Interdependent Hitting Times

Jaap H. Abbring Yifan Yu

2022 DSE Conference
Methodology and Applications of
Dynamic Structural Models with Strategic Interactions

Interdependent Durations

Complementarities, sorting on characteristics, and common shocks may all lead to synchronization in optimal stopping

- Honoré & de Paula 18: Retirement of couples
- de Paula 09: Desertion from the Union Army
- Literature on risky behavior: Adolescents dropping out of high school; starting the use of alcohol, drugs, or cigarettes
- Participation in social welfare programs, bank runs, migration, marriage, divorce
- Complementaries in retail entry and exit

Distinction complementarities and other mechanisms matters

Distinct mechanisms underlying dependent behavior come with distinct policy implications (e.g. Manski's 93 "social multipliers")

Interdependent Hitting Times

Specification, identification, and estimation of models of multiple durations that are outcomes of stopping games in which payoffs vary with

- common shocks (spectrally-negative Lévy process)
- observed and unobserved agents' characteristics
- other agents' stopping actions (with a focus on strategic complementarity)

Main idea: Such games reduce to econometric models of interdependent hitting times

Background

- Econometrics of stopping games: de Paula 09, Honoré and de Paula 10 & 18, Lin and Liu 21
- Econometric duration analysis: Abbring and Van den Berg 03, Abbring 12
- Theory: Stokey 09, Boyarchenko and Levendorskiĭ 07, Simon and Stinchcombe 89, Murto 04, ...
- Computation: Doraszelski and Judd 12, Arcidiacono et al. 16, ...

A Simple Synchronization Game general theory

- Shopping mall with anchor stores A and B (Vitorino 12)
- Stores continuously contemplate exit, with infinite horizons
  - As long as both active, they decide on exit simultaneously
  - As soon as i exits mall, store j can respond immediately
  - If j does not exit at same time, it can leave at any future time
- Flow payoffs store $i \in \{A, B\}$ at time t if
  - a joint anchor: $R_i^J - C_i^J \exp(\gamma_i Y_t)$
  - a lone anchor i: $R_i^L - C_i^L \exp(\gamma_i Y_t)$
  - exited: 0

  with $R_i^J \geq R_i^L > 0$ and $C_i^L \geq C_i^J > 0$ (complementarity); $\gamma_i > 0$; and $Y_t = \mu t + \sigma W_t$ a mall-level Brownian motion with $\mu \geq 0$ and $\sigma > 0$
- Complete information: At t, both stores know $(R_i^J, R_i^L, C_i^J, C_i^L, \gamma_i)$, $i \in \{A, B\}$; $\{Y_\tau; 0 \leq \tau \leq t\}$; and their mall survival histories
- Pure Markov strategies: Store i leaves when Y_t hits $\mathcal{Y}_i^J$ (if both still in mall) or $\mathcal{Y}_i^L$ (if i is lone anchor)

Equilibrium

Agents $i \in \{A, B\}$ choose pure Markov strategies that maximize payoffs discounted at rate $\rho_i > \psi(\gamma_i) \equiv \mu\gamma_i + \frac{\sigma^2}{2}\gamma_i^2 > 0$ in a Markov perfect equilibrium

Lone anchor subgame

Standard single-agent optimal stopping problem

- Exit when Y_t hits $\mathcal{Y}_i^L = [\overline{Y}_i^L, \infty)$, with

$$\overline{Y}_i^L \equiv \gamma_i^{-1} \ln\left(\frac{C_i^L}{R_i^L} \cdot \frac{\Lambda(\rho_i)}{\Lambda(\rho_i) - \gamma_i} \cdot \frac{\rho_i - \psi(\gamma_i)}{\rho_i}\right)$$

where

$$\Lambda(\rho_i) \equiv \frac{-\mu + \sqrt{\mu^2 + 2\rho\sigma^2}}{\sigma^2} > \gamma_i$$

solves $\psi\left(\Lambda(\rho_i)\right) = \rho_i$

Equilibrium

Joint mall exit decisions

Suppose joint anchors also use threshold rules

- Single agent with joint anchor payoffs would exit when Y_t hits

$$\overline{Y}_i^J \equiv \gamma_i^{-1} \ln \left(\frac{C_i^J}{R_i^J} \cdot \frac{\Lambda(\rho_i)}{\Lambda(\rho_i) - \gamma_i} \cdot \frac{\rho_i - \psi(\gamma_i)}{\rho_i} \right) \geq \overline{Y}_i^L$$

- Two classes of MPE:
  - Sequential exit: $\max \overline{Y}^L > \min \overline{Y}^J$
  - Simultaneous exit: $\max \overline{Y}^L \leq \min \overline{Y}^J$

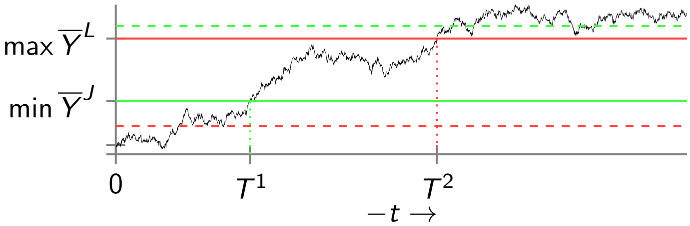
  where $\max \overline{Y}^L \equiv \max\{\overline{Y}_A^L, \overline{Y}_B^L\}$ and $\min \overline{Y}^J \equiv \min\{\overline{Y}_A^J, \overline{Y}_B^J\}$

Equilibrium

Sequential exit if $\max \overline{Y}^L > \min \overline{Y}^J$

Heterogeneity dominates complementarity $\overline{Y}_i^L \leq \overline{Y}_i^J < \overline{Y}_j^L \leq \overline{Y}_j^J$

- Multiple equilibria: $\mathcal{Y}_i^J = \left[\overline{Y}_i^J, \infty\right)$, but only required that $\mathcal{Y}_j^J \cap [0, \overline{Y}_j^L) = \emptyset$
- Unique outcome: First store exits when Y_t hits $\overline{Y}_i^J = \min \overline{Y}^J$; second store exits when Y_t subsequently hits $\overline{Y}_j^L = \max \overline{Y}^L$

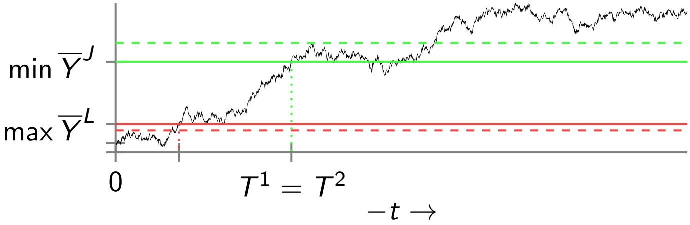


Equilibrium

Simultaneous exit if $\max \overline{Y}^L \leq \min \overline{Y}^J$

Complementarity dominates heterogeneity $\Rightarrow$ synchronization

- Multiple equilibria: Stores may fail to coordinate on best outcome (plus ...)
- Multiple outcomes: Exit jointly when Y_t hits $\overline{Y} \in [\max \overline{Y}^L, \min \overline{Y}^J]$
- Natural equilibrium refinement would pick $\overline{Y} = \min \overline{Y}^J$



Formal Analysis and Generalizations

General two-player game general theory

We provide a formal analysis of a game with

- general payoffs $u_i^J(Y_t) \geq u_i^L(Y_t)$ that are monotone in Y_t
- spectrally-negative Lévy processes $\{Y_t\}$, characterized by semiparametric ψ and Λ

that includes the simple example as a special case

The simple example with Lévy-driven payoffs

Note that the simple example extends directly to Lévy processes, with the semiparametric Laplace exponents ψ and Λ replacing the Brownian motion ones in

$$\overline{Y}_i^s = \gamma_i^{-1} \ln \left(\frac{C_i^s}{R_i^s} \cdot \frac{\Lambda(\rho_i)}{\Lambda(\rho_i) - \gamma_i} \cdot \frac{\rho_i - \psi(\gamma_i)}{\rho_i} \right); \qquad i \in \{A, B\}, \ s \in \{L, J\}$$

Formal Analysis and Generalizations

Extension to more than two players

Consider a version of the game in which $|\mathcal{I}| \geq 2$ agents $i \in \mathcal{I}$ have payoffs

- $u_i^s(Y_t)$ when they are among s survivors and
- 0 once they have exited,

with $u_i^{|\mathcal{I}|} \geq \cdots \geq u_i^2 \geq u_i^1$ (complementarity)

Like the simple example, this game can be solved in terms of corresponding "single-agent" thresholds

$$\overline{Y}_i^{|\mathcal{I}|} \geq \cdots \geq \overline{Y}_i^2 \geq \overline{Y}_i^1$$

Formal Analysis and Generalizations

Equilibrium duration outcomes with more than two players

Exit occurs in $W \leq |\mathcal{I}|$ "waves" $w \in \{1, 2, \ldots, W\}$, with one or more agents simultaneously exiting at each hitting time of an increasing sequence of thresholds

Wave w starts with $|\mathcal{S}^w| \geq 1$ survivors $\mathcal{S}^w \subseteq \mathcal{I}$ (note that $\mathcal{S}^1 = \mathcal{I}$) and sees agents $\mathcal{E}^w \equiv \bigcup_k \mathcal{E}^w_k$ exit when $\{Y_t\}$ hits $\min_{i \in \mathcal{S}^w} \overline{Y}_i^{|\mathcal{S}^w|}$, where

- $\mathcal{E}^w_1$ is the set of firms $j \in \mathcal{S}^w$ such that $\overline{Y}_j^{|\mathcal{S}^w|} = \min_{i \in \mathcal{S}^w} \overline{Y}_i^{|\mathcal{S}^w|}$

and for $k = 2, \ldots, |\mathcal{S}^w|$: If $\mathcal{E}^w_{k-1} = \emptyset$ then $\mathcal{E}^w_k = \emptyset$; otherwise,

- $\mathcal{E}^w_k$ is the set of firms $j \in \mathcal{S}^w \backslash \bigcup_{l=1}^{k-1} \mathcal{E}^w_l$ such that $\overline{Y}_j^{|\mathcal{S}^w \backslash \bigcup_{l=1}^{k-1} \mathcal{E}^w_l|} \leq \min_{i \in \mathcal{S}} \overline{Y}_i^{|\mathcal{S}|}$

Wave w's survivors $\mathcal{S}^{w+1} \equiv \mathcal{S}^w \backslash \mathcal{E}^w$ (if any) continue to wave $w+1$

Empirical Model

Extension mixed hitting times (MHT) model (Abbring 12) to interdependent durations

Specification

Sample of games, with data on (T^1, T^2) (possibly augmented with agent identities) and (common and agent-specific) covariates $x \equiv (x_A, x_B)$ for each game

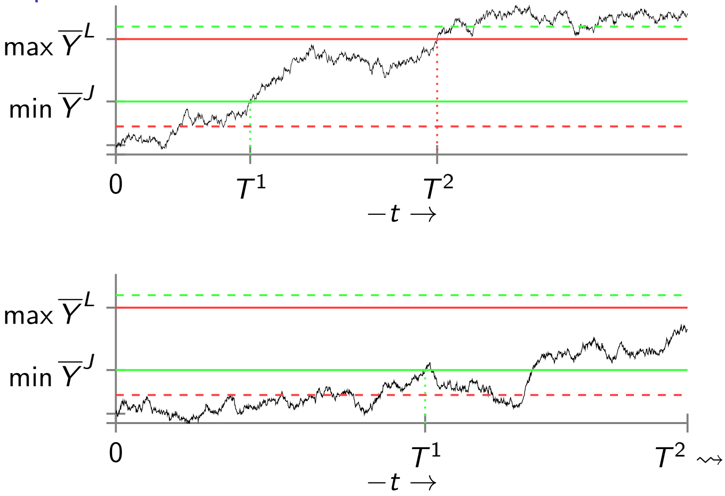
- We assume (T^1, T^2) are equilibrium outcomes from games driven by spectrally-negative Lévy processes, with primitives and thresholds that may vary across agents (and thus games) with both x and unobservables $V \equiv (V_A, V_B)$
- We consider two classes of reduced forms of e.g.

$$\overline{Y}_i^s = \gamma_i^{-1} \ln \left(\frac{C_i^s}{R_i^s} \cdot \frac{\Lambda(\rho_i)}{\Lambda(\rho_i) - \gamma_i} \cdot \frac{\rho_i - \psi(\gamma_i)}{\rho_i} \right); \quad i \in \{A, B\}, \; s \in \{L, J\};$$

  a multiplicative one, $\overline{Y}_i^s = \varphi^s(x_i) V_i$, and an additive one, $\overline{Y}_i^s = \varphi^s(x_i) + V_i$; with x, $V \sim G$, and $\{Y_t\}$ independent
- With grouped (in terms of V) data, we can allow for general dependence on x

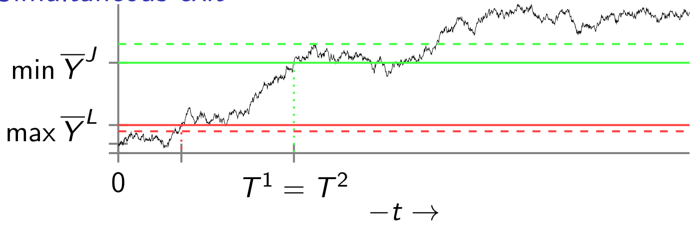
Empirical Model

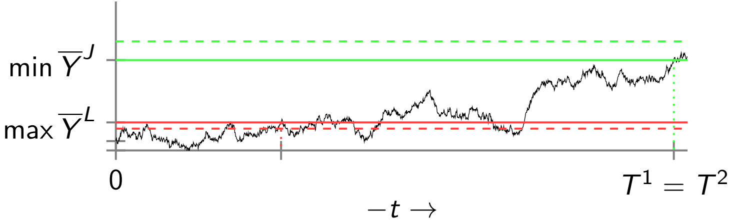
Sequential exit



Empirical Model

Simultaneous exit





Identification

Following Abbring (12)'s analysis of the MHT model, use that the first time $T(y)$ a spectrally-negative Lévy process $\{Y_t\}$ hits a threshold $y \geq 0$ has Laplace transform

$$\mathcal{L}_{T(y)}(s) \equiv \mathbb{E}\left[\exp(-sT(y))\right] = \exp\left(-\Lambda(s)y\right)$$

Identification strategy 1

1. Identify φ^J, Λ, and distribution min $V \equiv \min\{V_A, V_B\}$ from T^1
   - Exploit (minimal) variation with x along $x_A = x_B$
   - With stratified data, no x needed
2. Use structure on G to identify G
   - E.g., assume that V_A and V_B are iid (random matching)
   - Alternatively, use variation with x to identify G
3. Identify φ^L from data on (probability of) sequential exits

In a second stage, determine remaining primitives using e.g. (Abbring 10)

$$\overline{Y}_i^s = \gamma_i^{-1} \ln\left(\frac{C_i^s}{R_i^s} \cdot \frac{\Lambda(\rho_i)}{\Lambda(\rho_i) - \gamma_i} \cdot \frac{\rho_i - \psi(\gamma_i)}{\rho_i}\right); \qquad i \in \{A, B\},\ s \in \{L, J\}$$

Identification

Following Abbring (12)'s analysis of the MHT model, use that the first time $T(y)$ a spectrally-negative Lévy process $\{Y_t\}$ hits a threshold $y \geq 0$ has Laplace transform

$$\mathcal{L}_{T(y)}(s) \equiv \mathbb{E}\left[\exp(-sT(y))\right] = \exp\left(-\Lambda(s)y\right)$$

Identification strategy 2

1. Identify φ^J, Λ, and distribution min $V \equiv \min\{V_A, V_B\}$ from T^1
   - Exploit (minimal) variation with x along $x_A = x_B$
   - With stratified data, no x needed
2. Identify φ^L from data on sequential exit durations
   - Identify (sub)distributions of $(\overline{Y}_i^J, \overline{Y}_j^L)$ on $\{\overline{Y}_i^J < \overline{Y}_j^L\}$
   - Use support conditions, in particular: There exist x, x' such that $\phi^J(x) \leq \phi^L(x')$
3. Identify G from data on sequential exits

In a second stage, determine remaining primitives using e.g. (Abbring 10)

$$\overline{Y}_i^s = \gamma_i^{-1} \ln\left(\frac{C_i^s}{R_i^s} \cdot \frac{\Lambda(\rho_i)}{\Lambda(\rho_i) - \gamma_i} \cdot \frac{\rho_i - \psi(\gamma_i)}{\rho_i}\right); \qquad i \in \{A, B\},\ s \in \{L, J\}$$

Estimation

Estimation of a model with two players with e.g. ML is easy, but applications often involve (many) more than two players

Simulation-based estimation

We use simulation methods to deal with the many possible exit scenarios

- MSL: May be more efficient and works well with few players
- MSM: May be less efficient, but works with very many players

Monte Carlo exercises

Preliminary results for simple parametric models show that

- MSL works well for 2 and 3 players
- MSM can handle as many as 100 players

Concluding Remarks

"Proof of concept" that results and methods for mixed hitting-time models and optimal stopping problems can be extended to synchronization games

Extensions

- Cooperative stopping
- Agent-specific shocks
- Games with strategic substitutes, such as wars of attrition

Primitives

- Agents A an B live in continuous time, with infinite horizons
- They are initially engaged in some activity
- At each time t, they first pass sequentially through two decision nodes
  1. "Joint": if both still active, they simultaneously decide on exit
  2. "Lone": if i still in, but j not, i decides on exit

  before payoffs

$$\begin{array}{ll} u_i^J(Y_t) & \text{if both are still active} \\ u_i^L(Y_t) & \text{if } i \text{ is still active and } j \text{ is not} \\ 0 & \text{if } i \text{ has exited} \end{array}$$

  accrue to each agent i; with $u_i^J \geq u_i^L$, Y_t a spectrally-negative Lévy process, and u_i^J and u_i^L decreasing
- Complete information: At each node, agents know parameters, $\{Y_\tau; 0 \leq \tau \leq t\}$, and exit histories up to (but excluding) that node
- Agents maximize expected payoffs discounted at sufficiently high rate ρ

Equilibrium

Markov perfect equilibrium

Agent i chooses pure Markov strategy $(\mathcal{Y}_i^J, \mathcal{Y}_i^L)$ that maximize her expected discounted payoffs given her partner j's strategy $(\mathcal{Y}_j^J, \mathcal{Y}_j^L)$

Proposition: Lone survivor subgame

In equilibrium, $\mathcal{Y}_i^L = [\overline{Y}_i^L, \infty)$, for a unique $\overline{Y}_i^L$. Similar result for auxiliary problem with payoffs u_i^J, with $\overline{Y}_i^J \geq \overline{Y}_i^L$

Theorem: Sequential exit ($\max \overline{Y}^L > \min \overline{Y}^J$)

If $\overline{Y}_i^L \leq \overline{Y}_i^J < \overline{Y}_j^L \leq \overline{Y}_j^J$ then $\inf \mathcal{Y}_i^J = \overline{Y}_i^J$ and $\inf \mathcal{Y}_j^J \geq \overline{Y}_j^L$

- Threshold equilibrium: $\mathcal{Y}_i^J = \left[\overline{Y}_i^J, \infty\right)$ and $\mathcal{Y}_j^J = [\overline{Y}_j, \infty)$ for some $\overline{Y}_j \geq \overline{Y}_j^L$
- First agent exits when Y_t hits $\overline{Y}_i^J = \min \overline{Y}^J$; second agent exits when Y_t subsequently hits $\overline{Y}_j^L = \max \overline{Y}^L$

Equilibrium

Theorem: Simultaneous exit ($\max \overline{Y}^L \leq \min \overline{Y}^J$)

Either

(i) $\inf \mathcal{Y}_i^J = \inf \mathcal{Y}_j^J = \overline{Y}$ for some $\overline{Y} \in [\max \overline{Y}^L, \min \overline{Y}^J)$ or

(ii) $\inf \mathcal{Y}_i^J = \min \overline{Y}^J$ and $\inf \mathcal{Y}_j^J \geq \min \overline{Y}^J$

- Threshold equilibrium: Either
  (i) $\mathcal{Y}_i^J = \mathcal{Y}_j^J = [\overline{Y}, \infty)$ for some $\overline{Y} \in [\overline{Y}_j^L, \overline{Y}_i^J)$ or
  (ii) $\mathcal{Y}_i^J = [\overline{Y}_i^J, \infty)$ and $\mathcal{Y}_j^J = [\overline{Y}_j, \infty)$ for some $\overline{Y}_j \in [\overline{Y}_i^J, \infty)$
- Both agents exit when Y_t hit some $\overline{Y} \in [\max \overline{Y}^L, \min \overline{Y}^J]$
- Here, we select $\overline{Y} = \min \overline{Y}^J$