Endogenous grid point methods (EGM and DCEGM)

DSE2024: Econometric Society Summer School in Dynamic Structural Econometrics

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Discrete and continuous choice?

In economics discrete and continuous choice co-exist

- How much to work + when to retire/become an entrepreneur
- How much to save + when to buy a house/car/durables
- Which car to buy + how much to drive

Often modeled separately using traditional solution methods:

- Discrete choice $\rightarrow$ optimization over finite set
- Continuous choice $\rightarrow$ first order conditions + concavity(?)
- Dynamic $\rightarrow$ dynamic programming (VFI,policy,time iterations)

Discrete and continuous choice?

In discrete-continuous choice models:

- Intrinsic non-concavity
- First order conditions not sufficient
- Kinks in value functions + discontinuities in policy functions

Traditional methods are not ideal

- Need global optimizer in each point of the state space
- Need to locate and keep track of kinks and discontinuities
- Need special numerical procedures for non-smooth objects

$\Rightarrow$ **Endogenous grid point methods**

Plan for the lecture

1. Original EGM for continuous choice only
   Only for particular (yet interesting and important) models (stochastic growth models, consumption-savings (buffer stock) models)
2. DC-EGM for discrete-continuous choice without taste shocks
   For models with one continuous and additional discrete choices
   Nasty and scary
3. DC-EGM for discrete-continuous choice with taste shocks
   For models with one continuous and additional discrete choices
   Structural taste shocks or logit smoothing
   Much better, possible to work with
4. Some words on multi-dimensional extensions and occasionally binding constraints

What is EGM?

The Method of Endogenous Gridpoints — fast method for solving dynamic stochastic consumption/savings problems

1. finite and infinite horizon
2. Strictly concave monotone and differentiable utility function
3. one continuous state variable (*wealth*) and one continuous choice (*consumption*)
4. particular structure of the law of motion for state variables (*intertemporal budget constraint*)
5. very well accommodate potentially binding borrowing constraints

DC-EGM for Discrete-Continuous problems

Expand the class of problems to be solved:

1. A1. Strictly concave monotone and differentiable utility function
2. Continuous state M_t with a particular motion rule
3. Additional (discrete) state variables st_t
   A2. Transition probabilities of st_t are independent of M_t
4. One continuous (c_t) and one* discrete choice variable d_t

Two flavors:

1. Without taste shocks: DC-EGM iterates on value function and policy function, produces exact solutions for the optimal thresholds for discrete decisions (discrete policy)
2. With taste shocks: DC-EGM iterates on discrete choice specific value and policy functions, produces choice probabilities for discrete alternatives

Learning outcomes = points to remember

1. If your model has one continuous (consumption) choice and additional discrete choices → Use DC-EGM
2. In regular cases DC-EGM avoids all root-finding operations
3. If utility is separable in continuous and discrete choices, DC-EGM deals very easily with credit constraints
4. Extreme value taste shocks → solution is much better behaved
5. Faster and more accurate than traditional approaches

EGM

Simple consumption/savings model (Phelps)

$$V_t(M_t) = \max_{0 \le c \le M_t} \left[u(c) + \beta E V_{t+1} \left(\tilde{R}(M_t - c) \right) \right]$$

M_t	cash-in-hand, all resources available at period t
$A_t = M_t - c_t$	assets at the end of period t (savings)
$\tilde{R}$	*deterministic* or *stochastic* return on savings
$u(c)$	utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \underset{\rho \to 0}{\rightarrow} log(c)$$

Analytic solution (Hakansson, 1970, Phelps, 1962)

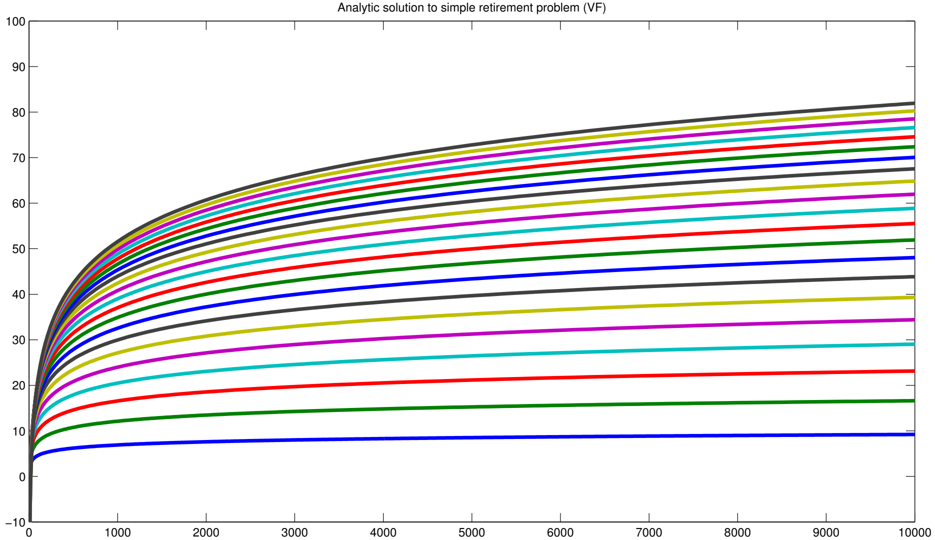
$$V_{T-t}(M) = \left[\frac{M^\rho}{\rho}\right]\left(\sum_{i=0}^{t} K^i\right)^{(1-\rho)} - \frac{1}{\rho}\left(\sum_{i=0}^{t} \beta^i\right)$$

$$V_{T-t}(M) \underset{\rho\to 0}{\rightarrow} \log(M)\left(\sum_{i=0}^{t} \beta^i\right) + K_t$$

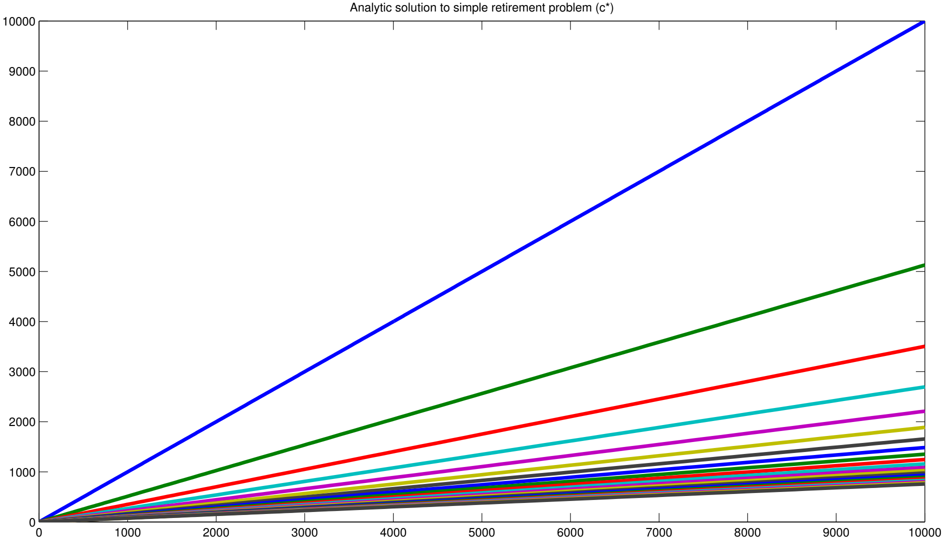
$$c_{T-t}(M) = M\left(\sum_{i=0}^{t} K^i\right)^{-1}$$

K and K_t are functions of primitives, $K \underset{\rho\to 0}{\rightarrow} \beta$

Analytic solution : value functions



Analytic solution : consumption rule



Simple consumption/savings model (Deaton)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} \left[u(c) + \beta E V_{t+1} \left(R(M_t - c) + \tilde{y} \right) \right]$$

M_t	cash-in-hand, all resources available at period t
$A_t = M_t - c_t$	assets at the end of period t (savings)
R	*deterministic* return on savings
$\tilde{y}$	*stochastic* income
$u(c)$	utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \underset{\rho \to 0}{\rightarrow} log(c)$$

No analytical solution!

Traditional approach : value function iterations

1. *Fix grid over M_t*. For every point on this grid:
2. In the terminal period calculate
   $V_T(M_T) = max_{0 \le c_T \le M_T} \{u(c_T)\}$ and
   $c_T^* = argmax_{0 \le c_T \le M_T} \{u(c_T)\}$
3. With $t+1$ value function at hand, proceed backward to period t and calculate
   $V_t(M_t) = max_{0 \le c_t \le M_t} \left\{ u(c_t) + \beta E V_{t+1}\left(\tilde{R}(M_t - c_t)\right) \right\}$
   and
   $c_t^* = argmax_{0 \le c_t \le M_t} \left\{ u(c_t) + \beta E V_{t+1}\left(\tilde{R}(M_t - c_t)\right) \right\}$
   using Bellman equation

1. Phelps and Deaton models
2. Run VFI solver

- See the `code/python` directory in the repository

Euler equation

Bellman equation: $V_t(M_t) = \max_{0 \le c_t \le M_t} \left[u(c_t) + \beta E V_{t+1} \left(\tilde{R}(M_t - c_t) \right) \right]$

F.O.C. for Bellman equation: $u'(c_t) = \beta E \left[\frac{\partial V_{t+\mathbf{1}}(M_{t+\mathbf{1}})}{\partial M_{t+\mathbf{1}}} \tilde{R} \right]$

Envelope theorem:

$$\frac{\partial V_t(M_t)}{\partial M_t} = \beta E \left[\frac{\partial V_{t+\mathbf{1}}(M_{t+\mathbf{1}})}{\partial M_{t+\mathbf{1}}} \tilde{R} \right] \Rightarrow \frac{\partial V_t(M_t)}{\partial M_t} = u'(c_t) \Rightarrow$$
$$\Rightarrow \frac{\partial V_{t+\mathbf{1}}(M_{t+\mathbf{1}})}{\partial M_{t+\mathbf{1}}} = u'(c_{t+1})$$

Euler equation to characterize the interior solutions: $u'(c_t) = \beta E \left[u'(c_{t+1}) \tilde{R} \right]$

Traditional approach : solving Euler equation

1. *Fix grid over M_t*. For every point on this grid:
2. In the terminal period calculate
   $c_T^* = argmax_{0 \leq c_T \leq M_T} \{u(c_T)\}$
3. With $t+1$ optimal consumption rule $c_{t+1}^*(M_{t+1})$ at hand, proceed backward to period t and calculate
   c_t from equation
   $u'(c_t) = \beta E\left[u'\left(c_{t+1}^*\left(\tilde{R}(M_t - c_t)\right)\right)\tilde{R}\right]$
   to recover $c_t^*(M_t)$
4. When M_t is small enough so credit constraint binds, the Euler equation does not hold, and special provisions are necessary

What if no root-finding is necessary?

With numerical optimization

- Relatively slow: iterative numerical optimization in each point of state space!
- Hard to find global optimum in non-convex problems
- Loss of accuracy due to the absence of the point where credit constraint starts to bind on the fixed grid

Even when using state-of-the-art solvers!

Without numerical optimiation

- Much faster: no iterative methods in each point of the state space
- More accurate: using analytical structure of the problem

Endogenous gridpoint method (EGM)

Carroll 2006 *Economics Letters*
The method of endogenous gridpoints for solving dynamic stochastic optimization problems.

Idea

- Instead of searching for optimal decision in each point of the state space (traditional approaches)
- Look for the state variable (level of assets) where arbitrary chosen decision (consumption $\rightarrow$ savings) would be optimal (EGM)

EGM algorithm

Start with $c^\star_T = M_T$. In each period $t = T, T-1, .., 1$:

EGM step

1. Take the next value A = current period savings $(= M_t - c_t)$ from fixed (or adaptive) grid
2. Intertemporal budget constraint: $A \rightarrow M_{t+1}$
   $M_{t+1} = \tilde{R}(M_t - c_t) = \tilde{R} \cdot A$
3. Policy function at period $t+1$: $M_{t+1} \rightarrow c_{t+1}$
   $c_{t+1} = c^\star_{t+1}(M_{t+1})$
4. Inverted Euler equation: $c_{t+1} \rightarrow c_t$
   $c_t = (u')^{-1}\left(\beta E\left[\tilde{R} \cdot u'\left(c^\star_{t+1}(M_{t+1})\right) | A\right]\right)$
5. Intratemporal budget constraint: $c_t + A = M_t \rightarrow c_t(M_t)$
   $M_t = c_t + A \rightarrow c^\star_t(M_t)$

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   $M_t = c_t + A \to c_t^\star(M_t)$

EGM step as parametric curve

$$u\big(c_t(M)\big) = \beta E\left[\tilde{R} \cdot u'\left(c_{t+1}\big(\tilde{R}A\big)\right)|A\right]$$

Given any policy function $c_0(M)$, an updated policy function $c(M)$ is given as a parameterized curve

$$\begin{cases} c = (u')^{-1}\Big(\beta E\left[\tilde{R} \cdot u'\left(c_0\big(\tilde{R}A\big)\right)\Big|A\right]\Big) \\ M = (u')^{-1}\Big(\beta E\left[\tilde{R} \cdot u'\left(c_0\big(\tilde{R}A\big)\right)\Big|A\right]\Big) + A \end{cases}$$

- A is a parameter that takes non-negative values

Matlab implementation (minimal.m)

```
[quadp quadw]=quadpoints(EXPN,0,1);
quadstnorm=norminv(quadp,0,1);
sgrid=linspace(0,MMAX,NM);
policy{TBAR}.w=[0 MMAX];
policy{TBAR}.c=[0 MMAX];
for it=TBAR-1:-1:1
 w1=Y+exp(quadstnorm*SIGMA)*(1+R)*sgrid;
 c1=interp1(policy{it+1}.w,policy{it+1}.c,w1,'linear'
 rhs=quadw'*(1./c1);
 policy{it}.c=[0 1./(DF*(1+R)*rhs)];
 policy{it}.w=[0 sgrid+policy{it}.c(2:end)];
end
```

Accuracy and speed of EGM

	Traditional Euler	EGM
Running time	37 sec.	0.11 sec.
Max abs error, $c_t^\star$	5e-9	4e-14
Mean abs error, $c_t^\star$	1.4e-12	1.5e-14
Max abs error, $V_t(M)$	39.466	15.163
Mean abs error, $V_t(M)$	2.5e-02	3.2e-02

1. Compare speed of VFI and EGM solvers
2. Simulate flat consumption path using VFI and EGM solutions

- See the `code/python` directory in the repository

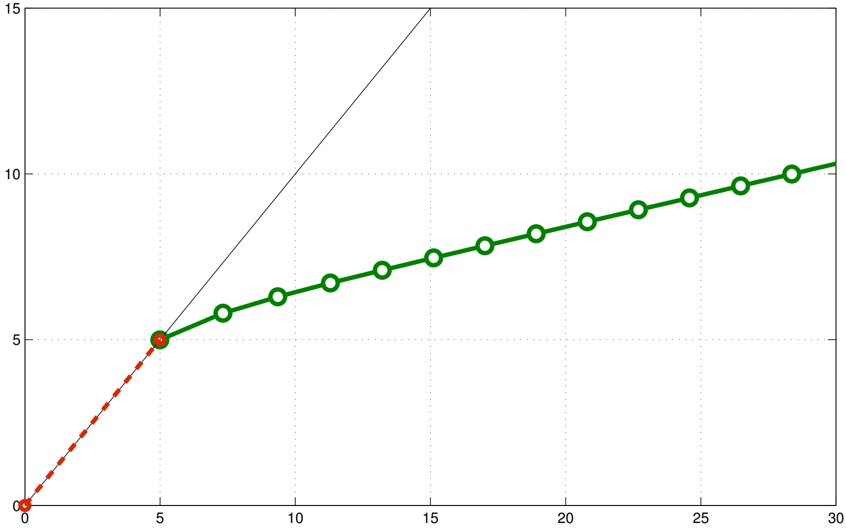
EGM and credit constraint

Theorem: Monotonicity of savings

Monotone and concave utility function $\Rightarrow$
end-of-period assets $A_t = M_t - c_t$ are non-decreasing in M_t

- With $A = 0$ the EGM loop recovers the value of cash-in-hand M_t^{cc} that bounds the credit constrained region
- For all $M_t < M_t^{cc}$ credit constrained binds $\Rightarrow c_t = M_t$
- Consumption rule in the credit constrained region is 45° line between $(0, 0)$ and (M_t^{cc}, M_t^{cc})
- As simple as "connect the dots" $(0, 0)$ and (M_t^{cc}, M_t^{cc})

EGM and credit constraint



Credit constraints and value function

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Inevitable when value functions have to be computed, but..

$M_t < M_t^{cc}$

$V_t(M) = u(M) + \beta EV_{t+1}(0)$

$EV_{t+1}(0)$ — expected value of ending period t with $A_t = 0$

- Value function has analytic form for $M_t < M_t^{cc}$!

DC-EGM

Generalization of EGM

Iskhakov, Jørgensen, Rust, Schjerning, QE 2017
The Endogenous Grid Method for Discrete-Continuous Dynamic Choice Models with (or without) Taste Shocks

- The DC-EGM paper
- Two flavors: with and without EV taste shocks
- Solution method made for empirical applications

Giulio Fella, RED 2014
A Generalized Endogenous Grid Method for Non-Smooth and Non-Concave Problems

- Identify the regions of the problem where Euler equation is not sufficient for optimality
- Use global optimization methods inside (VFI) and EGM outside
- Similar to DC-EGM without taste shocks

Simple retirement model

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} \max\limits_{0 \le c \le M_t} u(c, \mathbb{R}) + \beta E V_{t+1}\left(\tilde{R}(M_t - c), \mathbb{R}\right) \\ \max\limits_{0 \le c \le M_t} u(c, \mathbb{W}) + \beta E V_{t+1}\left(\tilde{R}(M_t - c) + y, \mathbb{W}\right) \end{array} \right\}$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \le c \le M_t} \left[u(c, \mathbb{R}) + \beta E V_{t+1}\left(\tilde{R}(M_t - c), \mathbb{R}\right) \right]$$

$\mathbb{R}$, $\mathbb{W}$ — retirement and working states st_t that evolve according to discrete choices $d \in \{\mathbb{R}, \mathbb{W}\}$

y — deterministic wage income

$$u(c, d) = \frac{c^\rho - 1}{\rho} - 1(d = \mathbb{W}) \underset{\rho \to 0}{\to} log(c) - 1(d = \mathbb{W})$$

Analytic solution

$$u(c) = log(c),\ R = 1 \quad \Rightarrow \quad c^{\star}_{T-t}(M, \mathbb{W}) =$$

$$\begin{cases}
M & \text{if} & M \leq y/\beta \\
(y+M)/(1+\beta) & \text{if} & y/\beta \leq M \leq \overline{M}^{l_1}_{T-t} \\
(2y+M)/(1+\beta+\beta^2) & \text{if} & \overline{M}^{l_1}_{T-t} \leq M \leq \overline{M}^{l_2}_{T-t} \\
\dots & \dots & \dots \\
((t-1)y+M)\left(\sum_{i=0}^{t-1}\beta^i\right)^{-1} & \text{if} & \overline{M}^{l_{t-2}}_{T-t} \leq M \leq \overline{M}^{l_{t-1}}_{T-t} \\
(ty+M)\left(\sum_{i=0}^{t}\beta^i\right)^{-1} & \text{if} & \overline{M}^{l_{t-1}}_{T-t} \leq M \leq \overline{M}^{r_1}_{T-t} \\
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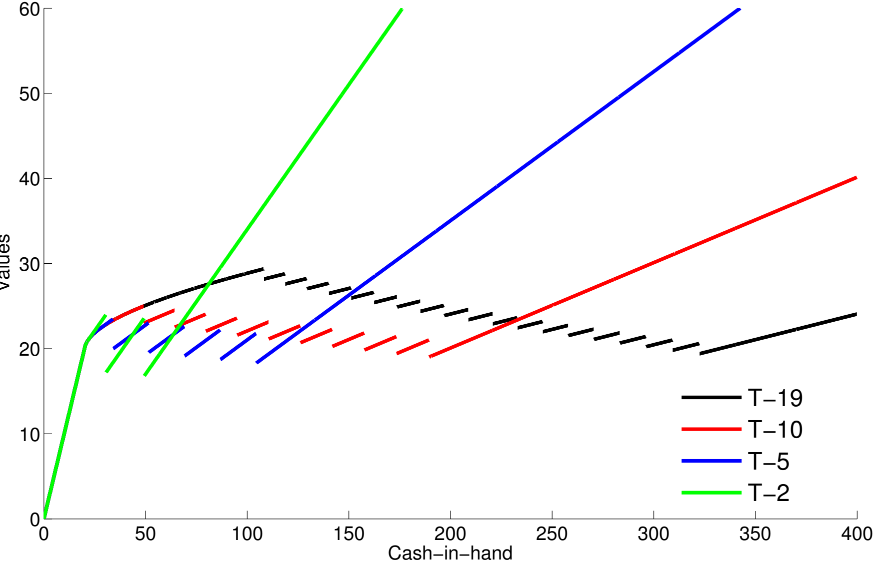
(y+M)\left(\sum_{i=0}^{t}\beta^i\right)^{-1} & \text{if} & \overline{M}^{r_{t-1}}_{T-t} \leq M \leq \overline{M}_{T-t} \\
M\left(\sum_{i=0}^{t}\beta^i\right)^{-1} & \text{if} & \overline{M}_{T-t} < M
\end{cases}$$

Analytic solution



How to approach discrete/continuos choice

The goal:

- Avoid root finding
- Keep efficient treatment of credit constraints

DC-EGM ver. 1.0

1. EGM step for each discrete choice d and every state st
2. Compute d-specific value functions and consumption rules
3. Compare the d-specific value functions to find optimal switching points (compute upper envelope)
4. Reconstruct overall consumption rule and value function from optimal switching points

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- No root finding!
- Efficient treatment of credit constraints (to be shown)
- Need to compute value functions
- Need to compute upper envelope

Is Euler equation still a necessary condition?

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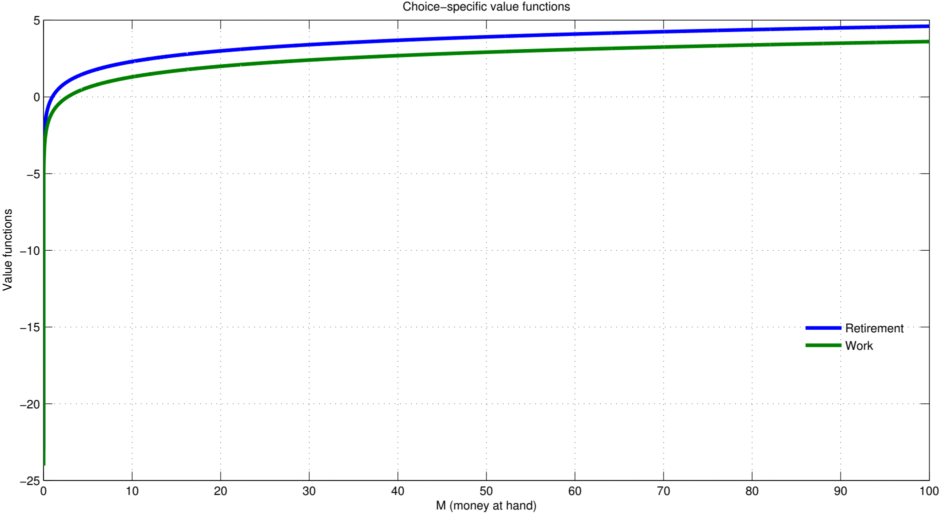
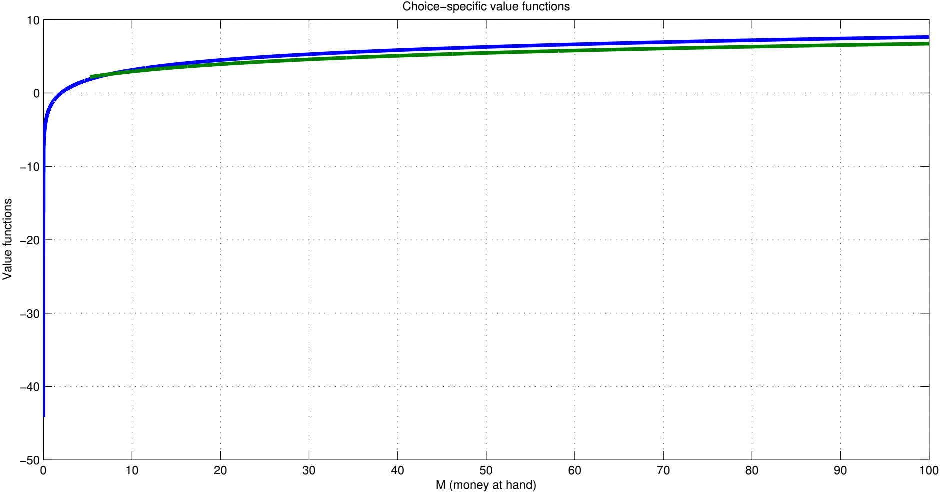
Clausen & Strub, 2010-2016
A General and Intuitive Envelope Theorem.

Show that Euler equation remains a necessary condition for the optimal continuous consumption.

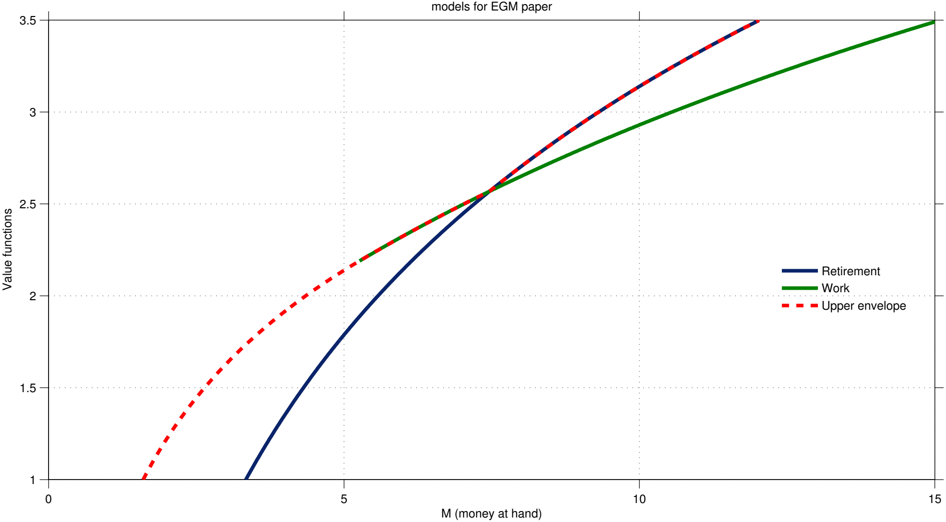
Period T : choice specific value functions



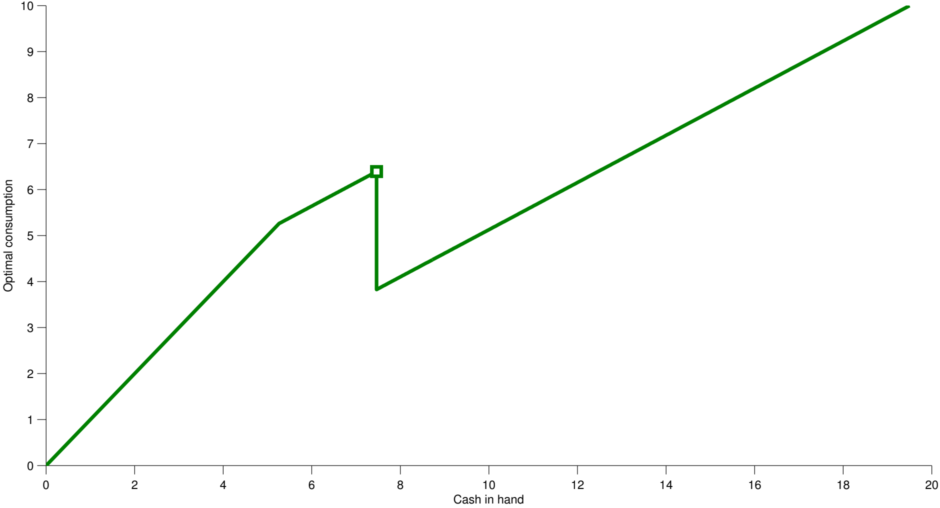
Period $T-1$: Choice specific VF



Period $T-1$: Choice specific VF



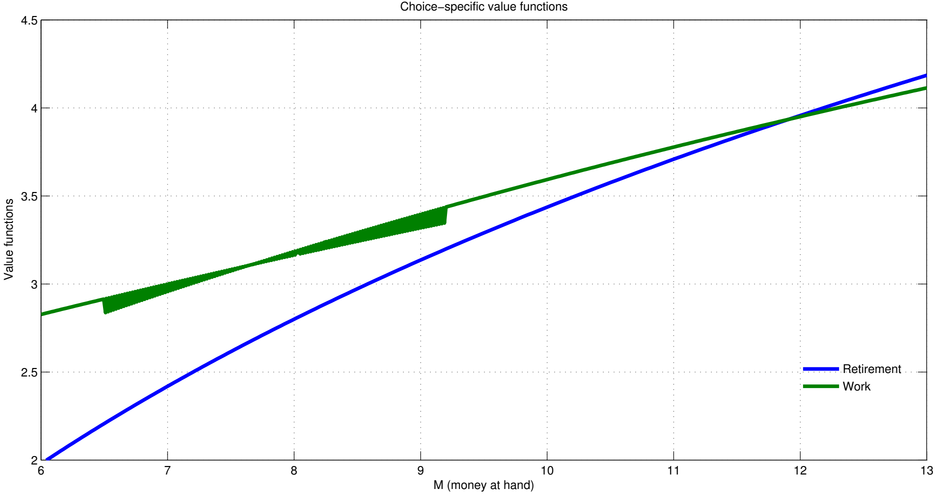
Period $T-1$: Optimal consumption



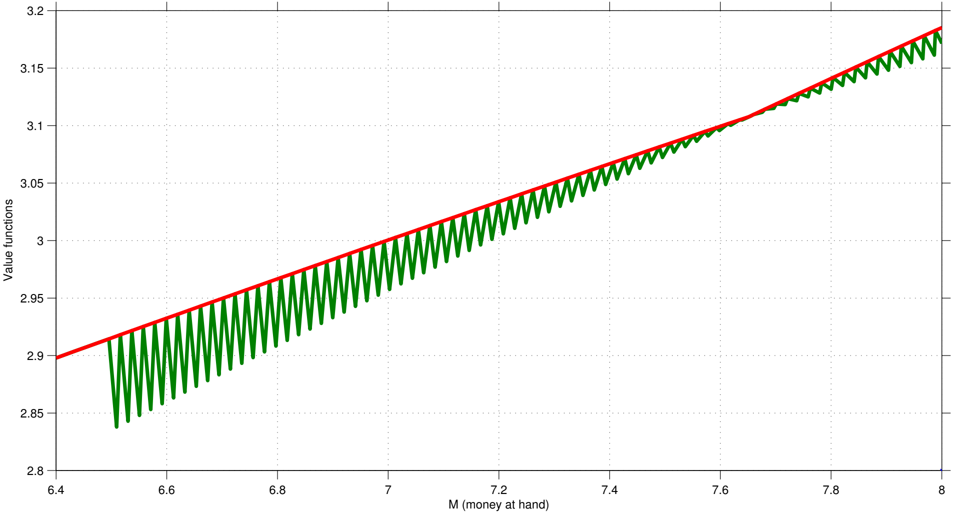
So, what is going on

1. d-specific value functions intersect
   (due to trade-off between income and disutility of work)
   $\Downarrow$
2. The upper envelope of the value functions has a kink
   and combined consumption function has a discontinuity

Period $T-2$: Choice specific VF



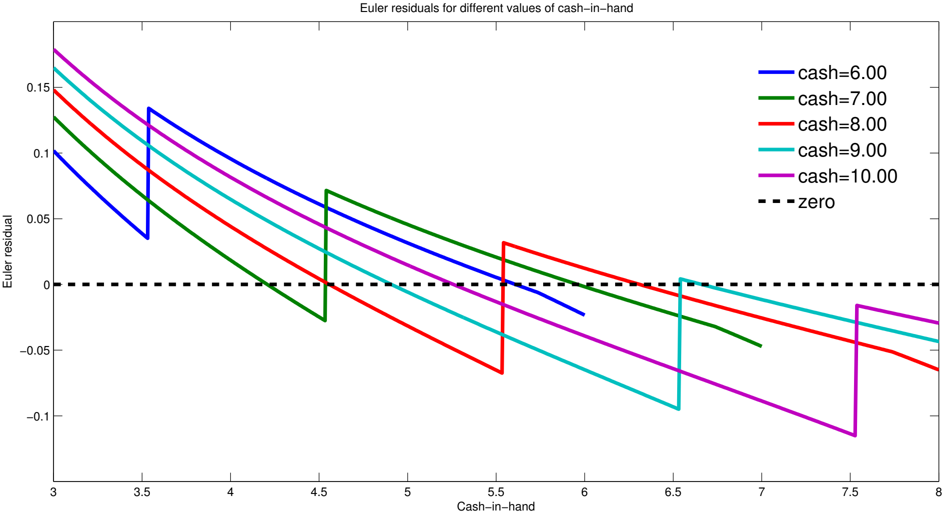
Period $T-2$: Secondary upper envelope



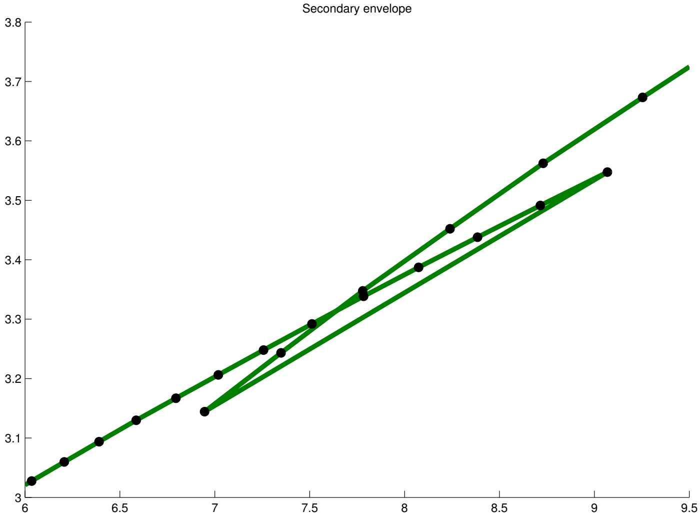
So, what is going on

1. d-specific value functions intersect
   (due to trade-off between income and disutility of work)
   $\Downarrow$
2. The upper envelope of the value functions has a kink
   and combined consumption function has a discontinuity
   $\Downarrow$
3. Derivative of the value function has a discontinuity
   at the kink
   $\Downarrow$
4. For some values of wealth (on endogenous grid) Euler equation has two solutions!
   If endogenous grid points are sorted $\rightarrow$ zigzag region

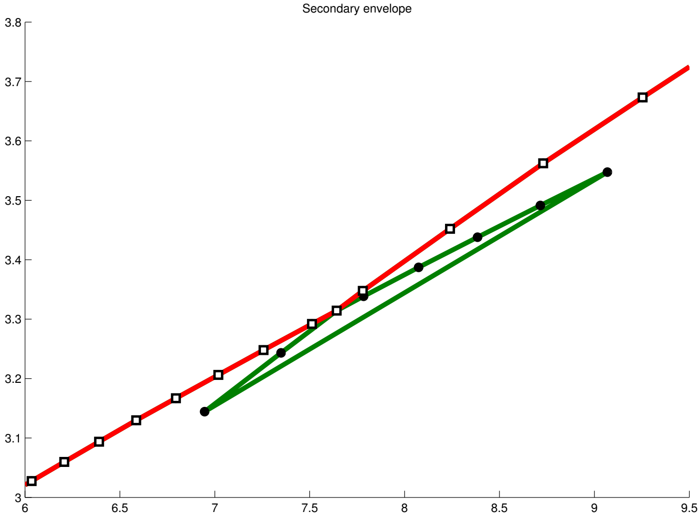
Multiple zeros of Euler residuals



Period $T-2$: Secondary upper envelope: detect



Period $T-2$: Secondary upper envelope: result



How to algorithmically detect "zigzag" regions?

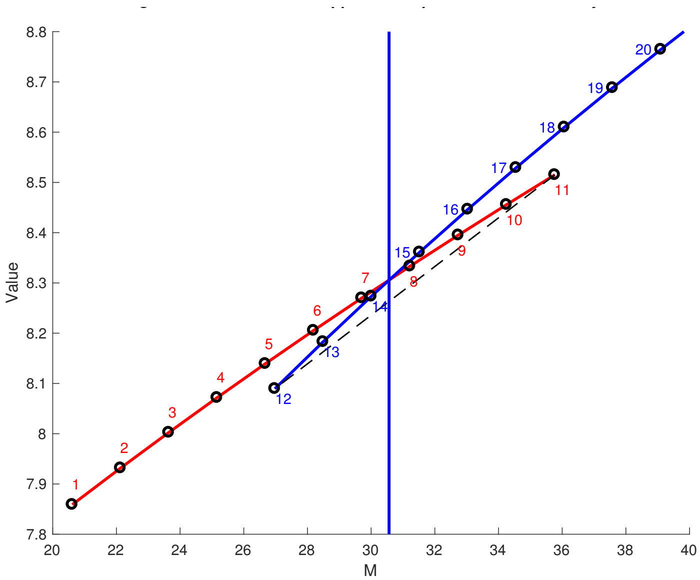
Theorem: monotonicity

Under weak regularity conditions on the utility function and intertemporal budget constraint, savings function is weakly increasing.
$A_t(M'_t) \geq A_t(M''_t)$ for every $M'_t \geq M''_t$ for all t.

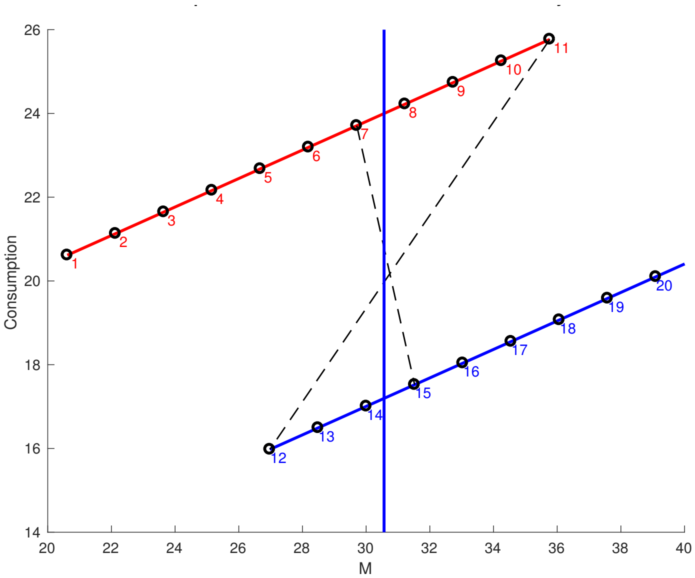
Note: savings function may still have "upward" jumps

1. Sort the exogenous grid over A in ascending order
2. Then the sequence of endogenous grid points over M has to be in ascending order as well as long as Euler equation is sufficient
3. Every time the endogenous grid "bends back" the endogenous grid is separated into subsets of points
4. Calculate the Upper envelope on the segments over the subsets
5. Delete suboptimal endogenous points
6. Find and add a kink point to the endogenous grid

What happens to optimal consumption?



What happens to optimal consumption?



Alternative: fast upper envelope scan (FUES)

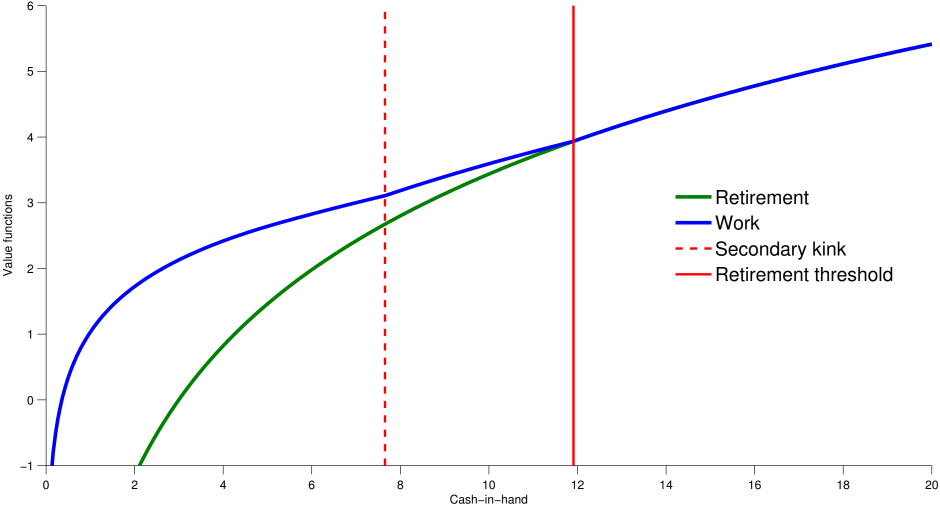
Loretti Dobrescu, Akshay Shanker (2022)
Fast Upper-Envelope Scan for Discrete-Continuous Dynamic Programming

Given the endogenous points of the value function and the corresponding optimal consumption:

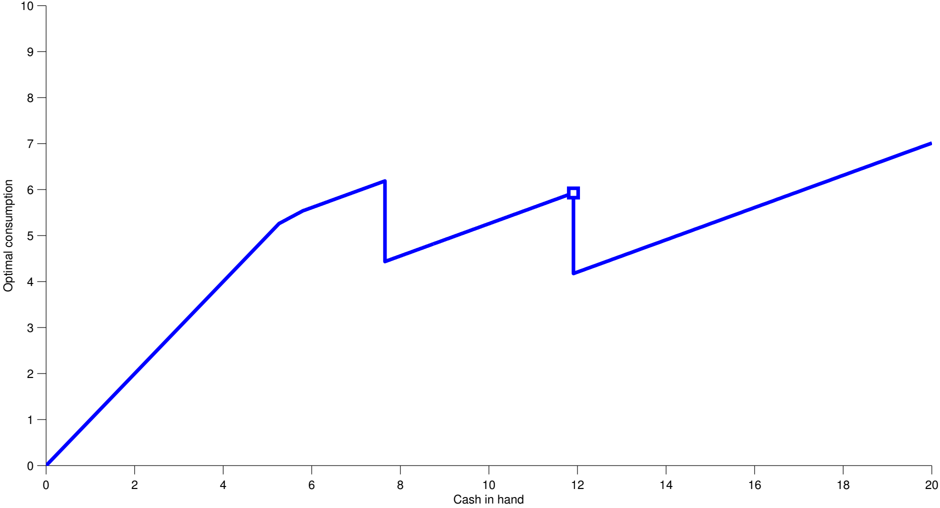
1. Sort the endogenous points in ascending order
2. Choose a jump threshold level for change in consumption
3. Given two consecutive points on the endogenous grid, if the value "function" decreses and in the same time consumption makes a jump, delete the point as inferior
4. Continue to the next point on the sorted endogenous grid

- Does not rely on the monotonicity of the savings function
- Can be generalized to higher dimensions
- But depends on a pre-set threshold which potentially leads to inaccuracies of the computed model solution

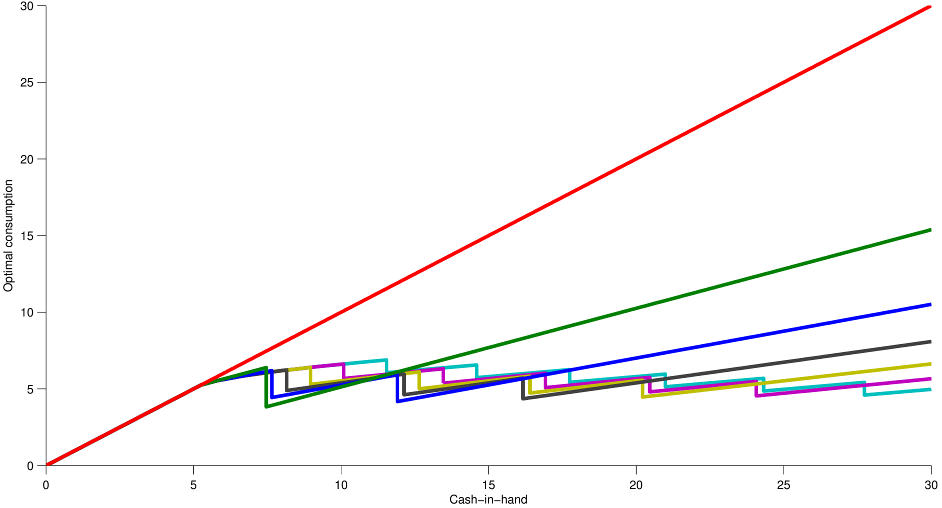
Period $T-2$: VF, primary and secondary kinks



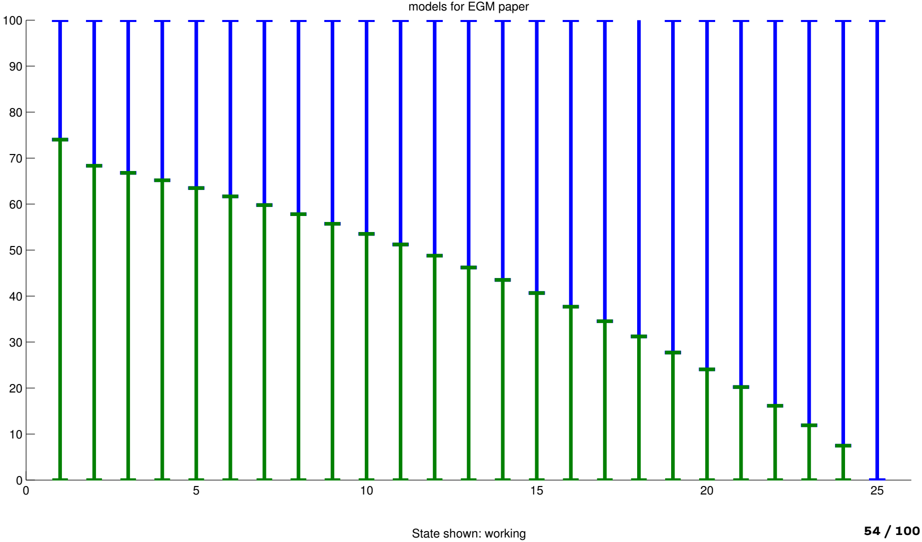
Period $T-2$: Optimal consumption



Optimal consumption (many periods)



Optimal retirement (many periods)



DC-EGM full algorithm

DC-EGM ver. 2.0

1. Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
2. EGM step for each discrete choice d and every state st
3. Compute d-specific value functions and consumption rules
4. Compute the "secondary" upper envelope over the "zig-zag" regions of the d-specific value functions and update the corresponding consumption rules
5. Compare the d-specific value functions to find optimal switching points (compute upper envelope)
6. Reconstruct overall consumption rule and value function from optimal switching points

Properties of the full solution

1. Value functions are non-concave and have kinks
2. Consumption functions have discontinuities
3. Discontinuities/kinks propagate through time and accumulate

This properties are attributes of the model itself.
Any solution method has to deal with these complexities.

DC-EGM matches the analytical solution perfectly!

1. Replicate the solution using `model_retirement.m`
2. Simulate the consumption path for $\beta R = 1$ and discuss the accuracy of the solutions

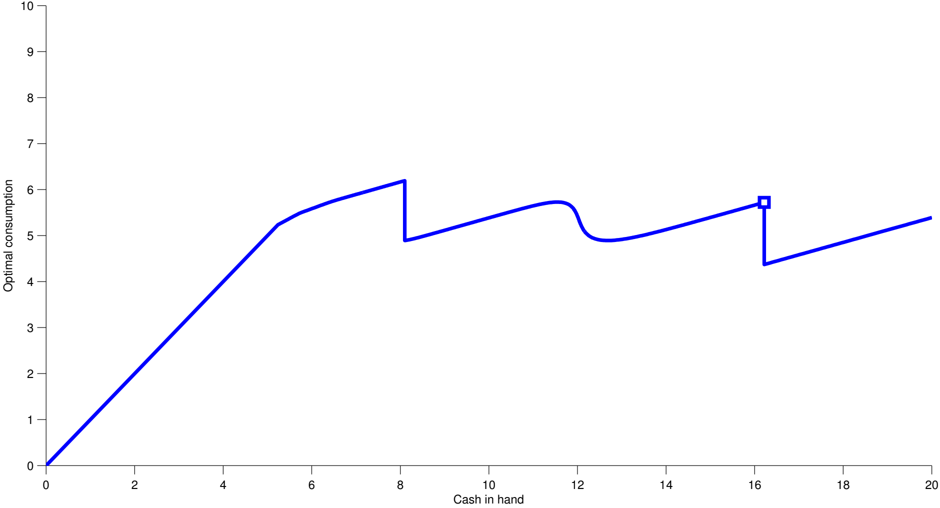
- See the `code/python` directory in the repository

Random returns $\tilde{R}$

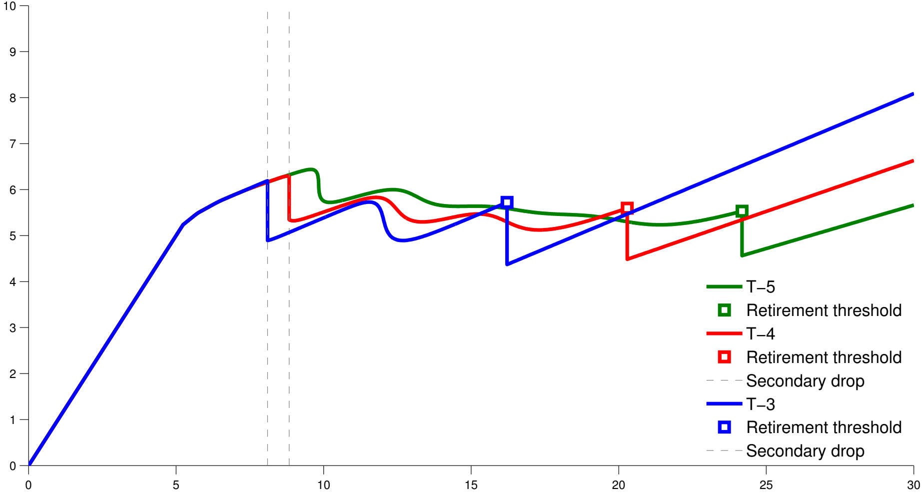
Random shocks do help, however:

- Smooth out secondary kinks only
- Primary kinks (switching between discrete options) remain
- May not smooth out all kinks: continuous but sharp declines in optimal consumption at t may lead to a discontinuity/kink at $t-1$
- Expectations in Euler equation have to be taken over discontinuous functions
  - More kinks/discontinuities from sloppy computation
  - Need to integrate over "continuous" intervals separately

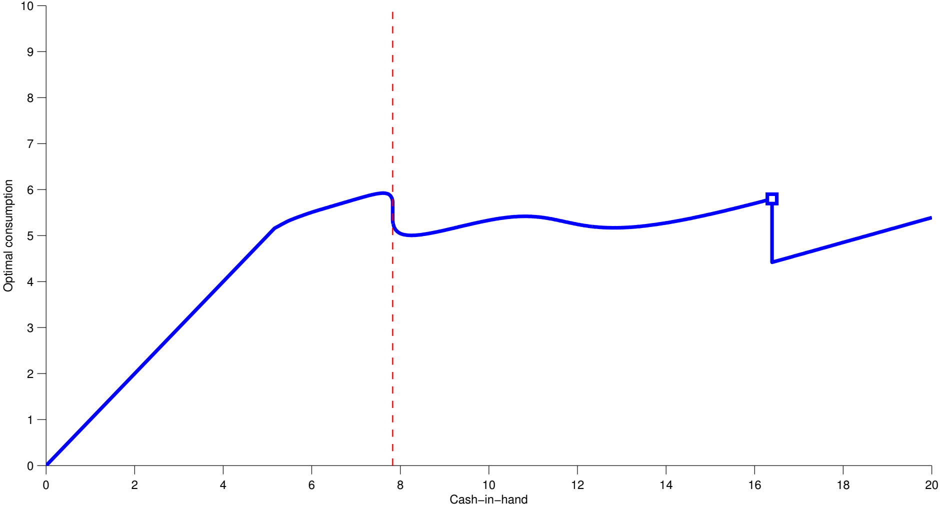
Period $T - 3$: Optimal consumption with $\sigma = 0.1$



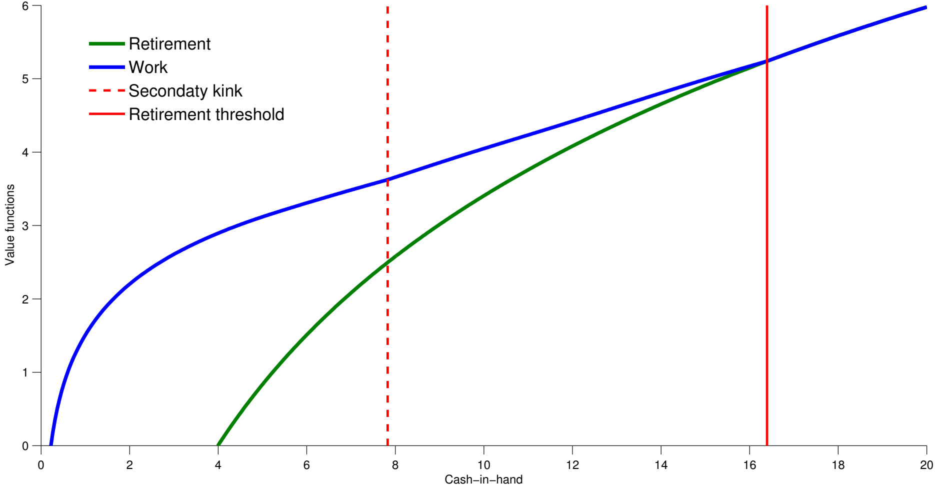
Before $T - 3$: Optimal consumption with $\sigma = 0.1$



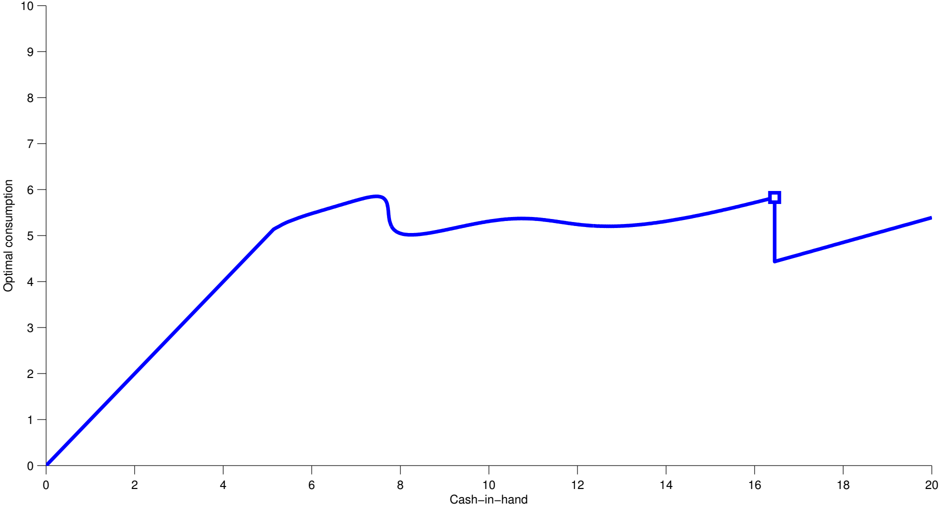
Period $T-3$: Optimal consumption with $\sigma = .2$



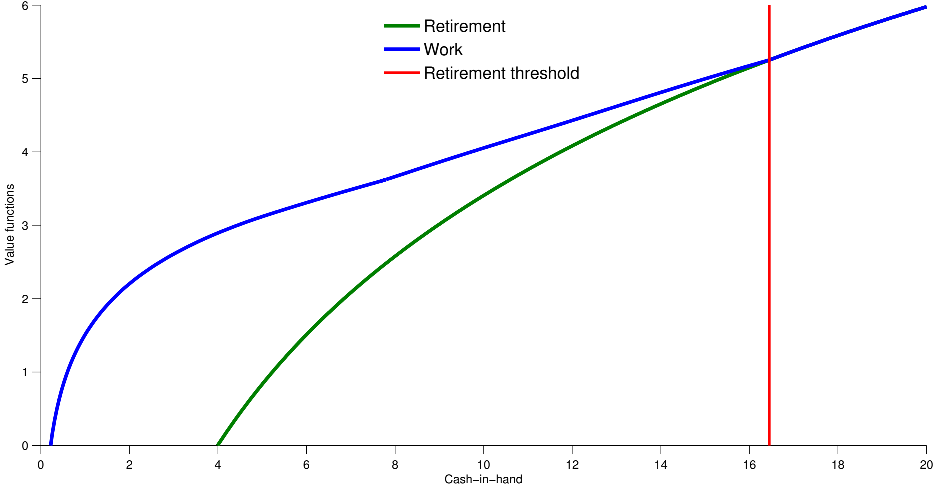
Period $T-3$: VF with $\sigma = .2$



Period $T-3$: Optimal consumption with $\sigma = .22$

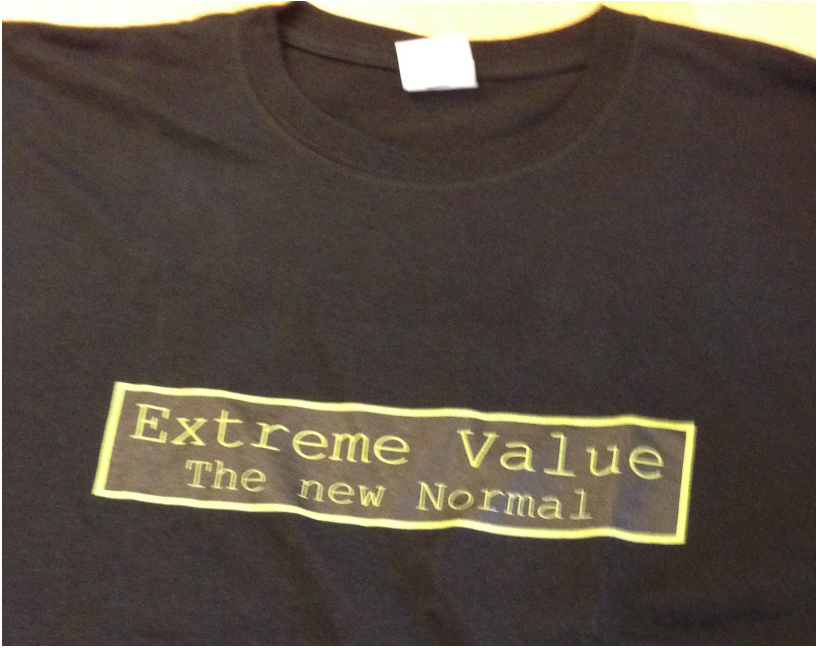


Period $T-3$: VF with $\sigma = .22$



Extreme value distributed taste shocks

- Smooth out primary kinks
- Extreme value distribution – closed form expectations and standard in empirical applications
- Two interchangeable interpretations
  - Structural: unobserved state variables
  - Logit smoothing: to streamline the solution
- Work together with other shocks in the model
  - EV taste shocks smooth out primary kinks
  - Random returns smooth out secondary kinks
- Complete smoothing is not guaranteed in general: secondary kinks may persist


Extreme Value
The new Normal

Retirement problem with taste shocks

Re-formulate in terms of choice specific value functions

$$V_t(M_t,\mathbb{W}) = \max \left\{ \begin{array}{l} v_t(M_t,\mathbb{W},\mathbb{R}) + \sigma\epsilon_{\mathbb{R}} \\ v_t(M_t,\mathbb{W},\mathbb{W}) + \sigma\epsilon_{\mathbb{W}} \end{array} \right\}$$

$$\begin{aligned} v_t(M_t,\mathbb{W},\mathbb{W}) &= \max_{0\leq c\leq M_t} \left[u(c) + \beta EV_{t+1}\left(\tilde{R}(M_t-c)+y,\mathbb{W}\right)\right] \\ v_t(M_t,\mathbb{W},\mathbb{R}) &= \max_{0\leq c\leq M_t} \left[u(c) + \beta EV_{t+1}\left(\tilde{R}(M_t-c),\mathbb{R}\right)\right] \end{aligned}$$

$$EV_{t+1}(x,\mathbb{W}) = \sigma \log\left[\exp\frac{v_{t+1}(x,\mathbb{W},\mathbb{W})}{\sigma} + \exp\frac{v_{t+1}(x,\mathbb{W},\mathbb{R})}{\sigma}\right]$$

$$V_t(M_t,\mathbb{R}) = \max_{0\leq c\leq M_t} \left[u(c) + \beta EV_{t+1}\left(\tilde{R}(M_t-c),\mathbb{R}\right)\right]$$

Smoothed Euler equation

Without taste shocks – "discontinuous" Euler equation:

$$u'(c_t) = \beta E\left[u'\left(c_{t+1}(\mathbb{W}/\mathbb{R})\right)\tilde{R}\right]$$

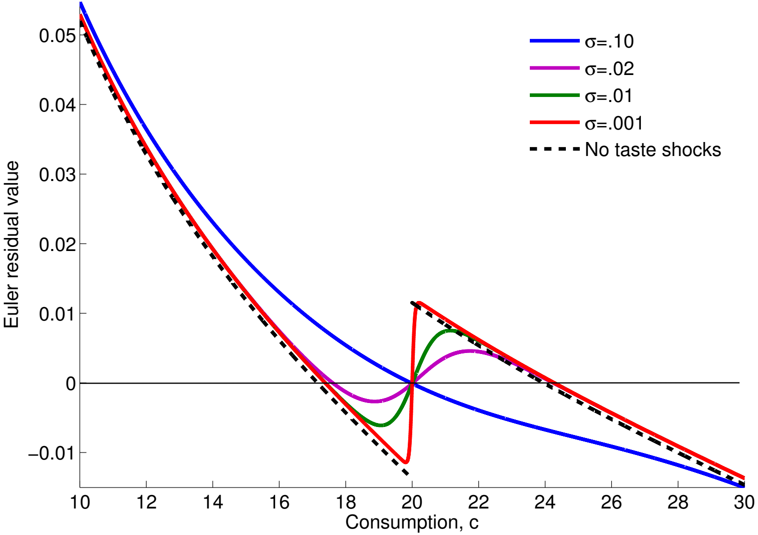
With EV taste shocks – smoothed Euler equation:

$$u'(c_t) = \beta E\left[P_{t+1}(\mathbb{W})u'\left(c_{t+1}(\mathbb{W})\right)\tilde{R} + P_{t+1}(\mathbb{R})u'\left(c_{t+1}(\mathbb{R})\right)\tilde{R}\right]$$

Choice probability

$$P_{t+1}(\mathbb{W}) = \frac{\exp\frac{v_{t+\mathbf{1}}(M_{t+\mathbf{1}},\mathbb{W},\mathbb{W})}{\sigma}}{\exp\frac{v_{t+\mathbf{1}}(M_{t+\mathbf{1}},\mathbb{W},\mathbb{W})}{\sigma} + \exp\frac{v_{t+\mathbf{1}}(M_{t+\mathbf{1}},\mathbb{W},\mathbb{R})}{\sigma}}$$

Smoothed Euler equation



DC-EGM with taste shocks

DC-EGM ver. 3.0

1. Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
2. EGM step for each discrete choice d and every state st
3. Compute d-specific value functions and consumption rules
4. Compute the "secondary" upper envelope over the "zig-zag" regions of the d-specific value functions and update the corresponding consumption rules
5. ~~Compare the d-specific value functions to find optimal switching points (compute upper envelope)~~
6. ~~Reconstruct overall consumption rule and value function from optimal switching points~~

DC-EGM with taste shocks

1. With EV taste shocks DC-EGM becomes simpler
2. The problem is re-formulated in terms of choice specific value functions
3. Calculation of *primary* upper envelope is replaced by calculation of logsum
4. Easier computation of expectations (due to less discontinuities)
5. More memory is required to store choice specific value functions

Extreme value Homotopy

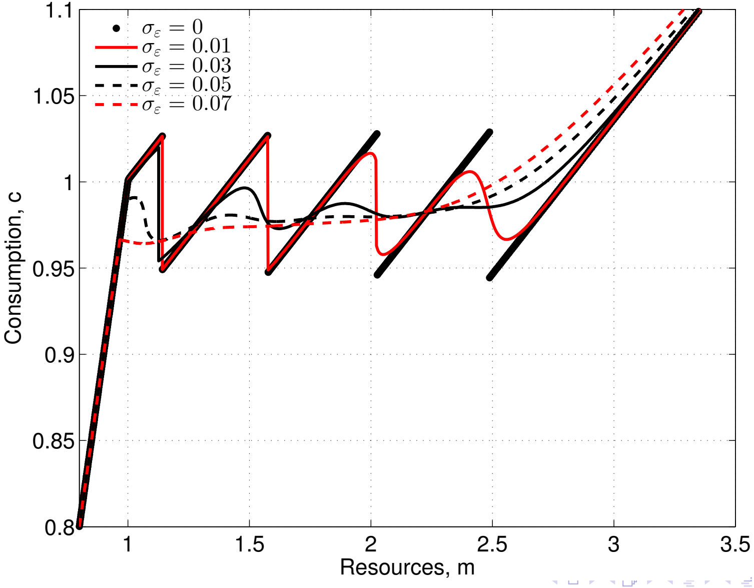
Theorem: approximation with logit smoother

Let σ be the scale of Type 1 extreme value taste shocks for the discrete choices in a DC problem with D choices. Then we have the following bound

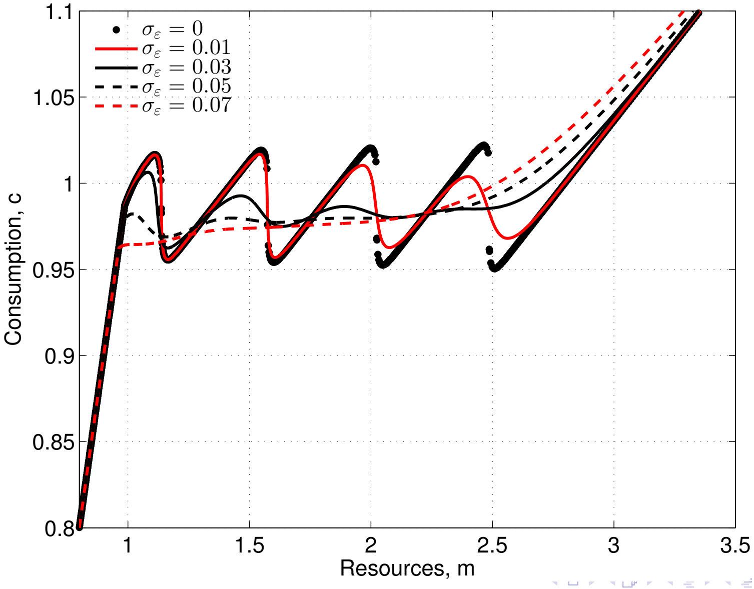
$$|EV_{\sigma,t}(s) - V_t(s)| \leq \sigma \left[\sum_{j=0}^{T-t} \beta^j \right] \log(D)$$

This implies that the extreme-value perturbed policy functions $c_{\sigma,t}(s,\epsilon)$ and $\delta_{\sigma,t}(s,\epsilon)$ converge pointwise to $c_t(s)$ and $\delta_t(s)$, the optimal continuous and discrete decision rules to a DP problem without any taste shocks as $\sigma \to 0$.

Optimal consumption with taste shocks only



Optimal consumption with random returns



Credit constraints

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Instead we "connect the dots" $(0, 0)$ and (M_t^{cc}, M_t^{cc})

  M_t^{cc} — level of wealth corresponding to $A_t = 0$

- Inevitable when value functions have to be computed
- If utility is additively separable in consumption and discrete choices (AS), the problem can be avoided entirely!

Credit constraints

Dealing with credit constraints

1. For each d_t compute M^{cc}_{t,d_t} correspond to zero savings
   EGM loop can be started from $A = 0$
   M^{cc}_{t,d_t} : $\forall M < M^{cc}_{t,d_t}$ $c^\star_t = M$
2. Value function for $M < M^{cc}_{t,d_t}$ has analytic form
   $V^{d_t}_t(M) = u(M, d_t) + \beta EV^0_{t+1}(d_t)$
   $EV^0_{t+1}(d_t)$ — expected value of ending period t with $A_t = 0$
3. (AS) $\Rightarrow V^{d_t}_t(M) = u(M) + v(d_t) + \beta EV^0_{t+1}(d_t)$
4. $V^{d_t}_t(M)$ do not intersect when $M < \min_{d_t}\left\{M^{cc}_{t,d_t}\right\}$
   $\Rightarrow$ No need to search for intersection points at nearly zero wealth
   $\Rightarrow$ Choice probabilities do not change

Credit constraints

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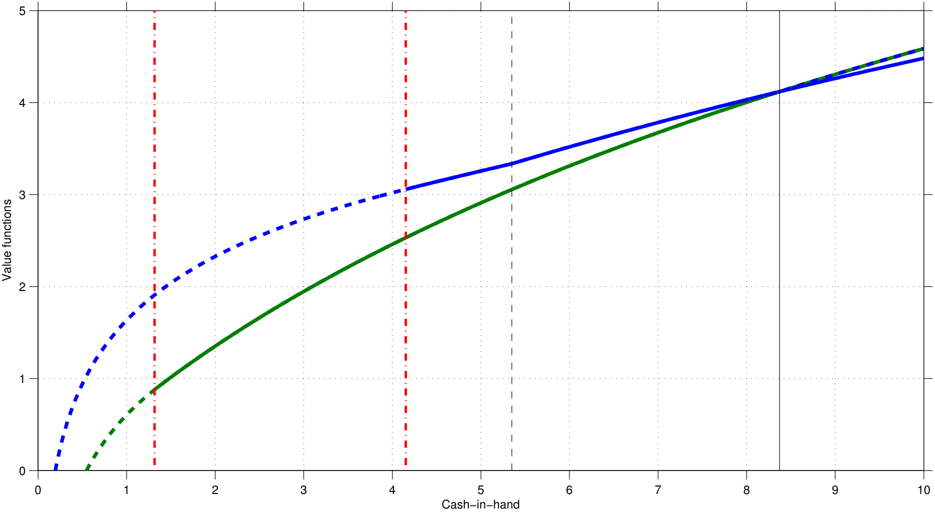
Pension benefit .25y



1. Solve the `model_retirement.m` with taste shocks
2. Simulate some consumption paths and distributions of retirement age

- See the `code/python` directory in the repository

Multi-dimensional generalizations

EGM + VFI

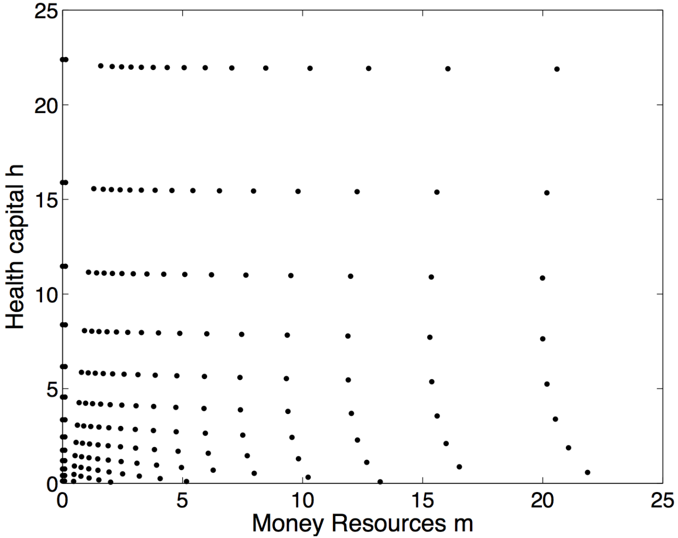
Barillas & Fernandez-Villaverde, JEDC 2007
A Generalization of the Endogenous Grid Method

1. Run EGM w.r.t. *one* choice keeping other controls fixed
2. Perform a VFI w.r.t. the rest of decision variables

Ludwig & Schön, Computational Economics, 2018
Endogenous Grids in Higher Dimensions: Delaunay Interpolation and Hybrid Methods

1. Solve the model of human capital investment + consumption/savings
2. Compare three approaches which differ by the interpolation method
3. Need to interpolate on irregular multidimensional grid

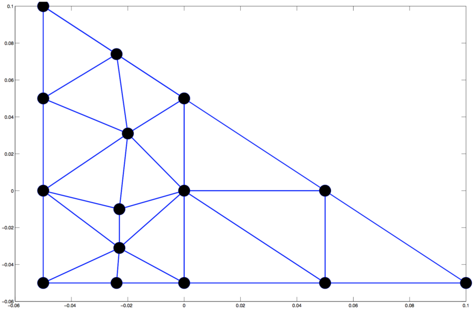
Multidimensional endogenous grid



Interpolation on the irregular grid

Johannes Brumm, Michael Grill, JEDC 2014
Computing equilibria in dynamic models with occasionally binding constraints

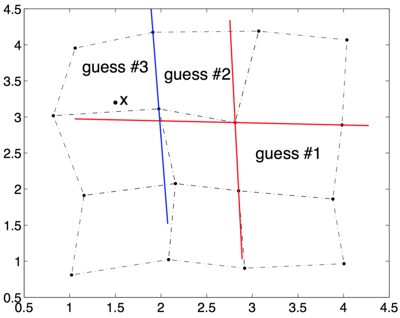
- Delaunay triangulation based interpolation



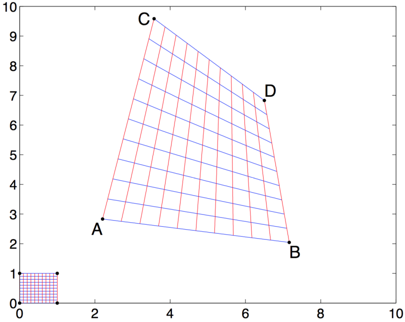
Interpolation on the irregular grid

Matthew White, JEDC 2015
The Method of Endogenous Gridpoints in Theory and Practice

- "Visibility walk" approach to allocate the non-linear rectangle containing the interpolation point
- Map non-linear rectangles into regular ones



(a) Identifying the sector by visibility walk

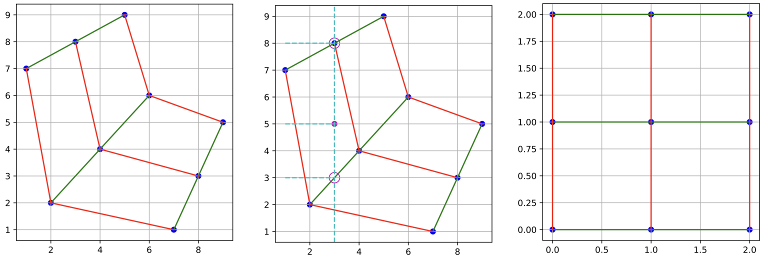


(b) Identifying relative coordinates

Interpolation on the irregular grid

Alan Lujan, 2024
EGM^n: The Sequential Endogenous Grid Method

- Improves on the interpolation scheme in White(2015)
- Interpolate using "index coordinate points" of the matrices of endogenous grids
- Multi-linear interpolation on non-linear grids afterwards

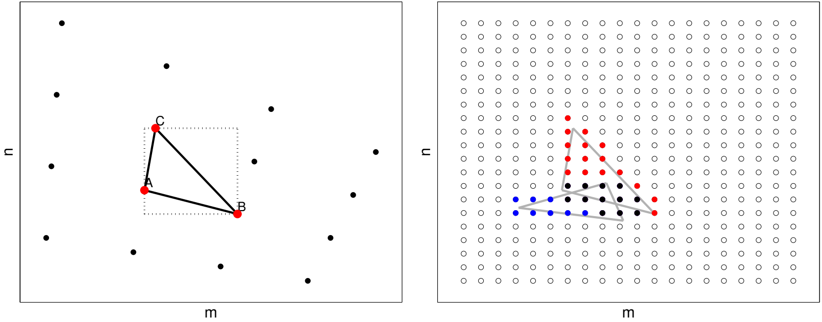


github.com/alanlujan91/multinterp

Interpolation on the irregular grid

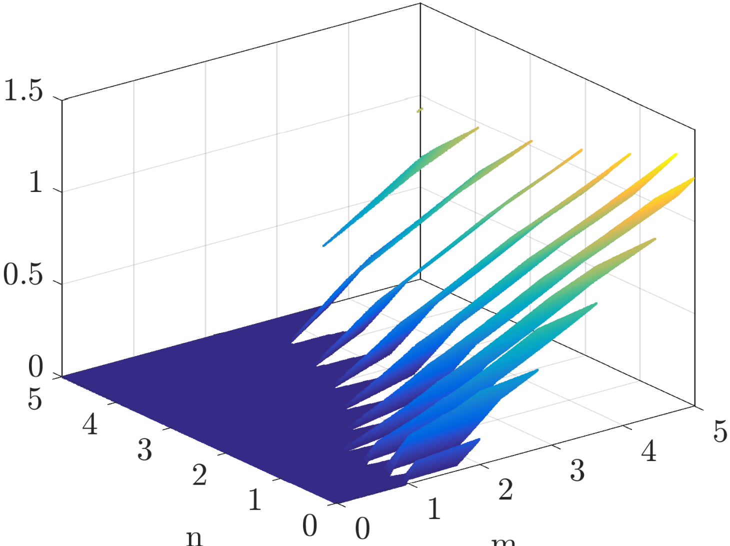
Jeppe Druedahl, Thomas Jørgensen, JEDC 2017
A General Endogenous Grid Method for Multi-Dimensional Models with Non-Convexities and Constraints

- Focus on occasionally binding constraints and for non-convexities
- Re-interpolate on regular grid while performing upper envelope



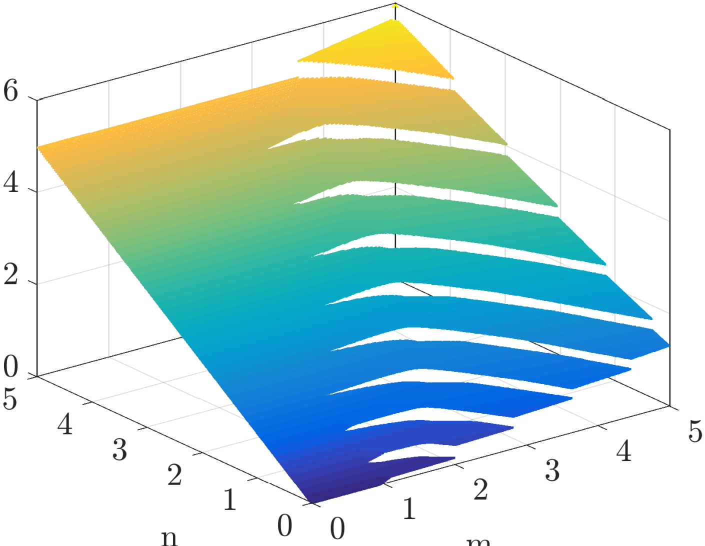
Consumption + pension contributions model

Pension fund contributions policy function



Consumption + pension contributions model

Next period pension wealth n



General theory on multidimensional EGM

Matthew White, JEDC 2015
The Method of Endogenous Gridpoints in Theory and Practice

- Invertibility condition for the system of non-linear equations

Jeppe Druedahl, Thomas Jørgensen, JEDC 2017
A General Endogenous Grid Method for Multi-Dimensional Models with Non-Convexities and Constraints

- Formulate the sufficient condition,
  i.e. particular mapping has to be an injection

Iskhakov, Econ Letters 2015
Multidimensional endogenous gridpoint method: solving triangular dynamic stochastic optimization problems without root-finding operations + Corrigendum

- Focus on analytical invertibility to avoid root-finding operations

Triangular example in $\mathbb{R}^3$

$$V_t(M_t) = \max_c \Big[u(c) + \beta E \big\{ V_{t+1}(M_{t+1}) \big| M_t, c \big\} \Big]$$

$$\begin{aligned}
M_t \in \mathbb{R}^3 \quad & \text{vector of resources at time } t \\
c_t \in \mathbb{R}^3 \quad & \text{utilization of resources} \\
A_t \in \mathbb{R}^3 \quad & \text{resources at the end of } t \text{ (post-decision states)} \\
A_t = g(M_t, c_t) \quad & \text{intra-temporal evolution} \\
M_{t+1} = f(A_t, \tilde{R}) \quad & \text{inter-temporal evolution (budget constraint)} \\
\tilde{R} \quad & \text{idiosyncratic stochastic shocks} \\
u : \mathbb{R}^3 \to \mathbb{R} \quad & \text{utility of current resource utilization}
\end{aligned}$$

Plan:

- Apply EGM ideas to discover the necessary assumptions on the fundamental of the problem that make it possible
- Abstract away from the budget constraint and other feasibility restrictions

Post-decision states

- Note, that we already assumed:

$$M_{t+1} = f\left(A_t, \tilde{R}\right) = f\left(g(M_t, c_t), \tilde{R}\right)$$

- allowing to replace the condition in the expectation in the Bellman equation to

$$E\left\{V_{t+1}(M_{t+1})\Big|M_t, c_t\right\} = E\left\{V_{t+1}(M_{t+1})\Big|A_t\right\}$$

- $\Leftrightarrow A_t$ are sufficient statistics for (M_t, c_t)

(TRI1) Assume that the vector of post-decition state variables follows

$$\begin{array}{rcccl} A_t^1 & = & g^1(M_t, c_t) & = & g^1(M_t^1, c_t^1) \\ A_t^2 & = & g^2(M_t, c_t) & = & g^2(M_t^1, c_t^1, M_t^2, c_t^2) \\ A_t^3 & = & g^3(M_t, c_t) & = & g^3(M_t^1, c_t^1, M_t^2, c_t^2, M_t^3, c_t^3) \end{array}$$

Bellman optimization and FOCs

$$V_t(M_t) = \max_c \left[u(c) + \beta E \left\{ V_{t+1} \left(\begin{array}{c} f^1(A_t^1, A_t^2, A_t^3, \tilde{R}) \\ f^2(A_t^1, A_t^2, A_t^3, \tilde{R}) \\ f^3(A_t^1, A_t^2, A_t^3, \tilde{R}) \end{array} \right) \middle| A_t \right\} \right]$$

The first order conditions are:

$$\begin{aligned}
u_1'(c) + \beta E \sum_{i=1}^{3} \sum_{j=1}^{3} (V_{t+1})_i' (f^i)_j' (\partial g^j / \partial c^1) &= 0 \\
u_2'(c) + \beta E \sum_{i=1}^{3} \sum_{j=2}^{3} (V_{t+1})_i' (f^i)_j' (\partial g^j / \partial c^2) &= 0 \\
u_3'(c) + \beta E \sum_{i=1}^{3} (V_{t+1})_i' (f^i)_3' (\partial g^3 / \partial c^3) &= 0
\end{aligned}$$

Envelope conditions

$$
\begin{aligned}
(V_t)'_1 &= \beta E \sum_{i=1}^{3} \sum_{j=1}^{3} (V_{t+1})'_i (f^i)'_j (\partial g^j / \partial M_t^1) \\
(V_t)'_2 &= \beta E \sum_{i=1}^{3} \sum_{j=2}^{3} (V_{t+1})'_i (f^i)'_j (\partial g^j / \partial M_t^2) \\
(V_t)'_3 &= \beta E \sum_{i=1}^{3} (V_{t+1})'_i (f^i)'_3 (\partial g^3 / \partial M_t^3)
\end{aligned}
$$

Combining these with the FOCs on the previous slide, we have

$$
\begin{aligned}
u'_3(c_t) &= -\frac{\partial g^3 / \partial c^3}{\partial g^3 / \partial M_t^3} (V_t)'_3 \\
u'_2(c_t) - u'_3(c_t) &= -\frac{\partial g^2 / \partial c^2}{\partial g^2 / \partial M_t^2} \left[(V_t)'_2 - (V_t)'_3 \right] \\
u'_1(c_t) - u'_2(c_t) &= -\frac{\partial g^1 / \partial c^1}{\partial g^1 / \partial M_t^1} \left[(V_t)'_1 - (V_t)'_2 \right]
\end{aligned}
$$

Euler equation

We have now expressed partial derivatives of the value function $(V_t)'_i$ as a linear functions of the marginal utilities $\mathcal{L}_t[u'(c_t)]$ with coefficients given by partial derivatives of $g^i(\bullet)$ functions

The system of Euler equations

$$
\begin{aligned}
u'_1(c_t^1, c_t^2, c_t^3) &= \beta E \sum_{i=1}^{3} \sum_{j=1}^{3} \mathcal{L}_{t+1}[u'(c_{t+1})](f^i)'_j(\partial g^j / \partial c^1) \\
u'_2(c_t^1, c_t^2, c_t^3) &= \beta E \sum_{i=1}^{3} \sum_{j=2}^{3} \mathcal{L}_{t+1}[u'(c_{t+1})](f^i)'_j(\partial g^j / \partial c^2) \\
u'_3(c_t^1, c_t^2, c_t^3) &= \beta E \sum_{i=1}^{3} \mathcal{L}_{t+1}[u'(c_{t+1})](f^i)'_3(\partial g^3 / \partial c^3)
\end{aligned}
$$

- interpolate $c_{t+1}(M_{t+1})$ from previous backward induction iteration
- RHS known, need to solve for c_t^1, c_t^2, c_t^3

Sufficient conditions for EGM to be applicable

(TRI2) Hessian of the utility function $u(c)$ is upper-triangular (up to permutations)
(*to allow for solution by back substitution*)

(CON) Utility function $u(c)$ is strictly concave
(*to allow for inversion of each Euler equation*)

(INV) Intra-temporal transition functions $g^i(M, c)$ have analytical inverses in the first argument $M = (g^i)^{-1}(A, c)$
(*to allow for constuction of endogenous grid*)

- Then the system of Euler equations can be solved by back substitution without root-finding operations by inversion
- Proof is constructive: follow the EGM step for a grid imposed over the post-decision state space

Conditions **TRI1**, **TRI2**, **CON**, **INV** are sufficient for the dynamic problem can be solvable (for interior solution) by the **Multidimensional EGM** algorithm

When are choices separable?

Alan Lujan, 2024
EGM^n: The Sequential Endogenous Grid Method

A fruitful idea coming out of the previous analysis:
Represent multiple dimensions of choice as a sequence of choices

$$\xrightarrow{\tilde{R}} \underbrace{c^1 \rightarrow c^2 \rightarrow c^3}_{\text{period } t} \xrightarrow{\tilde{R}} \underbrace{c^1 \rightarrow c^2 \rightarrow c^3}_{\text{period } t\text{+}1} \xrightarrow{\tilde{R}}$$

1. each choice is made subject to a pre-decision state M^i
2. post-decision states are given by $A^i = g^i(M^i, c^i)$ where g^i is a deterministic function
3. pre- and post-decision states are chained such that $M^{i+1} = A^i$
4. random shocks are realized in transition from t to $t+1$

Then **TRI1** is satisfied: multidimentional EGM may apply

What if the marginal utilities are not invertible analytically?

Hallengreen, Jørgensen, Olesen, 2024
The Endogenous Grid Method without Analytical Inverse Marginal Utility

A possible approach is:

1. Precompute marginal utilities on a fixed grid
2. Interpolate the function using any appropriate method
3. Use numerical inverse in the EGM step, effectively "swapping "

- The rest of the algorithm is unchanged
- Can be applied to both one- and multi-dimensional problems

Estimating life cycle models using endogenous gridpoint methods

What to do with EGM methods

We can solve many problems of this type $\Rightarrow$

1. Fast solver for important problems with discrete/continuous choice $\rightarrow$
   - calibration
   - structural estimation with your favourite method
   - NFXP: inner loop to solve the model, outer loop to optimize the objective function
2. Use the solver repeatedly in some "outer loop" $\rightarrow$
   - individual heterogeneity : solve the model for each individual in the sample
   - unobserved heterogeneity : random effects
   - flexibility of distributional assumptions

EGM vs. MPEC

Jørgensen, 2012 *Economics Letters*
Structural Estimation of Continuous Choice Models: Evaluating EGM and MPEC.

Deaton consumption/savings model in infinite horizon, MC experiment with ML on synthetic data, 1 structural parameter

β		RMSE	Time
.70	EGM	0.002	0.1 sec.
	MPEC	0.049	112.4 sec.
.95	EGM	0.006	1.9 sec.
	MPEC	0.009	93.7 sec.
.99	EGM	0.000	5.0 sec.
	MPEC	0.000	30.9 sec.

Points to take home

1. EGM and DC-EGM is fast and accurate solution methods
2. No root-finding operations in regular case
3. Efficient with credit constraint

4. Deterministic discrete-continuous problems are hard:
5. Kinks in value functions, discontinuous policy functions
6. Snowball effect in the accumulation of kinks over time

7. With EV taste shocks the problem is alleviated
8. EV taste shocks can be structural or added for smoothing
9. Facilitate estimation using discrete choice data