipe Callier [1990], who argued that the household ROSCA is Pareto improving because it allows consumers, on average, to get an indivisible consumer durable earlier than in the absence of the institution. Subsequently, Timothy J. Besley, Stephen Coate, and Glenn C. Loury [1993, 1994] have provided elegant and in-depth theoretical analyses of ROSCAs, focusing mostly on the randomly-rotating household variety, where they considered consumers who seek to make one-time pur-

[3]Elsewhere, Thomas Cope and Donald V. Kurtz [1980] have investigated default in the context of a hụi-like mechanism conducted in Mexico. We discuss this mechanism in further detail below.

chases of indivisible durable goods, such as bicycles and the like. Rogier van den Brink and Jean-Paul Chavas [1997] have also contributed to this literature with special reference to Africa. Abhijit V. Banerjee, Besley, and Timothy W. Guinnane [1994] constructed a theoretical model and developed an empirical test of a related institution, the credit cooperative, which developed in Germany in the nineteenth century; Michael Prinz [2002] has also contributed to this literature.

Besley and Alec R. Levenson [1996] as well as Levenson and Besley [1996] have reported careful and detailed empirical analyses of household ROSCAs in Taiwan, investigating the importance of these informal credit institutions in helping people who have perceived limited credit worthiness to make large purchases of consumer durables. Charles W. Calomiras and Indira Rajaraman [1998] have focused on an alternative role of ROSCAs, at least in India: instead of an institution that just facilitates the purchase of large indivisible consumer goods, it is an institution that also provides insurance against unforeseen events, such as funerals. Alternatively, by focusing on household ROSCAs with random rotation, Siwan Anderson and Jean-Marie Baland [2002] have emphasized the importance of the institution in Kenya, Africa to help women protect their savings from their husbands, some of whom have been known to spend surplus funds on cigarettes and alcohol, instead of saving for their children's educations, for example.

In this paper, we investigate business bidding ROSCAs, which we feel have been relatively neglected in the literature, perhaps because they are computationally somewhat tedious, particularly in environments containing private information. Ping Sing Kuo [1993] was the first to investigate bidding ROSCAs using modern game-theoretic methods. Subsequently, Jens Kovsted and Peter Lyk-Jensen [1999] couched the solution in terms of dynamic programming with a finite horizon. In the model of Kovsted and Lyk-Jensen [1999], however, the discount rate is a fixed constant that is different from the rate of return of a particular bidder. What does a fixed discount rate imply? Well, that agents can borrow and lend potentially infinite amounts at a fixed interest rate. In many developing countries, no option to borrow at a fixed discount rate exists. Also, the bid functions derived by Kovsted and Lyk-Jensen [1999] are just a sequence of bids; in short, information revealed in earlier rounds is ignored. Thus, for example, in the second round, a participant's optimal bid is not a number to be interpreted as his bid conditional on not winning in the first round. Instead, in the second round, a participant would want to condition his new bid on the observed winning bid of the first round. Thus, a second-round bid is a function of the winning bid of the first. In general, a bid is a function of the past history of winning bids as well as a bidder's own rate-of-return. Kuo [2002] later extended his research to examine the effects of default.

Most recent research concerning bidding ROSCAs, particularly empirical research, has been undertaken by Stefan Klonner—specifically, that first reported in his doctoral dissertation, Klonner [2001], and then in Klonner [2002, 2003a,b, 2008] as well as Klonner and Ashok S. Rai[2005]. In the work of Klonner that is closest to ours, he examines outcomes at second-price auctions because that institution generated his data. In our work, we investigate first-price auctions, which are somewhat different, at least technically. As we develop our theoretical and empirical framework below, we shall compare and contrast the work of these researchers with ours.

In this paper, we make at least three important contributions: First, we present a detailed game-theoretic analysis of an extant ROSCA that has not been studied as extensively as other ROSCAs. In the course of this analysis, we discuss some limitations of previous research concerning this ROSCA, and then rectify them. Second, within the theory of auctions, interdependent values have been studied in considerable generality. For example, see the pathbreaking research of Roger B.Myerson [1981] as well as Paul R. Milgrom and Robert J. Weber [1982]. Among practitioners who employ auctions, however, a common perception is that this generality translates into the outcome that anything can happen—a notion akin to a result in general equilibrium theory, where all one really knows about the excess demand correspondence is that it satisfies Walras' law and is homogeneous of degree zero in prices. By employing a standard auction model, within the symmetric private-values paradigm, in which the outcome of interdependence is quite explicit, we illustrate the rich patterns that can obtain in the presence of even the simplest interdependence. One can view this as a negative result, or the *raison d'etre* for conducting structural-econometric empirical research, namely, to reduce the scope of possible outcomes in an effort to inform policy. Third, we present empirical evidence concerning apparent rates-of-return to small investments—evidence suggesting that even small amounts of capital can garner remarkable monetary benefits.

In the remaining six sections of this paper, we present a summary of the following research: in section **??**, we use the theory of non-cooperative games under incomplete information to construct a series of simple theoretical models of the hụi as a sequential first-price, sealed-bid auction within the symmetric independent private-values paradigm. In that section, we also investigate some properties of the equilibrium bid and optimal value functions and then use solved numerical examples to illustrate key properties of the perfect Bayesian equilibrium that we have constructed. We relegate to an appendix our theoretical investigation of hụi in which two types of economic agents bid—those we refer to as *borrowers*, and those we refer to as *lenders*. Subsequently, in section 3, we describe data collected for a sample of hụi held in the Melbourne suburb Footscray, during the early 2000s, while in section 4, we use the theoretical model of section **??** to develop an empirical specification. Specifically, in section 4, we demonstrate that our extension of the standard first-price, sealed-bid auction model, within a symmetric independent private-values environment, is non-parametrically identified, at least in the second-to-last round of the hụi. Unfortunately, during our field work, we were only able to gather a very small sample of twenty-two hụi, so non-parametric estimation is out of the question. Thus, in order to proceed, we are forced to make an important parametric assumption—that the rates-of-return are distributed according to a lognormal random variable. In section 5, we report empirical results obtained by confronting the structural econometric model of section 4 with the field data from section 3, while in section 6, we investigate two simple policy experiments—one a shift to a lottery, which is how the dice-shaking version of the hụi is implemented in Mexico as well as many other parts of the world, while the other involving a shift to a second-price, sealed-bid format. Any other details too cumbersome to be included in the text of the paper (for example, our proof that a model of a second-price, sealed-bid hụi is non-parametrically unidentified) have been collected in the appendix at the end of the paper, too.

2 Theoretical Model

Before we develop our formal theoretical model, we devote some space to describing the environment within which we imagine economic actors making decisions. Consider a community in which many economic actors get investment opportunities. In this community, we take seriously the maintained assumption that there are no alternative ways in which to borrow or to lend, so our model has no constant rate of time preference. Implicit in the assumption of a constant rate of time preference is the fact that economic actors can borrow and lend at this rate. We imagine a world in which, if economic actors cannot get capital, then their potential investment opportunity produces nothing. In addition, there is no way to get a rate-of-return on savings. Thus, our framework is different from Kovsted and Lyk-Jensen [1999] as well as Klonner [2008] who assumed a constant discount rate.

Within this environment, bankers begin hụi. The motivation of hụi bankers is unclear as they do not appear to benefit financially from organizing hụi, but they appear to bear some risk. For example, keeping large sums of cash at one's home invites home invasion. Members of communities in which hụi are used extensively claim that the hụi bankers do it out of community spirit. We can neither confirm nor refute this claim. In fact, we remain silent on the motivation of hụi bankers.

Typically, however, bankers have a target number of participants in an hụi. The reasons bankers give for this target number can vary a bit, but the main reason appears to be that the number of participants in an hụi determines the duration of the hụi: bankers do not seem to want to manage hụi whose durations are longer than about five years, so fifty or sixty is usually the maximum number of participants chosen by bankers.

In our imagined environment, economic actors encounter investment opportunities that they would like to exploit, but for which they have insufficient capital to fund—e.g., the one-time purchase of an expensive machine whose seller is unwilling to extend credit. Based on the rate-of-return to his potential investment opportunity, an economic actor joins an hụi. When he joins the hụi, the number of other participants in the hụi as well as the terms of the hụi are complete information. Unknown to him are the rates-of-return of the potential investment opportunities of his opponent participants in the hụi: like the rate-of-return to his potential investment opportunity, these are the private of his opponent participants in the hụi.

Within this environment, we assume that a participant seeks to do the best he can given the limited resources at his disposal. All economic actors are assumed to make the decision to participate in an hụi freely. In our empirical framework, we impose the restriction that all participants in an hụi satisfy an individual-rationality constraint concerning their rates-of-return. Having chosen to participate in an hụi, we assume that the objective of a participant is to maximize the expected monetary return from the duration of the hụi, conditional on the behaviour of his opponents. Because borrowing at a financial institution is really not an option for hụi participants and because many in the community are reticent to deposit money in banks, the opportunity cost is effectively zero. Thus, for any participant, all decisions are made *vis-à-vis* the rate-of-return of his potential investment opportunity.

2.1 First Model: One-Time Draws

With this imagined environment as a backdrop, we should now like to develop a model of equilibrium behaviour in an hụi. Consider a set $\{0, 1, 2, \ldots, N\}$ of $(N + 1)$ players: the banker plus N potential borrowers and lenders. At the beginning of the hụi-auction game, each participant deposits u with player 0, the banker. We assume that each participant $n = 1, 2, \ldots, N$ receives an independent random draw R from a cumulative distribution function of returns $F_R^0(r)$ that has support $[\underline{r}, \bar{r}]$, with corresponding probability density function $f_R^0(r)$ that is strictly positive on $[\underline{r}, \bar{r}]$. We interpret participant n's draw r_n as that participant's *rate-of-return* on an investment opportunity, and assume that this draw is his private information in the sense that each participant knows his draw, but not those of his opponents. All that participants know about the draws of their opponents is that those draws are independent and from the same distribution $F_R^0(\cdot)$. Initially, we assume that the rates-of-return for the hụi are drawn just once, in the initial period, when the total sum Nu is deposited.[4] In that period, and in each period thereafter, an auction is held to decide who will win that round of the hụi and what bid discount will be paid. In each round of the hụi, the auction is conducted using the first-price, sealed-bid format, after which only the winning bid is revealed.

For an hụi having N rounds, we introduce the following notation to denote the ordered rates-of-return of participants, from largest to smallest:

$$r_{(1)} \geq r_{(2)} \geq \cdots \geq r_{(N)}$$

and

$$w_1,\ w_2,\ \cdots\ ,\ w_{N-1},\ w_N = 0$$

to denote the winning bids in the N rounds of the hụi. We have imposed the universally-observed outcome that w_N, the winning bid in the final round of all hụi, is always zero. In addition, although this is rarely stated, the reserve price in any round of an hụi is also zero.

Given the description of the hụi in the introduction, we can deduce that participants will exit the hụi acccording to their rate-of-return draws—the highest first, then the second-highest, and so forth. Note, too, that, once the first round allocation has been determined, then the decision problem changes: in short, the highest-valued rate-of-return participant has been removed from the pool. From a decision-theoretic perspective, however, none of the remaining $(N-1)$ participants has learned anything about the rates-of-return of their remaining opponents, save that they are all less than $r_{(1)}$. In short, the remaining rate-of-return draws, conditional on having observed the highest-valued draw, are independent as well as identically distributed.[5]

How does a participant determine how much to bid—effectively, in which round of the hụi to exit? We can couch the solution to this problem in terms of the solution to a dynamic programme. For a representative participant, this dynamic programming problem has two state variables: t, the round of the hụi, and r, the realization of his draw from the distribution of rates-of-return. We seek to construct a sequence of optimal policy (equilibrium bid) functions $\{\sigma_t\}_{t=1}^{N}$. In round t, the optimal policy function σ_t maps the rate-of-return state R into the real line. We begin by describing the problem intuitively.

In round t, the value function of participating in this round as well as all later ones can be decomposed into the expected value of winning the current round plus the discounted expected continuation value of the game, should one lose this round. Thus, the value function of a participant having rate-of-return draw r can be written as

$$V(r,t) = \max_{<s>} \left[tu + (N-t)(u-s) - u \sum_{i=1}^{N-t} \frac{1}{(1+r)^i} \right] \Pr\left(\text{win}\middle| s, w_1, w_2, \ldots, w_{t-1}\right) +$$
$$\text{Discounted Expected Continuation Value.}$$

[4] An alternative assumption, which we shall investigate later, is that, in each successive round, the remaining participants get a new sample of independent draws from $F_R^0(\cdot)$. Yet a third assumption would involve shocks to the initial draws over time for the remaining participants.

[5] How can the highest-valued draw $r_{(1)}$ be deduced? Well, in the first round, it is

$$r_{(1)} = \sigma_1^{-1}(w_1)$$

where w_1 is the winning bid in the first round, while $\sigma_1(\cdot)$ is the symmetric perfect Bayesian equilibrium bid function for first round, which we shall construct later in this section.

Here, the $tu + (N-t)(u-s)$ represents the capital raised in the current period if the hụi has been won, while the $-u\sum_{i=1}^{N-t}(1+r)^{-i}$ represents the current-valued obligations of what must be repaid, discounted using the participant's cost-of-funds, r, the rate-of-return on his potential investment.

We construct the $\{\sigma_t\}_{t=1}^{N}$ as well as $V^*(r,t)$ recursively. The solution to the bidding problem in the last round is easily found: since the reserve price in each round is zero, because he faces no competitors, the last participant need only bid zero for any rate-of-return. Thus, the optimal policy function, for all feasible R, is

$$\sigma_N(r) = 0.$$

Hence, in the last round, N, for any feasible value of R,

$$V^*(r,N) = Nu.$$

Consider now a representative participant in the second-to-last round who has rate-of-return r and who faces only one other opponent. What is

$$\Pr\left(\mathsf{win}\middle|s, w_1, w_2, \ldots, w_{N-2}\right)?$$

Suppose the participant's opponent is using a monotonically increasing function $\hat{\sigma}_{N-1}(r)$. The participant wins when his bid is higher than his opponent's because his rate-of-return is higher than the sole remaining opponent—in short,

$$\Pr\left(\mathsf{win}\middle|s, w_1, w_2, \ldots, w_{N-2}\right) = \frac{F_R^0\left[\hat{\sigma}_{N-1}^{-1}(s)\right]}{F_R^0\left[r_{(N-2)}\right]} \equiv G_R^{N-2}\left[\hat{\sigma}_{N-1}^{-1}(s)\right]$$

where $G_R^{N-2}(\cdot)$ has corresponding probability density function $g_R^{N-2}(\cdot)$ on $[\underline{r}, r_{(N-2)}]$. Why is the upper bound of support $r_{(N-2)}$? Well, to get to this round of the game, all of the higher types must have already won. Of course, knowing the rates-of-return of all those types is unnecessary: $r_{(N-2)}$, the rate-of-return of the winner in the previous round, round $(N-2)$, is sufficient.

What structure does the "Discounted Expected Continuation Value" have? Well, in the last round of the hụi,

$$V^*(r,N) = Nu,$$

so one part is

$$\frac{V^*(r,N)}{(1+r)} = \frac{Nu}{(1+r)},$$

the discounted value of the last round of the hụi. Also, if the participant loses, then he also earns $(W_{N-1} - u)$, which is the winning bid of his opponent in the second-to-last round of the hụi, minus what that participant contributed to the hụi in that round. Of course, W_{N-1} is a random variable, which always exceeds s, the choice variable of the bidder, because the bidder lost after tendering s. Thus,

$$V(r,N-1) = \max_{<s>} \left[(N-1)u + (u-s) - \frac{u}{(1+r)}\right] G_R^{N-2}\left[\hat{\sigma}_{N-1}^{-1}(s)\right] + \int_{\hat{\sigma}_{N-1}^{-1}(s)}^{r_{(N-2)}} \left(\left[\hat{\sigma}_{N-1}(x) - u\right] + \frac{Nu}{(1+r)}\right) g_R^{N-2}(x)\,\mathrm{d}x.$$

The above expression warrants some explanation. The integral on the right-hand side of the equal sign represents the discounted expected continuation value should the participant lose this round of the hụi. A participant loses this round when his opponent bid more than him because that opponent has an higher rate-of-return. Hence, the term $\hat{\sigma}_{N-1}^{-1}(s)$ in the lower bound of integration. The term $g_R^{N-2}(x)$ represents the probability density function of the rate-of-return of the opponent.

The following first-order condition is a necessary condition for an optimum:

$$\frac{\mathrm{d}V(r,N-1)}{\mathrm{d}s} = \left[(N-1)u + (u-s) - \frac{u}{(1+r)}\right] g_R^{N-2}\left[\hat{\sigma}_{N-1}^{-1}(s)\right] \frac{\mathrm{d}\hat{\sigma}_{N-1}^{-1}(s)}{\mathrm{d}s} - \\ G_R^{N-2}\left[\hat{\sigma}_{N-1}^{-1}(s)\right] - \left[(s-u) + \frac{Nu}{(1+r)}\right] g_R^{N-2}\left[\hat{\sigma}_{N-1}^{-1}(s)\right] \frac{\mathrm{d}\hat{\sigma}_{N-1}^{-1}(s)}{\mathrm{d}s} = 0.$$

In a symmetric perfect Bayesian equilibrium, $s = \hat{\sigma}_{N-1}(r)$ and, by monotonicity, $\mathrm{d}\hat{\sigma}_{N-1}^{-1}(s)/\mathrm{d}s$ equals $1/[\mathrm{d}\hat{\sigma}_{N-1}(r)/\mathrm{d}r]$, so the first-order condition above can be re-written as the following ordinary differential equation:

$$\frac{\mathrm{d}\sigma_{N-1}(r)}{\mathrm{d}r} + \frac{2f_R^0(r)}{F_R^0(r)}\sigma_{N-1}(r) = \left[\frac{r(N+1)u}{(1+r)}\right]\frac{f_R^0(r)}{F_R^0(r)}.$$

The initial condition is $\sigma_{N-1}(\underline{r})$ equal $\underline{r}u$: when a participant has the lowest possible rate-of-return, he bids the value of that rate-of-return in terms of the hụi deposit u. Later, we assume $\underline{r}$ is zero, so the initial condition will be zero. In any case,

$$\begin{aligned}\sigma_{N-1}(r) &= \frac{\int_{\underline{r}}^{r}\left[\frac{x(N+1)u}{(1+x)}\right]F_R^0(x)f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^2} + \underline{r}u \\ &= \left[\frac{\int_{\underline{r}}^{r}\left[\frac{x(N+1)}{(1+x)}\right]F_R^0(x)f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^2} + \underline{r}\right]u \\ &\equiv \sigma_{N-1,1}(r)u.\end{aligned}$$

In other words, $\sigma_{N-1}(\cdot)$ is homogeneous of degree one in u. Here, the notation $\sigma_{N-1,1}(\cdot)$ is used to denote that this is a bid function when u is one, a "unit" bid function. Also,

$$V^*(r,N-1) = \left[Nu - \sigma_{N-1}(r) - \frac{u}{(1+r)}\right]G_R^{N-2}(r) + \\ \int_r^{r_{(N-2)}}\left(\left[\sigma_{N-1}(x) - u\right] + \frac{Nu}{(1+r)}\right)g_R^{N-2}(x)\,\mathrm{d}x,$$

which is homogeneous of degree one in u, too.

Consider round $(N-2)$ next. Now,

$$V(r,N-2) = \max_{<s>}\left[(N-2)u + 2(u-s) - \sum_{i=1}^{2}\frac{u}{(1+r)^i}\right]G_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right]^2 + \\ \int_{\hat{\sigma}_{N-2}^{-1}(s)}^{r_{(N-3)}}\left(\left[\hat{\sigma}_{N-2}(x) - u\right] + \frac{V^*(r,N-1)}{(1+r)}\right)2G_R^{N-3}(x)g_R^{N-3}(x)\,\mathrm{d}x$$

where

$$G_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right] = \frac{F_R^0\left[\hat{\sigma}_{N-2}^{-1}(s)\right]}{F_R^0\left[r_{(N-3)}\right]},$$

with corresponding probability density function $g_R^{N-3}(\cdot)$ on $[\underline{r}, r_{(N-3)}]$. The above expression also warrants some explanation. In particular, what about the term $G_R^{N-3}[\cdot]^2$? In this round, there are two opponents, so $G_R^{N-3}[\cdot]^2$ is the cumulative distribution function of the maximum of their two rates-of-return, while $2G_R^{N-3}(\cdot)g_R^{N-3}(\cdot)$ is the probability density function of that maximum.

At a stationary point, the following first-order condition obtains:

$$\begin{aligned}\frac{\mathrm{d}V(r,N-2)}{\mathrm{d}s} = &\left[(N-2)u + 2(u-s) - u\sum_{i=1}^{2}\frac{1}{(1+r)^i}\right] \times \\ &2G_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right]g_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right]\frac{\mathrm{d}\hat{\sigma}_{N-2}^{-1}(s)}{\mathrm{d}s} - 2G_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right]^2 - \\ &\left[(s-u) + \frac{V^*(r,N-1)}{(1+r)}\right] 2G_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right]g_R^{N-3}\left[\hat{\sigma}_{N-2}^{-1}(s)\right] = 0.\end{aligned}$$

Again, in a symmetric perfect Bayesian equilibrium, $s = \hat{\sigma}_{N-2}(r)$ and, by monotonicity, $\mathrm{d}\hat{\sigma}_{N-2}^{-1}(s)/\mathrm{d}s$ equals $1/[\mathrm{d}\hat{\sigma}_{N-2}(r)/\mathrm{d}r]$, so the first-order condition above can be re-written as the following ordinary differential equation:

$$\frac{\mathrm{d}\hat{\sigma}_{N-2}(r)}{\mathrm{d}r} + \frac{3f_R^0(r)}{F_R^0(r)}\hat{\sigma}_{N-2}(r) = \left[(N+1)u - u\sum_{i=1}^{2}\frac{1}{(1+r)^i} - \frac{V^*(r,N-1)}{(1+r)}\right]\frac{f_R^0(r)}{F_R^0(r)}.$$

The solution has the same initial condition as above, so

$$\begin{aligned}\sigma_{N-2}(r) &= \frac{\int_{\underline{r}}^{r}\left((N+2)u - \frac{(1+x)}{x}\left[1 - \frac{1}{(1+x)^3}\right]u - \frac{V^*(x,N-1)}{(1+x)}\right)F_R^0(x)^2 f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^3} + \underline{r}u \\ &= \left[\frac{\int_{\underline{r}}^{r}\left((N+2) - \frac{(1+x)}{x}\left[1 - \frac{1}{(1+x)^3}\right] - \frac{V_1^*(x,N-1)}{(1+x)}\right)F_R^0(x)^2 f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^3} + \underline{r}\right]u \\ &\equiv \sigma_{N-2,1}(r)u\end{aligned}$$

where $\sigma_{N-2,1}(\cdot)$ is the unit bid function, and $V_1^*(\cdot,\cdot)$ is the "unit" value function. Here, we have used the fact that

$$\sum_{i=0}^{k}\frac{1}{(1+r)^i} = \frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{k+1}}\right].$$

In general, for rounds $t = 2, 3, \ldots, N-1$, we have

$$\begin{aligned}V(r,t) = \max_{<s>} &\left[tu + (N-t)(u-s) - \sum_{i=1}^{N-t}\frac{u}{(1+r)^i}\right]G_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t} + \\ &\int_{\hat{\sigma}_t^{-1}(s)}^{r_{(t-1)}}\left(\left[\hat{\sigma}_t(x) - u\right] + \frac{V^*(r,t+1)}{(1+r)}\right)(N-t)G_R^{t-1}(x)^{N-t-1}g_R^{t-1}(x)\,\mathrm{d}x\end{aligned}$$

where

$$G_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right] = \frac{F_R^0\left[\hat{\sigma}_t^{-1}(s)\right]}{F_R^0\left[r_{(t-1))}\right]},$$

with corresponding probability density function $g_R^{t-1}(\cdot)$ on $[\underline{r}, r_{(t-1)}]$. At a stationary point, the following first-order condition obtains:

$$\begin{aligned}\frac{\mathrm{d}V(r,t)}{\mathrm{d}s} = &\left[tu + (N-t)(u-s) - u\sum_{i=1}^{N-t}\frac{1}{(1+r)^i}\right]\times \\ &(N-t)G_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t-1} g_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right]\frac{\mathrm{d}\hat{\sigma}_t^{-1}(s)}{\mathrm{d}s} - (N-t)G_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t} - \\ &\left[(s-u) + \frac{V^*(r,t+1)}{(1+r)}\right](N-t)G_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t-1} g_R^{t-1}\left[\hat{\sigma}_t^{-1}(s)\right]\frac{\mathrm{d}\hat{\sigma}_t^{-1}(s)}{\mathrm{d}s} = 0,\end{aligned}$$

so the first-order condition can now be re-written as the following ordinary differential equation:

$$\frac{\mathrm{d}\hat{\sigma}_t(r)}{\mathrm{d}r} + \frac{(N-t+1)f_R^0(r)}{F_R^0(r)}\hat{\sigma}_t(r) = \left[(N+1)u - u\sum_{i=1}^{N-t}\frac{1}{(1+r)^i} - \frac{V^*(r,t+1)}{(1+r)}\right]\frac{f_R^0(r)}{F_R^0(r)}$$

which has solution

$$\begin{aligned}\sigma_t(r) &= \frac{\int_{\underline{r}}^{r}\left((N+2)u - \frac{(1+x)}{x}\left[1 - \frac{1}{(1+x)^{N-t+1}}\right]u - \frac{V^*(x,t+1)}{(1+x)}\right)F_R^0(x)^{(N-t)}f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^{(N-t+1)}} + \underline{r}u \\ &= \left[\frac{\int_{\underline{r}}^{r}\left((N+2) - \frac{(1+x)}{x}\left[1 - \frac{1}{(1+x)^{N-t+1}}\right] - \frac{V_1^*(x,t+1)}{(1+x)}\right)F_R^0(x)^{(N-t)}f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^{(N-t+1)}} + \underline{r}\right]u \\ &\equiv \sigma_{t,1}(r)u.\end{aligned}$$

The structure of the value function in the first round of the hụi is slightly different: in particular, because no previous bids have been observed, the upper bound of integration is now $\bar{r}$, the upper bound of support of R. Thus,

$$\begin{aligned}V(r,1) = \max_{<s>}\; &\left[u + (N-1)(u-s) - \sum_{i=1}^{N-1}\frac{u}{(1+r)^i}\right]F_R^0\left[\hat{\sigma}_1^{-1}(s)\right]^{N-1} + \\ &\int_{\hat{\sigma}_1^{-1}(s)}^{\bar{r}}\left(\left[\hat{\sigma}_1(x) - u\right] + \frac{V^*(r,2)}{(1+r)}\right)(N-1)F_R^0(x)^{(N-2)}f_R^0(x)\,\mathrm{d}x.\end{aligned}$$

In the equation above, we have noted that $G_R^0(\cdot)$ is simply $F_R^0(\cdot)$. At a stationary point, the following first-order condition obtains:

$$\begin{aligned}\frac{\mathrm{d}V(r,1)}{\mathrm{d}s} = &\left[u + (N-1)(u-s) - u\sum_{i=1}^{N-1}\frac{1}{(1+r)^i}\right]\times \\ &(N-1)F_R^0\left[\hat{\sigma}_1^{-1}(s)\right]^{N-2} f_R^0\left[\hat{\sigma}_1^{-1}(s)\right]\frac{\mathrm{d}\hat{\sigma}_1^{-1}(s)}{\mathrm{d}s} - (N-1)F_R^0\left[\hat{\sigma}_1^{-1}(s)\right]^{N-1} - \\ &\left[(s-u) + \frac{V^*(r,2)}{(1+r)}\right](N-1)F_R^0\left[\hat{\sigma}_1^{-1}(s)\right]^{N-2} f_R^0\left[\hat{\sigma}_1^{-1}(s)\right]\frac{\mathrm{d}\hat{\sigma}_1^{-1}(s)}{\mathrm{d}s} = 0,\end{aligned}$$

so the first-order condition can be re-written as the following ordinary differential equation:

$$\frac{\mathrm{d}\hat{\sigma}_1(r)}{\mathrm{d}r} + \frac{N f_R^0(r)}{F_R^0(r)}\hat{\sigma}_1(r) = \left[(N+1)u - u\sum_{i=1}^{N-1}\frac{1}{(1+r)^i} - \frac{V^*(r,2)}{(1+r)}\right]\frac{f_R^0(r)}{F_R^0(r)},$$

which has solution

$$\begin{aligned}
\sigma_1(r) &= \frac{\int_{\underline{r}}^{r}\left((N+2)u - \frac{(1+x)}{x}\left[1-\frac{1}{(1+x)^N}\right]u - \frac{V^*(x,2)}{(1+x)}\right)F_R^0(x)^{(N-1)}f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^N} + \underline{r}u \\
&= \left[\frac{\int_{\underline{r}}^{r}\left((N+2) - \frac{(1+x)}{x}\left[1-\frac{1}{(1+x)^N}\right] - \frac{V_1^*(x,2)}{(1+x)}\right)F_R^0(x)^{(N-1)}f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^N} + \underline{r}\right]u \\
&\equiv \sigma_{1,1}(r)u.
\end{aligned}$$

This theoretical model has some strong similarities to one developed in Harris et al. [1995]. In that paper, Harris et al. [1995], showed that a subgame-perfect equilibrium need not exist in a model very similar to the one developed above. We believe that a finite time horizon in conjunction with a recursive structure allows us to focus on a pure-strategy equilibrium, which is unique.

2.1.1 Properties of Equilibrium Bid and Optimal Value Functions

For rounds $t = 1, 2, \ldots, N-1$, denoting $\bar{r}$ by $r_{(0)}$, the unit value function is

$$\begin{aligned}
V_1^*(r,t) = &\left((N+1) - \frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] - (N-t)\sigma_{t,1}(r)\right)G_R^{t-1}(r)^{N-t} + \\
&\int_r^{r_{(t-1)}}\left(\left[\sigma_{t,1}(x) - u\right] + \frac{V_1^*(r,t+1)}{(1+r)}\right)(N-t)G_R^{t-1}(x)^{N-t-1}g_R^{t-1}(x)\,\mathrm{d}x.
\end{aligned}$$

As demonstrated above, the value function is homogeneous of degree one in u, which means that

$$V^*(r,t) = V_1^*(r,t)u.$$

Thus, all calculations can be done in terms of a unit bid and unit value functions $\sigma_{t,1}(r)$ and $V_1^*(r,t)$, and then just multiplied u to get $\sigma_t(r)$ and $V^*(r,t)$, respectively. In the empirical part of our research, when different hụi have different deposit sums, this simplifies matters considerably. Of course, when the numbers of rounds in hụi differ, there is no easy way to adjust for that.

2.2 Second Model: Perfect Renewal

As it stands, one problem with the theoretical model is that it cannot generate the pattern in figure 1, which is a sequence of bids across rounds of an actual hụi. In other words, under the model as specified above, the winning bids cannot rise across successive rounds of the hụi because the participants exit in order of rate-of-return, from highest to smallest, and the number of opponents fall as the hụi proceeds.

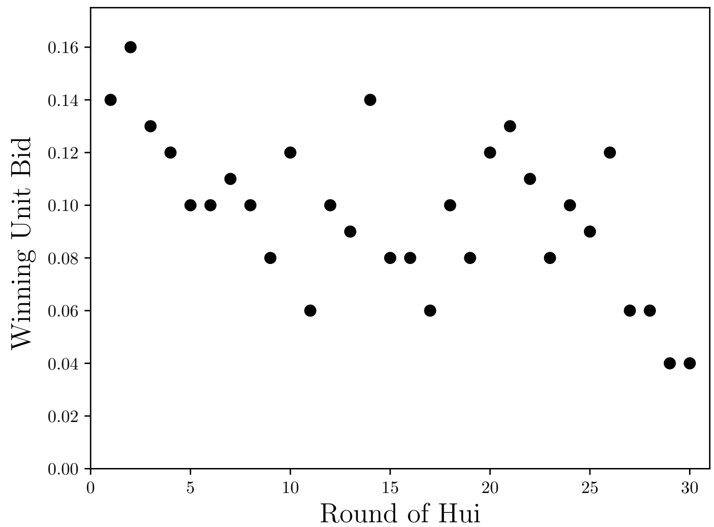

Figure 1: Winning Discount versus Rounds

How could such a saw-tooth pattern be generated in an equilibrium model of the hụi? One straightforward way to reconcile the observed bidding outcomes with a model having the above structure is to allow the remaining participants in any round of the hụi to get new random draws from the cumulative distribution function $F_R^0(r)$. Under this assumption, in rounds $t = 1, 2, \ldots, N-1$, the value function is

$$V(r,t) = \max_{<s>} \left[tu + (N-t)(u-s) - \sum_{i=1}^{N-t} \frac{u}{(1+r)^i} \right] F_R^0\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t} + \int_{\hat{\sigma}_t^{-1}(s)}^{\bar{r}} \left(\left[\hat{\sigma}_t(x) - u\right] + \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right] \right) (N-t) F_R^0(x)^{N-t-1} f_R^0(x)\, \mathrm{d}x.$$

Note that the upper bounds of support no longer depend on previous order statistics of rates-of-return. Also, because new draws are obtained in each period, one must take the expectation of the discounted continuation value function over all feasible values of R. At a stationary point, the following first-order condition obtains:

$$\frac{\mathrm{d}V(r,t)}{\mathrm{d}s} = \left((N+1)u - (N-t+1)s - u\sum_{i=1}^{N-t} \frac{1}{(1+r)^i} - \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right] \right) \times (N-t) F_R^0\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t-1} f_R^0\left[\hat{\sigma}_t^{-1}(s)\right] \frac{\mathrm{d}\hat{\sigma}_t^{-1}(s)}{\mathrm{d}s} - (N-t) F_R^0\left[\hat{\sigma}_t^{-1}(s)\right]^{N-t} = 0,$$

so the first-order condition can be re-written as the following ordinary differential equation:

$$\frac{\mathrm{d}\hat{\sigma}_t(r)}{\mathrm{d}r} + \frac{(N-t+1) f_R^0(r)}{F_R^0(r)} \hat{\sigma}_t(r) = \left((N+1)u - u\sum_{i=1}^{N-t} \frac{1}{(1+r)^i} - \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right] \right) \frac{f_R^0(r)}{F_R^0(r)}. \quad (1)$$

which has solution

$$
\begin{aligned}
\sigma_t(r) &= \frac{\int_{\underline{r}}^{r}\left((N+2)u - \frac{(1+x)}{x}\left[1-\frac{1}{(1+x)^{N-t+1}}\right]u - \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right]\right)F_R^0(x)^{(N-t)}f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^{(N-t+1)}} + \underline{r}u \\
&= \left[\frac{\int_{\underline{r}}^{r}\left((N+2) - \frac{(1+x)}{x}\left[1-\frac{1}{(1+x)^{N-t}}\right] - \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right]\right)F_R^0(x)^{(N-t)}f_R^0(x)\,\mathrm{d}x}{F_R^0(r)^{(N-t+1)}} + \underline{r}\right]u \\
&\equiv \sigma_{t,1}(r)u.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
V_1^*(r,t) = {} & \left((N+1) - \frac{(1+r)}{r}\left[1-\frac{1}{(1+r)^{N-t+1}}\right] - (N-t)\sigma_{t,1}(r)\right)F_R^0(r)^{N-t} + \\
& \int_r^{\overline{r}}\left(\left[\sigma_{t,1}(x)-1\right] + \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right]\right)(N-t)F_R^0(x)^{N-t}f_R^0(x)\,\mathrm{d}x.
\end{aligned}
$$

Building $V_1^*(r,t)$ recursively is much simpler under this model than under the previous one. Specifically,

$$
\begin{aligned}
V_1^*(r,N) &= N \\
V_1^*(r,N-1) &= \left[\left((N+1) - \frac{(1+r)}{r}\left[1-\frac{1}{(1+r)^2}\right]\right) - \sigma_{N-1,1}(r)\right]F_R^0(r) + \\
&\quad \int_r^{\overline{r}}\left(\left[\sigma_{N-1,1}(x)-1\right] + \mathbb{E}\left[\frac{N}{(1+R)}\right]\right)F_R^0(x)\,\mathrm{d}x \\
\vdots &= \vdots \\
V_1^*(r,t) &= \left[\left((N+1) - \frac{(1+r)}{r}\left[1-\frac{1}{(1+r)^{N-t+1}}\right]\right) - (N-t)\sigma_{t,1}(r)\right]F_R^0(r)^{N-t} + \\
&\quad \int_r^{\overline{r}}\left(\left[\sigma_{t,1}(x)-1\right] + \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right]\right)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x \\
\vdots &= \vdots \\
V_1^*(r,1) &= \left[\left((N+1) - \frac{(1+r)}{r}\left[1-\frac{1}{(1+r)^{N}}\right]\right) - (N-1)\sigma_{1,1}(r)\right]F_R^0(r)^{N-1} + \\
&\quad \int_r^{\overline{r}}\left(\left[\sigma_{1,1}(x)-1\right] + \mathbb{E}\left[\frac{V_1^*(R,2)}{(1+R)}\right]\right)(N-1)F_R^0(x)^{N-2}f_R^0(x)\,\mathrm{d}x.
\end{aligned}
$$

Under this alternative assumption, increases in the winning discount bid across rounds of an hụi can obtain because an unusually high rate-of-return draw in a later round may occur and this event can more than make-up for the decrease in equilibrium bidding behaviour that obtains because there are fewer participants in later rounds, and the option values are higher in later rounds of the hụi, holding R constant. Nevertheless, the trend in the winning discount bid should, on average, be downward sloping across rounds of the hụi, as it is in figure 1.

Because this seems like a reasonable model, we sought to investigate its properties numerically, as a prelude to our structural econometric analysis. We chose to work within the Beta family of random variables, scaled to the interval $[\underline{r}, \overline{r}]$, where we have assumed that $\underline{r} = 0$ and $\overline{r} = 0.15$. A Beta-distributed random variable Y, having support on $[0, 1]$, has pdf

$$f_Y(y; \theta_1, \theta_2) = \frac{y^{\theta_1 - 1} y^{\theta_2 - 1}}{\mathrm{B}(\theta_1, \theta_2)} \quad 0 < y < 1,\ \theta_1 > 0,\ \theta_2 > 0.$$

Here,

$$\mathrm{B}(\theta_1, \theta_2) = \frac{\Gamma(\theta_1)\Gamma(\theta_2)}{\Gamma(\theta_1 + \theta_2)},$$

with

$$\Gamma(\theta) = \int_0^\infty x^{\theta - 1} \exp(-x)\, \mathrm{d}x.$$

We introduce the following linear transformation:

$$R = \frac{\overline{r} - Y}{\overline{r}}.$$

Note that the above pdf of Y does not have strictly positive support at the end points. To remedy this problem, we mix this distribution with the uniform using the mass π, which in practice is 10^{-6}.

2.2.1 Some Solved Examples

When N is twenty, while u is one, and R is distributed as a mixed $\mathrm{B}(\theta_1, \theta_2)$ on the interval $[0, \overline{r}]$, the pdf of R is

$$f_R^0(r; \boldsymbol{\theta}, \overline{r}, \pi) = \pi + (1 - \pi) \frac{\left(\frac{r}{\overline{r}}\right]^{\theta_1 - 1} \left(1 - \frac{r}{\overline{r}}\right)^{\theta_2 - 1}}{\overline{r}\mathrm{B}(\theta_1, \theta_2)} \quad 0 < r < \overline{r},\ \theta_1 > 0,\ \theta_2 > 0,$$

where we collect θ_1 and θ_2 in the vector $\boldsymbol{\theta}$. Based on this, we solved for the perfect Bayesian equilibrium bid functions $\{\sigma_{t,1}(r)\}_{t=1}^N$. In figures 3, 5, and 7, we have graphed these bid functions versus the state variable R under three parameterizations—specifically $(2, 10, 0.15)$, $(2, 2, 0.15)$, and $(5, 2, 0.15)$, respectively. In figures 2, 4, and 6 are depicted the pdfs under those parametrizations. Note that the equilibrium bid functions in different rounds of the hụi can cross, which means that winning bids need not fall monotonically across rounds.

A disturbing feature of these bid function is that, for some parameter combinations and some low values of R, the bid functions appear not to be monotonic, which is inconsistent with the hypothesis under which they were derived. This appears in Figure 3, where the bid functions are truncated at zero: In reality, the differential equation is initially negative, and then positive—not monotonic—which signals a problem.

What is economic interpretation of this non-monotonicity problem? Well, with renewed draws, a low rate-of-return participant actually seeks to lose the auction: Given a low enough draw, the worst thing that can happen to such a participant would be to win the auction since he can expect to do much better in the future with a new draw. This lack of persistence implies that low rate-of-return participants seek to submit negative bids, which are against the rules.

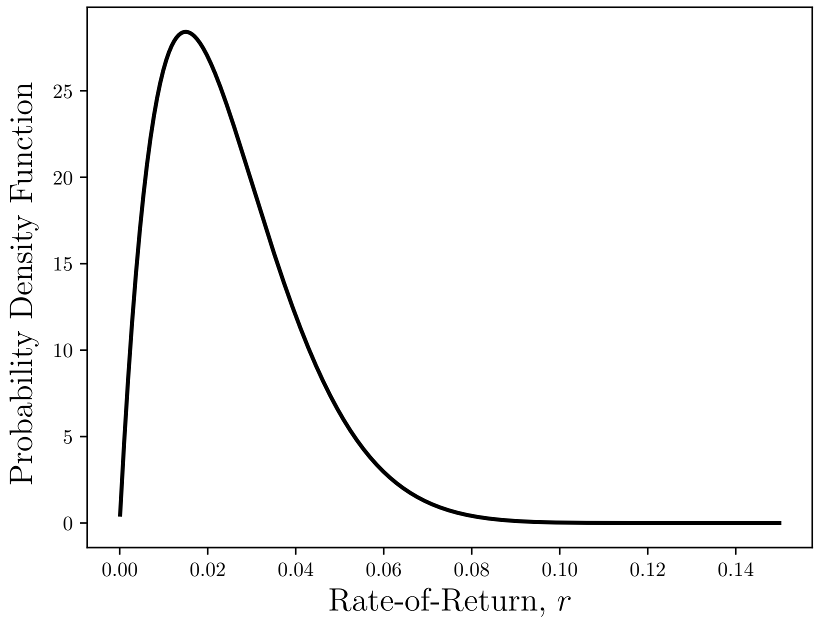

Figure 2: B(2,10) Probability Density Function on [0.0.15]

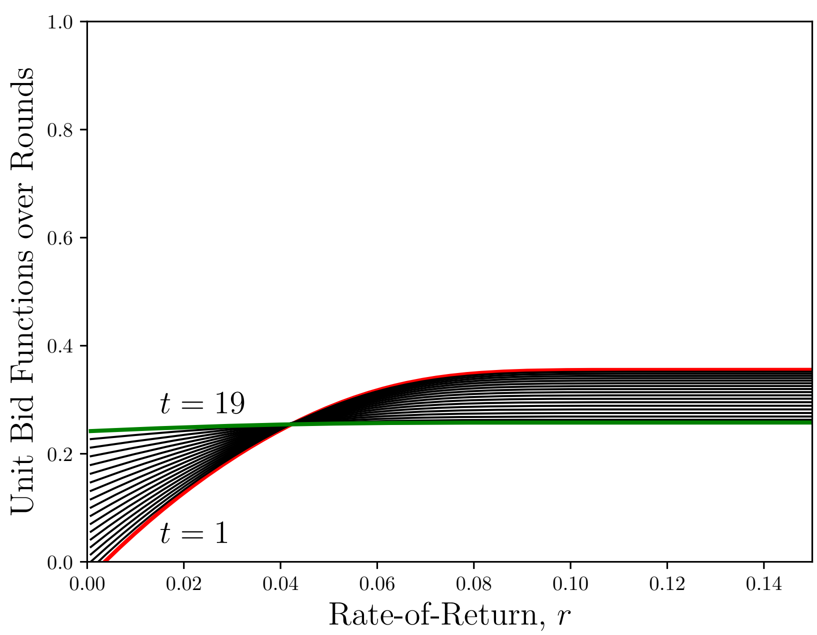

Figure 3: Equilibrium Bid Functions for B(2,10), $N = 20$, and $u = 1$

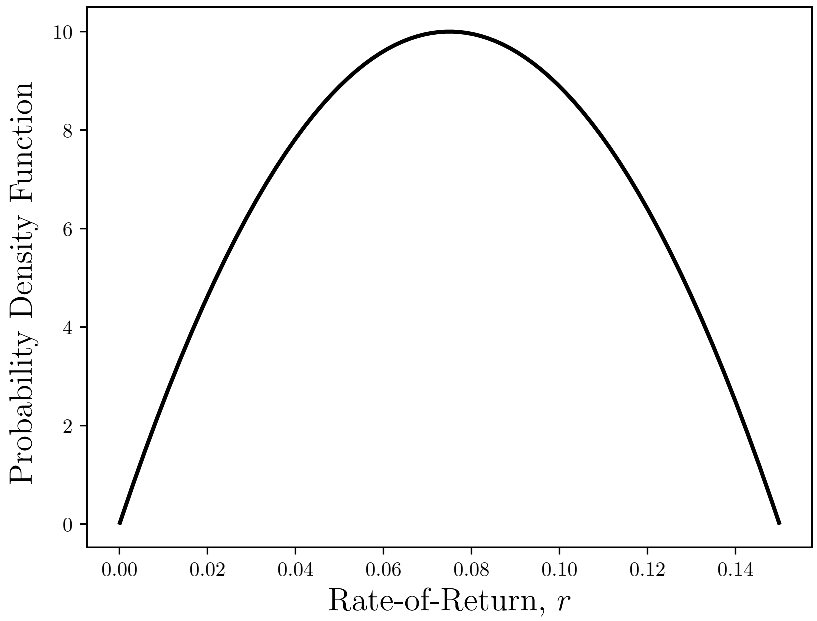

Figure 4: B(2,2) Probability Density Function on [0.0.15]

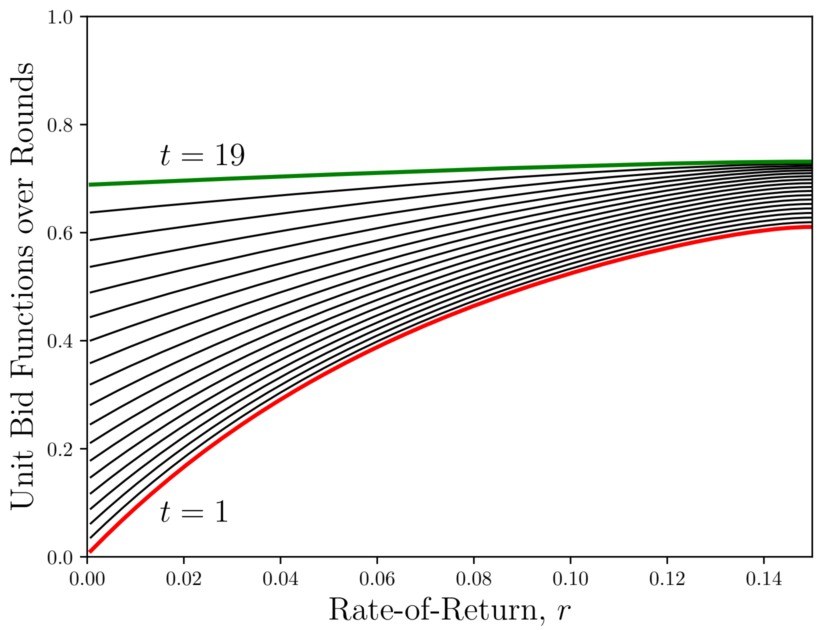

Figure 5: Equilibrium Bid Functions for B(2,2), $N = 20$, and $u = 1$

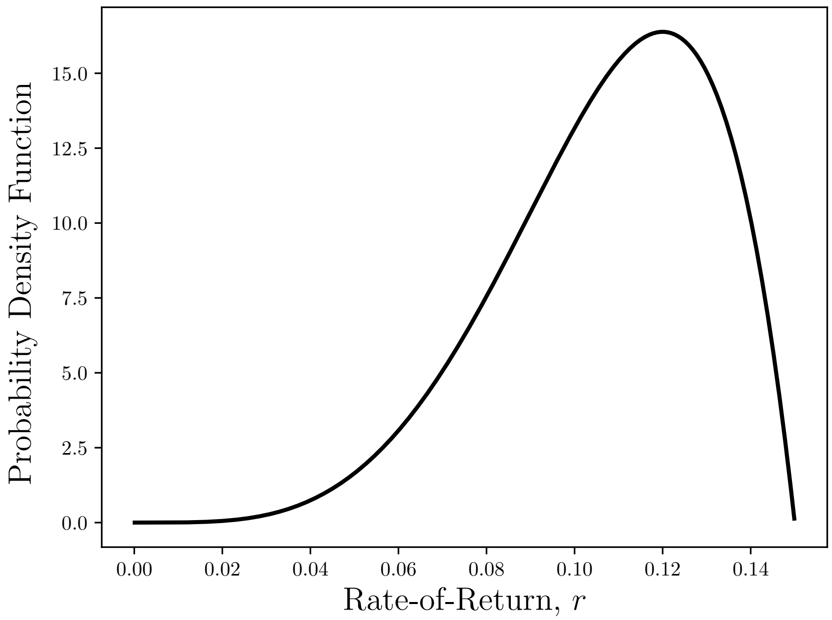

Figure 6: B(5,2) Probability Density Function on [0.0.15]

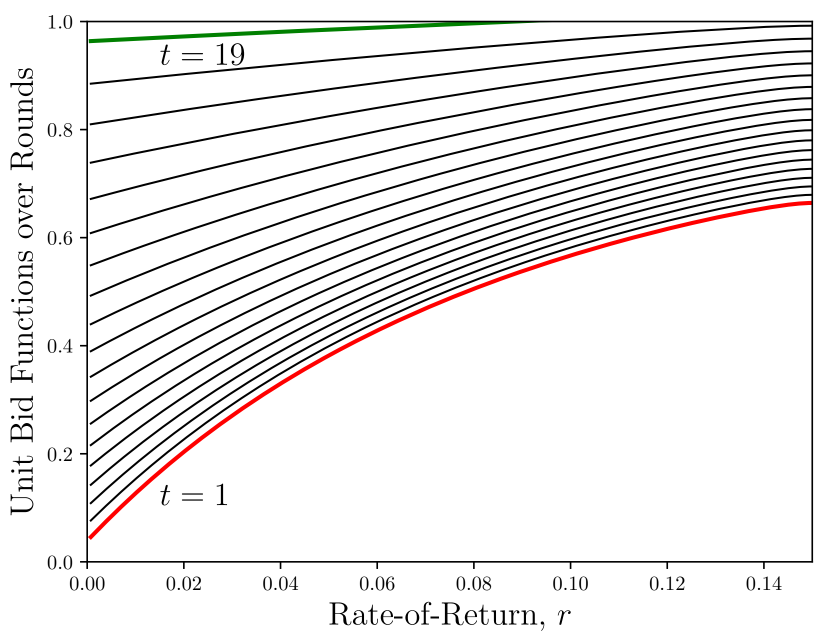

Figure 7: Equilibrium Bid Functions for B(5,2), $N = 20$, and $u = 1$

2.3 Third Model: Markov Model

The first two models introduced above involve the extremes—either a one-time draw of all returns before the hụi begins or new draws after each completed round of the hụi. A model having a structure that is between these two might involve the following: Imagine that each participant n gets an initial, random draw $R_{0,n}$ from $F_R^0(\cdot)$, and then between the close of the last round and the opening of the next round that draw is shocked by some random variable $U_{t,n}$ for $t = 1, 2, \ldots, N - 1$. Without too much lost of generality, suppose the shocks are multiplicative. In words then, $R_{0,n}$ is the return to participant n for the first round of the hụi, while $R_{0,n}U_{1,n}$ is the return to that participant for the second round of the hụi, and so forth. Suppressing the n subscript in the interest of notational parsimony, the return for the $(t+1)^{\text{st}}$ round of the hụi is then

$$R_t = R_0 \prod_{\tau=1}^{t} U_\tau.$$

Assume that the Us for all participants over all rounds are drawn independently from the same distribution, which has mean one and a finite variance. One candidate distribution would be the joint lognormal distribution, which is sometimes referred to as Galton's distribution. In that case, conditional on the known draw r_0, the sum in the following equation has a convenient Gaussian distribution:

$$\log R_t = \log r_0 + \sum_{\tau=1}^{T} \log U_\tau \equiv \log r_0 + \sum_{\tau=1}^{T} \varepsilon_\tau,$$

which is centered about $\log r_0$. Unfortunately, r_0 is unknown, so we must formally model its distribution. A natural candidate distribution for R_0 would be a translation of the Beta distribution to the interval $[\underline{r}, \overline{r}]$, but the convolution of the logarithm of the Beta and the Gaussian does not have a closed-form, which makes implementation arduous. However, under the assumption that $\log R_0$ is jointly Gaussian with the εs, too, but independent of them, one can derive in closed-form the distribution of R_t.

In figure 8, we depict a particular parametrization of Galton's distribution as the solid line and a particular parameterization of the Beta as the dashed line, which suggests that qualitatively the difference may be small. Under Galton's distribution, the pdf of initial returns is

$$f_R^0(r|\mu_R, \sigma_R) = \frac{1}{r\sqrt{2\pi\sigma_R^2}} \exp\left[-\frac{(\log r - \mu_R)^2}{2\sigma_R^2}\right],$$

while under the scaled Beta assumption, that pdf is

$$f_R^0(r|\theta_1, \theta_2, \overline{r}) = \frac{\left(\frac{r}{\overline{r}}\right)^{\theta_1 - 1}\left[1 - \left(\frac{r}{\overline{r}}\right)\right]^{\theta_2 - 1}}{\overline{r}\mathrm{B}(\theta_1, \theta_2)} \quad 0 \le r \le \overline{r},$$

where

$$\mathrm{B}(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

with

$$\Gamma(a) = \int_0^\infty u^{a-1} \exp(-u)\, \mathrm{d}u.$$

In the figure, $\mu_R = -3$, while $\sigma_R = 1$, while $\theta_1 = 2$, $\theta_2 = 8$, and $\overline{r} = 0.15$. Although the advantages in terms of tractability may not be obvious now, on the computational side, particularly in terms of quadrature, they are huge; this will become clear in later sections of the paper.

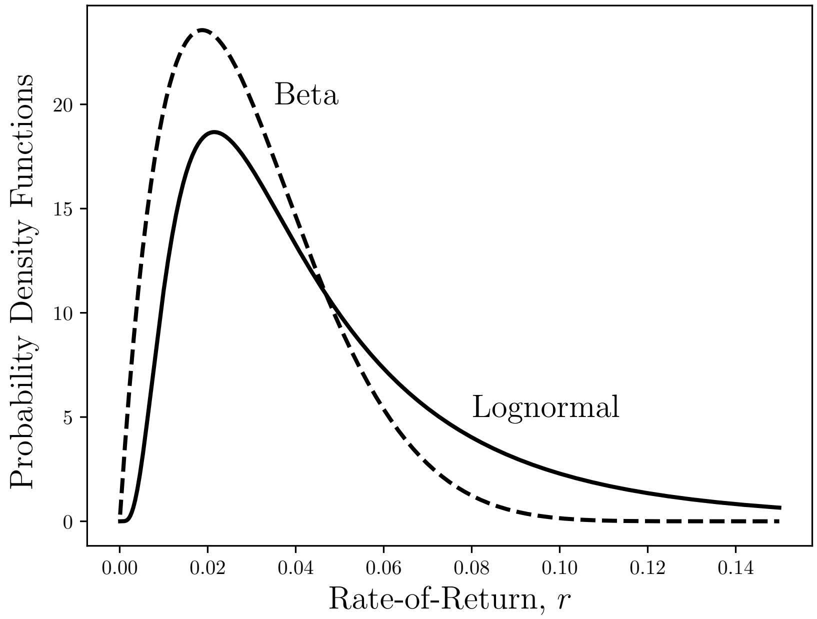

Figure 8: Beta versus Galton's Distribution

Under the lognormal assumptions, when the variance of ε is denoted by σ_ε^2, and the mean of $\log R_0$ is denoted by μ_R, and its the corresponding variance is denoted by σ_R^2, R_t is distributed lognormally, where

$$\mathbb{E}(\log R_t) = \mu_R - t\frac{\sigma_\varepsilon^2}{2}$$

and

$$\mathbb{V}(\log R_t) = \sigma_R^2 + t\sigma_\varepsilon^2.$$

We refer to this model as the *random-walk specification*, in contrast to the *autoregressive specification*, where

$$[\log(R_t) - \log(r_0)] = \rho[\log(r_{t-1}) - \log(r_0)] + \varepsilon_t \quad |\rho| < 0.$$

In the autoregressive specification,

$$\mathbb{E}(\log R_t) = \mu_R - \frac{\sigma_\varepsilon^2}{2}\frac{(1-\rho^t)}{(1-\rho)}$$

and

$$\mathbb{V}(\log R_t) = \sigma_R^2 + \frac{\sigma_\varepsilon^2(1-\rho^{2t})}{(1-\rho^2)}.$$

To understand the effect of the shocks, we assumed some representative values for μ_R, σ_R^2, and σ_ε^2, when the number of participants is twenty. Specifically, we assumed that the average return is five percent, while ninety-five percent of the mass is less than fifteen percent, which implies that μ_R is around -3, when σ_R is around one. We assumed that the bulk of the variation in rates-of-return is due to the inherent variability of R, not because of the shocks, so we chose a value of σ_ε of one quarter, which means around two thirds of the shocks are less than twenty-five

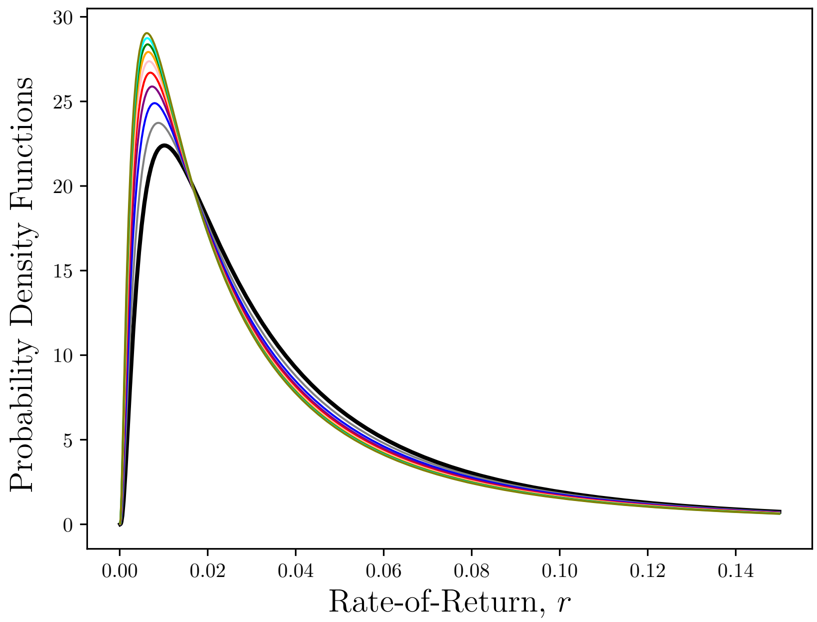

Figure 9: Effect of Autoregressive Shocks on Distribution

percent in magnitude, while ninety-five percent of them are less than fifty percent in magnitude. In order for the expectation of U to be one, the mean of ε must be -0.03125. In the autoregressive specification, we set ρ to be one half. Under these parameters, we present in figures 9 and 10 the pdfs of successively later rounds in the hụi as indexed by $t = 1, 2, \ldots, N-1$. In both figures, the lowest of the pdfs (the thick black lines) is that of $f_R^0(r)$, while as the rounds continue, the shocks cause the pdfs (the thinner coloured lines) to rise in concentration around the initial r_0, albeit the distribution is shifted a bit to the left.

2.4 Analysis of Equilibrium Differential Equations

In this subsection, we present an analysis of the equilibrium differential equation. In order to save on notation, we eliminate subscripts and superscripts on the probability density and cumulative distribution functions. We also substitute the letters ℓ and v for

$$\ell = \frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right]$$

and

$$v = \frac{V_1^*(r, t+1)}{(1+r)} \quad \text{or} \quad v = \mathbb{E}\left[\frac{V_1^*(R, t+1)}{(1+R)}\right],$$

respectively. Thus, we can write an equilibrium differential equation as

$$\begin{aligned}\frac{\mathrm{d}\sigma_t}{\mathrm{d}r} &= [(N+2-\ell) - v]\,\frac{f}{F} - \sigma_t \frac{f}{F} \\ &= (\theta - \sigma_t)\frac{f}{F}.\end{aligned}$$

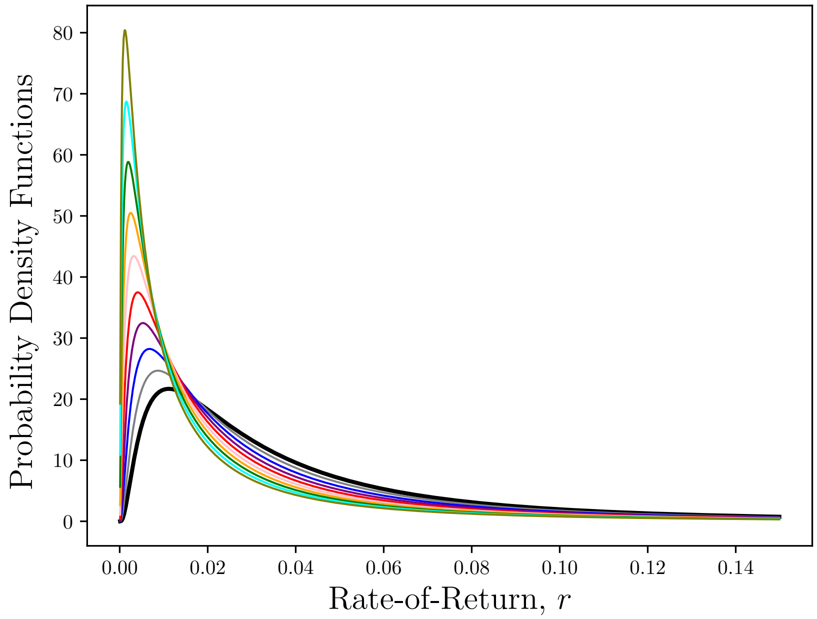

Figure 10: Effect of Random-Walk Shocks on Distribution

Now, suppose that $\frac{f}{F}$ is a constant. Then,

$$\begin{aligned}\frac{\mathrm{d}\sigma_t}{\mathrm{d}r} &= \alpha(\theta - \sigma_t) \\ \frac{\mathrm{d}}{\mathrm{d}r}\left[\exp(\alpha r)\sigma_t\right] &= \alpha \exp(\alpha r)\sigma_t + \exp(\alpha r)\alpha(\theta - \sigma_t) \\ &= \alpha \exp(\alpha r)\theta.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{r_j}^{r_j+h} \mathrm{d}\left[\exp(\alpha r)\sigma_t\right] &= \int_{r_j}^{r_j+h} \alpha \exp(\alpha r)\theta \,\mathrm{d}r \\ &= \exp[\alpha(r_j + h)]\sigma_t(r_j + h) - \exp(\alpha r_j)\sigma_t(r_j) \\ &= \theta\left[\exp(\alpha r)\right]_{r_j}^{r_j+h}.\end{aligned}$$

Therefore,

$$\sigma_t(r_j + h) = \exp(-\alpha h)\sigma_t(r_j) + \theta\left[1 - \exp(-\alpha h)\right].$$

Consider as an example,

$$\begin{aligned}\frac{\mathrm{d}\sigma_t}{\mathrm{d}r} &= \left[(N + 2 - \ell) - v\right]\frac{f}{F} - 2\sigma_t\frac{f}{F} \\ &= (\theta - 2\sigma_t)\alpha\left[\frac{\mathrm{d}}{\mathrm{d}r}\exp(2\alpha r)\sigma_t(r)\right] \\ &= \sigma_t(r)\frac{\mathrm{d}}{\mathrm{d}r}\exp(2\alpha r) + \exp(2\alpha r)\frac{\mathrm{d}\sigma_t(r)}{\mathrm{d}r}\end{aligned}$$

$$
\begin{aligned}
&= 2\alpha \exp(2\alpha r)\sigma_t(r) + \alpha \exp(2\alpha r)\left[\theta - 2\sigma_t(r)\right] \\
&= \alpha\theta \exp(2\alpha r),
\end{aligned}
$$

so

$$
\begin{aligned}
\int_{r_j}^{r_j+h} \mathrm{d}\left[\exp(2\alpha r)\sigma_t(r)\right] &= \int_{r_j}^{r_j+h} \alpha \exp(2\alpha r)\theta \,\mathrm{d}r \\
&= \exp[2\alpha(r_j+h)]\sigma_t(r_j+h) - \exp(2\alpha r_j)\sigma_t(r_j) \\
&= \frac{\theta}{2}\left[\exp(2\alpha r)\right]_{r_j}^{r_j+h}.
\end{aligned}
$$

Therefore,

$$
\sigma_t(r_j+h) = \exp(-2\alpha h)\sigma_t(r_j) + \frac{\theta}{2}\left[1 - \exp(-2\alpha h)\right].
$$

In general,

$$
\sigma_t(r_j+h) = \exp(-t\alpha h)\sigma_t(r_j) + \frac{\theta}{t}\left[1 - \exp(-t\alpha h)\right] \quad t = 1, 2, \ldots, N-1.
$$

What does it all mean? Well, there is a strong attractor to this equilibrium differential equation, and this attractor gets stronger as the rounds of the hụi proceed. In practical terms, the equilibrium bid functions in later round will be weakly higher at the right-hand part of the interval $[\underline{r}, \bar{r}]$ than in early rounds.

2.5 Numerical Solution of First-Order Ordinary Differential Equations

Consider the following first-order ordinary differential equation for σ as a function of r:

$$
\frac{\mathrm{d}\sigma(r)}{\mathrm{d}r} = D(r, \sigma). \tag{2}
$$

Several different numerical methods exist to solve differential equations like (2). The simplest of the finite difference methods is, of course, *Euler's method*: starting at r_0, an initial r—say, $\underline{r}$, where $\sigma(\underline{r})$ is $\underline{r}$ in our case—the value of $\sigma(\underline{r}+h)$ can then be approximated by the value of $\sigma(\underline{r})$ plus the step h multiplied by the slope of the function, which is the derivative of $\sigma(r)$, evaluated at $\underline{r}$. This is simply a first-order Taylor-series expansion, so

$$
\sigma(\underline{r}+h) \approx \sigma(\underline{r}) + h \left.\frac{\mathrm{d}\sigma(r)}{\mathrm{d}r}\right|_{r=\underline{r}} = \sigma(\underline{r}) + hD[\underline{r}, \sigma(\underline{r})].
$$

Denoting this approximate value by σ_1, and the initial value by σ_0, we have

$$
\sigma_1 = \sigma(\underline{r}) + h \left.\frac{\mathrm{d}\sigma(r)}{\mathrm{d}r}\right|_{r=\underline{r}} = \sigma(\underline{r}) + hD[\underline{r}, \sigma(\underline{r})] = \sigma_0 + hD(r_0, \sigma_0) = \sigma_0 + hD_0. \tag{3}
$$

If one can calculate the value of $\mathrm{d}\sigma/\mathrm{d}r$ at r using equation (2), then one can generate an approximation for the value of σ at r equal $(\underline{r}+h)$ using equation (3). One can then use this new value of σ, at $(\underline{r}+h)$, to find $\mathrm{d}\sigma/\mathrm{d}r$ (at the new r) and repeat. When $D(r, \sigma)$ does not change too quickly, the method can generate an approximate solution of reasonable accuracy. For example, on an infinite-precision computer, the local truncation error is $\mathcal{O}(h^2)$, while the global error is $\mathcal{O}(h)$—first-order accuracy.

When the differential equation changes very quickly in response to a small step h, then it is referred to as a *stiff* differential equation. To solve stiff differential equations using Euler's method, h must be very small, which means that Euler's methods will take a long time to compute an accurate solution. Although this may not be an issue when one just wants to do this once, in empirical work concerning auctions, one may need to solve the differential equation thousands (even millions) of times.

Perhaps the most well-known generalization of Euler's method is a family of methods referred to collectively as *Runge–Kutta methods*. Of all the members in this family, the one most commonly used is the fourth-order method, sometimes referred to as *RK4*. Under RK4,

$$\sigma_{k+1} = \sigma_k + h\frac{1}{6}(d_1 + 2d_2 + 2d_3 + d_4)$$

where

$$\begin{aligned}
d_1 &= D\left(r_k, \sigma_k\right) \\
d_2 &= D\left(r_k + \frac{1}{2}h, \sigma_k + \frac{1}{2}hd_1\right) \\
d_3 &= D\left(r_k + \frac{1}{2}h, \sigma_k + \frac{1}{2}hd_2\right) \\
d_4 &= D\left(r_k + h, \sigma_k + hd_3\right).
\end{aligned}$$

Thus, the next value σ_{k+1} is determined by the current one σ_k, plus the product of the step size h and an estimated slope. The estimated slope is a weighted average of slopes: d_1 is the slope at the left endpoint of the interval; d_2 is the slope at the midpoint of the interval, using Euler's method along with slope d_1 to determine the value of σ at the point $(r_k + \frac{1}{2}h)$; d_3 is again the slope at the midpoint, but now using the slope d_2 is used to determine σ; and d_4 is the slope at the right endpoint of the interval, with its σ value determined using d_3. Assuming the Lipschitz condition is satisfied, the local truncation error of the RK4 method is $\mathcal{O}(h^5)$, while the global truncation error is $\mathcal{O}(h^4)$, which is a huge improvement over Euler's method. Note, too, that if $D(\cdot)$ does not depend on σ, so the differential equation is equivalent to a simple integral, then RK4 is simply *Simpson's rule*, a well-known and commonly-used quadrature rule.

Like Euler's method, however, Runge–Kutta methods do not always perform well on stiff problems; for more on this, see Ernst Hairer and Gerhard Wanner [1996]. Note, too, that neither the method of Euler nor the methods of Runge–Kutta use past information to improve the approximation as one works to the right.

In response to these limitations, numerical analysts have pursued a variety of other strategies. For a given h, these alternative methods are more accurate than Euler's method, and may have a small error constant than Runge–Kutta methods as well. Some of the alternative methods are referred to as *multi-step methods*. Under multi-step methods, one again starts from an initial point $\underline{r}$ and then takes a small step h forward in r to find the next value of σ. The difference is that, unlike Euler's method (which is a single-step method that refers only to one previous point and its derivative at that point to determine the next value), multi-step methods use some intermediate points to obtain a higher-order approximation of the next value. Multi-step methods gain efficiency by keeping track of as well as using the information from previous steps rather than discarding it. Specifically, multi-step methods use the values of the function at several previous points as well as the derivatives (or some of them) at those points.

Linear multi-step methods are special cases in the class of multi-step methods. As the name suggests, under these methods, a linear combination of previous points and derivative values is used to approximate the solution. Denote by m the number of previous steps used to calculate the next value. Denote the desired value at the current stage by σ_{k+m}. A linear multi-step method has the following general form:

$$\begin{aligned}
&\sigma_{k+m} + a_{m-1}\sigma_{k+m-1} + a_{m-2}\sigma_{k+m-2} + \cdots + a_0\sigma_k \\
&\quad = h\left[b_m D(r_{k+m}, \sigma_{k+m}) + b_{m-1}D(r_{k+m-1}, \sigma_{k+m-1}) + \cdots + b_0 D(r_k, \sigma_k)\right].
\end{aligned}$$

The values chosen for $a_0, \ldots, a_{m-1}$ and $b_0, \ldots, b_m$ determine the solution method; a numerical analyst must choose these coefficients. Often, many of the coefficients are set to zero. Sometimes, the numerical analyst chooses the coefficients so they will interpolate $\sigma(r)$ exactly when $\sigma(r)$ is a k^{th} order polynomial. When b_m is nonzero, the value of σ_{k+m} depends on the value of $D(r_{k+m}, \sigma_{k+m})$, and the equation for σ_{k+m} must be solved iteratively, perhaps using Newton's method, or some other method.

A simple linear, multi-step method is the *Adams–Bashforth two-step method*. Under this method,

$$\sigma_{k+2} = \sigma_{k+1} + h\frac{3}{2}D(r_{k+1}, \sigma_{k+1}) - h\frac{1}{2}D(r_k, \sigma_k).$$

To wit, a_1 is -1, while b_2 is zero, and b_1 is $\frac{3}{2}$, while b_0 is $-\frac{1}{2}$. However, to implement Adams–Bashforth, one needs two values (σ_{k+1} and σ_k) to compute the next value σ_{k+2}. In a typical initial-value problem, only one value is provided; in our case, for example, $\sigma(\underline{r})$ or σ_0 equals $\underline{r}$ or r_0 is the only condition provided. One way to circumvent this lack of information is to use the σ_1 computed by Euler's method as the second value. With this choice, the Adams–Bashforth two-step method yields a candidate approximating solution.

For other values of m, John C. Butcher [2003] has provided explicit formulas to implement the Adams–Bashforth methods. Again, assuming the Lipschitz condition is satisfied, the local truncation error of the Adams–Bashforth two-step method is $\mathcal{O}(h^3)$, while the global truncation error is $\mathcal{O}(h^2)$. (Other Adams–Bashforth methods have local truncation errors that are $\mathcal{O}(h^5)$ and global truncation errors that are $\mathcal{O}(h^4)$, and are, thus, competitive with RK4.)

In addition to Adams–Bashforth, two other families are also used: first, Adams–Moulton methods and, second, backward differentiation formulas (BDFs).

Like Adams–Bashforth methods, the Adams–Moulton methods have a_{m-1} equal -1 and the other a_is equal to zero. However, where Adams–Bashforth methods are explicit, Adams–Moulton methods are implicit. For example, when m is zero, under Adams–Moulton,

$$\sigma_k = \sigma_{k-1} + hD(r_k, \sigma_k), \tag{4}$$

which is sometimes referred to as the *backward Euler method*, while when m is one,

$$\sigma_{k+1} = \sigma_k + h\frac{1}{2}\left[D(r_{k+1}, \sigma_{k+1}) + D(r_k, \sigma_k)\right], \tag{5}$$

which is sometimes referred to as the *trapezoidal rule*. Note that these equations only define the solutions implicitly; that is, equations (4) and (5) must be solved numerically for σ_k and σ_{k+1}, respectively.

BDFs constitute the main other way to solve ordinary differential equations. BDFs are linear multi-step methods which are especially useful when solving stiff differential equations. From above, we know that, given equation (2), for step size h, a linear multi-step method can, in general, be written as

$$\begin{aligned}\sigma_{k+m} &+ a_{m-1}\sigma_{k+m-1} + a_{m-2}\sigma_{k+m-2} + \cdots + a_0\sigma_k \\ &= h\left[b_m D(r_{k+m}, \sigma_{k+m}) + b_{m-1} D(r_{k+m-1}, \sigma_{k+m-1}) + \cdots + b_0 D(r_k, \sigma_k)\right].\end{aligned}$$

BDFs involve setting b_i to zero for any i other than m, so a general BDF is

$$\sigma_{k+m} + a_{m-1}\sigma_{k+m-1} + a_{m-2}\sigma_{k+m-2} + \cdots + a_0\sigma_k = hb_m D_{k+m}$$

where D_{k+m} denotes $D(r_{k+m}, \sigma_{k+m})$. Note that, like Adams–Moulten methods, BDFs are implicit methods as well: one must solve nonlinear equations at each step—typically, using Newton's method, but some other method could be used as well. Thus, the methods can be computationally burdensome. However, the evaluation of σ at r_{k+m} in $D(\cdot)$ is an effective way in which to discipline approximate solutions to stiff differential equations.

The principal numerical difficulty with solving the ordinary differential equation (1) is that it does not satisfy the Lipschitz condition at the left endpoint $\underline{r}$ because, at that point,

$$\frac{f_R^0(r)}{F_R^0(r)} = \frac{f_R^0(r)}{\int_{\underline{r}}^{r} F_R^0(u)\,\mathrm{d}u}$$

is unbounded. One strategy to avoid this problem would be to analyze the equilibrium inverse-bid function $\varphi(s)$ which equals $\sigma^{-1}(s)$. In this case, one obtains an ordinary differential equation of the following form:

$$\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} = p(s)\varphi(s)\frac{F_R^0[\varphi(s)]}{f_R^0[\varphi(s)]} + q(s)\frac{F_R^0[\varphi(s)]}{f_R^0[\varphi(s)]} = C(s, \varphi) \tag{6}$$

where $p(s)$ and $q(s)$ are known functions, and where the initial value involves $\varphi(\bar{s})$ equalling $\bar{r}$. In this formulation, however, $\bar{s}$ is unknown, so the problem is sometimes referred to as a *free boundary-value problem*, which can be solved using the method of *backward shooting* (*reverse shooting*). Under backward (reverse) shooting, one specifies an initial guess for $\bar{s}$, and then solves the system backward (in reverse) toward $\varphi(\underline{r})$, which must equal $\underline{r}$ at the left endpoint using any of the methods we have described above. However, Gadi Fibich and Nir Gavish [2011] have demonstrated that, for this problem, backward shooting methods are numerically unstable. Despite the numerical problems, we have had some success in solving differential equations like (1).

3 Field Data

A former hụi banker, now retired, has graciously provided us with a small sample of bids from twenty-two hụi, which were held in the early 2000s in a suburb of Melbourne, Australia. As part of our agreement with this banker, we can say very little more than this. Specifically, we cannot provide demographic characteristics of the participants, nor can we describe the activities in which the funds from the hụi were invested. The reason is obvious: would you want your banker sharing your private information with us? One of the reasons the banker felt comfortable with giving us

Table 2: Sample Descriptive Statistics

Variable	Sample Size	Mean	St.Dev.	Minimum	Median	Maximum
u_h	22	345.45	126.22	200	300	500
N_h	22	35.95	11.00	21	36	51
w_{ht}	769	58.98	11.00	5	40	150
$w_{ht,1}$	769	0.1579	0.0764	0.0200	0.1500	0.4500

these data is that they are more than fives years old. We can, however, describe the important economic variables for the sample of hụi. The hụi had Ns between 21 and 51 participants, while the deposits u were between \$200 and \$500. In figure 1, we depicted the winning bids, across successive rounds, for one of the hụi; the patterns of winning bids in other hụi are qualitatively similar. In table 2, we present the sample descriptive statistics over all of the hụi.

3.1 Internal Rates-of-Return

4 Econometric Model

In a very influential paper, Emmanuel Guerre, Isabelle Perrigne, and Quang H. Vuong [2000] (hereafter, GPV) introduced a clever trick to invert the bid function at single-object, first-price auctions and, thus, to recover the unobserved type from the observed action as well as its distribution in a non-cooperative auction game with incomplete private-valued information. In this section, we first demonstrate how to make use of this trick in the case of the hụi and, thus, demonstrate that this model is non-parametrically identified. Subsequently, we note that implementing GPV requires more data than we have been able to gather. Because we would like to implement our theoretical model using field data, we are forced to make a parametric assumption to develop an empirical specification which we estimate using the methods developed by Stephen G. Donald and Harry J. Paarsch [1993, 1996].

To begin, we outline the basic framework of GPV: consider a single-object auction at which N potential buyers vy to win the good for sale. Suppose each gets an independent draw V from the cumulative distribution of values $F_V(v)$ that has support $[\underline{v}, \overline{v}]$, with corresponding probability density function $f_V(v)$ that is strictly positive on $[\underline{v}, \overline{v}]$. Because the potential buyers are *ex ante* symmetric, we can focus on the decision problem of player 1. Player 1, who has valuation draw v, is assumed to maximize, by choice of bid s, the following expect profit from winning the auction:

$$\mathbb{E}[\pi(s)] = (v - s)\Pr(\mathsf{win}|s).$$

But what is $\Pr(\mathsf{win}|s)$? Well, player 1 wins when all of his opponents bid less than him, so

$$\Pr(\mathsf{win}|s) = \Pr[(S_2 < s) \cap \ldots \cap (S_N < s)].$$

Now, because the draws of potential buyers are independent,

$$\Pr(\mathsf{win}|s_1) = \Pr(S_2 < s)\Pr(S_3 < s)\cdots\Pr(S_N < s) = \prod_{n=2}^{N}\Pr(S_n < s).$$

To analyze this case, focus on symmetric, perfect Bayesian equilibria. To construct an equilibrium, as in section 2, suppose that the $(N-1)$ opponents of player 1 are using a common bidding rule

$\hat{\sigma}(V)$, which is monotonically increasing in V: potential buyers who have high values bid more than those who have low values.

The probability of player 1 winning with bid s equals the probability that every other opponent bids lower because each has a lower value, so

$$\Pr(\mathsf{win}|s) = F_V\left[\hat{\sigma}^{-1}(s)\right]^{N-1}.$$

Given that player 1's value v is determined before the bidding, his choice of bid s has only two effects on his expected profit

$$(v-s)F_V\left[\hat{\sigma}^{-1}(s)\right]^{(N-1)}.$$

The higher is s, the higher is $F_V\left[\hat{\sigma}^{-1}(s)\right]^{(N-1)}$, which is player 1's probability of winning the auction, but the lower is the pay-off following a win $(v-s)$.

Maximizing behaviour implies that the optimal bidding strategy solves the following necessary first-order condition:

$$-F_V\left[\hat{\sigma}^{-1}(s)\right]^{(N-1)} +$$
$$(v-s)(N-1)F_V\left[\hat{\sigma}^{-1}(s)\right]^{(N-2)} f_V\left[\hat{\sigma}^{-1}(s)\right]\frac{\mathrm{d}\hat{\sigma}^{-1}(s)}{\mathrm{d}s} = 0.$$

In a symmetric perfect Bayesian equilibrium, $s = \hat{\sigma}(v)$ and, again, under monotonicity, $\mathrm{d}\hat{\sigma}^{-1}(s)/\mathrm{d}s$ equals $[1/\hat{\sigma}'(v)]$, so the equilibrium solution is characterized by the following ordinary differential equation:

$$\sigma'(v) = [v - \sigma(v)]\frac{(N-1)f_V(v)}{F_V(v)} \tag{7}$$

where $\sigma'(v)$ is a short-hand notation for $\mathrm{d}\sigma(v)/\mathrm{d}v$. The above equilibrium differential equation came from differentiating the following exact equilibrium solution with respect to v:

$$\sigma(v) = v - \frac{\int_{\underline{v}}^{v} F_V(u)^{N-1}\,\mathrm{d}u}{F_V(v)^{N-1}}.$$

Note, too, that even though $v \in [\underline{v}, \overline{v}]$, $\sigma(v) \in [\underline{s}, \overline{s}]$ where $\overline{s} = \sigma(\overline{v}; F_V, N) < \overline{v}$. In short, the support of S depends on the distribution $F_V(\cdot)$ as well as N. This fact will be important later when we come to implement our parametric empirical specification.

Consider now the cumulative distribution function of an equilibrium bid $G_S(s)$ and its corresponding probability density function $g_S(s)$. Recall that, when $S = \sigma(V)$ is a monotonic function of V,

$$G_S(s) = \Pr(S \le s) = \Pr\left[\sigma^{-1}(S) \le \sigma^{-1}(s)\right] = \Pr(V \le v) = F_V(v).$$

Also,

$$\begin{aligned}
g_S(s)\,\mathrm{d}s &= f_V(v)\,\mathrm{d}v \\
g_S(s) &= f_V(v)\,\frac{\mathrm{d}v}{\mathrm{d}s} \\
g_S(s) &= \frac{f_V(v)}{\sigma'(v)} \\
g_S(s) &= \frac{f_V\left[\sigma^{-1}(s)\right]}{\sigma'\left[\sigma^{-1}(s)\right]}.
\end{aligned}$$

Thus, re-arranging equation (7) yields

$$v = s + \frac{F_V(v)\sigma'(v)}{(N-1)f_V(v)} = s + \frac{G_S(s)}{(N-1)g_S(s)}. \tag{8}$$

In short, the unobserved value v can be identified from the observed bid s as well as its distribution $G_S(s)$ which yields its density $g_S(s)$. Thus, if one is willing to substitute non-parameteric estimates of $G_S(s)$ and $g_S(s)$ into equation (8), then one can get an estimate of the unobserved v corresponding to an observed s, which one can then use to estimate the cumulative distribution and probability density functions $F_V(v)$ and $f_V(v)$.

Using a parallel reasoning, introduce $G_S^t(s)$ and $g_S^t(s)$ to denote the distribution of equilibrium bids in round t of a hụi having N rounds with deposit u. Denote by $s_{t,1}$ the unit bid in round t of a hụi having deposit u; in other words, $s_{t,1}$ is (s_t/u). Now, focus on the perfect Bayesian equilibrium differential equation

$$\begin{aligned}
\frac{\mathrm{d}\sigma_t(r)}{\mathrm{d}r} &= \Bigg((N+1)u - u\frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] - \\
&\qquad \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right] - (N-t+1)\sigma_t(r)\Bigg)\frac{f_R^0(r)}{F_R^0(r)} \\
1 &= \Bigg((N+1)u - u\frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] - \\
&\qquad \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right] - (N-t+1)\sigma_t(r)\Bigg)\frac{f_R^0(r)}{F_R^0(r)}\,\frac{\mathrm{d}r}{\mathrm{d}\sigma_t(r)} \\
1 &= \Bigg((N+1)u - u\frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] - \\
&\qquad \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right] - (N-t+1)\sigma_t(r)\Bigg)\frac{g_S^t(s_t)}{G_S^t(s_t)} \\
\frac{G_S^t(s_t)}{g_S^t(s_t)} &= \Bigg((N+1) - \frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] - \\
&\qquad \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right] - (N-t+1)\sigma_{t,1}(r)\Bigg)u \\
\frac{G_S^t(s_t)}{ug_S^t(s_t)} &= \Bigg((N+1) - \frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] - \\
&\qquad \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right] - (N-t+1)s_{t,1}\Bigg) \\
(N+1) - (N-t+1)s_{t,1} - \frac{G_S^t(s_t)}{ug_S^t(s_t)} &= \left(\frac{1+r}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] + \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right]\right) \\
(N+1) - (N-t+1)s_{t,1} - \frac{G_{S,1}^t(s_{t,1})}{g_{S,1}^t(s_{t,1})} &= \left(\frac{1+r}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right] + \mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right]\right)
\end{aligned}$$

where $G^t_{S,1}(\cdot)$ and $g^t_{S,1}(\cdot)$ denote the cumulative distribution and probability density functions of equilibrium unit bids in round t. Now, the left-hand side of the above expression is a function of observables—viz.,

$$(N+1) - (N-t+1)s_{t,1} - \frac{G^t_{S,1}(s_{t,1})}{g^t_{S,1}(s_{t,1})},$$

while the right-hand side is the sum of a known function of r—viz,. the

$$\frac{1+r}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right],$$

and an unknown function of R—viz., the

$$\frac{V_1^*(R,t+1)}{(1+R)},$$

whose structure depends on the unknown $F_R^0(\cdot)$ itself, except in one case:

$$V_1^*(R,N) = N.$$

However,

$$\mathbb{E}\left[\frac{V_1^*(R,t+1)}{(1+R)}\right] \neq \frac{N}{(1+r)},$$

unless we assume that current rate-of-return is used to discount, instead of the average of next-period's draw. Suppose that the current rate-of-return r is used to discount. Then

$$\begin{aligned}
(N+1) - 2s_{N-1,1} - \frac{G^{N-1}_{S,1}(s_{N-1,1})}{g^{N-1}_{S,1}(s_{N-1,1})} &= \left(\frac{1+r}{r}\left[1 - \frac{1}{(1+r)^2}\right] + \frac{N}{(1+r)}\right) \\
(N+1) - 2s_{N-1,1} - \frac{G^{N-1}_{S,1}(s_{N-1,1})}{g^{N-1}_{S,1}(s_{N-1,1})} &= \frac{2+r+N}{(1+r)},
\end{aligned}$$

so r is uniquely identified by observables in the second-to-last round; its distribution can be non-parametrically estimated using the observed winning bids in the second-to-last round.

Of course, the alert reader will note that, in the second-to-last round, one only observes the winning bid, the maximum of the two bids in that round. Denote by $G^{N-1}_{W,1}(w)$ the cumulative distribution function of the winning unit bid in the second-to-last round of the hụi and by $g^{N-1}_{W,1}(w)$ its corresponding probability density function. Now,

$$W_{N-1,1} = \max(S_{N-1,1,1}, S_{N-1,1,2}),$$

so

$$G^{N-1}_{W,1}(w) = G^{N-1}_{S,1}(w)^2,$$

or

$$G^{N-1}_{S,1}(s) = \sqrt{G^{N-1}_{W,1}(s)},$$

and

$$g^{N-1}_{W,1}(w) = 2G^{N-1}_{S,1}(w)g^{N-1}_{S,1}(w),$$

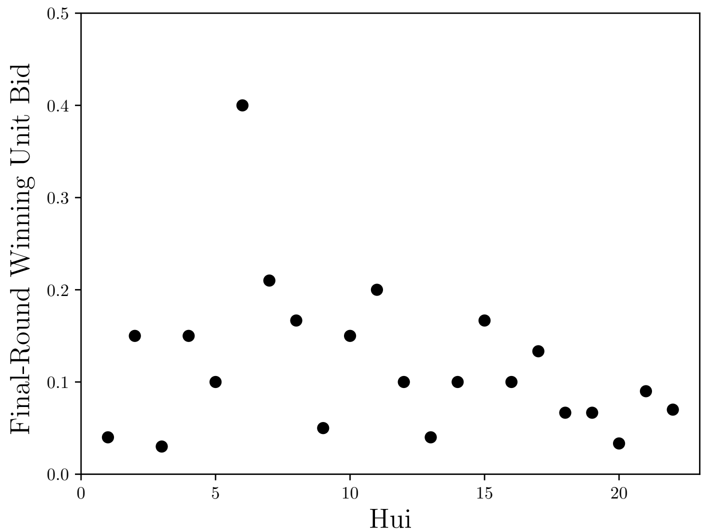

Figure 11: Final Round Unit Winning Discount Bids

so

$$g_{S,1}^{N-1}(s) = \frac{g_{W,1}^{N-1}(s)}{2G_{S,1}^{N-1}(s)} = \frac{g_{W,1}^{N-1}(s)}{2\sqrt{G_{W,1}^{N-1}(s)}}.$$

In short, the model is non-parametrically identified in the second-to-last round.

Unfortunately, we have found it difficult to gather more than a small sample of data from hụi held in Melbourne in the 2000s. As mentioned, in the previous section, we have data from twenty-two hụi. For each of the last rounds of those hụi, we have plotted, in figure 11, the unit winning discounts. In figure 12, we present the GPV kernel-smoothed estimate of $f_R^0(r)$ (the solid line) as well as the maximum-likelihood (ML) estimate assuming a Beta distribution (the dashed line).

In light of this dearth of data and in order to implement our theoretical model, we have been forced to make a parametric assumption concerning the distribution of R. In particular, we assume that R is distributed $\mathrm{B}(\theta_1, \theta_2)$ on the interval $[0, \theta_3]$. We recognize that this three-parameter family of distributions is restrictive.

Consider $\{(u_h, N_h, w_{1,h}, w_{2,h}, \ldots, w_{N_h-1})\}_{h=1}^{H}$, a sample of H hụi, indexed by $h = 1, 2, \ldots, H$. Under our second informational assumption,

$$w_{t,h} = \sigma_t\left[r_{(1:N_h-t+1)}; \boldsymbol{\theta}, u_h, N_h\right]$$

where we have now made explicit the dependence of the winning bid discounts on both u_h and N_h as well as $\boldsymbol{\theta}$. Denote the cumulative distribution function of $R_{(1:N_h-t+1)}$ for participants at hụi h by

$$F_{(1:N_h-t+1)}(r; \boldsymbol{\theta}, N_h, t) = (N_h - t + 1)\int_0^{F_R^0(r;\boldsymbol{\theta})} x^{N_h-t}\,\mathrm{d}x$$

and its probability density function by

$$f_{(1:N_h-t+1)}(r; \boldsymbol{\theta}, N_h, t) = (N_h - t + 1)F_R^0(r; \boldsymbol{\theta})^{N_h-t} f_R^0(r; \boldsymbol{\theta}).$$

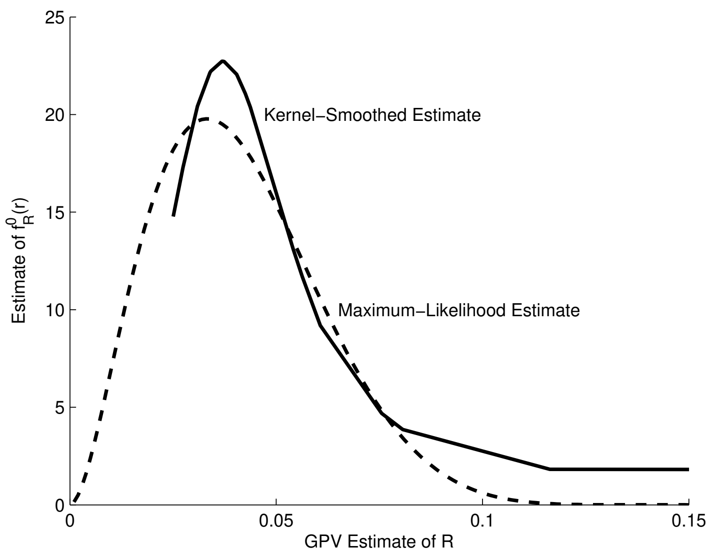

Figure 12: Kernel-Smoothed and Maximum-Likelihood Estimates of $f_R^0(r)$

Now, the probability density function of the winning bid in round t of hụi h is then

$$f_{W_{t,h}}(w;\boldsymbol{\theta},u_h,N_h,t) = \frac{f_{(1:N_h-t+1)}\left[\sigma_t^{-1}(w;\boldsymbol{\theta},u_h,N_h);\boldsymbol{\theta},N_h,t\right]}{\sigma_t'\left[\sigma_t^{-1}(w;\boldsymbol{\theta},u_h,N_h);,\boldsymbol{\theta},u_h,N_h,t\right]}.$$

Thus, collecting the u_hs in the vector $\boldsymbol{u}$, the N_h in the vector $\boldsymbol{N}$, and the $w_{t,h}$s in the vector $\boldsymbol{w}$, the logarithm of the likelihood function can be written as

$$\mathcal{L}(\boldsymbol{\theta};\boldsymbol{u},\boldsymbol{N},\boldsymbol{w}) = \sum_{h=1}^{H}\sum_{t=1}^{N_h-1}\Bigg[\log\Big(f_{(1:N_h-t+1)}\left[\sigma_t^{-1}(w_{t,h};\boldsymbol{\theta},u_h,N_h);\boldsymbol{\theta},N_h,t\right]\Big) - \\ \log\Big(\sigma_t'\left[\sigma_t^{-1}(w_{t,h};\boldsymbol{\theta},u_h,N_h);\boldsymbol{\theta},u_h,N_h,t\right]\Big)\Bigg].$$

To estimate this empirical specification, we proceeded as follows:

0. set $k=0$ and initialize $\boldsymbol{\theta}$ at $\tilde{\boldsymbol{\theta}}^k$;

1. solve for $\tilde{\sigma}_{t,1}^k(r) = \sigma_{t,1}(r;\tilde{\boldsymbol{\theta}}^k,N_h)$ and $\tilde{V}_1^k(r,t)$ for $t=1,2,\ldots,N_h-1$ and $h=1,2,\ldots H$;

2. for each $w_{t,h}$ in $\boldsymbol{w}$, then solve $(w_{t,h}/u_h) = \tilde{\sigma}_{t,1}^k\left[\tilde{r}_{(1:N_h-t+1)}^k\right]$—viz., find the $\tilde{r}_{(1:N_h-t+1)}^k$ consistent with $\tilde{\boldsymbol{\theta}}^k$;

3. form the logarithm of the likelihood function for iteration k and maximize it with respect to $\boldsymbol{\theta}$, taking into account the following constraints:

$$\frac{w_{t,h}}{u_h} \le \tilde{\sigma}_{t,1}^k(\bar{r}) \quad t=1,2,\ldots,N_h-1;\ h=1,2,\ldots,H;$$

4. check for an improvement in the objective function: if no improvement obtains, then stop, otherwise increment k and update $\tilde{\boldsymbol{\theta}}^k$ and return to step 1.

5 Empirical Results

6 Policy Experiments

An alternative way in which to conduct the auction in each round of the hụi would be to use a second-price rule. This could be done in a number of different ways, which are not outcome equivalent, even under our assumed information structure. Under other information structures, such as ones having affiliated rate-of-return draws, even greater differences could obtain.

In the first case, we propose that, at the beginning of the hụi, each participant is required to report a present discounted value of the income stream from his investment, given his rate-of-return. The winner is the participant with the highest-valued report, but he only pays the highest of his opponents' reports. How to implement this as a second-price, sealed-bid auction?

Consider the following structure: in the first round of the hụi, when a bid b is charged, denote by

$$
\begin{aligned}
\hat{v}(r_n, b) &= u + (N-1)(u-b) - u \sum_{i=1}^{N-1} \frac{1}{(1+r_n)^i} \\
&= 2u + (N-1)(u-b) - u \sum_{i=0}^{N-1} \frac{1}{(1+r_n)^i} \\
&= (N+1)u - (N-1)b - u \frac{(1+r_n)\left[1 - \frac{1}{(1+r_n)^N}\right]}{r_n}
\end{aligned}
$$

the present discounted value of participant n's investment opportunity where the discounting is done using his own rate-of-return r_n. Note, too, that participant n is indifferent between winning the auction with a bid b_n and getting nothing, when b_n solves

$$
\hat{v}(r_n, b_n) = 0 = (N+1)u - (N-1)b_n - u \frac{(1+r_n)\left[1 - \frac{1}{(1+r_n)^N}\right]}{r_n}
$$

so

$$
\begin{aligned}
b_n &= \frac{(N+1) - \frac{(1+r_n)}{r_n}\left[1 - \frac{1}{(1+r_n)^N}\right]}{(N-1)} u \\
&\equiv \beta_{1,1}(r_n; N) u
\end{aligned}
$$

where $\beta_{1,1}(r_n; N)$ is the unit bid function in a hụi having N rounds, when the return is r_n. Under the second-price, sealed-bid format, all N participants would submit their bids $\{b_n\}_{n=1}^N$. These bids would then be ordered, so

$$
b_{(1)} \geq b_{(2)} \geq \cdots \geq b_{(N)} \geq b_{(N+1)} = 0,
$$

and the hụi would then end. The winner of round t would be the participant who tendered $b_{(t)}$, and he would pay $b_{(t+1)}$, with the last participant's paying the implicit reserve price $b_{(N+1)}$—viz., zero.

What about holding a sequence of second-price, sealed-bid auctions, instead of holding just one in the first round? Well, one feature of the second-price auction is that, when the winner is determined in the first round, the participant with the second-highest rate-of-return, is made aware of his place in the order. How? When a participant has the second-highest rate-of-return, this information is confirmed to him because he sees his bid as the winning bid in the first round. Thus, this participant is asymmetrically informed *vis-à-vis* his opponents. Even within the independent private-values paradigm, this differential information release has relevance: to wit, it is not a dominant strategy for each participant to reveal the truth.

Suppose that we shut-down this asymmetry of information. How? Let us assume that before each round of the hụi new rates-of-return are drawn independently from $F_R^0(r)$ for the remaining participants. In each round of the hụi, a second-price, sealed-bid auction is conducted to determine who will win that round of the hụi, and what bid discount will be paid; only the winning bid is revealed.

In round t of an hụi, we denote the realized ordered bids, from largest to smallest, by

$$b_{(1)} \geq b_{(2)} \geq \cdots \geq b_{(N-t+1)},$$

and the random variables by

$$B_{(1)} \geq B_{(2)} \geq \cdots \geq B_{(N-t+1)}.$$

The winner is the participant with the highest bid $b_{(1)}$, but he pays what his nearest opponent tendered $b_{(2)}$. Prior to bidding, however, no participant knows $B_{(2)}$, the highest bid of his opponents: $B_{(2)}$ is a random variable.

How does a participant determine how much to bid—effectively, in which round of the hụi to exit? Again, we can couch the solution to this problem in terms of the solution to a dynamic programme. For a representative participant, this dynamic programming problem has two state variables: t, the round of the hụi, and r, the realization of his draw from the distribution of rates-of-return. We seek to construct a sequence of optimal policy (equilibrium bid) functions $\{\beta_t\}_{t=1}^N$. In round t, the optimal policy function β_t maps the rate-of-return state R into the real line. We begin by describing the problem intuitively.

In round t, the value function of participating in this round as well as all later ones can be decomposed into the expected value of winning the current round plus the expected discounted continuation value of the game, should one lose this round, so

$$V(r,t) = \max_{<b>} \Big(\text{Expected Value of Winning, Given Bid } b\Big) + \Big(\text{Expected Discounted Continuation Value}\Big).$$

When he wins round t of the hụi, a participant earns $[tu + (N-t)(u - B_{(2)})]$, which represents the capital raised in the current period. However, he owes $-u\sum_{i=1}^{N-t}(1+r)^{-i}$, which represents the current-valued obligations of what must be repaid, discounted using the participant's cost-of-funds, r, the rate-of-return on his potential investment. As mentioned above, $B_{(2)}$ is a random variable. Thus, it needs to be integrated out. To this end, one needs to derive the joint probability density function of the highest two order statistics from an independent and identically-distributed sample of size M, which equals $(N-t+1)$, the number of participants in round t of the hụi. Denoting the highest order statistic by Y and the second-highest one by X, the joint probability density function of X and Y is

$$f_{12}(x,y) = \begin{cases} \frac{M!}{(M-1-1)!(M-(M-1)-1)!(M-M)!} F_R^0(x)^{M-2} f_R^0(x) f_R^0(y) & x < y \\ 0 & x \geq y. \end{cases}$$

We construct the $\{\beta_t\}_{t=1}^{N}$ as well as $V^*(r,t)$ recursively. The solution to the bidding problem in the last round is easily found: since the reserve price in each round is zero, because he faces no competitors, the last participant need only bid zero for any rate-of-return. Thus, the optimal policy function, for all feasible R, is

$$\beta_N(r) = 0.$$

Hence, in the last round, N, for any feasible value of R,

$$V^*(r,N) = Nu.$$

Consider now a representative participant in the second-to-last round who faces only one other opponent. Suppose the participant's opponent is using a monotonically increasing function $\hat{\beta}_{N-1}(r)$. The participant wins when his bid is higher than his opponent's because his rate-of-return is higher than the sole remaining opponent. Although the price he pays is random, under risk neutrality, the expected value of winning round $(N-1)$, so M is $[N-(N-1)+1]$ or two, is

$$\int_{\underline{r}}^{\hat{\beta}_{N-1}^{-1}(b)} \int_{\underline{r}}^{y} \left((N-1)u + [u - \hat{\beta}_{N-1}(x)] - u\frac{1}{(1+r)} \right) 2f_R^0(x)\,\mathrm{d}x\, f_R^0(y)\,\mathrm{d}y.$$

On the other hand, when he loses, the expected discounted continuation value is

$$\int_{\hat{\beta}_{N-1}^{-1}(b)}^{\bar{r}} \int_{\underline{r}}^{y} \left((b-u) + \mathbb{E}\left[\frac{Nu}{(1+R)}\right] \right) 2f_R^0(x)\,\mathrm{d}x\, f_R^0(y)\,\mathrm{d}y.$$

The above expression warrants some explanation. In the last round of the hụi, the value of the optimal programme is

$$V^*(r,N) = Nu,$$

so for some realization r, its discounted value is

$$\frac{V^*(r,N)}{(1+r)} = \frac{Nu}{(1+r)}.$$

But, by assumption, new rates-of-return are drawn in each successive round for remaining participants, so its expectation is

$$\mathbb{E}\left[\frac{Nu}{(1+R)}\right].$$

Also, when a participant loses the second-to-last round of the hụi, his losing bid determines what he earns. Hence, the term $(b-u)$, which is his losing bid in the second-to-last round of the hụi, minus what he contributed to the hụi in that round. As we shall see below, however, this is a special feature of the second-to-last round. Bringing all of this together yields

$$\begin{aligned} &V(r,N-1) = \\ &\max_{<b>} \int_{\underline{r}}^{\hat{\beta}_{N-1}^{-1}(b)} \int_{\underline{r}}^{y} \left((N-1)u + [u - \hat{\beta}_{N-1}(x)] - u\frac{1}{(1+r)} \right) 2f_R^0(x)\,\mathrm{d}x\, f_R^0(y)\,\mathrm{d}y\, + \\ &\quad \int_{\hat{\beta}_{N-1}^{-1}(b)}^{\bar{r}} \int_{\underline{r}}^{y} \left((b-u) + \mathbb{E}\left[\frac{Nu}{(1+R)}\right] \right) 2f_R^0(x)\,\mathrm{d}x\, f_R^0(y)\,\mathrm{d}y. \end{aligned}$$

The following first-order condition is a necessary condition for an optimum:

$$
\begin{aligned}
&\frac{\mathrm{d}V(r,N-1)}{\mathrm{d}b} = \\
&\int_{\underline{r}}^{y} \left((N-1)u + [u - \hat{\beta}_{N-1}(x)] - u\frac{1}{(1+r)} \right) 2f_R^0(x)\ \mathrm{d}x\ f_R^0\left[\hat{\beta}_{N-1}^{-1}(b)\right] \frac{\mathrm{d}\hat{\beta}_{N-1}^{-1}(b)}{\mathrm{d}b} - \\
&\int_{\underline{r}}^{y} \left((b-u) + \mathbb{E}\left[\frac{Nu}{(1+R)} \right] \right) 2f_R^0(x)\ \mathrm{d}x\ f_R^0\left[\hat{\beta}_{N-1}^{-1}(b)\right] \frac{\mathrm{d}\hat{\beta}_{N-1}^{-1}(b)}{\mathrm{d}b} + \\
&\int_{\hat{\beta}_{N-1}^{-1}(b)}^{\overline{r}} \int_{\underline{r}}^{y} 2f_R^0(x)\ \mathrm{d}x\ f_R^0(y)\ \mathrm{d}y = 0.
\end{aligned}
$$

In a symmetric perfect Bayesian equilibrium, $b = \hat{\beta}_{N-1}(r)$ and, by monotonicity, $\mathrm{d}\hat{\beta}_{N-1}^{-1}(b)/\mathrm{d}b$ equals $1/[\mathrm{d}\hat{\beta}_{N-1}(r)/\mathrm{d}r]$, so the first-order condition above can be re-written as the following non-linear differential equation:

$$
\begin{aligned}
&\int_{\underline{r}}^{r} \left((N-1)u + [u - \hat{\beta}_{N-1}(x)] - u\frac{1}{(1+r)} \right) 2f_R^0(x)\ \mathrm{d}x\ \frac{f_R^0(r)\ \mathrm{d}r}{\mathrm{d}\hat{\beta}_{N-1}(r)} - \\
&\int_{\underline{r}}^{r} \left(\left[\hat{\beta}_{N-1}(x) - u\right] + \mathbb{E}\left[\frac{Nu}{(1+R)} \right] \right) 2f_R^0(x)\ \mathrm{d}x\ \frac{f_R^0(r)\ \mathrm{d}r}{\mathrm{d}\hat{\beta}_{N-1}(r)} + \left[1 - F_R^0(r)^2\right] = 0,
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\beta}_{N-1}(r)}{\mathrm{d}r} = &\frac{\int_{\underline{r}}^{r} \left(\left[\hat{\beta}_{N-1}(x) - u\right] + \mathbb{E}\left[\frac{Nu}{(1+R)}\right] \right) 2f_R^0(x)\ \mathrm{d}x\ f_R^0(r)}{\left[1 - F_R^0(r)^2\right]} - \\
&\frac{\int_{\underline{r}}^{r} \left[Nu - \hat{\beta}_{N-1}(x) - u\frac{1}{(1+r)} \right] 2f_R^0(x)\ \mathrm{d}x\ f_R^0(r)}{\left[1 - F_R^0(r)^2\right]}.
\end{aligned}
$$

The initial condition is $\beta_{N-1}(\underline{r})$ equal $\underline{r}u$: when a participant has the lowest possible rate-of-return, he bids the value of that rate-of-return in terms of the hụi deposit u. This differential equation can only be solved numerically. Later, we assume $\underline{r}$ is zero, so the initial condition will be zero.

Like $\sigma_{N-1}(\cdot)$, $\beta_{N-1}(\cdot)$ is homogeneous of degree one in u. For later use, we denote a bid function when u is one, a "unit" bid function, by $\beta_{N-1,1}(\cdot)$. Also,

$$
\begin{aligned}
V^*(r,N-1) = &\int_{\underline{r}}^{r} \int_{\underline{r}}^{y} \left[Nu - \beta_{N-1}(x) - u\frac{1}{(1+r)} \right] 2f_R^0(x)\ \mathrm{d}x\ f_R^0(y)\ \mathrm{d}y + \\
&\int_{r}^{\overline{r}} \int_{\underline{r}}^{y} \left([\beta_{N-1}(x) - u] + \mathbb{E}\left[\frac{Nu}{(1+R)} \right] \right) 2f_R^0(x)\ \mathrm{d}x\ f_R^0(y)\ \mathrm{d}y.
\end{aligned}
$$

which is homogeneous of degree one in u, too.

Consider now round $(N-2)$, so M is three. In this case,

$$
\begin{aligned}
&V(r,N-2) = \\
&\quad \max_{<b>} \int_{\underline{r}}^{\hat{\beta}_{N-2}^{-1}(b)} \int_{\underline{r}}^{y} \left((N-2)u + 2[u - \hat{\beta}_{N-2}(x)] - u\sum_{i=1}^{2}\frac{1}{(1+r)^i} \right) \times
\end{aligned}
$$

$$6F_R(x)f_R^0(x)\ \mathrm{d}x\ f_R^0(y)\ \mathrm{d}y\ +$$

$$\int_{\hat{\beta}_{N-2}^{-1}(b)}^{\bar{r}}\int_{\underline{r}}^{y}\left(\left[0.5\times\hat{\beta}_{N-2}(x)+0.5\times b-u\right]+\mathbb{E}\left[\frac{V^*(R,N-1)}{(1+R)}\right]\right)\times$$

$$6F_R^0(x)f_R^0(x)\ \mathrm{d}x\ f_R^0(y)\ \mathrm{d}y.$$

The above expression also warrants some explanation: specifically, the presence of $\hat{\beta}_{N-2}(x)$ in the second integral as well as the 0.5 multiplying it, and b, demand discussion. In round $(N-1)$, this is simply b because, if a participant loses, then his action determines what he is paid. In round $(N-2)$, however, a losing participant's action only determines what he is paid with some probability. Under the sampling scheme assumed above, the probability that one's bid determines what one is paid is $[1/(M-1)]$, in this case one-half; the probability that one of the losing opponents determines what one is paid is $[(M-2)/(M-1)]$, in this case also one-half. The following first-order condition is a necessary condition for an optimum:

$$\frac{\mathrm{d}V(r,N-2)}{\mathrm{d}b} = \int_{\underline{r}}^{y}\left((N-2)u+2[u-\hat{\beta}_{N-2}(x)]-u\sum_{i=1}^{2}\frac{1}{(1+r)^i}\right)\times$$

$$6F_R^0(x)f_R^0(x)\ \mathrm{d}x\ f_R^0\left[\hat{\beta}_{N-2}^{-1}(b)\right]\frac{\mathrm{d}\hat{\beta}_{N-2}^{-1}(b)}{\mathrm{d}b}\ -$$

$$\int_{\underline{r}}^{y}\left(\left[0.5\times\hat{\beta}_{N-2}(x)+0.5\times b-u\right]+\mathbb{E}\left[\frac{V^*(R,N-1)}{(1+R)}\right]\right)\times$$

$$6F_R^0(x)f_R^0(x)\ \mathrm{d}x\ f_R^0\left[\hat{\beta}_{N-2}^{-1}(b)\right]\frac{\mathrm{d}\hat{\beta}_{N-2}^{-1}(b)}{\mathrm{d}b}\ +$$

$$0.5\times\int_{\hat{\beta}_{N-2}^{-1}(b)}^{\bar{r}}\int_{\underline{r}}^{y}6F_R^0(x)f_R^0(x)\ \mathrm{d}x\ f_R^0(y)\ \mathrm{d}y = 0.$$

In a symmetric perfect Bayesian equilibrium, $b = \hat{\beta}_{N-2}(r)$ and, by monotonicity, $\mathrm{d}\hat{\beta}_{N-2}^{-1}(b)/\mathrm{d}b$ equals $1/[\mathrm{d}\hat{\beta}_{N-2}(r)/\mathrm{d}r]$, so the first-order condition above can be re-written as the following nonlinear differential equation:

$$\int_{\underline{r}}^{y}\left((N-2)u+2[u-\hat{\beta}_{N-2}(x)]-u\sum_{i=1}^{2}\frac{1}{(1+r)^i}\right)\times$$

$$6F_R^0(x)f_R^0(x)\ \mathrm{d}x\ \frac{f_R^0(r)\ \mathrm{d}r}{\mathrm{d}\hat{\beta}_{N-2}(r)}\ -$$

$$\int_{\underline{r}}^{y}\left(\left[\hat{\beta}_{N-2}(x)-u\right]+\mathbb{E}\left[\frac{V^*(R,N-1)}{(1+R)}\right]\right)\times$$

$$6F_R^0(x)f_R^0(x)\mathrm{d}x\ \frac{f_R^0(r)\ \mathrm{d}r}{\mathrm{d}\hat{\beta}_{N-2}(r)}+0.5\times[1-F_R^0(r)^3]=0$$

or

$$\frac{\mathrm{d}\hat{\beta}_{N-2}(r)}{\mathrm{d}r} =$$

$$\frac{\int_{\underline{r}}^{r}\left(\left[\hat{\beta}_{N-2}(x)-u\right]+\mathbb{E}\left[\frac{V^*(R,N-1)}{(1+R)}\right]\right)12F_R^0(x)f_R^0(x)\ \mathrm{d}x\ f_R^0(r)}{\left[1-F_R^0(r)^3\right]}\ -$$

$$\frac{\int_{\underline{r}}^{r}\left((N+1)u-\hat{\beta}_{N-2}(x)-\frac{u(1+r)}{r}\left[1-\frac{1}{(1+r)^3}\right]\right)12F_R^0(x)f_R^0(x)\,\mathrm{d}x\,f_R^0(r)}{\left[1-F_R^0(r)^3\right]}.$$

Consider now any other round t where

$$V(r,t)=\max_{<b>}\int_{\underline{r}}^{\hat{\beta}_t^{-1}(b)}\int_{\underline{r}}^{y}\left(tu+(N-t)[u-\hat{\beta}_t(x)]-u\sum_{i=1}^{N-t}\frac{1}{(1+r)^i}\right)\times$$
$$(N-t+1)(N-t)F_R(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x\,f_R^0(y)\,\mathrm{d}y+$$
$$\int_{\hat{\beta}_t^{-1}(b)}^{\bar{r}}\int_{\underline{r}}^{y}\left(\left[\pi_t\hat{\beta}_t(x)+(1-\pi_t)b-u\right]+\mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right]\right)\times$$
$$(N-t+1)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x\,f_R^0(y)\,\mathrm{d}y.$$

Here, $(1-\pi_t)$ equals $[1/(N-t)]$. The following first-order condition is a necessary condition for an optimum:

$$\frac{\mathrm{d}V(r,t)}{\mathrm{d}b}=\int_{\underline{r}}^{y}\left(tu+(N-t)[u-\hat{\beta}_t(x)]-u\sum_{i=1}^{N-t}\frac{1}{(1+r)^i}\right)\times$$
$$(N-t+1)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x\,f_R^0\left[\hat{\beta}_t^{-1}(b)\right]\frac{\mathrm{d}\hat{\beta}_t^{-1}(b)}{\mathrm{d}b}-$$
$$\int_{\underline{r}}^{y}\left(\left[\pi_t\hat{\beta}_t(x)+(1-\pi_t)b-u\right]+\mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right]\right)\times$$
$$(N-t+1)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x\,f_R^0\left[\hat{\beta}_{N-2}^{-1}(b)\right]\frac{\mathrm{d}\hat{\beta}_{N-2}^{-1}(b)}{\mathrm{d}b}+$$
$$\int_{\hat{\beta}_t^{-1}(b)}^{\bar{r}}\int_{\underline{r}}^{y}(1-\pi_t)(N-t+1)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x\,f_R^0(y)\,\mathrm{d}y=0.$$

In a symmetric perfect Bayesian equilibrium, $b=\hat{\beta}_t(r)$ and, by monotonicity, $\mathrm{d}\hat{\beta}_t^{-1}(b)/\mathrm{d}b$ equals $1/[\mathrm{d}\hat{\beta}_t(r)/\mathrm{d}r]$, so the first-order condition above can be re-written as the following nonlinear differential equation:

$$\int_{\underline{r}}^{r}\left(tu+(N-t)[u-\hat{\beta}_t(x)]-u\sum_{i=1}^{N-t}\frac{1}{(1+r)^i}\right)\times$$
$$(N-t+1)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\,\mathrm{d}x\,\frac{f_R^0(r)\,\mathrm{d}r}{\mathrm{d}\hat{\beta}_t(r)}-$$
$$\int_{\underline{r}}^{r}\left(\left[\hat{\beta}_t(x)-u\right]+\mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right]\right)\times$$
$$(N-t+1)(N-t)F_R^0(x)^{N-t-1}f_R^0(x)\mathrm{d}x\,\frac{f_R^0(r)\,\mathrm{d}r}{\mathrm{d}\hat{\beta}_t(r)}+(1-\pi_t)[1-F_R^0(r)^{N-t}]=0$$

or

$$\frac{\mathrm{d}\hat{\beta}_t(r)}{\mathrm{d}r}=$$

Table 3: Net Cash Flow: *Tanda* (Lottery) Version of Hụi used in Mexico

Bidder/Round	Banker	1	2	3	4	Final
Banker	$1,200	−$300	−$300	−$300	−$300	$0
1	−300	1,200	−$300	−300	−300	0
2	−300	−300	1,200	−300	−300	0
3	−300	−300	−300	1,200	−300	0
4	−300	−300	−300	−300	1,200	0

$$\frac{\int_{\underline{r}}^{r} \left(\left[\hat{\beta}_t(x) - u\right] + \mathbb{E}\left[\frac{V^*(R,N-1)}{(1+R)}\right]\right) (N-t+1)(N-t)^2 F_R^0(x)^{N-t-1} f_R^0(x)\,\mathrm{d}x\, f_R^0(r)}{\left[1 - F_R^0(r)^{N-t+1}\right]} -$$
$$\frac{\int_{\underline{r}}^{r} \left((N+1)u - \hat{\beta}_{N-2}(x) - \frac{u(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right]\right) (N-t+1)(N-t)^2 F_R^0(x)^{N-t-1} f_R^0(x)\,\mathrm{d}x\, f_R^0(r)}{\left[1 - F_R^0(r)^{N-t+1}\right]}.$$

Although the differential equations derived above are nonlinear, first-order differential equations, they can be written as linear, second-order differential equations.

Another alternative is not to hold an auction at all, but rather to hold a lottery in each round of the hụi. In fact, this is how the hụi is conducted in some parts of Mexico, where it is apparently referred to as the *tanda*. In Mexico, under the rules of the *tanda*, each participant deposits u with the banker at the beginning. The winner in the first round, and each subsequent round, is chosen at random from the pool of remaining participants. In this case, following the example from table 1 of the introduction, the cash payouts are summarized in table 3.

How to value this institution? Well, one way would be to calculate the expected value of each round, under random sampling, as well as the average value of the *tanda* using $f_R^0(r)$, and then to compare this with the average value of the other two institutions.

In general, there are N potential pay-out streams which, when valued by a random participant's rate-of-return r, have the following present discounted value:

$$\begin{aligned}
P_1(r) &= -u + Nu - \frac{u}{(1+r)} - \frac{u}{(1+r)^2} - \cdots - \frac{u}{(1+r)^{(N-1)}} \\
P_2(r) &= -u - u + \frac{Nu}{(1+r)} - \frac{u}{(1+r)^2} - \cdots - \frac{u}{(1+r)^{(N-1)}} \\
\vdots &= \vdots \\
P_{N-1}(r) &= -u - u - \frac{u}{(1+r)} - \cdots + \frac{Nu}{(1+r)^{N-2}} - \frac{u}{(1+r)^{N-1}} \\
P_N(r) &= -u - u - \frac{u}{(1+r)} - \cdots - \frac{u}{(1+r)^{N-2}} + \frac{Nu}{(1+r)^{N-1}}.
\end{aligned}$$

Now,

$$\mathbb{E}\left[P_t(R)\right] = \int_{\underline{r}}^{\bar{r}} P_t(r) f_R^0(r)\,\mathrm{d}r.$$

Because the lottery assigns participants at random to these investment streams, the average value to investments allocate under the *tanda* rule is

$$\frac{1}{N}\sum_{t=1}^{N} \mathbb{E}\left[P_t(R)\right].$$

Depending on the informational assumption, we can compare this to

$$\frac{1}{N}\sum_{t=1}^{N}\mathbb{E}\left\{P_t\left[R_{(t:N)}\right]\right\}$$

and

$$\frac{1}{N}\sum_{t=1}^{N}\mathbb{E}\left\{P_t\left[R_{(1:N-t)}\right]\right\}$$

to get some notion concerning how much is gained by ordering the investments optimally by rate-of-return.

7 Summary and Conclusions

Using the theory of non-cooperative games under incomplete information, we have analyzed the hụi—a borrowing and lending institution used by Vietnamese immigrants in Australia and New Zealand, in particular, but in other parts of the world as well. Essentially, the hụi is a sequential, double auction among the participants in a collective. Within the symmetric independent private-values paradigm, we constructed the perfect Bayesian equilibrium of a sequential, first-price, sealed-bid auction game and then investigated the properties of the equilibrium using numerical methods. We also demonstrated that this model is non-parametrically identified, at least in the second-to-round of the hụi. Subsequently, we used this structure to interpret field data gathered from a sample of hụi held in Melbourne, Australia during the early 2000s. We also investigated two simple policy experiments—one involving a shift to a second-price, sealed-bid format and the other a shift to a lottery, which is how a mechanism like the hụi is implemented in Mexico. Under the second-price, sealed-bid format we constructed a perfect Bayesian equilibrium and demonstrated that, unlike the first-price modle, this model is non-parametrically unidentified, even in the second-to-round of the hụi. Unlike in single-object auctions within the IPVP, pay-off equivalence does not exist under either of our independent private-values assumptions. Although it is obvious that the hụi will do much better in allocating capital efficiently than the random allocation under the *tanda*, our estimates provide some notion of the efficiency gain from using the hụi.

As a economic institution, the hụi obviously facilitates inter-temporal smoothing, and appears implementable under primitive market conditions, such as those present in developing countries. Presumably, the structure of the hụi accommodates an informational asymmetry that conventional banks cannot.

The hụi is one way in which an overlapping generations model can be implemented in practice. By and large, there are two kinds of immigrants participating in the hụi: first, young immigrants who have difficulty raising capital through conventional financial institutions, probably because they do not have credit histories long enough to make them credit worthy; second, older immigrants who, for various reasons, may not trust depositing their savings at conventional financial institutions.

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A. Appendix

In this appendix, we present calculations too cumbersome for inclusion in the text of the paper as well as describe the creation of the data set used.

A.1 Borrowers and Lenders

In this section of the appendix, we expand the model to admit two types of participants in the hụi, those whom we refer to as *borrowers*, and those whom we refer to as *lenders*. For notational parsimony, we refer to the value of the hụi in a representative round t to a participant as v, instead of writing out

$$(N+1)u - \frac{(1+r)}{r}\left[1 - \frac{1}{(1+r)^{N-t+1}}\right]u - \mathbb{E}\left[\frac{V^*(R,t+1)}{(1+R)}\right].$$

We imagine two different urns from which rates-of-return are drawn. Intuitively, the borrowers have a distribution of rates-of-returns which is everywhere to the right of the distribution of that for the lenders. However, in any round, it is possible that a borrower gets a draw that is below that of some of the lenders: such is the nature of random draws. Below, we are going to represent the unobserved rate-of-return heterogeneity as heterogeneity in values. Without loss of generality, we assume that the borrowers are type 1, while the lenders are type 2.

Thus, consider two urns $F_1(v)$ and $F_2(v)$. Suppose there are K potential opponents, of which an unknown K_1 are potential *borrowers*, while K_2 are potential *lenders* where $(K_1 + K_2)$ equals K, where in the models considered above K is $(N-t)$. Suppose the number of type 1 opponents is distributed binomially, having the following probability mass function:

$$p_M(m; K, \alpha) = \binom{K}{m}\alpha^m(1-\alpha)^{K-m}, \quad 0 < \alpha \leq 1,\ m = 0, 1, 2, \ldots, K.$$

Now, expected profit to a type $i = 1, 2$ bidder having value v who submits s_i is

$$\pi_i(v, s_i) = (v - s_i)\Pr(\mathsf{win}|s_i).$$

Suppose a potential bidder of type $i = 1, 2$ bids is using an increasing monotonic function $\hat{\sigma}_i(v)$ where $\hat{\sigma}_i'(v) > 0$. Conditional on m,

$$\Pr(\mathsf{win}|s_1, m) = F_1\left[\hat{\sigma}_1^{-1}(s_1)\right]^m F_2\left[\hat{\sigma}_2^{-1}(s_1)\right]^{K-m}$$

and

$$\Pr(\mathsf{win}|s_2, m) = F_1\left[\hat{\sigma}_1^{-1}(s_2)\right]^m F_2\left[\hat{\sigma}_2^{-1}(s_2)\right]^{K-m},$$

while

$$\sum_{m=0}^{K}\Pr(\mathsf{win}|s_1, m)p_M(m; K, \alpha) =$$

$$\sum_{m=0}^{K} F_1\left[\hat{\sigma}_1^{-1}(s_1)\right]^m F_2\left[\hat{\sigma}_2^{-1}(s_1)\right]^{K-m}\binom{K}{m}\alpha^m(1-\alpha)^{K-m} =$$
$$\left(\alpha F_1\left[\hat{\sigma}_1^{-1}(s_1)\right]+(1-\alpha)F_2\left[\hat{\sigma}_2^{-1}(s_1)\right]\right)^K$$

and

$$\sum_{m=0}^{K}\Pr(\mathsf{win}|s_2,m)p_M(m;K,\alpha) =$$
$$\sum_{m=0}^{K} F_1\left[\hat{\sigma}_1^{-1}(s_2)\right]^m F_2\left[\hat{\sigma}_2^{-1}(s_2)\right]^{K-m}\binom{K}{m}\alpha^m(1-\alpha)^{K-m} =$$
$$\left(\alpha F_1\left[\hat{\sigma}_1^{-1}(s_2)\right]+(1-\alpha)F_2\left[\hat{\sigma}_2^{-1}(s_2)\right]\right)^K,$$

so

$$\pi_i(v,s_i) = (v-s_i)\left(\alpha F_1\left[\hat{\sigma}_1^{-1}(s_i)\right]+(1-\alpha)F_2\left[\hat{\sigma}_2^{-1}(s_i)\right]\right)^K.$$

Now,

$$\begin{aligned}\frac{\partial\pi_i(v,s_i)}{\partial s_i} &= -(\cdot)^K+(v-s_i)\left[K(\cdot)^{K-1}\frac{\partial(\cdot)}{\partial s_i}\right]\\ &= -(\cdot)^K+\\ &\qquad (v-s_i)K(\cdot)^{K-1}\left[\alpha f_1(\cdot)\frac{\mathrm{d}\hat{\sigma}_1^{-1}(\cdot)}{\mathrm{d}s_i}+(1-\alpha)f_2(\cdot)\frac{\mathrm{d}\hat{\sigma}_2^{-1}(\cdot)}{\mathrm{d}s_i}\right]\\ &= 0,\end{aligned}$$

so, at an equilibrium,

$$\begin{aligned}1 &= \left\{\frac{K\left[\alpha f_1(v)\frac{1}{\sigma_1'(v)}+(1-\alpha)f_2(v)\frac{1}{\sigma_2'(v)}\right]}{[\alpha F_1(v)+(1-\alpha)F_2(v)]}\right\}[v-\sigma_i(v)]\\ \sigma_1'(v)\sigma_2'(v) &= \left\{\frac{K\left[\alpha f_1(v)\sigma_2'(v)+(1-\alpha)f_2(v)\sigma_1'(v)\right]}{[\alpha F_1(v)+(1-\alpha)F_2(v)]}\right\}[v-\sigma_i(v)]\end{aligned}$$

Thus,

$$\left\{\frac{K\left[\alpha f_1(v)\sigma_2'(v)+(1-\alpha)f_2(v)\sigma_1'(v)\right]}{[\alpha F_1(v)+(1-\alpha)F_2(v)]}\right\}[v-\sigma_1(v)] =$$
$$\left\{\frac{K\left[\alpha f_1(v)\sigma_2'(v)+(1-\alpha)f_2(v)\sigma_1'(v)\right]}{[\alpha F_1(v)+(1-\alpha)F_2(v)]}\right\}[v-\sigma_2(v)].$$

In short,

$$\sigma_1(v)=\sigma_2(v)$$

which, we shall denote $\sigma_1(\cdot)$, for the first round.

The thing is that α evolves across rounds. Suppose that α is initially α_1. When a bidder wins the auction, there is one less potential buyer of type i, depending on who won, a type 1 or a type 2. If it is a type 1 bidder who won, then

$$\alpha_{2|1} = \alpha_1 - \frac{1}{K},$$

while if it is a type 2 bidder who won, then

$$\alpha_{2|2} = \alpha_1 + \frac{1}{K}.$$

What is the probability of either of these events? Well, when a winning bid w_1 is observed in round 1, then the relative likelihood of these events is determined by

$$\begin{aligned} &\gamma_2(w_1) = \\ &\quad \sum_{m=0}^{K} \left(\frac{F_1\left[\sigma_1^{-1}(w_1)\right]^m}{F_1\left[\sigma_1^{-1}(w_1)\right]^m + F_2\left[\sigma_1^{-1}(w_1)\right]^{K-m}} \right] p_M(m; K, \alpha_1), \end{aligned}$$

so

$$\begin{aligned} \alpha_2(w_1) &= \alpha_{2|1}\gamma_2(w_1) + \alpha_{2|2}\left[1 - \gamma_2(w_1)\right] \\ &= \left(\alpha_1 - \frac{1}{K}\right)\gamma_2(w_1) + \left(\alpha_1 + \frac{1}{K}\right)\left[1 - \gamma_2(w_1)\right]. \end{aligned}$$

Similarly, after a winning bid w_2 is observed in the second round, then

$$\alpha_{3|1}(w_1) = \alpha_2(w_1) - \frac{1}{(K-1)},$$

while if it is a type 2 bidder who won, then

$$\alpha_{3|2}(w_1) = \alpha_2(w_1) + \frac{1}{(K-1)}.$$

What is the probability of either of these events? Now, the relative likelihood of these events is determined by

$$\begin{aligned} &\gamma_3(w_1, w_2) = \\ &\quad \sum_{m=0}^{K-1} \left(\frac{F_1\left[\sigma_2^{-1}(w_2)\right]^m}{F_1\left[\sigma_2^{-1}(w_2)\right]^m + F_2\left[\sigma_2^{-1}(w_2)\right]^{K-m}} \right) p_M(m; K-1, \alpha_2), \end{aligned}$$

so

$$\alpha_3(w_1, w_2) = \alpha_{3|1}(w_1)\gamma_3(w_2) + \alpha_{3|2}(w_1)\left[1 - \gamma_3(w_2)\right].$$

In general, in round t, having observed winning bids $(w_1, w_2, \ldots, w_{t-1})$ in the previous $(t-1)$ rounds,

$$\alpha_{t|1}(w_1, w_2, \ldots, w_{t-2}) = \alpha_{t-1}(w_1, w_2, \ldots, w_{t-2}) - \frac{1}{(K-t+2)},$$

while if it is a type 2 bidder who won, then

$$\alpha_{t|2}(w_1, w_2, \ldots, w_{t-2}) = \alpha_{t-1}(w_1, w_2, \ldots, w_{t-2}) + \frac{1}{(K-t+2)}$$

where the probability of either of these events is determined by

$$\gamma_t(w_1, w_2, \ldots, w_{t-1}) = \sum_{m=0}^{K-t+2} \left(\frac{F_1\left[\sigma_{t-1}^{-1}(w_{t-1})\right]^m}{F_1\left[\sigma_{t-1}^{-1}(w_{t-1})\right]^m + F_2\left[\sigma_{t-1}^{-1}(w_{t-1})\right]^{K-t+2-m}} \right) p_M(m; K-t+2, \alpha_{t-1}),$$

so

$$\alpha_t(w_1, w_2, \ldots, w_{t-1}) = \alpha_{t|1}(w_1, w_2, \ldots, w_{t-2})\gamma_t(w_1, w_2, \ldots, w_{t-1}) + \alpha_{t|2}(w_1, w_2, \ldots, w_{t-2})\left[1 - \gamma_t(w_1, w_2, \ldots, w_{t-1})\right].$$

A.2 Second-Price, Sealed-Bid Hụi is Non-Parametrically Unidentified

We assume that before each round of the hụi new rates-of-return are drawn independently from $F_R^0(r)$ for the remaining participants. In each round of the hụi, a second-price, sealed-bid auction is conducted to determine who will win that round of the hụi, and what bid discount will be paid; only the winning bid is revealed. For a first-price, sealed-bid hụi, we established non-parametric identification in the second-to-last round. That is where we shall begin here, too: if we cannot establish non-parametric identification in the second-to-last round, then such identification cannot be established in earlier rounds either because (through backward induction) behaviour in those rounds conditions on that in the second-to-last round.

Consider a representative participant in the second-to-last round who faces only one other opponent. In a symmetric perfect Bayesian equilibrium, when u is one, the first-order condition for expected discounted-value maximization can be re-written as the following differential equation:

$$\int_{\underline{r}}^{r} \left((N-1) + [1 - \beta_{N-1,1}(x)] - \frac{1}{(1+r)} \right) 2 f_R^0(x)\, \mathrm{d}x\, \frac{f_R^0(r)\, \mathrm{d}r}{\mathrm{d}\beta_{N-1,1}(r)} - \int_{\underline{r}}^{r} \left([\beta_{N-1,1}(x) - 1] + \mathbb{E}\left[\frac{N}{(1+R)}\right] \right) 2 f_R^0(x)\, \mathrm{d}x\, \frac{f_R^0(r)\, \mathrm{d}r}{\mathrm{d}\beta_{N-1,1}(r)} + \left[1 - F_R^0(r)^2\right] = 0.$$

Denote by $G_{B,1}^{N-1}(b)$ the population cumulative distribution function of unit bids observed in the second-to-last round, and by $g_{B,1}^{N-1}(b)$ the corresponding population probability density function. Now,

$$G_{B,1}^{N-1}(b) = \Pr(B \le b) = \Pr\left[\beta_{N-1,1}^{-1}(B) \le \beta_{N-1,1}^{-1}(b)\right] = \Pr(R \le r) = F_R^0(r).$$

Also,

$$g_{B,1}^{N-1}(b)\, \mathrm{d}b = f_R^0(r)\, \mathrm{d}r$$
$$g_{B,1}^{N-1}(b) = f_R^0(r)\, \frac{\mathrm{d}r}{\mathrm{d}b}$$

$$
\begin{aligned}
g_{B,1}^{N-1}(b) &= \frac{f_R^0(r)}{\beta'_{N-1,1}(r)} \\
g_{B,1}^{N-1}(b) &= \frac{f_R^0\left[\beta_{N-1,1}^{-1}(b)\right]}{\beta'_{N-1,1}\left[\beta_{N-1,1}^{-1}(b)\right]}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_{\underline{r}}^{r} \left((N-1) + [1-\beta_{N-1,1}(x)] - \frac{1}{(1+r)} \right) 2 f_R^0(x)\, \mathrm{d}x\, g_{B,1}^{N-1}(b) - \\
&\int_{\underline{r}}^{r} \left([\beta_{N-1,1}(x) - 1] + \mathbb{E}\left[\frac{N}{(1+R)} \right] \right) 2 f_R^0(x)\, \mathrm{d}x\, g_{B,1}^{N-1}(b) + \left[1 - G_{N-1,1}^{N-1}(b)^2 \right] = 0,
\end{aligned}
$$

which can be re-written in terms of observables on the left-hand side as

$$
\begin{aligned}
\frac{\left[1 - G_{N-1,1}^{N-1}(b)^2\right]}{g_{B,1}^{N-1}(b)} &= \int_{\underline{r}}^{r} \left(2\beta_{N-1,1}(x) + \mathbb{E}\left[\frac{N}{(1+R)} \right] + \frac{1}{(1+r)} - (N+1) \right) 2 f_R^0(x)\, \mathrm{d}x \\
&= \left(\frac{1}{(1+r)} + \mathbb{E}\left[\frac{N}{(1+R)} \right] - (N+1) \right) 2F_R^0(r) + \int_{\underline{r}}^{r} 4\beta_{N-1,1}(x)\, f_R^0(x)\, \mathrm{d}x.
\end{aligned}
$$

Now, if we are willing to change the timing of discounting, as we did in the first-price, sealed-bid case, then this gives rise to

$$
\begin{aligned}
\frac{\left[1 - G_{N-1,1}^{N-1}(b)^2\right]}{g_{B,1}^{N-1}(b)} &= \left[\frac{1}{(1+r)} + \frac{N}{(1+r)} - (N+1) \right] 2F_R^0(r) + \int_{\underline{r}}^{r} 4\beta_{N-1,1}(x)\, f_R^0(x)\, \mathrm{d}x \\
&= \left[\frac{r(N+1)}{(1+r)} \right] 2G_{B,1}^{N-1}(b) + \int_{\underline{r}}^{r} 4\beta_{N-1,1}(x)\, f_R^0(x)\, \mathrm{d}x.
\end{aligned}
$$

Were the integral on the right-hand-side not there, then we could establish non-parametric identification in the second-price, sealed-bid hụi. Unfortunately, the integral is there. In short, its presence implies that the second-price, sealed-bid hụi is non-parametrically unidentified.

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